PRICING THE GUARANTEED LIFETIME WITHDRAWAL BENEFIT (GLWB) IN A VARIABLE ANNUITY CONTRACT

(Settore scientifico-disciplinare: SECS-S/06)

DOTTORANDA
MARIANGELA SCORRANO

COORDINATORE
PROF. GIANNI BOSI

SUPERVISORE DI TESI
PROF.SSA ANNA RITA BACINELLO

CO-SUPERVISORE DI TESI
DOTT. MASSIMILIANO KAUCIC

ANNO ACCADEMICO 2013/2014
Contents

Introduction v

1 Variable Annuity products 1

1.1 Introduction to annuities . . . . . . . . . . . . . . . . . . . . . . . . . 1
1.2 History and development of the Annuity market . . . . . . . . . . . . 6
1.3 Variable annuity products: the GMxB features . . . . . . . . . . . . 8
1.4 Risks underlying Variable Annuities . . . . . . . . . . . . . . . . . . 21
1.5 Variable annuities during the recent market Crisis . . . . . . . . . . . 26
1.6 Variable annuities around the world . . . . . . . . . . . . . . . . . . 30

1.6.1 U.S.A . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30
1.6.2 Japan . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 36
1.6.3 Europe . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 41
1.7 A brief literature review . . . . . . . . . . . . . . . . . . . . . . . . . 44

2 The GLWB option: the valuation model 47

2.1 The structure of the contract . . . . . . . . . . . . . . . . . . . . . . . 47
2.2 The valuation model . . . . . . . . . . . . . . . . . . . . . . . . . . . 50

2.2.1 The financial market . . . . . . . . . . . . . . . . . . . . . . . . 51
2.2.2 The mortality model . . . . . . . . . . . . . . . . . . . . . . . . 56
2.2.3 The combined model . . . . . . . . . . . . . . . . . . . . . . . . 60
2.2.4 The valuation formula: two valuation perspectives . . . . . . . 61
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>The deterministic model: numerical results</td>
<td>67</td>
</tr>
<tr>
<td>3.1</td>
<td>Numerical results</td>
<td>67</td>
</tr>
<tr>
<td>4</td>
<td>Generalization of the pricing model: stochastic interest rate</td>
<td>83</td>
</tr>
<tr>
<td>4.1</td>
<td>Stochastic interest rates models</td>
<td>83</td>
</tr>
<tr>
<td>4.2</td>
<td>The CIR model</td>
<td>85</td>
</tr>
<tr>
<td>4.3</td>
<td>Numerical results</td>
<td>87</td>
</tr>
<tr>
<td>5</td>
<td>Generalization of the pricing model: stochastic volatility</td>
<td>97</td>
</tr>
<tr>
<td>5.1</td>
<td>Stochastic volatility models</td>
<td>98</td>
</tr>
<tr>
<td>5.2</td>
<td>Numerical results</td>
<td>100</td>
</tr>
<tr>
<td>6</td>
<td>The multi-factor model</td>
<td>107</td>
</tr>
<tr>
<td>6.1</td>
<td>Multi-factor models</td>
<td>107</td>
</tr>
<tr>
<td>6.2</td>
<td>Heston-CIR hybrid pricing model</td>
<td>107</td>
</tr>
<tr>
<td>7</td>
<td>Conclusion</td>
<td>115</td>
</tr>
<tr>
<td>A</td>
<td>An introduction to Stochastic Calculus</td>
<td>117</td>
</tr>
<tr>
<td>A.1</td>
<td>Preliminary notions</td>
<td>117</td>
</tr>
<tr>
<td>A.2</td>
<td>Stochastic Processes</td>
<td>119</td>
</tr>
<tr>
<td>A.2.1</td>
<td>The Wiener process</td>
<td>121</td>
</tr>
<tr>
<td>A.3</td>
<td>Stochastic Differential Equations</td>
<td>122</td>
</tr>
<tr>
<td>A.4</td>
<td>Numerical approaches to SDEs</td>
<td>124</td>
</tr>
<tr>
<td>A.4.1</td>
<td>Time discrete approximations</td>
<td>125</td>
</tr>
<tr>
<td>A.4.2</td>
<td>Convergence of SDE solvers</td>
<td>127</td>
</tr>
</tbody>
</table>

Bibliography                                     133
Acknowledgements

This thesis has been submitted to the University of Trieste in fulfillment of the requirements for the Doctoral degree in “Insurance and Finance: Mathematics and Management” at the Department of Economics, Business, Mathematical and Statistical Sciences “Bruno De Finetti”. I want to start expressing a sincere acknowledgement to my supervisors Prof.ssa Anna Rita Bacinello and Dott. Massimiliano Kaucic, for their guidance and encouragement. Special thanks go to Prof. Gianni Bosi, and to his predecessor, Prof. Vicig Paolo, who provided me with the opportunity to embark on this journey. Finally, a special thanks to my parents for their support, patience and love, for giving me the possibility to study.
Introduction

The purpose of life insurance is to provide financial security to policyholders and their families. Traditionally, this security has been guaranteed through a lump sum payable according to the death or survival of the insured. Against the payment of one or more premiums for the duration of the contract, the policyholder was entitled to the insured sum. These annuities used to provide policyholders with a good return in bull markets, since the guaranteed amount was determined by the policyholder’s age and the level of interest rates. However, in the last decades, as interest rates declined, these contracts became less appealing. Insurance markets around the world have begun to change. Increasing life expectancy, as well as reduction of state retirement pensions in several countries have led to the rapid growth of new needs among consumers, and to the subsequent introduction of new products. The public has become more aware of investment opportunities outside the insurance sector and is increasingly trying to seize all the benefits of equity investment in conjunction with mortality protection. Over the last years, the competition with alternative investment vehicles offered by the financial industry has generated substantial innovation in the design of life products and in the range of provided benefits. In particular, equity-linked policies have become ever more popular, exposing policyholders to financial markets and providing them with different ways to consolidate investment performance over time as well as protection against mortality-related risks. Interesting examples of such contracts are variable annuities. First introduced in 1952 in the United States, this kind of policy experienced remarkable growth in Europe, especially during the last decade, characterized by bearish financial markets.
and relatively low interest rates. The success of these contracts is due to the presence of tax incentives, but mainly to the possibility of underwriting several rider benefits that provide protection of the policyholder’s savings for the period before and after retirement. These forms of guarantees fall in two main categories: living benefits and death benefits. In particular, the provision of more and more attractive living benefits, in order to meet consumers’ new needs, has been an important factor in the success of variable annuities.

