

Resolution of the ideal sheaf of a generic union of conics in \mathbb{P}^3 : I

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ABSTRACT. *We work over an algebraically closed field \mathcal{K} of characteristic zero. Let Y be the generic union of $r \geq 2$ skew conics in $\mathbb{P}_{\mathcal{K}}^3$, \mathcal{I}_Y its ideal sheaf and v the least integer such that $h^0(\mathcal{I}_Y(v)) > 0$. We first establish a conjecture (concerning a maximal rank problem) which allows to compute, by a standard method, the minimal free resolution of \mathcal{I}_Y if $r \geq 5$ and $\frac{v(v+2)(v+3)}{12v+2} < r < \frac{(v+1)(v+2)(v+3)}{12v+6}$. At the second time, we give the first part of the proof of that conjecture.*

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1. Introduction

We work over an algebraically closed field \mathcal{K} of characteristic zero. We denote by \mathbb{P}^3 the projective space $\text{Proj}(\mathcal{K}[x_0, x_1, x_2, x_3])$ of dimension 3 over \mathcal{K} , and by \mathcal{O} its structural sheaf.

For $a \in \mathbb{N}, m \in \mathbb{Z}$, and for a coherent sheaf \mathcal{F} on \mathbb{P}^3 , we put:

$$a\mathcal{O}(m) = \underbrace{\mathcal{O}(m) \oplus \cdots \oplus \mathcal{O}(m)}_{a \text{ times}}, \mathcal{F}(m) = \mathcal{F} \otimes \mathcal{O}(m), h^i(\mathcal{F}(m)) = \dim_{\mathcal{K}} H^i(\mathcal{F}(m)).$$

It is well known (Hilbert's syzygies theorem) that the graded $\mathcal{K}[x_0, x_1, x_2, x_3]$ -module, $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(\mathcal{F}(n))$, has a minimal graded free resolution of length at most 4. After sheafifying, we get a minimal free resolution of \mathcal{F} :

$$0 \rightarrow \mathcal{E}_4 \rightarrow \mathcal{E}_3 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0, \quad (1)$$

where each \mathcal{E}_j is of the form $\bigoplus_{i=1}^{N_j} a_{ij} \mathcal{O}(-n_{ij})$, with $N_j, n_{ij}, a_{ij} \in \mathbb{N}$.

However, if one wants to get more information about the N_j 's, the n_{ij} 's and the a_{ij} 's, many problems arise, namely the postulation problem (see [2, 3,

9, 10] and references therein). So, one cannot always calculate completely that resolution.

Let v be the least integer such that $h^0(\mathcal{F}(v)) \neq 0$ and consider Conditions (C_1) , (C_2) and (C_3) below:

(C_1) \mathcal{F} is $v + 1$ -regular and $h^0(\mathcal{F}(k)) \cdot h^1(\mathcal{F}(k)) = 0$, for any $k \in \mathbb{Z}$,

(C_2) $h^0(\Omega \otimes \mathcal{F}(k + 1)) \cdot h^1(\Omega \otimes \mathcal{F}(k + 1)) = 0$, for any $k \in \mathbb{Z}$,

(C_3) $h^0(\Omega^* \otimes \mathcal{F}(k + 1)) \cdot h^1(\Omega^* \otimes \mathcal{F}(k + 1)) = 0$, for any $k \in \mathbb{Z}$,

where Ω (resp. Ω^*) is the cotangent bundle (resp. the tangent bundle) over \mathbb{P}^3 .

The following facts (illustrated in Proposition 2.6 for a particular case) are well known:

- If Conditions (C_1) and (C_2) are both satisfied with $h^1(\Omega \otimes \mathcal{F}(v + 1)) \neq 0$, then one knows exactly \mathcal{E}_0 , \mathcal{E}_1 and \mathcal{E}_2 in (1).
- If (C_1) and (C_2) are satisfied with $h^1(\Omega \otimes \mathcal{F}(v + 1)) = 0$, then we need Condition (C_3) to get our target.

In the case where \mathcal{F} is the ideal sheaf of a generic union of r skew lines in \mathbb{P}^3 , Condition (C_1) holds (see [7]). M. Idà proved ([9]) that Condition (C_2) holds also if $r \neq 4$. We do not know whether Condition (C_3) may be satisfied. So, the minimal free resolution of \mathcal{F} is well known, for infinitely many (but not for all) values of r .

Now, if \mathcal{F} is a general instanton bundle (with Chern classes $c_1 = 0$ and $c_2 > 0$), then (see [6, 13, 14]) Conditions (C_1) , (C_2) and (C_3) are all satisfied and we know completely the resolution of \mathcal{F} , without exception.

The case of a general stable bundle \mathcal{F} of rank two, on \mathbb{P}^3 (with $c_1 = -1$ and $c_2 = 2p \geq 6$), is not yet completely solved: Conditions (C_1) and (C_2) hold (see [6, 15]), but Condition (C_3) is not proved to be true.

In this paper, we are interested in the ideal sheaf \mathcal{I}_Y of the generic union $Y := Y_r$ of r skew conics in \mathbb{P}^3 , with $r \in \mathbb{N}^*$. E. Ballico showed ([2]) that Condition (C_1) holds if $r \geq 5$. We conjecture that Condition (C_2) would be also satisfied (see Conjecture 1.1) for any $r \in \mathbb{N}^*$, and we will give the first part of its proof.

Note that if $\mathcal{F} = \mathcal{I}_Y$, then (C_2) (resp. (C_3)) means that the natural (restriction) map $r_Y(n) : H^0(\Omega(n)) \rightarrow H^0(\Omega(n)|_Y)$ (resp. $r_Y^*(n) : H^0(\Omega^*(n)) \rightarrow H^0(\Omega^*(n)|_Y)$) has *maximal rank* (i.e., it is injective or surjective). So, we may establish our conjecture as:

CONJECTURE 1.1. *Let Y be the generic union of r skew conics in \mathbb{P}^3 , $r \in \mathbb{N}^*$, and let Ω be the cotangent bundle on \mathbb{P}^3 . Then for any integer n , the natural map from $H^0(\Omega(n))$ to $H^0(\Omega(n)|_Y)$ has maximal rank.*

We remark that (see Theorem 5.2 in [5], p. 228) there exists a positive integer n_0 (depending on Ω and Y) such that $h^1(\Omega(n) \otimes \mathcal{I}_Y) = 0$, for any $n \geq n_0$. Therefore, the restriction map $r_Y(n)$ is always surjective for any such n . We also get: $h^0(\Omega(n) \otimes \mathcal{I}_Y) = h^0(\Omega(n)) = 0$, for any $n \leq 1$. Our Conjecture is then true for $n \notin \{2, \dots, n_0 - 1\}$.

We give in Section 3, the main idea to prove such a maximal rank problem. But before that, we recall (Section 2) the standard method to get the minimal free resolution of \mathcal{I}_Y . Section 4 is devoted to notations, definitions and several results which are necessary to our (first part of the) proof in Section 5. Finally, we give in Section 6 some Maple programs which help us for computations.

2. Standard method

We adapt here the standard method to our situation where \mathcal{F} is the ideal sheaf \mathcal{I}_Y of the generic union Y of r skew conics in \mathbb{P}^3 . In this case, the form of the minimal free resolution of \mathcal{I}_Y is:

$$0 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{I}_Y \rightarrow 0 \tag{2}$$

where for $0 \leq j \leq 3$, $\mathcal{E}_j = \bigoplus_{i=1}^{N_j} a_{ij} \mathcal{O}(-n_{ij})$, with $N_j, n_{ij}, a_{ij} \in \mathbb{N}$.

We need the two following lemmata.

LEMMA 2.1. *i) For any $k \in \mathbb{N}$, one has:*

$$h^0(\mathcal{I}_Y(k)) - h^1(\mathcal{I}_Y(k)) = \binom{k+3}{3} - (2k+1)r, \quad h^2(\mathcal{I}_Y(k)) = h^3(\mathcal{I}_Y(k-3)) = 0.$$

ii) If $r \geq 5$ then:

- a) $h^0(\mathcal{I}_Y(k)) \cdot h^1(\mathcal{I}_Y(k)) = 0$ for any $k \in \mathbb{Z}$,
- b) $h^0(\mathcal{I}_Y(k)) = \max(0, \binom{k+3}{3} - (2k+1)r)$ for any $k \in \mathbb{Z}$,
- c) $v = \min\{m \in \mathbb{N} / \binom{m+3}{3} - (2m+1)r \geq 1\} \geq 5$,
- d) $h^1(\mathcal{I}_Y(v)) = 0, h^2(\mathcal{I}_Y(v-1)) = 0$ and $h^3(\mathcal{I}_Y(v-2)) = 0$.

Proof. i): consider cohomologies in the exact sequence:

$$0 \rightarrow \mathcal{I}_Y(l) \rightarrow \mathcal{O}(l) \rightarrow \mathcal{O}_Y(l) \rightarrow 0,$$

and remark that

$$h^2(\mathcal{I}_Y(l)) = h^1(\mathcal{O}_Y(l)) = r \cdot h^1(\mathcal{O}_{\mathbb{P}^1}(2l)) = r \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(-2l-2)) = 0 \text{ if } l \geq 0,$$

$$\text{and } h^3(\mathcal{I}_Y(l)) = h^3(\mathcal{O}(l)) = h^0(\mathcal{O}(-l-4)) = 0 \text{ if } l \geq -3.$$

ii): a) is obtained from [2]. Parts b), c) and d) immediately follow. □

Now, put $\mathbb{I}_k = H^0(\mathcal{I}_Y(k))$ and $\mathbb{I} = \bigoplus_{k \geq 0} \mathbb{I}_k$, the homogeneous ideal of Y .

