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A Nonstandard Proof of the Banach-Steinhaus Theorem

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Abstract. *The Banach-Steinhaus theorem, also known as Uniform Boundedness Principle, has a standard proof a little bit too long. In this article we will give a real short proof using the nonstandard analysis technique.*

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1. Introduction

The nonstandard analysis is a technique rather than a subject, it can be used in different subjects. Any result that can be proved with the nonstandard technique can also be proved with the standard technique, the difference might be the lenght, difficulty, accessibility, intuition, etc.

On 1961 Abraham Robinson showed that a nonstandard model of the formal theory of analysis provides the hyperreal numbers, furthermore, it provides a fundamental facts about infinitesimals in mathematics. He showed that the ideas and methods of this model clarified the notion of "infinitely small" and "infinitely big". What Robinson did was to exhibit a proper extension, $^*R$, of the real numbers system R with the same "formal" properties as R.

After Robinson's discovery, it was clear that the method could be applied with the same success to other mathematical structure, in analysis, topology, differential equations or algebra.

There are three important tools of the nonstandard analysis: The Transfer Principle (TP), The Concurrence Principle (CP) and the Internal Principle (IP).

The TP states that the same assertions of the formal language that are true in the standard universe, are true in the nonstandard universe.

The CP is a logical technique that guarantees that the nonstandard structure contains "what we need", completions, compactifications, cartesian products, etc.

The IP deals with certain collection of elements of the nonstandard structure, called internals, it includes the standard ones. A set S of the nonstandard universe is internal if S itself is an element of the nonstandard universe. A sentence expressed only with internal elements, in the nonstandard structure, can also be expressed in the standard structure and, both are true or both are false.

The reader, interested in nonstandard techniques and their applications, can take a look to any book from the references and will be surprised by the little he needs to learn this technique in order to apply it correctly.

2. Preliminaries

$E[t]$ will denote a topological vector space E (real or complex) with a Hausdorff locally convex topology t. $(E[t])'$ will denote the topological dual space of E, in other words, the vector space of all continuous funcionals on $E[t]$

DEFINITION 2.1. *For any subset* $S \subset E$ *and* $T \subset (E[t])'$

$$\begin{aligned} S^o &= \{y \in E' \mid |y(x)| \leq 1 \;\forall\, x \in S\} \\ T^o &= \{x \in E \mid |y(x)| \leq 1 \;\forall\, y \in T\} \end{aligned}$$

DEFINITION 2.2. ${}^*S$ *and* ${}^*T$ *will denote, respectively, the nonstandard structures of* S *and* T *inside the nonstandard superstructure, [1]*

NOTE 1. $S \subset {}^*S$. $S = {}^*S \Leftrightarrow S$ *is finite*

DEFINITION 2.3. ω *and* μ *will denote, respectively, the weak and the Mackey topologies on* E. σ *will denote the weak-star topology on* $(E[t])'$

NOTE 2. *$\omega < t < \mu$ are, repectively, the weakest and the strongest locally convex topologies with the same dual $(E[t])'$ as t. On the other hand, $(E[\sigma])'' = E$*

NOTE 3. *If $t_1 < t_2$ are two topologies on E, t_2-bounded $\Rightarrow$ t_1-bounded [2]*

3. Nonstandard Proof

THEOREM 3.1 (Banach-Steinhaus). *Every subset of E weakly bounded (ω-bounded) is bounded with respect to the Mackey topology (μ-bounded)*

Proof. Let $A \subset E$ be ω-bounded, in other words, for each $y \in E'$ there exists $r_y \in R^+$ such that $|y(x)| \leq r_y \; \forall x \in A$ ($A \subset \{\frac{1}{r_y} y\}^o$)

Let $K \subset E'$ be σ-compact

Claim: There exists $r \in R^+$ such that $A \subset (\frac{1}{r}K)^o$

Suppose that such r does not exist, i.e , for each $n \in N$ there is $x_n \in A$ such that $x_n \notin (\frac{1}{n}K)^o$

The following standard assertion is true

$$(\forall n \in N)(\exists x_n \in A)(\exists y_n \in K)(|y_n(x_n)| > n)$$

By the Transfer Principle [1], the following nonstandard assertion is true

$$(\forall n \in {}^*N)(\exists x_n \in {}^*A)(\exists y_n \in {}^*K)(|y_n(x_n)| > n) \tag{1}$$

Fix some $n \in {}^*N \setminus N$, hence,

$$r < n \; \forall r \in R^+ \tag{2}$$

By the σ-compactness of K there exists $z_n \in K$ such that the following internal sentence is true

$$(\forall x \in A)(|(y_n - z_n)(x)| \leq 1)$$

By the Transfer Principle, the following sentence is true

$$(\forall x \in {}^*A)(|(y_n - z_n)(x)| \leq 1) \tag{3}$$

Since A is ω-bounded there exists $r \in R^+$ such that the following sentence is true

$$(\forall\, x \in A)(|z_n(x)| \leq r)$$

By the Transfer Principle, the following nonstandard sentence is true

$$(\forall\, x \in {}^*A)(|z_n(x)| \leq r) \tag{4}$$

From (1), (3) and (4) we obtain

$$n < |y_n(x_n)| \leq |(y_n - z_n)(x_n)| + |z_n(x_n)| \leq 1 + r \in R^+$$

which contradicts (2).

So that, there exists $r \in R^+$ such that $A \subset (\frac{1}{r}K)^o$, in other words, the set A is bounded in E for the topology μ. □

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References

[1] M. Davis, *Applied nonstandard analysis*, John Wiley Publishing Co., U.S.A. (1977).

[2] J. Horvath, *Topological vector spaces and distributions*, Addison-Wesley Publishing Co., U.S.A. (1966).

[3] W.A.J. Luxemburg, *What is nonstandard analysis*, Foundations of Mathematics, Amer. Math. Monthly **80** (1973).

[4] A. Robinson, *Nonstandard analysis*, Studies in Logic and the Foundations of Mathematics, North-Holland, The Netherlands (1966).

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