In this thesis, we focus on the Guaranteed Lifetime Withdrawal Benefit (GLWB) rider. This option offers the security of guaranteed capital and covers the longevity risk, while providing flexible payout options. Moreover, it enables policyholders to take advantage of continuous participation in the value appreciation of the fund investments (by means of “ratchet” mechanisms, which link the increase of the account value with the one in the guaranteed amounts). For the affluent baby-boomers who are now approaching retirement, such variable annuities have been particularly appealing as protection against market and longevity risks, when making the transition from the accumulation to the decumulation phase.

In this thesis, we propose a valuation model for the policy using tractable financial and stochastic mortality processes in a continuous time framework. We analyze the policy considering two points of view, the policyholder’s and the insurer’s, and assuming a static approach, in which policyholders each year withdraw just the guaranteed amount. In particular, we have based ourselves on the model proposed in the paper “Systematic mortality risk: an analysis of guaranteed lifetime withdrawal benefits in variable annuities” by M. C. Fung, K. Ignatieva and M. Sherris (2014), with the aim of generalizing it later on. The valuation, indeed, is performed in a Black and Scholes economy: the sub-account value is assumed to follow a geometric Brownian motion, thus with a constant volatility, and the term structure of interest rates is assumed to be constant. These hypotheses, however, do not find justification in the financial markets. In order to consider a model better reflecting the market, we seek to weaken these misspecifications with the introduction of a stochastic process for the term structure of interest rates and for the volatility of the underlying account. We
address these two hypotheses separately at first, and then afterwards. As part of our analysis, we implement the theoretical model using a Monte Carlo approach. To this end, we have created ad hoc codes based on the programming language MATLAB, exploiting its fast matrix-computation facilities.

The work is organized as follows.

Chapter 1. This chapter has an introductory purpose and aims at presenting the basic structures of annuities in general and of variable annuities in particular. We offer an historical review of the development of the VA contracts and describe the embedded guarantees. We examine the main life insurance markets in order to highlight the international developments of VAs and their growth potential. In the last part we retrace the main academic contributions on the topic.

Chapter 2. Among the embedded guarantees, we focus in particular on the Guaranteed Lifetime Withdrawal Benefit (GLWB) rider. We analyze a valuation model for the policy basing ourselves on the one proposed by M. Sherris (2014). We introduce the two components of the model: the financial market, on the one hand, and the mortality intensity on the other. We first describe them separately, and subsequently we combine them into the insurance market model. In the second part of the chapter we describe the valuation formula considering the GLWB from two perspectives, the policyholder’s and the insurer’s.

Chapter 3. Here we implement the theoretical model creating ad hoc codes with the programming language MATLAB. Our numerical experiments use a Monte Carlo approach: random variables have been simulated by MATLAB high level random number generator, whereas concerning the approximation of expected values, scenario-based averages have been evaluated by exploiting MATLAB fast matrix-computation facilities. Sensitivity analyses are conducted in order to investigate the relation between the fair fee rate and important financial and demographic factors.

Chapter 4. The assumption of deterministic interest rates, which can be acceptable for short-term options, is not realistic for medium or long-term contracts such as life insurance products. GLWB contracts are investment vehicles with a long-term horizon and, as such, they are very sensitive to interest rate movements, which are
uncertain by nature. A stochastic modeling of the term structure is thus appropriate. In this chapter, therefore, we propose a generalization of the deterministic model allowing interest rates to vary randomly. A Cox-Ingersoll-Ross model is introduced. Sensitivity analyses have been conducted.

**Chapter 5.** Empirical studies of stock price returns show that volatility exhibits “random” characteristics. Consequently, the hypothesis of a constant volatility is rather “counterfactual”. In order to consider a more realistic model, we introduce the stochastic Heston process for the volatility. Sensitivity analyses have been conducted.

**Chapter 6.** In this chapter we price the GLWB option considering a stochastic process for both the interest rate and the volatility. We present a numerical comparison with the deterministic model.

**Chapter 7.** Conclusions are drawn.

**Appendix.** This section presents a quick survey of the most fundamental concepts from stochastic calculus that are needed to proceed with the description of the GLWB’s valuation model.
Chapter 1

Variable Annuity products

1.1 Introduction to annuities

Among all the hurdles that investors face when saving for retirement, the most challenging is perhaps the risk of running out of money before they die. Analyses show that the years just before and after retirement are a critical phase for the savings accumulated by investors throughout their working life. Traditional asset investments and the consequent sustainable retirement income streams are very sensitive to market fluctuations; a downturn in the markets could reduce savings to a level that will not provide sufficient revenue, and there may be not enough time for investors to recover their losses. Another remarkable source of uncertainty is the survival threshold. The fact that a retiree doesn’t know his/her date of death makes it harder to choose a consumption profile. Indeed, if he/she consumes relatively little in the first few years of retirement, he/she will make adequate provisions for a very long life. There is a chance, however, that he/she will die with a large sum of remaining capital. Alternatively, if the individual consumes excessively in the short term, he/she might need to reduce consumption later if he/she lives longer than expected. As a consequence, the fear of out-living one’s assets drives some investors to adopt unnecessary frugal lifestyles; while, at the opposite extreme, some investors will spend too much depleting their savings early in their retirement years.
Annuities can solve the retiree’s consumption problem, avoiding these extreme outcomes through a proper combination of traditional investment products (such as mutual funds) and insurance products that can offer a guaranteed income stream for life.

At a basic level, annuities are financial contracts (usually offered by insurance companies) that, in return for an initial capital payment, assure a steady stream of income for an agreed-upon span of time. They can provide periodic payouts for a fixed number of years (annuities certain), or for the duration of one or more people (the annuitants’) lives (life annuities).

Annuities are sometimes referred to as “reverse life insurance”. In fact, with life insurance, the policyholder pays the insurer each year until he or she dies, after which the insurance company pays a lump sum to the insured’s beneficiaries. With annuities, instead, the lump sum payment is from the annuitant to the insurance company before the annuity payout begins, and the annuitant receives regular payouts from the insurer until death.

The annuity payout rate rises based on the annuitant’s prospective mortality risk and on the rate of return that the annuity provider can earn on invested assets. Younger individuals (the same applies to women), because they are expected to receive payments for a longer time period, receive lower annuity payouts than older (or men) annuitants do for a given amount of capital invested.