We get by Castelnuovo-Mumford Lemma ([11, p. 99]) and by Lemma 2.1:

LEMMA 2.2. *If $r \geq 5$, the sheaf \mathcal{I}_Y is $v + 1$ -regular, $\mathbb{I}_k = (0)$ if $k < v$ and \mathbb{I} is generated by $\mathbb{I}_v \oplus \mathbb{I}_{v+1}$.*

As consequences, we know more about the minimal free resolution of \mathcal{I}_Y , for $r \geq 5$:

COROLLARY 2.3. (see [14] and [9, Proposition 7.2.1]) *If $r \geq 5$, then the \mathcal{O} -modules $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ involved in (2) are*

$$\begin{aligned}\mathcal{E}_0 &= \alpha_1 \mathcal{O}(-v) \oplus \beta_1 \mathcal{O}(-v-1), \\ \mathcal{E}_1 &= \alpha_2 \mathcal{O}(-v-1) \oplus \beta_2 \mathcal{O}(-v-2), \\ \mathcal{E}_2 &= \alpha_3 \mathcal{O}(-v-2) \oplus \beta_3 \mathcal{O}(-v-3),\end{aligned}$$

where

$$(\star) : \begin{cases} \alpha_1 = h^0(\mathcal{I}_Y(v)), \\ \beta_1 = h^1(\Omega \otimes \mathcal{I}_Y(v+1)), \\ \alpha_2 = h^0(\Omega \otimes \mathcal{I}_Y(v+1)), \\ \beta_2 = h^1(\Omega^* \otimes \mathcal{I}_Y(v-2)), \\ \alpha_3 = h^0(\Omega^* \otimes \mathcal{I}_Y(v-2)), \\ \beta_3 = h^1(\mathcal{I}_Y(v-1)), \\ \alpha_2 - \beta_1 = 4h^0(\mathcal{I}_Y(v)) - h^0(\mathcal{I}_Y(v+1)), \\ \alpha_3 - \beta_2 = \alpha_2 - \beta_1 - \beta_3 - \alpha_1 + 1, \text{ by considering ranks.} \end{cases}$$

COROLLARY 2.4. *We suppose that $r \geq 5$.*

- i) If $r_Y(v+1)$ has maximal rank, then \mathcal{E}_0 is completely known.*
- ii) If $r_Y(v+1)$ is injective but not surjective, then $\mathcal{E}_0, \mathcal{E}_1$ and \mathcal{E}_2 are completely known.*
- iii) If $r_Y(v+1)$ is surjective and if $r_Y^*(v-2)$ has maximal rank, then $\mathcal{E}_0, \mathcal{E}_1$ and \mathcal{E}_2 are completely known.*

Proof. i): The integers α_1 and β_3 are already known. We see that $r_Y(v+1)$ has maximal rank if and only if $\alpha_2 \beta_1 = 0$. We can precise the exact value of β_1 since $\alpha_2 - \beta_1 = 4h^0(\mathcal{I}_Y(v)) - h^0(\mathcal{I}_Y(v+1))$.

ii): In this case, $\beta_1 \neq 0$ and $\alpha_2 = 0$. Thus, by minimality, $\alpha_3 = 0$. We obtain β_2 from (\star) in Corollary 2.3.

iii): We get $\beta_1 = 0$ and $\alpha_3 \beta_2 = 0$. Again, (\star) gives the values of α_2, α_3 and β_2 . \square

Relations between r and v are given by

LEMMA 2.5. *i) One has:*

$$\frac{v(v+1)(v+2)}{12v-6} \leq r < \frac{(v+1)(v+2)(v+3)}{12v+6}.$$

ii) If $\alpha_2 \beta_1 = 0$, then

$$\beta_1 \neq 0 \iff \beta_1 > 0 \iff \frac{v(v+2)(v+3)}{12v+2} < r < \frac{(v+1)(v+2)(v+3)}{12v+6}.$$

Proof. i): One has, from Lemma 2.1:

$$\begin{aligned} h^0(\mathcal{I}_Y(v)) > 0, \quad h^1(\mathcal{I}_Y(v)) = 0, \quad h^0(\mathcal{I}_Y(v-1)) = 0 \text{ and } h^1(\mathcal{I}_Y(v-1)) \geq 0, \\ \binom{v+3}{3} - (2v+1)r = h^0(\mathcal{I}_Y(v)) - h^1(\mathcal{I}_Y(v)) = h^0(\mathcal{I}_Y(v)) > 0, \\ \binom{v+2}{3} - (2v-1)r = h^0(\mathcal{I}_Y(v-1)) - h^1(\mathcal{I}_Y(v-1)) = -h^1(\mathcal{I}_Y(v-1)) \leq 0. \end{aligned}$$

ii): $\beta_1 > 0$ and $\alpha_2 = 0$. Hence we get from (\star) in Corollary 2.3:

$$\begin{aligned} (6v+1)r - \frac{v(v+2)(v+3)}{2} &= \binom{v+4}{3} - (2v+3)r - 4\binom{v+3}{3} + 4(2v+1)r \\ &= h^0(\mathcal{I}_Y(v+1)) - 4h^0(\mathcal{I}_Y(v)) \\ &= \beta_1 > 0. \end{aligned}$$

□

Proposition 2.6 follows from Corollary 2.4 and Lemma 2.5.

PROPOSITION 2.6. *Let Y be the generic union of $r \geq 5$ skew conics in \mathbb{P}^3 . If $r_Y(v+1)$ has maximal rank and if $\frac{v(v+2)(v+3)}{12v+2} < r < \frac{(v+1)(v+2)(v+3)}{12v+6}$, then \mathcal{I}_Y has the following minimal free resolution:*

$$0 \rightarrow \beta_3 \mathcal{O}(-v-3) \rightarrow \beta_2 \mathcal{O}(-v-2) \rightarrow \alpha_1 \mathcal{O}(-v) \oplus \beta_1 \mathcal{O}(-v-1) \rightarrow \mathcal{I}_Y \rightarrow 0,$$

where:

$$\begin{cases} \alpha_1 = \frac{1}{6}(v+1)(v+2)(v+3) - (2v+1)r, \\ \beta_1 = (6v+1)r - \frac{1}{2}v(v+2)(v+3), \\ \beta_2 = (6v-1)r - \frac{1}{2}v(v+1)(v+3), \\ \beta_3 = (2v-1)r - \frac{1}{6}v(v+1)(v+2). \end{cases}$$

REMARK 2.7. i) The first twenty values of r and the corresponding values of v , for which Proposition 2.6 holds, are:

r	5	6	9	11	13	15	18	20	23	26
v	5	6	8	9	10	11	12	13	14	15

r	29	32	35	39	42	43	46	47	50	51
v	16	17	18	19	20	20	21	21	22	22

Hence, $v_1 = 5$ is the minimal value of n in Conjecture 1.1, that we shall consider. However, it is natural to treat also the case $n \leq 4$ (see Section 5.4).

- ii) If $r \in \{2, 3, 4\}$, then \mathcal{I}_Y does not satisfy Condition (C_1) in Section 1 (see [2]). So we cannot apply Proposition 2.6. In that case, the minimal free resolution of \mathcal{I}_Y would be obtained by direct (but delicate) computations. We will do it in the future.

3. How to prove Conjecture 1.1?

A maximal rank problem (depending on a natural number n) can be proved by using the so called *Horace method* (see Section 3.1 and [8]). It is an induction proof (on n) where each step requires more or less sophisticated conditions (equations and inequations satisfied by many integers), called *adjusting conditions* (see e.g. the hypotheses of Proposition 4.12). If n is sufficiently large, then those conditions are not difficult to realize, whereas for “small” values of n , one must verify them case by case: the *initial cases*. A priori, for each n (large or not), many complicated calculations arise (see e.g. [9] or [13]). So we often use Maple computations.

3.1. The Horace method (see [8])

We omit here to recall the notion of specialization of a subscheme (see e.g. [15, Section 3.1]).

Let E be a bundle on a quasi-projective scheme T and let Z be a subscheme of T . We consider the restriction map $\rho : H^0(E) \rightarrow H^0(E|_Z)$. We say that:

- Z is numerically E -settled if $h^0(E) = h^0(E|_Z)$,
- Z is E -settled if ρ has maximal rank.

If Δ is a Cartier divisor on T and Z_s is a specialization of Z , then we put:

$$Z'' = Z_s \cap \Delta \text{ (trace of } Z_s \text{ on } \Delta),$$

$Z' = \text{res}_\Delta Z_s$ (residual scheme: scheme such that its ideal sheaf is the kernel of the natural morphism $\mathcal{O} \rightarrow \mathcal{H}om(\mathcal{I}_{Z_s}, \mathcal{O}_\Delta)$).

From the residual exact sequence (cf. [8, p. 353]):

$$0 \rightarrow I_{Z'}(-\Delta) \rightarrow I_Z \rightarrow I_{Z'', \Delta} \rightarrow 0,$$

we get the following lemmata.

LEMMA 3.1. *If Z_s is numerically E -settled, then, Z' is numerically $E(-\Delta)$ -settled if and only if Z'' is numerically $E|_{\Delta}$ -settled.*

In this case, we say that Z_s is a (E, Δ) -adjusted specialization of Z .

LEMMA 3.2. *Let i be a natural number. If $h^i(E(-\Delta) \otimes I_{Z'}) = 0$ (condition called d egue) and if $h^i(E \otimes I_{Z'', \Delta}) = 0$ (d ime), then $h^i(E \otimes I_Z) = 0$.*

REMARK 3.3. i) We call *adjusting conditions*, the conditions for which, the specialization Z_s of Z is numerically E -settled.

ii) We say that one *exploits* a divisor if one applies the Horace method with it.

iii) Again, to prove the d egue and the d ime, we may apply the Horace method and so on... It leads, after a finite number of steps, to simpler statements, because for each “d egue”, the bundle degree decreases, and for each “d ime”, the subscheme dimension decreases.

3.2. A first step of the proof

Conjecture 1.1 says that, for any integer n , the natural map $r_Y(n)$ from $H^0(\Omega(n))$ to $H^0(\Omega(n)|_Y)$ has maximal rank, Ω being the cotangent bundle over \mathbb{P}^3 .

As mentioned at the end of Section 1, the map $r_Y(n)$ is injective if $n \leq 1$ and it is surjective if $n \geq n_0$, for some $n_0 \in \mathbb{N}^*$. It remains then the case: $2 \leq n \leq n_0 - 1$. For $2 \leq n \leq 4$, see Section 5.4.1. Now, we suppose that $n \geq 5$. We would like to apply exactly the idea described in [9]. We put:

$$\mathcal{X}^* = \mathbb{P}(\Omega), \quad L_n = \mathcal{O}_{\mathcal{X}^*}(1) \otimes \pi^* \mathcal{O}(n), \quad Y^* = \pi^{-1}(Y),$$

where $\pi : \mathcal{X}^* \rightarrow \mathbb{P}^3$ is the canonical projection.

We remark that L_n is a bundle of rank 1, so we may define (Section 5.1) a subscheme $T^*(n)$, not depending on r , contained in Y^* or containing Y^* , such that $h^0(L_n) = h^0(L_n|_{T^*(n)})$ (see [9, Section 1.1]).

Let $\rho_n : H^0(L_n) \rightarrow H^0(L_n|_{T^*(n)})$ be the restriction map. If ρ_n is bijective and if $Y^* \subset T^*(n)$ (resp. $Y^* \supset T^*(n)$), then $r_Y(n)$ is surjective (resp. injective). So, we get Conjecture 1.1. The bijectivity of ρ_n is equivalent to $H(n) : H^0(L_n \otimes \mathcal{I}_{T^*(n)}) = 0$, where $\mathcal{I}_{T^*(n)}$ is the ideal sheaf of $T^*(n)$.

The equality $H(n)$ is proved by using the *Horace method*. For that, we build another subscheme $T'^*(n)$ of \mathcal{X}^* such that $h^0(L_n) = h^0(L_n|_{T'^*(n)})$, in such a manner that if the natural map $\rho'_{n-2} : H^0(L_{n-2}) \rightarrow H^0(L_{n-2}|_{T'^*(n-2)})$ is bijective, then we get $H(n)$. We remark also that the bijectivity of ρ'_n is equivalent to $H'(n) : H^0(L_n \otimes \mathcal{I}_{T'^*(n)}) = 0$, and $H'(n)$ may be proved by the Horace method, and so on...

In Section 5, we define the schemes $T^*(n)$ and $T'^*(n-2)$ and we prove the implication: $H'(n-2) \Rightarrow H(n)$ for any $n \geq 5$. Unfortunately, contrary to what happened in [9] and [15], the statement $H'(n)$ is more difficult to prove because the adjusting conditions are more complicated. We shall try to look more carefully at this situation, in a forthcoming paper, in order to complete the proof of this Conjecture.

