Bounded variation solutions of capillarity-type equations

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Ph.D. Student: Sabrina Rivetti
Ph.D. Program Director: Prof. Vincenzo Armenio
Thesis Supervisor: Prof. Pierpaolo Omari

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To my father,
who is always with me,
who lives in me,
here and from the stars.

To Spotty
and
to all my beloved animals.
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I am sitting here in the middle of the last lines that complete a work that has been exciting, demanding and full of challenges, successes and falls. In this spirit my mind flies to the place where the most basic source of my life energy resides: my Family. Especially my Mum, who has always been able to pick up all my pieces and put them back together. Without my family’s strength and their love, this thesis would end up here.

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Abstract

We investigate by different techniques, the solvability of the capillarity-type problem

\[\begin{align*}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) &= f(x, u) \quad \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial\Omega,
\end{align*}\]

(1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function, \(n\) is the unit outer normal to \(\partial\Omega\) and \(\kappa : \partial\Omega \to \mathbb{R}\) is a bounded function. Since our approach is variational, the natural context where this problem has to be settled is the space \(BV(\Omega)\) of bounded variation functions. Solutions of (1) are defined as subcritical points of the action functional

\[\int_{\Omega} \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} \kappa v d\mathcal{H}_{N-1} - \int_{\Omega} F(x, v) dx.\]

More precisely, we say that \(u \in BV(\Omega)\) is a solution of (1) if

\[\int_{\Omega} \sqrt{1 + |Dv|^2} \geq \int_{\Omega} \sqrt{1 + |Du|^2} - \int_{\partial\Omega} \kappa(v-u) d\mathcal{H}_{N-1} + \int_{\Omega} f(x, u)(v-u) dx,\]

for all \(v \in BV(\Omega)\).

We first introduce a lower and upper solution method for problem (1) in the space of bounded variation functions. We prove the existence of solutions in the case where the lower solution is smaller than the upper solution. A solution, bracketed by the given lower and upper solutions, is obtained as a local minimizer of the associated functional without any assumption on the boundedness of the right-hand side \(f\). In this context we also prove order stability results for the minimum and the maximum solution lying between the given lower and upper solutions.

Next we develop an asymmetric version of the Poincaré inequality in the space of bounded variation functions. Namely, we single out in the plane a curve \(\mathcal{C} = \mathcal{C}(\Omega)\) made up of all pairs \((\mu, \nu)\) such that every \(u \in BV(\Omega)\), with

\[\mu \int_{\Omega} u^+ dx - \nu \int_{\Omega} u^- dx = 0,\]

also satisfies

\[\mu \int_{\Omega} u^+ dx + \nu \int_{\Omega} u^- dx \leq \int_{\Omega} |Du|.\]
Several properties of the curve $C$ are then derived and basically relying on these results, we discuss the solvability of the capillarity problem (1), assuming a suitable control on the interaction of the supremum and the infimum of the function $f$ with the curve $C$. Non-existence and multiplicity results are investigated as well.

The case of dimension $N = 1$, which sometimes presents a different behaviour, is also discussed. In particular, we provide an existence result which recovers the case of non-ordered lower and upper solutions, placing a control on $f$ with respect to the curve $C$. 
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Introduction

The aim of this work is to investigate the solvability of the capillarity-type problem

\[
\begin{align*}
-\text{div}\left(\nabla u/\sqrt{1 + |\nabla u|^2}\right) &= f(x, u) \quad \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\), \(f: \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function, \(n\) is the unit outer normal to \(\partial \Omega\) and \(\kappa: \partial \Omega \to \mathbb{R}\) is a bounded function.

The beginning of the long story of the capillarity equation is linked to the names of Young and Laplace who first introduced a mathematical formulation in the early years of the nineteenth century. In the seminal paper [111], Young presented the first conceptual derivation of the equation, simply based on the total balance of the forces acting on the free surface interface that separates a liquid and a gas bounded into a rigid surface. Let us think to the most common case of a glass tube with water and air. A few years later, in [76], Laplace gave a mathematically primitive formulation of the equation for the three dimensional surface parameterized by \((x, y, u(x, y))\). The equation derived following the theoretical deductions of Young was exactly

\[
\text{div}(Tu) = \lambda u + \eta,
\]

where \[
Tu = \left(\frac{u_x}{\sqrt{1 + |\nabla u|^2}}, \frac{u_y}{\sqrt{1 + |\nabla u|^2}}\right).
\]

Here \(\eta\) is a constant depending only on the volume constraint and \(\lambda\) is the originally called “capillarity constant”

\[
\lambda = \frac{\rho g}{\sigma}
\]

with \(\rho\) the density of the liquid, \(g\) the gravitational acceleration and \(\sigma\) the surface tension. For what concerns the boundary conditions at the contact line between the free surface and the support surface, it was envisioned that the contact angle \(\gamma\) satisfies the relation

\[
Tu \cdot n = \cos \gamma
\]

where \(\gamma\) depends only on the materials and anything else such as the shape of the wall, the external forces or other parameters. For the first structured and rigorous derivation
of the equation of capillarity, one has to wait until 1830. In [56], following the Principle of Virtual Work formulated by Bernoulli in 1717, Gauss described the configuration of the contact interface as a stationary point for the mechanical energy of the system.

Although at that time the thermodynamics and the modern hydrodynamics had not yet been invented, all the endless literature produced in the almost a century and a half following (see, e.g., [96], [47], [107], [108], [57], [101], [27], [28], [30], [12]) ends up agreeing with the initial model, derived by Young, Laplace and Gauss, that introduced the mathematical concept of mean curvature that now underlies the entire theory of capillarity. In [52], R. Finn in 2001 writes “[...] the original working hypothesis of discontinuous jump from one fluid to another at the free surface, as introduced by Young, Laplace and Gauss, appears to be justified on theoretical grounds, as it has been experimentally in almost two centuries of practical application under diverse conditions.”

Once stated that the study of capillarity-type problems is mathematically founded on the \( N \)-dimensional mean curvature operator

\[
\frac{1}{N} \text{div}\left( \nabla u / \sqrt{1 + |\nabla u|^2} \right),
\]

several authors investigated how this operator models all such a kind of phenomena in which a surface tension appears. According with [65], [110] and [51], the problem of a liquid drop resting on a horizontal plane, in a uniform gravitational field, is modeled by

\[
-\text{div}\left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) = \lambda u + \eta.
\]

Two cases arise in this context. If the horizontal plane is below the drop, we speak about a sessile drop and hence \( \lambda < 0 \) as discussed in [109] and [50]. Otherwise, if the horizontal plane is above the drop, we are in presence of a pendent drop that corresponds to \( \lambda > 0 \), as explained in [29] or [69].

From a pure mathematical point of view, the study of the equation of capillarity falls within the general problem of finding a function \( u \) whose graph has mean curvature which is prescribed by a given function \( f(x, u) \), that is \( u \) satisfies

\[
-\text{div}\left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) = f(x, u).
\]  

The literature concerning this equation is very huge: indeed it engaged mathematicians in the last 100 years, starting with the case when \( f = 0 \), which corresponds to the study of minimal surfaces.

With respect to problem (2), the established techniques can mainly be divided into two different currents. On the one hand, the techniques for nonlinear elliptic partial differential equations have been extensively applied to the study of surface of prescribed mean curvature, within the aim of finding classical solutions. Starting with [71], [99], [74], [70], [106], several works are devoted to derive gradient estimates and eventually to prove existence by using topological degree methods. In this context, due to the
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regularity required on the solutions, some strong conditions on the regularity of the data are needed. On the other hand, this problem was faced in a variational frame via measure-theoretic methods, in particular by De Giorgi [41], [42], [43] and his school [84], [85], [17]. The starting idea of De Giorgi was to define a hypersurface in $\mathbb{R}^N$ as the boundary of a measurable set $E$ whose characteristic function $\chi_E$ has distributional derivative that is a vector Radon measure of locally finite total variation. These sets are called Caccioppoli sets. In this formulation, the $(N-1)$-dimensional area is defined as the total variation of $D\chi_E$. By this approach, once existence is proven, a much more difficult task is to establish the regularity of the obtained hypersurfaces. The paper [44] paved the way to the $BV$-approach to the mean curvature equations and many authors carried on this perspective [86], [47], [57], [60], [54], [62], [64].

More recently, starting from [97], the mean curvature operator has also been introduced in order to describe flux limited diffusion phenomena. Indeed, it was observed that in realistic diffusion processes, characterized for small gradients by a linear dependence of the flux on gradients, the response of the flux to an increase of gradients is expected to slow down and ultimately to approach saturation. Accordingly, P. Roseanu and other authors proposed to replace the classical reaction diffusion equation

$$-\Delta u = f(x, u)$$

with equation (2), as in [73], [22] and [21]. It was also pointed out that, when the saturation of the diffusion is incorporated into these processes, it may cause a fundamental change in the morphology of the responses as they may exhibit discontinuities when the potential of $f$ exceeds a critical threshold.

Within this scenario, we present in this work some results concerning existence, non-existence, multiplicity and stability of $BV$-solutions for the problem

$$\begin{cases}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = f(x, u) + h(x) & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega,
\end{cases} \quad (3)$$

devoting special attention to the case where solutions are not global minimizer of a suitably defined action functional; however without facing the although relevant issue of the regularity of the obtained solutions.

The leading hypotheses, throughout this work, are the following

(h0) $\Omega$ is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial\Omega$;
(h1) $h \in L^p(\Omega)$, for some $p > N$, and $\kappa \in L^\infty(\partial\Omega)$;
(h2) there exists a constant $\rho > 0$ such that

$$\left| \int_B h \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq (1 - \rho) \int_\Omega |D\chi_B|$$

for every Caccioppoli set $B \subseteq \Omega$;
(h$_3$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous and, for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable; moreover, there exist constants $a > 0$ and $q \in [1, 1^*]$, with $1^* = \frac{N}{N-1}$ if $N \geq 2$, and $1^* = +\infty$ if $N = 1$, and a function $b \in L^p(\Omega)$, with $p > N$, such that

$$|f(x, s)| \leq a|s|^{q-1} + b(x)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

It is considerable to point out that condition (h$_2$) has been introduced in [61], where it was shown to be necessary for the existence of a solution $u \in C^2(\overline{\Omega})$ of the problem

$$\begin{cases}
-\text{div}\left(\nabla u / \sqrt{1 + |\nabla u|^2}\right) = h(x) & \text{in } \Omega, \\
-\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega.
\end{cases}$$

Actually a weaker condition than (h$_2$) is necessary for the existence of a solution in the setting of $BV$-functions, namely

$$\left| \int_B h \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq \int_{\Omega} |D\chi_B|$$

for every Caccioppoli set $B \subseteq \Omega$.

Formally (3) is the Euler-Lagrange equation of the functional

$$\mathcal{H}(v) = \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, dx - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}^{N-1} - \int_{\Omega} F(x, v) \, dx$$

where $F(x, s) = \int_0^s f(x, \xi) \, d\xi$. The functional $\mathcal{H}$ is well-defined in the space $W^{1,1}(\Omega)$. Yet this space, which could be a natural candidate where to settle the problem, is not a favourable framework to deal with critical point theory. Therefore, following the ideas of De Giorgi, Miranda, Giusti and other authors, we replace the space $W^{1,1}(\Omega)$ with the space $BV(\Omega)$ of bounded variation functions, i.e., we consider the relaxed functional $\mathcal{I} : BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{I}(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} \, dx - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}^{N-1} - \int_{\Omega} F(x, v) \, dx.$$

Here, for any $v \in BV(\Omega)$, the area functional $\int_{\Omega} \sqrt{1 + |Dv|^2}$ is defined by

$$\int_{\Omega} \sqrt{1 + |Dv|^2} = \sup \left\{ \int_{\Omega} \left( v \sum_{i=1}^{N} \frac{\partial w_i}{\partial x_i} + w_{N+1} \right) \, dx : w_i \in C^1_0(\Omega) \right\},$$

for $i = 1, 2, \ldots, N + 1$ and $\left\| \sum_{i=1}^{N+1} w_i \right\|_\infty \leq 1 \right\}$.
For convenience, we set

\[ J(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} - \int_{\Omega} hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}^{N-1}. \]

The functional \( J : BV(\Omega) \to \mathbb{R} \) is the sum of the area functional and of a linear term that takes into account of the non-homogeneity of the problem. This functional, by the continuity of the trace map from \( BV(\Omega) \) to \( L^1(\partial \Omega) \), is convex and Lipschitz-continuous, but not differentiable in \( BV(\Omega) \). Hence, the functional \( I \) is not differentiable in \( BV(\Omega) \) as well. However, as \( I \) is the sum of a convex term, coming from the mean curvature operator, and of a differentiable one, which is the potential associated to the problem, it is natural to define a critical point as the solution of a suitable variational inequality. Namely, we give the following:

**Definition of solution.** We say that a function \( u \in BV(\Omega) \) is a solution of problem (3), if \( u \) satisfies

\[ J(v) - J(u) \geq \int_{\Omega} f(x, u)(v - u) \, dx \quad (4) \]

for every \( v \in BV(\Omega) \).

To make more transparent this definition of solution, one can glimpse the usual variational approach noticing that \( u \) is a solution of (3), if and only if \( u \) is a minimizer in \( BV(\Omega) \) of the functional \( K_u : BV(\Omega) \to \mathbb{R} \) defined by

\[ K_u(v) = J(v) - \int_{\Omega} f(x, u)v \, dx. \]

Hence an alternative but equivalent perspective to look at this type of solution follows from [10] where it was pointed out that \( u \in BV(\Omega) \) is a minimizer of \( K_u \), if and only if

\[ \int_{\Omega} \frac{(Du)^s(D\phi)^s}{\sqrt{1 + |(Du)^s|^2}} \, dx + \int_{\Omega} S \left( \frac{Du}{|Du|} \right) \frac{D\phi}{|D\phi|} |D\phi|^s \]

\[ = \int_{\Omega} (f(x, u) + h)\phi \, dx - \int_{\partial \Omega} \kappa \phi \, d\mathcal{H}^{N-1} \]

holds for every \( \phi \in BV(\Omega) \) such that \( |D\phi|^s \) is absolutely continuous with respect to \( |Du|^s \). Here, \( \mu^a \, dx + \mu^s \) denotes the Lebesgue decomposition of a Radon measure \( \mu \) in its absolutely continuous part \( \mu^a \), and its singular part \( \mu^s \) with respect to the \( N \)-dimensional Lebesgue measure in \( \mathbb{R}^N \). Moreover, \( S \) is the projection onto the unit sphere of \( \mathbb{R}^N \), i.e., \( S(\xi) = |\xi|^{-1}\xi \) if \( \xi \in \mathbb{R}^N \setminus \{0\} \) and \( S(\xi) = 0 \) if \( \xi = 0 \).

Finally, we point out that we cannot in general expect that bounded variation solutions of (3) are more regular; indeed, even simple one-dimensional examples can be constructed possessing only discontinuous solutions. For a discussion of this matter, we refer, e.g., to [47], [86], [58], [61], [53], [93] and [92].

Keeping in mind the basic physical distinction that lies under the different models described above, we perform the study of problem (3), by focusing on the two different
behaviours of the right-hand side $f$, which roughly correspond to the case of a sessile drop, when the right-hand side is decreasing, and to the case of a pendent drop, when the right-hand side is increasing.

In the former case it is natural to implement a method of lower and upper solutions, inspired to what have been done in [79], [90] and [93]. To do this, we introduce two notions, of increasing generality, of lower and upper solutions for problem (3) starting from weaker lower and upper solutions and arriving to the more general definition of $BV$-lower and $BV$-upper solution. In accordance with the variational inequality that defines the $BV$-solution, we have the following:

**Definition of $BV$-lower and $BV$-upper solutions.** We say that a function $\alpha \in BV(\Omega)$ is a $BV$-lower solution of problem (3) if

$$J(\alpha + z) - J(\alpha) \geq \int_{\Omega} f(x, \alpha) z \, dx$$

for all $z \in BV(\Omega)$ with $z \leq 0$. The same relation, for $z \in BV(\Omega)$ with $z \geq 0$, defines a $BV$-upper solution.

Once investigated the expected relation between the classical and the $BV$-formulation of lower and upper solutions, we prove an existence results for problem (3) in the case of well-ordered $BV$-lower and $BV$-upper solutions. As usual, a solution bracketed between the given lower and upper solutions is obtained by minimizing a suitably modified functional. In this situation, we are also able to produce results about the compactness of the sets of such a kind of solutions. In particular we prove the existence of a minimum solution $v$ and of a maximum solution $w$ lying between $\alpha$ and $\beta$.

Recall that the cases faced with the lower and upper solutions method, are linked with the model for the sessile drop, that actually resembles, in some physical meaning, a stable profile. From this heuristic observation, it comes the idea of detecting certain stability properties of solutions of problem (3) by the use of lower and upper solutions. In particular we prove the order stability, as defined in [68], of the minimum and of maximum solution lying between a pair of lower and upper solutions $\alpha$ and $\beta$ satisfying $\alpha \leq \beta$. Namely, starting from a lower solution we define recursively an increasing sequence of lower solutions that converges to the minimal solution of the problem. Similarly, starting from an upper solution we define recursively a decreasing sequence of upper solutions that converges to the maximal solution of the problem. It is also worth noting that our stability conclusions are obtained without assuming any additional regularity condition, like, e.g., Lipschitz continuity, on $f$, as it is usually done in other cases in order to associate with the considered problem an order preserving operator (see, e.g., [2], [68]).

In order to illustrate these results, we consider few sample applications. A mathematically meaningful example (a sort of paradigm for nonlinear analysis) is the capil-
larity equation with a periodic right-hand side, like
\[
\begin{aligned}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) &= A \sin u + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n \sqrt{1 + |\nabla u|^2} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

An existence result via well-ordered lower and upper solutions can be easily obtained here, provided that \( A \geq \|h\|_\infty \). In this case the constants \( \pi/2 \) and \( 3\pi/2 \) play the role of lower and upper solutions, respectively. Actually, the existence of a couple of non-well-ordered lower and upper solutions, e.g., \( \pi/2 \) and \( -\pi/2 \), suggests that a second solution, not differing from the other by an integer multiple of \( 2\pi \), should exist, like in the semilinear case. Unfortunately, at the moment there is no result that allows to face this situation for the prescribed mean curvature equation in arbitrary dimension. Nevertheless, a variational argument, based on a three-solutions theorem, allows to get the existence of two different solutions for any \( A \in \mathbb{R} \), under the condition \( \int_\Omega h \, dx = 0 \).

We start from two solutions which differ by \( 2\pi \) and we proceed as in the proof of the well-ordered existence theorem, the solutions playing the role of well-ordered lower and upper solution. The modified functional we get, is coercive and bounded from below and the two solutions are two minimizers at the same critical level (actually global minimizers). Then a third solution, with a different critical value, is found by using a non-smooth version of the classical mountain pass lemma. This version of the mountain pass lemma, we adapt here to the \( BV \)-setting, takes inspiration from \([83], [72], [78], [91]\) and it does not requires the validity of the Palais-Smale condition in \( BV(\Omega) \).

This result will be the main tool for discussing the more challenging circumstance when the right-hand side of the equation may be increasing. As already noticed, this is related to the modelling of the pendent drop. In this frame minimization techniques do not generally work, as the functional \( I \) may be unbounded from above and from below. It is worthy to point out that the study of this situation for problem (3) in a sufficiently wide generality has been faced only in the last decade. We will restrict ourselves to the case where the right-hand side \( f \) is a bounded function, but the positive and the negative part may have significantly different sizes. In this context an asymmetric version of the Poincaré inequality will be one of the main actors.

It is well-known that the classical Poincaré inequality in \( BV(\Omega) \) ensures the existence of a constant \( c = c(\Omega) > 0 \) such that every \( u \in BV(\Omega) \), with \( \int_\Omega u \, dx = 0 \), satisfies
\[
c \int_\Omega |u| \, dx \leq \int_\Omega |Du|.
\] (5)
The largest constant \( c = c(\Omega) \) for which (5) holds is called the Poincaré constant and is variationally characterized by
\[
c = \min \left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v \, dx = 0, \int_\Omega |v| \, dx = 1 \right\}.
\]
Clearly, any minimizer yields the equality in (5). Aimed to the study of problem (3), instead of the total variation we consider the more general functional \( \mathcal{L} : BV(\Omega) \to \mathbb{R} \),
defined as

\[ \mathcal{L}(v) = \int_{\Omega} |Dv| - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, dH_{N-1} \]

for every \( v \in BV(\Omega) \) and we prove an asymmetric counterpart of the Poincaré inequality (5), where \( u^+ \) and \( u^- \) weigh differently, i.e., the ratio \( r = \frac{\int_{\Omega} u^+ \, dx}{\int_{\Omega} u^- \, dx} \) is not necessarily 1. We shall remark that the study of the Poincaré and of the Sobolev-Poincaré inequalities, as well as of theirs variants and generalizations, is relevant by itself. Indeed this is a very classical topic in functional analysis and it is still a field of very active research in various different directions as one can see in [20], [14], [18], [26] and recently [102].

Namely, inspired from [39] and [31], we show that for each \( r > 0 \) there exist constants \( \mu = \mu(r, \Omega) > 0 \) and \( \nu = \nu(r, \Omega) > 0 \), with \( \nu = r\mu \), such that every \( u \in BV(\Omega) \), with

\[ \mu \int_{\Omega} u^+ \, dx - \nu \int_{\Omega} u^- \, dx = 0, \]

satisfies

\[ \mu \int_{\Omega} u^+ \, dx + \nu \int_{\Omega} u^- \, dx \leq \mathcal{L}(u). \] (6)

The constants \( \mu \) and \( \nu \) are variationally characterized by

\[ \mu = \min \left\{ \mathcal{L}(v) : v \in BV(\Omega), \int_{\Omega} v^+ \, dx - r \int_{\Omega} v^- \, dx = 0, \int_{\Omega} v^+ \, dx + r \int_{\Omega} v^- \, dx = 1 \right\} \] (7)

and

\[ \nu = \min \left\{ \mathcal{L}(v) : v \in BV(\Omega), r^{-1} \int_{\Omega} v^+ \, dx - \int_{\Omega} v^- \, dx = 0, r^{-1} \int_{\Omega} v^+ \, dx + \int_{\Omega} v^- \, dx = 1 \right\}, \] (8)

respectively. Clearly, any minimizer in (7), or (8), yields the equality in (6).

This construction allows us to single out in the plane a curve \( C = C(\Omega) \) made up of all pairs \((\mu, \nu) = (\mu(r, \Omega), \nu(r, \Omega))\) defined by (7) and (8), by letting \( r \) vary in \( \mathbb{R}^+_0 \). The way as the curve \( C \) is defined resembles a similar method used in [32], [39] in order to get a variational characterization of the first non-trivial branch of the Fučík spectrum of the Laplacian with either periodic, or Dirichlet, or Neumann boundary conditions.

From formulas (7) and (8) several properties of the curve \( C \) can be derived:

(i) \( C \) is symmetric with respect to the diagonal if \( h = 0 \) and \( \kappa = 0 \);

(ii) \( C \) is continuous, as both functions \( \mu(r) \) and \( \nu(r) \) are continuous;

(iii) \( C \) is strictly decreasing, in the sense that \( \mu(r) \) is strictly decreasing and \( \nu(r) \) is strictly increasing;
(iv) if $N \geq 2$, $C$ is asymptotic to the lines $\mu = 0$ and $\nu = 0$, that is $\lim_{r \to 0^+} \mu(r) = +\infty$ and $\lim_{r \to 0^+} \nu(r) = 0$;

(v) if $N = 1$, $C$ is asymptotic to the lines $\mu = \bar{\mu} > 0$ and $\nu = \bar{\nu} > 0$, that is $\lim_{r \to 0^+} \mu(r) = \bar{\mu}$ and $\lim_{r \to 0^+} \nu(r) = \bar{\nu}$.

The discrepancy occurring in the asymptotic behaviour of $C$ between the case $N = 1$ and $N \geq 2$ is due to the existence in higher dimension of functions having arbitrarily large oscillation and arbitrarily small variation.

Figure 1: Qualitative behaviour of curve $C$.

Here we stress how the discussion of the one-dimensional can be deeper and more complete. Indeed, the specific properties of the 1-dimensional bounded variation functions, make possible to state a much more complete characterization for the curve $C$ under the assumption $h = 0$ and $\kappa = 0$, where $N = 1$. We have

$$C = \left\{ (\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T} \right\}$$

and for any fixed $(\mu, \nu) \in C$, every $v \in BV(0, T)$, such that

$$\mu \int_0^T v^+ \, dx - \nu \int_0^T v^- \, dx = 0,$$

also satisfies

$$\mu \int_0^T v^+ \, dx + \nu \int_0^T v^- \, dx \leq \int_{[0,T]} |Dv|.$$  

Moreover, as the curve $C$ is characterized by an equation, we provide a description of the related eigenfunctions. We prove that the equality in (9) is attained if and only if
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$v$ is a positive multiple either of $\varphi$, or of $\varphi(T - \cdot)$, with

$$\varphi(x) = \begin{cases} \frac{1}{2} \frac{\sqrt{\mu} + \sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} & \text{if } 0 < x < \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T, \\ \frac{1}{2} \frac{\sqrt{\mu} + \sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} & \text{if } \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T \leq x < T. \end{cases}$$

Our analysis here makes use of rearrangement techniques.

Finally, we point out that the properties of the curve $C$ resemble those of the first non-trivial curve of the Fučík spectrum of the $p$-Laplace operator, with $p > 1$, as described in [33], [32], [39] and [31].

Once a complete description of $C$ is established, taking into account the crucial conclusions (iv) and (v), we are in condition of discussing the solvability of the capillarity problem (3) for bounded $f$. To clarify the underlying theme of our ideas, let us consider for a while the problem where the function $f$, at the right-hand side of the equation in (3), does not depend on $u$, that is

$$\begin{cases} -\text{div} \left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) = f(x) & \text{in } \Omega, \\ -\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = 0 & \text{on } \partial\Omega \end{cases}$$

(10)

with $f \in L^\infty(\Omega)$ given. It is easy to see that (10) may have a solution only if

$$\int_\Omega f \, dx = 0.$$

A simple minimization argument based on the classical Poincaré inequality (5) shows that, assuming $\int_\Omega f \, dx = 0$, (10) has a solution if

$$\|f\|_\infty < c$$

and it may have no solution if

$$\|f\|_\infty > c.$$

If we write $f = f^+ - f^-$, then the condition $\int_\Omega f \, dx = 0$ reads

$$\int_\Omega f^+ \, dx - \int_\Omega f^- \, dx = 0$$

and the condition $\|f\|_\infty < c$ can be expressed by requiring that both

$$\text{ess sup}_\Omega f^+ < c \quad \text{and} \quad \text{ess sup}_\Omega f^- < c.$$

Looking at elementary one-dimensional examples, where $f$ is a piecewise constant function, one is led to guess that the existence of a solution of (10) can still be guaranteed
even if \( \text{ess sup}_\Omega f^+ \) is large, provided that \( \text{ess sup}_\Omega f^- \) is sufficiently small, allowing in this way asymmetric perturbations \( f \).

Indeed, consider the curve \( C \) in the case \( h = 0 = \kappa \) and let us denote by \( \mathcal{A} \) the component of \((\mathbb{R}_0^+ \times \mathbb{R}_0^+) \setminus C\) lying “below” \( C \) and by \( \mathcal{B} \) the component lying “above” \( C \). Then, basically relying on the asymmetric Poincaré inequality (6), we prove that, assuming \( \int_\Omega f \, dx = 0 \) and setting

\[
\text{ess sup}_\Omega f^+ = \mu \quad \text{and} \quad \text{ess sup}_\Omega f^- = \nu,
\]

problem (10) has a solution if \( (\mu, \nu) \in \mathcal{A} \) and it may have no solution if \( (\mu, \nu) \in \mathcal{B} \).

Figure 2: Existence vs Non-Existence.

We first establish various technical tools that will be extensively used in the sequel. In particular, we prove some coercivity results for the functional \( \mathcal{L} \) on suitable cones of \( BV(\Omega) \). Second, we prove some simple non-existence results, which will justify the assumptions we are going to place later on the function \( f \) in order to achieve the solvability of problem (3). In particular, we show that there exist functions \( e \in L^\infty(\Omega) \), with \( \int_\Omega e \, dx = 0 \), and \( g : \mathbb{R} \to \mathbb{R} \) continuous, bounded and strictly monotone, with

\[
\lim_{s \to -\infty} g(s) < 0 < \lim_{s \to +\infty} g(s)
\]

or

\[
\lim_{s \to -\infty} g(s) > 0 > \lim_{s \to +\infty} g(s),
\]

such that problem

\[
\begin{cases}
-\text{div}\left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = g(u) + c(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = 0 \quad \text{on } \partial\Omega
\end{cases}
\]
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has no solution. This means that results in the spirit of [75], [1] do not carry over, as they stand, to this context.

Keeping in mind these facts we perform the study of the more general problem (3), assuming suitable conditions like

$$\lim_{s \to \pm \infty} \int_{\Omega} F(x, s) \, dx = +\infty \quad (11)$$

or

$$\lim_{s \to \pm \infty} \int_{\Omega} F(x, s) \, dx = -\infty. \quad (12)$$

In the frame of semilinear problems these assumptions are usually referred to as Ahmad-Lazer-Paul conditions after the seminal paper [1]. We couple these assumptions with the asymmetric two-sided restriction

$$\text{ess sup}_{\Omega \times \mathbb{R}} f^+(x, s) < \mu \quad \text{and} \quad \text{ess sup}_{\Omega \times \mathbb{R}} f^-(x, s) < \nu \quad (13)$$

for some $$(\mu, \nu) \in C$$. Assumptions (12) and (13) imply the boundedness from below and the coercivity of the associated functional; whereas assumptions (11) and (13) yield a mountain pass geometry. Then the already cited non-smooth version of the classical mountain pass lemma ensures the existence of a solution of problem (3). This technique has been inspired by [83], [72], [78] and [91].

It is worthwhile to observe that the above cited non-existence results show that condition (12), which requires that $$f$$ lies in a suitable sense to the left of the first eigenvalue $$\lambda_1 = 0$$ of the 1-Laplace operator with Neumann boundary conditions (see, e.g., [25]), and the boundedness of $$f$$, with bounds unrelated to $$C$$, do not guarantee solvability.

Therefore it may have some interest to find conditions which allow to drop assumption (13): to achieve this, integral conditions should be replaced by pointwise conditions. Indeed, in order to get rid of condition (13), we replace the Ahmad-Lazer-Paul condition (12) with the following stronger Hammerstein-type condition (cf. [67] and [86]): there exists $$\zeta \in L^1(\Omega)$$, with $$\zeta(x) \geq 0$$ for a.e. $$x \in \Omega$$ and $$\zeta(x) > 0$$ on a set of positive measure, such that

$$\liminf_{s \to \pm \infty} \frac{F(x, s)}{|s|} \geq \zeta(x) \quad \text{uniformly a.e. in } \Omega.$$

Then we show that this condition yields the existence of a solution of (3), without any further assumption on $$f$$. We just notice that the Ahmad-Lazer-Paul condition (11) is implied by a Hammerstein-type condition assumed to the right of $$\lambda_1$$: there exists $$\zeta \in L^1(\Omega)$$, with $$\zeta(x) \geq 0$$ for a.e. $$x \in \Omega$$ and $$\zeta(x) > 0$$ on a set of positive measure, such that

$$\liminf_{s \to \pm \infty} \frac{F(x, s)}{|s|} \geq \zeta(x) \quad \text{uniformly a.e. in } \Omega.$$

However, in this case, assumption (13) cannot be dropped.

xv
Finally, we also study the existence of multiple solutions of problem (3), when \( \int_{\Omega} F(x, s) \, dx \) exhibits an oscillatory behaviour. In particular, we show that infinitely many solutions exist assuming, in addition to (13), the conditions
\[
\liminf_{s \to \pm \infty} \int_{\Omega} F(x, s) \, dx = -\infty \quad \text{and} \quad \limsup_{s \to \pm \infty} \int_{\Omega} F(x, s) \, dx = +\infty.
\]
We point out that multiplicity results, under oscillatory conditions on the potential \( F \), have been already considered for other boundary value problems associated with the equation
\[-\Delta_p u = f(x, u),\]
where \( \Delta_p \) is the \( p \)-Laplace operator, with \( p > 1 \) (see, e.g., [49], [94], [66], [88], [89] and the references therein).

In accordance to what was already highlighted, we conclude our work with a brief chapter devoted to some additional results that can be obtained in the one-dimensional case. Firstly, we perform a brief discussion of some explicit conditions which yield the existence of lower and upper solutions, assuming suitable condition on \( f \).

As mentioned before, in this setting we are able to prove the existence of a solution if \( \alpha \leq \beta \) by putting a control on \( f \) with respect to the curve \( \mathcal{C} \) as we did in [93] for the periodic problem. The approach of the proof is perturbative: the solution of problem (3) is obtained as limit in \( BV(0, T) \) of solutions of an approximating sequence of regularized problems. In this context a stronger notion of lower and upper solutions is needed and no localization information is obtained. It remains an open question to prove this result by a more direct method, which could probably allow to overcome such limitations. Nevertheless, these existence results yield rather general and flexible tools to investigate the solvability of the 1-dimensional capillarity problem.

Finally we show how in dimension \( N = 1 \), under hypotheses of Ahmad-Lazer-Paul type, the two-sided condition on \( f \) can be replaced by the one-sided conditions, i.e.
\[
\text{ess sup}_{[0, T] \times \mathbb{R}} f^+(t, s) < \frac{1}{2T} \quad \text{or} \quad \text{ess sup}_{[0, T] \times \mathbb{R}} f^-(t, s) < \frac{1}{2T}.
\]
The proofs are essentially the same but here it is allowed \( f \) to be unbounded from either from below or from above, respectively.

**Work in progress.** We have recently started the study of the evolutionary counterpart of the capillarity equation with the aim of exploring more deeply the stability properties of the stationary problem for which we already proved some results concerning the order stability. The evolutionary mean curvature equation is a challenging nonlinear parabolic problem which belongs to the class of degenerate equations studied, e.g., in [8], [5], [6]. In the case of linear potential, the existence of evolutionary surfaces with prescribed mean curvature was firstly faced in [80] in the framework of Sobolev spaces. Subsequently, the case of a convex potential has been performed in [59] and [46] for bounded variation solutions. It is worthy to point out that, especially with regard
to the $BV$-setting, this problem has not yet been explored exhaustively as highlighted by the very recent work [15].

By now we have been able to produce just an existence result for the initial value problem in presence of a couple of well-ordered lower and upper solutions: the proof is presented in Appendix A.
Notations

As usual we set

\[ \mathbb{R}^+_0 \] the open interval \([0, +\infty[\),

\[ \mathbb{R}^+ \] the closed interval \([0, +\infty[\),

\[ u \wedge v = \min\{u, v\} \] the pointwise minimum of the real functions \(u\) and \(v\),

\[ u \vee v = \max\{u, v\} \] the pointwise maximum of the real functions \(u\) and \(v\),

\[ u^+ = u \vee 0 \] the positive part of the function \(u\),

\[ u^- = -(u \wedge 0) \] the negative part of the function \(u\),

\[ u \ast v \] the convolution of \(u\) and \(v\),

\[ \|u\|_\infty \] the \(L^\infty\)-norm of \(u\),

\[ \mathcal{H}_k \] the \(k\)-dimensional Hausdorff measure, with \(k \in \mathbb{N}\),

\[ 1^* = \frac{N}{N-1} \] if \(N \geq 2\),

\[ 1^* = +\infty \] if \(N = 1\).

If \(E(\subseteq \mathbb{R}^N)\) is measurable, we set

\[ |E| \] the \(N\)-dimensional measure of \(E\),

\[ \chi_E \] the characteristic function of set \(E\).

If \(\Omega(\subseteq \mathbb{R}^N)\) is an open set and \(E(\subseteq \Omega)\) is a Caccioppoli set, we define

\[ \operatorname{Per}(E) \] the perimeter of \(E\) in \(\Omega\), defined as \(\operatorname{Per}(E) = \int_\Omega |D\chi_E|\).
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If \( v, u : E(\subseteq \mathbb{R}^N) \rightarrow \mathbb{R} \), we denote

- \( \{v > 0\} \) the set \( \{x \in E : v(x) > 0\} \),
- \( \{v \geq 0\} \) the set \( \{x \in E : v(x) \geq 0\} \),
- \( \{v = 0\} \) the set \( \{x \in E : v(x) = 0\} \),
- \( u \leq v \) if \( u(x) \leq v(x) \) for a.e. \( x \in E \),
- \( u < v \) if \( u \leq v \) and \( u(x) < v(x) \) in a subset of \( E \) having positive measure.

For any \( v \in BV(\Omega) \), we set

\[
Dv = (Dv)^a dx + (Dv)^s
\]
is the Lebesgue decomposition of the measure \( Dv \),

- \( (Dv)^a dx \) the absolutely continuous part,
- \( (Dv)^s \) the singular part with respect to the \( N \)-dimensional Lebesgue measure in \( \mathbb{R}^N \),
- \( |Dv| \) the absolute variation of the measure \( Dv \),
- \( \frac{Dv}{|Dv|} \) the density function of \( Dv \) with respect to its absolute variation \( |Dv| \).
Chapter I

Preliminaries

I.1 General results for BV-functions

This first section is devoted to recall some general definitions and properties of bounded variation functions. For a complete discussion we refer to [3],[63] and [112].

Functions of bounded variation

**Definition of \( BV(\Omega) \).** Let \( \Omega \subset \mathbb{R}^N \) be an open set. We say that \( u \in L^1(\Omega) \) is a function of bounded variation if the distributional derivatives of \( u \) are representable by a finite Radon measure in \( \Omega \). Namely, a function \( u \in L^1(\Omega) \) belongs to \( BV(\Omega) \), if and only if there exists a finite Radon vector measure \( Du = (D_1u, \ldots, D_Nu) \) in \( \Omega \), such that

\[
\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = -\int_\Omega \varphi \, D_i u
\]

for all \( \varphi \in C_0^\infty(\Omega) \) and \( i = 1, \ldots, N \).

If \( u \in BV(\Omega) \), the total variation of the measure \( Du \) is

\[
|Du|(\Omega) = \sup \left\{ \int_\Omega u \, \text{div} \psi : \psi \in C_0^\infty(\Omega, \mathbb{R}^N), \|\psi\|_{\infty} \leq 1 \right\}.
\]

As usual, if \( u \in BV(\Omega) \), we denote the total variation \( |Du|(\Omega) \) as

\[
\int_\Omega |Du|.
\]

The space \( BV(\Omega) \) endowed with the norm

\[
\|u\|_{BV} := \|u\|_{L^1} + |Du|(\Omega)
\]

is a Banach space. Notice that the space \( W^{1,1}(\Omega) \) is a subset of \( BV(\Omega) \) and the norm \( \| \cdot \|_{BV} \) restricted to \( W^{1,1} \) coincides with the \( \| \cdot \|_{W^{1,1}} \) norm, as

\[
\int_\Omega |Du| = \int_\Omega |\nabla u| \, dx
\]

for each \( u \in W^{1,1}(\Omega) \).
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Definition of perimeter of a set. Let $\Omega$ be an open set in $\mathbb{R}^N$ and let $E$ be a Borel set. The perimeter of $E$ in $\Omega$ is defined as

$$P(E, \Omega) = |D\chi_E|_{\Omega} = \sup \left\{ \int_\Omega \chi_E \text{div} \phi \, dx : \phi \in C^1_0(\Omega, \mathbb{R}^N), \|\phi\|_{\infty} \leq 1 \right\},$$

where $\chi_E$ is the characteristic function of $E$, i.e., the perimeter of $E$ in $\Omega$ is defined to be the total variation of its characteristic function on that open set.

Definition of Caccioppoli set. We say that a Borel set $E$ is a Caccioppoli set if it has finite perimeter in every open, bounded set $\Omega$ of $\mathbb{R}^N$, i.e.,

$$P(E, \Omega) < +\infty.$$ 

Therefore a Caccioppoli set has a characteristic function which is a function of $BV(\Omega)$ for all $\Omega \subset \mathbb{R}^N$ open and bounded sets.

General properties of $BV(\Omega)$

Now we state some results, concerning the space $BV(\Omega)$, that will be used throughout this work.

**Proposition I.1.1** (A characterization property). Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. A function $u$ belongs to $BV(\Omega)$ if and only if there exists a sequence of smooth functions $(u_n)_{n \in \mathbb{N}} \subset C^\infty(\Omega)$ such that

$$\|u_n - u\|_{L^1(\Omega)} \to 0 \quad \text{and} \quad \lim_{n \to +\infty} \int_\Omega |\nabla u_n| \, dx = \int_\Omega |Du|.$$

**Proof.** See [3, Theorem 3.9].

We remark that, differently from the usual Sobolev spaces, the space $C^\infty(\Omega)$ is “not dense” in the strong topology (see, e.g., [63, Example 1.4]).

**Proposition I.1.2** (Embedding result). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial \Omega$. Then the embedding $BV(\Omega) \hookrightarrow L^{1^*}(\Omega)$ is continuous and the embedding $BV(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $p \in [1, 1^*[$.

**Proof.** See [3, Corollary 3.49].

**Proposition I.1.3** (A Poincaré inequality). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial \Omega$. There exists a constant $c = c(\Omega)$ such that for all $u \in BV(\Omega)$

$$\int_\Omega |u - \bar{u}| \, dx \leq c \int_\Omega |Du|,$$

where $\bar{u} = \int_\Omega u \, dx$. 

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**Proof.** See [3, Theorem 3.44].

**Proposition I.1.4** (Trace operator). Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial \Omega$. Then there exists a bounded linear trace operator

$$T : BV(\Omega) \to L^1(\partial \Omega, \mathcal{H}_{N-1})$$

such that, for all $u \in BV(\Omega)$

$$\int_{\Omega} u \operatorname{div} \varphi \, dx = -\int_{\Omega} \varphi \, Du + \int_{\partial \Omega} (\varphi \cdot n) T u \, d\mathcal{H}_{N-1}$$

for all $\varphi \in C^\infty(\mathbb{R}^N)$, where $n$ denotes the exterior unit normal to $\partial \Omega$ and $\mathcal{H}_{N-1}$ is the Hausdorff $N-1$-dimensional measure. Moreover the trace operator from $BV(\Omega) \to L^1(\partial \Omega)$ is continuous.

**Proof.** See [3, Theorem 3.87].

**Some properties of the area functional in $BV(\Omega)$**

**Definition of the area functional in $BV(\Omega)$.** Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. For each $u \in BV(\Omega)$ we define the area functional as

$$\int_{\Omega} \sqrt{1 + |Du|^2} = \int_{\Omega} \sqrt{1 + |(Du)^a|^2} \, dx + \int_{\Omega} |(Du)^s|,$$

where $Du = (Du)^a \, dx + (Du)^s$ being the decomposition of the measure $Du$ in its absolutely continuous and singular parts with respect to the $N$-dimensional Lebesgue measure.

Equivalently, we have

$$\int_{\Omega} \sqrt{1 + |Du|^2} = \sup \left\{ \int_{\Omega} \left( u \sum_{i=1}^{N} \frac{\partial w_i}{\partial x_i} + w_{N+1} \right) \, dx : w_i \in C^1_0(\Omega) \right\}$$

for $i = 1, 2, \ldots, N + 1$ and $\left\| \sum_{i=1}^{N+1} w_i^2 \right\|_\infty \leq 1$. \hspace{1cm} (I.1)

**Proposition I.1.5** (Basic inequality). Let $\Omega$ be a bounded domain of $\mathbb{R}^N$. For all $u \in BV(\Omega)$ we have

$$\int_{\Omega} |Du| \leq \int_{\Omega} \sqrt{1 + |Du|^2} \leq \int_{\Omega} |Du| + |\Omega|.$$ 

**Proof.** This relation can be easily obtained using definition (I.1). For a detailed proof, see [63, p. 160].

**Proposition I.1.6** (Lower semicontinuity). Let $\Omega$ be a domain of $\mathbb{R}^N$. If $(u_n)_n$ is a sequence in $BV(\Omega)$ converging in $L^1_{\text{loc}}(\Omega)$ to a function $u \in BV(\Omega)$, then

$$\int_{\Omega} \sqrt{1 + |Du|^2} \leq \liminf_{n \to +\infty} \int_{\Omega} \sqrt{1 + |Du_n|^2}.$$ 

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Proof. See [63, Theorem 14.2].

**Proposition I.1.7** (Approximation property). Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). For any given \( v \in BV(\Omega) \) there exists a sequence \((v_n)_n\) in \( W^{1,1}(\Omega) \) such that

\[
\lim_{n \to +\infty} v_n = v \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega,
\]

\[
\lim_{n \to +\infty} \int_\Omega |\nabla v_n| \, dx = \int_\Omega |Dv|.
\]

and

\[
\lim_{n \to +\infty} \int_\Omega \sqrt{1 + |\nabla v_n|^2} \, dx = \int_\Omega \sqrt{1 + |Dv|^2}.
\]

Proof. See [10, Fact 3.3, p.491].

**Proposition I.1.8** (One-sided approximation property). Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \). For any given \( u \in BV(\Omega) \) there exist sequences \((v_n)_n\) and \((w_n)_n\) in \( W^{1,1}(\Omega) \) such that, for all \( n \), \( v_n \geq u \) and \( w_n \leq u \),

\[
\lim_{n \to +\infty} v_n = \lim_{n \to +\infty} w_n = u \text{ in } L^q(\Omega)
\]

for each \( q \in [1,1^*] \) and a.e. in \( \Omega \). Moreover

\[
\lim_{n \to +\infty} \int_\Omega |\nabla v_n| \, dx = \lim_{n \to +\infty} \int_\Omega |\nabla w_n| \, dx = \int_\Omega |Du| \quad (I.2)
\]

and

\[
\lim_{n \to +\infty} \int_\Omega \sqrt{1 + |\nabla v_n|^2} \, dx = \lim_{n \to +\infty} \int_\Omega \sqrt{1 + |\nabla w_n|^2} \, dx = \int_\Omega \sqrt{1 + |Du|^2}. \quad (I.3)
\]

Proof. We only prove the existence of the sequence \((v_n)_n\) as the result about the sequence \((w_n)_n\) can be similarly verified. By [24, Theorem 3.3, p. 370] there exists a sequence \((v_n)_n\) in \( W^{1,1}(\Omega) \) with \( v_n \geq v \), such that \( \lim_{n \to +\infty} v_n = u \) in \( L^1(\Omega) \) and a.e. in \( \Omega \), for which (I.3) holds. By [10, Fact 3.1, p. 490], (I.2) also holds. As the sequence \((v_n)_n\) is bounded in \( BV(\Omega) \) we can extract a subsequence, we still denote by \((v_n)_n\), such that \((v_n)_n\) converges to \( u \) in \( L^q(\Omega) \) for each \( q \in [1,1^*] \), a.e. in \( \Omega \) and (I.2) and (I.3) hold.