There are two possible phases for an annuity, the first one in which customers deposit and accumulate money into an account (the deferral or accumulation phase), and another phase in which they receive payments for some period of time (the annuity or income phase). The policy, hence, has both a savings component (especially when it exhibits long accumulation phases) and an insurance component. During the accumulation phase, purchase payments are made and allocated to a number of investment options. The money invested will increase or decrease over time, depending on the fund’s performance. At the beginning of the payout phase, policyholders receive their purchase payments plus investment income and gains (if any) as a lump-sum payment, or they may choose to receive them as a stream of payments at regular
intervals (generally monthly). In the latter case, they may have a number of choices of how long the payments will last. Under most annuity contracts, the policyholder can choose to have his/her annuity payments last for a period that he/she sets (such as 20 years) or for an indefinite period (such as his/her lifetime or the lifetime of the annuitant and his/her spouse or other beneficiary). During the payout phase, the annuity contract may permit to choose between receiving payments that are fixed in amount or payments that vary based on the performance of the mutual fund investment options. The amount of each periodic payment will depend, in part, on the time period selected for receiving payments.

Annuity contracts with a deferral phase always have an annuity phase and are called deferred annuities. In this case, usually the onset of annuity payments is deferred until the annuity owner retires. These policies are therefore structured to meet the investor’s need to contribute and accumulate capital over his/her working life to build a sizable income stream for retirement. Sometimes, when establishing a deferred annuity, an investor may transfer a large sum of assets from another investment account, such as a pension plan. In this way the investor begins the accumulation phase with a large lump-sum contribution, followed by smaller periodic contributions. Types of deferred annuities include those with single, periodic or flexible premium. When an investor deposits a single lump sum and the annuity benefits are deferred until a later date, the annuity is called a single-premium deferred annuity. An annuity contract can also allow an investor to make periodic payments on a scheduled basis, either monthly, quarterly or annually during the policy’s accumulation phase. The annuity pays out its benefits at the end of this phase or possibly some years afterward and is referred to as a periodic-payment deferred annuity. The multiple premium payments could represent a saving plan for an individual who plans to use an annuity to draw down accumulated resources. The flexible-premium deferred annuity, instead, permits annuitants to make cash contributions at times of their choosing and allows the premiums accumulated value to be converted into an annuity at some future date or specified age of the annuitant.

An annuity contract may also be structured so that it has only the annuity phase;
such a contract is called an immediate annuity. It is paid in completely up front and payments start immediately (hence the name). Typically this type of annuity is chosen in case of a one-time payment of a large amount of capital, such as lottery winnings or inheritance and by investors who need immediate income from their annuity. Immediate annuities are a good option for investors who are already retired, but they are attractive also for people very close to retirement because they provide a guaranteed rate of return rather than risking their nest egg on the open market.

Regarding the return derivable from this kind of contract, another classification can be made:

- fixed annuities. They provide at least a guaranteed minimum rate of investment return over the length of the annuity. Since the rate is guaranteed, it typically won’t be as high as it would otherwise be if the same amount of money is invested in the stock market or mutual funds. The advantage to having a fixed annuity is that the return is guaranteed, and therefore relatively risk free, so most investors who choose this option aren’t looking to strike it rich;

- variable annuities. With a variable annuity, contract owners are able to choose from a wide range of investment options called sub accounts, each of which generally invests in shares of single underlying mutual funds, such as equity funds, bond funds, funds that combine equities and bonds, actively managed funds, index funds, domestic funds, international funds, etc. The investment return of variable annuities fluctuates. During the accumulation phase, the contract value varies based on the performance of the underlying sub accounts chosen. During the payout phase of a deferred variable annuity (and throughout the entire life of an immediate variable annuity), the amount of the annuity payments may fluctuate, again based on how the portfolio performs;

- indexed annuities. An indexed annuity operates as a combination of fixed and variable annuities. In fact, it is a fixed annuity that typically provides the contract owner with an investment return that is a function of the change in the level of an
Table 1.1: Classification of annuities’ contracts

<table>
<thead>
<tr>
<th>Method of paying premiums</th>
<th>Number of lives covered</th>
<th>Waiting period for benefits to begin</th>
<th>Nature of payouts</th>
</tr>
</thead>
<tbody>
<tr>
<td>- single premium</td>
<td>- one</td>
<td>- none</td>
<td>- fixed annuity</td>
</tr>
<tr>
<td>- fixed annual premium</td>
<td>- more than one (joint life, joint and survivor annuities)</td>
<td>- some waiting period (deferred annuity)</td>
<td>- variable annuity</td>
</tr>
<tr>
<td>- flexible premium</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

An important distinction among annuity products concerns the nature of the payout stream. Historically, most annuities provided fixed nominal payouts. They distributed a given principal across many periods, but they didn’t provide a constant real (i.e. adjusted for inflation) payout stream if the price level changed. Even modest inflation rates, in fact, can reduce the real value of annuity payouts. Variable annuities are designed to solve this problem. Indeed, they offer the opportunity to link payouts to the returns on an underlying asset portfolio. If the underlying assets provide a hedge against inflation, so will the payouts on the variable annuity. Because variable annuities are defined in part by the securities that back them, they are more complex than fixed annuities. In spite of their complexity, however, they have become one of the most rapidly growing annuity products in recent years. They merge the most attractive commercial features of unit-linked and participating life insurance contracts: dynamic investment opportunities, protection against financial risks and benefits in case of early death. Further, they offer modern solutions in regard of the post-retirement income, trying to arrange a satisfactory trade-off between annuitisation needs and bequest preferences. This work will focus on this class of annuities.
1.2 History and development of the Annuity market