4. Preliminary results

In the rest of the paper, Q denotes a smooth quadric surface in \mathbb{P}^3 , Ω the cotangent bundle over \mathbb{P}^3 , $\bar{\Omega}$ the restriction of Ω on Q , $\mathcal{X}^* = \mathbb{P}(\Omega)$.

$\pi : \mathcal{X}^* \rightarrow \mathbb{P}^3$, $p_1, p_2 : Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ are the canonical projections.

We put: $C^* = \pi^{-1}(C)$ for a subscheme C of \mathbb{P}^3 , and for two integers a and b :

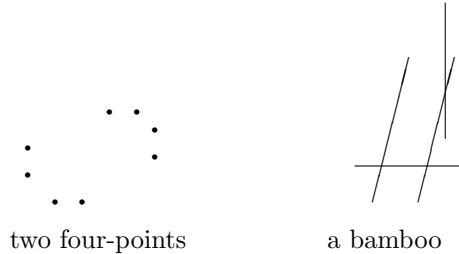
$$\mathcal{O}_Q(a, b) = p_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b), \quad \bar{\Omega}(a, b) = \Omega \otimes \mathcal{O}_Q(a, b), \quad \bar{\Omega}(a) = \bar{\Omega}(a, a), \\ K_{a,b} = \mathcal{O}_{Q^*}(1) \otimes \pi^* \mathcal{O}_Q(a, b), \quad K_a = K_{a,a}.$$

We denote by $[\frac{a}{b}]$ the quotient (by Euclidean division) of a by b , and by $\{\frac{a}{b}\}$ the remainder.

4.1. Definitions (see [1] and [9])

- A s-point is a point of \mathcal{X}^* .
- A d-point represents two s-points lying in a same fiber $\pi^{-1}(x)$, $x \in \mathbb{P}^3$.
- A t-point (resp. t-curve) represents three non-collinear points lying in a same fiber $\pi^{-1}(x)$ (resp. inverse image of a curve in \mathbb{P}^3 , under π).
- A grille of type (p, q) is a set of pq points of Q , which are the intersection of p lines of type $(1, 0)$ and q lines of type $(0, 1)$.
- A *four-point* is a set of 4 points, $[P] = \{P_1, \dots, P_4\} \subset Q$, such that $P_1, P_2 \in \ell \setminus \ell'$ and $P_3, P_4 \in \ell' \setminus \ell$, for some lines $\ell, \ell' \subset Q$ of type $(1, 0)$ and $(0, 1)$. In other words, P_1, \dots, P_4 are cocyclic but 3 by 3 non collinear. For example, the intersection of Q with a degenerate conic transverse to Q , such that the singular point does not lie on Q , is a four-point.
- A *bamboo* (see [2]) is a union of 4 lines L_1, \dots, L_4 such that: $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \leq 1$.
- the first infinitesimal neighborhood of a point x in \mathbb{P}^3 , denoted by $\xi(x)$, is the subscheme of \mathbb{P}^3 , having \mathcal{I}_x^2 as ideal sheaf.
- A triple-point (resp. double-point) is a subscheme of \mathbb{P}^3 , supported by a point having ideal locally defined by $(x_1, x_2)^2$ (resp. by (x_1^2, x_2)) in $\mathcal{K}[x_1, x_2]$. For example, $\xi(x) \cap Q$ is a triple-point of Q if $x \in Q$.
- A t-first infinitesimal neighborhood (resp. a t-grille) is the inverse image of a first infinitesimal neighborhood (resp. of a grille), under π .

- We say that t-points, d-points, and s-points are *collinear* (resp. *cocyclic*) in Q^* if their projections on Q lie on the same line (resp. same conic).



4.2. Examples of specialization

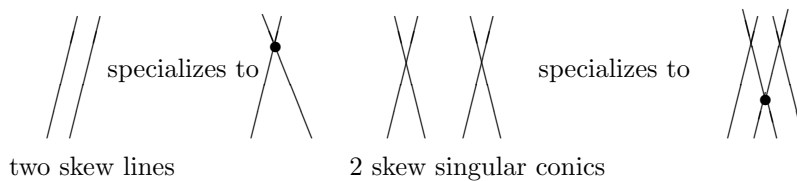
We give some specializations, traces and residual schemes which are useful in Sections 4.3, 5.2 and 5.4 (see also [15, Section 4.4]).

LEMMA 4.1. *i) The trace (resp. residual scheme) of a finite union of subschemes equals the union of traces (resp. of residual schemes).*

ii) If ℓ and ℓ' are two lines in Q , intersecting at the point x , then $\ell \cup \ell' \cup \xi(x)$ is a specialization of two skew lines in \mathbb{P}^3 . Moreover, the residual scheme $\text{res}_Q(\ell \cup \ell' \cup \xi(x))$ equals $\{x\}$.

iii) If $[L] = (L_1, \dots, L_4)$ is a bamboo and if $\{x\} = L_2 \cap L_3$, then the union $[L] \cup \xi(x)$ is a specialization of two skew conics in \mathbb{P}^3 .

Proof. i): see [8], 4.4. ii): see [7], 2.1.1. iii) follows from i) and ii). □



LEMMA 4.2. (see [15, Lemme 4.2])

i) If ℓ is a line and if $x \in \ell$, then $\ell \cap \xi(x)$ is a double-point and $\text{res}_\ell(\xi(x))$ is the (simple) point x .

ii) If C is a rational curve of type $(1, 2)$ on Q and if $x \in C$, then $C \cap \xi(x)$ is a double-point and $\text{res}_C(\xi(x))$ is the (simple) point x .

4.3. Lemmata on the quadric Q

First, we recall some general results which we can apply in Lemma 4.10 and in Proposition 4.12. Let E be a bundle on a quasi-projective scheme T and let Z be a subscheme of T . We denote by $\pi : \mathbb{P}(E) \rightarrow T$ the canonical projection. For a subscheme W of $\mathbb{P}(E)$, let $\pi(W)$ be the subscheme (of T), of ideal sheaf $\pi^{\#-1}(\pi_* I_W)$, where $\pi^{\#}$ is the canonical morphism from \mathcal{O}_T to $\pi_* \mathcal{O}_{\mathbb{P}(E)}$.

LEMMA 4.3. *One has:*

- i) $\pi^{-1}(Z) \cong \mathbb{P}(E|_Z)$, $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\pi^{-1}(Z)} \cong \mathcal{O}_{\mathbb{P}(E|_Z)}(1)$, $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(1)) \cong E$,
- ii) $\pi^* I_Z \cong I_{\pi^{-1}Z}$,
- iii) if W is a subscheme of $\mathbb{P}(E)$, then: $\pi_* I_W \cong I_{\pi(W)}$.

Proof. i): see [9, p. 21] and [5, Proposition 7.11, p.162].

ii): π has smooth fibers so the functor π^* is exact. Thus, it suffices to apply it, on the exact sequence: $0 \rightarrow I_Z \rightarrow \mathcal{O}_T \rightarrow \mathcal{O}_Z \rightarrow 0$ and to consider the exact sequence: $0 \rightarrow I_{\pi^{-1}Z} \rightarrow \mathcal{O}_{\mathbb{P}(E)} \rightarrow \mathcal{O}_{\pi^{-1}Z} \rightarrow 0$.

iii): since π is proper and has connected fibers, $\pi^{\#}$ is an isomorphism. Therefore, $\pi_* I_W \cong \pi^{\#-1}(\pi_* I_W) = I_{\pi(W)}$. \square

COROLLARY 4.4. *One has, for $n, a, b \in \mathbb{N}^*$ and for any subscheme C of Q :*

$$h^0(L_n) = h^0(\Omega(n)), \quad h^0(K_{a,b}) = h^0(\overline{\Omega}(a, b)), \quad h^0(K_{a,b}|_{\pi^{-1}(C)}) = h^0(\overline{\Omega}(a, b)|_C).$$

Proof. The projection formula (see [5], p. 124) and Lemma 4.3 give:

$$h^0(L_n) = h^0(\pi_*(L_n)) = h^0(\pi_*(\mathcal{O}_{\mathcal{X}^*}(1) \otimes \pi^* \mathcal{O}(n))) = h^0(\Omega(n)).$$

Similarly, we get: $h^0(K_{a,b}) = h^0(\overline{\Omega}(a, b))$ and $h^0(K_{a,b}|_{\pi^{-1}(C)}) = h^0(\overline{\Omega}(a, b)|_C)$. \square

LEMMA 4.5. (see [12, p. 8], [9, Section 3-1] and [4])

Let $n, a, b \in \mathbb{N}^$ and let C be a rational curve, of type $(1, n)$ on Q . Then*

- i) $h^0(\Omega(n)) = \frac{(n^2 - 1)(n + 2)}{2}$, $h^0(K_{a,b}) = h^0(\overline{\Omega}(a, b)) = 3ab - a - b - 1$.
- ii) $\overline{\Omega}(a, b)|_C \cong 2\mathcal{O}_{\mathbb{P}^1}((a - 1)n + b - 2) \oplus \mathcal{O}_{\mathbb{P}^1}((a - 2)n + b)$.

LEMMA 4.6. i) *If H is a plane in \mathbb{P}^3 , then $\Omega|_H \cong \Omega_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$.*

ii) *If D is a line in \mathbb{P}^3 , then $\Omega|_D \cong \overline{\Omega}|_D \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus 2\mathcal{O}_{\mathbb{P}^1}(-1)$.*

LEMMA 4.7. *Let $n, a, b, \tau, \delta, \epsilon \in \mathbb{N}$ and let D be a line on Q . Then*

- i) $h^0(K_{a,b|D^*}) = h^0(\overline{\Omega}(a,b)|_D) = h^0(\overline{\Omega}_D(b)) = 3b - 1$ if D is of type $(1,0)$.
- ii) $h^0(K_{a,b|D^*}) = h^0(\overline{\Omega}(a,b)|_D) = h^0(\overline{\Omega}_D(a)) = 3a - 1$ if D is of type $(0,1)$.
- iii) *If $S^* \subset \mathcal{X}^*$ is a union of τ t-points, δ d-point and ϵ s-point, then*
 $h^0(L_{n|S^*}) = 3\tau + 2\delta + \epsilon = h^0(K_{a,b|S^*})$.

LEMMA 4.8. *Let $a, b, a', b' \in \mathbb{N}$ and let C, C' be two distinct curves on Q , of type (a,b) and (a',b') . Then*

- i) $\#(C \cap C') = ab' + a'b$.
- ii) $\#(C \cap [P]) \leq 3$, $\#(C' \cap [P]) \leq 4$, $\#(\ell \cap [P]) \leq 2$, if C is of type $(1,2)$, C' of type $(1,1)$, ℓ of type $(1,0)$ and $[P]$ a four-point.

Corollary 4.4 and Lemma 4.5 imply:

COROLLARY 4.9. *Let $n, a, b \in \mathbb{N}^*$ and let C be a rational curve on Q . Then*

- i) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)|_C) = 3(a+b-2) + 1$ if C is of type $(1,1)$.
- ii) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)|_C) = 3(2a+b-3)$ if C is of type $(1,2)$.
- iii) $h^0(K_{a,b|C^*}) = h^0(\overline{\Omega}(a,b)|_C) = 3(2b+a-3)$ if C is of type $(2,1)$.