I.2 Variational setting for the capillarity-type problem

In this work, under assumptions:

\((h_0)\) \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) having a Lipschitz boundary \( \partial \Omega \);

\((h_1)\) \( h \in L^p(\Omega) \), for some \( p > N \), and \( \kappa \in L^\infty(\partial \Omega) \);
(h12) there exists a constant $\rho > 0$ such that
\[
\left| \int_B h \, dx - \int_{\partial \Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq (1 - \rho) \int_\Omega |D\chi_B|
\]
for every Caccioppoli set $B \subseteq \Omega$.

we will consider the capillarity-type problem
\[
\begin{cases}
- \text{div} \left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) = f(x, u) + h(x) \quad \text{in } \Omega, \\
- \nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = \kappa(x) \quad \text{on } \partial \Omega.
\end{cases}
\tag{I.4}
\]

In order to clarify this general discussion, at the moment we assume the stronger hypothesis

(h13) $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, i.e., for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and, for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \rightarrow \mathbb{R}$ is measurable; moreover, there exist a constant $a > 0$ and a function $b \in L^p(\Omega)$, with $p > N$, such that
\[
|f(x, s)| \leq a |s|^{q-1} + b(x)
\]
for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, with $q = \frac{p}{p-1}$.

Case-by-case weaker conditions will be assumed. Moreover, notice that condition (h13) obviously holds if $f$ satisfies the Carathéodory conditions and
\[
\text{ess sup}_{\Omega \times \mathbb{R}} |f(x, s)| < +\infty.
\]

Remark I.2.1 Condition (h12) implies in particular that
\[
\int_\Omega h \, dx - \int_{\partial \Omega} \kappa \, d\mathcal{H}^{N-1} = 0.
\]

Further comments and remarks on hypothesis (h12) can be found in the dedicated Section I.2.

Some properties of functionals $\mathcal{L}$ and $\mathcal{J}$

Taking in mind that our goal is to solve problem (I.4), we define a functional $\mathcal{L} : BV(\Omega) \rightarrow \mathbb{R}$ by setting
\[
\mathcal{L}(v) = \int_\Omega |Dv| - \int_\Omega hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}^{N-1}
\]
for every $v \in BV(\Omega)$.  

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Analogously we define a functional \( J : BV(\Omega) \to \mathbb{R} \) by setting

\[
J(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}_{N-1} \tag{1.5}
\]

for every \( v \in BV(\Omega) \).

We prove now some basic properties of functionals \( L \) and \( J \) that will be often used throughout this work.

**Proposition I.2.1.** Assume \((h_{10}), (h_{11})\) and \((h_{12})\). The functional \( L \) is continuous, positively homogeneous of degree 1 and is invariant under constant shifts, i.e., \( L(v+r) = L(v) \) for every \( v \in BV(\Omega) \) and \( r \in \mathbb{R} \). Thus, there exists a constant \( \sigma > 0 \) such that

\[
L(v) \leq \sigma \int_{\Omega} |Dv|
\]

for every \( v \in BV(\Omega) \).

**Proof.** The continuity of functional \( L \) is a direct consequence of the continuity of the trace map from \( BV(\Omega) \) to \( L^1(\partial\Omega) \) (see [3, Theorem 3.87]). The homogeneity is trivial and the invariance under translation is a direct consequence of Remark I.2.1. Take any \( v \in BV(\Omega) \) and write \( v = \bar{v} + \tilde{v} \) with \( \bar{v} = \int_{\Omega} v \, dx \) and \( \int_{\Omega} \tilde{v} \, dx = 0 \). By Poincaré-Wirtinger inequality, (see [3, Theorem 3.44]), there exists a constant \( \sigma > 0 \) such that

\[
L(v) = L(\bar{v}) \leq \|\bar{v}\|_{BV} \leq \sigma \int_{\Omega} |D\bar{v}| = \sigma \int_{\Omega} |Dv|.
\]

**Proposition I.2.2.** Assume \((h_{10}), (h_{11})\) and \((h_{12})\). The functional \( J \) is continuous, convex and it is invariant under constant shifts, i.e., \( J(v+r) = J(v) \) for every \( v \in BV(\Omega) \) and \( r \in \mathbb{R} \).

**Proof.** The continuity of functional \( J \) follows from the continuity of the trace map from \( BV(\Omega) \) to \( L^1(\partial\Omega) \) as stated in Proposition I.1.4. The invariance under translation is a direct consequence of Remark I.2.1. \( \square \)

**Proposition I.2.3.** Assume \((h_{10}), (h_{11})\) and \((h_{12})\). Then, for all \( v \in BV(\Omega) \), we have

\[
L(v) \geq \rho \int_{\Omega} |Dv|, \tag{1.6}
\]

with \( \rho \) defined in \((h_{12})\).

**Proof.** The proof of this result closely follows the argument in [61, Lemma 2.1]. Fix any \( v \in BV(\Omega) \). For each \( t \in \mathbb{R} \), define the set

\[
E_t = \{x \in \Omega : v(x) > t\}
\]
and the function $\varphi_{E_t} \in BV(\Omega)$ by
\[
\varphi_{E_t}(x) = \begin{cases} 
\chi_{E_t}(x) & \text{if } t > 0, \\
\chi_{E_t}(x) - 1 = -\chi_{\Omega \setminus E_t}(x) & \text{if } t \leq 0.
\end{cases}
\]

Then the representation
\[
v(x) = \int_{-\infty}^{+\infty} \varphi_{E_t}(x) \, dt
\]
holds for a.e. $x \in \Omega$. Hence we can write
\[
\int_{\partial \Omega} \kappa v \, dH_{N-1} \, dv - \int_{\Omega} hv \, dx = \int_{\partial \Omega} \kappa \int_{-\infty}^{+\infty} \varphi_{E_t}(x) \, dt \, dH_{N-1} - \int_{\Omega} h \int_{-\infty}^{+\infty} \varphi_{E_t}(x) \, dt \, dx
\]
\[
= \int_{-\infty}^{0} \left( \int_{\partial \Omega} \kappa \varphi_{E_t}(x) \, dH_{N-1} - \int_{\Omega} h \varphi_{E_t}(x) \, dx \right) \, dt
\]
\[
+ \int_{0}^{+\infty} \left( \int_{\partial \Omega} \kappa \varphi_{E_t}(x) \, dH_{N-1} - \int_{\Omega} h \varphi_{E_t}(x) \, dx \right) \, dt
\]
\[
= \int_{-\infty}^{0} \left( \int_{\partial \Omega} \kappa \chi_{\Omega \setminus E_t} \, dH_{N-1} - \int_{\Omega \setminus E_t} \kappa \varphi_{E_t}(x) \, dx \right) \, dt
\]
\[
+ \int_{0}^{+\infty} \left( \int_{\partial \Omega} \kappa \chi_{E_t}(x) \, dH_{N-1} - \int_{E_t} \kappa \varphi_{E_t}(x) \, dx \right) \, dt.
\]

Using (h12) and the coarea formula [13, Theorem 10.3], we get
\[
\int_{\partial \Omega} \kappa v \, dH_{N-1} - \int_{\Omega} hv \, dx \geq -(1 - \rho) \left( \int_{-\infty}^{0} \int_{\Omega} \left| D\chi_{\Omega \setminus E_t} \right| \, dt + \int_{0}^{+\infty} \int_{\Omega} \left| D\chi_{E_t} \right| \, dt \right)
\]
\[
= -(1 - \rho) \left( \int_{-\infty}^{0} \int_{\Omega} \left| D\chi_{E_t} \right| \, dt + \int_{0}^{+\infty} \int_{\Omega} \left| D\chi_{E_t} \right| \, dt \right)
\]
\[
= -(1 - \rho) \int_{-\infty}^{+\infty} \int_{\Omega} \left| D\chi_{E_t} \right| \, dt = -(1 - \rho) \int_{\Omega} |Dv|.
\]

The last computation yields (I.6). □

By the inequality proved in Proposition I.1.5 the following result is immediate.

**Corollary I.2.4.** Assume (h10), (h11) and (h12). Then, for all $v \in BV(\Omega)$, we have
\[
J(v) \geq \rho \int_{\Omega} |Dv|,
\]
with $\rho$ defined in (h12).

**Proposition I.2.5.** Assume (h10), (h11) and (h12). The functional $L : BV(\Omega) \to \mathbb{R}$ is lower semicontinuous with respect to the $L^p$-convergence in $BV(\Omega)$ with $q = \frac{p}{p-1}$, i.e., if $(v_n)_n$ is a sequence in $BV(\Omega)$ converging in $L^q(\Omega)$ to a function $v \in BV(\Omega)$, then
\[
L(v) \leq \liminf_{n \to +\infty} L(v_n).
\]
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Proof. Let \((v_n)_n\) be a sequence in \(BV(\Omega)\) converging in \(L^q(\Omega)\) to a function \(v \in BV(\Omega)\).
It follows from [61, Lemma 2.2] that for every \(\delta > 0\) there exists a constant \(C_\delta\) such that, for all \(w \in BV(\Omega)\),
\[
\left| \int_{\partial \Omega} \kappa w \, dH_{N-1} \right| \leq (1 - \frac{\rho}{2}) \int_{S_\delta} |Dw| + C_\delta \int_{S_\delta} |w| \, dx,
\]
where
\[
S_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \delta \}.
\]
Fix \(\delta > 0\). Then we can write
\[
L(v) - L(v_n) = \int_{\Omega} |Dv| - \int_{\Omega} |Dv_n| - \int_{\Omega} h(v - v_n) \, dx + \int_{\partial \Omega} \kappa(v - v_n) \, dH_{N-1}
\]
\[
\leq \int_{\Omega} |Dv| - \int_{\Omega} |Dv_n| - \int_{\Omega} h(v - v_n) \, dx
\]
\[
+ \int_{S_\delta} |D(v - v_n)| + C_\delta \int_{S_\delta} |v - v_n| \, dx.
\]
Since \((v_n)_n\) converges to \(v\) in \(L^q(\Omega)\) we get
\[
L(v) - \liminf_{n \to +\infty} L(v_n) \leq \limsup_{n \to +\infty} \left( \int_{\Omega} |Dv| - \int_{\Omega} |Dv_n| + \int_{S_\delta} |D(v - v_n)| \right)
\]
\[
\leq \limsup_{n \to +\infty} \left( \int_{\Omega} |Dv| - \int_{\Omega} |Dv_n| + \int_{S_\delta} |Dv_n| + \int_{S_\delta} |Dv| \right)
\]
\[
\leq \int_{\Omega \setminus S_\delta} |Dv| - \liminf_{n \to +\infty} \int_{\Omega \setminus S_\delta} |Dv_n| + 2 \int_{S_\delta} |Dv|.
\]
By the lower semicontinuity of the total variation with respect to the \(L^q\)-convergence (see, e.g., [3, Remark 3.5]) in \(BV(\Omega)\), we obtain
\[
L(v) \leq \liminf_{n \to +\infty} L(v_n) + 2 \int_{S_\delta} |Dv|
\]
for all \(\delta > 0\). The conclusion follows letting \(\delta \to 0\), as \(\bigcap_{\delta > 0} S_\delta = \emptyset\). \(\square\)

Proposition I.2.6. Assume \((h_0)\), \((h_1)\) and \((h_2)\). The functional \(J : BV(\Omega) \to \mathbb{R}\) is lower semicontinuous with respect to the \(L^q\)-convergence in \(BV(\Omega)\) with \(q = \frac{p}{p-1}\), i.e., if \((v_n)_n\) is a sequence in \(BV(\Omega)\) converging in \(L^q(\Omega)\) to a function \(v \in BV(\Omega)\), then
\[
J(v) \leq \liminf_{n \to +\infty} J(v_n).
\]

Proof. The proof follows combining Proposition I.1.6 and Proposition I.2.5. For the details we refer to [61, Proposition 2.1]. \(\square\)
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Proposition I.2.7 (Lattice property). Assume \((h_{0})\) and \((h_{1})\). For every \(u, v \in BV(\Omega)\),
\[
\mathcal{J}(u \vee v) + \mathcal{J}(u \wedge v) \leq \mathcal{J}(u) + \mathcal{J}(v).
\]

Proof. We first recall that \(BV(\Omega)\) is a lattice \([4]\). Then, also using \([105, \text{Theorem 1.56}]\), we see that, for every \(u, v \in W^{1,1}(\Omega)\),
\[
\int_{\Omega} \sqrt{1 + \left| \nabla (u \vee v) \right|^{2}} \, dx + \int_{\Omega} \sqrt{1 + \left| \nabla (u \wedge v) \right|^{2}} \, dx = \int_{\Omega} \sqrt{1 + \left| \nabla u \right|^{2}} \, dx + \int_{\Omega} \sqrt{1 + \left| \nabla v \right|^{2}} \, dx.
\]

Take now \(u, v \in BV(\Omega)\). The approximation property and the semicontinuity result, stated in \([10, \text{p. 491, p. 498}]\) and Proposition I.2.6, easily yield
\[
\int_{\Omega} \sqrt{1 + \left| D(u \vee v) \right|^{2}} + \int_{\Omega} \sqrt{1 + \left| D(u \wedge v) \right|^{2}} \leq \int_{\Omega} \sqrt{1 + \left| Du \right|^{2}} + \int_{\Omega} \sqrt{1 + \left| Dv \right|^{2}}.
\]

As we have
\[
\int_{\Omega} h(u \vee v) \, dx - \int_{\partial \Omega} \kappa (u \vee v) \, d\mathcal{H}_{N-1} + \int_{\Omega} h(u \wedge v) \, dx - \int_{\partial \Omega} \kappa (u \wedge v) \, d\mathcal{H}_{N-1} = \int_{\Omega} hu \, dx - \int_{\partial \Omega} \kappa u \, d\mathcal{H}_{N-1} + \int_{\Omega} hv \, dx - \int_{\partial \Omega} \kappa v \, d\mathcal{H}_{N-1},
\]
the conclusion follows. \(\Box\)

**BV-Solutions for capillarity-type problems**

First of all we have to introduce a suitable definition of solutions for problem (I.4) in the context of bounded variation functions.

We start noticing that, formally, (I.4) is the Euler-Lagrange equation of the functional
\[
\mathcal{H}(v) = \int_{\Omega} \sqrt{1 + \left| Dv \right|^{2}} - \int_{\Omega} hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}_{N-1} - \int_{\Omega} F(x, v) \, dx
\]
where \(F(x, s) = \int_{0}^{s} f(x, \xi) \, d\xi\). The functional \(\mathcal{H}\) is well-defined in the space \(W^{1,1}(\Omega)\). Yet this space, which could be a natural candidate where to settle the problem, is not a favourable framework to deal with critical point theory. Since our approach will be variational it is well-known (see, e.g., \([47, 61, 53]\)) that the natural context where this problem has to be settled is the space \(BV(\Omega)\) of bounded variation functions.

Namely, we consider the functional \(I : BV(\Omega) \to \mathbb{R}\) defined by
\[
I(v) = \int_{\Omega} \sqrt{1 + \left| Dv \right|^{2}} - \int_{\Omega} hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}_{N-1} - \int_{\Omega} F(x, v) \, dx \quad (I.8)
\]
Moreover, by (h_{i3}), we define the functional \( F : BV(\Omega) \to \mathbb{R} \) as
\[
F(v) = \int_{\Omega} F(x,v) \, dx
\]
and hence, by (I.5), we have
\[
I(v) = J(v) - F(v)
\]
for all \( v \in BV(\Omega) \). Clearly, \( F \) is of class \( C^1 \) over the space \( BV(\Omega) \). Moreover, by Proposition I.2.2, \( J \) is convex and Lipschitz-continuous (see [24, p. 362]) in \( BV(\Omega) \), but not differentiable as it follows from [10, Theorem 2.4]. Hence, the relaxed functional \( I \) is not differentiable in \( BV(\Omega) \). As in [103], since \( I \) is the sum of a convex term and of a \( C^1 \) term, it is natural to say that \( u \in BV(\Omega) \) is a critical point of \( I \) if
\[
0 \in \partial I(u),
\]
or equivalently,
\[
F'(u) = f(\cdot, u) \in \partial J(u),
\]
where \( \partial I(u) \) and \( \partial J(u) \) are the subdifferentials of \( I \) and of \( J \) at \( u \), respectively. Accordingly, the following definition of a \( BV \)-solution is adopted.

**Definition of solution.** We say that a function \( u \in BV(\Omega) \) is a solution of problem (I.4) if \( u \) satisfies
\[
J(v) - J(u) \geq \int_{\Omega} f(x,u)(v-u) \, dx, \tag{1.9}
\]
for every \( v \in BV(\Omega) \).

**Remark I.2.2** Note that \( u \) is a solution of (I.4) if and only if \( u \) is a minimizer in \( BV(\Omega) \) of the functional \( K_u : BV(\Omega) \to \mathbb{R} \) defined by
\[
K_u(v) = J(v) - \int_{\Omega} f(x,u)v \, dx.
\]

**Remark I.2.3** It follows from [10] that \( u \in BV(\Omega) \) satisfies the variational inequality (I.9), for every \( v \in BV(\Omega) \), if and only if
\[
\int_{\Omega} \frac{(Du)^a (D\phi)^a}{\sqrt{1 + |(Du)^a|^2}} \, dx + \int_{\Omega} S \left( \frac{Du}{|Du|} \right) \frac{D\phi}{|D\phi|} |D\phi|^s \, dx
\]
\[
= \int_{\Omega} (f(x,u) + h)\phi \, dx - \int_{\partial\Omega} \kappa \phi \, dH_{N-1}
\]
holds for every \( \phi \in BV(\Omega) \) such that \( |D\phi|^s \) is absolutely continuous with respect to \( |Du|^s \). Here \( S \) is the \( N \)-dimensional sign function, i.e., \( S(\xi) = |\xi|^{-1} \xi \) if \( \xi \in \mathbb{R}^N \setminus \{0\} \) and \( S(\xi) = 0 \) if \( \xi = 0 \).
Once provided the definition of solution we discuss briefly the solvability of some particular cases for the capillarity-type equation.

Assume \((h_0)\) and let us start with the simpler homogeneous problem where the right-hand side of the equation in (I.4) does not depend on \(u\), i.e.,

\[
\begin{cases}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = h(x) & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = 0 & \text{on } \partial\Omega,
\end{cases}
\]  

with \(h \in L^\infty(\Omega)\) given. It is easy to see that (I.10) may have a solution in \(BV(\Omega)\) only if \(\int_{\Omega} h \, dx = 0\). Indeed, a simple minimization argument, based on the classical Poincaré inequality, (see Proposition I.1.3) shows that, assuming \(\int_{\Omega} h \, dx = 0\), problem (I.10) has a solution if

\[\|h\|_\infty < c\]

and it may have no solution if

\[\|h\|_\infty > c\]

where \(c\) is the Poincaré constant.

Let us consider now the non-homogeneous problem. Assume \((h_0), (h_1), (h_2)\) and consider the generic problem

\[
\begin{cases}
-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = h(x) & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega.
\end{cases}
\]  

Accordingly with the given notion of \(BV\)-solution, we remark that a function \(u \in BV(\Omega)\) is a solution of problem (I.11) if \(u\) is a minimum point for the functional \(J\) defined in (I.5), i.e.,

\[J(u) \leq J(v),\]

for every \(v \in BV(\Omega)\). Hence we have the following existence result, as in [61].

**Proposition I.2.8.** Assume \((h_0), (h_1), (h_2)\). Then problem (I.11) has a solution \(w \in BV(\Omega)\) with \(\int_{\Omega} w \, dx = 0\).

**Proof.** Define a closed subspace of \(BV(\Omega)\) by setting

\[W = \left\{ w \in BV(\Omega) : \int_{\Omega} w \, dx = 0 \right\}.
\]

By the Poincaré inequality, stated in Proposition I.1.3, \(W\) is a Banach space if endowed with the norm

\[\|w\|_W = \int_{\Omega} |Dw|.
\]

Moreover, the functional \(J\) restricted to \(W\) is bounded from below and coercive, by Proposition I.2.3, and lower semicontinuous with respect to the \(L^q\)-convergence, with \(q = \frac{p}{p-1}\), in \(W\) (see Proposition I.2.6). Hence, \(J\) has a minimum in \(W\) and any corresponding minimizer is a solution of (I.11).
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This is the starting point for the study of the solvability of problem (I.4). In Chapter II and Chapter IV, our leading idea will be to find out conditions on the nonlinear perturbation $f$ such that the solvability of (I.11) is maintained passing to problem (I.4).

Remarks on the hypotheses

Motivations

First of all, we point out that condition $(h_{I2})$ has been introduced in [61], where it was shown to be necessary for the existence of a solution $u \in C^2(\Omega)$ of the problem

$$
\begin{aligned}
&-\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = h(x) \quad \text{in } \Omega, \\
&-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa(x) \quad \text{on } \partial\Omega.
\end{aligned}
$$

(I.13)

Let us verify that the weaker condition

$$
\left| \int_B h \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}_{N-1} \right| \leq \int_{\Omega} |D\chi_B|,
$$

(I.14)

for every Caccioppoli set $B \subseteq \Omega$, is necessary for the existence of a solution $u \in BV(\Omega)$ of (I.13). Indeed, if $B \subseteq \Omega$ is any Caccioppoli set, taking $v = u + \chi_B$ as a test function in (I.12), we easily get

$$
\int_{\Omega} h \chi_B \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}_{N-1} 
\leq \int_{\Omega} \sqrt{1 + |D(u + \chi_B)|^2} - \int_{\Omega} \sqrt{1 + |Du|^2} \leq \int_{\Omega} |D\chi_B|.
$$

Similarly, taking $v = u - \chi_B$ as a test function in (I.12), we obtain

$$
-\int_{\Omega} h \chi_B \, dx + \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}_{N-1} \leq \int_{\Omega} |D\chi_B|.
$$

Hence (I.14) follows.

Some explicit conditions

We produce now some more explicit assumptions that imply the validity of $(h_{I2})$. Let us set

$$
\alpha = \inf \left\{ \int_{\Omega} |Dw| : w \in BV(\Omega), \int_{\Omega} w \, dx = 0, \|w\|_{L^1} = 1 \right\}
$$

(I.15)

and

$$
\beta = \inf \left\{ \int_{\Omega} |Dw| : w \in BV(\Omega), \int_{\Omega} w \, dx = 0, \int_{\partial\Omega} |w| \, d\mathcal{H}_{N-1} = 1 \right\}.
$$

(I.16)
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By the Sobolev-Poincaré inequality (see [3, Remark 3.50]) and the continuity of the trace embedding (see Proposition I.1.4), we infer that $\alpha > 0$ and $\beta > 0$. A discussion of situations where the infimum in (I.15), or in (I.16), is attained can be found, e.g., in [7], [18] and [26].

Proposition I.2.9. Assume $(h_0)$, $(h_1)$,

$$
(h_{1}^{2'}) \int_{\Omega} h \, dx = \int_{\partial \Omega} \kappa \, d\mathcal{H}_{N-1}
$$

and

$$
(h_{1}^{2''})\quad \alpha^{-1}\|h\|_{L^N} + \beta^{-1}\|\kappa\|_\infty < 1.
$$

Then $(h_1^2)$ holds.

Proof. By $(h_{1}^{2'})$, Hölder inequality, (I.15) and (I.16) we get, for every $w \in BV(\Omega)$,

$$
\int_{\Omega} hw \, dx - \int_{\partial \Omega} \kappa \, w \, d\mathcal{H}_{N-1}
= \int_{\Omega} h \left( w - \frac{1}{|\Omega|} \int_{\Omega} w \, dx \right) \, dx - \int_{\partial \Omega} \kappa \left( w - \frac{1}{|\Omega|} \int_{\Omega} w \, dx \right) \, d\mathcal{H}_{N-1}
\leq (\alpha^{-1}\|h\|_{L^N} + \beta^{-1}\|\kappa\|_\infty) \int_{\Omega} |Dw|.
$$

Hence, using $(h_{1}^{2''})$, we obtain $(h_1^2)$ with $1 - \rho = \alpha^{-1}\|h\|_{L^N} + \beta^{-1}\|\kappa\|_\infty < 1$. \qed

Let us set, for $p > N$,

$$
X_p(\Omega) = \{ z \in L^\infty(\Omega, \mathbb{R}^N) \mid \text{div} \, z \in L^p(\Omega) \}.
$$

For $u \in BV(\Omega)$ and $z \in X_p(\Omega)$, let $[z, n] \in L^\infty(\partial \Omega)$ be the weak trace on $\partial \Omega$ of the component of $z$ along the outer normal $n$ to $\partial \Omega$ and $(Du, z)$ the Radon measure defined in [9]. Recall that Green’s formula

$$
\int_{\Omega} u \, \text{div} \, z \, dx = \int_{\partial \Omega} [z, n] \, u \, d\mathcal{H}_{N-1} - \int_{\Omega} (Du, z)
$$

holds [9, Theorem 1.9]. Using this formula and the method of Lagrange multipliers, it is proved in [25] that to the eigenvalue $c_2$ there correspond eigenfunctions $\varphi \in BV(\Omega)$, for which there exists $z \in X_p(\Omega)$, with $\|z\|_\infty \leq 1$,

$$
-\text{div} z \in c_2 \text{ Sgn}(\varphi),
$$

$$
(D\varphi, z) = |D\varphi|,
$$

$$
[z, n] = 0 \quad \mathcal{H}_{N-1}\text{-a.e. on } \partial \Omega.
$$

In this functional setting, we have the following result.
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Proposition I.2.10. Assume \((h_10), (h_11)\) and

\((h_12'')\) there exists \(z \in X_p(\Omega)\), with \(p > N\), such that \(\text{div} \, z = h\) a.e. in \(\Omega\), \([z, n] = \kappa\) \(H_{N-1}\)-a.e. on \(\partial\Omega\) and \(\|z\|_\infty < 1\).

Then \((h_12)\) holds.

Proof. Using Theorem 1.9 and Corollary 1.6 in [9], we get, for every \(w \in BV(\Omega)\),

\[
\int_\Omega hw \, dx - \int_{\partial\Omega} \kappa w \, dH_{N-1} = \int_\Omega w \, \text{div} \, z \, dx - \int_{\partial\Omega} [z, n] w \, dH_{N-1} = -\int_\Omega (Dw, z) \leq \|z\|_\infty \int_\Omega |Dw|.
\]

Hence, by \((h_12'')\), we obtain \((h_12)\) with \(1 - \rho = \|z\|_\infty\). □

The 1-dimensional case

In the case of dimension \(N = 1\) condition \((h_11)\) can be weakened in most cases by assuming the simpler condition \((h_11')\)

\(h \in L^1(\Omega);\)

moreover, some further conditions can be stated that imply assumption \((h_12)\).

Proposition I.2.11. Assume \(N = 1\) and let \(\Omega = ]0, T[\). Assume further \((h_11'), (h_12')\), that is,

\[
\int_0^T h \, dx = \kappa(0) + \kappa(T),
\]

and

\((h_12'')\) \(\|h\|_{L^1} < 1\) and \(\|\kappa\|_{L^1} < 1\).

Then \((h_12)\) holds.

Proof. Take \(\rho > 0\) such that both \(\|h\|_{L^1} < 1 - \rho\) and \(\|\kappa\|_{L^1} < 1 - \rho\). Let \(B \subseteq ]0, T[\) be any Cacciopoli set. In case \(B = ]0, T[\) or \(B = \emptyset\) the inequality \((h_12)\) is trivially satisfied. In case \(0 \notin B\) and \(T \notin B\) we have \(\int_{]0, T[} |D\chi_B| \geq 2\) and \(\int_{\partial[0, T]} \kappa \chi_B \, dH_0 = 0\); hence the inequality in \((h_12)\) is satisfied as

\[
\left| \int_B h \, dx - \int_{\partial[0, T]} \kappa \chi_B \, dH_0 \right| \leq \|h\|_{L^1} \leq (1 - \rho) \int_{]0, T[} |D\chi_B|.
\]

In case \(0 \in B\), \(T \in B\) and \(B \neq ]0, T[\) we have \(\int_{]0, T[} |D\chi_B| \geq 2\) and \(\int_{\partial[0, T]} \kappa \chi_B \, dH_0 = \kappa(0) + \kappa(T);\) hence the inequality in \((h_12)\) is satisfied as

\[
\left| \int_B h \, dx - \int_{\partial[0, T]} \kappa \chi_B \, dH_0 \right| = \left| \int_B h \, dx - \int_0^T h \, dx \right| \leq \|h\|_1 \leq (1 - \rho) \int_{]0, T[} |D\chi_B|.
\]
Suppose now that $0 \in \bar{B}$ and $T \not\in \bar{B}$. Then we have $\int_{[0,T]} |D\chi_B| \geq 1$. Observe that, by \((h_12')\),
\[
2\left( \int_0^T h^+ \, dx - \kappa(0) \right) = \int_0^T h^+ \, dx - \kappa(0) + \int_0^T h^- \, dx + \kappa(T)
\]
\[
= \int_0^T |h| \, dx - \kappa(0) + \kappa(T)
\]
\[
\leq 2(1 - \rho)
\]
and hence
\[
\int_0^T h^+ \, dx - \kappa(0) \leq 1 - \rho.
\]
A similar computation yields
\[
\int_0^T h^- \, dx + \kappa(0) \leq 1 - \rho.
\]
Therefore we obtain
\[
\left| \int_B h \, dx - \int_{\partial[0,T]} \kappa\chi_B \, d\mathcal{H}_0 \right| = \left| \int_B h^+ \, dx - \int_B h^- \, dx - \kappa(0) \right|
\]
\[
\leq \max \left\{ \int_0^T h^+ \, dx - \kappa(0), \int_0^T h^- \, dx + \kappa(0) \right\} \leq 1 - \rho.
\]
The case $0 \not\in \bar{B}$ and $T \in \bar{B}$ is treated similarly.

I.3 A mountain pass lemma

This last section is devoted to state and prove a non-smooth version of the classical mountain pass lemma, without the Palais-Smale condition and adapted to the $BV$-setting.

Lemma I.3.1 (A mountain pass lemma). Assume \((h_10), (h_11), (h_12)\) and \((h_13)\). Let $x_0, x_1 \in BV(\Omega)$ be given. Set
\[
\Gamma = \{ \gamma \in C^0([0,1],BV(\Omega)) : \gamma(0) = x_0, \gamma(1) = x_1 \}.
\]
Suppose that
\[
c_I = \inf_{\gamma \in \Gamma} \max_{\xi \in [0,1]} I(\gamma(\xi)) > \max\{I(x_0), I(x_1)\},
\]
where $I$ is defined in \((I.8)\). Then there exist sequences $(\gamma_k)_k$, $(v_k)_k$, $(\varepsilon_k)_k$, with $\gamma_k \in \Gamma$, $v_k \in BV(\Omega)$ and $\varepsilon_k \in \mathbb{R}$ such that $\lim_{k \to +\infty} \varepsilon_k = 0$, satisfying for each $k$
\[
c_I - \frac{1}{k} \leq I(v_k) \leq \max_{\xi \in [0,1]} I(\gamma_k(\xi)) \leq c_I + \frac{1}{k}, \tag{I.17}
\]
I. PRELIMINARIES

\[ \min_{\xi \in [0,1]} \| v_k - \gamma_k(\xi) \|_{BV} \leq \frac{1}{k} \]  \hspace{1cm} (I.18)

and, for all \( v \in BV(\Omega) \),

\[ J(v) - J(v_k) \geq \int_{\Omega} f(x,v_k)(v - v_k) \, dx + \varepsilon_k \| v - v_k \|_{BV}. \] \hspace{1cm} (I.19)

**Proof.** In [72, Theorem 5.1] a mountain pass theorem is proved for a continuous functional \( \Phi \) on a complete metric space \( X \), with distance \( d \). In such a setting, a critical point of \( \Phi \) is defined as a point \( x \in X \) such that \( \delta(\Phi,x) = 0 \), where \( \delta(\Phi,x) \) is the regularity constant of \( \Phi \) at \( x \). We recall (see [72, Definition 5.1]) that \( x \in X \) is a \( \delta \)-regular point of \( \Phi \) if there is a neighbourhood \( U \) of \( x \), \( \alpha > 0 \) and a continuous mapping \( \Psi : U \times [0,\alpha] \to X \) such that, for all \( (w,t) \in U \times [0,\alpha] \), \( d(\Psi(u,t),u) \leq t \) and \( \Phi(u) - \Phi(\Psi(u,t)) \geq \delta t \). The regularity constant of \( \Phi \) at \( x \) is \( \delta(\Phi,x) = \sup\{\delta : \Phi \text{ is } \delta \text{-regular at } x\} \).

If \( x \) is not a \( \delta \)-regular point of \( \Phi \) for any \( \delta > 0 \) we say that \( x \) is a critical point of \( \Phi \) according to [72, Definition 5.1] and we set \( \delta(\Phi,x) = 0 \).

In our situation we have \( X = BV(\Omega) \) and \( \Phi = I \).

Claim. \( u \in BV(\Omega) \) is a critical point according to [72, Definition 5.1] if and only if \( 0 \) is an element of the subgradient of \( I \) at \( u \), i.e., \( u \) satisfies (I.9) for all \( v \in BV(\Omega) \).

Suppose first that \( u \in BV(\Omega) \) is not a critical point according to [72, Definition 5.1]. We prove that \( u \) does not satisfy (I.9) for some \( v \in BV(\Omega) \). Indeed, \( u \) is a \( \delta \)-regular point of \( I \) for some \( \delta > 0 \), i.e., there exist \( \alpha > 0 \), a neighbourhood \( U \) of \( u \) and a continuous mapping \( \Psi : U \times [0,\alpha] \to BV(\Omega) \) such that, for all \( (w,t) \in U \times [0,\alpha] \), \( \| \Psi(w,t) - w \|_{BV} \leq t \) and \( \delta(\Psi(u,t)) \geq \delta t \). The regularity constant of \( \Phi \) at \( x \) is

\[ \delta(\Phi,x) = \sup\{\delta : \Phi \text{ is } \delta \text{-regular at } x\}. \]

Using the fact that the functional \( F \) is of class \( C^1 \) in \( BV(\Omega) \), we have

\[ \mathcal{J}(\Psi(u,t)) - \mathcal{J}(u) = \mathcal{I}(\Psi(u,t)) - \mathcal{I}(u) + \int_{\Omega} F(x,\Psi(u,t)) \, dx - \int_{\Omega} F(x,u) \, dx \]

\[ \leq -\delta t + \int_{\Omega} f(x,u)(\Psi(u,t) - u) \, dx + t\eta(t) \]

where \( \lim_{t \to 0^+} \eta(t) = 0 \), thus yielding, for \( t \) sufficiently small,

\[ \mathcal{J}(\Psi(u,t)) - \mathcal{J}(u) < \int_{\Omega} f(x,u)(\Psi(u,t) - u) \, dx. \]

Therefore, \( u \) does not satisfy (I.9) for \( v = \Psi(u,t) \).

Suppose now that it is false that \( u_0 \in BV(\Omega) \) satisfies

\[ \mathcal{J}(v) - \mathcal{J}(u_0) \geq \int_{\Omega} f(x,u_0)(v - u_0) \, dx \]
for all $v \in BV(\Omega)$. We shall prove that $u_0$ is not a critical point according to [72, Definition 5.1]. We are assuming that there exist $\delta > 0$ and $w \in BV(0, T)$ satisfying

$$J(w) - J(u_0) \leq \int_\Omega f(x, u_0)(w - u_0) \, dx - 2\delta.$$ 

By the continuity in $BV(\Omega)$ of $J$ and of the map $u \mapsto \int_\Omega f(x, u)(w - u) \, dx$, there exists a bounded neighbourhood $U$ of $u_0$ in $BV(\Omega)$ such that

$$J(w) - J(u) \leq \int_\Omega f(x, u)(w - u) \, dx - \delta$$

holds for all $u \in U$.

Take $\alpha$, with $0 < \alpha < \inf_{u \in U} \|w - u\|_{BV}$, and set $\tilde{\delta} = \inf_{u \in U} \|w - u\|_{BV} > 0$. Then we have

$$J(u + t \frac{w - u}{\|w - u\|_{BV}}) - J(u) \leq \frac{t}{\|w - u\|_{BV}}(J(w) - J(u))$$

$$\leq \int_\Omega f(x, u)t \frac{w - u}{\|w - u\|_{BV}} \, dx - \frac{t}{\|w - u\|_{BV}}\tilde{\delta}$$

$$\leq t\left(\int_\Omega f(x, u)\frac{w - u}{\|w - u\|_{BV}} \, dx - \tilde{\delta}\right)$$

for all $u \in U$ and all $t \in [0, \alpha]$. Define $\Psi : U \times [0, \alpha] \to BV(\Omega)$ by

$$\Psi(u, t) = u + t \frac{w - u}{\|w - u\|_{BV}}.$$ (I.20)

Then we have $\|\Psi(u, t) - u\|_{BV} = t$. Moreover, possibly reducing $U$, there exists a function $\eta$ satisfying $\lim_{t \to 0^+} \eta(t) = 0$ uniformly with respect to $u \in U$, such that

$$I(\Psi(u, t)) - I(u) = J(\Psi(u, t)) - J(u) - \int_\Omega F(x, \Psi(u, t)) \, dx + \int_\Omega F(x, u) \, dx$$

$$\leq t\left(\int_\Omega f(x, u)\frac{w - u}{\|w - u\|_{BV}} \, dx - \tilde{\delta}\right) - \int_\Omega f(x, u)t \frac{w - u}{\|w - u\|_{BV}} \, dx + t\eta(t)$$

$$= t\left(-\tilde{\delta} + \eta(t)\right),$$ (I.21)

for all $u \in U$ and all $t \in [0, \alpha]$. Possibly reducing $\alpha$, we may assume that $\eta(t) \leq \frac{1}{2}\tilde{\delta}$ for all $t \in [0, \alpha]$. Therefore, (I.21) yields $I(u) - I(\Psi(u, t)) \geq \frac{1}{2}\tilde{\delta}$ for all $u \in U$ and all $t \in [0, \alpha]$, thus showing that $u_0$ is a $\frac{1}{2}\tilde{\delta}$-regular point of $I$. This concludes the proof of the claim.

Fix $k \geq 1$ and pick any $\gamma_k \in \Gamma$ such that

$$\max_{\xi \in [0,1]} I(\gamma_k(\xi)) \leq c_I + \frac{1}{k}.$$
According to Ekeland’s variational principle (see, e.g., [83]), there is $\tilde{\gamma}_k \in \Gamma$ such that
\[
\max_{\xi \in [0,1]} \|\tilde{\gamma}_k(\xi) - \gamma_k(\xi)\|_{BV} \leq \frac{1}{\sqrt{k}},
\]
\[
\max_{\xi \in [0,1]} \mathcal{I}(\tilde{\gamma}_k(\xi)) \leq \max_{\xi \in [0,1]} \mathcal{I}(\gamma_k(\xi)) \leq c_\mathcal{I} + \frac{1}{k}
\]
and
\[
\max_{\xi \in [0,1]} \mathcal{I}(\gamma(\xi)) > \max_{\xi \in [0,1]} \mathcal{I}(\tilde{\gamma}_k(\xi)) - \frac{1}{\sqrt{k}} \max_{\xi \in [0,1]} \|\gamma(\xi) - \tilde{\gamma}_k(\xi)\|_{BV}
\]
for all $\gamma \in \Gamma$ with $\gamma \neq \tilde{\gamma}_k$. In the proof of [72, Theorem 5.1], it is shown that there exists $\xi_k \in [0,1]$ such that
\[
c_\mathcal{I} - \frac{1}{k} \leq \mathcal{I}(\tilde{\gamma}_k(\xi_k))
\]
and
\[
\delta(\mathcal{I}, \tilde{\gamma}_k(\xi_k)) \leq \frac{1}{\sqrt{k}}.
\]
Set $v_k = \tilde{\gamma}_k(\xi_k)$. Then, (I.17) and
\[
\min_{\xi \in [0,1]} \|v_k - \gamma_k(\xi)\|_{BV} \leq \frac{1}{\sqrt{k}}
\]
are satisfied for all $k$. In particular,
\[
\lim_{k \to +\infty} \mathcal{I}(v_k) = c_\mathcal{I}.
\]
We claim that there exists a sequence $(\varepsilon_k)_k$ in $\mathbb{R}$, with $\lim_{k \to +\infty} \varepsilon_k = 0$, satisfying (I.19) for each $k$ and all $v \in BV(\Omega)$. By contradiction, suppose that there exist $k$, $\varepsilon_k < -2\delta(\mathcal{I}, v_k)$, $\sigma > 0$ and $w \in BV(\Omega)$ satisfying
\[
\mathcal{J}(w) - \mathcal{J}(v_k) \leq \int_\Omega f(x, v_k)(w - v_k) \, dx + \varepsilon_k \|w - v_k\|_{BV} - 2\sigma.
\]
Arguing as in the claim above, we can actually assume that there exists a bounded neighbourhood $U$ of $v_k$ in $BV(\Omega)$ such that
\[
\mathcal{J}(w) - \mathcal{J}(u) \leq \int_{\Omega} f(x, u)(w - u) \, dx + \varepsilon_k \|w - u\|_{BV} - \sigma
\]
holds for all $u \in U$. Take $\alpha$, with $0 < \alpha < \inf_{u \in U} \|w - u\|_{BV}$. Then, for all $u \in U$ and all $t \in [0, \alpha]$, we have
\[
\mathcal{J}\left(u + t \frac{w - u}{\|w - u\|_{BV}}\right) - \mathcal{J}(u) \leq \int_{\Omega} f(x, u)t \frac{w - u}{\|w - u\|_{BV}} \, dx + t\varepsilon_k.
\]
Define $\Psi : U \times [0, \alpha] \to BV(\Omega)$ as in (I.20). Then we have $\|\Psi(u, t) - u\|_{BV} = t$. Moreover, arguing as above, we can find a function $\eta$ satisfying, possibly reducing $U$, $\lim_{t \to 0^+} \eta(t) = 0$ uniformly with respect to $u \in U$, and
\[
\mathcal{I}(\Psi(u, t)) - \mathcal{I}(u) = \mathcal{J}(\Psi(u, t)) - \mathcal{J}(u) - \int_\Omega F(x, \Psi(u, t)) \, dx + \int_\Omega F(x, u) \, dx
\]
\[
\leq t(\varepsilon_k + \eta(t)).
\]
Possibly reducing $\alpha$, we may assume that $\eta(t) \leq -\frac{\epsilon_k}{2}$ for all $t \in [0, \alpha]$. Then $\mathcal{I}(u) - \mathcal{I}(\Psi(u, t)) \geq \frac{1}{2}|\epsilon_k|$ for all $t \in [0, \alpha]$, thus showing that $v_k$ is a $\frac{1}{2}|\epsilon_k|$-regular point of $\mathcal{I}$, contradicting the assumption $\epsilon_k < -2\delta(\mathcal{I}, v_k)$.

We proved that the sequences $(\gamma_k)_k$, $(v_k)_k$ and $(\epsilon_k)_k$, with $\lim_{k \to +\infty} \epsilon_k = 0$, satisfy, for each $k$, (I.17), (I.22) and, for all $v \in BV(\Omega)$, (I.19). A relabelling of the subsequence with indexes $k^2$ finally satisfies all these properties and (I.18) as well.
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Chapter II

Solvability via lower and upper solutions method

We develop in this chapter a lower and upper solutions method, in the spirit of [90] and [93], for the problem

\[
\begin{cases}
-\text{div}\left(\nabla u \sqrt{1 + |\nabla u|^2}\right) = f(x, u) + h(x) & \text{in } \Omega, \\
-\nabla u \cdot n \sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial\Omega.
\end{cases}
\] (II.1)

The result deals with the case where the lower solution is smaller than the upper solution. The approach is variational and provides the existence of a solution bracketed by the given lower and upper solutions.

Hereafter we assume

\((h_{II1})\) \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) having a Lipschitz boundary \(\partial\Omega\);

\((h_{II2})\) \(h \in L^p(\Omega)\), for some \(p > N\), and \(\kappa \in L^\infty(\partial\Omega)\);

\((h_{II3})\) there exists a constant \(\rho > 0\) such that

\[ \left| \int_B h \, dx - \int_{\partial\Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq (1 - \rho) \int_\Omega |D\chi_B| \]

for every Caccioppoli set \(B \subseteq \Omega\);

\((h_{II4})\) \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) satisfies the Carathéodory conditions, i.e., for a.e. \(x \in \Omega\), \(f(x, \cdot) : \mathbb{R} \to \mathbb{R}\) is continuous and, for every \(s \in \mathbb{R}\), \(f(\cdot, s) : \Omega \to \mathbb{R}\) is measurable.

We recall that functional \(\mathcal{J} : BV(\Omega) \to \mathbb{R}\) is defined as

\[ \mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} - \int_\Omega hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}^{N-1} \]

for every \(v \in BV(\Omega)\). According with the discussion in Section I.2, we give the following definition.
II. SOLVABILITY VIA LOWER AND UPPER SOLUTIONS METHOD

Definition of solution. We say that a function \( u \in BV(\Omega) \) is a solution of problem (II.1) if \( f(\cdot, u) \in L^N(\Omega) \) and

\[
\mathcal{J}(v) - \mathcal{J}(u) \geq \int_{\Omega} f(x, u)(v - u) \, dx, \tag{II.2}
\]

for every \( v \in BV(\Omega) \).

II.1 Lower and upper solutions

In this section we present different definitions of lower and upper solutions of increasing generality. A discussion of relations intercurring between different definitions is provided.

\( W^{2,p} \)-lower solution and \( W^{2,p} \)-upper solution.

We say that a function \( \alpha \in W^{1,\infty}(\Omega) \) is a \( W^{2,p} \)-lower solution of problem (II.1), for some \( p \geq 1 \), if there exist functions \( \alpha_1, \ldots, \alpha_m \in W^{2,p}(\Omega) \) such that \( \alpha = \alpha_1 \lor \cdots \lor \alpha_m \) and, for each \( i = 1, \ldots, m \), \( f(\cdot, \alpha_i) \in L^N(\Omega) \) and

\[
\left\{ \begin{array}{l}
- \text{div}\left( \nabla \alpha_i / \sqrt{1 + |\nabla \alpha_i|^2} \right) \leq f(x, \alpha_i) + h(x) \quad \text{in } \Omega, \\
- \nabla \alpha_i \cdot n / \sqrt{1 + |\nabla \alpha_i|^2} \geq \kappa(x) \quad \text{on } \partial \Omega.
\end{array} \right. \tag{II.3}
\]

We say that a function \( \beta \in W^{1,\infty}(\Omega) \) is a \( W^{2,p} \)-upper solution of problem (II.1), for some \( p \geq 1 \), if there exist functions \( \beta_1, \ldots, \beta_n \in W^{2,p}(\Omega) \) such that \( \beta = \beta_1 \land \cdots \land \beta_n \) and, for each \( j = 1, \ldots, n \), \( f(\cdot, \beta_j) \in L^N(\Omega) \) and

\[
\left\{ \begin{array}{l}
- \text{div}\left( \nabla \beta_j / \sqrt{1 + |\nabla \beta_j|^2} \right) \geq f(x, \beta_j) + h(x) \quad \text{in } \Omega, \\
- \nabla \beta_j \cdot n / \sqrt{1 + |\nabla \beta_j|^2} \leq \kappa(x) \quad \text{on } \partial \Omega.
\end{array} \right. \tag{II.3}
\]

\( BV \)-lower and \( BV \)-upper solutions.