Although annuities have only existed in their present form for a few decades, the idea of paying out a stream of income to an individual or family dates clear back to the Roman Empire. The Latin word “annua” means annual income and ancient Roman contracts known as *annua* promised an individual a stream of payments for a specified period of time, or possibly for life, in return for an up-front payment. The Roman speculator and jurist Gnaeus Domitius Annius Ulpianis is cited as one of the earliest dealers of these annuities, and he is also credited with creating the very first actuarial life table. Roman soldiers were paid annuities as a form of compensation for military service. During the Middle Ages, annuities were used by feudal lords and kings to help cover the heavy costs of their constant wars and conflicts with each other. At that time, annuities were offered in the form of a *tontine*. In return for an initial lump-sum payment, purchasers received life annuities. The amount of the payments was increased each year for the survivors. In fact, when investors eventually died off, their payments ceased and were redistributed to the remaining investors, with the last investor finally receiving the entire pool. This provided investors the incentive of not only receiving payments, but also the chance to “win” the entire pool if they could outlive their peers. The tontine thus combined insurance with an element of lottery-style gambling. European countries continued to offer annuity arrangements in later centuries to fund wars, provide for royal families and for other purposes. They were popular investments among the wealthy at that time, due mainly to the security they offered, which most other types of investments did not provide. Up until this point, annuities cost the same for any investors, regardless of their age or gender. However, issuers of these instruments began to see that their annuitants generally had longer life expectancies than the population at large and started to adjust their pricing structures accordingly. Annuities came to America in 1759 in the form of a retirement pool for church pastors in Pennsylvania. These annuities were funded
by contributions from both church leaders and their congregations, and provided a lifetime stream of income for both ministers and their families. They became the forerunners of modern widow and orphan benefits. Benjamin Franklin left the cities of Boston and Philadelphia each an annuity in his will; incredibly, the Boston annuity continued to pay out until the early 1990s, when the city finally decided to stop receiving payments and take a lump-sum distribution of the remaining balance. But the concept of annuities was slow to catch on with the general public in the United States because the majority of the population at that time felt that they could rely on their extended families to support them in their old age. Instead, annuities were used chiefly by attorneys and executors of estates who had to employ a secure means of providing for beneficiaries as specified in the will and testament of their deceased clients. Annuities did not become commercially available to individuals until 1812, when the “Pennsylvania Company for insurance on Lives and Granting Annuities” was founded and began marketing ready-made contracts to the public. During the Civil War, the Union government used annuities to provide an alternate form of compensation to soldiers instead of land. President Lincoln supported this plan as a means of helping injured and disabled soldiers and their families, but annuity premiums only accounted for 1.5% of all life insurance premiums collected between 1866 and 1920. Annuity growth began to slowly increase during the early 20th century as the percentage of multigenerational households in America declined. The stock market crash of 1929 marked the beginning of a period of tremendous growth for these vehicles as the investing public sought safe havens for their hard-earned cash. The first variable annuity was unveiled in 1952, and many new features, riders and benefits have been incorporated into both fixed and variable contracts ever since. Indexed annuities first made their appearance in the late 1980s and early 1990s, and these products have grown more diverse and sophisticated as well. Despite their original conceptual simplicity, modern annuities are complex products that have also been among the most misunderstood, misused and abused products in the financial marketplace, and they have had more than their fair share of negative publicity from the media.
1.3 Variable annuity products: the GMxB features

As variable annuities (VAs) are essentially a quite new product class, an industry standard definition does not yet exist. Ledlie et al. (2008) define them as *unit-linked or managed fund vehicles which offer optional guarantee benefits as a choice for the customer*. They are generally issued with a single premium (lump sum) or single recurrent premiums. The total amount of premiums is also named the principal of the contract or the invested amount. Apart from some upfront costs, premiums are entirely invested into a well diversified reference portfolio. In USA the National Association of Variable Annuity Writers explain that “with a variable annuity, contract owners are able to choose from a wide range of investment options called sub accounts, enabling them to direct some assets into investment funds that can help keep pace with inflation, and some into more conservative choices. Sub accounts are similar to mutual funds that are sold directly to the public in that they invest in stocks, bonds, and money market portfolios”. Customers can therefore influence the risk-return profile of their investment by choosing from a selection of different mutual funds, from more conservative to more dynamic asset combinations.  

Unlike in unit-linked, with profit or participating policies, reference funds backing variable annuities are not required to replicate the guarantees selected by the policyholder, as these are hedged by specific assets. Therefore, reference fund managers have more flexibility in catching investment opportunities. During the contract’s lifespan, its value may increase, or decrease, depending on the performance of the reference portfolio, thus policyholders are provided with equity participation. Under the terms and conditions specified by the contract, the insurer promises to make periodic payments to the client on preset future dates. These payments are usually determined as a fixed or variable percentage of the invested premium. The possibility of fluctuating payments is both an attraction (it provides potential protection against rising con-

\[1\] From the insurer’s perspective, the buyer’s portfolio choice can have a substantial impact on the profitability of the variable annuity. Individuals could increase risk and return in their portfolios to the point that the guarantee becomes unprofitable for the insurers. This is the reason why many actual prospectus of offered VAs restrict investment choices for their buyers.
sumer prices) and, for some potential buyers, a disadvantage (the nominal payout stream is not certain).

Table 1.2 shows how a basic VA contract operates. Suppose that the policyholder invests part of his/her savings in an immediate variable annuity contract depositing an amount of money equal, for the sake of simplicity, to $100 on the 1st February 2000. Among the available funds for the investment portfolio, the policyholder chooses to invest the entire premium in the Nikkei 225 index. Assume that the fixed annual withdrawal rate provided by the contract is equal to 7%, so the insured has the possibility to withdraw $7 every year until the drawdown process will exhaust, sooner or later, the VA sub-account. Fulfilling the financial needs of the client, the insurance company makes monthly payments which amount to $(7/12) = $0.58333. In the table, the first column reports the life span of the contract split in monthly installments. In the second column historical prices of the index are recorded; in particular we have considered the closing prices recorded on the first day of each month and adjusted to consider possible dividends payable. In the third column we have calculated the monthly index yield rate through the formula $r = (P_t - P_{t-1})/P_{t-1}$, where $P_t$ is the index price at time $t$, $P_{t-1}$ is the index price at immediately before month. The later columns recorded the value of the VA sub-account before (Vbefore) and after (Vafter) withdrawals and the amount of periodical withdrawals (VAs). The example underlines the effective dependence of the VA sub-account’s value on the investment portfolio’s performance, it being understood that the policyholder is still alive. For example, at the end of the first month considered, due to the positive return of the index, the value of the account increases from $100 to $(100(1+0.0309)) = $103.091. From this amount we have to deduct the monthly withdrawal, so that the account value becomes equal to $(103.091-0.5833) = $102.508. At the end of the third month, instead, the negative performance of the investment portfolio drives down the value of the account. We notice that on March 2009 the account value before withdrawals amounts to $0.09547. This sum of money is not sufficient to cover the periodic withdrawal equal to $0.58333. A standard VA policy, in this situation, runs out. The total withdrawals amount to $(109 \times 0.5833 + 0.09547)= $63.092. It’s well-rendered
Table 1.2: How a basic variable annuity works