LEMMA 4.10. *In one of the following cases, Z is (numerically) E -settled.*

- a) $E = K_{a,b|C^*}$, where C is a conic in Q and Z the generic union, in C^* , of $a+b-2$ t-points (counted with multiplicity) and one s-point;
- b) $E = K_{a,b|\ell^*}$, where ℓ is a line in Q , of type $(1,0)$ and Z the generic union, in ℓ^* , of $b-1$ t-points (counted with multiplicity) and one d-point;
- c) $E = K_1$, $Z = \emptyset$;
- d) $E = K_2$, Z : a generic union of 2 t-points and 1 s-point;
- e) $E = K_3$, Z : a generic union of 6 t-points and 1 d-point, such that at most 3 are cocyclic with the d-point.

Proof. We see, from Lemma 4.7 and Corollary 4.9, that for each case, $h^0(E) = h^0(E|_Z)$. So, Z is numerically E -settled. It remains to show that $H^0(E \otimes I_Z) = 0$.

a) and b): see [9, p. 23-24].

c): it follows from the fact: $h^0(E) = h^0(\overline{\Omega}(1,1)) = 0$.

d): Z specializes to a union of 2 t-points and 1 s-point lying on a t-conic C^* . We exploit $\Delta = C^*$. The residual scheme Z' is the empty scheme and $E(-\Delta) = K_1$. Hence we get the dègue from c). The trace Z'' consists of 2 t-points and 1 s-point. Moreover, we get by Lemma 4.3: $E|_\Delta \cong K_{2|C^*}$. Thus,

the dîme follows from a).

e): We may get a specialization of Z by putting the 6 t-points on a t-curve C^* (C of type $(1, 2)$ on Q). We exploit C^* . The trace Z'' consists of 6 t-points and $E|_{C^*} \cong K_{3|C^*} \cong 3\mathcal{O}_{\mathbb{P}^1}(5)$. So the dîme is true.

The residual scheme Z' consists of 1 d-point and $E(-C^*) = K_{2,1}$. The dègue follows from b) and c), by exploiting a t-line ℓ^* of type $(1, 0)$ passing through the d-point. \square

REMARK 4.11. Proposition 4.12 is crucial in the proof of the statement in Section 5: $H'(n-2) \Rightarrow H(n)$. The following notations will be useful to show it. For $f, h, i, \ell \in \mathbb{N}$ such that $1 \leq i \leq 3$, $f = i + 3\ell$, set:

$$a = f + h, \quad b = f + 2h, \quad \mu_{\max}(a, b) = \begin{cases} a + b - 3 & \text{if } a + b \equiv 0 \pmod{3} \\ a + b - 2 & \text{if } a + b \not\equiv 0 \pmod{3} \end{cases}$$

$$V_{\max}(f, h) = \sum_{k=1}^{\ell} (v_{1k}(f) + v_{2k}(f)) + \sum_{k=1}^h v_k^*(f, h),$$

$$M_{\max}(f, h) = \sum_{k=1}^{\ell} (m_{1k}(f) + m_{2k}(f)) + \sum_{k=1}^h m_k^*(f, h).$$

The choice of the integers $v_{1k}(f), m_{1k}(f), v_{2k}(f), m_{2k}(f), \dots$ will allow us to exploit t-rational curves of type $(1, 2)$ and $(2, 1)$. We give below their different values.

- Case $f + h \leq 3$: $V_{\max}(f, h) = M_{\max}(f, h) = 0$.
- Case $4 \leq f \leq 6$ and $h = 0$: $v_{11} = i - 1, m_{11} = i, v_{21} = m_{21} = 0$.
- Case $f \geq 7$ and $h = 0$: $v_{11} = i - 1, m_{11} = i, v_{21} = m_{21} = 0$, and for $2 \leq k \leq \ell$, $v_{1k} = i + 3k - 4, m_{1k} = 3, v_{2k} = i + 3k - 6, m_{2k} = 3$, $V_{\max}(f, 0) = i - 1 + (\ell - 1)(f + i - 4)$, $M_{\max}(f, 0) = i + 6(\ell - 1) = 2f - i - 6$.
- Case $f = 1$ and $h \geq 3$: $v_k^* = 0, m_k^* = 2k - 2$ for $1 \leq k \leq h$, $V_{\max}(1, h) = 0$, $M_{\max}(1, h) = h(h - 1)$.
- Case $(f \geq 2$ and $h \geq 2)$ or $(f \geq 3$ and $h = 1)$: $v_k^* = f - 2, m_k^* = 2k$ for $1 \leq k \leq h$, $V_{\max}(f, h) = V_{\max}(f, 0) + (f - 2)h$, $M_{\max}(f, h) = M_{\max}(f, 0) + h(h + 1)$.

PROPOSITION 4.12. Let $f, h, a, b, v, m, u, \mu, \delta, \epsilon \in \mathbb{N}$ such that

$$\begin{cases} 1 \leq f \leq a = f + h \leq b = f + 2h < 2a, \\ \delta + \epsilon \leq 1, \\ v \leq V_{\max}(f, h), \\ m \leq M_{\max}(f, h), \\ \mu \leq \mu_{\max}(a, b), \\ 3ab - a - b - 1 = 12v + 9m + 3u + 3\mu + 2\delta + \epsilon. \end{cases}$$

We consider the generic union $F(a, b) \subset Q^*$ of m t -triple-points, v t -four-points, u t -points and μ t -points, δ d -point et ϵ s -point which are cocyclic. Then $F(a, b)$ is $K_{a,b}$ -settled.

Proof. By construction, $F(a, b)$ is numerically $K_{a,b}$ -settled:

$$h^0(K_{a,b}) = 3ab - a - b - 1 = 12v + 9m + 3u + 3\mu + 2\delta + \epsilon = h^0(K_{a,b}|_{F(a,b)}).$$

The proof is similar to that of Lemma 3.3.1 in [9]. Denote by $R(f, h)$ the statement: “the scheme $F(f + h, f + 2h)$ is $K_{f+h, f+2h}$ -settled”. The main idea is as follows.

If $h \geq 1$, then pass from $R(f, h)$ to $R(f, 0)$ by exploiting h times, a t -rational curve of type $(1, 2)$. In other words, prove $R(f, k)$ by induction on k , for $0 \leq k \leq h$.

Now, we have to prove $R(f, 0)$. If $f \geq 4$, then set $f = i + 3\ell$ where $\ell = \lfloor \frac{f-1}{3} \rfloor$, $i = f - 3\ell \in \{1, 2, 3\}$. Pass from $R(f, 0)$ to $R(i, 0)$ by exploiting alternately ℓ times, two t -rational curves of types $(1, 2)$ and $(2, 1)$. Here, we also use an inductive proof.

- *Proof of $R(i, 0)$ (case $a = b = f = i \in \{1, 2, 3\}$):*

One has: $M_{\max}(i, 0) = V_{\max}(i, 0) = 0$ and thus $m = v = 0$.

- The case $i = 1$ follows from Lemma 4.10-c).
- If $i = 2$, then $\epsilon = 1, \delta = 0, u + \mu = 2$. So, we may suppose that $u = 2$ and $\mu = 0$. Lemma 4.10-d) gives our result.
- If $i = 3$, then $\epsilon = 0, \delta = 1, u + \mu = 6, \mu \leq \mu_{\max}(3, 3) = 3$. $R(3, 0)$ is true by Lemma 4.10-e).

- *Proof of $R(f, 0)$, $f = i + 3\ell \geq 4$ (case $a = b = f \geq 4$):*

We denote by $\tilde{R}(i, k)$ the statement $R(i + 3k, 0)$, for $0 \leq k \leq \ell$. We prove it by induction on k . The case $k = 0$ corresponds to $f \in \{1, 2, 3\}$ and is just treated. We refer to Notations in Remark 4.11.

We suppose that $k \geq 1$ and $\tilde{R}(i, k - 1)$ is true. We denote by C' (resp. by Γ) a rational curve on Q , of type $(1, 2)$ (resp. the conic passing through the cocyclic t -points). Put $\tilde{f} = \tilde{f}_k = f - 3(\ell - k) = i + 3k$. We take $\mu_1 = \min(\mu, 3)$, $v_1 = \min(v, v_{1k}(f))$ and $m_1 = \min(m, m_{1k}(f))$. Let $u_1 \in \mathbb{N}$ such that $u_1 \leq u$ and

$$3v_1 + 2m_1 + u_1 + \mu_1 = 3\tilde{f} - 3. \tag{3}$$

We define the two following subschemes of Q^* , \tilde{F}_1 and \tilde{F}_2 as follows.

\tilde{F}_1 is the union of $v - v_1$ t -four-points, $m - m_1$ t -triple-points, $u - u_1 + v_1$ t -points and $\mu - \mu_1$ t -points, δ d -point and ϵ s -point which are cocyclic.

\tilde{F}_2 consists of $u_1 + 3v_1$ t -points lying on C'^* , μ_1 t -points on $C'^* \cap \Gamma^*$ and the t -infinitesimal neighborhoods of m_1 points on C' .

The two subschemes $F(\tilde{f}, \tilde{f})$ and $\tilde{F}_1 \cup \tilde{F}_2$ have the same number of t -triple-points: $m = (m - m_1) + m_1$. Moreover, the v_1 t -points of \tilde{F}_1 together with

the $3v_1$ t-points of \tilde{F}_2 form a specialization of v_1 t-four-points of $F(\tilde{f}, \tilde{f})$. It follows that $F(\tilde{f}, \tilde{f})$ generalizes $\tilde{F}_1 \cup \tilde{F}_2$.

We exploit C'^* . The trace Z'' consists of m_1 t-double-points and $3v_1 + u_1 + \mu_1$ t-points. Moreover, one has: $\pi_*(K_{\tilde{f}|C'^*}) \cong 3\mathcal{O}_{\mathbb{P}^1}(3\tilde{f} - 4)$. By Corollary 4.9 and Equality (3), Z'' is numerically $K_{\tilde{f}|C'^*}$ -settled:

$$h^0(K_{\tilde{f}|C'^*}) = h^0(\pi_*(K_{\tilde{f}|C'^*})) = 3(3\tilde{f} - 3) = 9v_1 + 6m_1 + 3u_1 + 3\mu_1 = h^0(K_{\tilde{f}|Z''}).$$

Hence, we get the dîme.

Now, we prove the dègue. One has $K_{\tilde{f}}(-C'^*) \cong K_{\tilde{f}-1, \tilde{f}-2}$. The residual scheme Z' is exactly the disjoint union of \tilde{F}_1 with m_1 t-points. By Lemma 3.1, it is numerically $K_{\tilde{f}-1, \tilde{f}-2}$ -settled:

$$3a'b' - a' - b' - 1 = 12(v - v_1) + 9(m - m_1) + 3u^* + 3(\mu - \mu_1) + 2\delta + \epsilon,$$

where $a' = \tilde{f} - 1$, $b' = \tilde{f} - 2$, $u^* = u - u_1 + m_1 + v_1$. Take

$$\mu_2 = \min(\mu - \mu_1, 3), v_2 = \min(v - v_1, v_{2k}(f)), m_2 = \min(m - m_1, m_{2k}(f)).$$

Let $u_2 \in \mathbb{N}$ such that $u_2 \leq u^*$ and

$$3v_2 + 2m_2 + u_2 + \mu_2 = 3\tilde{f} - 8. \quad (4)$$

Consider a rational curve C'' of type (2, 1) on Q . As above, Z' may specialize to the disjoint union of $F(\tilde{f} - 3, \tilde{f} - 3)$ with $u_2 + 3v_2$ t-points lying on C''^* , with μ_2 t-points on $C''^* \cap \Gamma^*$ and with the t-infinitesimal neighborhood of m_2 points on C'' . We exploit C''^* . The trace consists of m_2 t-double-points and $3v_2 + u_2 + \mu_2$ t-points. Since $\pi_*(K_{\tilde{f}-1, \tilde{f}-2|C''^*}) \cong 3\mathcal{O}_{\mathbb{P}^1}(3\tilde{f} - 9)$, Equality (4) implies the dîme.