We say that a function \( \alpha \in BV(\Omega) \) is a \( BV \)-lower solution of problem (II.1) if there exist functions \( \alpha_1, \ldots, \alpha_m \in BV(\Omega) \) such that \( \alpha = \alpha_1 \lor \cdots \lor \alpha_m \) and, for each \( i = 1, \ldots, m \), \( f(\cdot, \alpha_i) \in L^N(\Omega) \) and

\[
\mathcal{J}(\alpha_i + z) - \mathcal{J}(\alpha_i) \geq \int_{\Omega} f(x, \alpha_i) z \, dx, \tag{II.4}
\]

for all \( z \in BV(\Omega) \) with \( z \leq 0 \). We say that a lower solution \( \alpha \) is proper if \( \alpha \) is not a solution of (II.1). Finally, we say that a lower solution \( \alpha \) is strict if every solution \( u \) of (II.1) with \( u \geq \alpha \) satisfies

\[
\text{ess inf}_{\Omega} (u - \alpha) > 0.
\]
II. Solvability via lower and upper solutions method

Similarly, we say that a function $\beta \in BV(\Omega)$ is a $BV$-upper solution of problem (II.1) if there exist functions $\beta_1, \ldots, \beta_n \in BV(\Omega)$ such that $\beta = \beta_1 \wedge \cdots \wedge \beta_n$ and, for each $j = 1, \ldots, n$, $f(\cdot, \beta_j) \in L^N(\Omega)$ and

$$J(\beta_j + z) - J(\beta_j) \geq \int_{\Omega} f(x, \beta_j)z \, dx,$$

(II.5)

for every $z \in BV(\Omega)$ with $z \geq 0$.

We say that an upper solution $\beta$ is proper if $\beta$ is not a solution of (II.1). Finally, we say that an upper solution $\beta$ is strict if every solution $u$ of (II.1) with $u \leq \beta$ satisfies

$$\text{ess sup}_\Omega (u - \beta) < 0.$$

**Remark II.1.1** A function $\alpha \in BV(\Omega)$, with $f(\cdot, \alpha) \in L^N(\Omega)$, is a $BV$-lower solution of problem (II.1), with $m = 1$, if and only if $\alpha$ minimizes the functional

$$v \mapsto J(v) - \int_{\Omega} f(x, \alpha)v \, dx$$
on the cone $\{v \in BV(\Omega) : v \leq \alpha\}$.

Similarly, $\beta \in BV(\Omega)$, with $f(\cdot, \beta) \in L^N(\Omega)$, is a $BV$-upper solution of problem (II.1), with $n = 1$, if and only if $\beta$ minimizes the functional

$$v \mapsto J(v) - \int_{\Omega} f(x, \beta)v \, dx$$
on the cone $\{v \in BV(\Omega) : v \geq \beta\}$.

This notion of lower and upper solutions has already been used in [63, Section 12] for dealing with classical solutions of the minimal surface equation, as well as in [79], [90], [93] for studying the Dirichlet, the Neumann and the periodic one-dimensional problems for the prescribed mean curvature equation in the setting of bounded variation functions.

**Remark II.1.2** A function $u \in BV(\Omega)$ is a $BV$-solution of problem (II.1) if and only if it is simultaneously a $BV$-lower solution of problem (II.1), with $m = 1$, and a $BV$-upper solution of problem (II.1), with $n = 1$.

In fact, if $u \in BV(\Omega)$ is a $BV$-solution then trivially it is a $BV$-lower solution and a $BV$-upper solution of problem (II.1). On the other hand, let $u \in BV(\Omega)$ be simultaneously a $BV$-lower solution and a $BV$-upper solution. By definition, for all $z \in BV(\Omega)$, we have

$$J(u - z^-) - J(u) \geq \int_{\Omega} f(x, u)(-z^-) \, dx$$

and

$$J(u + z^+) - J(u) \geq \int_{\Omega} f(x, u)z^+ \, dx.$$
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Observe that $u + z^+ = u \lor (u + z)$ and $u - z^- = u \land (u + z)$. By Proposition I.2.7, we get

$$
\mathcal{J}(u + z^+) + \mathcal{J}(u - z^-) - 2\mathcal{J}(u) = \mathcal{J}(u \lor (u + z)) + \mathcal{J}(u \land (u + z)) - 2\mathcal{J}(u) \\
\leq \mathcal{J}(u + z) - \mathcal{J}(u).
$$

Then, summing up the two previous relations, we can conclude

$$
\mathcal{J}(u + z) - \mathcal{J}(u) \geq \int_{\Omega} f(x, u) z \, dx
$$

for all $z \in BV(\Omega)$, i.e., $u$ is a $BV$-solution of (II.1).

**Proposition II.1.1.** Suppose that $\alpha = \alpha_1 \lor \cdots \lor \alpha_m$ is a $W^{2,1}$-lower solution of problem (II.1) Then $\alpha$ is a $BV$-lower solution of problem (II.1).

**Proof.** We may suppose $m = 1$. Let $z \in W^{1,1}(\Omega)$ be such that $z \leq 0$. Multiplying the first inequality in (II.3) by $z$ and using the divergence theorem, we obtain

$$
\int_{\Omega} f(x, \alpha) z \, dx + \int_{\Omega} hz \, dx \leq \int_{\Omega} \frac{\nabla \alpha \cdot \nabla z}{1 + |\nabla \alpha|^2} \, dx - \int_{\partial \Omega} \frac{\nabla \alpha \cdot n}{1 + |\nabla \alpha|^2} z \, dH_{N-1} \\
\leq \int_{\Omega} \frac{\nabla \alpha \cdot \nabla z}{1 + |\nabla \alpha|^2} \, dx + \int_{\partial \Omega} \kappa z \, dH_{N-1}.
$$

Using the convexity of the function $s \mapsto \sqrt{1 + s^2}$, we get

$$
\int_{\Omega} f(x, \alpha) z \, dx \leq \int_{\Omega} \sqrt{1 + |\nabla (\alpha + z)|^2} \, dx - \int_{\Omega} \sqrt{1 + |\nabla \alpha|^2} \, dx \\
- \int_{\Omega} hz \, dx + \int_{\partial \Omega} \kappa z \, dH_{N-1}
$$

and then

$$
\mathcal{J}(\alpha + z) - \mathcal{J}(\alpha) \geq \int_{\Omega} f(x, \alpha) z \, dx
$$

for all $z \in W^{1,1}(\Omega)$ with $z \leq 0$. Now, let $z \in BV(\Omega)$ be such that $z \leq 0$. Set $v = \alpha + z$. By Proposition I.1.8 there exists a sequence $(w_n)_n$ in $W^{1,1}(\Omega)$ such that, for every $n$,

$$
w_n \leq v,
$$

$$
\lim_{n \to +\infty} w_n = v \quad \text{in } L^q(\Omega)
$$

with $q \in [1, 1^*)$ and a.e. in $\Omega$, and

$$
\lim_{n \to +\infty} \int_{\Omega} \sqrt{1 + |\nabla w_n|^2} \, dx = \int_{\Omega} \sqrt{1 + |\nabla v|^2} \, dx.
$$

By [61, Theorem 2.11, p. 37] we have that

$$
\lim_{n \to +\infty} \int_{\partial \Omega} \kappa w_n \, dH_{N-1} = \int_{\partial \Omega} \kappa v \, dH_{N-1}.
$$
and then \( \lim_{n \to +\infty} \mathcal{J}(w_n) = \mathcal{J}(v) \). Set, for each \( n \), \( z_n = w_n - \alpha \); we have \( z_n \in W^{1,1}(\Omega) \) and \( z_n \leq z \leq 0 \). Moreover, \( \lim_{n \to +\infty} z_n = z \) in \( L^q(\Omega) \) and a.e. in \( \Omega \). Hence we get, using the Lebesgue convergence theorem,

\[
\mathcal{J}(\alpha + z) = \mathcal{J}(v) = \lim_{n \to +\infty} \mathcal{J}(w_n) = \lim_{n \to +\infty} \mathcal{J}(\alpha + z_n)
\]

\[
\geq \lim_{n \to +\infty} \int_{\Omega} f(x, \alpha) z_n \, dx + \mathcal{J}(\alpha) = \int_{\Omega} f(x, \alpha) z \, dx + \mathcal{J}(\alpha),
\]

i.e., \( \alpha \) is a BV-lower solution of problem (II.1).

\( \square \)

A similar result can be proved for upper solutions.

**Proposition II.1.2.** Suppose that \( \beta = \beta_1 \wedge \cdots \wedge \beta_n \) is a \( W^{2,1} \)-upper solution of problem (II.1). Then \( \beta \) is a BV-upper solution of problem (II.1).

**Proposition II.1.3.** Let \( \alpha \) be a BV-lower solution of problem (II.1). Assume that \( m = 1 \) and \( \alpha \in W^{2,1}(\Omega) \). Then \( \alpha \) is a \( W^{2,1} \)-lower solution of problem (II.1).

**Proof.** Fix \( z \in W^{1,1}(\Omega) \), with \( z \leq 0 \). From (II.4) we have, for every \( s > 0 \),

\[
\int_{\Omega} f(x, \alpha) z \, dx \leq \frac{1}{s} (\mathcal{J}(\alpha + sz) - \mathcal{J}(\alpha))
\]

\[
= \frac{1}{s} \left( \int_{\Omega} \sqrt{1 + |\nabla(\alpha + sz)|^2} \, dx - \int_{\Omega} \sqrt{1 + |\nabla \alpha|^2} \, dx \right)
\]

\[
- \int_{\Omega} h z \, dx + \int_{\partial \Omega} \kappa z \, dH_{N-1}.
\]

We recall that the area functional restricted to \( W^{1,1}(\Omega) \) is Gateaux differentiable as shown in [40] and hence, letting \( s \to 0^+ \) in the last inequality, we get

\[
\int_{\Omega} \frac{\nabla \alpha \cdot \nabla z}{\sqrt{1 + |\nabla \alpha|^2}} \, dx - \int_{\Omega} h z \, dx + \int_{\partial \Omega} \kappa z \, dH_{N-1} \geq \int_{\Omega} f(x, \alpha) z \, dx.
\]

Using the divergence theorem, we get

\[
\int_{\Omega} \frac{\nabla \alpha \cdot \nabla z}{\sqrt{1 + |\nabla \alpha|^2}} \, dx = -\int_{\Omega} \text{div} \left( \nabla \alpha / \sqrt{1 + |\nabla \alpha|^2} \right) z \, dx
\]

\[
+ \int_{\partial \Omega} \left( \nabla \alpha \cdot n / \sqrt{1 + |\nabla \alpha|^2} \right) z \, dH_{N-1}
\]

and then

\[
\int_{\Omega} f(x, \alpha) z \, dx + \int_{\Omega} h z \, dx \leq -\int_{\Omega} \text{div} \left( \nabla \alpha / \sqrt{1 + |\nabla \alpha|^2} \right) z \, dx
\]

\[
+ \int_{\partial \Omega} \left( \nabla \alpha \cdot n / \sqrt{1 + |\nabla \alpha|^2} \right) z \, dH_{N-1} + \int_{\partial \Omega} \kappa z \, dH_{N-1}.
\]
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By choosing $z \in W^{1,1}(\Omega)$ with compact support, we obtain
\[-\text{div}\left(\nabla \alpha / \sqrt{1 + |\nabla \alpha|^2}\right) \leq f(x, \alpha) + h(x)\]
a.e. in $\Omega$. Moreover, by a standard construction we can choose $z \in W^{1,1}(\Omega)$ such that the quantity
\[\int_{\Omega} \text{div}\left(\nabla \alpha / \sqrt{1 + |\nabla \alpha|^2}\right) z \, dx + \int_{\Omega} f(x, \alpha) z \, dx + \int_{\Omega} h z \, dx\]
is as small as we wish. Since $z \leq 0$, we can conclude
\[-\nabla \alpha \cdot n / \sqrt{1 + |\nabla \alpha|^2} \geq \kappa(x)\]
a.e. on $\partial \Omega$.

A similar result can be proved for upper solutions.

**Proposition II.1.4.** Let $\beta$ be a $BV$-upper solution of problem (II.1). Assume that $n = 1$ and $\beta \in W^{2,1}(\Omega)$. Then $\beta$ is a $W^{2,1}$-upper solution of problem (II.1).

II.2 Existence result in case of well-ordered lower and upper solutions

Here we present an existence theorem in the presence of a couple of well-ordered $BV$-lower and $BV$-upper solutions. Other results in this direction can be found in [79], [95], where the Dirichlet case is treated.

**Theorem II.2.1.** Assume $(h_{II0})$, $(h_{II1})$, $(h_{II2})$, $(h_{II3})$,

$(h_{II4})$ there exist a $BV$-lower solution $\alpha$ and a $BV$-upper solution $\beta$ of the problem (II.1) satisfying $\alpha \leq \beta$

and

$(h_{II5})$ there exists $\gamma \in L^p(\Omega)$, for some $p > N$, such that $|f(x, s)| \leq \gamma(x)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ such that

\[s \in \left[\min_{i=1,\ldots,m} \alpha_i(x), \max_{j=1,\ldots,n} \beta_j(x)\right].\]

Then the problem (II.1) has at least one solution $u \in BV(\Omega)$ such that

$\alpha \leq u \leq \beta$ and $\mathcal{I}(u) = \min_{\alpha \leq v \leq \beta} \mathcal{I}(v)$

where

$\mathcal{I}(v) = J(v) - \int_{\Omega} [F(x, v) - F(x, \alpha)] \, dx$
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for all \( v \in BV(\Omega) \) and \( F(x, s) = \int_0^s f(x, \xi) \, d\xi \).

Moreover, there exist solutions \( v, w \) of the problem (II.1), with \( \alpha \leq v \leq w \leq \beta \), such that every solution \( u \) of problem (II.1), with \( \alpha \leq u \leq \beta \), satisfies \( v \leq u \leq w \).

**Proof.** The argument follows some lines that are rather standard the context quasilinear elliptic problems, even though some modifications are needed.

Set \( q = \frac{p}{p-1} \in ]1, 1^*] \). Let \( \alpha = \alpha_1 \land \cdots \land \alpha_m \) and \( \beta = \beta_1 \land \cdots \land \beta_n \) where, for each \( i = 1, \ldots, m \), \( \alpha_i \) satisfies (II.4) for all \( z \in BV(\Omega) \) with \( z \leq 0 \) and, for all \( j = 1, \ldots, n \), \( \beta_j \) satisfies (II.5) for all \( z \in BV(\Omega) \) with \( z \geq 0 \).

**Step 1. A modified problem.** Let us set, for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \),

\[
Q(s) = \begin{cases} \frac{q}{2} s^2 & \text{if } |s| \leq 1, \\ |s|^q + \frac{q-2}{2} & \text{if } |s| > 1. \end{cases} \tag{II.6}
\]

Moreover, we set

\[
h_i(x, s) = \begin{cases} f(x, \alpha_i(x)) + Q'(\alpha_i(x)) & \text{if } s < \alpha_i(x), \\ f(x, s) + Q'(s) & \text{if } s \geq \alpha_i(x), \end{cases}
\]

\[
k_j(x, s) = \begin{cases} f(x, \beta_j(x)) + Q'(\beta_j(x)) & \text{if } s > \beta_j(x), \\ f(x, s) + Q'(s) & \text{if } s \leq \beta_j(x), \end{cases}
\]

for \( i = 1, \ldots, m \), \( j = 1, \ldots, n \), and

\[
\ell(x, s) = \begin{cases} \max_{i=1,\ldots,m} h_i(x, s) & \text{if } s < \alpha(x), \\ f(x, s) + Q'(s) & \text{if } \alpha(x) \leq s \leq \beta(x), \\ \min_{j=1,\ldots,n} k_j(x, s) & \text{if } s > \beta(x). \end{cases}
\]

Clearly, \( Q \) is of class \( C^1 \) and \( \ell \) satisfies the Carathéodory conditions (hII3) and (hII5). Notice that, for every \( s \in \mathbb{R} \),

\[
Q(s) \geq |s|^q - 1 \tag{II.7}
\]

and, by construction, there exists a function \( \lambda \in L^p(\Omega) \) such that, for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \),

\[
|\ell(x, s)| \leq \lambda(x).
\]

Setting \( L(x, s) = \int_0^s \ell(x, \xi) d\xi \), we have

\[
|L(x, s)| \leq \lambda(x)|s| \tag{II.8}
\]

for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \).
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Let us consider the modified problem
\[
\begin{aligned}
-\text{div} \left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) &= \ell(x, u) - Q'(u) + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial \Omega.
\end{aligned}
\] (II.9)

Of course, a solution of the problem (II.9) is a function \( u \in BV(\Omega) \) such that
\[
\mathcal{J}(v) - \mathcal{J}(u) \geq \int_{\Omega} (\ell(x, u) - Q'(u))(v - u) \, dx
\] (II.10)
for every \( v \in BV(\Omega) \).

**Step 2. Existence of solutions of the modified problem.** Define a functional \( K : BV(\Omega) \to \mathbb{R} \) by setting
\[
K(v) = \mathcal{J}(v) + \int_{\Omega} Q(v) \, dx - \int_{\Omega} L(x, v) \, dx
\]
for every \( v \in BV(\Omega) \).

We want to show the existence of a minimum point for the functional \( K \) on the space \( BV(\Omega) \).

**Claim 1.** \( \inf_{v \in BV(\Omega)} K(v) > -\infty \) and \( \lim_{\|v\|_{BV} \to +\infty} K(v) = +\infty \). Using Corollary I.2.4, the relations (II.7), (II.8) and standard inequalities, we can find constants \( d_1, d_2 > 0 \) such that
\[
K(v) = \mathcal{J}(v) + \int_{\Omega} Q(v) \, dx - \int_{\Omega} L(x, v) \, dx \\
\geq \rho \int_{\Omega} |Dv| + \|v\|_{L^q}^q - |\Omega| - \int_{\Omega} \lambda(x)|v| \, dx \\
\geq \rho \int_{\Omega} |Dv| + \|v\|_{L^q}^q - |\Omega| - \|\lambda\|_{L^p} \|v\|_{L^q} \\
\geq d_1 \|v\|_{BV} - d_2
\]
for every \( v \in BV(\Omega) \) and where \( \rho \) is defined in (hII2). This yields the conclusions of Claim 1.

**Claim 2.** There exists \( \min_{v \in BV(\Omega)} K(v) \). Let \( (u_n) \) be a minimizing sequence. Claim 1 implies that \( (u_n) \) is bounded in \( BV(\Omega) \). Hence, by Proposition I.1.2, there is a subsequence of \( (u_n) \), which we still denote by \( (u_n) \), and a function \( u \in BV(\Omega) \) such that \( \lim_{n \to +\infty} u_n = u \) in \( L^q(\Omega) \). We have
\[
\liminf_{n \to +\infty} \mathcal{J}(u_n) \geq \mathcal{J}(u),
\]
as, by Proposition I.2.6, \( \mathcal{J} \) is lower semicontinuous with respect to the \( L^q \)-convergence in \( BV(\Omega) \), and moreover
\[
\lim_{n \to +\infty} \int_{\Omega} (Q(u_n) - L(x, u_n)) \, dx = \int_{\Omega} (Q(u) - L(x, u)) \, dx,
\]

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as the functionals \( v \mapsto \int_\Omega Q(v) \, dx \) and \( v \mapsto \int_\Omega L(x, v) \, dx \) are continuous in \( L^q(\Omega) \).

Hence, we conclude that

\[
\inf_{v \in BV(\Omega)} \mathcal{K}(v) = \lim_{n \to +\infty} \mathcal{K}(u_n) \geq \mathcal{K}(u),
\]

that is, \( \mathcal{K}(u) = \min_{v \in BV(\Omega)} \mathcal{K}(v) \).

Observe that any minimizer \( u \) of \( \mathcal{K} \) satisfies (II.10) for every \( v \in BV(\Omega) \). Indeed, let \( v \in BV(\Omega) \) and take \( s \in ]0, 1[ \). By the convexity of \( \mathcal{J} \) we obtain

\[
(1-s) \mathcal{J}(u) + s \mathcal{J}(v) - \mathcal{J}(u) \geq \mathcal{J}((1-s)u + sv) - \mathcal{J}(u)
\]

\[
\geq - \int_\Omega (Q(u + s(v-u)) - L(x, u + s(v-u))) - (Q(u) - L(x, u)) \, dx.
\]

Hence we obtain, dividing by \( s \),

\[
\mathcal{J}(v) - \mathcal{J}(u) \geq - \int_\Omega \frac{1}{s} (Q(u + s(v-u)) - L(x, u + s(v-u))) - (Q(u) + L(x, u)) \, dx
\]

and, letting \( s \to 0^+ \), we get

\[
\mathcal{J}(v) - \mathcal{J}(u) \geq - \int_\Omega (Q'(u) - \ell(x, u)) (v - u) \, dx,
\]

i.e., (II.10) holds. Therefore we conclude that the modified problem (II.9) has at least one solution.

**Step 3.** Any solution \( u \) of (II.9) satisfies \( \alpha \leq u \leq \beta \). Let us show that \( u \leq \beta \); by a similar argument one sees that \( u \geq \alpha \). Fix \( j \in \{1, \ldots, n\} \) and prove that \( u \leq \beta_j \). Take \( v = u \wedge \beta_j = u - (u - \beta_j)^+ \) as a test function in (II.10). We obtain

\[
\mathcal{J}(u \wedge \beta_j) - \mathcal{J}(u) \geq - \int_\Omega (\ell(x, u) - Q'(u))(u - \beta_j)^+ \, dx
\]

\[
\geq - \int_\Omega (k_j(x, \beta_j) - Q'(u)) (u - \beta_j)^+ \, dx \tag{II.11}
\]

\[
= - \int_\Omega f(x, \beta_j)(u - \beta_j)^+ \, dx + \int_\Omega (Q'(u) - Q'(\beta_j))(u - \beta_j)^+ \, dx.
\]

Taking \( z = (u - \beta_j)^+ \) as a test function in (II.5), we have, as \( u \vee \beta_j = \beta_j + (u - \beta_j)^+ \),

\[
\mathcal{J}(u \vee \beta_j) - \mathcal{J}(\beta_j) \geq \int_\Omega f(x, \beta_j)(u - \beta_j)^+ \, dx. \tag{II.12}
\]

Summing (II.11) and (II.12) and using Proposition 1.2.7, we find

\[
0 \geq \mathcal{J}(u \wedge \beta_j) + \mathcal{J}(u \vee \beta_j) - \mathcal{J}(\beta_j) - \mathcal{J}(u)
\]

\[
\geq \int_0^\infty (Q'(u) - Q'(\beta_j))(u - \beta_j)^+ \, dx \geq 0.
\]

As \( Q' \) is strictly increasing, we conclude that \( (u - \beta_j)^+ = 0 \) a.e. in \( \Omega \) and therefore \( u \leq \beta_j \).
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Step 4. There is a solution \( u \) of the problem (II.1) such that

\[
\alpha \leq u \leq \beta \quad \text{and} \quad I(u) = \min_{\alpha \leq v \leq \beta} I(v).
\]

Let \( u \) be a solution of the problem (II.9). As \( u \) is such that \( \alpha \leq u \leq \beta \), we have \( \ell(\cdot, u) - Q'(u) = f(\cdot, u) \) and hence \( u \) is a solution of the problem (II.1). Further, as for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \), with \( \alpha(x) \leq s \leq \beta(x) \), we have

\[
L(x, s) = L(x, \alpha) + F(x, s) - F(x, \alpha) + Q(s) - Q(\alpha).
\]

Hence we get for every \( v \in BV(\Omega) \), with \( \alpha \leq v \leq \beta \)

\[
K(v) = J(v) + \int_{\Omega} Q(v) \, dx - \int_{\Omega} [L(x, \alpha) + F(x, v) - F(x, \alpha) + Q(v) - Q(\alpha)] \, dx
= J(v) - \int_{\Omega} L(x, \alpha) \, dx - \int_{\Omega} [F(x, v) - F(x, \alpha)] \, dx + \int_{\Omega} Q(\alpha) \, dx.
\]

Since \( \int_{\Omega} L(x, \alpha) \, dx \) and \( \int_{\Omega} Q(\alpha) \, dx \) are constants, and \( u \) minimizes \( K \), we conclude that \( u \) minimizes \( I \) on the set of all \( v \in BV(\Omega) \), with \( \alpha \leq v \leq \beta \).

Step 5. Existence of extremum solutions. Let us set

\[
S = \{ u \in BV(\Omega) : u \text{ is a solution of (II.1) such that } \alpha \leq u \leq \beta \}.
\]

Claim 1. \( S \) is a compact subset of \( L^q(\Omega) \). Let \( (u_n)_n \) be a sequence in \( S \). For each \( n \), we have

\[
||u_n||_{L^q} \leq |||\alpha|| \vee ||\beta||||_{L^q} (\text{II.13})
\]

and, since each \( u_n \) is a solution of (II.1), we have

\[
J(v) - \int_{\Omega} f(x, u_n) v \, dx \geq J(u_n) - \int_{\Omega} f(x, u_n) u_n \, dx (\text{II.14})
\]

for every \( v \in BV(\Omega) \). Recall that hypothesis \((h_{II5})\) implies that there exists \( \gamma \in L^p(\Omega) \) such that

\[
|f(x, u_n)| \leq \gamma(x) (\text{II.15})
\]

for a.e. \( x \in \Omega \) and every \( n \in \mathbb{N} \). Hence, taking \( v = 0 \) in (II.14), by (II.13) and (II.15) we get

\[
J(u_n) \leq J(0) + \int_{\Omega} f(x, u_n) u_n \, dx \leq |\Omega| + \|f(\cdot, u_n)||_{L^p} ||u_n||_{L^q}
\leq |\Omega| + \|\gamma||_{L^p} ||\alpha|| \vee ||\beta||||_{L^q}.
\]

By Corollary I.2.4, and (II.13), we conclude that \( (u_n)_n \) is bounded in \( BV(\Omega) \). Therefore, there is a subsequence of \( (u_n)_n \), which we still denote by \( (u_n)_n \), and a function \( u \in
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$BV(\Omega)$ such that $\lim_{n \to +\infty} u_n = u$ in $L^q(\Omega)$ and a.e. in $\Omega$. Hence we have $\alpha \leq u \leq \beta$ and by Proposition I.2.6 we get

$$\mathcal{J}(u) \leq \liminf_{n \to +\infty} \mathcal{J}(u_n).$$

Moreover, (II.13), (II.15) and the Lebesgue convergence theorem yield

$$\lim_{n \to +\infty} \int_{\Omega} f(x, u_n) \, dx = \int_{\Omega} f(x, u) \, dx$$

for every $v \in BV(\Omega)$ and

$$\lim_{n \to +\infty} \int_{\Omega} f(x, u_n) u_n \, dx = \int_{\Omega} f(x, u) u \, dx.$$

Thus we conclude that

$$\mathcal{J}(v) - \int_{\Omega} f(x, u) v \, dx \geq \mathcal{J}(u) - \int_{\Omega} f(x, u) u \, dx$$

for every $v \in BV(\Omega)$. Therefore $u \in S$. This proves the compactness of $S$ in $L^q(\Omega)$.

**Claim 2.** $S$ has a minimum element $v$ and a maximum element $w$. Let us prove the existence of min $S$; a similar argument shows the existence of max $S$. For each $u \in S$ define the closed subset of $L^q(\Omega)$

$$C_u = \{v \in S : v \leq u\}.$$

The family $(C_u)_{u \in S}$ has the finite intersection property. Indeed, if $u_1, u_2 \in S$, then it is easy to see that $u_1 \land u_2$ is an upper solution of problem (II.1), with $\alpha \leq u_1 \land u_2$. Hence, there is a solution $u$ of problem (II.1), with $\alpha \leq u \leq u_1 \land u_2 \leq \beta$; i.e., $u \in C_{u_1} \cap C_{u_2}$.

The compactness of $S$ implies that there exists $v \in S$ such that $v \in \bigcap_{u \in S} C_u$; that is, $v \leq u$ for every $u \in S$.

**Remark II.2.1** If instead of (hII4) we assume the stronger condition

(hII4* ) there exist a $BV$-lower solution $\alpha$ and a $BV$-upper solution $\beta$ of the problem (II.1) satisfying $\alpha \leq \beta$ with $\alpha_i, \beta_j \in L^\infty(\Omega)$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$,

then in this case we can replace assumption (hII5) with

(hII3') $f : \Omega \times [-\rho, \rho] \to \mathbb{R}$ satisfies the $L^p$-Carathéodory conditions, for some $p > N$, i.e., for a.e. $x \in \Omega$, $f(x, \cdot) : [-\rho, \rho] \to \mathbb{R}$ is continuous, and for every $s \in [-\rho, \rho]$, $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable, where

$$\rho > \max_{i=1, \ldots, m; j=1, \ldots, n} \{\|\alpha_i\|_\infty, \|\beta_j\|_\infty\};$$

there exists $\gamma \in L^p(\Omega)$ such that $|f(x, s)| \leq \gamma(x)$ for a.e. $x \in \Omega$ and every $s \in [-\rho, \rho]$.  

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Remark II.2.2 Let the assumptions of Theorem II.2.1 be satisfied. Assume $N = 1$ and let $\Omega = [0,T]$. If a solution $u$ of problem (II.1), such that $\alpha \leq u \leq \beta$ and

$$I(u) = \min_{v \in BV(0,T)} \ I(v),$$

also satisfies

$$\text{ess inf}_{[0,T]} (u - \alpha) > 0 > \text{ess sup}_{[0,T]} (u - \beta),$$

then $u$ is a local minimum point of $I$ in $BV(0,T)$. Indeed, due to the embedding of $BV(0,T)$ in $L^\infty(0,T)$, there exists a number $\delta > 0$ such that, if $v \in BV(0,T)$ satisfies $\|u - v\|_{BV} < \delta$, then $\alpha \leq v \leq \beta$ and hence

$$I(u) = \min_{\|u - v\|_{BV} < \delta} I(v).$$

II.3 Examples

In order to illustrate the applicability of Theorem II.2.1, we discuss here few sample applications.

Proposition II.3.1. Assume $(h_{II0})$ and $(h_{II3})$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the $L^p$-Carathéodory conditions, i.e., for a.e. $x \in \Omega$, $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous and, for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable.

Moreover for each $r > 0$, there exists $\gamma \in L^p(\Omega)$ such that $|f(x,s)| \leq \gamma(x)$ for a.e. $x \in \Omega$ and every $s \in [-r,r]$.

Suppose further $h = 0$ and $\kappa = 0$. If $\alpha \in \mathbb{R}$ is such that $f(x,\alpha) \geq 0$ a.e. in $\Omega$, then $\alpha$ is a $W^{2,\infty}$-lower solution of problem (II.1). Similarly, if $\beta \in \mathbb{R}$ is such that $f(x,\beta) \leq 0$ a.e. in $\Omega$, then $\beta$ is a $W^{2,\infty}$-upper solution of problem (II.1).

We use now Theorem II.2.1 to recover an existence result for a particular class of capillarity-type problem. It is clear that in this case much more information on the regularity of the (unique) solution are known (see, e.g., [58]).

Proposition II.3.2. Assume $\Omega$ is a bounded domain in $\mathbb{R}^N$ of class $C^2$. Take any $\kappa \in L^\infty(\partial \Omega)$ with $\|\kappa\|_\infty < 1$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the $L^\infty$-Carathéodory conditions. If $g$ is of class $C^1$ with respect to the second variable and

$$\text{ess sup}_{\Omega \times \mathbb{R}} \frac{\partial g}{\partial s} < 0,$$

then problem

$$\begin{cases}
-\text{div}\left(\nabla u/\sqrt{1 + |\nabla u|^2}\right) = g(x,u) & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial \Omega
\end{cases}$$

has at least one $BV$-solution.
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Proof. In order to construct a couple of well-ordered $BV$-lower and upper solutions we need the following technical result.

Claim 1. There exists a function $F \in C^2(\omega; \mathbb{R})$, where $\omega$ is an open set with $\overline{\Omega} \subset \omega$, such that

$$\Omega = \{ x \in \mathbb{R}^N : F(x) < 0 \}, \quad \partial \Omega = \{ x \in \mathbb{R}^N : F(x) = 0 \}$$

(II.18)

and

$$\nabla F(x) = n(x) |\nabla F(x)|$$

for all $x \in \partial \Omega$. Since $\Omega$ is a bounded domain of class $C^2$, by the compactness of $\partial \Omega$, there exists a finite open cover $\{ U_i \}_{i=1}^n$ of $\partial \Omega$ such that for each $i = 1, \ldots, n$ there exists $g_i : U_i \to \mathbb{R}$, of class $C^2$ satisfying

$$\Omega \cap U_i = \{ x \in U_i : g_i(x) < 0 \},$$
$$\partial \Omega \cap U_i = \{ x \in U_i : g_i(x) = 0 \},$$
$$\Omega \setminus U_i = \{ x \in U_i : g_i(x) > 0 \}.$$ 

Since $\{ U_i \}_{i=1}^n$ is a finite open cover of $\partial \Omega$ we can find an open set $U_0$ with $\overline{U_0} \subset \Omega$ such that $U_0 \cup U_1 \cup \cdots \cup U_n$ is an open cover of $\Omega$. Set $g_0 : U_0 \to \mathbb{R}$ as $g_0(x) = -1$ for all $x \in U_0$. Let $\{ \lambda_i \}_{i=0}^n$ be a partition of unity subordinate to the open cover $\{ U_i \}_{i=0}^n$ and let us define an open set $\omega = U_0 \cup U_1 \cup \cdots \cup U_n$ and a function $F : \omega \to \mathbb{R}$ by

$$F(x) = \sum_{i=0}^n \lambda_i(x) g_i(x).$$

Notice that $F$ is of class $C^2$ and satisfies (II.18). Moreover, for all $x \in \partial \Omega$ we have

$$\frac{\nabla F(x)}{|\nabla F(x)|} = n(x),$$

(II.19)

where $n(x)$ is the unit outer normal vector to $\partial \Omega$.

We show that there exists a suitable choice of constants $a, c \in \mathbb{R}$ such that the function $\alpha : \Omega \to \mathbb{R}$ defined by

$$\alpha(x) = aF(x) - c$$

(II.20)

is a $W^{2,\infty}$-lower solution for the problem (II.17). By (II.16), there exist constants $\varepsilon > 0$, $k_1$ and $k_2$ such that

$$g(x,s) \leq -\varepsilon s + k_1$$

for a.e. $x \in \Omega$ and every $s \geq 0$, and

$$g(x,s) \geq -\varepsilon s + k_2$$

for a.e. $x \in \Omega$ and every $s < 0$. 

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We observe that

\[
\frac{\nabla \alpha}{\sqrt{1 + |\nabla \alpha|^2}} = \frac{a \nabla F}{\sqrt{1 + a^2|\nabla F|^2}}
\]

and simple computations yield

\[
-\text{div} \left( \frac{\nabla \alpha}{\sqrt{1 + |\nabla \alpha|^2}} \right) = \frac{a \nabla \left( \sqrt{1 + a^2|\nabla F|^2} \right) \cdot \nabla F}{1 + a^2|\nabla F|^2} - \frac{a \Delta F}{\sqrt{1 + a^2|\nabla F|^2}}
\]

Moreover, as \( F \in C^0(\overline{\Omega}) \), for each \( a \in \mathbb{R} \), we have

\[
g(x, \alpha(x)) = g(x, aF(x) - c) \geq -\varepsilon(aF(x) - c) + k_2
\]

for a.e. \( x \in \Omega \) and every sufficiently large \( c > 0 \).

Since the function \( F \in C^2(\omega) \), for each \( a \in \mathbb{R} \), possibly taking \( c > 0 \) even larger, we get

\[
a \nabla \left( \sqrt{1 + a^2|\nabla F|^2} \right) \cdot \nabla F \geq \frac{a \nabla F}{\sqrt{1 + a^2|\nabla F|^2}} \leq g(x, aF(x) - c)
\]

for all \( x \in \overline{\Omega} \), i.e.,

\[
-\text{div} \left( \frac{\nabla \alpha}{\sqrt{1 + |\nabla \alpha|^2}} \right) \leq g(x, \alpha).
\]

In order to conclude that \( \alpha \) is a \( W^{2,\infty} \)-lower solution for the problem (II.17), we have to show that there exists \( a \in \mathbb{R} \) such that

\[
-\frac{a \nabla F \cdot n}{\sqrt{1 + a^2|\nabla F|^2}} \geq \kappa(x)
\]

for all \( x \in \partial \Omega \). Set \( \kappa = ||\kappa||_\infty < 1 \). Then, by (II.19), we can take \( a < 0 \) sufficiently large, such that

\[
-\frac{a \nabla F \cdot \nabla F}{\sqrt{1 + a^2|\nabla F|^2}} = -\frac{a |\nabla F|}{\sqrt{1 + a^2|\nabla F|^2}} \geq \kappa
\]

holds a.e. in \( \partial \Omega \). This implies that the function \( \alpha \) defined in (II.20) is a \( W^{2,\infty} \)-lower solution of problem (II.17) with \( a < 0 \) and large \( c > 0 \). Similar computations show that there exists \( \beta : \Omega \to \mathbb{R} \), defined as

\[
\beta(x) = a'F(x) + c'
\]

that is, a \( W^{2,\infty} \)-upper solution of (II.17) with \( a' > 0 \) and large \( c' > 0 \). Finally a suitable choice of large constants \( c, c' > 0 \) guarantees that \( \alpha \leq \beta \).

Now we are in condition of applying Theorem II.2.1 and we conclude that there exists a solution \( u \in BV(\Omega) \cap L^\infty(\Omega) \) of problem (II.17).
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Another example is the capillarity-type equation with a periodic right-hand side.

**Proposition II.3.3.** Assume $(h_{II0})$, and take any $h \in L^\infty(\Omega)$, such that $\int_\Omega h \, dx = 0$. If $A \geq \|h\|_\infty$, then problem

\[
\begin{align*}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= A \sin u + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]

(II.21)

has at least a solution.

**Proof.** By hypothesis, we have

\[ A + h(x) \geq 0 \quad \text{and} \quad -A + h(x) \leq 0 \]

for a.e. $x \in \Omega$. Hence, defining $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega \to \mathbb{R}$ as

\[ \alpha(x) = \frac{\pi}{2} \quad \text{and} \quad \beta(x) = \frac{3\pi}{2}, \]

it is easy to verify that $\alpha$ and $\beta$ are respectively lower and upper solutions of problem (II.21). Then Theorem II.2.1 gives us the existence of a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ such that $\frac{\pi}{2} \leq u \leq \frac{3\pi}{2}$. It is clear that for all $k \in \mathbb{Z}$ the functions defined by

\[ u_k = u + 2k\pi \]

are solutions of (II.21). \qed

In the proof of this last result it is easy to see that a couple of non-well-ordered lower and upper solutions, given by

\[ \alpha = \frac{\pi}{2} \quad \text{and} \quad \beta = -\frac{\pi}{2}, \]

also exists. At the moment there are no known results that allow to face this case for the capillarity equation, except in the one dimensional case: this result is presented in Chapter V. Anyway, in this specific case we can improve Propositions II.3.3 proving the existence of a solution of (II.21) for any given $A \in \mathbb{R}$, if $\int_\Omega h \, dx = 0$. Moreover, by a three-solution-type result, we are able to find a second solution, not differing from the other one by an integer multiple of $2\pi$.

We start providing a lemma concerning the existence of a third solution in the presence of two solutions at the same critical level.

**Proposition II.3.4.** Assume $(h_{II0})$, $(h_{II1})$, $(h_{II2})$, $(h_{II3})$ and $(h_{II6})$ there exist $u_1, u_2 \in BV(\Omega)$ solutions of (II.1), such that $u_1 < u_2$ and

\[ I(u_1) = I(u_2), \]

where $I : BV(\Omega) \to \mathbb{R}$ is defined as

\[ I(v) = \int_\Omega \sqrt{1 + |Dv|^2} - \int_\Omega hv \, dx + \int_{\partial\Omega} \kappa v \, d\mathcal{H}^{N-1} - \int_\Omega F(x,v) \, dx. \]

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Suppose moreover
\((h_{15}')\) there exists \(\gamma \in L^p(\Omega)\), for some \(p > N\), such that
\[ |f(x,s)| \leq \gamma(x) \]
for a.e. \(x \in \Omega\) and every \(s \in [u_1(x), u_2(x)]\).

Then problem (II.1) has at least one solution \(u \in BV(\Omega)\) such that
\[ u_1 < u < u_2. \]

Proof. Set \(q = \frac{p}{p - 1} \in [1, 1^*].\)

\textbf{Step 1. A modified problem.} As in Step 1 of Theorem II.2.1, taking \(u_1\) and \(u_2\) as a lower solution and an upper solution, respectively, we can consider the modified problem
\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = g(x,u) - Q'(u) + h & \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa & \text{on } \partial \Omega,
\end{cases}
\]
where
\[
g(x,s) = \begin{cases} 
  f(x,u_1(x)) + Q'(u_1(x)) & \text{if } s < u_1(x), \\
  f(x,s) + Q'(s) & \text{if } u_1(x) \leq s \leq u_2(x), \\
  f(x,u_2(x)) + Q'(u_2(x)) & \text{if } s > u_2(x)
\end{cases}
\]
and \(Q\) is defined as in (II.6). We remark that a solution of (II.22) is a function \(u \in BV(\Omega)\) such that
\[
\mathcal{J}(v) - \mathcal{J}(u) \geq \int_{\Omega} (g(x,u) - Q'(u))(v - u) \, dx
\]
for every \(v \in BV(\Omega)\). Set \(G(x,s) = \int_0^s g(x,\xi) d\xi\) and let us define \(\bar{I} : BV(\Omega) \to \mathbb{R}\) as
\[
\bar{I}(v) = \mathcal{J}(v) + \int_{\Omega} Q(v) \, dx - \int_{\Omega} G(x,v) \, dx.
\]
Obviously, a critical point of \(\bar{I}\) is a function \(u \in BV(\Omega)\) satisfying the variational inequality (II.24), i.e., it is a solution of (II.22). By (II.23), it follows
\[
\bar{I}(u_1) = \bar{I}(u_2)
\]
and, for the sake of simplicity, set \(C = \bar{I}(u_1) = \bar{I}(u_2)\).

As in Step 2 of Theorem II.2.1, applying Corollary I.2.4, the functional \(\bar{I}\) is bounded from below and coercive on \(BV(\Omega)\). Indeed, we have the existence of \(c_1, c_2 > 0\) such that
\[
\bar{I}(v) = \mathcal{J}(v) + \int_{\Omega} Q(v) \, dx - \int_{\Omega} G(x,v) \, dx \geq c_1\|v\|_{BV} - c_2
\]
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for all $v \in BV(\Omega)$ and hence $\bar{I}$ has a global minimizers $\bar{u} \in BV(\Omega)$, i.e.,

$$\min_{u \in BV(\Omega)} \bar{I}(u) = \bar{I}(\bar{u}).$$

Step 2. Existence of a third solution of the modified problem. We start observing that, if the global minimizer $\bar{u} \in BV(\Omega)$ is such that $\bar{I}(\bar{u}) < C$, then $\bar{u}$ satisfies (II.24) and hence it is a solution of (II.22) such that $\bar{u} \neq u_1$ and $\bar{u} \neq u_2$.

On the other side, if $\bar{I}(\bar{u}) = C$, we have that both $u_1$ and $u_2$ are global minimizer of $I$ on the space $BV(\Omega)$. In this setting let us define

$$\Gamma = \{ \gamma \in C^0([0, 1], BV(\Omega)) : \gamma(0) = u_1, \gamma(1) = u_2 \}$$

and set

$$c_T := \inf_{\gamma \in \Gamma} \max_{\xi \in [0, 1]} \bar{I}(\gamma(\xi)).$$

By the previous considerations, we have that $c_T \geq C$ and hence we need to investigate two different cases.

Case $c_T > C$. Accordingly, we are in condition to apply Lemma I.3.1 that yields the existence of sequences $(v_k)_k$ and $(\varepsilon_k)_k$, with $v_k \in BV(\Omega)$ and $\varepsilon_k \in \mathbb{R}$ such that

$$\lim_{k \to +\infty} \bar{I}(v_k) = c_T$$

and, for all $v \in BV(\Omega)$,

$$J(v) - J(v_k) \geq \int_{\Omega} (g(x, v_k) - Q'(v_k))(v - v_k) \, dx + \varepsilon_k \|v - v_k\|_{BV}. \tag{II.27}$$

By (II.26) and (II.25) we conclude that the sequence $(v_n)_n$ is uniformly bounded in $BV(\Omega)$ and hence there exist a subsequence, we still denote by $(v_k)_k$, and a function $u \in BV(\Omega)$, such that $
lim_{k \to +\infty} v_k = u$ in $L^q(\Omega)$ and a.e. in $\Omega$. The Lebesgue convergence theorem implies

$$\lim_{k \to +\infty} g(x, v_k) - Q'(v_k) = g(x, u) - Q'(u)$$

in $L^p(\Omega)$ and for a.e. $x \in \Omega$. Hence, for any fixed $v \in BV(\Omega)$, we have

$$\lim_{k \to +\infty} \int_{\Omega} (g(x, v_k) - Q'(v_k))(v - v_k) \, dx = \int_{\Omega} (g(x, u) - Q'(u))(v - u) \, dx.$$

Moreover the lower semicontinuity of $J$ with respect to the $L^q$-convergence in $BV(\Omega)$ implies

$$\lim_{k \to +\infty} J(v_k) \geq J(u)$$

and

$$\lim_{k \to +\infty} \varepsilon_k \|v - v_k\|_{BV} = 0.$$
Thus we get, passing to the inferior limit in (II.27),

$$J(v) - \int_{\Omega} (g(x, u) - Q'(u)) (v - u) \, dx$$

$$= J(v) - \lim_{k \to +\infty} \int_{\Omega} (g(x, v_k) - Q'(v_k))(v - v_k) \, dx - \lim_{k \to +\infty} \varepsilon_k \|v - v_k\|_{BV}$$

$$\geq \liminf_{k \to +\infty} J(v_k) \geq J(u)$$

and hence

$$J(v) - J(u) \geq \int_{\Omega} (g(x, u) - Q'(u))(v - u) \, dx$$

for all $v \in BV(\Omega)$, that is, $u$ is a solution of (II.22). Taking $v = u$ in (II.27), we get for all $k$

$$J(u) - \int_{\Omega} (g(x, v_k) - Q'(v_k))(u - v_k) \, dx - \varepsilon_k \|u - v_k\|_{BV} \geq J(v_k)$$

and hence

$$J(u) = \lim_{k \to +\infty} \left( J(u) - \int_{\Omega} (g(x, v_k) - Q'(v_k))(u - v_k) \, dx - \varepsilon_k \|u - v_k\|_{BV} \right)$$

$$\geq \limsup_{k \to +\infty} J(v_k).$$

Since, on the other hand,

$$J(u) \leq \liminf_{k \to +\infty} J(v_k),$$

we conclude that

$$\lim_{k \to +\infty} J(v_k) = J(u).$$

Observe that

$$\lim_{k \to +\infty} \int_{\Omega} Q(v_k) \, dx - \int_{\Omega} G(x, v_k) \, dx = \int_{\Omega} Q(u) \, dx - \int_{\Omega} G(x, u) \, dx,$$

as the functionals $v \to \int_{\Omega} Q(v) \, dx$ and $v \to \int_{\Omega} G(x, v) \, dx$ are continuous in $L^q(\Omega)$. Thus we obtain

$$c \bar{I} = \lim_{k \to +\infty} \bar{I}(v_k) = \lim_{k \to +\infty} J(v_k) - \lim_{k \to +\infty} \int_{\Omega} Q(v_k) \, dx - \int_{\Omega} G(x, v_k) \, dx$$

$$= J(u) - \int_{\Omega} Q(u) \, dx - \int_{\Omega} G(x, u) \, dx$$

$$= \bar{I}(u)$$

and hence $u \neq u_1$ and $u \neq u_2$. 