<table>
<thead>
<tr>
<th>t</th>
<th>P</th>
<th>r</th>
<th>Vbefore</th>
<th>VAs</th>
<th>Vafter</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 2000</td>
<td>1995.52</td>
<td>0.030913692</td>
<td>103,0913692</td>
<td>0.583333333</td>
<td>102,5080359</td>
</tr>
<tr>
<td>March 2000</td>
<td>20337.32</td>
<td>0.018928311</td>
<td>104,4483399</td>
<td>0.583333333</td>
<td>103,8650066</td>
</tr>
<tr>
<td>April 2000</td>
<td>17973.7</td>
<td>-0.11622082</td>
<td>91,79373036</td>
<td>0.583333333</td>
<td>91,21039703</td>
</tr>
<tr>
<td>May 2000</td>
<td>16332.45</td>
<td>-0.091313975</td>
<td>82,88161307</td>
<td>0.583333333</td>
<td>82,29827974</td>
</tr>
<tr>
<td>June 2000</td>
<td>17411.05</td>
<td>0.066040306</td>
<td>87,73328334</td>
<td>0.583333333</td>
<td>87,14995001</td>
</tr>
<tr>
<td>July 2000</td>
<td>15727.49</td>
<td>-0.096694915</td>
<td>78,7229993</td>
<td>0.583333333</td>
<td>78,13965966</td>
</tr>
<tr>
<td>August 2000</td>
<td>16861.26</td>
<td>0.072088426</td>
<td>83,77262474</td>
<td>0.583333333</td>
<td>83,18929141</td>
</tr>
<tr>
<td>September 2000</td>
<td>15747.26</td>
<td>-0.066068609</td>
<td>77,69309061</td>
<td>0.583333333</td>
<td>77,10975278</td>
</tr>
<tr>
<td>October 2000</td>
<td>14539.6</td>
<td>-0.076690167</td>
<td>71,19619711</td>
<td>0.583333333</td>
<td>70,61286378</td>
</tr>
<tr>
<td>November 2000</td>
<td>14648.51</td>
<td>0.007490577</td>
<td>71,1417949</td>
<td>0.583333333</td>
<td>70,55846157</td>
</tr>
<tr>
<td>December 2008</td>
<td>8512.27</td>
<td>-0.007544614</td>
<td>1,872332397</td>
<td>0.583333333</td>
<td>1,289999064</td>
</tr>
<tr>
<td>January 2009</td>
<td>8859.56</td>
<td>0.040798753</td>
<td>1,341588618</td>
<td>0.583333333</td>
<td>0.758255285</td>
</tr>
<tr>
<td>February 2009</td>
<td>7994.05</td>
<td>-0.09769221</td>
<td>0.68417965</td>
<td>0.583333333</td>
<td>0.100846316</td>
</tr>
<tr>
<td>March 2009</td>
<td>7568.42</td>
<td>-0.05324335</td>
<td>0.095476921</td>
<td>0.095476921</td>
<td>-</td>
</tr>
<tr>
<td>April 2009</td>
<td>8109.53</td>
<td>0.071495768</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

the risks underlying the contract. A prolonged negative performance of the reference portfolio during the lifespan of the contract could preempt its end and consequently reduce the total withdrawals received by the policyholder. The same happens for example if the annuitant dies few years after the contract’s drafting, unlike his/her expectations. Just to face these risks that the VA market has begun to develop, insomuch as this class of annuities has achieved resounding success among investors. Many other features, in fact, contributed to make these products attractive.

The demand for this kind of policy is supported by two main factors: the preferential tax treatment and the possibility of underwriting several rider benefits that provide a protection of the policyholder’s savings account for the period before and after retirement.

The success of variable annuities is no doubt due to the presence of tax incentives, introduced by governments to support the development of individual pension solutions.
and contain public expenditure. From a tax perspective, annuities have the following key features that until now were not available with other investment products:

- tax deferability of investment earnings until the commencement of withdrawals; interests, dividends and capital gains that accrue on assets held in variable annuity accounts, in fact, are not taxed until the policyholder receives its payouts. Thus, the sum invested in the contract can grow faster. This tax-advantaged treatment is not offered to variable annuities that are owned by a non-natural person, such as a corporation or certain trusts. In most USA traded VAs, when the policyholder takes his/her money out of a variable annuity, however, he/she will be taxed on the earnings at ordinary income tax rates rather than capital gains rates that might be lower. Moreover, if taken prior to age 59\(\frac{1}{2}\), withdrawals may be subject to a 10% federal additional tax. In general, when a variable annuity is part of a retirement plan, the benefits of tax deferral will outweigh the costs only if the tax rate in the decumulation phase is lower than in the accumulation phase and the variable annuity is hold as a long-term investment to meet retirement and other long-range goals;

- favorable tax treatment of annuity income payments through the determination of an exclusion ratio to allow for a portion of each payment to be considered return of principal and a portion to be considered return of taxable investment earnings;

- tax-free transfer of funds between VA investment options. “Section 1035 exchanges” of the Internal Revenue Code in the USA allows a policyholder to make a direct transfer of accumulated funds in one annuity policy into another annuity policy without creating a taxable event; an individual can exchange one company’s product for another’s and the earnings from the original investment will remain tax deferred until the annuity owner withdraws money from the variable annuity contract;

- protection of VA assets from the insurance company’s creditors in the event of an insurer bankruptcy. The funds invested in a variable annuity contract are held in designed “sub accounts” that are kept separate from the insurance company’s
other assets. So the assets are not subject to claims by the insurance company’s creditors if it became insolvent.

In respect of traditional life insurance products, the main feature of variable annuities is the possibility of enjoying of a large variety of benefits represented by guarantees against investment and mortality/longevity risks. In particular, these products are designed to guarantee a minimum performance level of the underlying, thus protecting the policyholder against market downfalls, both in case of death and in case of life. Over the years, the guarantees offered on variable annuity products have evolved as the market has adapted to meet customer needs. While the vast majority of current variable annuities offers a death benefit rider as a default feature, more sophisticated designs include a variety of living benefit riders. Available guarantees are usually referred to as GMxB, where “x” stands for the class of benefits involved. A first classification, as above mentioned, is between:

- Guaranteed Minimum Death Benefits (GMDB);
- Guaranteed Minimum Living Benefits (GMLB).

The Guaranteed Minimum Death Benefit (GMDB) is usually available during the accumulation period even if some insurers are willing to provide them also after retirement, up to some maximum age (say, 75 years). It addresses the concern that the policyholder may die before all payments are made. If it happens, the beneficiary receives a death benefit equal to the current asset value of the contract or, if higher, the guaranteed amount, which typically is the amount of premiums paid by the deceased policyholder accrued at the guaranteed rate.