The residual scheme is $F(\tilde{f} - 3, \tilde{f} - 3)$. By Lemma 3.1, it is numerically $K_{\tilde{f}-3, \tilde{f}-3}$ -settled: $3\tilde{a}\tilde{b} - \tilde{a} - \tilde{b} - 1 = 12\tilde{v} + 9\tilde{m} + 3\tilde{u} + 3\tilde{\mu} + 2\delta + \epsilon$, where

$$\begin{aligned} \tilde{v} &= v - v_1 - v_2 = \max(0, v - v_{1k}(f) - v_{2k}(f)) \leq V_{\max}(\tilde{f} - 3, 0), \\ \tilde{a} &= \tilde{b} = \tilde{f} - 3, \quad \tilde{m} = m - m_1 - m_2 \leq M_{\max}(\tilde{f} - 3, 0), \\ \tilde{\mu} &= \mu - \mu_1 - \mu_2 = \max(0, \mu - 6) \leq \mu_{\max}(\tilde{f}, \tilde{f}) - 6 = \mu_{\max}(\tilde{f} - 3, \tilde{f} - 3), \\ \tilde{u} &= u^* - u_2 + m_2 + v_2 = u - u_1 - u_2 + m_1 + v_1 + m_2 + v_2. \end{aligned}$$

Therefore, $\tilde{a}, \tilde{b}, \tilde{m}, \tilde{v}, \tilde{u}, \tilde{\mu}, \delta$ and ϵ satisfy all the hypotheses of Proposition 4.12. The dègue is the statement $\tilde{R}(i, k - 1)$. It is true by inductive assumption.

- *Proof of $R(f, h)$ (the general case):*

We necessarily have: $2 \leq a < b \leq 2a - 1$. We recall that $a = f + h$, $b = f + 2h$ where $f = 2a - b \geq 1$, $h = b - a \geq 1$, and $R(f, k)$ is the statement:

“ $F(f+k, f+2k)$ is $K_{f+k, f+2k}$ -settled”. We prove it by induction on k , for $0 \leq k \leq h$. The proof is similar to the previous one.

The case $k=0$ corresponds to $a=b=f$ and has been already done. We suppose that $k \geq 1$ and $R(f, k-1)$ is true. We denote by C' a rational curve on Q , of type $(1, 2)$. Set

$$\mu_1 = \min(\mu, 3), \quad v_1 = \min(v, v_k^*(f, h)), \quad m_1 = \min(m, m_k^*(f, h)).$$

Let $u_1 \in \mathbb{N}$ such that $u_1 \leq u$ and

$$3v_1 + 2m_1 + u_1 + \mu_1 = 3f + 4k - 3. \quad (5)$$

We consider the disjoint union \tilde{F} of $F(f+k-1, f+2k-2)$ with $u_1 + 3v_1 + \mu_1$ t-points lying on C'^* and the t-infinitesimal neighborhoods of m_1 points on C' . We see that \tilde{F} is a specialization of $F(f+k, f+2k)$. We exploit C'^* . The trace Z'' consists of m_1 t-double-points and $u_1 + 3v_1 + \mu_1$ t-points. Corollary 4.9 and Equality (5) give:

$$h^0(K_{f+k, f+2k|_{C'^*}}) = 3(3f+4k-3) = 9v_1 + 6m_1 + 3u_1 + 3\mu_1 = h^0(K_{f+k, f+2k|_{Z''}}).$$

Hence, Z'' is numerically $K_{f+k, f+2k|_{C'^*}}$ -settled and we get the dîme.

The residual scheme Z' is exactly $F(f+k-1, f+2k-2)$, $K_{f+k, f+2k}(-C'^*)$ is isomorphic to $K_{f+k-1, f+2k-2}$. Again, from Lemma 3.1, Z' is numerically $K_{f+k-1, f+2k-2}$ -settled. As before, we see that all the hypotheses of Proposition 4.12 are satisfied. The dègue is then true, by inductive assumption. \square

COROLLARY 4.13. *We consider the subscheme $F(a, b)$ of Proposition 4.12. Let $c, d_1, d_2, n \in \mathbb{N}^*$ and let G be the union, in Q^* , of c t-conics, d_1 t-lines of type $(1, 0)$ and d_2 t-lines of type $(0, 1)$, such that $G \cap F(a, b) = \emptyset$. We suppose that $J = G \cup F(a, b)$ is numerically K_n -settled and $a + c + d_1 = b + c + d_2 = n$. Then J is K_n -settled.*

Proof. Since any conic on Q is of type $(1, 1)$, we see that the ideal sheaf \mathcal{I}_G of G is isomorphic to $\pi^* \mathcal{O}_Q(-c - d_1, -c - d_2)$. Hence, we get: $H^0(K_n \otimes \mathcal{I}_J) = H^0(K_{a,b} \otimes \mathcal{I}_{F(a,b)}) = 0$ by Proposition 4.12. \square

5. Proof of $H'(n-2) \Rightarrow H(n)$, $n \geq 5$

5.1. The subscheme $T^*(n)$

We define $T^*(n)$ as the generic union of $\lambda(n)$ disjoint t-conics, and $\tau(n)$ t-points, $\delta(n)$ d-points, $\epsilon(n)$ s-point which are cocyclic. We see that:

$$\begin{aligned} T^*(n) \text{ is numerically } L_n\text{-settled} &\iff h^0(L_n) = h^0(L_n|_{T^*(n)}), \\ \text{if } S^* \text{ is a s-point (resp. d-point, t-point, t-line, t-conic, t-bamboo),} & \\ \text{then } h^0(L_n|_{S^*}) = 1 \text{ (resp. } 2, 3, 3n-1, 6n-5, 2(6n-5)-3 = 12n-13). & \end{aligned}$$

It follows that: $h^0(L_n|_{T^*(n)}) = \lambda(n)(6n - 5) + 3\tau(n) + 2\delta(n) + \epsilon(n)$.

Thus, in order to get $T^*(n)$ numerically L_n -settled, we may take:

$$\lambda(n) = \left\lceil \frac{h^0(L_n)}{6n - 5} \right\rceil, \quad \tau(n) = \left\lceil \frac{s(n)}{3} \right\rceil, \quad 2\delta(n) + \epsilon(n) = \left\lfloor \frac{s(n)}{3} \right\rfloor, \quad \delta(n), \epsilon(n) \in \{0, 1\},$$

$$\text{where: } h^0(L_n) = h^0(\Omega(n)) = \frac{(n^2 - 1)(n + 2)}{2}, \quad \text{and } s(n) = \left\lfloor \frac{h^0(L_n)}{6n - 5} \right\rfloor.$$

We must prove the statement $H(n) : H^0(L_n \otimes I_{T^*(n)}) = 0$ by the Horace method. We shall build a specialization $T_s(n)$ of $T^*(n)$ and show that $H^0(L_n \otimes I_{T_s(n)}) = 0$.

5.2. Specialization of $T^*(n)$ - The subscheme $T'^*(n - 2)$

We define $T_s(n)$ as a union of:

- s_1 t-conics in general position,
- s_2 t-bamboos,
- t_1 degenerate t-conics: one of the lines of each of them is contained in Q and is of type $(1, 0)$,
- t_2 degenerate t-conics: one of the lines of each of them is contained in Q and is of type $(0, 1)$,
- c t-conics in Q^* ;
- the t-first infinitesimal neighborhood (cf. 4.1) of $c^2 - c$ intersection points of c conics,
- the t-first infinitesimal neighborhood of s_2 triple-points, among the intersection points, with Q , of the s_2 bamboos,
- the t-first infinitesimal neighborhood of $t_1 t_2$ intersection points of $t_1 + t_2$ lines in Q ,
- the t-first infinitesimal neighborhood of $(t_1 + t_2)c$ intersection points, with c conics, of $t_1 + t_2$ lines,
- the t-first infinitesimal neighborhood of τ' cocyclic t-points, where $\tau' \leq \tau(n)$ and $\tau' \leq (t_1 + c) + (t_2 + c) = t_1 + t_2 + 2c$,
- $(\tau(n) - \tau')$ t-points, $\delta(n)$ d-point and $\epsilon(n)$ s-point lying on a t-conic in Q^* .

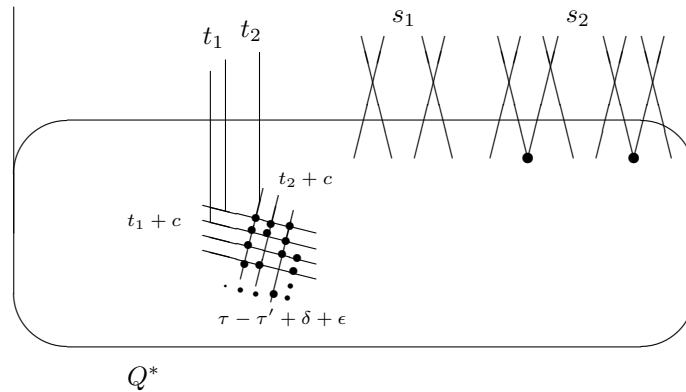
The integers $s_1, s_2, t_1, t_2, c, \tau', p_1, q_1$ are chosen in such a manner that the subscheme $T_s(n)$ is a (L_n, Q^*) -adjusted specialization of $T^*(n)$ (cf. Lemma 3.1). We may then use the Horace method by exploiting the divisor Q^* . In this case, we denote by $T'^*(n - 2)$ the residual scheme of $T_s(n)$. It consists of:

- s_1 disjoint t-conics, s_2 disjoint t-bamboos, $t_1 + t_2$ disjoint t-lines and
- $(t_1 + c)(t_2 + c) - c + \tau'$ t-points lying on a t-grille of type (p_1, q_1) .

Since $L_n(-Q^*) = L_{n-2}$, the (L_n, Q^*) -adjusting condition gives:

$$h^0(L_{n-2}) = (6n - 17)s_1 + (3n - 7)(t_1 + t_2) + (12n - 37)s_2 + 3[(t_1 + c)(t_2 + c) - c + \tau'].$$

We prove $H^0(L_n \otimes I_{T_s(n)}) = 0$. We exploit Q^* . The dègue is the statement $H'(n - 2) : H^0(L_{n-2} \otimes \mathcal{I}_{T'^*(n-2)}) = 0$, which is true by hypothesis.