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Case \( c_T = C \). By definition of \( c_T \), there exists a sequence \((\gamma_k)_k \subset \Gamma \) such that
\[
\lim_{k \to +\infty} \left( \max_{\xi \in [0,1]} \mathcal{I}(\gamma_k(\xi)) \right) = C.
\]
Let us define
\[
\gamma = \frac{1}{2} \|u_1 - u_2\|_{L^q} > 0.
\]
As for each \( k \), the curve \( \gamma_k \) is a continuous function from \([0,1]\) to \( L^q(\Omega) \), then there exists \( \xi_k \in [0,1] \) such that
\[
\|\gamma_k(\xi_k) - u_2\|_{L^q} = \gamma.
\]
Set \( v_k = \gamma_k(\xi_k) \) and notice that
\[
\lim_{k \to +\infty} \mathcal{I}(v_k) = C.
\]
Therefore \((v_k)_k\) is a minimizing sequence of \( \mathcal{I} \) which, by (II.25), is bounded in \( BV(\Omega) \).

Hence, possibly passing to a subsequence that we still denote by \((v_k)_k\), this sequence converges in \( L^q(\Omega) \) to some function \( \bar{u} \in BV(\Omega) \) with \( \|\bar{u} - u_1\|_{L^q} = \gamma \).

As in the previous case, the lower semicontinuity of \( \mathcal{I} \) with respect to the \( L^q(\Omega) \) convergence, implies that \( \bar{u} \) is a global minimizer for \( \mathcal{I} \) and therefore it is a solution of (II.22) which differs from both \( u_1 \) and \( u_2 \).

Step 3. Every critical point \( u \in BV(\Omega) \) of \( \mathcal{I} \) satisfies \( u_1 \leq u \leq u_2 \) and hence is a critical point of \( \mathcal{I} \). Let us show now that \( u \leq u_2 \), similarly one can prove \( u \geq u_1 \). Since \( u \) satisfies (II.22), taking \( v = u \wedge u_2 = u - (u - u_2)^+ \) as a test function in (II.24), we obtain
\[
\mathcal{J}(u \wedge u_2) - \mathcal{J}(u) \geq - \int_\Omega (g(x,u) - Q'(u))(u - u_2)^+ \, dx
\]
\[
= - \int_\Omega f(x,u_2)(u - u_2)^+ \, dx + \int_\Omega (Q'(u) - Q'(u_2))(u - u_2)^+ \, dx. \tag{II.28}
\]
Since \( u_2 \) is a solution of (II.1), taking \( v = u \vee u_2 = u_2 + (u - u_2)^+ \) as a test function in (II.2), we have
\[
\mathcal{J}(u \vee u_2) - \mathcal{J}(u_2) \geq \int_\Omega f(x,u_2)(u - u_2)^+ \, dx. \tag{II.29}
\]

Summing (II.28) and (II.29) and using Proposition I.2.7, we get
\[
0 \geq \mathcal{J}(u \vee u_2) + \mathcal{J}(u \wedge u_2) - \mathcal{J}(u) - \mathcal{J}(u_2)
\]
\[
\geq \int_\Omega (Q'(u) - Q'(u_2))(u - u_2)^+ dx \geq 0.
\]
As \( Q' \) is strictly increasing, we conclude that \( (u - u_2)^+ = 0 \) and therefore \( u \leq u_2 \).

Similar arguments show \( u \geq u_1 \) and hence, since \( u \neq u_1 \) and \( u \neq u_2 \), we finally get \( u_1 < u < u_2 \). As \( g(x,s) - Q'(s) = f(x,s) \) for a.e. \( x \in \Omega \) and every \( s \) such that \( u_1(x) \leq s \leq u_2(x) \), we have that \( u \) is a solution of problem (II.1). \( \square \)
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The following variant of Proposition II.3.3 holds.

**Proposition II.3.5.** Assume $(h_{II0})$, $(h_{II1})$ and $(h_{II2})$. Then, for any $A \in \mathbb{R}$, problem

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= A \sin u + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial \Omega
\end{aligned}
\]  

(II.30)

has two solution, not differing from each other by an integer multiple of $2\pi$.

**Proof.** Let us consider the functional $I : BV(\Omega) \to \mathbb{R}$ associated to (II.30) and defined as

\[
I(v) = J(v) + A \int_{\Omega} \cos v \, dx
\]

for all $v \in BV(\Omega)$. By Corollary I.2.4, there holds

\[
I(v) \geq \rho \int_{\Omega} |Dv| - A|\Omega|
\]  

(II.31)

for all $v \in BV(\Omega)$ and hence $I$ is bounded from below. Take $(v_n)_n$ a minimizing sequence in $BV(\Omega)$. By (II.31), we have that there exists $C > 0$ such that, for each $n$,

\[
\int_{\Omega} |Dv_n| \leq C.
\]

Define the sequence $(k_n)_n$ in $\mathbb{Z}$, such that

\[
0 \leq v_n - k_n2\pi < 2\pi
\]

for each $n$, and set $w_n = v_k - 2k_n\pi$. By construction, the sequence $(w_n)_n$ is bounded in $L^1(\Omega)$ and, since $I(v_n) = I(w_n)$ for each $n$, (II.31) implies that $(w_n)_n$ is a bounded minimizing sequence for $I$ in $BV(\Omega)$. By Proposition I.1.2, there is a subsequence of $(w_n)_n$, which we still denote by $(w_n)_n$, and a function $u \in BV(\Omega)$ such that

\[
\lim_{n \to +\infty} w_n = u \text{ in } L^0(\Omega).
\]

We have

\[
\liminf_{n \to +\infty} J(w_n) \geq J(u),
\]

as, by Proposition I.2.6, $J$ is lower semicontinuous with respect to the $L^1$-convergence in $BV(\Omega)$, and moreover

\[
\lim_{n \to +\infty} \int_{\Omega} \cos w_n \, dx = \int_{\Omega} \cos u \, dx,
\]

as the functional $v \mapsto \int_{\Omega} \cos v \, dx$ is continuous in $L^q(\Omega)$. Hence, we conclude that

\[
\inf_{v \in BV(\Omega)} I(v) = \lim_{n \to +\infty} I(w_n) \geq I(u),
\]
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that is, \( I(u) = \min_{v \in BV(\Omega)} I(v) \). Finally, \( u \) is a solution of (II.30). Notice that, by Remark I.2.1, we have a sequence of infinitely many distinct solutions, defined as

\[ u_k = u + 2\pi k \]

for every \( k \in \mathbb{Z} \). For any couple of solutions of the previous type, \( u_1 \) and \( u_2 \) such that \( u_1 = u_2 - 2\pi \), we can apply Proposition II.3.4. Hence we have the existence of a solution of (II.30) different from \( u_k \) for all \( k \in \mathbb{Z} \).

\[ \square \]

II.4 Order Stability

In this section we study the order stability for the problem (II.1) assuming the existence of a lower and an upper solution. First of all, we introduce the following concept of stability, adapted from [68, Chapter I].

**Order stability from below.**

We say that a solution \( u \) of problem (II.1) is order stable (respectively strictly order stable) from below if there exists a sequence \( (\alpha_n)_n \) of lower solutions (respectively proper lower solutions) such that, for each \( n \),

\[ \alpha_n < \alpha_{n+1} \quad \text{and} \quad \lim_{n \to +\infty} \alpha_n = u \]

in \( L^q(\Omega) \) for some \( q > 1 \).

**Order stability from above.**

We say that a solution \( u \) of problem (II.1) is order stable (respectively strictly order stable) from above if there exists a sequence \( (\beta_n)_n \) of upper solutions (respectively proper upper solutions) such that, for each \( n \),

\[ \beta_n > \beta_{n+1} \quad \text{and} \quad \lim_{n \to +\infty} \beta_n = u \]

in \( L^q(\Omega) \) for some \( q > 1 \).

We point out that our conclusions are got without assuming any additional regularity condition, like, e.g., Lipschitz continuity, on \( f \), as it is usually required in order to associate with the considered problem an order preserving operator (see, e.g., [2], [68]). In order to avoid regularity assumptions on function \( f \), from a technical point of view, the following lemma from [37] is crucial.

**Lemma II.4.1.** Assume \( (h_{\Omega 0}) \). Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfy the \( L^p \)-Carathéodory conditions. Then, for each \( \rho > 0 \), there exists an \( L^p \)-Carathéodory function \( g : \Omega \times [-\rho, \rho] \times [-\rho, \rho] \to \mathbb{R} \) such that

(i) \( g(x, \cdot, r) : [-\rho, \rho] \to \mathbb{R} \) is strictly increasing for a.e. \( x \in \Omega \) and every \( r \in [-\rho, \rho] \);
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(ii) \( g(x, s, \cdot) : [-\rho, \rho] \to \mathbb{R} \) is strictly decreasing for a.e. \( x \in \Omega \) and every \( s \in [-\rho, \rho] \);

(iii) \( g(x, s, r) = -g(x, r, s) \) for a.e. \( x \in \Omega \) and every \( (s, r) \in [-\rho, \rho] \times [-\rho, \rho] \);

(iv) for a.e. \( x \in \Omega \) and every \( (s, r) \in [-\rho, \rho] \times [-\rho, \rho] \), with \( r < s \), we have

\[
|f(x, s) - f(x, r)| < g(x, s, r).
\]  \( \text{(II.32)} \)

**Proof.** See [37, Lemma 2.1] and also [36, Proposition 2.3].

**Theorem II.4.2.** Assume \((h_{11}0), (h_{11}1), (h_{11}2)\) and \((h_{11}3)\). Let \( v \in BV(\Omega) \) be a solution of problem \((\text{II.1})\) such that \( v \in L^\infty(\Omega) \). Suppose that there exists a \( BV \)-lower solution \( \alpha = \tilde{\alpha}_1 \lor \cdots \lor \tilde{\alpha}_m \), of problem \((\text{II.1})\), with \( v > \alpha \) and such that \( \tilde{\alpha}_i \in L^\infty(\Omega) \) for each \( i = 1, \ldots, m \). Moreover suppose that there is no solution \( u \) of problem \((\text{II.1})\) satisfying \( \alpha \leq u < v \). Then \( v \) is strictly order stable from below.

**Proof.** Fix \( \rho > \max_{i=1,\ldots,m} \{ \|\tilde{\alpha}_i\|_\infty, \|v\|_\infty \} \). Since the function \( f \) satisfies the assumptions of Lemma II.4.1, we can consider the \( L^p \)-Carathéodory function \( g : \Omega \times [-\rho, \rho] \times [-\rho, \rho] \to \mathbb{R} \) such that \((i), (ii), (iii)\) and \((iv)\) hold. Let us consider now the modified problem

\[
\begin{aligned}
-\text{div}
\left(
\sqrt{\frac{1}{1 + |\nabla u|^2}} \nabla u
\right)
+ g(x, u, \alpha)
&= f(x, \alpha) + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n \sqrt{1 + |\nabla u|^2}
&= \kappa(x) \quad \text{on } \partial \Omega.
\end{aligned}
\]  \( \text{(II.33)} \)

Observe that a solution \( u \) of problem \((\text{II.33})\) is a function \( u \in BV(\Omega) \) satisfying

\[
\mathcal{J}(u + z) - \mathcal{J}(u) \geq \int_\Omega (f(x, \alpha) - g(x, u, \alpha)) z \, dx
\]  \( \text{(II.34)} \)

for all \( z \in BV(\Omega) \).

**Claim.** The problem \((\text{II.33})\) has a unique \( BV \)-solution \( \alpha_1 \), satisfying \( \alpha < \alpha_1 < v \), which is a proper \( BV \)-lower solution of the problem \((\text{II.1})\).

We first show that problem \((\text{II.33})\) has at most one solution. Suppose that both \( u_1, u_2 \in BV(\Omega) \) are solutions of problem \((\text{II.33})\), \( \alpha < u_1 < v \) and \( \alpha < u_2 < v \). Then we have

\[
\mathcal{J}(u_2) - \mathcal{J}(u_1) \geq \int_\Omega (f(x, \alpha) - g(x, u_1, \alpha))(u_2 - u_1) \, dx
\]

and

\[
\mathcal{J}(u_1) - \mathcal{J}(u_2) \geq \int_\Omega (f(x, \alpha) - g(x, u_2, \alpha))(u_1 - u_2) \, dx.
\]

Summing up these two relations we get

\[
0 \geq \int_\Omega (g(x, u_1, \alpha) - g(x, u_2, \alpha))(u_1 - u_2) \, dx.
\]
II. Solvability via lower and upper solutions method

Since the function \( g(x,\cdot,r) : [-\rho,\rho] \to \mathbb{R} \) is strictly increasing for a.e. \( x \in \Omega \) and every \( r \in [-\rho,\rho] \), we conclude that
\[
\int_{\Omega} \left( g(x,u_1,\alpha) - g(x,u_2,\alpha) \right) (u_1 - u_2) \, dx \geq 0
\]
and hence \( u_1 = u_2 \).

Next we prove that the problem (II.33) has a solution. Let us verify that \( \alpha = \tilde{\alpha}_1 \vee \cdots \vee \tilde{\alpha}_m \) is a BV-lower solution of problem (II.33), that is, for each \( j = 1,\ldots,m \),
\[
\mathcal{J}(\tilde{\alpha}_j + z) - \mathcal{J}(\tilde{\alpha}_j) \geq \int_{\Omega} \left( f(x,\alpha) - g(x,\tilde{\alpha}_j,\alpha) \right) z \, dx
\]
for all \( z \in BV(\Omega) \) with \( z \leq 0 \). Indeed, as \( \tilde{\alpha}_j \leq \alpha \), we have, by (II.32),
\[
f(\cdot,\tilde{\alpha}_j) \leq f(\cdot,\alpha) + g(\cdot,\alpha,\tilde{\alpha}_j) = f(\cdot,\alpha) - g(\cdot,\tilde{\alpha}_j,\alpha)
\]
and hence, as \( \alpha \) is a BV-lower solution of problem (II.1), we conclude
\[
\mathcal{J}(\tilde{\alpha}_j + z) - \mathcal{J}(\tilde{\alpha}_j) \geq \int_{\Omega} f(x,\tilde{\alpha}_j) z \, dx \geq \int_{\Omega} \left( f(x,\alpha) - g(x,\tilde{\alpha}_j,\alpha) \right) z \, dx
\]
for all \( z \in BV(\Omega) \) with \( z \leq 0 \). This means that \( \alpha \) is a BV-lower solution of problem (II.33). Similarly we verify that \( v \) is a BV-upper solution of problem (II.33). Since \( \alpha < v \), by (II.32), we have
\[
f(\cdot,v) > f(\cdot,\alpha) - g(\cdot,v,\alpha) \tag{II.35}
\]
and hence, as \( v \) is a solution of problem (II.1),
\[
\mathcal{J}(v + z) - \mathcal{J}(v) \geq \int_{\Omega} f(x,v) z \, dx \geq \int_{\Omega} \left( f(x,\alpha) - g(x,v,\alpha) \right) z \, dx
\]
for all \( z \in BV(\Omega) \) with \( z \geq 0 \). Theorem II.2.1 and Remark II.2.1 yield the existence of a solution \( \alpha_1 \) of problem (II.33) such that \( \alpha \leq \alpha_1 \leq v \).

Let us prove that \( \alpha < \alpha_1 \). Suppose, by contradiction, that \( \alpha_1 = \alpha \). Then, by relation (II.34), we get
\[
\mathcal{J}(\alpha + z) - \mathcal{J}(\alpha) \geq \int_{\Omega} \left( f(x,\alpha) - g(x,\alpha,\alpha) \right) z \, dx = \int_{\Omega} f(x,\alpha) z \, dx
\]
for all \( z \in BV(\Omega) \), i.e., \( \alpha \) is also a solution of problem (II.1), thus contradicting the assumption that \( \alpha \) is a proper BV-lower solution.

Similarly, we verify that \( \alpha_1 < v \). Suppose by contradiction that \( \alpha_1 = v \). Then, testing relation (II.34) against \( z = -1 \) and against \( z = 1 \), by Remark I.2.1, we obtain
\[
\int_{\Omega} \left( f(x,\alpha) - g(x,v,\alpha) \right) \, dx = 0.
\]
Similarly, testing
\[ J(v + z) - J(v) \geq \int_{\Omega} f(x, v) z \, dx \]
against \( z = -1 \) and against \( z = 1 \), we get
\[ \int_{\Omega} f(x, v) \, dx = 0. \]
Hence, we have
\[ \int_{\Omega} (f(x, v) - f(x, \alpha) + g(x, v, \alpha)) \, dx = 0, \]
with \( \alpha < v \), thus contradicting (II.35) or (II.32).

Finally, we observe that \( \alpha_1 \) is a BV-lower solution of problem (II.1). Indeed, if we take \( z \in \text{BV}(\Omega) \) with \( z \leq 0 \), using (II.32), we have
\[ J(\alpha_1 + z) - J(\alpha_1) \geq \int_{\Omega} (f(x, \alpha) - h(x, \alpha_1, \alpha)) z \, dx \geq \int_{\Omega} f(x, \alpha_1) z \, dx. \]
As problem (II.1) has no solution \( u \) of the type \( \alpha < u < v \), we conclude that \( \alpha_1 \) is proper.

Recursively we define a sequence \( (\alpha_n)_n \), where \( \alpha_0 = \alpha, \alpha_1 \) has been constructed in the above claim and, for every \( n \geq 1 \), \( \alpha_{n+1} \) is the unique BV-solution of problem
\[ \begin{cases} -\text{div}(\nabla u/\sqrt{1 + |\nabla u|^2}) + g(x, u, \alpha_n) = f(x, \alpha_n) + h(x) & \text{in } \Omega, \\ -\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = \kappa(x) & \text{on } \partial \Omega, \end{cases} \]
with \( \alpha_n < \alpha_{n+1} < v \). This means that
\[ J(\alpha_{n+1} + z) - J(\alpha_{n+1}) \geq \int_{\Omega} (f(x, \alpha_n) - g(x, \alpha_{n+1}, \alpha_n)) z \, dx \]
holds for all \( z \in \text{BV}(\Omega) \). Arguing as above one can see that the sequence \( (\alpha_n)_n \) is well-defined and, for each \( n \), \( \alpha_n \) is a proper BV-lower solution of problem (II.1), satisfying \( \alpha_n < \alpha_{n+1} < v \).

Finally we verify that \( (\alpha_n)_n \) converges to \( v \) in \( L^q(\Omega) \), where \( q = \frac{p}{p-1} \). Since the sequence \( (\alpha_n)_n \) is increasing and uniformly bounded in \( L^\infty(\Omega) \), it converges in \( L^q(\Omega) \) and a.e. in \( \Omega \) to some function \( \tilde{v} \), with \( \alpha < \tilde{v} \leq v \). Since there exists \( \gamma \in L^p(\Omega) \) such that, for all \( n \),
\[ |g(x, \alpha_{n+1}(x), \alpha_n(x))| \leq \gamma(x) \]
and
\[ \lim_{n \to +\infty} g(x, \alpha_{n+1}(x), \alpha_n(x)) = g(x, \tilde{v}(x), \tilde{v}(x)) = 0 \]
for a.e. \( x \in \Omega \), we get
\[ \lim_{n \to +\infty} g(\cdot, \alpha_{n+1}, \alpha_n) = 0. \]
in $L^p(\Omega)$. By $(h_{II}3)$ and by the Lebesgue convergence theorem we also have
\[
\lim_{n \to +\infty} f(\cdot, \alpha_n) = f(\cdot, \tilde{v})
\]
in $L^p(\Omega)$. Fix any $w \in BV(\Omega)$. Since, for each $n$
\[
\mathcal{J}(w) - \mathcal{J}(\alpha_{n+1}) \geq \int_{\Omega} \left( f(x, \alpha_n) - g(x, \alpha_{n+1}, \alpha_n) \right) (w - \alpha_{n+1})\, dx,
\]
passing to the limit and using Proposition I.2.6, we conclude that
\[
\mathcal{J}(w) - \mathcal{J}(\tilde{v}) \geq \int_{\Omega} f(x, \tilde{v}) (w - \tilde{v})\, dx.
\]
Accordingly, $\tilde{v}$ is a solution of problem (II.1), satisfying $\alpha \leq \tilde{v} \leq v$, and hence $\tilde{v} = v$. □

In a completely similar way we can prove the following symmetric result.

**Theorem II.4.3.** Assume $(h_{II}0)$, $(h_{II}1)$, $(h_{II}2)$ and $(h_{II}3'')$. Let $w \in BV(\Omega)$ be a solution of the problem (II.1) such that $w \in L^\infty(\Omega)$. Suppose that there exists a $BV$-upper solution $\beta = \beta_1 \land \cdots \land \beta_n$ of problem (II.1), with $w < \beta$ and such that $\beta_j \in L^\infty(\Omega)$ for each $j = 1, \ldots, n$. Moreover assume that there is no solution $u$ of problem (II.1) satisfying $w < u \leq \beta$. Then $w$ is strictly order stable from above.

Combining Theorem II.4.2 and Theorem II.4.3 yields the order stability of the minimum and the maximum solutions of problem (II.1), lying between a pair of lower and upper solutions $\alpha$ and $\beta$, with $\alpha, \beta \in L^\infty(\Omega)$ and $\alpha \leq \beta$.

**Corollary II.4.4.** Assume $(h_{II}0)$, $(h_{II}1)$, $(h_{II}2)$, $(h_{II}3'')$ and $(h_{II}4\infty)$. Suppose further that $\alpha$ and $\beta$ are proper lower and upper solutions, respectively. Then the minimum solution $v$ and the maximum solution $w$ in $[\alpha, \beta]$ of problem (II.1) are strictly order stable from below and strictly order stable from above, respectively.
II. SOLVABILITY VIA LOWER AND UPPER SOLUTIONS METHOD
Chapter III

An asymmetric Poincaré inequality

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \), with a Lipschitz boundary \( \partial \Omega \). The classical Poincaré inequality in \( \text{BV}(\Omega) \) (see, e.g., [3, Theorem 3.44]) asserts that there exists a constant \( c = c(\Omega) > 0 \) such that every \( u \in \text{BV}(\Omega) \), with \( \int_\Omega u \, dx = 0 \), satisfies

\[
c \int_\Omega |u| \, dx \leq \int_\Omega |Du|.
\] (III.1)

The largest constant \( c = c(\Omega) \) for which (III.1) holds is called the Poincaré constant and it is variationally characterized by

\[
c = \min \left\{ \int_\Omega |Dv| : v \in \text{BV}(\Omega), \int_\Omega v \, dx = 0, \int_\Omega |v| \, dx = 1 \right\}.
\]

Clearly, any minimizer yields the equality in (III.1).

The aim of this chapter is to prove an asymmetric counterpart of the Poincaré inequality (III.1), where \( u^+ \) and \( u^- \) weigh differently, i.e., the ratio \( r = \frac{\int_\Omega u^+ \, dx}{\int_\Omega u^- \, dx} \) is not necessarily 1. Namely, we will show that for each \( r > 0 \) there exist constants \( \mu = \mu(r, \Omega) > 0 \) and \( \nu = \nu(r, \Omega) > 0 \), with \( \nu = r \mu \), such that every \( u \in \text{BV}(\Omega) \), with \( \mu \int_\Omega u^+ \, dx - \nu \int_\Omega u^- \, dx = 0 \), satisfies

\[
\mu \int_\Omega u^+ \, dx + \nu \int_\Omega u^- \, dx \leq \int_\Omega |Du|.
\] (III.2)

It is easy to see that constants \( \mu \) and \( \nu \) are variationally characterized by

\[
\mu = \min \left\{ \int_\Omega |Dv| : v \in \text{BV}(\Omega), \int_\Omega v^+ \, dx - r \int_\Omega v^- \, dx = 0, \quad \int_\Omega v^+ \, dx + r \int_\Omega v^- \, dx = 1 \right\}.
\] (III.3)
and
\[ \nu = \min \left\{ \int_{\Omega} |Dv| : v \in BV(\Omega), r^{-1} \int_{\Omega} v^+ dx - \int_{\Omega} v^- dx = 0, \right. \]
\[ \left. r^{-1} \int_{\Omega} v^+ dx + \int_{\Omega} v^- dx = 1 \right\}, \tag{III.4} \]
respectively. Clearly, any minimizer in (III.3), or (III.4), yields the equality in (III.2).

This construction will allow us to single out in the plane a curve \( C = C(\Omega) \) made up of all pairs \((\mu, \nu) = (\mu(r, \Omega), \nu(r, \Omega))\) defined by (III.3) and (III.4), by letting \( r \) vary in \( \mathbb{R}_0^+ \).

### III.1 A Poincaré-type inequality

Throughout this chapter we assume that

\( (h_{III0}) \) \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) having a Lipschitz boundary \( \partial \Omega \);

\( (h_{III1}) \) \( h \in L^p(\Omega) \), for some \( p > N \), and \( \kappa \in L^\infty(\partial \Omega) \);

\( (h_{III2}) \) there exists a constant \( \rho > 0 \) such that

\[ \left| \int_B h \, dx - \int_{\partial \Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq (1 - \rho) \int_{\Omega} |D\chi_B| \]

for every Caccioppoli set \( B \subseteq \Omega \).

Notice that in dimension \( N = 1 \) condition \( (h_{III1}) \) can be weakened in most cases to

\( (h_{III1}') \) \( h \in L^1(\Omega) \) and \( \kappa \in L^\infty(\partial \Omega) \).

Here we want to prove an asymmetric version of the Poincaré inequality (III.1) which involves the functional \( \mathcal{L} \). Namely, we will show that for each \( r > 0 \) there exist constants \( \mu = \mu(r) > 0 \) and \( \nu = \nu(r) > 0 \), which also depend on \( \Omega, h \) and \( \kappa \), such that every \( u \in BV(\Omega) \), with

\[ \mu \int_{\Omega} u^+ dx - \nu \int_{\Omega} u^- dx = 0, \tag{III.5} \]
satisfies

\[ \mu \int_{\Omega} u^+ dx + \nu \int_{\Omega} u^- dx \leq \mathcal{L}(u). \tag{III.6} \]

For each \( r > 0 \) we define \( \mu \) and \( \nu \) through the variational formulas

\[ \mu = \mu(r) = \inf \left\{ \mathcal{L}(v) : v \in BV(\Omega), \int_{\Omega} v^+ dx - r \int_{\Omega} v^- dx = 0, \right. \]
\[ \left. \int_{\Omega} v^+ dx + r \int_{\Omega} v^- dx = 1 \right\} \tag{III.7} \]
and
\[ \nu = \nu(r) = \inf \left\{ \mathcal{L}(v) : v \in BV(\Omega), r^{-1} \int_{\Omega} v^+ dx - \int_{\Omega} v^- dx = 0, \right\} \]
\[ r^{-1} \int_{\Omega} v^+ dx + \int_{\Omega} v^- dx = 1 \right\}. \quad \text{(III.8)} \]

Proposition I.2.3 implies that \( \mu \geq 0 \) and \( \nu \geq 0 \). For convenience we also set, for any \( r > 0 \),
\[ C_r = \left\{ v \in BV(\Omega) : \int_{\Omega} v^+ dx = \frac{1}{2} \right\} \]

Note that the constraints in (III.7) and in (III.8) can be equivalently expressed by requiring \( v \in C_r \) and \( r^{-1} v \in C_r \), respectively.

**Proposition III.1.1** (Minimum properties). Assume \((h_{\text{III}}0), (h_{\text{III}}1)\) and \((h_{\text{III}}2)\). Then, for each \( r > 0 \), we have
\[ \mu(r) = \min \{ \mathcal{L}(v) : v \in C_r \} \quad \text{and} \quad \nu(r) = \min \{ \mathcal{L}(v) : r^{-1} v \in C_r \}, \quad \text{(III.9)} \]

with \( \mu(r) > 0 \) and \( \nu(r) = r \mu(r) \).

**Proof.** Fix \( r > 0 \). Let us show that the functional \( \mathcal{L} \) has a minimum in the set \( C_r \). Let \((v_n)_n\) be a minimizing sequence in \( C_r \), that is,
\[ \lim_{n \to +\infty} \mathcal{L}(v_n) = \mu(r) = \inf \{ \mathcal{L}(v) : v \in C_r \} \geq 0. \]

By Proposition I.2.3 the sequence \((v_n)_n\) is bounded in \( BV(\Omega) \) and hence there exists a subsequence, that we still denote by \((v_n)_n\), which converges in \( L^q(\Omega) \), with \( q = \frac{p}{p-1} \), to some \( v \in BV(\Omega) \). We have \( v \in C_r \) and, by Proposition I.2.5,
\[ \mathcal{L}(v) \leq \liminf_{n \to +\infty} \mathcal{L}(v_n). \]

This implies that \( \mathcal{L}(v) = \mu(r) \).

Moreover, we have \( \mu(r) = \mathcal{L}(v) > 0 \). Indeed, suppose by contradiction that \( \mu(r) = 0 \) and hence \( \mathcal{L}(v) = 0 \). Proposition I.2.3 implies that \( \int_{\Omega} |Dv| = 0 \) and therefore, by [3, Proposition 3.2], \( v \) is constant, which is impossible as \( v \in C_r \). Similar conclusions can be achieved for \( \nu(r) \).

Finally, let \( v \in BV(\Omega) \) be such that \( r^{-1} v \in C_r \) and \( \mathcal{L}(v) = \nu(r) \). Setting \( u = r^{-1} v \), we have \( u \in C_r \) and \( \mu(r) \leq \mathcal{L}(u) = r^{-1} \nu(r) \). Conversely, if \( u \in C_r \) is such that \( \mathcal{L}(u) = \mu(r) \), setting \( v = ru \), we have \( r^{-1} v \in C_r \) and \( \nu(r) \leq \mathcal{L}(v) = r \nu(r) \). Thus the conclusion follows.

**Remark III.1.1** It is clear that if \( v \) is a minimizer in (III.9), corresponding to \( \mu(r) \) or to \( \nu(r) \), then \( v \) satisfies (III.5) and yields the equality in (III.6). Conversely, if \( v \) satisfies (III.5) and yields the equality in (III.6), then it is a positive multiple of a minimizer in (III.9), corresponding to \( \mu(r) \) or to \( \nu(r) \).
III. AN ASYMMETRIC POINCARÉ INEQUALITY

Proposition III.1.2 (Asymmetric Poincaré inequality). Assume \((h_{III0}), (h_{III1})\) and \((h_{III2})\). For each \(r > 0\) let \(\mu = \mu(r)\) and \(\nu = \nu(r)\) be defined by \((III.9)\). Then, every \(v \in BV(\Omega)\), for which \((III.5)\) holds, that is
\[
\mu \int_{\Omega} v^+ \, dx - \nu \int_{\Omega} v^- \, dx = 0,
\]
also satisfies \((III.6)\), that is
\[
\mu \int_{\Omega} v^+ \, dx + \nu \int_{\Omega} v^- \, dx \leq \mathcal{L}(v).
\]

Proof. The conclusion follows from Proposition III.1.1.

Remark III.1.2 If \(v \in BV(\Omega)\) satisfies \((III.5)\) for some \((\mu, \nu) \in \mathcal{C}\), then we have, in particular,
\[
\int_{\Omega} |v| \, dx \leq \frac{1}{2} (\frac{1}{\mu} + \frac{1}{\nu}) \mathcal{L}(v).
\]

III.2 The curve \(\mathcal{C}\) and its properties

In this section we are concerned with the study of the functions \(r \mapsto \mu(r)\) and \(r \mapsto \nu(r)\), and of the planar curve parametrized by them
\[
\mathcal{C} = \{(\mu(r), \nu(r)) : r \in \mathbb{R}_0^+\}.
\]

Proposition III.2.1 (Symmetry in case \(h = 0\) and \(\kappa = 0\)). Assume \((h_{III0})\) and suppose that \(h = 0\) and \(\kappa = 0\). Then, for each \(r > 0\), we have \(\mu(r^{-1}) = \nu(r)\); in particular, \(\mathcal{C}\) is symmetric with respect to the diagonal.

Proof. Let \(v \in C_{r^{-1}}\) be such that \(\int_{\Omega} |Dv| = \mu(r^{-1})\). Set \(u = -v\). We have \(r^{-1}u \in C_r\) and hence \(\nu(r) \leq \int_{\Omega} |Du| = \mu(r^{-1})\). Conversely, if \(u \in BV(\Omega)\) is such that \(r^{-1}u \in C_r\) and \(\int_{\Omega} |Du| = \nu(r)\), setting \(v = -u\), we have \(v \in C_{r^{-1}}\) and hence \(\mu(r^{-1}) \leq \int_{\Omega} |Dv| = \nu(r)\). Thus we conclude that \(\nu(r) = \mu(r^{-1})\).

Proposition III.2.2 (Continuity). Assume \((h_{III0}), (h_{III1})\) and \((h_{III2})\). Then, the functions \(r \mapsto \mu(r)\) and \(r \mapsto \nu(r)\) are continuous.

Proof. The proof is divided into two steps.

Step 1. The function \(r \mapsto \mu(r)\) is lower semicontinuous. Fix \(r \in \mathbb{R}_0^+\) and take any sequence \((r_n)\) in \(\mathbb{R}_0^+\) with \(\lim_{n \to +\infty} r_n = r\). We can suppose that
\[
\liminf_{n \to +\infty} \mu(r_n) = \bar{\mu} \in [0, +\infty],
\]
because otherwise the conclusion is trivial. For each \(n\), let \(v_n \in C_{r_n}\) be such that \(\mu(r_n) = \mathcal{L}(v_n)\). Since
\[
\|v_n\|_{L^1} = \frac{1}{2} (1 + \frac{1}{r_n})
\]
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for every \( n \), the sequence \( (v_n)_n \) is bounded in \( L^1(\Omega) \). Moreover, we can extract a subsequence of \( (v_n)_n \), still denoted by \( (v_n)_n \), such that \( (\mathcal{L}(v_n))_n \) converges to \( \bar{\mu} \). Proposition I.2.3 then implies that \( (v_n)_n \) is bounded in \( BV(\Omega) \). Hence, possibly passing to a further subsequence, we can suppose that \( (v_n)_n \) converges in \( L^q(\Omega) \), with \( q = \frac{p}{p-1} \), to some function \( v \in C_r \). Thus Proposition I.2.5 yields

\[
\mu(r) \leq \mathcal{L}(v) \leq \liminf_{n \to +\infty} \mathcal{L}(v_n) = \bar{\mu} = \liminf_{n \to +\infty} \mu(r_n).
\]

Step 2. The function \( r \mapsto \mu(r) \) is upper semicontinuous. Fix \( r \in \mathbb{R}_0^+ \) and take any sequence \( (r_n)_n \) in \( \mathbb{R}_0^+ \) with \( \lim_{n \to +\infty} r_n = r \). Let \( v \in C_r \) be such that \( \mu(r) = \mathcal{L}(v) \). Define a sequence \( (v_n)_n \) by setting for each \( n \)

\[
v_n = v^+ - \frac{r}{r_n} v^-.
\]

We have \( v_n \in C_{r_n} \) and hence \( \mu(r) \leq \mathcal{L}(v_n) \). Since for all \( n \)

\[
\|v_n - v\|_{BV} = |1 - \frac{r}{r_n}| \|v^-\|_{BV},
\]

the sequence \( (v_n)_n \) converges to \( v \) in \( BV(\Omega) \). The continuity of the functional \( \mathcal{L} \) in \( BV(\Omega) \) finally yields

\[
\mu(r) = \mathcal{L}(v) = \lim_{n \to +\infty} \mathcal{L}(v_n) \geq \limsup_{n \to +\infty} \mu(r_n).
\]

Hence we conclude that the function \( r \mapsto \mu(r) \) is continuous.

The continuity of the map \( r \mapsto \nu(r) \) follows from the relation \( \nu(r) = r \mu(r) \). \( \square \)

**Proposition III.2.3** (Monotonicity). Assume \((h_{III}0)\), \((h_{III}1)\) and \((h_{III}2)\). Then, the function \( r \mapsto \mu(r) \) is strictly decreasing and the function \( r \mapsto \nu(r) \) is strictly increasing.

**Proof.** We only show that the map \( r \mapsto \mu(r) \) is strictly decreasing; a similar argument allows us to prove that the function \( r \mapsto \nu(r) \) is strictly increasing. Fix \( r, s \in \mathbb{R}_0^+ \), with \( r < s \), and let \( u \in C_r \) be such that \( \mu(r) = \mathcal{L}(u) \). Let us define a function \( \phi : \mathbb{R}^+ \to \mathbb{R} \) by setting

\[
\phi(\xi) = \int_{\Omega} ((u + \xi)^+ - s(u + \xi)^-) \, dx.
\]

It is easy to see that \( \phi \) is continuous. As \( u \in C_r \), it follows that

\[
\phi(0) = \int_{\Omega} (u^+ - su^-) \, dx < \int_{\Omega} (u^+ - ru^-) \, dx = 0.
\]

Moreover, we have

\[
\lim_{\xi \to +\infty} \phi(\xi) = +\infty,
\]

as, on the one hand, Fatou’s lemma implies that

\[
\lim_{\xi \to +\infty} \int_{\Omega} (u + \xi)^+ \, dx = +\infty
\]
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and, on the other hand,
\[ \sup_{\xi \in \mathbb{R}^+} \int_{\Omega} (u + \xi)^- \, dx \leq \int_{\Omega} u^- \, dx. \]

As a consequence, there exists \( \bar{\xi} \in \mathbb{R}^+_0 \) such that \( \phi(\bar{\xi}) = 0 \), i.e.,
\[ \int_{\Omega} (u + \bar{\xi})^+ \, dx = s \int_{\Omega} (u + \bar{\xi})^- \, dx. \]

Set
\[ m = 2 \int_{\Omega} (u + \bar{\xi})^+ \, dx > 2 \int_{\Omega} u^+ \, dx = 1 \]
and define a function \( v \in BV(\Omega) \) by
\[ v = \frac{u + \bar{\xi}}{m}. \]

A simple calculation shows that
\[ \int_{\Omega} v^+ \, dx = \frac{1}{2} \quad \text{and} \quad \int_{\Omega} v^- \, dx = \frac{1}{2s}, \]
that is, \( v \in C_s \). By Remark I.2.1, we get
\[ \mu(s) \leq \mathcal{L}(v) = \frac{1}{m} \left( \int_{\Omega} |D(u + \bar{\xi})| - \int_{\Omega} h(u + \bar{\xi}) \, dx + \int_{\partial \Omega} \kappa(u + \bar{\xi}) \, d\mathcal{H}_{N-1} \right) \]
\[ = \frac{1}{m} \left( \mathcal{L}(u) - \int_{\Omega} h \xi \, dx + \int_{\partial \Omega} \kappa \xi \, d\mathcal{H}_{N-1} \right) \]
\[ = \frac{1}{m} \mathcal{L}(u) = \frac{1}{m} \mu(r) < \mu(r), \]
which yields the conclusion. \( \square \)

**Proposition III.2.4** (Asymptotic behaviour). Assume \((h_{III0}), (h_{III1})\) and \((h_{III2})\). Then, we have
\[ \lim_{r \to 0^+} \mu(r) = +\infty \quad \text{and} \quad \lim_{r \to +\infty} \nu(r) = +\infty. \]

**Proof.** We start with the following variant of the Poincaré inequality (III.1).

**Claim.** For each \( \alpha \in [0,1] \) there exists a constant \( C = C(\alpha, \Omega) \) such that every \( v \in BV(\Omega) \), with
\[ |\{v = 0\}| \geq \alpha |\Omega|, \quad \text{(III.11)} \]
also satisfies
\[ \int_{\Omega} |v| \, dx \leq C \int_{\Omega} |Dv|. \quad \text{(III.12)} \]
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Proof of the claim. Take $\alpha \in [0, 1]$ and let $v \in BV(\Omega)$ satisfy (III.11). Set $\Omega_0 = \{ v = 0 \}$, $\tilde{\Omega} = \Omega \setminus \Omega_0$, and $\tilde{v} = \frac{1}{|\tilde{\Omega}|} \int_{\tilde{\Omega}} v \, dx$. The Poincaré inequality (III.1) yields

$$
\int_{\tilde{\Omega}} |\tilde{v}| \, dx = |\tilde{v}| |\tilde{\Omega}| = \frac{\tilde{\Omega}}{|\Omega|} \left| \int_{\Omega} v \, dx \right| \leq \frac{\tilde{\Omega}}{|\Omega|} \int_{\Omega} |v| \, dx
$$

and then

$$
\int_{\Omega} |v| \, dx \leq \frac{1}{c} \int_{\Omega} |Dv| + \frac{|\tilde{\Omega}|}{|\Omega|} \int_{\Omega} |v| \, dx.
$$

where $c = c(\Omega)$ is the Poincaré constant. We have

$$
\int_{\tilde{\Omega}} |\tilde{v}| \, dx = |\tilde{v}| |\tilde{\Omega}| = \frac{|\tilde{\Omega}|}{|\Omega|} \left| \int_{\Omega} v \, dx \right| \leq \frac{|\tilde{\Omega}|}{|\Omega|} \int_{\Omega} |v| \, dx
$$

and then

$$
\int_{\Omega} |v| \, dx \leq \frac{1}{c} \int_{\Omega} |Dv| + \frac{|\tilde{\Omega}|}{|\Omega|} \int_{\Omega} |v| \, dx.
$$

Since

$$
\frac{|\tilde{\Omega}|}{|\Omega|} = \frac{|\Omega| - |\Omega_0|}{|\Omega|} \leq 1 - \alpha,
$$

it follows that (III.12) holds, setting $C = \frac{1}{\alpha c}$. This concludes the proof of the claim.

The monotonicity of the function $r \mapsto \mu(r)$ implies that there exists

$$
\lim_{r \to 0^+} \mu(r) = \bar{\mu} \in [0, +\infty].
$$

Suppose by contradiction that $\bar{\mu} < +\infty$. Take any sequence $(r_n)_n$ in $\mathbb{R}_0^+$ such that $\lim_{n \to +\infty} r_n = 0$ and let $(v_n)_n$ be a sequence in $BV(\Omega)$ such that, for each $n$, $v_n \in C_{r_n}$ and

$$
\mu(r_n) = \mathbb{L}(v_n).
$$

We have

$$
\int_{\Omega} v_n^+ \, dx = \frac{1}{2},
$$

for all $n$, and

$$
\lim_{n \to +\infty} \int_{\Omega} v_n^- \, dx = \lim_{n \to +\infty} \frac{1}{2r_n} = +\infty. \quad (III.13)
$$

Moreover, as $\bar{\mu}$ is finite, Proposition I.2.3 and the lattice property of $BV(\Omega)$ (see Proposition I.2.7) yield the existence of a constant $M$ such that

$$
\int_{\Omega} |Dv_n^-| + \int_{\Omega} |Dv_n^+| \leq \int_{\Omega} |Dv_n| \leq M
$$

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for all \( n \). Accordingly, the sequence \((v_n^+)\) is bounded in \( BV(\Omega)\) and therefore, possibly passing to a subsequence still denoted by \((v_n^+)\), it converges in \( L^q(\Omega)\), with \( q = \frac{p}{p-1} \), and pointwise a.e. in \( \Omega \) to a function \( v \in BV(\Omega) \), with \( v \geq 0 \), such that

\[
\int_{\Omega} v \, dx = \frac{1}{2}.
\]

Let us prove that the measures of the essential supports of the functions \( v_n^+ \) remain bounded away from 0. Indeed, the pointwise convergence a.e. in \( \Omega \) of \((v_n^+)\) to \( v \) implies that

\[
\liminf_{n \to +\infty} \chi_{\{v_n > 0\}}(x) \geq \chi_{\{v > 0\}}(x)
\]

for a.e. \( x \in \Omega \). Then Fatou’s lemma yields

\[
\liminf_{n \to +\infty} |\{v_n > 0\}| = \liminf_{n \to +\infty} \int_{\Omega} \chi_{\{v_n > 0\}} \, dx
\]

\[
\geq \int_{\Omega} \liminf_{n \to +\infty} \chi_{\{v_n > 0\}} \, dx \geq \int_{\Omega} \chi_{\{v > 0\}} \, dx = |\{v > 0\}|.
\]

Set

\[
m = |\{v > 0\}| > 0.
\]

As \( \{v_n^+ > 0\} \subseteq \{v_n^- = 0\} \), we get, for all sufficiently large \( n \),

\[
|\{v_n^- = 0\}| \geq |\{v_n^+ > 0\}| \geq \frac{1}{2}m.
\]

Applying the claim above with \( \alpha = \frac{1}{2} \frac{m}{|\Omega|} \) to \( v_n^- \), we get

\[
\int_{\Omega} |v_n^-| \, dx \leq c(\alpha) \int_{\Omega} |Dv_n^-| \leq CM
\]

for all sufficiently large \( n \), thus contradicting (III.13).

By a similar argument we can prove that \( \lim_{r \to +\infty} \nu(r) = +\infty \). \( \square \)

**Proposition III.2.5** (Asymptotic behaviour in dimension \( N \geq 2 \)). Assume \( N \geq 2 \) and suppose that \((h_{III0})\), \((h_{III1})\) and \((h_{III2})\) hold. Then, we have

\[
\lim_{r \to +\infty} \mu(r) = 0 \quad \text{and} \quad \lim_{r \to 0^+} \nu(r) = 0.
\]

**Proof.** We prove that \( \lim_{r \to +\infty} \mu(r) = 0 \); a similar argument shows that \( \lim_{r \to 0^+} \nu(r) = 0 \). The monotonicity of the function \( r \mapsto \mu(r) \) implies that there exists

\[
\lim_{r \to +\infty} \mu(r) = \inf_{r \in \mathbb{R}_0^+} \mu(r).
\]

Let us prove that

\[
\inf_{r \in \mathbb{R}_0^+} \mu(r) = 0, \quad (\text{III.14})
\]
that is, for every \( \eta > 0 \) there exist \( r \in \mathbb{R}_0^+ \) and \( v \in C_r \) such that

\[
\mathcal{L}(v) = \int_{\Omega} |Dv| - \int_{\Omega} hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}^{N-1} < \eta. \tag{III.15}
\]

Fix any \( \eta > 0 \). Pick \( x_0 \in \Omega \) and denote by \( B_\delta \) the closed ball centered at \( x_0 \) of radius \( \delta > 0 \). Take \( \delta > 0 \) so small that \( B_\delta \subset \Omega \). Pick constants \( a, b > 0 \) and define a function \( v \in BV(\Omega) \) by

\[
v = -a\chi_{B_\delta} + b(1 - \chi_{B_\delta}) = b - (a + b)\chi_{B_\delta}.
\]

We have

\[
\int_{\Omega} |Dv| = (a + b) \int_{\Omega} |D\chi_{B_\delta}|
\]

and, by Remark I.2.1,

\[
- \int_{\Omega} hv \, dx + \int_{\partial \Omega} \kappa v \, d\mathcal{H}^{N-1} = (a + b) \left( \int_{\Omega} h\chi_{B_\delta} \, dx - \int_{\partial \Omega} \kappa\chi_{B_\delta} \, d\mathcal{H}^{N-1} \right).
\]

Then, using (hIII2), we get

\[
\mathcal{L}(v) = (a + b) \left( \int_{\Omega} |D\chi_{B_\delta}| + \int_{\Omega} h\chi_{B_\delta} \, dx - \int_{\partial \Omega} \kappa\chi_{B_\delta} \, d\mathcal{H}^{N-1} \right)
\leq (a + b) \left( \int_{\Omega} |D\chi_{B_\delta}| + (1 - \rho) \int_{\Omega} |D\chi_{B_\delta}| \right)
\leq 2(a + b) \int_{\Omega} |D\chi_{B_\delta}|.
\]

From [48, Theorem 5.4.1], we have

\[
\int_{\Omega} |D\chi_{B_\delta}| = N\delta^{N-1}\omega_N,
\]

where \( \omega_N \) is the volume of the unit ball in \( \mathbb{R}^N \). The function \( v \) belongs to \( C_r \), for some \( r \in \mathbb{R}_0^+ \), if and only if

\[
b(|\Omega| - \delta^N\omega_N) = \int_{\Omega} u^+ \, dx = \frac{1}{2} \quad \text{and} \quad a\delta^N\omega_N = \int_{\Omega} u^- \, dx = \frac{1}{2r}.
\]

Moreover, \( v \) satisfies (III.15) if

\[
2(a + b)N\delta^{N-1}\omega_N < \eta. \tag{III.16}
\]

Take \( a = 1 \) and set

\[
r = \frac{1}{2\delta^N\omega_N} \quad \text{and} \quad b = \frac{1}{2(|\Omega| - \delta^N\omega_N)}.
\]

As \( N \geq 2 \) we have \( \lim_{\delta \to 0} \delta^{N-1} = 0 \). Hence plugging \( a \) and \( b \) in (III.16) yields

\[
2N\omega_N \left( 1 + \frac{1}{2(|\Omega| - \delta^N\omega_N)} \right) \delta^{N-1} < \eta.
\]
where, as \( N \geq 2 \), the left-hand side goes to 0 letting \( \delta \to 0 \). Accordingly, a number \( r > 0 \) and a function \( v \in C_r \) have been found such that (III.15) holds. Thus (III.14) follows.