In contrast, living benefits can be described as wealth-preservation or wealth-decumulation products as they enable the policyholder to preserve wealth during the drawdown period. There are three common types of living benefit riders:

- Guaranteed Minimum Accumulation Benefits (GMAB);
- Guaranteed Minimum Income Benefits (GMIB);
- Guaranteed Minimum Withdrawal Benefits (GMWB).
The Guaranteed Minimum Accumulation Benefit (GMAB) is designed as a wealth-accumulation product, available prior to retirement. It guarantees that the final contract value at the end of the accumulation phase will not fall below a specific level regardless of the actual investment performance. This type of guarantee is particularly enticing to younger investors.

The other living benefits focus on the decumulation or payout phase of a variable annuity.

The Guaranteed Minimum Income Benefit (GMIB) rider is designed to provide the investor with a base amount of lifetime income at retirement, which is at least as valuable as the account value of the investments at the point of conversion. Triggering this guarantee is similar to purchasing an annuity in the traditional sense.

The Guaranteed Minimum Withdrawal Benefit (GMWB) riders guarantee that a certain percentage (usually 5% to 7%) of the invested premium can be withdrawn annually until the entire amount is completely recovered, regardless of market performance. So periodical withdrawals are allowed even if the account value reduces to zero because of bad investment performances. The contract may include clauses that serve to discourage excessive withdrawal. For example, when the policyholder withdraws at a higher rate than that contractually specified, the guarantee level could be reset to the minimum of the prevailing guarantee level and the account value. Also a percentage penalty charge could be applied on the excessive portion of the withdrawal amount. In this work we will refer to this last rider, and to be more precise, to its ultimate version, represented by the Guaranteed Lifetime Withdrawal Benefit.

To facilitate the understanding of the GMWB policy, in Table 1.3 we consider its workings through a numerical example. Consider the same policy described in Table 1.2 adding a GMWB rider. With this supplementary option, the policyholder preserves the possibility to withdraw the same amount until the contract maturity, also in case of a market drawdown. So, in the example, at the end of the 109th period (March 2009), the guarantee becomes effective and ensures a stream of monthly payments equal to $0.5833 until the initial premium has been totally recouped. This
happens \((100/0.5833) = 171.44\) months \((171.44/12 = 14.28\) years\) after the initiation of the contract. Therefore, adding a GMWB rider to a standard variable annuity contract, the policyholder is really provided against the negative performance of the investment portfolio.

Note that a key advantage of the Guaranteed Withdrawal Benefit (GWB) feature over other VA based options available in the market is that in GWB the underlying investment can continue to have market exposure even when the withdrawals start and thus has a greater growth opportunity. In contrast, with other widely marketed VA based options such as the GMAB or the GMIB/GAO, the underlying investment is effectively annuitized or invested in fixed income instruments upon maturity.

As a result of rising life expectancies as well as increases in lifestyle and healthcare costs, retirement lifespans have become both longer and more expensive. At the same time, with the social security system under considerable stress, the idea that individuals and households need to plan for their own retirement is gaining traction. To satisfy these new needs insurance companies have started offering a lifetime benefit feature with GMWB, enabling the investor to simultaneously manage both financial as well as longevity related risks. GMWB with lifetime withdrawals is commonly known as “Guaranteed Lifetime Withdrawal Benefits” (GLWB) or “Guaranteed Withdrawal Benefits (GWB) for life”. This rider guarantees policyholders the possibility of withdrawing an annual amount (typically 4% to 7%) of their guaranteed protection amount (GLWB Base) for their entire lifetime, no matter how the investments in the sub-accounts perform. It’s the only product that combines longevity protection with withdrawal flexibility, hence it is seen as a “second-generation” guarantee. The guarantee can concern one or two lives (typically spouses). Each annual withdrawal does not exceed some maximum value, but it is evident that the total amount of withdrawals is not limited, depending on the policyholder’s lifetime. Annual withdrawals of about 5% of the (single initial) premium are commonly guaranteed for insured aged 60+. In case of death any remaining fund value is paid to the insured’s dependants. In deferred versions of the contract, the product is fund linked during the deferment and the account value at the end of this period, or a
Table 1.3: How works a variable annuity with a GMWB rider

<table>
<thead>
<tr>
<th>t</th>
<th>P</th>
<th>r</th>
<th>Vbefore</th>
<th>VAs</th>
<th>Vafter</th>
<th>GMWB</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 2000</td>
<td>19959,52</td>
<td>0,030913692</td>
<td>103,0913692</td>
<td>0,583333333</td>
<td>102,5080359</td>
<td>0,583333333</td>
</tr>
<tr>
<td>March 2000</td>
<td>20337,32</td>
<td>0,018928311</td>
<td>104,4483399</td>
<td>0,583333333</td>
<td>103,8650066</td>
<td>0,583333333</td>
</tr>
<tr>
<td>April 2000</td>
<td>17973,7</td>
<td>-0,11622082</td>
<td>91,79373036</td>
<td>0,583333333</td>
<td>91,21039703</td>
<td>0,583333333</td>
</tr>
<tr>
<td>May 2000</td>
<td>16332,45</td>
<td>-0,091313975</td>
<td>82,88161307</td>
<td>0,583333333</td>
<td>82,29827974</td>
<td>0,583333333</td>
</tr>
<tr>
<td>June 2000</td>
<td>17411,05</td>
<td>0,066040306</td>
<td>87,7328334</td>
<td>0,583333333</td>
<td>87,1495001</td>
<td>0,583333333</td>
</tr>
<tr>
<td>July 2000</td>
<td>15727,49</td>
<td>-0,096694915</td>
<td>78,722993</td>
<td>0,583333333</td>
<td>78,13965966</td>
<td>0,583333333</td>
</tr>
<tr>
<td>August 2000</td>
<td>16861,26</td>
<td>0,072088426</td>
<td>83,77262474</td>
<td>0,583333333</td>
<td>83,18929141</td>
<td>0,583333333</td>
</tr>
<tr>
<td>September 2000</td>
<td>15747,26</td>
<td>-0,06068609</td>
<td>77,60309061</td>
<td>0,583333333</td>
<td>77,10975728</td>
<td>0,583333333</td>
</tr>
<tr>
<td>October 2000</td>
<td>14539,6</td>
<td>-0,076690167</td>
<td>71,19619711</td>
<td>0,583333333</td>
<td>70,61286378</td>
<td>0,583333333</td>
</tr>
<tr>
<td>November 2000</td>
<td>14648,51</td>
<td>0,007490577</td>
<td>71,1417949</td>
<td>0,583333333</td>
<td>70,55846157</td>
<td>0,583333333</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>December 2008</td>
<td>8512,27</td>
<td>-0,007544614</td>
<td>1,872332397</td>
<td>0,583333333</td>
<td>1,288999064</td>
<td>0,583333333</td>
</tr>
<tr>
<td>January 2009</td>
<td>8859,56</td>
<td>0,040798753</td>
<td>1,341588618</td>
<td>0,583333333</td>
<td>0,758255285</td>
<td>0,583333333</td>
</tr>
<tr>
<td>February 2009</td>
<td>7994,05</td>
<td>-0,09769221</td>
<td>0,68417965</td>
<td>0,583333333</td>
<td>0,100846316</td>
<td>0,583333333</td>
</tr>
<tr>
<td>March 2009</td>
<td>7568,42</td>
<td>-0,05324335</td>
<td>0,095476921</td>
<td>0,095476921</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>April 2009</td>
<td>8109,53</td>
<td>0,071495768</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>December 2013</td>
<td>15661,87</td>
<td>0,093099915</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>January 2014</td>
<td>16291,31</td>
<td>0,040189326</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>February 2014</td>
<td>14914,53</td>
<td>-0,08451085</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>March 2014</td>
<td>14841,07</td>
<td>-0,004925398</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>April 2014</td>
<td>14827,83</td>
<td>-0,000892119</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
<tr>
<td>May 2014</td>
<td>14304,11</td>
<td>-0,03532007</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0,583333333</td>
</tr>
</tbody>
</table>
guaranteed amount if greater, is treated like a single premium paid for an immediate GLWB.