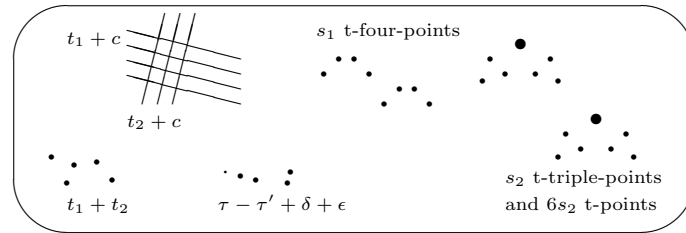
We prove now the dîme. We obtain the following facts:

- the s_1 t-conics of $T_s(n)$ meet Q^* in s_1 t-four-points,
- the s_2 t-bamboos meet Q^* in $6s_2$ t-points and in s_2 t-triple-points.

Thus, the trace $T_s(n) \cap Q^*$ is the subscheme J described in Corollary 4.13 with:

$$a = n - c - t_1, \quad b = n - c - t_2, \quad v = s_1, \quad m = s_2, \quad u = 6s_2 + t_1 + t_2, \quad \mu = \tau(n) - \tau'.$$

$T_s(n) \cap Q^*$:



Furthermore, $L_n|_{Q^*}$ is isomorphic to K_n and J is, by construction (see Lemma 3.1), numerically K_n -settled. One has: $h^0(K_n) = h^0(K_n|_J)$, which is equivalent to:

$$(E_1) : 3ab - a - b - 1 = 12s_1 + 27s_2 + 3(t_1 + t_2) + 3(\tau(n) - \tau') + 2\delta + \epsilon.$$

LEMMA 5.1. *If $t_1, t_2, c, a = n - c - t_1$ and $b = n - c - t_2$ satisfy Equation (E_1) , then $\tau(n) - \mu_{\max}(a, b) \leq t_1 + t_2 + 2c$.*

Proof. We know that $s(n) = \left\{ \frac{h^0(L_n)}{6n-5} \right\} \leq 6n - 6$ and $\tau(n) = \left[\frac{s(n)}{3} \right] \leq 2n - 2$. Moreover, one has: $\tau(n) = 2n - 2 \Rightarrow (\delta = \epsilon = 0) \Rightarrow (a + b \equiv 2 \pmod{3})$.

- If $a + b \equiv 0 \pmod{3}$, then

$$\tau(n) \leq 2n - 3 \text{ and } \tau(n) - \mu_{\max}(a, b) = \tau(n) - (a + b - 3) \leq t_1 + t_2 + 2c.$$

- If $a + b \not\equiv 0 \pmod{3}$, then

$$\tau(n) \leq 2n - 2 \text{ and } \tau(n) - \mu_{\max}(a, b) = \tau(n) - (a + b - 2) \leq t_1 + t_2 + 2c.$$

□

We suppose that $(a, b) \neq (1, 1)$. According to the hypotheses of Proposition 4.12, the integers $s_1, s_2, t_1, t_2, c, \tau', p_1, q_1, a, b, f, h, u, \mu$ must satisfy:

$$(\star\star) : \begin{cases} \lambda(n) = s_1 + 2s_2 + t_1 + t_2 + c, & t_1 \geq t_2 \\ a = n - c - t_1, & b = n - c - t_2 \\ 2 \leq a \leq b \leq 2a - 1, & h = b - a \geq 0, f = 2a - b \geq 1 \\ s_1 \leq V_{\max}(f, h), & s_2 \leq M_{\max}(f, h) \\ p_1 = c + t_1 \text{ if } \tau' = 0, & p_1 = c + t_1 + 1 \text{ otherwise} \\ q_1 = c + t_2 \text{ if } \tau' = 0, & q_1 = c + t_2 + 1 \text{ otherwise} \\ \tau(n) - \tau' \leq \mu_{\max}(a, b), & 0 \leq \tau' \leq \min(t_1 + t_2 + 2c, \tau(n)). \end{cases}$$

It remains then to prove the existence of s_1, s_2, t_1, \dots satisfying Equation (E_1) and Conditions $(\star\star)$ above.

5.3. Choice for the integers s_1, s_2, t_1, \dots

We would like to know the orders of magnitude of integers involved in the definitions of $T^*(n)$, $T_s(n)$ and of $T'^*(n-2)$, for sufficiently large values of n . We shall prove (Proposition 5.5) that we may take $n \geq 25$ but $n \notin \Lambda = \{26, 27, 30, 31, 33, 34, 37, 38, 43, 45, 48, 51, 55, 72\}$. For $2 \leq n \leq 24$ or for $n \in \Lambda$, see Section 5.4.

In the subscheme $T^*(n)$, four integers occur: $\lambda(n)$, $\tau(n)$, $\delta(n)$ and $\epsilon(n)$.

One has:

$$\lambda(n) = \left[\frac{h^0(L_n)}{6n-5} \right] \text{ with } h^0(L_n) = \frac{n^3 + 2n^2 - n - 2}{2}, \text{ so: } \lambda(n) \sim \frac{n^2}{12} + \frac{17n}{72},$$

$$\tau(n) = \left[\frac{s(n)}{3} \right] \text{ with } s(n) = \left\{ \frac{h^0(L_n)}{6n-5} \right\} < 6n - 5, \text{ so: } \tau(n) \leq 2n - 2,$$

$$2\delta(n) + \epsilon(n) = \left\{ \frac{s(n)}{3} \right\} \text{ with } 0 \leq \delta(n) + \epsilon(n) \leq 1.$$

In the subscheme $T_s(n)$, we must estimate five integers: s_1, s_2, t_1, t_2 and c . The adjusting condition gives:

$$3ab - a - b - 1 = 12s_1 + 27s_2 + \dots \text{ with } a = n - c - t_1, b = n - c - t_2.$$

We take: $12s_1 \sim 3ab \sim n^2$, $c \sim 2t_1 \sim \frac{n}{3}$, $0 \leq t_2 \leq 2$, since $\lambda(n) \sim \frac{n^2}{12} + \frac{17n}{72}$.
More precisely, we obtain

PROPOSITION 5.2. *The following integers, if they exist, satisfy Equation (E₁):*

$$\begin{aligned} t_1 &= \left[\frac{n}{6}\right] + \theta, \quad c = \left[\frac{n}{3}\right], \quad t_2 = \left\{ \frac{2n+1+s(n)-t_1-2c}{3} \right\} \\ 3s_2 &= \max(0, B(n, \theta), B(n, \theta) + 3(\tau(n) - \mu_{\max}(a, b))), \quad 3\tau' = 3s_2 - B(n, \theta), \\ s_1 &= \lambda(n) - t_1 - t_2 - c - 2s_2, \end{aligned}$$

where $a = n - c - t_1$, $b = n - c - t_2 \geq 5$ and

$$\begin{cases} \theta = 3 \left[\frac{\theta_1}{3}\right], \quad \theta_1 = \min\left(\left[\frac{3\tau(n) + A(n)}{3b-10}\right], \left[\frac{3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n)}{3b-13}\right]\right), \\ B(n, \theta) = A(n) - (3(n-t_2-c) - 10)\theta = A(n) - (3b-10)\theta, \\ A(n) = 3(n-c - \left[\frac{n}{6}\right])b - 2n - 1 - s(n) - 12\lambda(n) + 14c + 10\left[\frac{n}{6}\right] + 10t_2. \end{cases}$$

Moreover, one has: $\tau(n) - \tau' \leq \mu_{\max}(a, b)$, $0 \leq \tau' \leq \min(t_1 + t_2 + 2c, \tau(n))$.

Proof. By direct computations, since $a = n - t_1 - c$ and $b = n - t_2 - c$, Equation (E₁) may be written as (E₂): $B(n, \theta) - 3s_2 + 3\tau' = 0$. The choice of t_2 is due to the fact:

$$a + b + 1 + s(n) \equiv a + b + 1 + 2\delta + \epsilon \equiv 0 \pmod{3}.$$

It follows that: $A(n) \equiv 0 \pmod{3}$ and $B(n, \theta) \equiv -(3b-10)\theta \equiv \theta \pmod{3}$.
Conditions (**) and Equation (E₂) give:

$$\begin{aligned} B(n, \theta) - 3s_2 = -3\tau' &\leq 0, \quad B(n, \theta) - 3s_2 = -3\tau' \geq -3 \min(t_1 + t_2 + 2c, \tau(n)), \\ B(n, \theta) - 3s_2 = -3\tau' &\leq -3(\tau(n) - \mu_{\max}(a, b)). \end{aligned}$$

Thus, we must have:

$$\begin{aligned} 3s_2 &\geq B(n, \theta), \quad B(n, \theta) \equiv 0 \pmod{3}, \quad 0 \leq 3s_2 \leq 3\tau(n) + B(n, \theta), \\ 0 &\leq 3s_2 \leq 3(t_1 + t_2 + 2c) + B(n, \theta), \quad 3s_2 \geq 3(\tau(n) - \mu_{\max}(a, b)) + B(n, \theta). \end{aligned}$$

Since $t_1 = \left[\frac{n}{6}\right] + \theta$, θ satisfies:

$$3\tau(n) + A(n) \geq (3b-10)\theta, \quad 3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n) \geq (3b-13)\theta.$$

It suffices then to take:

$$\begin{aligned} \theta &= 3 \left[\frac{\theta_1}{3}\right], \quad 3s_2 = \max(0, B(n, \theta), 3(\tau(n) - \mu_{\max}(a, b)) + B(n, \theta)), \\ \tau' &= s_2 - \frac{1}{3} B(n, \theta), \\ \text{with } \theta_1 &= \min\left(\left[\frac{3\tau(n) + A(n)}{3b-10}\right], \left[\frac{3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n)}{3b-13}\right]\right) \text{ and } b \geq 5. \end{aligned}$$

Now, we check that: $\tau' \geq 0, \tau(n) - \tau' \leq \mu_{\max}(a, b), \tau' \leq \min(t_1 + t_2 + 2c, \tau(n))$.
The first two inequalities follow from the facts:

$$3s_2 \geq B(n, \theta) \text{ and } 3s_2 \geq 3(\tau(n) - \mu_{\max}(a, b)) + B(n, \theta).$$

It remains to prove the third one. Since $3b - 10 \geq 3b - 13 \geq 1$ and $\theta \leq \theta_1$, one has: $(3b - 10)\theta \leq 3\tau(n) + A(n), (3b - 13)\theta \leq 3(\lfloor \frac{n}{6} \rfloor + t_2 + 2c) + A(n)$.
Therefore, $B(n, \theta)$ satisfies: $3 \min(t_1 + t_2 + 2c, \tau(n)) \geq -B(n, \theta)$, because

$$\begin{aligned} 3(t_1 + t_2 + 2c) + A(n) &= 3(\theta + \lfloor \frac{n}{6} \rfloor + t_2 + 2c) + A(n) \geq (3b - 10)\theta, \\ 3\tau(n) + B(n, \theta) &\geq 0 \text{ and } 3(t_1 + t_2 + 2c) + B(n, \theta) \geq 0. \end{aligned}$$

- If $3s_2 = 0$, then $3\tau' = -B(n, \theta) \leq 3 \min(\tau(n), t_1 + t_2 + 2c)$.
- If $3s_2 = B(n, \theta)$, then $\tau' = 0 \leq \min(\tau(n), t_1 + t_2 + 2c)$.
- If $3s_2 = B(n, \theta) + 3(\tau(n) - \mu_{\max}(a, b))$, then

$$\begin{aligned} 3\tau' &= 3s_2 - B(n, \theta) = 3(\tau(n) - \mu_{\max}(a, b)) \leq 3\tau(n), \\ 3\tau' &= 3(\tau(n) - \mu_{\max}(a, b)) \leq 3(t_1 + t_2 + 2c) \text{ by Lemma 5.1.} \end{aligned}$$

□

Proposition 5.5 allows us to determine all values of n for which Equation (E_1) and Conditions $(\star\star)$ hold. We shall use the following results for its proof.