In the 1-dimensional case, the asymptotic behaviour of the curve \( C \) is deeply different from the high dimensional one. This discrepancy is due to the impossibility in dimension \( N = 1 \) of having functions with arbitrarily large oscillation and arbitrarily small variation. Indeed we have the following result.

Proposition III.2.6 (Asymptotic behaviour in dimension \( N = 1 \)). Assume \( N = 1 \) and let \( \Omega = [0,T] \). Suppose that \((h_{III}1')\) and \((h_{III}2)\) hold. Then we have

\[
\lim_{r \to +\infty} \mu(r) > 0 \quad \text{and} \quad \lim_{r \to 0^+} \nu(r) > 0.
\]

Proof. We prove that \( \lim_{r \to +\infty} \mu(r) > 0 \); a similar argument shows that \( \lim_{r \to 0^+} \nu(r) > 0 \).

The monotonicity of the function \( r \mapsto \mu(r) \) implies that there exists

\[
\lim_{r \to +\infty} \mu(r) = \bar{\mu} \in [0, +\infty[.
\]

Assume by contradiction that \( \bar{\mu} = 0 \). Take any sequence \((r_n)n\) in \( \mathbb{R}^+ \) such that \( r_n \to +\infty \) and consider a corresponding sequence of functions \((v_n)n\) in \( BV(0,T) \) such that

\[
\int_0^T v_n^+ \, dx = \frac{1}{2} \quad \text{and} \quad \int_0^T v_n^- \, dx = \frac{1}{2r_n} \quad \text{(III.17)}
\]

and

\[
\mu(r_n) = \mathcal{L}(v_n).
\]

From Proposition I.2.3 we get

\[
\lim_{n \to +\infty} \int_{[0,T]} |Dv_n| = 0.
\]

The inequality

\[
\text{ess sup}_{[0,T]} v - \text{ess inf}_{[0,T]} v \leq \int_{[0,T]} |Dv_n|,
\]

which holds for all \( v \in BV(0,T) \) (see, e.g., [23, Chapter 2.3]), implies

\[
\lim_{n \to +\infty} \left( \text{ess sup}_{[0,T]} v_n - \text{ess inf}_{[0,T]} v_n \right) = 0.
\]

The conditions in (III.17) imply that

\[
\text{ess inf}_{[0,T]} v_n \leq 0 \leq \text{ess sup}_{[0,T]} v_n
\]

and therefore

\[
\lim_{n \to +\infty} \text{ess sup}_{[0,T]} v_n = 0,
\]

which yields a contradiction with the first condition in (III.17). \( \Box \)
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The case $h = 0$ and $\kappa = 0$. It is clear that if $h = 0$ and $\kappa = 0$, then taking $r = 1$ we get

$$\mu(1) = \nu(1) = \min \left\{ \int_\Omega |Dv| : v \in BV(\Omega), \int_\Omega v\,dx = 0, \int_\Omega |v|\,dx = 1 \right\} = c.$$ 

We want to compare the Poincaré constant $c$ with the second eigenvalue $c_2$ of the 1-Laplace operator with Neumann boundary conditions as defined in [25]. To this aim we recall the variational characterization of $c_2$ provided therein. Let $A$ be a closed and symmetric subset of a Banach space and denote by $\gamma(A)$ its Kransnoselskii genus. We recall that $\gamma(A) \geq 2$ if no continuous odd function $g : A \to \mathbb{R} \setminus \{0\}$ exists. Set

$$\mathcal{F}_2 = \{ A \subseteq L^1(\Omega) : A \text{ closed}, A = -A, \gamma(A) \geq 2 \}.$$

Set $S = \{ v \in L^1(\Omega) : \|v\|_{L^1} = 1 \}$ and define a functional $\mathcal{E} : L^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(v) = \begin{cases} \int_\Omega |D(v)| & \text{if } v \in S \cap BV(\Omega), \\ +\infty & \text{if } v \in L^1(\Omega) \setminus (S \cap BV(\Omega)) \end{cases}.$$

Then from [25] we have

$$c_2 = \inf_{A \in \mathcal{F}_2} \sup_{v \in A} \mathcal{E}(v). \quad (III.18)$$

**Proposition III.2.7.** Assume $(h_{III0})$ and suppose that $h = 0$ and $\kappa = 0$. Then, we have $c \leq c_2$.

**Proof.** Pick any $A \in \mathcal{F}_2$. We want to prove that

$$c \leq \sup_{v \in A} \mathcal{E}(v). \quad (III.19)$$

We may assume $A \subseteq S \cap BV(\Omega)$, otherwise the inequality is trivially satisfied. Observe that $\int_\Omega v_0\,dx = 0$ for some $v_0 \in A$. Indeed, otherwise, we would have $A = A^- \cup A^+$, with

$$A^- = \{ v \in A : \int_\Omega v\,dx < 0 \} \quad \text{and} \quad A^+ = \{ v \in A : \int_\Omega v\,dx > 0 \},$$

and we could define an odd continuous function $g : A \to \mathbb{R} \setminus \{0\}$ by setting $g(v) = \chi_{A^+} - \chi_{A^-}$, thus contradicting the assumption $\gamma(A) \geq 2$. Therefore we have $c \leq \int_\Omega |Dv_0| = \mathcal{E}(v_0)$ and thus (III.19) follows. Since (III.19) holds for all $A \in \mathcal{F}_2$, we conclude that $c \leq c_2$.

**III.3 Characterization of the curve $C$ in dimension $N = 1$**

In this section we deal with the case of dimension $N = 1$ in which we are able to provide an explicit characterization of the curve $C$. The proof makes use of rearrangement techniques.
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Proposition III.3.1 (Characterization of \( C \) in dimension \( N = 1 \)). Assume \( N = 1 \) and let \( \Omega = [0,T[. \) Suppose that \( h = 0 \) and \( \kappa = 0 \). Then we have

\[
C = \left\{ (\mu,\nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T} \right\}.
\]

In particular, for any fixed \( (\mu,\nu) \in C \), every \( v \in BV(0,T) \) such that

\[
\mu \int_0^T v^+ \, dx - \nu \int_0^T v^- \, dx = 0
\]

also satisfies

\[
\mu \int_0^T v^+ \, dx + \nu \int_0^T v^- \, dx \leq \int_{[0,T[} |Dv|.
\]

Moreover, the equality is attained if and only if \( u \) is a positive multiple either of \( \varphi \) or of \( \varphi(T-\cdot) \), with

\[
\varphi(x) = \begin{cases}
1 & \text{if } 0 < x < \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T, \\
-1 & \text{if } \frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu}} T \leq x < T.
\end{cases}
\]

(III.20)

Proof. Part 1. We prove that, if \( \mu,\nu \in \mathbb{R}_0^+ \) are such that

\[
\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T},
\]

then every \( v \in BV(0,T) \) such that

\[
\mu \int_0^T v^+ \, dx - \nu \int_0^T v^- \, dx = 0,
\]

(III.21)

also satisfies

\[
\mu \int_0^T v^+ \, dx + \nu \int_0^T v^- \, dx \leq \int_{[0,T[} |Dv|;
\]

(III.22)

the equality being attained if and only if \( u \) is a positive multiple either of \( \varphi \) or of \( \varphi(T-\cdot) \), with \( \varphi \) defined by (III.20).

The proof of this part is divided into four steps.

Step 1. A monotone decreasing rearrangement. Take \( v \in BV(0,T) \). Set, for each \( t \in \mathbb{R} \),

\[
E_t = \{ x \in [0,T[ : v(x) > t \}
\]

and

\[
E_t^* = [0,|E_t|] \cap [0,T[.
\]

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The monotone decreasing rearrangement of $v$ (see, e.g., [87, Chapter I]) is the function $v^* \in BV(0,T)$ defined, for a.e. $x \in ]0,T[$, by

$$v^*(x) = \sup\{t \in \mathbb{R} : x \in E_t^\ast\} = \sup\{t \in \mathbb{R} : x \leq |E_t|\}.$$  

By [87, Theorem 1.1], we have

$$\int_0^T (v^*)^+ \, dx = \int_0^T v^+ \, dx \quad \text{and} \quad \int_0^T (v^*)^- \, dx = \int_0^T v^- \, dx. \quad (III.23)$$

Moreover, the Polya-Szegö inequality

$$\int_{]0,T[} |Dv^*| \leq \int_{]0,T[} |Dv| \quad (III.24)$$

holds. This follows by first observing that for each $t \in \mathbb{R}$

$$\text{Per}(E_t^\ast) \leq \text{Per}(E_t). \quad (III.25)$$

Indeed, if $\text{Per}(E_t) = 0$, then $\int_{]0,T[} |D\chi_{E_t}| = 0$ and $\chi_{E_t}$ is constant a.e. in $]0,T[$. Thus we infer that, up to a set of measure 0, either $E_t = ]0,T[$ and hence $E_t^\ast = ]0,T[$, or $E_t = \emptyset$ and hence $E_t^\ast = \emptyset$; therefore, in both cases, $\text{Per}(E_t^\ast) = 0$. Accordingly, (III.25) follows observing that $\text{Per}(E_t^\ast) \leq 1$. Then, by (III.25), the coarea formula (see [13, Theorem 10.3]) yields

$$\int_{]0,T[} |Dv^*| = \int_{]0,T[} \text{Per}(E_t^\ast) \, dt \leq \int_{]0,T[} \text{Per}(E_t) \, dt = \int_{]0,T[} |Dv|.$$  

**Step 2.** The inequality (III.22) holds for all decreasing functions $v : ]0,T[ \to \mathbb{R}$ satisfying (III.21). Let $v : ]0,T[ \to \mathbb{R}$ be a decreasing function satisfying (III.21) and

$$\mu \int_0^T v^+ \, dx + \nu \int_0^T v^- \, dx = 1, \quad (III.26)$$

or equivalently

$$\int_0^T v^+ \, dx = \frac{1}{2\mu} \quad \text{and} \quad \int_0^T v^- \, dx = \frac{1}{2\nu}. \quad (III.27)$$

Let $T_0 \in ]0,T]$ be such that $v(x) \geq 0$ a.e. in $[0,T_0]$ and $v(x) \leq 0$ a.e. in $[T_0,T]$. It is clear that $\text{ess sup}_0^{T_0} v \geq \frac{1}{2\mu T_0}$ and $\text{ess inf}_0^{T} v \leq \frac{1}{2\nu(T-T_0)}$ and therefore

$$\int_{]0,T[} |Dv| = \text{ess sup}_0^{T_0} v - \text{ess inf}_0^{T} v \geq \frac{1}{2\mu T_0} + \frac{1}{2\nu(T-T_0)} \geq \frac{1}{2T} \left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right)^2 = 1,$$

as the minimum of the function $\xi \mapsto \frac{1}{2\xi} + \frac{1}{2\nu(T-T_0)}$ in $]0,T]$ is attained at $\frac{\sqrt{\nu}}{\sqrt{\mu} + \sqrt{\nu T}}$ and is equal to $\frac{1}{2T} \left(\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}}\right)^2$. Hence (III.22) holds, as $v$ satisfies (III.26). The general
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classification, for all decreasing functions \( v : [0, T] \to \mathbb{R} \) satisfying (III.21), follows by homogeneity.

**Step 3.** The inequality (III.22) holds for all functions \( v \in BV(0, T) \) satisfying (III.21). Let \( v \in BV(0, T) \) satisfy (III.21) and let \( v^* \) be the decreasing rearrangement of \( v \) as defined in Step 1, which satisfies (III.21) as well. By (III.24), (III.23) and Step 2 we have

\[
\int_{[0,T]} |Dv| \geq \int_{[0,T]} |Dv^*| \geq \mu \int_0^T (v^*)^+ dx + \nu \int_0^T (v^*)^- dx = \mu \int_0^T v^+ dx + \nu \int_0^T v^- dx.
\]

**Step 4.** The equality is attained both in (III.21) and in (III.22) if and only if \( v \) is a positive multiple either of \( \varphi \) or of \( \varphi(T - \cdot) \), with \( \varphi \) defined by (III.20). It is easily checked by a direct inspection that if \( v \) is a positive multiple of \( \varphi \) or of \( \varphi(T - \cdot) \), then it satisfies the equality both in (III.21) and in (III.22).

Let us prove the converse implication. Assume that \( v \in BV(0, T) \) satisfies the equality both in (III.21) and in (III.22). Possibly rescaling \( v \), we can suppose that (III.27) holds too and hence

\[
\int_{[0,T]} |Dv| = \mu \int_0^T v^+ dx + \nu \int_0^T v^- dx = 1.
\]

Let \( v^* \) be the decreasing rearrangement of \( v \), as defined in Step 1. We have

\[
\int_0^T (v^*)^+ dx = \frac{1}{2\mu} \quad \text{and} \quad \int_0^T (v^*)^- dx = \frac{1}{2\nu},
\]

as well as

\[
1 = \int_{[0,T]} |Dv| \geq \int_{[0,T]} |Dv^*| \geq \mu \int_0^T (v^*)^+ dx + \nu \int_0^T (v^*)^- dx = 1.
\]

Hence, by the coarea formula, we have

\[
\int_{-\infty}^{+\infty} \text{Per}(E_t) \, dt = \int_{[0,T]} |Dv| = \int_{[0,T]} |Dv^*| = \int_{-\infty}^{+\infty} \text{Per}(E^*_t) \, dt
\]

and, by (III.25), we conclude that, for a.e. \( t \in \mathbb{R} \),

\[
\text{Per}(E_t) = \text{Per}(E^*_t).
\]

Therefore, up to a set of measure 0, \( E_t \) is an interval having either 0 or \( T \) as one of its endpoints. Namely, we have that either \( E_t = [0, |E_t|] \cap [0, T] \) for a.e. \( t \in \mathbb{R} \), or \( E_t = [T - |E_t|, T] \cap [0, T] \) for a.e. \( t \in \mathbb{R} \). Then the representation formula (I.7) implies that either \( v = v^* \) or \( v = v^*(T - \cdot) \).
Suppose that \( v = v^* \). Let \( T_0 \in ]0,T[ \) be such that \( v(x) \geq 0 \) a.e. in \([0,T_0]\) and \( v(x) \leq 0 \) a.e. in \([T_0,T]\). As \( \text{ess sup}_v \geq \frac{1}{2\mu T_0} \) and \( \text{ess inf}_v \leq -\frac{1}{2\nu(T-T_0)} \), we have

\[
1 = \int_{]0,T[} |Dv| = \text{ess sup}_v - \text{ess inf}_v \geq \frac{1}{2\mu T_0} + \frac{1}{2\nu(T-T_0)} \geq \frac{1}{2T} \left( \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \right)^2 = 1,
\]

and therefore \( \text{ess sup}_v = \frac{1}{2\mu T_0} \), \( \text{ess inf}_v = -\frac{1}{2\nu(T-T_0)} \) and \( T_0 = \sqrt{\frac{\mu}{\mu+\nu}}T \). Thus we conclude that \( v(x) = \frac{1}{2\mu T_0} \) a.e. in \([0,T_0]\) and \( v(x) = -\frac{1}{2\nu(T-T_0)} \) a.e. in \([T_0,T]\), i.e., \( v = \varphi \).

Similarly, we show that if \( v = v^*(T-\cdot) \), then \( v = \varphi(T-\cdot) \).

**Part 2.** We have

\[
C = \left\{ (\mu,\nu) \in \mathbb{R}^+_0 \times \mathbb{R}^+_0 : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T} \right\}.
\]

Assume that \( \mu,\nu \in \mathbb{R}^+_0 \) satisfy \( \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T} \). Then, setting \( r = \frac{\nu}{\mu} \), we have that \( \mu \varphi \in C_r \), with \( \varphi \) defined in (III.20), and

\[
\mu = \mu \int_{]0,T[} |D\varphi| = \mu \min \left\{ \int_{]0,T[} |Dv| : v \in BV(0,T), \int_0^T v^+ \, dx = \frac{1}{2\mu}, \int_0^T v^- \, dx = \frac{1}{2\nu} \right\} = \min \left\{ \int_{]0,T[} |Dv| : v \in C_r \right\}.
\]

Thus we conclude that \((\mu,\nu) \in C\).

Conversely, suppose that \((\mu,\nu) \in C\). Take \( \tilde{\mu}, \tilde{\nu} \in \mathbb{R}^+_0 \) such that \( \frac{1}{\sqrt{\tilde{\mu}}} + \frac{1}{\sqrt{\tilde{\nu}}} = \sqrt{2T} \) and \( \tilde{\nu} = r = \frac{\nu}{\mu} \). We know from the previous step that

\[
\tilde{\mu} = \min \left\{ \int_{]0,T[} |Dv| : v \in C_r \right\} = \mu.
\]

Thus we conclude that \( \mu = \tilde{\mu} \) and \( \nu = \tilde{\nu} \).

**Remark III.3.1** Under the assumptions of Proposition III.3.1 we have that \((\tilde{\frac{T}{\mu}}, \tilde{\frac{T}{\nu}}) \in C\), where \( \tilde{\frac{T}{\mu}} \) is the second eigenvalue \( c_2 \) of the one-dimensional 1-Laplace operator with Neumann boundary conditions in \( ]0,T[ \), defined by (III.18) and explicitly calculated in [25]. Moreover, \( C \) is asymptotic to the lines \( \mu = \frac{1}{2T} \) and \( \nu = \frac{1}{2T} \).
III. AN ASYMMETRIC POINCARÉ INEQUALITY
Chapter IV

Solvability of capillarity-type problems with asymmetric perturbations

In this chapter we collect various statements concerning non-existence, existence and multiplicity of bounded variation solutions of a class of capillarity-type problems with asymmetric perturbations. In particular, we are concerned with the solvability of the problem
\[
\begin{align*}
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= f(x, u) + h(x) \quad \text{in } \Omega, \\
- \nabla u \cdot n / \sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial \Omega.
\end{align*}
\]

(IV.1)

Hereafter we assume

\begin{enumerate}
\item[(h_{IV}0)] $\Omega$ is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial \Omega$;
\item[(h_{IV}1)] $h \in L^p(\Omega)$, for some $p > N$, and $\kappa \in L^\infty(\partial \Omega)$;
\item[(h_{IV}2)] there exists a constant $\rho > 0$ such that
\[
\left| \int_B h \, dx - \int_{\partial \Omega} \kappa \chi_B \, d\mathcal{H}^{N-1} \right| \leq (1 - \rho) \int_\Omega |D\chi_B|
\]
for every Caccioppoli set $B \subseteq \Omega$;
\item[(h_{IV}3)] $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., for a.e. $x \in \Omega$, $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous and, for every $s \in \mathbb{R}$, $f(\cdot, s) : \Omega \to \mathbb{R}$ is measurable; moreover, there exist constants $a > 0$ and $q \in [1, 1^*]$ and a function $b \in L^p(\Omega)$, with $p > N$, such that
\[
|f(x, s)| \leq a|s|^{q-1} + b(x)
\]
for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.
\end{enumerate}
IV. SOLVABILITY OF CAPILLARITY-TYPE PROBLEMS WITH ASYMMETRIC PERTURBATIONS

Remark IV.0.2 Whenever \((h_{IV1})\) and \((h_{IV3})\) are assumed simultaneously, we suppose that \(q = \frac{1}{p - 1}\). We also notice that condition \((h_{IV3})\) obviously holds if \(f\) satisfies the Carathéodory conditions and

\[
\text{ess sup}_{\Omega \times \mathbb{R}} |f(x, s)| < +\infty.
\]

This is the situation that will occur for the most in the sequel.

As done before, we set \(F(x, s) = \int_0^s f(x, \xi) \, d\xi\) and we consider the functional \(I : BV(\Omega) \to \mathbb{R}\) defined by

\[
I(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, dH_{N-1} - \int_{\Omega} F(x, v) \, dx.
\]

For convenience, we recall the definition of functionals \(L : BV(\Omega) \to \mathbb{R}\), i.e.,

\[
L(v) = \int_{\Omega} |Dv| - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, dH_{N-1}
\]

and \(J : BV(\Omega) \to \mathbb{R}\), i.e.,

\[
J(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} - \int_{\Omega} hv \, dx + \int_{\partial\Omega} \kappa v \, dH_{N-1}.
\]

In accordance with the discussion performed in Chapter I, the following notion of solution is adopted.

**Definition of solution.** We say that a function \(u \in BV(\Omega)\) is a solution of problem \((IV.1)\) if \(u\) satisfies

\[
J(v) - J(u) \geq \int_{\Omega} f(x, u)(v - u) \, dx \tag{IV.2}
\]

for every \(v \in BV(\Omega)\).

We recall that \(u\) is a solution of \((IV.1)\) if and only if \(u\) is a minimizer in \(BV(\Omega)\) of the functional \(K_u : BV(\Omega) \to \mathbb{R}\) defined by \(K_u(v) = J(v) - \int_{\Omega} f(x, u) v \, dx\).

IV.1 Technical results

In this Section we state some results which will be repeatedly used in the sequel.

**Proposition IV.1.1** (A continuous projection). Fix \(\mu, \nu \in \mathbb{R}_0^+\). Then, for each \(v \in L^1(\Omega)\) there exists a unique \(P(v) \in \mathbb{R}\) such that

\[
\mu \int_{\Omega} (v - P(v))^+ \, dx - \nu \int_{\Omega} (v - P(v))^- \, dx = 0. \tag{IV.3}
\]

The map \(P : L^1(\Omega) \to \mathbb{R}\) such that \(v \mapsto P(v)\) is a continuous projection.
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**Proof.** Consider the continuous function \( h : L^1(\Omega) \times \mathbb{R} \to \mathbb{R} \) defined by

\[
h(v, s) = \mu \int_{\Omega} (v - s)^+ \, dx - \nu \int_{\Omega} (v - s)^- \, dx.
\]

Observe that, for each \( v \in L^1(\Omega) \), the function \( h(v, \cdot) : \mathbb{R} \to \mathbb{R} \) is strictly decreasing. Using Fatou’s lemma we easily verify that

\[
\lim_{s \to -\infty} h(v, s) = \lim_{s \to -\infty} \mu \int_{\Omega} (v - s)^+ \, dx = +\infty
\]

and

\[
\lim_{s \to +\infty} h(v, s) = \lim_{s \to +\infty} \nu \int_{\Omega} (v - s)^- \, dx = -\infty.
\]

Hence, by the strict monotonicity of \( h(v, \cdot) \) and the intermediate value theorem, there exists a unique \( P(v) \in \mathbb{R} \) such that \( h(v, P(v)) = 0 \), i.e., (IV.3) holds. Moreover we clearly have \( P \circ P = P \).

Let us prove that \( P \) is continuous. Fix \( v_0 \in L^1(\Omega) \) and pick \( \varepsilon > 0 \). Since \( h(v_0, \cdot) \) is strictly decreasing and \( h(v_0, P(v_0)) = 0 \) we have

\[
h(v_0, P(v_0) + \varepsilon) < 0 < h(v_0, P(v_0) - \varepsilon).
\]

By the continuity, for any fixed \( s \in \mathbb{R} \), of the map \( h(\cdot, s) : L^1(\Omega) \to \mathbb{R} \), we can find a neighbourhood \( V \) of \( v_0 \) such that

\[
h(v, P(v_0) + \varepsilon) < 0 < h(v, P(v_0) - \varepsilon)
\]

for all \( v \in V \). Again by the strict monotonicity of the real function \( h(v, \cdot) \) and the intermediate value theorem, we conclude that, for every \( v \in V \), the unique point \( P(v) \) such that \( h(v, P(v)) = 0 \) belongs to the interval \( [P(v_0) - \varepsilon, P(v_0) + \varepsilon] \). This shows the continuity of \( P \) at \( v_0 \).

**Proposition IV.1.2** (A coercivity property over cones). Assume \((h_{IV}0), (h_{IV}1), (h_{IV}2), (h_{IV}3)\) and

\((h_{IV}4)\) there exists \((\mu, \nu) \in C\) such that

\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x, s) < \mu \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x, s) > -\nu.
\]

Then there exists \( \eta > 0 \) such that

\[
\mathcal{I}(w + r) \geq \eta \int_{\Omega} |Dw| - \int_{\Omega} F(x, r) \, dx
\]

for every \( r \in \mathbb{R} \) and \( w \in \mathcal{W} \), where

\[
\mathcal{W} = \left\{ w \in BV(\Omega) : \mu \int_{\Omega} w^+ \, dx - \nu \int_{\Omega} w^- \, dx = 0 \right\}.
\]

\[\text{(IV.4)}\]
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Proof. By \((h_{IV}4)\) there exists \(\vartheta \in ]0,1[\) such that
\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x,s) \leq \vartheta \mu \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x,s) \geq -\vartheta \nu.
\]
For any given \(r \in \mathbb{R}\) and \(w \in W\) we have, by Proposition III.1.2, Proposition I.2.3 and \((h_{IV}2)\),
\[
\mathcal{I}(w + r) = \mathcal{J}(w) - \int_{\Omega} F(x,w + r) \, dx
\]
\[
= \mathcal{J}(w) - \int_{\Omega} \left( F(x,w + r) - F(x,r) \right) dx - \int_{\Omega} F(x,r) \, dx
\]
\[
= \mathcal{J}(w) - \int_{\Omega} \left( \int_{r}^{r+w(x)} f(x,s) \, ds \right) \text{sgn}(w^+) \, dx
\]
\[
+ \int_{\Omega} \left( \int_{r}^{r+w(x)} f(x,s) \, ds \right) \text{sgn}(w^-) \, dx - \int_{\Omega} F(x,r) \, dx
\]
\[
\geq \mathcal{L}(w) - \vartheta \mu \int_{\Omega} w^+ \, dx - \vartheta \nu \int_{\Omega} w^- \, dx - \int_{\Omega} F(x,r) \, dx
\]
\[
\geq \left( 1 - \vartheta \right) \mathcal{L}(w) - \int_{\Omega} F(x,r) \, dx
\]
\[
\geq \left( 1 - \vartheta \right) \rho \int_{\Omega} |Dw| - \int_{\Omega} F(x,r) \, dx.
\]
Hence the conclusion follows. \(\Box\)

Remark IV.1.1 Condition \((h_{IV}4)\) can be replaced by the apparently more general assumption
\((h_{IV}4')\) there exists \((\mu, \nu) \in C\) such that either
\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x,s) < \mu \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x,s) \geq -\nu
\]
or
\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x,s) \leq \mu \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x,s) > -\nu.
\]
Indeed, the properties of the curve \(C\) stated in Proposition III.2.2 and in Proposition III.2.3 guarantee the existence of a point \((\tilde{\mu}, \tilde{\nu}) \in C\) which can be possibly different from \((\mu, \nu)\), such that
\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x,s) < \tilde{\mu} \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x,s) > -\tilde{\nu}.
\]

Proposition IV.1.3 (A positive definite homogeneous form). Assume \((h_{IV}0)\), \((h_{IV}1)\) and \((h_{IV}2)\). Let \(\zeta \in L^p(\Omega)\), with \(p > N\), be such that \(\zeta(x) \leq 0\) a.e. in \(\Omega\) and \(\zeta(x) < 0\) on a set of positive measure. Then there exists \(\delta > 0\) such that
\[
\mathcal{L}(v) - \int_{\Omega} |v| \, dx \geq \delta \|v\|_{BV}
\]
for all \(v \in BV(\Omega)\).
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*Proof.* Possibly replacing \( \zeta \) with \(-1 \lor \zeta\), we can assume \( \zeta \in L^\infty(\Omega)\). Define \( K : BV(\Omega) \to \mathbb{R} \) by

\[
K(v) = \mathcal{L}(v) - \int_\Omega \zeta |v| \, dx.
\]

Note that \( K(v) \geq 0 \) for all \( v \in BV(\Omega) \) and \( K(v) = 0 \) if and only if \( v = 0 \). Indeed, if \( K(v) = 0 \) then \( \mathcal{L}(v) = 0 \) and hence, by Proposition I.2.3, \( \int_\Omega |Dv| = 0 \). By [3, Proposition 3.2], \( v \) is constant a.e. in \( \Omega \); therefore we easily conclude that \( v = 0 \). In order to prove the thesis we suppose, by contradiction, that there exists a sequence \((w_n)_n\) in \( BV(\Omega) \) such that, for each \( n \),

\[
0 \leq K(w_n) < \frac{1}{n} \| w_n \|_{BV}
\]

and hence, setting \( v_n = \frac{w_n}{\| w_n \|_{BV}} \),

\[
0 \leq K(v_n) < \frac{1}{n}.
\]

We also have, by Proposition I.2.3,

\[
K(v_n) = \mathcal{L}(v_n) + \int_\Omega |v_n| \, dx - \int_\Omega (\zeta + 1)|v_n| \, dx \\
\geq \rho \left( \int_\Omega |Dv_n| + \int_\Omega |v_n| \, dx \right) - \int_\Omega (\zeta + 1)|v_n| \, dx \\
= \rho - \int_\Omega (\zeta + 1)|v_n| \, dx.
\]

Since the sequence \((v_n)_n\) is bounded in \( BV(\Omega) \), there exists a subsequence, we still denote by \((v_n)_n\), which converges in \( L^q(\Omega) \), with \( q = \frac{p}{p-1} \), to some \( v \in BV(\Omega) \). In particular we have

\[
\lim_{n \to +\infty} \int_\Omega \zeta |v_n| \, dx = \int_\Omega \zeta |v| \, dx.
\]

As we have, by (IV.5), \( \lim_{n \to +\infty} K(v_n) = 0 \) and hence, by (IV.6), \( \lim_{n \to +\infty} \int_\Omega (\zeta + 1)|v_n| \, dx \geq \rho > 0 \), we conclude that \( v \neq 0 \) and therefore \( K(v) > 0 \). The lower semicontinuity of \( K \) with respect to the \( L^q \)-convergence finally yields

\[
0 < K(v) \leq \liminf_{n \to +\infty} K(v_n) = 0,
\]

which is a contradiction.

**IV.2 Existence versus non-existence**

In order to make more transparent our statements we assume in this subsection that \( h = 0 \) and \( \kappa = 0 \), so that the functional \( \mathcal{L} \) is just the total variation. However,
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similar conclusions hold in the general case as well. We also write \( f \) in the form
\[
f(x,s) = g(x,s) + e(x).
\]

Our first result shows that the existence of solutions is guaranteed in the case where \( g = 0 \) and \( e \) lies, in some sense, “below” the curve \( C \) defined in (III.10).

**Proposition IV.2.1.** Assume \((h_{IV}0)\). Fix \((\mu, \nu) \in C\). Then for every \( e \in L^\infty(\Omega) \), with \( \int_\Omega e \, dx = 0 \), \( \text{ess sup } e < \mu \) and \( \text{ess inf } e > -\nu \), the problem
\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = e(x) & \text{in } \Omega, \\
-\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = 0 & \text{on } \partial \Omega
\end{cases}
\]
(IV.7)

has a solution \( w \in BV(\Omega) \) with \( \mu \int_\Omega w^+ \, dx - \nu \int_\Omega w^- \, dx = 0 \).

**Proof.** For every \( v \in BV(\Omega) \), let us write \( v = w + \mathcal{P}(v) \), with \( \mathcal{P} \) defined in Proposition IV.1.1. The condition \( \int_\Omega e \, dx = 0 \) implies that
\[
I(v) = I(w) = \int_\Omega \sqrt{1 + |Dw|^2} - \int_\Omega ew \, dx.
\]
By Proposition IV.1.2, there exists \( \eta > 0 \) such that
\[
I(v) \geq \eta \int_\Omega |Dv|
\]
and hence the functional \( I \) is bounded from below. Let \( (v_n)_n \) be a minimizing sequence and set \( w_n = v_n - \mathcal{P}(v_n) \). Clearly, \( (w_n)_n \) is a minimizing sequence too. Using again Proposition IV.1.2 and Proposition I.2.1 combined with Remark III.1.2, we see that \( (w_n)_n \) is bounded in \( BV(\Omega) \) and hence it has a subsequence converging in \( L^1(\Omega) \) to some \( w \in \mathcal{W} \), where \( \mathcal{W} \) is defined in (IV.4). The usual semicontinuity argument shows that \( w \) is a minimizer and therefore a solution of (IV.7).

**Remark IV.2.1** It is evident that the condition \( \int_\Omega e \, dx = 0 \) is necessary for the existence of solutions of (IV.7); hence if either \( \text{ess sup } e \leq 0 \) and \( \text{ess inf } e < 0 \), or \( \text{ess sup } e \geq 0 \) and \( \text{ess inf } e > 0 \), then no solution of (IV.7) may exist.

Our second result shows that the existence of solutions is not guaranteed if \( f \) lies, in some sense, “above” the curve \( C \) defined in (III.10).

**Proposition IV.2.2.** Assume \((h_{IV}0)\). Fix \((\mu, \nu) \in C\). Then there exist functions \( e \in L^\infty(\Omega) \), with \( \int_\Omega e \, dx = 0 \) and either \( \text{ess sup } e > \mu \) and \( \text{ess inf } e \leq -\nu \), or \( \text{ess sup } e \geq \mu \) and \( \text{ess inf } e < -\nu \), and a constant \( \gamma > 0 \) such that, for any function \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying the Carathéodory conditions and \( \text{ess sup } g(x,s) \leq \gamma \), the problem
\[
\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = g(x,u) + e(x) & \text{in } \Omega, \\
-\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = 0 & \text{on } \partial \Omega
\end{cases}
\]
(IV.8)
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has no solution.

Proof. Let \( \varphi \in BV(\Omega) \setminus \{0\} \) be such that

\[
\mu \int_{\Omega} \varphi^+ dx - \nu \int_{\Omega} \varphi^- dx = 0 \quad \text{and} \quad \int_{\Omega} |D\varphi| = \mu \int_{\Omega} \varphi^+ dx + \nu \int_{\Omega} \varphi^- dx.
\]

Pick \( \rho, \sigma \in \mathbb{R}_0^+ \) such that \( \sigma \int_{\Omega} \text{sgn}(\varphi^-) dx = \rho \int_{\Omega} \text{sgn}(\varphi^+) dx, \rho > \mu \) and \( \sigma > \nu \). Define \( e \in L^\infty(\Omega) \) by setting

\[
e = \rho \text{sgn}(\varphi^+) - \sigma \text{sgn}(\varphi^-).
\]

Clearly, we have \( \int_{\Omega} e \, dx = 0 \). Take \( \gamma > 0 \) and any function \( g: \Omega \times \mathbb{R} \to \mathbb{R} \) satisfying the Carathéodory conditions and \( \text{ess sup}_{\Omega \times \mathbb{R}} |g(x, s)| \leq \gamma \). Take any \( u \in BV(\Omega) \), using Proposition I.1.5, we compute for \( k \in \mathbb{R}_0^+ \)

\[
J(k\varphi) - \int_{\Omega} (g(x, u) + e) k\varphi \, dx \leq |\Omega| + \int_{\Omega} |Dk\varphi| + \gamma \int_{\Omega} |k\varphi| \, dx - \int_{\Omega} ek\varphi \, dx \]
\[
= |\Omega| + k \int_{\Omega} |D\varphi| + k\gamma \int_{\Omega} |\varphi| \, dx - k \int_{\Omega} (\rho \text{sgn}(\varphi^+) - \sigma \text{sgn}(\varphi^-)) (\varphi^+ - \varphi^-) \, dx \]
\[
= |\Omega| + k \left( \int_{\Omega} |D\varphi| - \mu \int_{\Omega} \varphi^+ dx - \nu \int_{\Omega} \varphi^- dx \right) \]
\[
- k(\rho - \mu) \int_{\Omega} \varphi^+ dx - k(\sigma - \nu) \int_{\Omega} \varphi^- dx + k\gamma \int_{\Omega} |\varphi| \, dx \]
\[
= |\Omega| - k \left( (\rho - \mu - \gamma) \int_{\Omega} \varphi^+ dx + (\sigma - \nu - \gamma) \int_{\Omega} \varphi^- dx \right).
\]

Hence we infer that

\[
\inf_{v \in BV(\Omega)} \left( J(v) - \int_{\Omega} (g(x, u) + e) v \, dx \right) = -\infty,
\]

provided that \( \gamma > 0 \) is taken so small that \( (\rho - \mu - \gamma) \int_{\Omega} \varphi^+ dx + (\sigma - \nu - \gamma) \int_{\Omega} \varphi^- dx > 0 \).

Therefore problem (IV.8) has no solution, according to Remark I.2.2. \( \square \)

Remark IV.2.2 In Proposition IV.2.2, we can replace the requirement

\[
\text{ess sup}_{\Omega \times \mathbb{R}} |g(x, s)| \leq \gamma,
\]

for some small constant \( \gamma \), with

\[
|g(x, s)| \leq \gamma(x),
\]

for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \), for some \( \gamma \in L^p(\Omega) \), with \( p > N \), having a small \( L^p \)-norm.
Remark IV.2.3 From Proposition IV.2.2 it follows, in particular, that there exist functions \( e \in L^\infty(\Omega) \), with \( \int_\Omega e \, dx = 0 \), and \( g : \mathbb{R} \to \mathbb{R} \) continuous, bounded and strictly monotone, with either
\[
\lim_{s \to -\infty} g(s) < 0 < \lim_{s \to +\infty} g(s),
\]
or
\[
\lim_{s \to -\infty} g(s) > 0 > \lim_{s \to +\infty} g(s),
\]
such that problem (IV.8) has no solution. This remark motivates the existence results we are going to present in the following section.

In Chapter V a sharper result of non-existence is performed in the 1-dimensional case.

IV.3 Existence results

In this section we shall prove existence of solutions of problem (IV.1). Let us observe that a necessary condition, in order a solution \( u \) exists, is that \( \int_\Omega f(x,u) \, dx = 0 \). This implies that \( f \), if it is not identically zero, must change sign in \( \Omega \times \mathbb{R} \). Here we shall assume some stronger assumptions implying this fact, namely, either an Ahmad-Lazer-Paul condition, or a Landesman-Lazer condition, or a Hammerstein-type condition, or a pointwise sign condition. These assumptions will generally be coupled with the non-interference condition with respect to the curve \( C \) expressed by (IV.4), or variations thereof. Account of the cases where this assumption can be omitted will be also given.

Ahmad-Lazer-Paul conditions

In this subsection we assume the coercivity, or the anticoercivity, on \( \mathbb{R} \) of the averaged potential map \( s \mapsto \int_\Omega F(x,s) \, dx \). If \( h = 0 \) and \( \kappa = 0 \), and then \( \mathcal{L} \) is the total variation, this assumption can be interpreted as a non-interference condition with the first eigenvalue \( c_1 = 0 \) of the 1-Laplace operator with Neumann boundary conditions as defined in [25]. This assumption will be coupled with a non-interference condition with the curve \( C \) defined in (III.10), as expressed by assumption (IV.4). It is worth noting that, in the light of the non-existence results stated in Section IV.2, and in particular of Remark IV.2.3, assumption (IV.4) cannot be omitted in this frame.

Our first result deals with the case where a coercivity condition is assumed on \( s \mapsto \int_\Omega F(x,s) \, dx \). In this case a solution is obtained by a min-max procedure.

**Theorem IV.3.1.** Assume (IV.0), (IV.1), (IV.2), (IV.3), (IV.4) and
\[
(IV.5) \quad \lim_{s \to \pm \infty} \int_\Omega F(x,s) \, dx = +\infty.
\]
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Then problem (IV.1) has at least one solution $u$ such that

$$- \min_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx \leq \mathcal{I}(u) \leq |\Omega| - \min_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx.$$  

Proof. The proof is divided into four steps.

Step 1. Mountain pass geometry. By (hIV3) the function $s \mapsto \int_{\Omega} F(x, s) \, dx$ is continuous on $\mathbb{R}$. Hence, using (hIV5), we can find $a^-, a^+, b \in \mathbb{R}$, with $a^- < b < a^+$, such that

$$\int_{\Omega} F(x, b) \, dx = \min_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx$$

and

$$\int_{\Omega} F(x, a^\pm) \, dx > |\Omega| + \int_{\Omega} F(x, b) \, dx.$$  

Set

$$\mathcal{S} = \{ v \in BV(\Omega) : \mathcal{P}(v) = b \} = \{ w + b : w \in \mathcal{W} \},$$

where $\mathcal{P}$ and $\mathcal{W}$ are defined in Proposition IV.1.1 and by (IV.4), respectively. By Proposition IV.1.2 we have

$$\inf_{v \in \mathcal{S}} \mathcal{I}(v) = \inf_{w \in \mathcal{W}} \mathcal{I}(w + b) \geq -\int_{\Omega} F(x, b) \, dx$$

and hence

$$\mathcal{I}(a^-) = |\Omega| - \int_{\Omega} F(x, a^-) \, dx < -\int_{\Omega} F(x, b) \, dx \leq \inf_{\mathcal{S}} \mathcal{I}(v), \quad (IV.9)$$

$$\mathcal{I}(a^+) = |\Omega| - \int_{\Omega} F(x, a^+) \, dx < -\int_{\Omega} F(x, b) \, dx \leq \inf_{\mathcal{S}} \mathcal{I}(v). \quad (IV.10)$$

Let us define

$$\Gamma = \{ \gamma \in C^0([0, 1], BV(\Omega)) : \gamma(0) = a^-, \gamma(1) = a^+ \}.$$  

For any $\gamma \in \Gamma$ the function $\mathcal{P} \circ \gamma : [0, 1] \to \mathbb{R}$ is continuous and satisfies

$$\mathcal{P}(\gamma(0)) = a^- < b < a^+ = \mathcal{P}(\gamma(1)).$$

Therefore there exists $\xi \in [0, 1]$ such that $\mathcal{P}(\gamma(\xi)) = b$, thus showing that $\gamma([0, 1]) \cap \mathcal{S} \neq \emptyset$, that is, the sets $\{a^-, a^+\}$ and $\mathcal{S}$ link. We set $u_0 = a^-$, $u_1 = a^+$ and

$$c_\mathcal{I} = \inf_{\gamma \in \Gamma} \max_{\xi \in [0, 1]} \mathcal{I}(\gamma(\xi)).$$

By (IV.9) and (IV.10) we have

$$c_\mathcal{I} \geq \inf_{\mathcal{S}} \mathcal{I}(v) > \max\{\mathcal{I}(u_0), \mathcal{I}(u_1)\}.$$  

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Accordingly, Lemma I.3.1 yields the existence of sequences \((v_k)_k\) and \((\varepsilon_k)_k\), with \(v_k \in BV(\Omega)\) and \(\varepsilon_k \in \mathbb{R}\), satisfying
\[
\lim_{k \to +\infty} \varepsilon_k = 0,
\]
(I.17), and in particular
\[
\lim_{k \to +\infty} J(v_k) = c_I,
\]
and (I.19), that is,
\[
J(v) - J(v_k) \geq \int_{\Omega} f(x, v_k)(v - v_k) \, dx + \varepsilon_k\|v - v_k\|_{BV},
\]
for each \(k\) and all \(v \in BV(\Omega)\).

Step 2. The sequence \((v_k)_k\) is bounded in \(BV(\Omega)\). By \((h_{IV}4)\) there exists \(\vartheta \in ]0, 1[\) such that
\[
\text{ess sup}_{\Omega \times \mathbb{R}} f(x, s) \leq \vartheta \mu \quad \text{and} \quad \text{ess inf}_{\Omega \times \mathbb{R}} f(x, s) \geq -\vartheta \nu.
\]
(IV.11)

For each \(k\) we set \(r_k = \mathcal{P}(v_k)\) and \(w_k = v_k - r_k \in W\). Taking \(v = v_k - w_k^+\) in (I.19), we have by \((h_{IV}2)\) and (IV.11)
\[
\mathcal{J}(w_k) - \mathcal{J}(-w_k^-) = \mathcal{J}(w_k) - \mathcal{J}(w_k - w_k^+) - \mathcal{J}(w_k - w_k^-) = \mathcal{J}(w_k) - \mathcal{J}(v_k - w_k^+)
\]
\[
\leq \int_{\Omega} f(x, v_k)w_k^+ \, dx - \varepsilon_k\|w_k^+\|_{BV}
\]
\[
\leq \vartheta \mu \int_{\Omega} w_k^+ \, dx - \varepsilon_k\|w_k^+\|_{BV}.
\]

Similarly, taking \(v = v_k + w_k^-\) in (I.19), we have
\[
\mathcal{J}(w_k) - \mathcal{J}(w_k^+) = \mathcal{J}(w_k) - \mathcal{J}(w_k + w_k^-) - \mathcal{J}(w_k - w_k^+)
\]
\[
= \mathcal{J}(w_k) - \mathcal{J}(v_k + w_k^-)
\]
\[
\leq -\int_{\Omega} f(x, v_k)w_k^- \, dx - \varepsilon_k\|w_k^-\|_{BV}
\]
\[
\leq -\vartheta \nu \int_{\Omega} w_k^- \, dx - \varepsilon_k\|w_k^-\|_{BV}.
\]

Summing up we obtain, by Proposition I.2.7 and Proposition III.1.2,
\[
\mathcal{L}(w_k) - |\Omega| \leq \mathcal{J}(w_k) - \mathcal{J}(0)
\]
\[
\leq \mathcal{J}(w_k) + \mathcal{J}(w_k) - \mathcal{J}(w_k^+) - \mathcal{J}(-w_k^-)
\]
\[
\leq \vartheta \mu \int_{\Omega} w_k^+ \, dx - \varepsilon_k\|w_k^+\|_{BV} + \vartheta \nu \int_{\Omega} w_k^- \, dx - \varepsilon_k\|w_k^-\|_{BV}
\]
\[
\leq \vartheta \mathcal{L}(w_k) + |\varepsilon_k|\|w_k\|_{BV}.
\]
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and hence, by Proposition I.2.3 and Remark III.1.2,

\[(1 - \vartheta)\mathcal{L}(w_k) \leq |\varepsilon_k||w_k|_{BV} + |\Omega|\]

\[= |\varepsilon_k|\left(\int_{\Omega} |Dw_k| + \int_{\Omega} |w_k| dx\right) + |\Omega|\]

\[\leq |\varepsilon_k|\left(\frac{1}{\rho} + \frac{1}{\mu} + \frac{1}{\nu}\right)\mathcal{L}(w_k) + |\Omega|.