As noted earlier, the Benefit Base is the figure the policyholder’s future guaranteed income payments will be based on. Typically, in all rider benefits described, it initially equals the contributions made to the annuity. However, during the lifetime of the contract, it could be a different sum because of additional features offered with GMxBs.

For example, if a roll-up is included within the policy, the annual guaranteed amount is increased by a fixed percentage every year during a certain time period but only if the policyholder has not started withdrawing money. Therefore, roll-ups are commonly used as an incentive to the policyholder not to withdraw money from the account in the first years. Usually the minimum rate of growth is 5% - 7%.

The amount guaranteed for withdrawal may also depend on the account value and increase during the policy lifespan if the fund’s assets perform well, allowing the policyholder to withdraw a higher amount than that initially guaranteed. This increase may either be permanent or be effective just for the single withdrawal. The step-up feature, in particular, can increase the benefit base amount if the VA account value after withdrawals is higher than the benefit base on specified dates. Typically, step-up dates are annually or every three or five years on the policy anniversary date. Therefore, step-ups only occur if the policyholder’s funds yield high performance and the account value has not been decreased heavily due to previous withdrawals. Common step-up features are, e.g., annual ratchet guarantees. Regarding this feature, four different product designs can be observed in the market (Kling et al. 2011):

- No Ratchet. In this case no ratchets or surplus exist at all; the guaranteed annual withdrawal is constant and does not depend on market movements;

- Lookback Ratchet. This alternative considers a withdrawal benefit base at outset given by the single premium paid. During the contract term, on each policy anniversary date, the benefit base is compared with the account value at that time. The higher value among them is taken as the new benefit base. So, if the account value at a certain date exceeds the previous benefit base, the guaranteed withdrawal
is increased accordingly to the preset percentage multiplied by the new withdrawal benefit base. This effectively means that the fund performance needs to compensate for policy charges and annual withdrawals in order to increase the guaranteed withdrawals. With this product design, increases in the guaranteed withdrawal amount are permanent, i.e. over time, the guaranteed withdrawal amount may only increase, never decrease;

- Remaining Withdrawal Benefit Base Ratchet. As in the previous case, the withdrawal benefit base at outset is given by the single premium paid. The withdrawal benefit base is however reduced by every guaranteed withdrawal. As before, on each policy anniversary, if the account value exceeds the benefit base, the new benefit base is increased to the account value. The guaranteed annual withdrawal is however increased by the preset percentage multiplied by the difference between the account value and the previous benefit base. This effectively means that, in order to cause an increase of guaranteed annual withdrawals, the fund performance needs to compensate for policy charges only but not for annual withdrawals. This ratchet mechanism, other things being equal, is therefore somewhat “richer” than the Lookback Ratchet. Therefore, typically the initially guaranteed withdrawal amount should be lower than with a product offering a Lookback Ratchet. As with the Lookback Ratchet design, increases in the guaranteed amount are permanent;

- Performance Bonus. For this alternative the withdrawal benefit base is never increased. On each policy anniversary date, in fact, if the current account value is greater than the current withdrawal benefit base, 50% of the difference is added to this year’s guaranteed amount as “performance bonus”. In contrast to the previous two designs, therefore, in this case the guaranteed withdrawal amounts remain unchanged. For the calculation of the withdrawal benefit base, only annual withdrawals are subtracted from the benefit base and not the performance bonus payments.

Another guarantee which may be attached to GMxB is the reset. In a GMAB
it gives the opportunity to renew the option when it reaches maturity, so it allows to postpone the maturity date; in a GMDB, instead, it allows the guaranteed withdrawal amount to equal the account value at some prior specified date, note as reset dates. In this case, unlike the ratchet guarantee, the minimum guaranteed amount may decrease if the account value falls off between two reset dates.

One of the most common options available in policies with a considerable savings component as variable annuities is the possibility to exit (surrender) the contract before maturity and to receive a lump sum (surrender value) reflecting the insured’s past contributions to the policy, minus any costs incurred by the company and possibly some charges. The idea is to boost sales by ensuring that the policyholder does not perceive insurance securities as an illiquid investment. At the same time, however, surrenders are not welcome by insurers, as they imply a reduction in the assets under management and may generate imbalances in the exposure to the mortality risk of remaining insureds (selective surrenders). For these reasons, there are often surrender penalties that apply if funds are withdrawn before a pre-specified time period, often seven years. These penalties, known as Contingent Deferred Surrender Charges (CDSC), can be several percent of the annuity’s value.

This is a simplified description of the basic design of the guarantees embedded in VAs; a complete description of all possible variants would be beyond the scope of this thesis, focused on the actuarial and financial valuation of this kind of contracts. Thus, some products offered in the market may have features different from those investigated above or may be a combination of two or more guarantees. The reader interested in a detailed overview of variable annuities could refer to Ledlie et al. (2008).