LEMMA 5.3. *We consider the natural numbers: $b, A(n), \theta_1, \theta$ and s_2 defined in Proposition 5.2. One has for $n \geq 68$:*

$$b \geq 5, -8n \leq A(n) \leq 6n, -5 \leq \theta_1 < 5, \theta \in \{-3, 0\} \text{ and } s_2 \leq 2n.$$

Proof. By standard bounding, we obtain:

$$\begin{cases} \frac{n}{3} - 1 \leq c < \frac{n}{3}, 0 \leq t_2 \leq 2, 0 \leq s(n) \leq 6n - 6, 0 \leq \tau(n) \leq 2n - 2, \\ \frac{2n}{3} - 2 \leq b = n - c - t_2 < \frac{2n}{3} + 1, \frac{n}{2} \leq a + \theta < \frac{n}{2} + 2, \\ \frac{n^2}{12} + \frac{17n}{72} - 2 \leq \lambda(n) \leq \lambda_{\max} = \frac{h^0(\bar{L}_n)}{6n - 5} \leq \frac{n^2}{12} + \frac{17n}{72} + 1. \end{cases} \quad (6)$$

So, $b \geq 5$ if $n \geq 11$. Set $n = 6\ell + w$ with $0 \leq w \leq 5$. Since $t_1 = \lfloor \frac{n}{6} \rfloor + \theta = \ell + \theta$, $t_2 \in \{0, 1, 2\}$ and $\lambda(n) = \frac{n^2}{12} + \frac{17n}{72} + \zeta$, for some $\zeta \in [-2, 1]$, simple calculations give: $-8n \leq A(n) \leq 6n$, because

$$A(n) = A_1(n) \text{ (resp. } A_1(n) - 21\ell - 6w + 3t_2 + 17) \text{ if } w \leq 2 \text{ (resp. if } w \geq 3),$$

$$\text{where } A_1(n) = (9w + 9 - 9t_2)\ell - s(n) - 12\zeta + 10t_2 + 2w^2 - \frac{29}{6}w - 3t_2 - 1.$$

By definition, $\theta = 3\left[\frac{\theta_1}{3}\right] \equiv 0 \pmod{3}$ and θ_1 satisfies:

$$\begin{aligned}\theta_1 &\leq \frac{3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n)}{3b - 13} \leq \frac{\left(\frac{5n}{2} + 6\right) + 6n}{3\left(\frac{2n}{3} - 2\right) - 13} = \frac{17n + 12}{4n - 38} < 5 \text{ if } n \geq 68, \\ \theta_1 &\geq \frac{A(n)}{3b - 10} \geq \frac{-8n}{3b - 10} \geq \frac{-8n}{3\left(\frac{2n}{3} - 2\right) - 10} = \frac{-8n}{2n - 16} \geq -5 \text{ if } n \geq 40.\end{aligned}$$

Hence, $\theta = 3\left[\frac{\theta_1}{3}\right] \leq \theta_1 < 5$, $\frac{\theta_1}{3} \geq -\frac{5}{3}$, and $\theta = 3\left[\frac{\theta_1}{3}\right] \geq 3 \times (-2) = -6$. We get $\theta \in \{-6, -3, 0, 3\}$. Now, we prove that $\theta \notin \{-6, 3\}$.

If $\theta = -6$, then $\theta_1 < -3$ so that $\theta_1 \leq -4$. Thus

$$\frac{3\tau(n) + A(n)}{3b - 10} < -3 \quad \text{or} \quad \frac{3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n)}{3b - 13} < -3.$$

i.e., $(3\tau(n) + A(n) + 9b - 30 < 0)$ or $(3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n) + 9b - 39 < 0)$.

It is impossible, if $n \geq 56$, by taking into account the above expressions of $A(n)$ and by the facts: $-2 \leq -s(n) + 3\tau(n) = -2\delta - \epsilon \leq 0$ and $b \sim 4\ell$. If $\theta = 3$, then $\theta_1 \geq 3$ and $\frac{3\tau(n) + A(n)}{3b - 10} \geq 3$. So, $3\tau(n) + A(n) - 9b + 30 \geq 0$, which is also impossible because $3\tau(n) + A(n)$ is at most of order 33ℓ but $-9b + 30 \sim -36\ell$.

It remains to prove that $s_2 \leq 2n$.

- If $\theta = -3$, then $(3\tau(n) + A(n) < 0)$ or $(3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n) < 0)$. Hence $A(n) < 0$ and $B(n, \theta) = A(n) + 3(3b - 10) < 9b - 30$.
 - If $\tau(n) - \mu_{\max}(a, b) \leq 0$, then $3s_2 = \max(0, B(n, \theta)) \leq 9b - 30 \leq 6n - 21$.
 - If $\tau(n) - \mu_{\max}(a, b) \geq 1$, then set $C(n, \theta) = B(n, \theta) + 3\tau(n) - 3\mu_{\max}(a, b)$. Note that $\mu_{\max}(a, b) \geq a + b - 3$ and from Inequalities (6), $a \geq \frac{n}{2} + 3$, $b \leq \frac{2n}{3} + 1$. If $A(n) + 3\tau(n) < 0$ then $C(n, \theta) \leq 9b - 30 - 3\mu_{\max}(a, b) \leq 6b - 21 \leq 4n$. If $A(n) + 3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) < 0$ then

$$C(n, \theta) \leq -3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + 9b - 30 + 3\tau(n) - 3\mu_{\max}(a, b) \leq 6n.$$

Thus, $3s_2 = \max(0, C(n, \theta)) \leq 6n$.

- If $\theta = 0$, then $B(n, \theta) = A(n) \leq 6n$ and $\theta_1 < 3$.
 - If $\tau(n) - \mu_{\max}(a, b) \leq 0$, then $3s_2 = \max(0, B(n, \theta)) \leq 6n$.
 - Now, we suppose that $\tau(n) - \mu_{\max}(a, b) \geq 1$. Since $\theta_1 < 3$, one has

$$3\tau(n) + A(n) < 3(3b - 10) \quad \text{or} \quad 3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + A(n) < 3(3b - 13).$$

Therefore $(C(n, \theta) \leq 9b - 30 - 3\mu_{\max}(a, b) \leq 6b - 21 \leq 4n)$

or $(C(n, \theta) \leq -3\left(\left[\frac{n}{6}\right] + t_2 + 2c\right) + 3(\tau(n) - \mu_{\max}(a, b)) + 9b - 39 \leq 6n)$.

So, $3s_2 = \max(0, C(n, \theta)) \leq 6n$. □

LEMMA 5.4. *Let a, b, c, t_1, t_2 be the integers defined in Proposition 5.2 and put $f = 2a - b = i + 3\ell$, $h = b - a$, $1 \leq i \leq 3$. Then for $n \geq 347$, one has:*

$$2n \leq M_{\max}(f, h) \text{ and } \frac{n^2}{12} - \frac{n}{4} + 9 \leq V_{\max}(f, h).$$

Proof. We get from Inequalities (6): $f \geq \frac{n}{3} - 1$ and $h \geq \frac{n}{6} - 6$. Thus $f, h \geq 2$ if $n \geq 48$ and

$$\begin{aligned} M_{\max}(f, h) &= 2f - i - 6 + h^2 + h \geq \frac{n^2}{36} - \frac{7n}{6} + 19 \geq 2n \text{ if } n \geq 108, \\ V_{\max}(f, h) &= i - 1 + (f + i - 4)(\ell - 1) + (f - 2)h \\ &\geq \frac{5n^2}{54} - \frac{7n}{2} + \frac{74}{3} \geq \frac{n^2}{12} - \frac{n}{4} + 9 \text{ if } n \geq 347. \end{aligned}$$

□

PROPOSITION 5.5. *If $n \geq 25$ and $n \notin \Lambda$, then the integers defined in Proposition 5.2 satisfy Equation (E_1) and Conditions $(\star\star)$.*

Proof. According to (the proof of) Proposition 5.2, it remains to prove, for such n , the existence of integers $s_1, s_2, t_1, t_2, c, \dots$ satisfying:

$$\begin{aligned} 5 \leq a = f + h \leq b = f + 2h < 2a, \quad s_1 \leq V_{\max}(f, h), \quad s_2 \leq M_{\max}(f, h), \\ \text{where } s_1 + 2s_2 + t_1 + t_2 + c = \lambda(n), \quad t_1 \geq t_2, \quad a = n - c - t_1, \quad b = n - c - t_2. \end{aligned}$$

From Inequalities (6) and from Lemmas 5.3 and 5.4, one has for $n \geq 347$:

$$\begin{aligned} \theta \in \{-3, 0\}, \quad \frac{n}{6} - 4 < t_1 \leq \frac{n}{6}, \quad h = b - a > \frac{n}{6} - 6 \geq 2, \quad 5 \leq \frac{n}{2} \leq a < \frac{n}{2} + 5, \\ 2 \leq \frac{n}{3} - 1 = 2\left(\frac{n}{2}\right) - \left(\frac{2n}{3} + 1\right) < f = 2a - b \leq \frac{n}{3} + 12, \quad s_2 \leq 2n \leq M_{\max}(f, h) \\ s_1 \leq \lambda_{\max} - \left(\frac{n}{6} - 7\right) - 0 - \left(\frac{n}{3} - 1\right) \leq \frac{n^2}{12} - \frac{n}{4} + 9 \leq V_{\max}(f, h). \end{aligned}$$

Conditions $(\star\star)$ are then satisfied, for any $n \geq 347$. By direct computations in Section 6.2, those conditions hold too, for $25 \leq n \leq 346$, except for $n \in \Lambda$. □

5.4. Initial cases

We recall that Y denotes the generic union of r skew conics, Q a smooth quadric surface in \mathbb{P}^3 , and Ω the cotangent bundle over \mathbb{P}^3 . In this section, we prove that:

- the map $r_Y(n) : H^0(\Omega(n)) \rightarrow H^0(\Omega(n)|_Y)$ has maximal rank if $2 \leq n \leq 4$,
- $H'(n - 2) \Rightarrow H(n)$ if $(5 \leq n \leq 24 \text{ or } n \in \Lambda)$.

5.4.1. Case $2 \leq n \leq 4$ • $n = 2$

We prove that $r_Y(2)$ is injective if $r = 1$, i.e., $h^0(\Omega(2) \otimes I_Y) = 0$ if Y is a conic. We exploit a plane H containing Y . The dègue: $h^0(\Omega(1)) = 0$, is satisfied. We obtain also the dîme: $h^0(\Omega(2)|_H \otimes I_Y) = 0$, since $h^0(\Omega(2)|_H \otimes I_Y) = h^0(\Omega(2)|_H \otimes \mathcal{O}_H(-2)) = h^0(\Omega|_H) = h^0(\Omega_H \oplus \mathcal{O}_H(-1)) = 0$. It follows that $r_Y(2)$ is injective for any $r \geq 1$.