As \(\lim_{k \to +\infty} \varepsilon_k = 0\), by Proposition I.2.3, there is a constant \(K > 0\) such that, for all \(k\),

\[\mathcal{L}(w_k) \leq K,

and hence,

\[\int_{\Omega} |Dw_k| \leq K,\]  \hspace{1cm} \text{(IV.12)}

\[\int_{\Omega} |w_k| dx \leq K,\]  \hspace{1cm} \text{(IV.13)}

\[J(w_k) \leq K.\]  \hspace{1cm} \text{(IV.14)}

From (I.17), (IV.13), (IV.14) we deduce, for all large \(k\),

\[c_I - 1 \leq \mathcal{I}(v_k) = \mathcal{I}(w_k + r_k)

= J(w_k) - \int_{\Omega} F(x, w_k + r_k) dx

\leq K - \int_{\Omega} \left(\int_{r_k}^{r_k + w_k(x)} f(x, s) ds\right) dx - \int_{\Omega} F(x, r_k) dx

\leq K + (\mu + \nu) \int_{\Omega} |w_k| dx - \int_{\Omega} F(x, r_k) dx

\leq K(1 + \mu + \nu) - \int_{\Omega} F(x, r_k) dx.

Using (hIV5) obtain

\[\sup_k |r_k| < +\infty.\]  \hspace{1cm} \text{(IV.15)}

Combining (IV.12), (IV.13) and (IV.15) we conclude

\[\sup_k \|v_k\|_{BV} < +\infty.

Step 3. Existence of a solution. Since the sequence \((v_k)\) is bounded in \(BV(\Omega)\) there exist a subsequence, that we still denote by \((v_k)\), and a function \(u \in BV(\Omega)\), such that \(\lim_{k \to +\infty} v_k = u\) in \(L^q(\Omega)\), with \(q = \frac{p}{p-1} \in [1, 1^*]\, and a.e. in \(\Omega\). Hence we have, by (hIV3),

\[\lim_{k \to +\infty} f(x, v_k(x)) = f(x, u(x)),\]
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for a.e. $x \in \Omega$, and, by the global boundedness of $f$ implied by $(h_{IV4})$,

$$\lim_{k \to +\infty} f(\cdot, v_k) = f(\cdot, u)$$

in $L^p(\Omega)$. Similarly, we have

$$\lim_{k \to +\infty} F(x, v_k(x)) = F(x, u(x)),$$

for a.e. $x \in \Omega$, and, as $F(\cdot, s)$ grows at most linearly with respect to $s$ uniformly a.e. in $\Omega$,

$$\lim_{k \to +\infty} \int_{\Omega} F(x, v_k) \, dx = \int_{\Omega} F(x, u) \, dx.$$

Moreover, the lower semicontinuity of $J$ with respect to the $L^q$-convergence in $BV(\Omega)$ implies

$$\liminf_{k \to +\infty} J(v_k) \geq J(u).$$

Finally, for any fixed $v \in BV(\Omega)$, we have

$$\lim_{k \to +\infty} \int_{\Omega} f(x, v_k)(v - v_k) \, dx = \int_{\Omega} f(x, u)(v - u) \, dx$$

and

$$\lim_{k \to +\infty} \varepsilon_k \|v - v_k\|_{BV} = 0.$$

Thus we get, passing to the inferior limit in (I.19),

$$J(v) - \int_{\Omega} f(x, u)(v - u) \, dx = J(v) - \lim_{k \to +\infty} \int_{\Omega} f(x, v_k)(v - v_k) \, dx$$

$$\geq \liminf_{k \to +\infty} J(v_k) \geq J(u),$$

and hence

$$J(v) - J(u) \geq \int_{\Omega} f(x, u)(v - u) \, dx$$

for all $v \in BV(\Omega)$, that is $u$ is a solution of (IV.1).

Step 4. A critical value estimate. Taking $v = u$ in (I.19), we get for all $k$

$$J(u) - \int_{\Omega} f(x, v_k)(u - v_k) \, dx - \varepsilon_k \|u - v_k\|_{BV} \geq J(v_k)$$

and hence

$$J(u) = \lim_{k \to +\infty} \left( J(u) - \int_{\Omega} f(x, v_k)(u - v_k) \, dx - \varepsilon_k \|u - v_k\|_{BV} \right) \geq \limsup_{k \to +\infty} J(v_k).$$
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Since, on the other hand,
\[ J(u) \leq \liminf_{k \to +\infty} J(v_k), \]
we conclude that
\[ \lim_{k \to +\infty} J(v_k) = J(u). \]
Thus we obtain
\[ c_I = \lim_{k \to +\infty} I(v_k) \]
\[ = \lim_{k \to +\infty} J(v_k) - \lim_{k \to +\infty} \int_{\Omega} F(x, v_k) \, dx = J(u) - \int_{\Omega} F(x, u) \, dx = I(u). \]
Taking as \( \gamma \) the segment joining \( a^- \) with \( a^+ \), we see that
\[ c_I \leq |\Omega| - \int_{\Omega} F(x, b) \, dx. \]
Since, on the other hand,
\[ c_I \geq \inf_{v \in S} I(v) \geq - \int_{\Omega} F(x, b) \, dx, \]
the conclusion follows.

Our second result deals with the case where an anti-coercivity condition is assumed on \( \int_{\Omega} F(x, s) \, dx \). In this case we find a solution which is a global minimizer.

**Theorem IV.3.2.** Assume \((h_{IV0}), (h_{IV1}), (h_{IV2}), (h_{IV3}), (h_{IV4})\) and

\((h_{IV6})\) \( \lim_{s \to \pm\infty} \int_{\Omega} F(x, s) \, dx = -\infty. \)

Then, problem (IV.1) has at least one solution \( u \), which is a global minimizer of \( I \) in \( BV(\Omega) \) and satisfies

\[ -\max_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx \leq I(u) \leq |\Omega| - \max_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx. \]  \hspace{1cm} (IV.16)

**Proof.** By condition \((h_{IV6})\) we can find \( b \in \mathbb{R} \) such that
\[ \int_{\Omega} F(x, b) \, dx = \max_{s \in \mathbb{R}} \int_{\Omega} F(x, s) \, dx. \]
We write every \( v \in BV(\Omega) \) in the form \( v = w + r \), with \( w \in W \) and \( r = \mathcal{P}(v) \), where \( W \) has been defined in (IV.4) and \( \mathcal{P} \) is the projection coming from in Proposition IV.1.1. Then Proposition IV.1.2 yields, by conditions \((h_{IV4})\),
\[ \mathcal{I}(v) = \mathcal{I}(w + r) \geq \eta \int_{\Omega} |Dw| - \int_{\Omega} F(x, r) \, dx \]  \hspace{1cm} (IV.17)
and hence
\[ \mathcal{I}(v) \geq \eta \int_{\Omega} |Dw| - \int_{\Omega} F(x, b) \, dx, \]  
(IV.18)
for some \( \eta > 0 \) and all \( v \in BV(\Omega) \). Therefore the functional \( \mathcal{I} \) is bounded from below and, by condition \((h_{IV}6)\), it is coercive. Let \((v_k)_k\) be a minimizing sequence and set, for each \( k \), \( w_k = v_k - r_k \), with \( r_k = \mathcal{P}(v_k) \). From (IV.18) we infer that
\[ \sup_k \int_{\Omega} |Dw_k| < +\infty. \]

Remark III.1.2 and Proposition I.2.1 also yield
\[ \sup_k \|w_k\|_{BV} < +\infty. \]

From (IV.17) we also get, by \((h_{IV}6)\),
\[ \sup_k |r_k| < +\infty. \]

Thus we have
\[ \sup_k \|v_k\|_{BV} < +\infty. \]

Hence there exist a subsequence of \((v_k)_k\), that we still denote by \((v_k)_k\), and \( u \in BV(\Omega) \) such that
\[ \lim_{k \to +\infty} v_k = u \text{ in } L^q(\Omega), \]
with \( q = \frac{p}{p-1} \in [1, 1^*] \), and a.e. in \( \Omega \). Therefore we conclude that
\[ \lim_{k \to +\infty} \int_{\Omega} F(x, v_k) \, dx = \int_{\Omega} F(x, u) \, dx, \]
\[ \liminf_{k \to +\infty} \mathcal{J}(v_k) \geq \mathcal{J}(u) \]
and finally
\[ \mathcal{I}(u) \leq \liminf_{k \to +\infty} \mathcal{I}(v_k) = \inf_{v \in BV(\Omega)} \mathcal{I}(v). \]

Therefore \( u \) is a global minimizer of \( \mathcal{I} \). Using the convexity of \( \mathcal{J} \) and the differentiability of the potential operator \( \mathcal{F} \) in \( BV(\Omega) \), we can easily prove that \( u \) satisfies (IV.2), for all \( v \in BV(\Omega) \), and hence it is a solution of problem (IV.1). Estimate (IV.18), which implies
\[ \mathcal{I}(u) \geq -\int_{\Omega} F(x, b) \, dx, \]
yields the first inequality in (IV.16), while the second inequality follows observing that \( \mathcal{I}(u) \leq \mathcal{I}(b) \). \( \square \)

**Remark IV.3.1** It is known (see [1], or [55]) that the Ahmad-Lazer-Paul conditions \((h_{IV}5)\) and \((h_{IV}6)\) are respectively implied by the Landesman-Lazer conditions
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\((h_{IV5}')\) there exists \(\ell \in L^1(\Omega)\) such that \(f(x,s)\sgn(s) \geq \ell(x)\) for a.e. \(x \in \Omega\) and every \(s \in \mathbb{R}\) and

\[
\int_\Omega \left( \limsup_{s \to -\infty} f(x,s) \right) \, dx < 0 < \int_\Omega \left( \liminf_{s \to +\infty} f(x,s) \right) \, dx
\]

and

\((h_{IV6}')\) there exists \(\ell \in L^1(\Omega)\) such that \(f(x,s)\sgn(s) \leq \ell(x)\) for a.e. \(x \in \Omega\) and every \(s \in \mathbb{R}\) and

\[
\int_\Omega \left( \liminf_{s \to -\infty} f(x,s) \right) \, dx > 0 > \int_\Omega \left( \limsup_{s \to +\infty} f(x,s) \right) \, dx.
\]

It is worthy to notice that, according to the non-existence results stated in Subsection IV.2 (see in particular Remark IV.2.3), assumption \((h_{IV4})\) cannot be omitted even if \((h_{IV5}')\), or \((h_{IV6}')\), is assumed in place of \((h_{IV5})\), or \((h_{IV6})\), respectively. Therefore it may have some interest to find conditions which allow to drop assumption \((h_{IV4})\).

As we are going to see, integral conditions should be replaced by pointwise conditions.

Hammerstein-type conditions

In this subsection we replace the Ahmad-Lazer-Paul condition \((h_{IV6})\) with the following Hammerstein-type condition, inspired from [67, 82]:

\((h_{IV6}'')\) there exists \(\zeta \in L^p(\Omega)\), with \(p > N\), \(\zeta(x) \leq 0\) for a.e. \(x \in \Omega\) and \(\zeta(x) < 0\) on a set of positive measure, such that

\[
\limsup_{s \to \pm\infty} \frac{F(x,s)}{|s|} \leq \zeta(x)
\]

uniformly for a.e. \(x \in \Omega\).

Clearly, assumption \((h_{IV6}'')\) implies \((h_{IV6})\). We stress again that in this case condition \((h_{IV4})\) can be dropped. The following result is related to some classical results in [47, 86].

**Theorem IV.3.3.** Assume \((h_{IV0})\), \((h_{IV1})\), \((h_{IV2})\), \((h_{IV3})\) and \((h_{IV6}'')\). Then problem \((IV.1)\) has at least one solution, which is a global minimizer of \(I\) in \(BV(\Omega)\).

**Proof.** By \((h_{IV3})\) the functional \(I\) is well-defined and continuous in \(BV(\Omega)\). Moreover, by \((h_{IV6}'')\), Proposition IV.1.3 implies the existence of a constant \(\delta > 0\) such that

\[
\mathcal{L}(v) - \int_\Omega \zeta |v| \, dx \geq \delta \|v\|_{BV}
\]

for all \(v \in BV(\Omega)\). Fix \(\varepsilon \in ]0,\delta[\). By \((h_{IV3})\) and \((h_{IV6}'')\), there is \(\gamma \in L^1(\Omega)\) such that

\[
F(x,s) \leq (\zeta(x) + \varepsilon) |s| + \gamma(x)
\]
for a.e. \( x \in \Omega \) and every \( s \in \mathbb{R} \). Hence we have
\[
I(v) \geq L(v) - \int_{\Omega} (\zeta + \varepsilon) |v| \, dx - \|\gamma\|_{L^1} \\
\geq \delta\|v\|_{BV} - \varepsilon\|v\|_{L^1} - \|\gamma\|_{L^1} \\
\geq (\delta - \varepsilon)\|v\|_{BV} - \|\gamma\|_{L^1}
\]
for all \( v \in BV(\Omega) \). Therefore the functional \( I \) is bounded from below and coercive in \( BV(\Omega) \). The usual lower semicontinuity argument (see, e.g., the proof of Theorem IV.3.2) shows that \( I \) has a global minimum. Since any minimizer \( u \) of \( I \) satisfies (IV.2) for every \( v \in BV(\Omega) \), we conclude that problem (IV.1) has at least one solution.

\[\square\]

**Remark IV.3.2** Condition \((h_{IV5})\) is implied by the counterpart of \((h_{IV6''})\), i.e., \((h_{IV5''})\) there exists \( \zeta \in L^1(\Omega) \), with \( \zeta(x) \geq 0 \) for a.e. \( x \in \Omega \) and \( \zeta(x) > 0 \) on a set of positive measure, such that
\[
\lim\inf_{s \to \pm\infty} \frac{F(x,s)}{|s|} \geq \zeta(x)
\]
uniformly for a.e. \( x \in \Omega \).

Accordingly a version of Theorem IV.3.1, where \((h_{IV5})\) is substituted with \((h_{IV5''})\), holds.

### IV.4 Multiplicity results

In this section we discuss the existence of multiple solutions. In particular we will show that under \((h_{IV4})\) the multiplicity of solutions can be proved, whenever the function \( s \mapsto \int_{\Omega} F(x,s) \, dx \) exhibits an oscillatory behaviour at infinity. For other multiplicity results, involving conditions on \( f \), we refer to [90, Section 3].

We start with a simple result where a solution is found by local minimization: this is a first step towards the proof of multiple solutions.

**Proposition IV.4.1.** Assume \((h_{IV0})\), \((h_{IV1})\), \((h_{IV2})\), \((h_{IV3})\), \((h_{IV4})\) and \((h_{IV7})\) there exist \( a^-, a^+ \in \mathbb{R} \), with \( a^- < a^+ \), such that
\[
\max_{s \in [a^-, a^+]} \int_{\Omega} F(x,s) \, dx > |\Omega| + \max \left\{ \int_{\Omega} F(x,a^-) \, dx, \int_{\Omega} F(x,a^+) \, dx \right\}.
\]
Then problem (IV.1) has at least one solution \( u \) such that
\[
-\max_{s \in [a^-, a^+]} \int_{\Omega} F(x,s) \, dx \leq I(u) \leq |\Omega| - \max_{s \in [a^-, a^+]} \int_{\Omega} F(x,s) \, dx
\]
and
\[\mathcal{P}(u) \in ]a^-, a^+[\]
\( \mathcal{P} \) being the projection defined in Proposition IV.1.1.
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Proof. From \((h_{IV}7)\) it follows that there exists \(b \in ]a^-, a^+[\) such that

\[
\int_{\Omega} F(x, b) \, dx = \max_{s \in [a^-, a^+]} \int_{\Omega} F(x, s) \, dx
\]

and

\[
\mathcal{I}(b) = |\Omega| - \int_{\Omega} F(x, b) \, dx < -\max \left\{ \int_{\Omega} F(x, a^-) \, dx, \int_{\Omega} F(x, a^+) \, dx \right\}.
\]

Define the set

\[
\mathcal{A} = \{ v \in BV(\Omega) : \mathcal{P}(v) \in ]a^-, a^+[\}
\]

which is open in \(BV(\Omega)\). By Proposition IV.1.2 we get

\[
\mathcal{I}(b) \geq \inf_{v \in \mathcal{A}} \mathcal{I}(v) \geq \min_{r \in [a^-, a^+]} \left( -\int_{\Omega} F(x, r) \, dx \right) = -\int_{\Omega} F(x, b) \, dx.
\]

Further, we have

\[
\inf_{w \in \mathcal{W}} \mathcal{I}(w + a^-) \geq -\int_{\Omega} F(x, a^-) \, dx > \mathcal{I}(b) \geq \inf_{v \in \mathcal{A}} \mathcal{I}(v) \quad \text{(IV.19)}
\]

and

\[
\inf_{w \in \mathcal{W}} \mathcal{I}(w + a^+) \geq -\int_{\Omega} F(x, a^+) \, dx > \mathcal{I}(b) \geq \inf_{v \in \mathcal{A}} \mathcal{I}(v) \quad \text{(IV.20)}
\]

Let \((v_k)_k\) be a sequence in \(\mathcal{A}\) such that

\[
\lim_{k \to +\infty} \mathcal{I}(v_k) = \inf_{v \in \mathcal{A}} \mathcal{I}(v).
\]

We write, for each \(k\), \(v_k = w_k + r_k\), with \(w_k \in \mathcal{W}\) and \(r_k \in ]a^-, a^+[\). Applying again Proposition IV.1.2 we get, for some \(\eta > 0\) and all large \(k\),

\[
\eta \int_{\Omega} |Dw_k| - \int_{\Omega} F(x, r_k) \, dx \leq \mathcal{I}(v_k) \leq \mathcal{I}(b) + 1
\]

and hence

\[
\sup_k \int_{\Omega} |Dw_k| < +\infty.
\]

Remark III.1.2 and Proposition I.2.1 finally yield

\[
\sup_k \|w_k\|_{BV} < +\infty.
\]

Hence there exist subsequences of \((w_k)_k\) and \((r_k)_k\), that we still denote by \((w_k)_k\) and \((r_k)_k\) respectively, \(w \in \mathcal{W}\) and \(r \in ]a^-, a^+[\) such that \(\lim_{k \to +\infty} w_k = w\) in \(L^q(\Omega)\), with \(q = \frac{p}{p-1}\), and a.e. in \(\Omega\), and \(\lim_{k \to +\infty} r_k = r\). Thus, setting \(u = w + r\), we have

\[
\lim_{k \to +\infty} \int_{\Omega} F(x, v_k) \, dx = \int_{\Omega} F(x, u) \, dx
\]
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and

$$\liminf_{k \to +\infty} J(v_k) \geq J(u).$$

Therefore we conclude that

$$\mathcal{I}(w + r) = \mathcal{I}(u) \leq \liminf_{k \to +\infty} \mathcal{I}(v_k) = \inf_{v \in A} \mathcal{I}(v).$$

From (IV.19) and (IV.20) we infer that \( r \in ]a^-, a^+[ \) and, hence, \( u \in A \) is a local minimizer of \( \mathcal{I} \), with

$$\int_{\Omega} F(x, b) \, dx \leq \mathcal{I}(u) \leq |\Omega| - \int_{\Omega} F(x, b) \, dx.$$

Finally, we can easily prove, using the convexity of \( J \) and the differentiability of the potential operator \( F \) in \( BV(\Omega) \), that \( u \) satisfies (IV.2), for all \( v \in BV(\Omega) \), and hence it is a solution of problem (IV.1).

Remark IV.4.1 Condition \((h_{IV7})\) is clearly implied by

$$\liminf_{s \to \pm \infty} \int_{\Omega} F(x, s) \, dx = -\infty.$$

We are in position of proving our first infinite multiplicity result, assuming the existence of a suitable gap between \( \liminf_{s \to +\infty} \int_{\Omega} F(x, s) \, dx \) and \( \limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx \).

Proposition IV.4.2. Assume \((h_{IV0}), (h_{IV1}), (h_{IV2}), (h_{IV3}), (h_{IV4})\) and

\((h_{IV8})\) \( \limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx > |\Omega| + \liminf_{s \to +\infty} \int_{\Omega} F(x, s) \, dx. \)

Then problem (IV.1) has a sequence \((u_n)\) of solutions such that

$$\limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx \leq \liminf_{n \to +\infty} \mathcal{I}(u_n) \leq |\Omega| - \limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx$$

and

$$\lim_{n \to +\infty} \mathcal{P}(u_n) = +\infty.$$

Proof. Condition \((h_{IV8})\) implies the existence of sequences \((a^-_n)\), \((a^+_n)\) and \((b_n)\), with

$$\lim_{n \to +\infty} a^-_n = \lim_{n \to +\infty} a^+_n = \lim_{n \to +\infty} b_n = +\infty$$

and

$$\lim_{n \to +\infty} \int_{\Omega} F(x, b_n) \, dx = \limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx,$$
such that, for each \( n \), \( a_n^- < b_n < a_n^+ \),

\[
\int_\Omega F(x, b_n) \, dx = \max_{s \in [a_n^-, a_n^+]} \int_\Omega F(x, s) \, dx > |\Omega| + \int_\Omega F(x, a_n^-) \, dx
\]

and

\[
\int_\Omega F(x, b_n) \, dx = \max_{s \in [a_n^-, a_n^+]} \int_\Omega F(x, s) \, dx > |\Omega| + \int_\Omega F(x, a_n^+) \, dx.
\]

Hence Proposition IV.4.1 yields, for each \( n \), the existence of a solution \( u_n \) of problem (IV.1), satisfying \( \mathcal{P}(u_n) \in ]a_n^-, a_n^+ [ \) and

\[
- \int_\Omega F(x, b_n) \, dx \leq I(u_n) \leq |\Omega| - \int_\Omega F(x, b_n) \, dx.
\]

Thus the conclusion follows.

\[\square\]

**Remark IV.4.2** A result similar to Proposition IV.4.2 holds, where condition \((h_{IV}8)\) is replaced by

\[
\limsup_{s \to -\infty} \int_\Omega F(x, s) \, dx > |\Omega| + \liminf_{s \to -\infty} \int_\Omega F(x, s) \, dx.
\]

In this case problem (IV.1) has a sequence \((u_n)\) of solutions such that

\[
- \limsup_{s \to -\infty} \int_\Omega F(x, s) \, dx \leq \lim_{n \to +\infty} I(u_n) \leq |\Omega| - \limsup_{s \to -\infty} \int_\Omega F(x, s) \, dx
\]

and

\[
\lim_{n \to +\infty} \mathcal{P}(u_n) = -\infty.
\]

**Remark IV.4.3** The following variant of Proposition IV.4.2 has been already proved in [90, Proposition 3.6], by using a lower and upper solution argument: under \((h_{IV}0), (h_{IV}3)\) and

\[
\liminf_{s \to +\infty} f(x, s) < 0 < \limsup_{s \to +\infty} f(x, s) \quad \text{uniformly a.e. in } \Omega,
\]

problem (IV.1), with \( h = 0 = \kappa \), has a sequence \((u_n)\) of solutions such that

\[
\lim_{n \to +\infty} \text{ess inf } u_n = +\infty.
\]

It is worth noting that here condition \((h_{IV}4)\) has not been assumed.

From Proposition IV.4.2 we immediately deduce the following statement.

**Theorem IV.4.3.** Assume \((h_{IV}0), (h_{IV}1), (h_{IV}2), (h_{IV}3), (h_{IV}4)\) and

\[
(h_{IV9}) \limsup_{s \to +\infty} \int_\Omega F(x, s) \, dx = +\infty > \liminf_{s \to +\infty} \int_\Omega F(x, s) \, dx.
\]
Then problem (IV.1) has a sequence \((u_n)_n\) of solutions such that
\[
\lim_{n \to +\infty} \mathcal{I}(u_n) = -\infty \quad \text{and} \quad \lim_{n \to +\infty} \mathcal{P}(u_n) = +\infty.
\]

**Remark IV.4.4** From Remark IV.4.2 we easily deduce that a result similar to Theorem IV.4.3 holds, where condition \((h_{IV9})\) is replaced by
\[
\lim sup_{s \to -\infty} \int_{\Omega} F(x, s) \, dx = +\infty > \lim inf_{s \to -\infty} \int_{\Omega} F(x, s) \, dx.
\]

In this case problem (IV.1) has a sequence \((u_n)_n\) of solutions such that
\[
\lim_{n \to +\infty} \mathcal{I}(u_n) = -\infty \quad \text{and} \quad \lim_{n \to +\infty} \mathcal{P}(u_n) = -\infty.
\]

Our next result is a local multiplicity result; its proof makes use of a min-max argument combined with a localization trick inspired by [66].

**Proposition IV.4.4.** Assume \((h_{IV0}), (h_{IV1}), (h_{IV2}), (h_{IV3}), (h_{IV4})\) and \((h_{IV10})\) there exist \(a^-, a^+, b^-, b^+\), with \(b^- < a^- < a^+ < b^+\), such that
\[
|\Omega| + \max \left\{ \int_{\Omega} F(x, b^-) \, dx, \int_{\Omega} F(x, b^+) \, dx \right\} < \int_{\Omega} F(x, b) \, dx
\]
\[
< \min \left\{ \int_{\Omega} F(x, a^-) \, dx, \int_{\Omega} F(x, a^+) \, dx \right\} - |\Omega|,
\]
where \(b \in [a^-, a^+]\) is such that
\[
\int_{\Omega} F(x, b) \, dx = \min_{s \in [a^-, a^+]} \int_{\Omega} F(x, s) \, dx.
\]

Then problem (IV.1) has at least three solutions \(u^{(1)}, u^{(2)}, u^{(3)}\) such that
\[
- \int_{\Omega} F(x, b) \, dx \leq \mathcal{I}(u^{(1)}) \leq |\Omega| - \int_{\Omega} F(x, b) \, dx,
\]
\[
\mathcal{I}(u^{(2)}) \leq |\Omega| - \int_{\Omega} F(x, a^-) \, dx, \quad \mathcal{I}(u^{(3)}) \leq |\Omega| - \int_{\Omega} F(x, a^+) \, dx,
\]
\[
\mathcal{P}(u^{(1)}) \in ]b^-, b^+[, \quad \mathcal{P}(u^{(2)}) \in ]b^-, b[, \quad \mathcal{P}(u^{(3)}) \in ]b, b^+[.
\]

**Proof.** The proof is divided into two parts.

**Part 1. Existence of the first solution** \(u^{(1)}\). By assumption \((h_{IV10})\) we have
\[
\int_{\Omega} F(x, b) \, dx + |\Omega| < \min \left\{ \int_{\Omega} F(x, a^-) \, dx, \int_{\Omega} F(x, a^+) \, dx \right\}.
\]
Step 1. Mountain pass geometry. As in Step 1 of the proof of Theorem IV.3.1, we set
\[ S = \{ v \in BV(\Omega) : \mathcal{P}(v) = b \} = \{ w + b : w \in \mathcal{W} \} , \]
where \( \mathcal{P} \) and \( \mathcal{W} \) are defined in Proposition IV.1.1 and by (IV.4), respectively,
\[ \Gamma = \{ \gamma \in C^0([0,1],BV(\Omega)) : \gamma(0) = a^-, \gamma(1) = a^+ \} \]
and
\[ c_I = \inf_{\gamma \in \Gamma} \max_{\xi \in [0,1]} I(\gamma(\xi)) . \]
Since, by Proposition IV.1.2,
\[ \inf_{v \in S} I(v) > -\int_{\Omega} F(x,b) \, dx \]
\[ \geq \max \left\{ -\int_{\Omega} F(x,a^-) \, dx, -\int_{\Omega} F(x,a^+) \, dx \right\} - |\Omega| \]
and \( \gamma([0,1]) \cap S \neq \emptyset \) for any \( \gamma \in \Gamma \), Lemma I.3.1 yields the existence of sequences \( (\gamma_k)_k \) and \( (\varepsilon_k)_k \), with \( \gamma_k \in \Gamma \), \( v_k \in BV(\Omega) \) and \( \varepsilon_k \in \mathbb{R} \), satisfying \( \lim_{k \to +\infty} \varepsilon_k = 0 \), condition (I.17), that is,
\[ c_I - \frac{1}{k} \leq I(v_k) \leq \max_{\xi \in [0,1]} I(\gamma_k(\xi)) \leq c_I + \frac{1}{k} , \]
for each \( k \), and condition (I.19), that is,
\[ J(v) - J(v_k) \geq \int_{\Omega} f(x,v_k)(v - v_k) \, dx + \varepsilon_k \| v - v_k \|_{BV} , \]
for each \( k \) and all \( v \in BV(\Omega) \). Notice that, by assumption \( (h_{IV.10}) \), we have
\[ \int_{\Omega} F(x,b) \, dx > |\Omega| + \max \left\{ \int_{\Omega} F(x,b^-) \, dx, \int_{\Omega} F(x,b^+) \, dx \right\} . \]
Step 2. The sequence \( (v_k)_k \) is bounded in \( BV(\Omega) \). Let us set
\[ B = \{ v \in BV(\Omega) : \mathcal{P}(v) \in [b^-, b^+] \} . \]
Observe that, for each \( k \), \( \gamma_k([0,1]) \cap B \neq \emptyset \). Taking as \( \gamma \) the segment joining \( a^- \) with \( a^+ \), we see that
\[ c_I \leq |\Omega| - \int_{\Omega} F(x,b) \, dx . \]
By \( (h_{IV.10}) \) and Proposition IV.1.2 we deduce that
\[ \inf_{w \in \mathcal{W}} \mathcal{I}(w + b^-) \geq -\int_{\Omega} F(x,b^-) \, dx > |\Omega| - \int_{\Omega} F(x,b) \, dx \geq c_I . \]
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and

$$\inf_{w \in W} \mathcal{I}(w + b^+) \geq - \int_{\Omega} F(x, b^+) \, dx > |\Omega| - \int_{\Omega} F(x, b) \, dx \geq c_\mathcal{I}.$$ 

Therefore, as $$\lim_{k \to +\infty} \max_{\xi \in [0,1]} \mathcal{I}(\gamma_k(\xi)) = c_\mathcal{I},$$ we obtain that, for large $$k$$, $$\gamma_k([0,1]) \subset \mathcal{B}$$. Set, for all $$k$$, $$r_k = \mathcal{P}(v_k)$$ and $$w_k = v_k - r_k$$, and recall (I.18), that is,

$$\min_{\xi \in [0,1]} \|v_k - \gamma_k(\xi)\|_{BV} \leq \frac{1}{k}.$$ 

By the continuity of the projection $$\mathcal{P}$$, guaranteed by Proposition IV.1.1, we infer that there exists a decreasing sequence $$(\eta_k)_k$$, with $$\lim_{k \to +\infty} \eta_k = 0$$, such that, for each $$k$$, $$r_k \in [b^- - \eta_k, b^+ + \eta_k]$$. Then Proposition IV.1.2 yields

$$\eta \int_{\Omega} |Du_k| \leq \mathcal{I}(v_k) + \int_{\Omega} F(x, r_k) \, dx \leq c_\mathcal{I} + 1 + \max_{s \in [b^- - 1, b^+ + 1]} \int_{\Omega} F(x, s) \, dx$$

for all large $$k$$. Thus we can conclude, by Remark III.1.2, that $$\sup_k \|u_k\|_{BV} < +\infty$$ and, hence, $$\sup_k \|v_k\|_{BV} < +\infty$$ as well. Arguing as in Step 3 in the proof of Theorem IV.3.1, we prove the existence of a solution $$u^{(1)}$$ of problem (IV.1) such that

$$- \min_{s \in [a^-, a^+]} \int_{\Omega} F(x, s) \, dx \leq \mathcal{I}(u^{(1)}) = c_\mathcal{I} \leq |\Omega| - \min_{s \in [a^-, a^+]} \int_{\Omega} F(x, s) \, dx.$$

Thanks to the continuity of $$\mathcal{P}$$ we have $$\mathcal{P}(u^{(1)}) \in [b^-, b^+]$$. By assumption $$(h_{IV.10})$$ and Proposition IV.1.2 we actually see that $$\mathcal{P}(u^{(1)}) \in [b^-, b^+]$$. 

Part 2. Existence of two further solutions $$u^{(2)}$$ and $$u^{(3)}$$. As we have

$$|\Omega| + \max \left\{ \int_{\Omega} F(x, b^-) \, dx, \int_{\Omega} F(x, b^+) \, dx \right\} < \int_{\Omega} F(x, a^-) \, dx \leq \max_{s \in [b^-, b^+]} \int_{\Omega} F(x, s) \, dx,$$

Proposition IV.4.1 yields the existence of a solution $$u^{(2)}$$ of problem (IV.1), satisfying

$$\mathcal{I}(u^{(2)}) \leq |\Omega| - \max_{s \in [b^-, b^+]} \int_{\Omega} F(x, s) \, dx \leq |\Omega| - \int_{\Omega} F(x, a^-) \, dx$$

and $$\mathcal{P}(u^{(2)}) \in [b^-, b^+]$$. Observe that $$u^{(2)} \neq u^{(1)}$$ because $$\mathcal{I}(u^{(1)}) > |\Omega| - \int_{\Omega} F(x, a^-) \, dx$$. Similarly we prove the existence of a solution $$u^{(3)}$$ of problem (IV.1), satisfying

$$\mathcal{I}(u^{(3)}) \leq |\Omega| - \max_{s \in [b^-, b^+]} \int_{\Omega} F(x, s) \, dx \leq |\Omega| - \int_{\Omega} F(x, a^+) \, dx$$

and $$\mathcal{P}(u^{(3)}) \in [b, b^+]$$, which is different from both $$u^{(1)}$$ and $$u^{(2)}$$. 

From Proposition IV.4.4 we easily derive the following statement.
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**Theorem IV.4.5.** Assume \( (h_{IV0}), (h_{IV1}), (h_{IV2}), (h_{IV3}), (h_{IV4}) \) and

\[
(h_{IV11}) \limsup_{s \to +\infty} \int_{\Omega} F(x, s) \, dx > -\infty = \liminf_{s \to -\infty} \int_{\Omega} F(x, s) \, dx.
\]

Then problem (IV.1) has two sequences \((u_n^{(1)})_n\) and \((u_n^{(2)})_n\) of solutions such that

\[
\lim_{n \to +\infty} I(u_n^{(1)}) = +\infty, \quad \limsup_{n \to +\infty} I(u_n^{(2)}) < +\infty
\]

and

\[
\lim_{n \to +\infty} P(u_n^{(2)}) = +\infty.
\]

**Proof.** By assumption \((h_{IV11})\) we can find sequences \((b_n^-)_n\), with \(\lim_{n \to +\infty} b_n^- = -\infty\), \((a_n^-)_n\), with \(\lim_{n \to +\infty} a_n^- = +\infty\), \((a_n^+_n)_n\) and \((b_n^+_n)_n\), such that \(b_n^- < a_n^- < a_n^+_n < b_n^+_n\) and

\[
|\Omega| + \max \left\{ \int_{\Omega} F(x, b_n^-) \, dx, \int_{\Omega} F(x, b_n^+) \, dx \right\} < \min_{s \in [a_n^-, a_n^+]} \int_{\Omega} F(x, s) \, dx
\]

\[
< \min \left\{ \int_{\Omega} F(x, a_n^-) \, dx, \int_{\Omega} F(x, a_n^+) \, dx \right\} - |\Omega|
\]

for every \(n\). Hence, Proposition IV.4.4 yields the conclusion. \(\square\)

**Remark IV.4.5** A result similar to Theorem IV.4.5 holds, where condition \((h_{IV11})\) is replaced by

\[
\limsup_{s \to -\infty} \int_{\Omega} F(x, s) \, dx > -\infty = \liminf_{s \to -\infty} \int_{\Omega} F(x, s) \, dx.
\]

In this case problem (IV.1) has two sequences \((u_n^{(1)})_n\) and \((u_n^{(2)})_n\) of solutions such that

\[
\lim_{n \to +\infty} I(u_n^{(1)}) = +\infty, \quad \limsup_{n \to +\infty} I(u_n^{(2)}) < +\infty
\]

and

\[
\lim_{n \to +\infty} P(u_n^{(2)}) = -\infty.
\]

Finally notice that, if we assume

\[
\limsup_{s \to -\infty} \int_{\Omega} F(x, s) \, dx > -\infty = \liminf_{s \to -\infty} \int_{\Omega} F(x, s) \, dx,
\]

then problem (IV.1) has three sequences \((u_n^{(1)})_n\), \((u_n^{(2)})_n\) and \((u_n^{(3)})_n\) of solutions such that

\[
\lim_{n \to +\infty} I(u_n^{(1)}) = +\infty, \quad \limsup_{n \to +\infty} I(u_n^{(2)}) < +\infty, \quad \limsup_{n \to +\infty} I(u_n^{(3)}) < +\infty
\]

and moreover

\[
\lim_{n \to +\infty} P(u_n^{(2)}) = +\infty \quad \text{and} \quad \lim_{n \to +\infty} P(u_n^{(3)}) = -\infty.
\]
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Chapter V

Additional conclusions in the 1-dimensional case

As we have already seen in the previous chapters, the 1-dimensional case sometimes presents a different behaviour. This is mainly due to the fact that, in dimension $N = 1$, the space $BV(\Omega)$ is continuously embedded in $L^\infty(\Omega)$. Therefore we consider separately the 1-dimensional counterpart of

$$
\begin{align*}
-\text{div}\left(\nabla u/\sqrt{1 + |\nabla u|^2}\right) &= f(x,u) + h(x) \quad \text{in } \Omega, \\
-\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} &= \kappa(x) \quad \text{on } \partial\Omega,
\end{align*}
$$

(V.1)

that has been studied in Chapter II and in Chapter IV.

Indeed, assuming $N = 1$, we are able to give some further results for the construction of lower and upper solutions of problem (V.1). Moreover, we also produce an existence result in case of non-well-ordered lower and upper solutions. On the other hand, as pointed out in Chapter III, the asymptotic behaviour of the curve $C$ in the case $N = 1$ differs from the case $N \geq 2$. Thanks to this property, we are in condition of proving results, similar to those of Chapter IV, where the usual two-sided conditions are replaced by one-sided ones.

Let us set $\Omega = [0, T[$, with $T > 0$, and from now on assume:

(h$_V$1) $h \in L^1(0, T)$ and $\kappa \in L^\infty(\partial[0, T[)$;

(h$_V$2) there exists a constant $\rho > 0$ such that

$$
\left| \int_B h \, dt - \int_{\partial[0, T[} \kappa \chi_B \, d\mathcal{H}_0 \right| \leq (1 - \rho) \int_{[0, T[} |D\chi_B|
$$

for every Caccioppoli set $B \subseteq [0, T[$;

(h$_V$3) $f : [0, T[ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for a.e. $t \in [0, T[$ and $f(\cdot, s) : [0, T[ \to \mathbb{R}$ is measurable for every $s \in \mathbb{R}$. 

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V. ADDITIONAL CONCLUSIONS IN THE 1-DIMENSIONAL CASE

For the sake of simplicity let us define \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) as

\[
\varphi(y) = \frac{y}{\sqrt{1 + y^2}} \tag{V.2}
\]

and set \( \psi : ]-1, 1[ \rightarrow \mathbb{R} \) as

\[
\psi(y) = \varphi^{-1}(y) = \frac{y}{\sqrt{1 - y^2}} \tag{V.3}
\]

We deal with problem

\[
\left\{
\begin{aligned}
-\frac{u'/\sqrt{1 + u'^2}}{\sqrt{1 + u'^2}} &= f(t, u) + h(t) \quad \text{in } ]0, T[, \\
\varphi(u'(0)) = \kappa(0), \quad -\varphi(u'(T)) = \kappa(T)
\end{aligned}
\right. \tag{V.4}
\]

that can be equivalently reformulated as

\[
\left\{
\begin{aligned}
-\frac{u'/\sqrt{1 + u'^2}}{\sqrt{1 + u'^2}} &= f(t, u) + h(t) \quad \text{in } ]0, T[, \\
u'(0) = \psi(\kappa(0)), \quad u'(T) = -\psi(\kappa(T)).
\end{aligned}
\right.
\]

As done before, we set \( F(t, s) = \int_0^s f(t, \xi) \, d\xi \) and we consider the functional \( \mathcal{I} : BV(0, T) \rightarrow \mathbb{R} \) defined by

\[
\mathcal{I}(v) = \int_{[0, T]} \sqrt{1 + |Dv|^2} - \int_0^T hv \, dt + \int_{\partial[0,T]} \kappa v \, dH_0 - \int_0^T F(t, v) \, dt.
\]

Moreover, we recall the definition of the functional \( \mathcal{J} : BV(0, T) \rightarrow \mathbb{R} \), i.e.,

\[
\mathcal{J}(v) = \int_{[0, T]} \sqrt{1 + |Dv|^2} - \int_0^T hv \, dt + \int_{\partial[0,T]} \kappa v \, dH_0.
\]

In accordance with the discussion performed in Chapter I, the following notion of solution is adopted.

**Definition of solution.** We say that a function \( u \in BV(0, T) \) is a solution of problem (V.4) if \( u \) satisfies

\[
\mathcal{J}(v) - \mathcal{J}(u) \geq \int_0^T f(t, u)(v - u) \, dt
\]

for every \( v \in BV(0, T) \).

Recall that a function \( u \) is a solution of (V.4) if and only if \( u \) is a minimizer in \( BV(0, T) \) of the functional \( K_u : BV(0, T) \rightarrow \mathbb{R} \) defined by \( K_u(v) = \mathcal{J}(v) - \int_0^T f(t, u)v \, dt \).
V.1 Technical results

Here we prove some technical results, specific of the case \( N = 1 \), that can be compared with those of Section IV.1. These ones will allow us to get sharper results in the sequel.

**Proposition V.1.1.** Assume \( N = 1 \) and let \( \Omega = \]0, T[ \). Then, for every \( u \in BV(0, T) \), we have

\[
\text{ess sup}_{\]0, T[} u - \text{ess inf}_{\]0, T[} u \leq \int_{\]0, T[} |Du| .
\] (V.5)

**Proof.** Let \( u \in BV(0, T) \). By Proposition I.1.7 there exists a sequence \((u_n)_n \) in \( W^{1,1}(0, T) \) such that

\[
\lim_{n \to +\infty} u_n = u \quad \text{in } L^1(0, T) \quad \text{and a.e. in } \]0, T[
\]

and

\[
\lim_{n \to +\infty} \int_0^T |u'_n| \, dt = \int_{\]0, T[} |Du| .
\]

By [23, Theorem 2.14], for any \( s, t \in \]0, T[ \) we have

\[
u_n(s) - u_n(t) = \int_t^s u'_n \, dt \leq \int_0^T |u'_n| \, dt
\]

and hence the conclusion follows passing to the limit for \( n \to +\infty \).

**Remark V.1.1** It is clear that equality in (V.5) is attained whenever \( u \) is monotone.

The last oscillation estimate, which relies on the continuous embedding of the space \( BV(0, T) \) in \( L^\infty(0, T) \), allows to prove a one-sided counterpart of the coercivity property stated in Proposition IV.1.2.

**Lemma V.1.2** (A coercivity property in dimension \( N = 1 \)). Suppose that \( N = 1 \) and let \( \Omega = \]0, T[ \). Assume \((hV1), (hV2)\),

\((hV3') \) \( f : \]0, T[ \times \mathbb{R} \to \mathbb{R} \) satisfies the \( L^1 \)-Carathéodory conditions, i.e., function \( f \) is a Carathéodory function such that, for every \( r > 0 \), there exists a function \( \gamma \in L^1(0, T) \) such that

\[
|f(t, s)| \leq \gamma(t)
\]

for a.e. \( t \in \]0, T[ \) and every \( s \in \mathbb{R} \) with \( |s| \leq r \).

and

\((hV4) \) there exists \( g \in L^1(0, T) \), with \( \|g^-\|_{L^1} < \rho \), with \( \rho \) defined in \((hV2)\), such that

\[
f(t, s) \geq g(t) \quad \text{for a.e. } t \in \]0, T[ \) and every \( s \in \mathbb{R} \).
Then there exists \( \eta > 0 \) such that

\[
I(v) \geq \eta \int_{0,T} |Dv| - \int_0^T F(t, \text{ess sup } v) \, dt
\]

for every \( v \in BV(0,T) \).

**Proof.** For any given \( v \in BV(0,T) \), by Corollary I.2.4 and Proposition V.1.1, we have

\[
I(v) = J(v) + \int_0^T \left( \int_{v(t)}^{\text{ess sup } v} f(t,s) \, ds \right) dt - \int_0^T F(t, \text{ess sup } v) \, dt
\]

\[
\geq \rho \int_{0,T} |Dv| - \int_0^T g^-(t)(\text{ess sup } v - v) \, dt - \int_0^T F(t, \text{ess sup } v) \, dt
\]

\[
\geq \rho \int_{0,T} |Dv| - \|g^-\|_{L^1} (\text{ess sup } v - \text{ess inf } v) - \int_0^T F(t, \text{ess sup } v) \, dt
\]

\[
\geq (\rho - \|g^-\|_{L^1}) \int_{0,T} |Dv| - \int_0^T F(t, \text{ess sup } v) \, dt.
\]

Hence the conclusion follows. \( \square \)

**Remark V.1.2** Suppose that \( N = 1 \) and let \( \Omega = [0,T] \). Assume \((h_V1), (h_V2), (h_V3')\) replace \((h_V4)\) with the symmetric assumption

\((h_V4')\) there exists \( g \in L^1(0,T) \), with \( \|g^+\|_{L^1} < \rho \), with \( \rho \) defined in \((h_V2)\), such that \( f(t,s) \leq g(t) \) for a.e. \( t \in [0,T] \) and every \( s \in \mathbb{R} \).

Then a symmetric conclusion of Lemma V.1.2 still holds, i.e., there exists \( \eta > 0 \) such that

\[
I(v) \geq \eta \int_{0,T} |Dv| - \int_0^T F(t, \text{ess inf } v) \, dt
\]

for every \( v \in BV(0,T) \).

Moreover, if \( h = 0 \) and \( \kappa = 0 \), then we can take \( \rho = 1 \) here and in Proposition V.1.2.

**V.2 Construction of BV-lower and BV-upper solutions**

We start producing some more explicit conditions on the function \( f \) which guarantee the existence of a \( BV \)-lower solution, or a \( BV \)-upper solution, assuming suitable conditions of Landesman-Lazer type (see [75] and [77]). In this setting it is convenient to split \( f \) as

\[
f(t,s) = g(t,s) - e(t), \quad \text{(V.6)}
\]

where \( e \in L^1(0,T) \). In the sequel we set \( \bar{e} = \frac{1}{T} \int_0^T e \, dt \) and \( \tilde{e} = e - \bar{e} \).
V. Additional conclusions in the 1-dimensional case

**Proposition V.2.1.** Assume $(hV_1)$, $(hV_2)$ and fix $(\mu, \nu) \in \mathcal{C}$. Then for every $e \in L^\infty(0, T)$, with $\int_0^T e \, dt = 0$ and

$$\text{ess sup } e < \mu, \quad \text{ess inf } e > -\nu,$$

the problem

$$\begin{cases}
-\left(u'/\sqrt{1+u'^2}\right)' = e(t) + h(t) & \text{ in } ]0, T[, \\
\varphi(u'(0)) = \kappa(0), & -\varphi(u'(T)) = \kappa(T)
\end{cases}$$

(V.7)

has a solution $w \in BV(0, T)$ with $\mu \int_0^T w^+ \, dt - \nu \int_0^T w^- \, dt = 0$.

**Proof.** The proof follows the same outline as that of Proposition IV.2.1

**Proposition V.2.2.** Assume $(hV_1)$, $(hV_2)$, $(hV_3')$. Take $e \in L^\infty(0, T)$ and define $g$ by (V.6). Assume further

$(hV_5')$ there exist $c \in \mathbb{R} \cup \{-\infty\}$ and $d \in \mathbb{R} \cup \{+\infty\}$, with $c < d$, such that

$$g(t,s) \geq \bar{e}$$

for a.e. $t \in ]0, T[$ and every $s \in ]c, d[$

and

$(hV_6)$ there exists $(\mu, \nu) \in \mathcal{C}$ and $\vartheta \in ]0, 1[$ such that

$$\text{ess inf } \bar{e} \geq -\vartheta \mu \quad \text{and} \quad \text{ess sup } \bar{e} \leq \vartheta \nu.$$

Finally suppose that

$$\frac{T}{\rho(1-\vartheta)} \leq \frac{d-c}{2},$$

(V.8)

where $\rho$ is defined in $(hV_2)$. Then there exists a $BV$-lower solution $\alpha$ of problem (V.4) such that $c \leq \alpha(t) \leq d$ for a.e. $t \in ]0, T[$.

**Proof.** First of all, we notice that Proposition V.2.1 applies to

$$\begin{cases}
-\left(u'/\sqrt{1+u'^2}\right)' = h(t) - \hat{e}(t) & \text{ in } ]0, T[, \\
\varphi(u'(0)) = \kappa(0), & -\varphi(u'(T)) = \kappa(T)
\end{cases}$$

(V.9)

and yields the existence of at least one solution $w \in BV(0, T)$ of problem (V.9), satisfying

$$\mu \int_0^T w^+ \, dt - \nu \int_0^T w^- \, dt = 0.$$
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We claim that such a solution satisfies
\[ \|w\|_{\infty} \leq \frac{T}{\rho(1 - \vartheta)}. \]  

(V.10)

Indeed \( w \) satisfies
\[ J(v) - J(w) \geq \int_0^T (-\bar{e})(v - w) \, dt \]
for every \( v \in BV(0,T) \) and, in particular taking \( v = 0 \), from Proposition III.1.2, we have
\[ L(w) \leq J(w) \leq J(0) + \int_0^T (\bar{e})_+ \, dt + \nu \int_0^T (\bar{e})_- \, dt \]
\[ \leq T + \vartheta \left( \mu \int_0^T w^+ \, dt + \nu \int_0^T w^- \, dt \right) \]
and hence, using Proposition V.1.1, estimate (V.10) follows. Next we show how to construct a lower solution \( \alpha \), with \( c \leq \alpha \leq d \) starting from this solution \( w \). In case \( c = -\infty \) or \( d = +\infty \), we can find a constant \( b \) such that, setting \( \alpha = w + b \), we have \( c \leq \alpha \leq d \). Otherwise, we define \( \alpha = \frac{1}{2}(c + d) + w \). We get, by (V.8), \( c \leq \alpha \leq d \) and, by (hV5),
\[ J(\alpha + z) - J(\alpha) = J(w + z) - J(w) \geq -\int_0^T \bar{e} \, z \, dt \geq \int_0^T g(t, \alpha) \, z \, dt - \int_0^T e \, z \, dt \]
for every \( z \in BV(0,T) \) with \( z \leq 0 \). Hence, we have that \( \alpha \) is a BV-lower solution of problem (V.4).

Remark V.2.1 We note that, in case \( c = -\infty \) or \( d = +\infty \), relation (V.8) is trivially satisfied.

Proposition V.2.3. Assume (hV1), (hV2) and (hV3'). Take \( e \in L^\infty(0,T) \) and define \( g \) by (V.6). Assume further
\( (hV5') \) there exist \( c \in \mathbb{R} \cup \{-\infty\} \) and \( d \in \mathbb{R} \cup \{+\infty\} \), with \( c < d \), such that
\[ g(t, s) \leq \bar{e} \]
for a.e. \( t \in ]0,T[ \) and every \( s \in ]c,d[ \),
\( (hV6) \) and suppose that (V.8) is satisfied. Then there exists a BV-upper solution \( \beta \) of problem (V.4) such that \( c \leq \beta(t) \leq d \) for a.e. \( t \in ]0,T[ \).

We show now that the two-sided bound on \( \bar{e} \) required by (hV6) can be replaced by a one-sided condition. To this aim, we need the following one-sided version of Proposition V.2.1 which is peculiar of the case \( N = 1 \).
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**Proposition V.2.4.** Assume \((h_{V1})\) and \((h_{V2})\). Then, for every \(g \in L^\infty(0, T)\), with \(\int_0^T g \, dt = 0\) and either \(\|g^-\|_{L^1} < \rho\) or \(\|g^+\|_{L^1} < \rho\), with \(\rho\) defined in \((h_{V2})\), problem \((V.7)\) has a solution \(w \in BV(0, T)\) with respectively \(\text{ess sup} \, w = 0\) or \(\text{ess inf} \, w = 0\).

**Proof.** Let us assume \(\|g^-\|_{L^1} < \rho\). The proof in case \(\|g^+\|_{L^1} < \rho\) is similar.

We set
\[
S = \left\{ v \in BV(0, T) : \text{ess sup}_{[0,T]} v = 0 \right\}.
\]

By Lemma V.1.2 and Proposition V.1.1, the functional \(I\) is bounded from below and coercive in \(S\). Arguing as in Proposition IV.2.1, we deduce that \(I\) has a global minimum at some \(w \in S\), which satisfies
\[
J(v) - J(w) \geq \int_0^T g(v - w) \, dt
\]
for every \(v \in S\) and, actually, for every \(v \in BV(0, T)\), as \(I(v + k) = I(v)\) for all \(k \in \mathbb{R}\). Hence \(w\) is a solution of \((V.7)\).

**Proposition V.2.5.** Assume \((h_{V1})\), \((h_{V2})\) and \((h_{V3'})\). Take \(e \in L^\infty(0, T)\) and define \(g\) by \((V.6)\). Assume further \((h_{V5})\),

\((h_{V7})\) there exists \(\vartheta \in ]0, 1[\) such that either \(\|\tilde{e}^+\|_{L^1} \leq \vartheta \rho\) or \(\|\tilde{e}^-\|_{L^1} \leq \vartheta \rho\), with \(\rho\) defined in \((h_{V2})\)

and suppose that \((V.8)\) is satisfied. Then there exists a \(BV\)-lower solution \(\alpha\) of problem \((V.4)\) such that \(c \leq \alpha(t) \leq d\) for a.e. \(t \in [0, T]\).

**Proof.** We shall assume \(\|\tilde{e}^+\|_{L^1} \leq \vartheta \rho\) in \((h_{V7})\). The proof in case \(\|\tilde{e}^-\|_{L^1} \leq \vartheta \rho\) is similar. By Proposition V.2.4 applied to \(g = -\tilde{e}\), we get \(w \in BV(0, T)\) solution of \((V.7)\) and satisfying
\[
\text{ess sup}_{[0,T]} w = 0.
\]

As in Proposition V.2.2, taking \(v = 0\) in the definition of solution, by Proposition I.1.5 and Proposition V.1.1, we get
\[
\mathcal{L}(w) \leq J(w) \leq J(0) + \int_0^T (-\tilde{e}) \, w \, dt
\]
\[
\leq J(0) - \int_0^T \tilde{e}^+ \, w \, dt
\]
\[
\leq T + \vartheta \rho \int_{[0,T]} |Dw|.
\]

By Proposition I.2.3, applying again Proposition V.1.1, we obtain
\[
\|w\|_{\infty} \leq \frac{T}{\rho(1 - \vartheta)}.
\]

The lower solution \(\alpha\) is finally constructed as in Proposition V.2.2. \(\square\)
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Proposition V.2.6. Assume \((h_1), (h_2)\) and \((h_3')\). Take \(e \in L^\infty(0,T)\) and define \(g\) by \((V.6)\). Suppose further that \((h_5')\) and \((h_7)\) hold and that \((V.8)\) is satisfied. Then there exists a \(BV\)-upper solution \(\beta\) of problem \((V.4)\) such that \(c \leq \beta(t) \leq d\) for a.e. \(t \in [0,T]\).

V.3 Non-well-ordered lower and upper solutions

Our next result deals with that case where \(\alpha\) and \(\beta\) may fail to satisfy the ordering condition \(\alpha \leq \beta\), assumed in Theorem II.2.1. We show that this restriction can be removed at the expense of assuming a stronger notion of lower and upper solutions, as well as of placing an additional control on \(f\), with respect to the curve \(C\) introduced in Chapter III.

Theorem V.3.1. Assume
\[(h_1'), h \in L^\infty(0,T)\) and \(\kappa \in L^\infty(\partial_0,T)\), with \(\|\kappa\|_\infty < 1\),
\((h_2)\) and \((h_3)\). Assume further
\[(h_8)\] there exist a \(W^{2,1}(0,T)\)-lower solution \(\alpha = \alpha_1 \lor \cdots \lor \alpha_m\) and a \(W^{2,1}(0,T)\)-upper solution \(\beta = \beta_1 \land \cdots \land \beta_n\) of \((V.4)\) such that, for each \(i = 1,\ldots,m\),
\[
\begin{align*}
-\left(\varphi(\alpha'_i)\right)' &\leq f(t,\alpha_i) + h(t), \\
\varphi(\alpha'_i(0)) &\geq \kappa(0), \\
\varphi(\alpha'_i(T)) &\geq \kappa(T)
\end{align*}
\]
and, for each \(j = 1,\ldots,n\),
\[
\begin{align*}
-\left(\varphi(\beta'_j)\right)' &\geq f(t,\beta_j) + h(t), \\
\varphi(\beta'_j(0)) &\leq \kappa(0), \\
\varphi(\beta'_j(T)) &\leq \kappa(T),
\end{align*}
\]
where \(\varphi\) is defined in \((V.2)\),

and
\[(h_6')\] there exists \((\mu,\nu) \in C\), such that
\[
\text{ess sup}_{[0,T] \times \mathbb{R}} f < \mu \quad \text{and} \quad \text{ess inf}_{[0,T] \times \mathbb{R}} f > -\nu.
\]

Then problem \((V.4)\) has at least one solution \(u \in BV(0,T)\).

Proof. In case \(\alpha \leq \beta\), Theorem II.2.1 guarantees the existence of a solution \(u \in BV(0,T)\) of problem \((V.4)\). Therefore, in the sequel, we may assume that there exists \(t_0 \in [0,T]\) such that
\[
\alpha(t_0) > \beta(t_0).
\]
Step 1. A modified problem. Let us consider the sequence of functions \((\varphi_n)_n\) in \(C^1(\mathbb{R})\) defined as
\[
\varphi_n(s) = \begin{cases} 
\varphi(s) & \text{if } |s| \leq n, \\
\text{linear} & \text{if } |s| > n
\end{cases}
\]
and notice that
\[
\varphi'_n(s) = \begin{cases} 
\varphi'(s) & \text{if } |s| \leq n, \\
\text{constant} & \text{if } |s| > n.
\end{cases}
\]
Let us observe that
\[
\varphi_n(s)s \geq \varphi(s)s \quad (V.11)
\]
for each \(n\) and every \(s \in \mathbb{R}\). Moreover, set \(\Phi_n(s) = \int_0^s \varphi_n(\xi) \, d\xi\). It is easy to verify that \(\Phi_n\) is convex and
\[
\Phi_n(s) \geq \sqrt{1 + s^2} \quad (V.12)
\]
for every \(s \in \mathbb{R}\).

For each \(n\) we consider the problem
\[
\begin{cases} 
-(\varphi_n(u'))' = f(t,u) + h(t) & \text{in } ]0,T[, \\
u'(0) = \psi(\kappa(0)), & u'(T) = -\psi(\kappa(T)).
\end{cases} \quad (V.13)
\]
This equation can be also written as
\[
\begin{cases} 
-u'' = g_n(t,u,u') & \text{in } ]0,T[, \\
\varphi(u'(0)) = \kappa(0), & -\varphi(u'(T)) = \kappa(T).
\end{cases} \quad (V.14)
\]
where, for each \(n\),
\[
g_n(t,s,\xi) = \frac{1}{\varphi_n'(|\xi|)} (f(t,s) + h(t)) \quad (V.15)
\]
for a.e. \(t \in ]0,T[\), every \(s \in \mathbb{R}\) and every \(\xi \in \mathbb{R}\). Note that, thanks to \((h_V1')\) and \((h_V6')\), we have, for each \(n\),
\[
|g_n(t,s,\xi)| \leq (1 + n^2)^{3/2} (\max\{\mu,\nu\} + \|h\|_{\infty}).
\]
Let us verify that, for large \(n\), \(\alpha\) and \(\beta\) are respectively a lower solution and an upper solution of \((V.13)\). As the space of functions \(W^{2,1}(0,T)\) is embedded in \(W^{1,\infty}(0,T)\), we can take \(\bar{n} \in \mathbb{N}\) such that
\[
\bar{n} > \max_{i=1,\ldots,m} \{ \|\alpha_i'\|_{\infty}, \|\beta_j'\|_{\infty}, |\psi(\kappa(0))|, |\psi(\kappa(T))| \}. \quad (V.16)
\]
Clearly, for every \(n > \bar{n}\) and each \(i = 1\ldots m\), we have
\[
\varphi_n(\alpha_i') = \varphi(\alpha_i') \quad (V.17)
\]

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and hence

\[-\alpha_i'' \leq g_n(t, \alpha_i, \alpha_i')\]

for a.e. \(t \in [0, T]\). Moreover, by the monotonicity of \(\psi\) defined in (V.3), we get

\[\alpha_i'(0) \geq \psi(\kappa(0)) \quad \text{and} \quad \alpha_i'(T) \leq \psi(-\kappa(T))\]

for each \(i = 1 \ldots m\) and hence \(\alpha\) is a lower solution for (V.13). Similarly, it follows that \(\beta\) is an upper solution for problem (V.14) for each \(n > \bar{n}\).

By [35, Chapter III], or [38, Theorem 3.1], we conclude that, for each \(n\) large, there exists a solution \(u_n \in W^{2,1}(0, T)\) of problem (V.13) such that

\[u_n(t_n') \leq \alpha(t_n') \quad \text{and} \quad u_n(t_n'') \geq \beta(t_n'')\]  

(V.18)

for some \(t_n', t_n'' \in [0, T]\).

**Step 2. Estimates.** Notice that, by (V.16), for \(n > \bar{n}\) we have

\[\varphi_n(u_n'(0)) = \kappa(0), \quad \varphi_n(u_n'(T)) = -\kappa(T)\]

and hence, for each \(n > \bar{n}\), \(u_n\) satisfies the following weak formulation of (V.13)

\[\int_0^T \varphi_n(u_n')v'\,dt + \int_{\partial[0,T]} \kappa v\,dH_0 = \int_0^T f(t, u_n)v\,dt + \int_0^T hv\,dt\]  

(V.19)

for all \(v \in H^1(0, T)\). Let \(\mathcal{P}\) be the projector operator defined in Proposition IV.1.1. Taking \(v = u_n - \mathcal{P}(u_n)\) as a test function in (V.19), we get

\[\int_0^T \varphi_n(u_n')u_n\,dt = \int_0^T f(t, u_n)(u_n - \mathcal{P}(u_n))\,dt + \int_0^T h u_n\,dt - \int_{\partial[0,T]} \kappa u_n\,dH_0.\]  

(V.20)

Assumption (hV6'), combined with Proposition IV.1.1 and Proposition III.1.2, yields the existence of a constant \(\vartheta \in [0, 1]\), independent of \(n\), such that

\[\int_0^T f(t, u_n)(u_n - \mathcal{P}(u_n))\,dt \leq \vartheta \left(\mu \int_0^T (u_n - \mathcal{P}(u_n))^+\,dt + \nu \int_0^T f(u_n - \mathcal{P}(u_n))^{-}\,dt\right)\]

\[\leq \vartheta \int_0^T |u_n'|\,dt - \vartheta \left(\int_0^T h u_n\,dt - \int_{\partial[0,T]} \kappa u_n\,dH_0\right).\]  

(V.21)

Hence, from (V.20) combined with (V.11) and (V.21), we get

\[\vartheta \int_0^T |u_n'|\,dt + (1 - \vartheta) \left(\int_0^T h u_n\,dt - \int_{\partial[0,T]} \kappa u_n\,dH_0\right)\]

\[\geq \int_0^T \varphi(u_n')u_n'\,dt \geq \int_0^T |u_n'|\,dt - cT\]
where
\[ c = \max_{s \in \mathbb{R}} \left( |s| - s^2 / \sqrt{1 + s^2} \right) > 0. \] (V.22)

Then we have
\[ [\vartheta + (1 - \vartheta)(1 - \rho)] \int_0^T |u_n'| dt \geq \int_0^T |u_n'| dt - cT \]
and we finally obtain
\[ \|u_n'\|_{L^1} \leq \frac{cT}{\rho(1 - \vartheta)}. \] (V.23)

Using (V.18) and (V.23) we obtain, for all \( t \in [0, T] \),
\[ u_n(t) = \int_{t_n}^t u_n'(s) ds + u_n(t_n') \leq \|u_n'\|_{L^1} + \alpha(t_n') \leq \frac{cT}{\rho(1 - \vartheta)} + \|\alpha\|_{\infty} \]
and
\[ u_n(t) = \int_{t_n}^t u_n'(s) ds + u_n(t_n'') \geq -\|u_n'\|_{L^1} + \beta(t_n'') \geq -\frac{cT}{\rho(1 - \vartheta)} - \|\beta\|_{\infty}, \]
which lead to
\[ \|u_n\|_{\infty} \leq \frac{cT}{\rho(1 - \vartheta)} + \max\{\|\alpha\|_{\infty}, \|\beta\|_{\infty}\}. \] (V.24)

This last estimate, combined with (V.23), yields
\[ \|u_n\|_{W^{1,1}} \leq K \] (V.25)
for some constant \( K \).

**Step 3. Existence of a solution.** Fix any \( w \in H^1(0, T) \). Taking \( v = w - u_n \) as a test function in (V.19), using the convexity of \( \Phi_n \), we get
\[ \int_0^T f(t, u_n)(w - u_n) dt \leq \int_0^T \Phi_n(w') dt - \int_0^T \Phi_n(u_n') dt \]
\[ - \int_0^T h(w - u_n) dt + \int_{\partial[0,T]} \kappa(w - u_n) d\mathcal{H}_0. \]

By (V.12) we have
\[ \int_0^T \Phi(w') dt - \int_0^T hw' dt + \int_{\partial[0,T]} \kappa w d\mathcal{H}_0 \geq \mathcal{J}(u_n) + \int_0^T f(t, u_n)(w - u_n) dt \]
for all \( w \in H^1(0, T) \). By (V.25) the sequence \((u_n)_n\) is bounded in \( W^{1,1}(0, T) \), then we can extract a subsequence, we still denote by \((u_n)_n\), converging with respect to the
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$L^1$-topology and a.e. in $[0,T]$ to a function $u \in BV(0,T)$. By $(hV3)$, $(hV6')$, (V.24) and the Lebesgue convergence theorem, we get

$$\lim_{n \to +\infty} \int_0^T f(t,u_n)(w-u_n)\,dt = \int_0^T f(t,u)(w-u)\,dt.$$  

Notice that the sequence $(\Phi_n(w'))_n$ is decreasing and for a.e. $t \in [0,T]$ the sequence $\Phi_n(w'(t))$ converges to $\Phi_n(w'(t))$, then applying again the Lebesgue convergence theorem, we get

$$\lim_{n \to +\infty} \int_0^T \Phi_n(w')\,dt = \int_0^T \sqrt{1+|w'|^2}\,dt.$$  

By Proposition I.2.6, we conclude

$$\mathcal{J}(u) \leq \liminf_{n \to +\infty} \mathcal{J}(u_n) \leq \mathcal{J}(w) - \lim_{n \to +\infty} \int_0^T f(t,u_n)(w-u_n)\,dt$$

$$= \mathcal{J}(w) - \int_0^T f(t,u)(w-u)\,dt,$$

that is,

$$\mathcal{J}(w) - \mathcal{J}(u) \geq \int_0^T f(t,u)(w-u)\,dt$$

for each $w \in H^1(0,T)$.

Fix any $v \in BV(0,T)$. Proposition I.1.7 and an additional regularization yield the existence of a sequence $(w_k)_k \in H^1(0,T)$, bounded in $W^{1,1}(0,T)$, such that $\lim_{k \to +\infty} w_k = v$ in $L^1(0,T)$ and a.e. in $[0,T]$ and $\lim_{k \to +\infty} \mathcal{J}(w_k) = \mathcal{J}(v)$. Arguing as above we see that

$$\lim_{k \to +\infty} \int_0^T f(t,u)(w_k-u)\,dt = \int_0^T f(t,u)(w-u)\,dt$$

and using again Proposition I.2.6 we conclude that

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \int_0^T f(t,u)(v-u)\,dt,$$

thus showing that $u$ is a solution of (V.4). \hfill \Box

We show now that the two-sided bound on $f$ required by $(hV6')$ can be replaced by a one-sided bound, as expressed by $(hV6'')$, or $(hV6'''$).

**Theorem V.3.2.** Assume $(hV1')$, $(hV2)$, $(hV3')$, $(hV8)$ and either

$(hV6'')$ there exists a measurable function $\ell : [0,T] \to \mathbb{R}$ such that $\|\ell^+\|_{L^1} < \rho$ and $f(t,s) \leq \ell(t)$ for a.e. $t \in [0,T]$ and every $s \in \mathbb{R}$, with $\rho$ defined in $(hV2)$, or

or
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\[ (h_{V6''}) \text{ there exists a measurable function } \ell : ]0, T[ \to \mathbb{R} \text{ such that } \|\ell^-\|_{L^1} < \rho \text{ and } f(t, s) \geq \ell(t) \text{ for a.e. } t \in ]0, T[ \text{ and every } s \in \mathbb{R}, \text{ with } \rho \text{ defined in } (h_{V2}). \]

Then problem \((V.4)\) has at least one solution \(u \in BV(0, T)\).

**Proof.** The proof is similar to that of Theorem V.3.1. Let us assume \((h_{V6''})\), in case of \((h_{V6''})\) the proof is the same. For each \(n\) we consider the modified problem \((V.13)\) and we define \(g_n\) as in \((V.15)\). By \((h_{V6''})\) we have

\[ |g_n(t, s, \xi)| \leq (1 + n^2)^{3/2}(\ell^+(t) + \|h\|_{\infty}) \]

for a.e. \(t \in ]0, T[, \) every \(s \in \mathbb{R} \) and every \(\xi \in \mathbb{R} \). By [35, Chapter 3], for each \(n\) there exists a solution \(u_n \in W^{2,1}(0, T)\) of problem \((V.13)\) satisfying \((V.18)\) for some \(t'_n, t''_n \in ]0, T[\). Let us take \(v = u_n - \text{ess inf } u_n\) as a test function in \((V.20)\). Notice that assumption \((h_{V6''})\) yields the existence of a constant \(\vartheta \in ]0, 1[\), independent of \(n\), such that

\[
\int_0^T f(t, u_n)(u_n - \text{ess inf } u_n) dt \leq \int_0^T \ell^+(t)(u_n - \text{ess inf } u_n) dt \\
\leq \|\ell^+\|_{L^1}(\text{ess sup } u_n - \text{ess inf } u_n) \\
\leq \vartheta \rho(\text{ess sup } u_n - \text{ess inf } u_n)
\]

and then Proposition V.1.1 implies

\[
\int_0^T f(t, u_n)(u_n - \text{ess inf } u_n) dt \leq \vartheta \rho \int_0^T |u'_n| dt.
\]

Proceeding as in the proof of Theorem V.3.1, we have

\[
[1 - \rho + \rho \vartheta] \int_0^T |u'_n| dt \geq \int_0^T |u'_n| dt - cT
\]

and hence

\[
\|u'_n\|_{L^1} \leq \frac{cT}{\rho(1 - \vartheta)},
\]

c being defined by \((V.22)\). The conclusion follows as well as in Theorem V.3.1. \(\square\)

**Examples.** We produce here two sample applications of the previous existence theorems; they can be compared with some statements obtained in [16, Sections 3, 4] and in [91, Section 3], for the periodic case but are independent of them.

**Example V.3.1.** Let \(h = 0\) and \(\kappa = 0\). Suppose \(f : [0, T] \times \mathbb{R} \to \mathbb{R} \) is continuous and satisfies either \((h_{V6'})\), or \((h_{V6''})\), or \((h_{V6''})\). Suppose that there exist \(a, b \in \mathbb{R}\) such that

\[ f(t, a) f(t, b) < 0 \quad \text{in } [0, T]. \]

Then problem \((V.4)\) has at least one solution.
V. ADDITIONAL CONCLUSIONS IN THE 1-DIMENSIONAL CASE

This statement follows by combining Proposition II.3.1 with Theorem V.3.1 or Theorem V.3.2. Of course, if \( a \leq b \), then we do not need to assume \((hV6')\), or \((hV6'')\), or \((hV6''')\) as already stated in Proposition II.3.1.

Example V.3.2. Assume \((hV1')\), \((hV2)\) and that \( f : [0, T] \times \mathbb{R} \to \mathbb{R} \) and \( e : [0, T] \to \mathbb{R} \) are continuous. Set \( g = f + e \) and \( \bar{e} = \frac{1}{T} \int_0^T e \, dt \). Suppose that there exists a constant \( c > 0 \) such that

\[
(g(t, s) - \bar{e}) \text{sgn}(s) \leq 0 \quad \text{for all } t \in [0, T] \text{ and every } s \text{ with } |s| \geq c.
\]

Assume finally that either \((hV6)\), or \((hV7)\) hold and \((V.8)\) is satisfied. Then problem \((V.4)\) has at least one solution.

This statement follows by combining Proposition V.2.2 and Proposition V.2.3, or Proposition V.2.5 and Proposition V.2.6, with Theorem II.2.1.

V.4 Remarks about non-existence

We point out that a condition like \((hV6)\) cannot be avoided in order to get the conclusion in Example V.3.2. This is a direct consequence of the non-existence result stated in Proposition IV.2.2. The same conclusions, regarding possible non-existence, can be somehow strengthened in the one-dimensional case as shown in the following result, where we assume \( h = 0 \) and \( \kappa = 0 \) and hence

\[
\mathcal{C} = \{(\mu, \nu) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T}\}.
\]

Proposition V.4.1. Assume \( N = 1 \) and let \( \Omega = [0, T] \). Fix \( \rho, \sigma \in \mathbb{R}_0^+ \) such that \( \frac{1}{\rho} + \frac{1}{\sigma} < T \) and set \( \tau = \frac{\sigma}{\rho + \sigma} T \). Then there exists \( \gamma \in L^1(0, T) \) such that for every \( e \in L^1(0, T) \), with \( \frac{1}{\tau} \int_0^\tau e \, dt = -\rho \) and \( \frac{1}{T-\tau} \int_\tau^T e \, dt = \sigma \) (and hence \( \int_0^T e \, dt = 0 \)), and for every \( g : [0, T] \times \mathbb{R} \to \mathbb{R} \) satisfying the Carathéodory conditions, with \(|g(t, s)| \leq \gamma(t)\) for a.e. \( t \in [0, T] \) and every \( s \in \mathbb{R} \), the problem

\[
\begin{align*}
-\left(\frac{u'}{\sqrt{1 + u'^2}}\right) &= g(t, u) + e(t) \quad \text{in } [0, T], \\
u'(0) &= u'(T) = 0
\end{align*}
\]

has no solution.

Proof. Let \( \varphi \in BV(0, T) \) be given by \( \varphi(t) = -1 \), if \( t \in [0, \tau[ \), and \( \varphi(t) = 1 \), if \( t \in ]\tau, T] \).
Take any $u \in BV(0,T)$ and compute, for $k \in \mathbb{R}_0^+$,

$$J(k\varphi) - \int_0^T (g(t,u) + e)k\varphi \, dt$$

$$\leq T + 2k + k\int_0^\tau e \, dt - k\int_\tau^T e \, dt + k\int_0^T |\gamma| \, dt$$

$$= T + 2k - k\rho \tau - k\sigma(T - \tau) + k\|\gamma\|_{L^1}$$

$$= T + 2k - 2k\frac{\rho}{\rho + \sigma}T + k\|\gamma\|_{L^1}$$

$$= T + 2k \left(1 - \frac{\rho}{\rho + \sigma}T + \frac{1}{2}\|\gamma\|_{L^1}\right).$$

Clearly, the last term tends to $-\infty$ as $k \to +\infty$, provided that $\frac{1}{2}\|\gamma\|_{L^1} \in [0, 2T/3 - 1]$. Therefore $u$ is not a solution of (V.26).

Remark V.4.1 Note that the curve

$$\Sigma_1 = \left\{ (\rho,\sigma) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\rho} + \frac{1}{\sigma} = T \right\}$$

is the limit, as $p \to 1^+$, of the curves

$$\Sigma_p = \left\{ (\rho,\sigma) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : \frac{1}{\rho^p} + \frac{1}{\sigma^p} = \frac{2T}{\pi_p} \right\},$$

where

$$\pi_p = \frac{2\pi(p-1)}{p\sin\left(\frac{\pi}{p}\right)}.$$ 

The curve $\Sigma_p$ is the first non-trivial branch of the Fučík spectrum of the 1-dimensional $p$-Laplace operator with Neumann boundary conditions.

Corollary V.4.2. Assume $N = 1$ and let $\Omega = [0,T[$. Fix $(\mu,\nu) \in \mathcal{C}$, i.e., $\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T}$. Then there exist $\gamma \in L^1(0,T)$ and $e \in C^\infty([0,T])$, with $\int_0^T e \, dt = 0$, $\text{ess inf} \, e < -\mu$ and $\text{ess sup} \, e > \nu$, such that for every $g : [0,T[ \times \mathbb{R} \to \mathbb{R}$ satisfying the Carathéodory conditions, with $|g(t,s)| \leq \gamma(t)$ for a.e. $t \in [0,T[$ and every $s \in \mathbb{R}$, problem (V.26) has no solution.

Proof. Pick $\eta > 1$ such that $\frac{1}{\mu} + \frac{1}{\nu} < \eta T$ and set $\rho = \eta \mu$ and $\sigma = \eta \nu$. Then Proposition V.4.1 easily yields the conclusion.

Remark V.4.2 Note that $\frac{1}{\sqrt{\rho}} + \frac{1}{\sqrt{\nu}} = \sqrt{2T}$ and $\frac{1}{\rho} + \frac{1}{\nu} = T$ if and only if $\mu = \nu = \frac{2}{T}$; in this case we can choose $\eta$ as close to 1 as we want, and hence $\|e\|_\infty$ as close as we want to the eigenvalue $\frac{2}{T}$ of the one-dimensional 1-Laplace operator with Neumann boundary conditions as defined in [25].
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V.5 Existence via one-sided conditions

In dimension $N = 1$, under hypotheses of Ahmad-Lazer-Paul-type, the two-sided condition on $f$, assumed in Theorem IV.3.1 and in Theorem IV.3.2, can be replaced by the one-sided condition $(h_{V4})$, or $(h'_{V4})$. This peculiarity is related to the asymptotic behaviour of the curve $C$ which differs in the case $N = 1$ from the case $N \geq 2$. The proofs are essentially the same, with the care of applying Proposition V.1.2 instead of Proposition IV.1.2. Note that condition $(h_{V4})$ allows $f$ to be unbounded from above and condition $(h'_{V4})$ allows $f$ to be unbounded from below.

**Theorem V.5.1.** Assume $(h_{V1})$, $(h_{V2})$, $(h_{V3'})$, either $(h_{V4})$ or $(h'_{V4})$ and

$$(h_{V9}) \lim_{s \to \pm\infty} \int_0^T F(t, s) \, dt = +\infty.$$ 

Then problem (V.4) has at least one solution.

**Proof.** The proof resembles to that of Theorem IV.3.1. Let us suppose that $(h_{V4})$ holds. A similar proof yields the same conclusion in case of $(h'_{V4})$.

**Step 1. Mountain pass geometry.** We set

$$S = \{v \in BV(0, T) : \text{ess sup}_{[0,T]} v = 0\}.$$ 

Using $(h_{V9})$ we can find $a^-, a^+ \in \mathbb{R}$, with $a^- < 0 < a^+$, such that

$$\min \left\{ \int_0^T F(t, a^-) \, dt, \int_0^T F(t, a^+) \, dt \right\} > T = T + \int_0^T F(t, 0) \, dt.$$ 

By Lemma V.1.2 we have

$$\inf_{v \in S} \mathcal{I}(v) \geq 0$$ 

and hence

$$\mathcal{I}(a^-) = T - \int_0^T F(t, a^-) \, dt < 0 \leq \inf_S \mathcal{I}(v),$$ 

$$\mathcal{I}(a^+) = T - \int_0^T F(t, a^+) \, dt < 0 \leq \inf_S \mathcal{I}(v).$$ 

We define

$$\Gamma = \{\gamma \in C^0([0,1], BV(0, T)) : \gamma(0) = a^-, \gamma(1) = a^+\}$$ 

and observe that $\gamma([0,1]) \cap S \neq \emptyset$, for all $\gamma \in \Gamma$. We set $u_0 = a^-$, $u_1 = a^+$,

$$c_\mathcal{I} = \inf_{\gamma \in \Gamma} \max_{\xi \in [0,1]} \mathcal{I}(\gamma(\xi))$$ 

and note that

$$c_\mathcal{I} > \max\{\mathcal{I}(u_0), \mathcal{I}(u_1)\}.$$ 

Lemma I.3.1 yields the existence of sequences $(v_k)_k$ and $(\varepsilon_k)_k$, with $v_k \in BV(0, T)$ and $\varepsilon_k \in \mathbb{R}$, satisfying $\lim_{k \to +\infty} \varepsilon_k = 0$, (I.23) and (I.19) for each $k$ and all $v \in BV(0, T)$.
Step 2. The sequence \((v_k)_k\) is bounded in \(BV(0,T)\). Fix any \(k\). By Proposition V.1.1 we have

\[
\|\text{ess sup}_{[0,T]} v_k - v_k\|_{BV} = \int_{[0,T]} |Dv_k| + \|\text{ess sup}_{[0,T]} v_k - v_k\|_{L^1}
\leq \int_{[0,T]} |Dv_k| + T (\text{ess sup}_{[0,T]} v_k - \text{ess inf}_{[0,T]} v_k)
\leq (1 + T) \int_{[0,T]} |Dv_k|.
\]

Hence, taking \(v = \text{ess sup}_{[0,T]} v_k\) as a test function in (I.19) we obtain, using (h\(V\)4) too,

\[
\rho \int_{[0,T]} |Dv_k| \leq \mathcal{L}(v_k) \leq \mathcal{J}(v_k)
\leq T - \int_0^T f(t,v_k)(\text{ess sup}_{[0,T]} v_k - v_k) \, dt - \varepsilon_k \|\text{ess sup}_{[0,T]} v_k - v_k\|_{BV}
\leq T + \int_0^T g^- (t)(\text{ess sup}_{[0,T]} v_k - \text{ess inf}_{[0,T]} v_k) \, dt + |\varepsilon_k|(1 + T) \int_{[0,T]} |Dv_k|
\leq T + (\|g^-\|_{L^1} + |\varepsilon_k|(1 + T)) \int_{[0,T]} |Dv_k|.
\]

This yields the existence of a constant \(K > 0\) such that, for all \(k\),

\[
\int_{[0,T]} |Dv_k| \leq K
\]

and

\[
\mathcal{J}(v_k) \leq K. \quad (V.27)
\]

Let us verify that the sequence \((\text{ess sup}_{[0,T]} v_k)_k\) is bounded from below. By contradiction, assume this is false; then we have \(\lim_{k \to +\infty} v_k = -\infty\) uniformly a.e. in \([0,T]\). From (h\(V\)9) it follows that

\[
\lim_{k \to +\infty} \int_0^T F(t,v_k) \, dt = +\infty. \quad (V.28)
\]

Combining (V.27) and (V.28) yields a contradiction with

\[
\lim_{k \to +\infty} \mathcal{I}(v_k) = c\mathcal{L}.
\]

Similarly we verify that the sequence \((\text{ess inf}_{[0,T]} v_k)_k\) is bounded from above. Therefore there exists \(R > 0\) such that

\[
\|v_k\|_{\infty} \leq \text{ess sup}_{[0,T]} v_k - \text{ess inf}_{[0,T]} v_k + R
\]
holds for all \(k\). Proposition V.1.1 then yields \(\sup_k \|v_k\|_{\infty} < +\infty\) and, hence we get \(\sup_k \|v_k\|_{BV} < +\infty\).
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Step 3. Existence of a solution. The existence of a solution $u$ of problem (V.4) is finally proved as in Step 3 in the proof of Theorem IV.3.1.
Appendix A

The evolutionary problem

With the aim of exploring more deeply the stability properties of the capillarity-type equation, we address here the study of the evolutionary problem. As a first step, in the spirit of [80] and [59], we prove an existence result for the evolutionary problem with initial datum in the space of $BV$-functions, in the presence of a couple of well-ordered lower and upper solutions.

Throughout this section we assume that

(h$_A$0) $Ω$ is a bounded domain in $\mathbb{R}^N$ having a Lipschitz boundary $\partial Ω$ and $T > 0$ is fixed.

We set $Q_T = Ω \times ]0,T[$ and $Σ_T = \partial Ω \times ]0,T[$. Suppose moreover

(h$_A$1) $f : Q_T \times \mathbb{R} \to \mathbb{R}$ satisfies the $L^\infty$-Carathéodory conditions, i.e., for a.e. $(x,t) ∈ Q_T$ $f(x,t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for every $s ∈ \mathbb{R}$ $f(\cdot,s) : Q_T \to \mathbb{R}$ is measurable and, for each $r > 0$, there exists $Γ ∈ \mathbb{R}$ such that $|f(x,t,s)| ≤ Γ$ for a.e. $(x,t) ∈ Q_T$ and every $s ∈ [−r,r]$.

We consider the evolutionary problem

\[
\begin{aligned}
& u_t - \text{div}(\nabla u/\sqrt{1+|\nabla u|^2}) = f(x,t,u) \quad \text{in } Q_T, \\
& -\nabla u \cdot n/\sqrt{1+|\nabla u|^2} = 0 \quad \text{on } Σ_T, \\
& u(x,0) = u_0 \quad \text{on } Ω,
\end{aligned}
\]

(A.1)

with $u_0 \in BV(Ω) \cap L^\infty(Ω)$. In the sequel $u_t$ will be also denoted by $\dot{u}$.

A.1 Preliminaries

We start with a suitable generalization of the approximation property stated in Lemma I.1.7 to functions with values in Banach spaces.
Lemma A.1.1. Assume $(hA0)$. Let $w \in L^2(0,T;BV(\Omega)) \cap L^p(Q_T)$, with $p \in [1, +\infty]$. There exists a sequence $(w_j)_j$ in $L^2(0,T;H^1(\Omega))$ such that
\[
\sup_j \|w_j\|_{L^p(Q_T)} < +\infty
\]
and, if $p < +\infty$, we have
\[
\lim_{j \to +\infty} w_j(t) = w(t) \quad \text{in } L^p(\Omega),
\]
for a.e. $t \in ]0,T[$. Moreover we have for a.e. $t \in ]0,T[$
\[
\lim_{j \to +\infty} \int_\Omega |\nabla w_j(t)| \, dx = \int_\Omega |Dw(t)|, \quad (A.2)
\]
and
\[
\lim_{j \to +\infty} J(w_j(t)) = J(w(t)). \quad (A.3)
\]

Proof. The proof closely follows the proof of [11, Teorema 1] and of [10, Fact 3.3]. Here we give the precise construction of the approximating regularized sequence. Fix any $\varepsilon > 0$ and define a sequence $(\Omega_i)_i$ of open sets with
\[
\Omega_i \subset \subset \Omega_{i+1} \quad \text{and} \quad \bigcup_{i=0}^{\infty} \Omega_i = \Omega,
\]
such that
\[
\int_{\Omega \setminus \Omega_0} |Dw| < \varepsilon \quad \text{and} \quad \int_{\partial \Omega_2} |Dw| = 0.
\]
Set for convenience $\Omega_{-1} = \emptyset$ and consider now the sequence $(A_i)_i$ of sets defined as
\[
A_i = \Omega_{i+1} \setminus \overline{\Omega_i},
\]
and let $(\phi_i)_i$ be a partition of the unity subordinate to the covering $(A_i)_i$, i.e.,
\[
\phi_i \in C_0^\infty(A_i), \quad 0 \leq \phi_i \leq 1
\]
for all $i = 1, 2, \ldots$ and
\[
\sum_{i=1}^{\infty} \phi_i = 1.
\]
Let $(\eta_i)_i$ be a family of positive symmetric mollifiers on $\Omega$, such that
\[
\text{supp } \eta_i \ast (w(t)\phi_i) \subset \Omega_{i+2} \setminus \overline{\Omega_{i-2}}
\]
for all $t \in ]0,T[$, as shown in [57, Appendix I]. We define the function $w_\varepsilon$ by setting
\[
w_\varepsilon = \sum_{i=1}^{\infty} \eta_i \ast (w\phi_i).
\]
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As the sum defining \( w_\varepsilon \) is locally finite, it clear that \( w_\varepsilon (t) \in C^\infty (\Omega) \) for a.e. \( t \in ]0, T[ \). Standard properties of mollifiers ensure that

\[
\lim_{\varepsilon \to 0} w_\varepsilon(t) = w(t) \quad \text{in} \quad L^p(\Omega)
\]

for a.e. \( t \in ]0, T[ \), if \( p < +\infty \). By definition of \( w_\varepsilon \) and using the regularity of the mollifiers, there exists a \( C_\varepsilon > 0 \) such that

\[
\| w_\varepsilon (t) \|_{H^1} \leq \| w(t) \|_{L^2} + C_\varepsilon \| \nabla \eta \|_{L^1} \| w(t) \|_{L^p}.
\]

for a.e. \( t \in ]0, T[ \) and where \( q = \frac{p}{p-1} \). Hence it follows \( w_\varepsilon \in L^2(0, T; H^1(\Omega)) \). Similarly, there exists \( K > 0 \), depending only on \( \| w \|_{L^p(Q_T)} \), such that

\[
\sup \| w_\varepsilon \|_{L^p} < K.
\]

To conclude this proof, we refer to [11, Teorema 1] that shows the validity of (A.2), whereas the proof of (A.3) is contained in [10, Fact 3.3].

**Lemma A.1.2.** Assume \( (h, A_0) \) and take any function \( G \in C^1(\mathbb{R}) \), with \( G' \in L^\infty (\mathbb{R}) \). If \( u \in H^1(0, T; L^2(\Omega)) \) then \( G \circ u \in H^1(0, T; L^2(\Omega)) \) and

\[
\frac{\partial}{\partial t} (G \circ u) = G'(u) \dot{u}.
\]

**Proof.** The proof of this result closely follows [105, Lemma 1.57]. First of all let us notice that \( G(u) \in L^2(0, T; L^2(\Omega)) \) and \( G'(u) \dot{u} \in L^2(0, T; L^2(\Omega)) \). Fix any \( t_1, t_2 \in ]0, T[ \) with \( t_1 < t_2 \). By standard regularization process, as in [104, Lemma 1.3], we have a sequence \( (u_n)_n \in C^\infty (0, T; L^2(\Omega)) \) such that

\[
\lim_{n \to +\infty} u_n \to u \quad \text{in} \quad L^2(t_1, t_2; L^2(\Omega)),
\]

\[
\lim_{n \to +\infty} \dot{u}_n \to \dot{u} \quad \text{in} \quad L^2(t_1, t_2; L^2(\Omega))
\]

and a.e. in \( \Omega \times ]t_1, t_2[ \). Moreover, for each \( n \) we have \( \frac{\partial}{\partial t} (G \circ u_n) = G'(u_n) \dot{u}_n \) and

\[
\lim_{n \to +\infty} G'(u_n) \to G'(u) \quad \text{a.e in} \quad \Omega \times ]t_1, t_2[.
\]

Notice that

\[
\int_{t_1}^{t_2} \int_{\Omega} |G(u_n) - G(u)|^2 \, dx \, dt \leq \| G' \|_{\infty}^2 \int_{t_1}^{t_2} \int_{\Omega} |u_n - u|^2 \, dx \, dt
\]

and moreover

\[
\int_{t_1}^{t_2} \int_{\Omega} |G'(u_n) \dot{u}_n - G'(u) \dot{u}|^2 \, dx \, dt \leq \int_{t_1}^{t_2} \int_{\Omega} |G'(u_n) (\dot{u}_n - \dot{u})|^2 \, dx \, dt
\]

\[
+ \int_{t_1}^{t_2} \int_{\Omega} |(G'(u_n) - G'(u)) \dot{u}|^2 \, dx \, dt
\]

\[
\leq \| G' \|_{\infty}^2 \int_{t_1}^{t_2} \int_{\Omega} |u_n - u|^2 \, dx \, dt
\]

\[
+ \int_{t_1}^{t_2} \int_{\Omega} |(G'(u_n) - G'(u))|^2 |\dot{u}|^2 \, dx \, dt.
\]
A. THE EVOLUTIONARY PROBLEM

Hence we get
\[
\lim_{n \to +\infty} G(u_n) = G(u) \quad \text{in } L^2(t_1, t_2; L^2(\Omega))
\]
and, by the Lebesgue convergence theorem,
\[
\lim_{n \to +\infty} \frac{\partial}{\partial t}(G \circ u_n) = G'(u)\dot{u} \quad \text{in } L^2(t_1, t_2; L^2(\Omega)).
\]
This shows that \( G \circ u \in H^1(t_1, t_2; L^2(\Omega)) \) and
\[
\frac{\partial}{\partial t}(G \circ u) = G'(u)\dot{u}
\]
a.e. in \([t_1, t_2]\[, for each \( t_1, t_2 \in ]0, T[ \) with \( t_1 < t_2 \). The conclusion of the lemma is now valid by definition of distributional derivative.

Lemma A.1.3. Assume \((h_30)\). If \( u \in H^1(0, T; L^2(\Omega)) \) then \( u^+, u^- \in H^1(0, T; L^2(\Omega)) \) and
\[
\frac{\partial}{\partial t}(u^+) = \chi_{u>0} \dot{u} \quad \frac{\partial}{\partial t}(u^-) = \chi_{u<0} \dot{u}
\]
where \( \chi_{u>0} \) and \( \chi_{u<0} \) respectively denote the characteristic functions of the subsets of \( Q_T \) where \( u > 0 \) and \( u < 0 \).