• $n = 3$

We prove that $r_Y(3)$ is injective if $r = 2$ and it is surjective if $r = 1$.

Injectivity of $r_Y(3)$: $H^0(\Omega(3) \otimes I_Y) = 0$ if Y is a union of two skew conics. By Lemma 4.1, Y specializes to a union of two (non-disjoint) conics in Q with the infinitesimal neighborhood (in \mathbb{P}^3) of their two intersection points. One exploits Q . The residual scheme Y'' is exactly two points. Hence, we get the dègue: $H^0((\Omega(1) \otimes I_{Y''})) = 0$. The trace Y' is a union of two conics (a curve of type $(2, 2)$ in Q). So, the dîme: $H^0(\Omega(3)|_Q \otimes I_{Y'}) = 0$ is also satisfied because: $h^0(\Omega(3)|_Q \otimes I_{Y'}) = h^0(\overline{\Omega}(1)) = 0$.

Surjectivity of $r_Y(3)$: $H^1(\Omega(3) \otimes I_Y) = 0$ if Y is a conic. We may suppose that $Y \subset Q$ and we exploit Q . We obviously get the dègue: $H^1(\Omega(1)) = 0$. Now, to prove the dîme: $H^1(\Omega(3)|_Q \otimes I_Y) = 0$, we remark that the trace (Y itself) is a curve of type $(1, 1)$ on Q . Thus, $h^1(\Omega(3)|_Q \otimes I_Y) = h^1(\overline{\Omega}(2)) = 0$.

• $n = 4$

We prove that $r_Y(4)$ is injective (resp. surjective) if $r = 3$ (resp. $r = 2$).

Injectivity of $r_Y(4)$: $H^0(\Omega(4) \otimes I_Y) = 0$ if Y is a union of 3 skew conics. Y specializes to a union of 2 (non disjoint) conics of Q , with the infinitesimal neighborhood (in \mathbb{P}^3) of their two intersection points, and one conic not contained in Q . One exploits Q . The residual scheme is a union of one conic and two points. Therefore, the dègue: $H^0(\Omega(2) \otimes I_{Y''}) = 0$ is verified (see case $n = 2$). The trace Y' consists of two conics and four points. The dîme: $H^0(\Omega(4)|_Q \otimes I_{Y'}) = 0$ is then equivalent to: $H^0(\overline{\Omega}(2) \otimes I_Z) = 0$, where Z is the union of those 4 points. In order to prove: $H^0(\overline{\Omega}(2) \otimes I_Z) = 0$, we exploit a conic C in Q , containing these 4 points: the dègue is trivial. We get the dîme since: $h^0(\overline{\Omega}(2)|_C \otimes I_Z) = h^0((2\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \otimes I_Z) = h^0(2\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$.

Surjectivity of $r_Y(4)$: $H^1(\Omega(4) \otimes I_Y) = 0$ if Y is a union of 2 skew conics. One exploits a plane H containing one of the 2 conics. The residual schema Y'' is a conic and the dègue: $H^1(\Omega(3) \otimes I_{Y''}) = 0$ is satisfied (see case $n = 3$). The trace Y' is a union of one conic and 2 points. The dîme is equivalent to: $H^1(\Omega(2)|_H \otimes I_{Z'}) = 0$, where Z' consists of 2 points (of Y'). To prove the last equality, one exploits a line passing through those 2 points. The dîme and dègue are trivial.

n	λ	τ	$2\delta + \epsilon$	c	s_1	s_2	t_1	t_2	τ'
5	3	3	0	2	0	0	1	0	0
6	4	5	1	4	0	0	0	0	3
7	5	10	1	2	1	1	0	0	2
8	7	4	2	3	1	1	1	0	2
9	8	16	0	8	0	0	0	0	16
10	10	14	2	2	2	1	3	1	5
11	12	16	0	4	4	2	0	0	6
12	14	21	0	2	4	1	4	2	9
13	17	6	1	5	4	4	0	0	0
14	19	19	2	5	8	3	0	0	4
15	22	11	1	4	6	3	4	2	11
16	25	6	2	5	10	3	2	2	3
17	28	6	2	4	1	9	5	0	0
18	31	12	1	7	14	5	0	0	0
19	34	24	2	7	15	6	0	0	3
20	38	6	1	7	27	0	2	2	5
21	41	33	0	8	25	4	0	0	9
22	45	27	0	7	30	2	2	2	9
23	49	27	2	8	34	2	2	1	12
24	53	36	0	9	41	1	1	0	10

n	c	s_1	s_2	t_1	t_2	τ'
26	8	49	0	3	2	1
27	9	49	2	3	2	2
30	11	61	3	2	2	9
31	7	44	13	9	1	2
33	12	80	1	2	2	27
34	14	90	0	0	0	5
37	9	59	22	10	0	23
38	16	113	0	0	0	14
43	18	145	0	1	0	27
45	18	160	0	1	0	0
48	20	183	0	0	0	9
51	21	207	0	0	0	39
55	23	241	0	1	0	14
72	27	415	0	7	0	1

5.4.2. Proof of $H'(n-2) \Rightarrow H(n)$, $5 \leq n \leq 24$ or $n \in \Lambda$

We give here some tables of integers involved in $T^*(n)$, $T_s^*(n)$ and in $T'^*(n-2)$. For $n \neq 20$, these integers are chosen (by Maple computations) in order to satisfy Equation (E_1) and Conditions $(\star\star)$. For example, the first row of the table means that $T'^*(3)$ does not contain any t-conic ($s_1 = 0$), any t-bamboo ($s_2 = 0$). It consists of $t_1 + t_2 = 1$ t-line and $(c+t_1)(c+t_2) - c + \tau' = 3 \cdot 2 - 2 + 0 = 4$ t-points on a t-grille of type (p_1, q_1) , where $p_1 = c + t_1 = 3$, $q_1 = c + t_2 = 2$. Note that, for each n , the corresponding 6-tuple $(c, s_1, s_2, t_1, t_2, \tau')$ may not be unique.

If $n = 20$, then one has $a = b = f = 11 = i + 3\ell$, $i = 2$, $\ell = 3$, $s_2 = 0$, $\mu = 1$. We cannot apply Proposition 4.12, since $v = s_1 = 27 > 19 = V_{\max}(f, 0)$.

However, by exploiting alternately 3 times t -rational curves of type $(1, 2)$ and $(2, 1)$ as in the proof of Proposition 4.12, and by taking $v_{11} = 10, v_{21} = 8, v_{12} = 7, v_{22} = 2, v_{13} = v_{23} = 0$, and $m_{1k} = m_{2k} = 0$ for any $1 \leq k \leq 3$, we see that the corresponding subscheme $F(a, b)$ is $K_{a,b}$ -settled.

6. Some Maple Programs

We give here the integers defined in Section 5.1 and in Remark 4.11:

$$f(n) = h^0(L_n) = h^0(\Omega(n)), \quad g(n) = h^0(L_n|_{C^*}) = h^0(\Omega(n)|_C) \text{ with } C \text{ a conic,}$$

$$\lambda(n), \quad s(n), \quad \tau(n), \quad \Delta(n) = 2\delta(n) + \epsilon(n), \quad \mu_{\max}(a, b), \quad V_{\max}(f, h), \quad M_{\max}(f, h).$$

restart:

```
f:=proc(n) (n**2-1)*(n+2)/2;end;
g:=proc(n) (6*n-5);end;
```

```
lambda:=proc(n) iquo(f(n),g(n));end;
s:=proc(n) irem(f(n),g(n));end;
tau:=proc(n) iquo(s(n),3);end;
Delta:=proc(n) irem(s(n),3);end;
```

```
mumax:=proc(a,b) if irem(a+b,3)=0 then a+b-3;else a+b-2;fi;end;
```

```
Vmax0:=proc(f) ell:=iquo(f-1,3):ii:=f-3*ell:if f<=3 then 0;
else if f<=6 then ii-1;else ii-1+(ell-1)*(f+ii-4);fi;fi;end;
Vmax:=proc(f,h) ell:=iquo(f-1,3):ii:=f-3*ell:if f+h<=3 then 0;
else if (f=1 and h>= 3) then 0; else Vmax0(f)+(f-2)*h; fi;fi;end;
```

```
Mmax0:=proc(f) ell:=iquo(f-1,3):ii:=f-3*ell: if f <= 3 then 0;
else if f<=6 then ii;else 2*f-ii-6;fi;fi;end;
Mmax:=proc(f,h) ell:=iquo(f-1,3):ii:=f-3*ell:if f+h<=3 then 0;
else if (f=1 and h >= 3) then h*(h-1); else Mmax0(f)+h*(h+1);
fi;fi;end;
```

6.1. Program 1

The function $List1(n)$ returns the list of $n, \lambda(n), \tau(n), \Delta(n), c, s_1, s_2, \dots$ if they satisfy Conditions $(\star\star)$ and Equation (E_1) in Section 5.2. It returns “impossible” if they do not. Note also that `EQUAL` is exactly Equation (E_1) .

```
List1:=proc(n) c:=iquo(n,3):t2:=irem(2*n+1+s(n)-iquo(n,6)-2*c,3):
b:=n-c-t2:lamb:=lambda(n):
A:=3*b*(n-c-iquo(n,6))-2*n-1-s(n)-12*lamb+14*c
+10*iquo(n,6)+10*t2:
```

```

theta1:=min(floor((3*tau(n)+A)/(3*b-10)),
floor((3*(iquo(n,6)+t2+2*c)+A)/(3*b-13))):
theta:=3*floor(theta1/3):
t1:=iquo(n,6)+theta:
a:=n-c-t1:
ef:=2*a-b:hh:=b-a:
iji:=ef-3*iquo(ef-1,3):
MUMAX:=mumax(a,b):
Bntheta:=A-(3*b-10)*theta:
troissdeux:=max(0,Bntheta,Bntheta+3*(tau(n)-MUMAX)):
s2:=troissdeux/3:s1:=lamb-2*s2-t1-t2-c:tauprim:=s2-Bntheta/3:
uu:=6*s2+t1+t2:muu:=tau(n)-tauprim:
EQUA1:=3*a*b-a-b-1-(12*s1+9*s2+3*uu+3*muu+Delta(n)):
VEmax:=Vmax(ef,hh):EMmax:=Mmax(ef,hh):
if EQUA1 = 0 and a <= b and b < 2*a and muu >= 0 and s1 >= 0 and
tauprim <= t1+t2+2*c and muu <= MUMAX and s2 <= EMmax and
s1 <= VEmax then [ene=n,lambdaa=lamb,TAU=tau(n),Deltaa=Delta(n),
C=c,es1=s1,es2=s2,te1=t1,te2=t2,Tauprime=tauprim,Theta=theta];
else impossible;fi;end;

```

6.2. Program 2

List2 returns the list of integers $n \in \{5, \dots, 346\}$ for which Equation (E_1) and Conditions ($\star\star$) are not satisfied. We see that it contains only integers n such that $5 \leq n \leq 24$ or $n \in \Lambda$.

```

ll:={}:for n from 5 to 346 do if evalb(List1(n)=impossible) then
ll:={op(ll),n};fi;od>List2=ll;

```

```

List2 = {5,6,7,8,9,10,12,15,16,17,18,19,20,21,22,23,24,
26,27,30,31,33,34,37,38,43,45,48,51,55,72}

```

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