Proof. The proof of this results closely follows [105, Theorem 1.56]. For any \( \varepsilon > 0 \) let us define the function \( G_\varepsilon \in C^1(\mathbb{R}) \) as
\[
G_\varepsilon(s) = \begin{cases} \sqrt{s^2 + \varepsilon^2} - \varepsilon & \text{for } s > 0, \\ 0 & \text{for } s \leq 0. \end{cases}
\]
Notice that for each \( \varepsilon > 0 \) we have \( G_\varepsilon' \in L^\infty(\mathbb{R}) \) and then we are in position to apply Lemma A.1.2, which yields \( G_\varepsilon \circ u \in H^1(0, T; L^2(\Omega)) \) and
\[
\frac{\partial}{\partial t}(G_\varepsilon \circ u) = G_\varepsilon'(u)\dot{u},
\]
i.e.,
\[
- \int_{Q_T} G_\varepsilon(u) \dot{\phi} \, dx \, dt = \int_{Q_T^+} \frac{\dot{u}}{\sqrt{u^2 + \varepsilon^2}} \phi \, dx \, dt
\]
for all \( \phi \in C_0^\infty(0, T; L^2(\Omega)) \), with \( Q_T^+ = \{(x, t) \in Q_T : u(x, t) > 0\} \). By letting \( \varepsilon \to 0 \) we get
\[
- \int_{Q_T} u^+ \dot{\phi} \, dx \, dt = \int_{Q_T^+} \dot{u} \phi \, dx \, dt
\]
for all \( \phi \in C_0^\infty(0, T; L^2(\Omega)) \) and this yields the conclusion. The same conclusion for \( u^- \) follows from the identity \( u^- = (-u)^+ \). \( \square \)
Lemma A.1.4. Assume \((h_A 0)\). If \(u \in H^1(0, T; L^2(\Omega))\) with \(u(x, 0) \leq 0\), then
\[
\int_{Q_T} \dot{u} u^+ \, dx \, dt \geq 0.
\]
Analogously, if \(u(x, 0) \geq 0\), then
\[
\int_{Q_T} \dot{u} u^- \, dx \, dt \leq 0.
\]
Proof. By Lemma A.1.3 we have \(u^+ \in H^1(0, T; L^2(\Omega))\) and
\[
\int_{Q_T} \dot{u} u^+ \, dx \, dt = \int_{Q_T} (\dot{u}^+) u^+ \, dx \, dt.
\]
We are in conditions to apply Lemma A.1.2 to the function \(u^+\) and hence we get
\[
\int_{Q_T} (\dot{u}^+) u^+ \, dx \, dt = \frac{1}{2} \int_0^T \frac{\partial}{\partial t} \int_{\Omega} |u^+|^2 \, dx \, dt
= \frac{1}{2} \left( \int_{\Omega} |u^+(x, T)|^2 \, dx - \int_{\Omega} |u^+(x, 0)|^2 \, dx \right)
= \frac{1}{2} \int_{\Omega} |u^+(x, T)|^2 \, dx
\]
which yields the required conclusion. The analogous conclusion for \(u^-\), follows from the identity \(u^- = (-u)^+\).

In accordance with the previous chapters, we define the functional
\[
\mathcal{J} : L^1(0, T; BV(\Omega)) \to L^1(0, T)
\]
by setting
\[
\mathcal{J}(v) = \int_{\Omega} \sqrt{1 + |Dv|^2}.
\]
Notice that this functional is well defined thanks to Lemma I.1.5.

Moreover, with reference to the discussion performed in Chapter I and to [59], [45] and [6], the following notion of solution is adopted.

Notion of solution. We say that a function \(u \in L^1(0, T; BV(\Omega)) \cap L^\infty(Q_T)\), such that \(\dot{u} \in L^2(Q_T)\) and \(u(x, 0) = u_0\), is a solution of (A.1) if
\[
\int_0^t \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \int_0^t \mathcal{J}(v) \, d\tau \geq \int_0^t \mathcal{J}(u) \, d\tau + \int_0^t \int_{\Omega} f(x, \tau, u)(v - u) \, dx \, d\tau
\]
for all \(v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T)\) and every \(t \in [0, T]\).

Notice that in particular \(u : [0, T] \to L^2(\Omega)\) is continuous and hence the initial value condition at \(t = 0\) is meaningful.
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Proposition A.1.5. Let $u \in L^1(0, T; BV(\Omega)) \cap L^\infty(Q_T)$, such that $\dot{u} \in L^2(Q_T)$, the following relations are equivalent

\begin{align*}
(i) & \quad \int_{\Omega} \dot{u}(v - u) \, dx + J(v) \geq J(u) + \int_{\Omega} f(x, t, u)(v - u) \, dx \quad (A.4) \\
& \quad \text{for all } v \in BV(\Omega) \cap L^2(\Omega) \text{ and a.e. in } [0, T]; \\
(ii) & \quad \int_0^t \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \int_0^t J(v) \, d\tau \\
& \quad \quad \geq \int_0^t J(u) \, d\tau + \int_0^t \int_{\Omega} f(x, \tau, u)(v - u) \, dx \, d\tau \\
& \quad \quad \text{for all } v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T) \text{ and every } t \in [0, T]; \\
(iii) & \quad \int_{Q_T} \dot{u}(v - u) \, dx \, d\tau + \int_0^T J(v) \, d\tau \\
& \quad \quad \geq \int_0^T J(u) \, d\tau + \int_{Q_T} f(x, \tau, u)(v - u) \, dx \, d\tau \quad (A.5) \\
& \quad \text{for all } v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T). 
\end{align*}

Proof. We show the following three implications.

(i) \Rightarrow (ii) Take any $v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T)$. For a.e. $\tau \in [0, T]$ the function $v(\tau) \in BV(\Omega) \cap L^2(\Omega)$ is an admissible test function in (A.4) and hence

\begin{align*}
\int_{\Omega} \dot{u}(\tau)(v(\tau) - u(\tau)) \, dx + J(v(\tau)) \\
& \quad \geq J(u(\tau)) + \int_{\Omega} f(x, \tau, u(\tau))(v(\tau) - u(\tau)) \, dx \quad (A.6) 
\end{align*}

for a.e. $\tau \in [0, T]$. As $v, \dot{u} \in L^2(Q_T)$ and $u \in L^\infty(Q_T)$, by (hA1) we have that the functions

\begin{align*}
\tau \mapsto \int_{\Omega} \dot{u}(\tau)(v(\tau) - u(\tau)) \, dx \quad \text{and} \quad \tau \mapsto \int_{\Omega} f(x, \tau, u(\tau))(v(\tau) - u(\tau)) \, dx 
\end{align*}

belong to $L^1(0, T)$. For any $t \in [0, T]$ we can integrate (A.6) between 0 and $t$ with respect to $\tau$ and we get exactly (ii).

(ii) \Rightarrow (iii) This implication follows just taking $t = T$ in (ii).
(iii) ⇒ (i) Fix $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Take any $v \in BV(\Omega) \cap L^2(\Omega)$ and consider the function $w : Q_T \to \mathbb{R}$ defined as

$$w(x, t) = \chi_{[t_1, t_2]}(t)v(x) + (1 - \chi_{[t_1, t_2]}(t))u(x, t).$$

Notice that $w(t) = v$ for $t \in [t_1, t_2]$ and $w(t) = u(t)$ for $t \not\in [t_1, t_2]$. Since $v \in BV(\Omega) \cap L^2(\Omega)$ and $u \in L^1(0, T; BV(\Omega)) \cap L^\infty(Q_T)$, it follows that $w \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T)$ and hence $w$ is an admissible test function in (A.5). Substituting, we get

$$\int_{t_1}^{t_2} \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \int_0^{t_1} \mathcal{J}(u) \, d\tau + \int_{t_1}^{t_2} \mathcal{J}(v) \, d\tau + \int_T^{t_2} \mathcal{J}(u) \, d\tau \geq \int_0^{t_1} \mathcal{J}(u) \, d\tau + \int_{t_1}^{t_2} \int_{\Omega} f(x, \tau, u)(v - u) \, dx \, d\tau$$

and then

$$\int_{t_1}^{t_2} \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \int_{t_1}^{t_2} \mathcal{J}(v) \, d\tau \geq \int_{t_1}^{t_2} \mathcal{J}(u) \, d\tau + \int_{t_1}^{t_2} \int_{\Omega} f(x, \tau, u)(v - u) \, dx \, d\tau$$

for every $v \in BV(\Omega) \cap L^2(\Omega)$ and all $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. In particular, for all $t \in [0, T]$ and every $\varepsilon > 0$ sufficiently small, we have

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \mathcal{J}(v) \geq \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathcal{J}(u) \, d\tau + \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \int_{\Omega} f(x, \tau, u)(v - u) \, dx \, d\tau. \quad (A.7)$$

Since all functions

$$\tau \mapsto \int_{\Omega} \dot{u}(v - u) \, dx,$$

$$\tau \mapsto \mathcal{J}(u),$$

$$\tau \mapsto \int_{\Omega} f(x, t, u)(v - u) \, dx$$

belong to $L^1(0, T)$, taking the limit in (A.7) for $\varepsilon \to 0$, the Lebesgue differentiation theorem (see, e.g., [98]) implies

$$\int_{\Omega} \dot{u}(v - u) \, dx + \mathcal{J}(v) \geq \mathcal{J}(u) + \int_{\Omega} f(x, t, u)(v - u) \, dx$$

for a.e. $t \in [0, T]$ and all $v \in BV(\Omega) \cap L^2(\Omega)$, i.e., (i) holds.

According to [45, Definition 2] and Chapter II, we give the following definition for $BV$-lower solution and $BV$-upper solution for an evolutionary problem like (A.1).
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BV-lower and BV-upper solutions.

- We say that a function \( \alpha \in L^1(0,T;BV(\Omega)) \cap L^\infty(Q_T) \) is a BV-lower solution of (A.1) if \( f(\cdot,\alpha) \in L^2(Q_T), \dot{\alpha} \in L^2(Q_T), \alpha(\cdot,0) \leq u_0 \) and

\[
\int_\Omega \dot{\alpha} z \, dx + J(\alpha + z) \geq J(\alpha) + \int_\Omega f(x,t,\alpha) z \, dx
\]

for all \( z \in BV(\Omega) \cap L^2(\Omega) \), with \( z \leq 0 \) and a.e. in \( ]0,T[ \).

- We say that a function \( \beta \in L^1(0,T;BV(\Omega)) \cap L^\infty(Q_T) \) is a BV-upper solution of (A.1) if \( f(\cdot,\beta) \in L^2(Q_T), \beta \in L^2(Q_T), \beta(\cdot,0) \geq u_0 \) and

\[
\int_\Omega \dot{\beta} z \, dx + J(\beta + z) \geq J(\beta) + \int_\Omega f(x,t,\beta) z \, dx
\]

(A.8)

for all \( z \in BV(\Omega) \cap L^2(\Omega) \), with \( z \geq 0 \) and a.e. in \( ]0,T[ \).

Remark A.1.1 Thanks to the equivalences stated in Proposition A.1.5, we have that a function \( \alpha \in L^1(0,T;BV(\Omega)) \cap L^\infty(Q_T) \) is a BV-lower solution of (A.1) if \( f(\cdot,\alpha) \in L^2(Q_T), \dot{\alpha} \in L^2(Q_T), \alpha \leq u_0 \) and

\[
\int_0^T \int_\Omega \dot{\alpha} z \, dx \, dt + \int_0^T \int_\Omega J(\alpha + z) \, d\tau \geq \int_0^T \int_\Omega J(\alpha) \, d\tau + \int_0^T \int_\Omega f(x,\tau,\alpha) z \, dx \, d\tau
\]

for all \( z \in L^1(0,T;BV(\Omega)) \cap L^2(Q_T) \) with \( z \leq 0 \) and every \( t \in [0,T] \) or equivalently

\[
\int_{Q_T} \dot{\alpha} z \, dx \, d\tau + \int_0^T \int_{Q_T} J(\alpha + z) \, d\tau \geq \int_0^T \int_{Q_T} J(\alpha) \, d\tau + \int_{Q_T} f(x,\tau,\alpha) z \, dx \, d\tau
\]

for all \( v \in L^1(0,T;BV(\Omega)) \cap L^2(Q_T) \) with \( z \leq 0 \).

The same equivalence holds also in the case of BV-upper solution.

Remark A.1.2 A function \( u \in L^1(0,T;BV(\Omega)) \) is a solution of (A.1) if and only if it is simultaneously a BV-lower solution and a BV-upper solution of (A.1). The proof of this fact is similar to the one of Remark II.1.2.

A.2 Existence result

Theorem A.2.1. Assume \((h_\Lambda 0), (h_\Lambda 1)\) and

\((h_\Lambda 2)\) there exist a BV-lower solution \( \alpha \) and a BV-upper solution \( \beta \) of problem (A.1) satisfying \( \alpha \leq \beta \).
Then problem (A.1) has at least one solution satisfying

\[ \alpha \leq u \leq \beta. \]

Proof. The proof is divided in several steps.

Step 1. A modified problem. We first modify problem (A.1) by adding, to both sides of the equation, the term \( cu \) with \( c \in \mathbb{R}_0^+ \). We get

\[
\begin{cases}
  u_t - \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + cu = g(x, t, u) & \text{in } Q_T, \\
  -\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = 0 & \text{on } \Sigma_T, \\
  u(x, 0) = u_0 & \text{on } \Omega,
\end{cases}
\]

with \( g(x, t, u) = f(x, t, u) + cu \). Let us define the function

\[ \bar{g}(x, t, s) = \begin{cases} 
  g(x, t, \alpha(x, t)) & \text{if } s < \alpha(x, t), \\
  g(x, t, s) & \text{if } \alpha(x, t) \leq s \leq \beta(x, t), \\
  g(x, t, \beta(x, t)) & \text{if } s > \beta(x, t)
\end{cases} \] (A.9)

and consider the problem

\[
\begin{cases}
  u_t - \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + cu = \bar{g}(x, t, u) & \text{in } Q_T, \\
  -\nabla u \cdot n/\sqrt{1 + |\nabla u|^2} = 0 & \text{on } \Sigma_T, \\
  u(x, 0) = u_0 & \text{on } \Omega.
\end{cases}
\] (A.10)

Notice that, by \((h_A 1)\) and \((h_A 2)\), there exists \( \Gamma > 0 \) such that

\[ |\bar{g}(x, t, s)| \leq \Gamma \] (A.11)

for a.e. \((x, t) \in Q_T\) and every \( s \in \mathbb{R} \).

Step 2. A perturbed problem. As the initial value \( u_0 \) belongs to \( BV(\Omega) \cap L^\infty(\Omega) \), by the approximation result stated Lemma A.1.1, there exists a sequence \((u_{0, k})_k\) in \( H^1(\Omega) \) such that

\[ \sup_k \|u_{0, k}\|_\infty < +\infty, \] (A.12)

\[ \lim_{k \to +\infty} u_{0, k} = u_0 \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega, \] (A.13)

\[ \lim_{k \to +\infty} \int_\Omega |\nabla u_{0, k}| \, dx = \int_\Omega |Du_0| \]

and

\[ \lim_{k \to +\infty} \int_\Omega \sqrt{1 + |\nabla u_{0, k}|^2} \, dx = \int_\Omega \sqrt{1 + |Du_0|^2}. \] (A.14)
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Define a sequence \((\varepsilon_k)_k\) in \(\mathbb{R}\) by setting
\[
\varepsilon_k = \frac{1}{\max \{k, \int_{\Omega} |\nabla u_{0,k}|^2 \, dx\}}.
\]
Notice that \(\varepsilon_k \to 0\) and
\[
\sup_{k \in \mathbb{N}} \varepsilon_k \int_{\Omega} |\nabla u_{0,k}|^2 \, dx \leq 1. \tag{A.15}
\]

For each \(k > 0\) consider the perturbed problem

\[
\begin{cases}
  u_t - \varepsilon_k \Delta u - \text{div} \left( \nabla u / \sqrt{1 + |\nabla u|^2} \right) + cu = \bar{g}(x, t, u) & \text{in } Q_T, \\
  -\nabla u \cdot n / \sqrt{1 + |\nabla u|^2} = 0 & \text{on } \Sigma_T, \\
  u(x, 0) = u_{0,k} & \text{on } \Omega.
\end{cases} \tag{A.16}
\]

Step 3. Existence of solutions for the perturbed problems. In order to solve (A.16) for any fixed \(k\), we apply a result contained in [81, Theorem 2.1, p. 323] to the problem (A.16) with respect to the space \(V = L^2(0, T; H^1(\Omega))\).

Adopting the notation in [81, p. 321], we set
\[
(A(u), v) = \int_{Q_T} A_0(x, t, u)v \, dx \, dt + \int_{Q_T} A_1(\nabla u) \cdot \nabla v \, dx \, dt \tag{A.19}
\]
for any \(u, v \in V\), with
\[
A_1(\xi) = \varepsilon_k \xi + \frac{\xi}{\sqrt{1 + |\xi|^2}},
\]
and
\[
A_0(x, t, s) = cs - \bar{g}(x, t, s).
\]

Observe that \(A_1 : \mathbb{R}^N \to \mathbb{R}\) is continuous and \(A_0\) is \(L^\infty\)-Carathéodory, i.e., for a.e. \((x, t) \in Q_T\), the function \(A_0(x, t, \cdot) : \mathbb{R}^N \to \mathbb{R}\) is continuous, for every \(\xi \in \mathbb{R}^N\), the function \(A_0(\cdot, \xi) : Q_T \to \mathbb{R}\) is measurable and, for each compact set \(K \subset \mathbb{R}^N\), there exists \(\gamma \in L^\infty(Q_T)\) such that \(|A_0(x, t, \xi)| \leq \gamma(x, t)\) for a.e. \((x, t) \in Q_T\) and every
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\( \xi \in K \). Notice that for any \( u \in V \) fixed, we have \( A_0(x,t,u), A_1(\nabla u) \in L^2(Q_T) \). Indeed, by (A.11), we get

\[
\|A_0(x,t,u)\|_{L^2} \leq c\|u\|_V + \Gamma
\]

and also

\[
\|A_1(\nabla u)\|_{L^2} \leq \varepsilon_k\|\nabla u\|_{L^2} + \left\| \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right\|_{L^2} 
\leq \varepsilon_k\|u\|_V + \sqrt{|\Omega|}.
\]

Observe that

\[
\lim_{|\xi| \to +\infty} \frac{A_1(\xi) \cdot \xi}{|\xi|} = +\infty
\]

and

\[
(A_1(\xi) - A_1(\xi^*)(\xi - \xi^*)) > 0
\]

for all \( \xi, \xi^* \in \mathbb{R}^N \) with \( \xi \neq \xi^* \). In order to apply [81, Theorem 2.1], we need to check that the operator \( A \) defined in (A.19) is coercive on the space \( V \), i.e., we want to show that, if \( \|u\|_V \to +\infty \), then

\[
\frac{(A(u), u)}{\|u\|_V} \to +\infty.
\]

Fix \( u \in V \), using (A.11) we have

\[
(A(u), u) = c \int_{Q_T} u^2 \, dx \, dt - \int_{Q_T} \bar{g}(x,t,u)u \, dx \, dt 
+ \varepsilon_k \int_{Q_T} |\nabla u|^2 \, dx \, dt + \int_{Q_T} \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} \, dx \, dt 
\geq c \int_{Q_T} u^2 \, dx \, dt + \varepsilon_k \int_{Q_T} |\nabla u|^2 \, dx \, dt - \Gamma \int_{Q_T} |u| \, dx \, dt
\]

and then the continuous Sobolev embedding

\[ V \hookrightarrow L^1(0,T; L^1(\Omega)), \]

implies the existence of \( k_1, k_2 > 0 \) such that

\[
\frac{(A(u), u)}{\|u\|_V} \geq k_1\|u\|_V - \Gamma \int_{Q_T} |u| \, dx \, dt \geq k_1\|u\|_V - k_2.
\]

This proves the coercivity of the operator \( A \) on the space \( V \). The hypotheses of [81, Theorem 2.1] are satisfied, then using also [81, Remark 2.4] and [19], we have that for all \( k \in \mathbb{N} \) there exists \( u_k \in V \) such that

\[
\frac{\partial u_k}{\partial t} \in L^2(0,T; (H^1(\Omega))^*),
\]

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which is a weak solution of problem \( (A.16) \), i.e., for all \( \varphi \in H^1(\Omega) \)
\[
\int_{\Omega} \dot{u}_k \varphi \, dx + \varepsilon_k \int_{\Omega} \nabla u_k \cdot \nabla \varphi \, dx + \int_{\Omega} \frac{\nabla u_k \cdot \nabla \varphi}{\sqrt{1 + |\nabla u_k|^2}} \, dx \\
+ c \int_{\Omega} u_k \varphi \, dx = \int_{\Omega} \bar{g}(x, t, u_k) \varphi \, dx
\]
\( (A.20) \)
a.e. in \( [0, T] \) and \( u_k(\cdot, 0) = u_{0,k} \). Moreover, since \( u_k \in L^2(0, T; H^1(\Omega)) \) and \( \frac{\partial u_k}{\partial t} \in L^2(0, T; (H^1(\Omega))^*) \), then \( u_k \in C^0([0, T]; L^2(\Omega)) \) (see, e.g., [34]) and hence
\[
\lim_{t \to 0} ||u_k(t) - u_{0,k}||_{L^2} = 0.
\]
The same arguments used in Lemma A.1.5 show that condition \( (A.20) \) is equivalent to requiring
\[
\int_{Q_T} \dot{u}_k \varphi \, dx \, dt + \varepsilon_k \int_{Q_T} \nabla u_k \cdot \nabla \varphi \, dx \, dt + \int_{Q_T} \frac{\nabla u_k \cdot \nabla \varphi}{\sqrt{1 + |\nabla u_k|^2}} \, dx \, dt \\
+ c \int_{Q_T} u_k \varphi \, dx \, dt = \int_{Q_T} \bar{g}(x, t, u_k) \varphi \, dx \, dt
\]
\( (A.21) \)
for all \( \varphi \in L^2(0, T; H^1(\Omega)) \).
Take any \( w \in L^2(0, T; H^1(\Omega)) \) and, for a.e. \( t \in [0, T] \), consider the test function \( w(t) - u_k(t) \) in \( (A.20) \). We get
\[
\int_{\Omega} \dot{u}_k (w - u_k) \, dx + \varepsilon_k \int_{\Omega} \nabla u_k \cdot (\nabla w - \nabla u_k) \, dx + \int_{\Omega} \frac{\nabla u_k \cdot (\nabla w - \nabla u_k)}{\sqrt{1 + |\nabla u_k|^2}} \, dx \\
+ c \int_{\Omega} u_k (w - u_k) \, dx = \int_{\Omega} \bar{g}(x, t, u_k) (w - u_k) \, dx
\]
a.e. in \( [0, T] \). Noticing that the function
\[
s \mapsto \varepsilon_k \frac{s^2}{2} + \sqrt{1 + s^2}
\]
is convex, we obtain
\[
\int_{\Omega} \dot{u}_k (w - u_k) \, dx + \frac{\varepsilon_k}{2} \int_{\Omega} |\nabla w|^2 \, dx - \frac{\varepsilon_k}{2} \int_{\Omega} |\nabla u_k|^2 \, dx \\
+ \int_{\Omega} \sqrt{1 + |\nabla w|^2} \, dx - \int_{\Omega} \sqrt{1 + |\nabla u_k|^2} \, dx \\
\geq \int_{\Omega} \bar{g}(x, t, u_k) (w - u_k) \, dx - c \int_{\Omega} u_k (w - u_k) \, dx
\]
for a.e. \( t \in [0, T] \) and then, using \( (A.17) \), we conclude
\[
\int_{\Omega} \dot{u}_k (w - u_k) \, dx + J_k(w) \geq J_k(u_k) + \int_{\Omega} (\bar{g}(x, t, u_k) - cu_k)(w - u_k) \, dx
\]
\( (A.22) \)
for all \( w \in L^2(0, T; H^1(\Omega)) \) and a.e. in \( [0, T] \).
Step 4. The sequence \((u_k)_k\) is uniformly bounded in \(L^\infty(Q_T)\). Our goal is to show that there exists a constant \(R > 0\) such that

\[
\|u_k\|_{L^\infty(Q_T)} < R
\]

for all \(k\). Fix \(R > 0\) and let us consider \(\varphi = (u_k - R)^+\). By Lemma A.1.3, \(\varphi\) is an ammissible test function in (A.21), since \(\varphi \in L^2(0,T;H^1(\Omega))\). Thus we get

\[
\int_{Q_T} \dot{u}_k(u_k - R)^+ \ dx \ dt + \varepsilon_k \int_{Q_T} |\nabla u_k|^2 \chi_{\{u_k > R\}} \ dx \ dt + \int_{Q_T} \frac{|\nabla u_k|^2 \chi_{\{u_k > R\}}}{\sqrt{1 + |\nabla u_k|^2}} \ dx \ dt
\]

\[
+ c \int_{Q_T} u_k(u_k - R)^+ \ dx \ dt = \int_{Q_T} \bar{g}(x,t,u_k)(u_k - R)^+ \ dx \ dt
\]

and therefore

\[
\int_{Q_T} \dot{u}_k(u_k - R)^+ \ dx \ dt + c \int_{Q_T} u_k(u_k - R)^+ \ dx \ dt \leq \int_{Q_T} \bar{g}(x,t,u_k)(u_k - R)^+ \ dx \ dt. \tag{A.23}
\]

We consider now the first term of the last inequality. Since \(u_0 \in L^\infty(\Omega)\), by (A.12), we can take the constant \(R\) sufficiently large, such that \((u_0 - R)^+ = 0\) and \((u_0 - R)^+ = 0\) for all \(k\). Then we are in conditions to apply Lemma A.1.4 to the function \(u_k - R\). This yields

\[
\int_{Q_T} \dot{u}_k(u_k - R)^+ \ dx \ dt \geq 0.
\]

Coming back to (A.23), for \(R\) sufficiently large, we get

\[
\int_{Q_T} (\bar{g}(x,t,u_k) - cu_k)(u_k - R)^+ \ dx \ dt \geq 0. \tag{A.24}
\]

We can further suppose that \(R > \|\beta\|_{\infty}\). By (A.9), for a.e \((x,t) \in Q_T\) such that \(u_k(x,t) > R > \|\beta\|_{\infty}\), we get

\[
\bar{g}(x,t,u_k(x,t)) - cu_k(x,t) = g(x,t,\beta(x,t)) - cu_k(x,t)
\]

\[
\leq f(x,t,\beta(x,t)) + c(\beta - u_k(x,t))
\]

\[
\leq \|f(\cdot,\cdot,\beta)\|_{\infty} - c(R - \|\beta\|_{\infty}) < 0
\]

for a sufficiently large \(R\). This last inequality, combined with (A.24), yields \((u_k - R)^+ = 0\). Similarly, one can prove \((u_k + R)^- = 0\) and hence

\[
\sup_k \|u_k\|_{\infty} \leq R. \tag{A.25}
\]

As a consequence, by (A.11), we also have

\[
\sup_k \|\bar{g}(x,t,u_k(x,t))\|_{\infty} < +\infty. \tag{A.26}
\]
Step 5. The sequence \((\dot{u}_k)_k\) is uniformly bounded in \(L^2(Q_T)\). The proof of this estimate closely follows [59, Lemma 2.1]. Fix \(k \in \mathbb{N}\) and consider \(u_k\) solution of (A.16). We have that \(u_k \in L^2(0,T; H^1(\Omega)) \cap L^\infty(Q_T)\), \(\dot{u}_k \in L^2(Q_T)\) and \(u_k\) satisfies (A.22). Let us consider, for all \(\eta > 0\), the function \(u_\eta\) solution of

\[
\begin{align*}
\eta \dot{u}_\eta + u_\eta &= u_k \quad \text{in } Q_T, \\
u_\eta(x,0) &= u_{0,k} \quad \text{on } \Omega.
\end{align*}
\]

It is easy to check that

\[
u_\eta = e^{-t/\eta}u_{0,k} + \frac{1}{\eta} \int_0^t e^{(\tau-t)/\eta} u_k(\tau) \, d\tau.
\]

We write \(u_\eta\) as a convex combination

\[
u_\eta = e^{-t/\eta}u_{0,k} + (1 - e^{-t/\eta}) \frac{1}{\eta(1 - e^{-t/\eta})} \int_0^t e^{(\tau-t)/\eta} u_k(\tau) \, d\tau
\]

and we use it as a test function in (A.22). Noticing that

\[
\int_{Q_T} (\dot{v} - \dot{u})(v - u) \, dx \, dt \geq 0
\]

for all \(u,v \in L^2(Q_T)\) such that \(\dot{u}, \dot{v} \in L^2(Q_T)\) with \(u(0) = v(0)\), we get

\[
\int_{Q_T} \dot{u}_\eta (u_\eta - u_k) \, dx \, dt + \int_0^T J_k(u_\eta) \, dt - \int_0^T J_k(u_k) \, dt \\
\geq \int_{Q_T} (\bar{g}(x,t,u_k) - cu_k)(u_\eta - u_k) \, dx \, dt.
\]

Using the convexity of \(J_k\) and the definition of \(u_\eta\), we have

\[
\eta \int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \leq \int_0^T J_k(u_\eta) \, dt - \int_0^T J_k(u_k) \, dt + \eta \int_{Q_T} (\bar{g}(x,t,u_k) - cu_k) \dot{u}_\eta \, dx \, dt
\]

\[
\leq J_k(u_{0,k}) \int_0^T e^{-t/\eta} \, dt + e^{t/\eta} \int_0^T (1 - e^{-t/\eta}) J_k \left( \frac{\int_0^t e^{(\tau-t)/\eta} u_k(\tau) \, d\tau}{\eta(1 - e^{-t/\eta})} \right) \, dt
\]

\[
- \int_0^T J_k(u_k) \, dt + \eta \int_{Q_T} (\bar{g}(x,t,u_k) - cu_k) \dot{u}_\eta \, dx \, dt.
\]

As

\[
\int_0^t e^{(\tau-t)/\eta} \, d\tau = 1,
\]

Jensen’s inequality yields

\[
J_k \left( \int_0^t \frac{e^{(\tau-t)/\eta}}{\eta(1 - e^{-t/\eta})} u_k(\tau) \, d\tau \right) \leq \frac{1}{\eta(1 - e^{-t/\eta})} \int_0^t e^{(\tau-t)/\eta} J_k(u_k(\tau)) \, d\tau.
\]
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Thus (A.28) becomes

\[
\eta \int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \leq \eta(1 - e^{-T\eta}) \mathcal{J}_k(u_{0,k}) + \frac{1}{\eta} \int_0^T \int_0^t e^{(\tau-t)/\eta} \mathcal{J}_k(u_k(\tau)) \, d\tau \, dt

- \int_0^T \mathcal{J}_k(u_k) \, dt + \eta \int_{Q_T} (\bar{g}(x,t,u_k) - cu_k) \dot{u}_\eta \, dx \, dt. \quad (A.29)
\]

Integrating by parts the second term on the right hand side of (A.29), we get

\[
\frac{1}{\eta} \int_0^T e^{-t/\eta} \int_0^t e^{\tau/\eta} \mathcal{J}_k(u_k(\tau)) \, d\tau \, dt \leq \int_0^T \mathcal{J}_k(u_k(t)) \, dt
\]

and we conclude from (A.29) that

\[
\int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \leq \mathcal{J}_k(u_{0,k}) + \int_{Q_T} (\bar{g}(x,t,u_k) - cu_k) \dot{u}_\eta \, dx \, dt.
\]

By (A.25) and (A.26), there exists a constant \( M > 0 \), depending only on \( \|\alpha\|_\infty, \|\beta\|_\infty, f \) and \( u_0 \), such that, for all \( k \) and every \( \eta > 0 \),

\[
\int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \leq \mathcal{J}_k(u_{0,k}) + M \left( \int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \right)^{1/2}
\]

and then

\[
\int_{Q_T} |\dot{u}_\eta|^2 \, dx \, dt \leq \sup_k \sqrt{\mathcal{J}_k(u_{0,k})} + M.
\]

Since \( \dot{u}_\eta \) is uniformly bounded in \( L^2(Q_T) \), passing to the limit for \( \eta \to 0 \) in (A.27), we have \( u_\eta \to u_k \) in \( L^2(Q_T) \). Moreover, as in [80, p. 351] or [59, p. 146], possibly passing to a subsequence, we still denote by \( u_\eta \), we have \( u_\eta \to u_k \) in \( H^1(0,T;L^2(\Omega)) \). Finally, using the lower semicontinuity of the norm of the space \( H^1(0,T;L^2(\Omega)) \), we get

\[
\int_{Q_T} |\dot{u}_k|^2 \, dx \, dt \leq \sup_k \sqrt{\mathcal{J}_k(u_{0,k})} + M.
\]

Finally, relation (A.18) allows us to conclude

\[
\sup_k \|\dot{u}_k\|_{L^2} < +\infty. \quad (A.30)
\]

Step 6. The sequence \( (\sqrt{\varepsilon_k}u_k)_k \) is uniformly bounded in \( L^2(0,T;H^1(\Omega)) \) and the sequence \( (u_k)_k \) is uniformly bounded \( L^2(0,T;W^{1,1}(\Omega)) \).

Take \( w = 0 \) in (A.22). By (A.25) and (A.26), there exist constants \( h_1, h_2 > 0 \), independent of \( k \), such that

\[
\mathcal{J}_k(u_k) = \int_\Omega \sqrt{1 + |\nabla u_k|^2} \, dx + \frac{\varepsilon_k}{2} \int_\Omega |\nabla u_k|^2 \, dx
\]

\[
\leq h_1 - \int_\Omega \dot{u}_k u_k \, dx
\]

\[
\leq h_1 + h_2 \|\dot{u}_k(t)\|_2
\]
a.e. in $]0,T[$. Then we have
\[
\int_0^T \left( \int \Omega |\nabla u_k| \, dx \right)^2 dt \leq \int_0^T (h_1 + h_2 \|\dot{u}_k(t)\|_2)^2 dt
\]
and
\[
\int_0^T \left( \int \Omega \varepsilon_k |\nabla u_k|^2 \, dx \right)^2 dt \leq \int_0^T (h_1 + h_2 \|\dot{u}_k(t)\|_2)^2 dt.
\]
Thus (A.30) yields to conclude that
\[
(u_k)_k \text{ is uniformly bounded in } L^2(0,T; W^{1,1}(\Omega))
\]
and
\[
(\sqrt{\varepsilon_k} u_k)_k \text{ is uniformly bounded in } L^2(0,T; H^1(\Omega)).
\]
Step 7. Convergence of the perturbed scheme. By (A.25) and (A.30), there exists $H > 0$ independent of $k$ such that
\[
\|\dot{u}_k\|_{L^2} + \|u_k\|_{\infty} \leq H
\]
and hence
\[
(u_k)_k \text{ is uniformly bounded in } H^1(0,T; L^2(\Omega)).
\]
Thus there exists a subsequence of $(u_k)_k$, we still denote by $(u_k)_k$, and
\[
u = L^2(0,T; W^{1,1}(\Omega))
\]
and
\[
(\sqrt{\varepsilon_k} u_k)_k \text{ is uniformly bounded in } L^2(0,T; H^1(\Omega)).
\]
Thus there exists a subsequence of $(u_k)_k$, we still denote by $(u_k)_k$, and $u \in L^2(Q_T)$ with $\dot{u} \in L^2(Q_T)$, such that
\[
\dot{u}_k \rightharpoonup \dot{u} \text{ in } L^2(Q_T)
\]
and
\[
u = L^2(0,T; W^{1,1}(\Omega))
\]
and hence, by (A.30) and (A.25), the Cauchy-Schwarz inequality implies that there is $K > 0$ such that
\[
\frac{1}{2} \|u_k(\cdot, \tau) - u_0\|_{L^2}^2 \leq K \tau^{1/2} + \frac{1}{2} \|u_{0,k} - u_0\|_{L^2}^2
\]
for every $\tau \in [0,T]$ and all $k$. The integration of the last inequality, with respect to $\tau$, yields
\[
\int_0^T \|u_k(\cdot, \tau) - u_0\|_{L^2}^2 \leq \frac{4K}{3} \tau^{3/2} + T \cdot \|u_{0,k} - u_0\|_{L^2}^2
\]
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for all \( t \in [0, T] \). By (A.32) we have

\[
  u_k \chi_{[0,t]} \rightharpoonup u \chi_{[0,t]} \quad \text{in} \quad L^2(Q_T)
\]

and hence it follows

\[
  \liminf_{k \to +\infty} \|u_k \chi_{[0,t]}\|_{L^2} \geq \|u \chi_{[0,t]}\|_{L^2}
\]

for all \( t \in [0, T] \). Using (A.13) and taking the inferior limit in both sides of (A.34), we conclude

\[
  \int_0^t \|u(x,\tau) - u_0\|_{L^2}^2 \leq \frac{4K_3}{3} t^{3/2},
\]

thus dividing both sides by \( t \), this relation yields \( u(0) = u_0 \).

Now we prove that

\[
  u_k(t) \rightharpoonup u(t) \quad \text{in} \quad L^2(\Omega)
\]

for a.e. \( t \in ]0, T[ \). Indeed, for all \( v \in L^2(\Omega) \) and a.e. \( t \in ]0, T[ \), we have

\[
  \int_0^t \int_\Omega \dot{u}_k(\tau) v \, dx \, d\tau = \int_\Omega (u_k(t) - u_{0,k})v \, dx
\]

and then

\[
  \int_0^t \int_\Omega \dot{u}(\tau) v \, dx \, d\tau = \lim_{k \to +\infty} \int_0^t \int_\Omega \dot{u}_k(\tau) v \, dx \, d\tau = \lim_{k \to +\infty} \int_\Omega (u_k(t) - u_{0,k})v \, dx
\]

\[
  = \lim_{k \to +\infty} \int_\Omega u_k(t)v \, dx - \int_\Omega u_0v \, dx.
\]

By (A.33), we conclude

\[
  \lim_{k \to +\infty} \int_\Omega u_k(t)v \, dx = \int_0^t \int_\Omega \dot{u}(\tau) v \, dx \, d\tau + \int_\Omega u_0v \, dx = \int_\Omega u(t)v \, dx
\]

for a.e. \( t \in ]0, T[ \).

Take any \( p \in [1, 1^*] \), with \( p \leq 2 \). Notice that \((u_k)_k\) is uniformly bounded in \( L^\infty(0, T; L^p(\Omega)) \cap L^1(0, T; W^{1,1}(\Omega)) \) and \((\dot{u}_k)_k\) is uniformly bounded in \( L^1(0, T; L^p(\Omega)) \). Since \( W^{1,1}(\Omega) \) is compactly embedded in \( L^p(\Omega) \), [100, Corollary 6] implies that \((u_k)_k\) is relatively compact in \( L^r(0, T; L^p(\Omega)) \) for each \( r \in [1, +\infty] \). Then we can extract a subsequence, we still denote by \((u_k)_k\), such that

\[
  u_k \to u \quad \text{in} \quad L^p(Q_T),
\]

\[
  |u_k(x,t)| \leq \ell(x,t) \quad \text{a.e. in} \quad Q_T, \quad \text{with} \quad \ell \in L^p(Q_T),
\]

\[
  u_k(x,t) \to u(x,t) \quad \text{a.e. in} \quad Q_T,
\]

\[
  u_k(t) \to u(t) \quad \text{in} \quad L^p(\Omega), \quad \text{for a.e.} \quad t \in ]0, T[,
\]

(A.36)
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\[ \int_{Q_T} |\nabla u_k| \, dx \, dt \leq C \quad \text{with } C > 0 \]

and by Proposition I.1.6, it follows \( u \in L^1(0, T; BV(\Omega)) \). Take any \( w \in L^2(0, T; H^1(\Omega)) \), integrating (A.22) between 0 and \( t \) with respect to \( \tau \), we get

\[ \int_0^t \int_\Omega \dot{u}_k(w - u_k) \, dx \, d\tau + \int_0^t J_k(w) \, d\tau \geq \int_0^t J_k(u_k) \, d\tau - c \int_0^t \int_\Omega u_k(w - u_k) \, dx \, d\tau \]

\[ + \int_0^t \int_\Omega \bar{g}(x, \tau, u_k)(w - u_k) \, dx \, d\tau. \quad \text{(A.37)} \]

First, let us recall that for a.e. \( t \in [0, T] \) we have

\[ \int_0^t \int_\Omega \dot{u}_k(w - u_k) \, dx \, d\tau = \int_0^t \int_\Omega \dot{u}_k w \, dx \, d\tau - \frac{1}{2} \int_\Omega |u_k(t)|^2 \, dx + \frac{1}{2} \int_\Omega |u_0| \, dx, \]

then by (A.31), (A.35) and (A.13), it follows

\[ \liminf_{k \to +\infty} \int_0^t \int_\Omega \dot{u}_k(w - u_k) \, dx \, d\tau \leq \int_0^t \int_\Omega \dot{u} w \, dx \, d\tau - \frac{1}{2} \int_\Omega |u(t)|^2 \, dx + \frac{1}{2} \int_\Omega |u_0| \, dx \]

\[ = \int_0^t \int_\Omega \dot{u}(w - u) \, dx \, d\tau. \quad \text{(A.38)} \]

By (A.17) we have

\[ \lim_{k \to +\infty} \int_0^t J_k(w) \, d\tau = \int_0^t J(w) \, d\tau \quad \text{(A.39)} \]

and applying Fatou’s Lemma, by (A.36), we can conclude

\[ \liminf_{k \to +\infty} \int_0^t J_k(u_k) \, d\tau \geq \liminf_{k \to +\infty} \int_0^t J(u_k) \, d\tau \geq \int_0^t J(u) \, d\tau. \quad \text{(A.40)} \]

By (A.11), (A.25) and Lebesgue convergence theorem we get

\[ \lim_{k \to +\infty} \int_0^t \int_\Omega \bar{g}(x, \tau, u_k)(w - u_k) \, dx \, d\tau = \int_0^t \int_\Omega \bar{g}(x, \tau, u)(w - u) \, dx \, d\tau \quad \text{(A.41)} \]

and moreover

\[ \lim_{k \to +\infty} c \int_0^t \int_\Omega u_k(w - u_k) \, dx \, d\tau = c \int_0^t \int_\Omega u(w - u) \, dx \, d\tau. \quad \text{(A.42)} \]

Finally, from (A.37), using (A.38), (A.39), (A.40), (A.41) and (A.42), we get for every \( t \in [0, T] \)

\[ \int_0^t \int_\Omega \dot{u}(w - u) \, dx \, d\tau + \int_0^t J(w) \, d\tau \geq \int_0^t J(u) \, d\tau - c \int_0^t \int_\Omega u(w - u) \, dx \, d\tau \]

\[ + \int_0^t \int_\Omega \bar{g}(x, \tau, u)(w - u) \, dx \, d\tau. \quad \text{(A.43)} \]
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for all \( w \in L^2(0, T; H^1(\Omega)) \).

Fix any \( v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T) \), by Lemma A.1.1, there exists a sequence \((v_n)_n\) in \( L^2(0, T; H^1(\Omega)) \) such that

\[
\sup_n \|v_n\|_{L^2} < +\infty \quad (A.44)
\]

and for a.e. \( t \in [0, T] \)

\[
\lim_{n \to +\infty} v_n(t) = v(t) \quad \text{in} \quad L^2(\Omega),
\]

\[
\lim_{n \to +\infty} \int_{\Omega} |\nabla v_n(t)| \, dx = \int_{\Omega} |Dv(t)|
\]

and

\[
\lim_{n \to +\infty} \mathcal{J}(v_n(t)) = \mathcal{J}(v(t)). \quad (A.45)
\]

By (A.44) there exists a subsequence, we still denote by \((v_n)_n\), such that

\[
v_n \rightharpoonup v \quad \text{in} \quad L^2(Q_T).
\]

Now, for each \( n \), the function \( v_n \) is an admissible test function in (A.43) and hence

\[
\int_0^t \int_{\Omega} \dot{u}(v_n - u) \, dx \, d\tau + \int_0^t \mathcal{J}(v_n) \, d\tau \geq \int_0^t \mathcal{J}(u) \, d\tau - c \int_0^t \int_{\Omega} u(v_n - u) \, dx \, d\tau
\]

\[
+ \int_0^t \int_{\Omega} \bar{g}(x, \tau, u)(v_n - u) \, dx \, d\tau. \quad (A.46)
\]

Using (A.45) and applying the Lebesgue convergence, from (A.46) we get

\[
\int_0^t \int_{\Omega} \dot{u}(v - u) \, dx \, d\tau + \int_0^t \mathcal{J}(v) \, d\tau \geq \int_0^t \mathcal{J}(u) \, d\tau - c \int_0^t \int_{\Omega} u(v - u) \, dx \, d\tau
\]

\[
+ \int_0^t \int_{\Omega} \bar{g}(x, \tau, u)(v - u) \, dx \, d\tau \quad (A.47)
\]

for all \( v \in L^1(0, T; BV(\Omega)) \cap L^2(Q_T) \). Hence \( u \) is a solution of problem (A.10).

**Step 8.** Any solution \( u \) of (A.10) satisfies \( \alpha \leq u \leq \beta \) and hen \( u \) is a solution of (A.1). Let us show that \( u \leq \beta \); by a similar argument we can prove that \( u \geq \alpha \). Take \( v = u \land \beta = u - (u - \beta)^+ \) as a test function in (A.47). By Proposition A.1.5, we obtain

\[
- \int_{\Omega} \dot{u}(u - \beta)^+ \, dx + \mathcal{J}(u \land \beta) - \mathcal{J}(u) \geq \int_{\Omega} (cu - \bar{g}(x, t, u))(u - \beta)^+ \, dx \quad (A.48)
\]

for a.e. \( t \in [0, T] \). Taking \( z = (u - \beta)^+ \) as a test function in (A.8), we have, as \( u \lor \beta = \beta + (u - \beta)^+ \),

\[
\int_{\Omega} \dot{\beta}(u - \beta)^+ \, dx + \mathcal{J}(u \lor \beta_j) - \mathcal{J}(\beta_j) \geq \int_{\Omega} f(x, t, \beta)(u - \beta)^+ \, dx \quad (A.49)
\]
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for a.e. $t \in ]0,T[$. Summing up (A.48) and (A.49) and using Proposition I.2.7, we find

$$0 \geq J(u \land \beta_j) + J(u \lor \beta_j) - J(\beta_j) - J(u)$$

$$\geq \int_\Omega (\dot{u} - \dot{\beta})(u - \beta)^+ \, dx + \int_\Omega (f(x, t, \beta) + cu - \bar{g}(x, t, u))(u - \beta)^+ \, dx.$$

Notice that, by (A.9), we get

$$\int_\Omega (f(x, t, \beta) + cu - \bar{g}(x, t, u))(u - \beta)^+ \, dx = \int_\Omega c(u - \beta)(u - \beta)^+ \, dx \geq 0$$

and, as shown in Lemma A.1.4, we have

$$\int_\Omega (\dot{u} - \dot{\beta})(u - \beta)^+ \, dx \geq 0.$$

The last two relations combined with

$$0 \geq \int_\Omega (\dot{u} - \dot{\beta})(u - \beta)^+ \, dx + \int_\Omega (f(x, t, \beta) + cu - \bar{g}(x, t, u))(u - \beta)^+ \, dx$$

imply $(u - \beta)^+ = 0$ and therefore $u \leq \beta$. As $\alpha \leq u \leq \beta$ we have $\bar{g}(x, t, u) = f(x, t, u) + cu$ and hence $u$ is a solution of (A.1). \qed
Bibliography