Insurance companies charge a fee for the offered benefits. Guarantees and asset management fees, administrative cost and other expenses are charged typically deducting a certain percentage of the underlying fund’s value from the policyholder’s funds account on an annual basis. Very rarely they are charged immediately as a single initial deduction. This improves the transparency of the contract, as any deduction to the policy account value must be reported to the policyholder. Some
guarantees can be added or removed, at policyholder’s discretion, when the contract is already in-force. Accordingly, the corresponding fees start or stop being charged. Unlike most “good” investments, VAs’ fees are quite high. For this reason, they used to receive heaps of bad press. Also investors don’t look kindly upon this aspect, because of the combination of investment management and insurance expenses substantially reduces their returns. Analyzing many VA contracts traded in USA it’s possible to classify insurance charge into many categories:

- mortality and expense risk charge (M&E). The M&E charge compensates the insurance company for insurance risks and other costs it assumes under the annuity contract. The fees for any optional death and/or living benefit the policyholder may select are described below and are not included in the M&E charge. M&E charges are assessed daily and typically range from 1.15% to 1.85% annually;

- administrative and distribution fees. These fees cover the costs associated with servicing and distributing the annuity. They include the costs of transferring funds between sub accounts, tracking purchase payments, issuing confirmations and statements as well as ongoing customer service. These fees are assessed daily and typically range from 0% to 0.35% annually;

- contract maintenance fee. It is an annual flat fee charged for record-keeping and administrative purposes. The fee typically ranges from $30 to $50 and is deducted on the contract anniversary. This fee is typically waived for contract values over $50,000;

- underlying sub account fees and expenses. Fees and expenses are also charged on the sub accounts. These include management fees that are paid to the investment adviser responsible for making investment decisions affecting investor’s sub accounts. This is similar to the investment manager’s fee in a mutual fund. Expenses include the costs of buying and selling securities as well as administering trades. These asset-based expenses will vary by sub account and typically range from 0.70% to 2.50% annually;
- contingent deferred sales charge (or surrender charge). Most variable annuities do not have an initial sales charge. This means that 100% of funds are used for immediate investment in the available sub accounts. However, insurance companies usually assess surrender charges to annuity owners who liquidate their contract (or make a partial withdrawal in excess of a specified amount) during the surrender period. The surrender charge is generally a percentage of the amount withdrawn and declines gradually during the surrender period. A typical surrender schedule has an initial surrender charge ranging from 7% to 9% and decreases each year that the contract is in force until the surrender charge reaches zero. Generally, the longer is the surrender schedule, the lower are the contract fees. Most contracts will begin a new surrender period for each subsequent purchase payment, specific to that subsequent purchase payment.

- fees and charges for other features. Special features offered by some variable annuities, such as a stepped-up death benefit, a guaranteed minimum income benefit, or long-term care insurance, often carry additional fees and charges.

First introduced in the US in the early 1970s, variable annuities quickly experienced remarkable growth. In recent years, they have become very popular life insurance products, able to address the long-term savings and retirement needs of a rapidly aging population. As individuals also become more heterogeneous in terms of their demand characteristics, there is growing recognition in the industry and by governments that existing retirement models have to be improved to better meet consumer needs. In particular, consumers require access to market returns in order to keep pace with the rising cost of living, but they also need to protect their assets and lifestyle from negative economic trends. Variable annuities represent a valuable compromise, and their commercial features make them attractive, providing a good opportunity for market development.

However, caution is necessary because variable annuities can have a high negative impact on the VA provider’s balance sheet. If the GWxB is significantly underpriced or raises the possibility of a debilitating loss for the underwriting company, then the related credit-worthiness issue should make potential clients skeptical of the product.
Therefore, for the long term sustainability of GWxBs it is important that the companies offering them remain profitable and viable. A proper risk management process is consequently needed. It usually requires several phases, from the risk identification, to its assessment, to the choice of (a mix of) risk management techniques. If fairly priced, the GWB for life option is an attractive retirement solution for investors as it allows them to manage the risks related to their own longevities, which cannot be mitigated at an individual level. Further, the ability to stay invested in the market while in retirement would allow investors to better cope with the inflation related risk, which becomes significant as the retirement lifespans get longer.

1.4 Risks underlying Variable Annuities

From the above description it is clear that several risks affect the performance of a VA contract. In addition to risks which are typically implied by life-insurance products, such as mis-selling risks, the risks arising from mis-specified policy conditions or other sales material, regulatory and accounting risks, etc., there are risks which are specific to VAs (Kalberer & Ravindran (2009)). These can be classified into three categories:

- shortfall risks. This category includes two kinds of risks. The first one concerns the possibility that the performance of the underlying asset is insufficient to cover the guarantees given a certain expected realisation of biometric\footnote{Underwriting risks covering everything related to human life conditions, e.g. death, disability, longevity, but also birth, marital status, age, and number of children (e.g. in collective pension schemes).} (or more general, insurance) risk, e.g. a certain pattern of expected deaths or surrender. The second kind of risk is linked to the chance that biometric factors insured (e.g. death) develop adversely such that even an asset performance able to cover the guarantees in the expected case becomes insufficient. This happens, for example, when longevity exceeds expectations. These two kinds of risks are evidently connected. So even if the biometric risk can be minimised
through diversification, the risk from the asset part has to be hedged through instruments that are dependent on both asset and biometric developments in a combined way, and such investments typically do not exist (and are definitely not liquid traded);

• pricing risk. It is the risk that the price of the guarantees may be inadequate. There are two methods to evaluate the premium charged by the insurance company:

  - a theoretical approach, through which the price should cover the theoretical value of the guarantee, consistent with the insights of financial economics. Such a premium, however, couldn’t consider frictional costs which arise in practice, like transaction costs, bid-offer spreads or simply the fact that the requirements of a theoretical model are rarely really fulfilled, e.g. continuous re-hedging, complete markets, no cost of capital, etc. So the premium charged could be insufficient to hedge the exposure properly. The model underlying the pricing might also be inadequate or insufficiently calibrated; this is called the “model risk”;

  - an hedging approach, through which the price should be just sufficient to cover the costs of hedging. But also in this case many risks emerge. It could happen that available hedging instruments used for pricing do not really replicate the exposure (replication risk). Moreover, not always prices of the hedging instruments are readily available and theoretical models have to be used to determine their prices, so the same problems presented in the previous case emerge. Also the policyholder behaviour has a hold on the premium charged by the insurance company. In fact, when determining the right price for the guarantee, a certain amount of policyholder lapsation\(^3\) is expected. If a policyholder lapses, then he/she loses his/her guarantees in most cases, such that a certain amount of

\(^3\) Lapsation of a life insurance policy is discontinuation of premium payment by the policyholder during the life span of the policy.
lapsation results in a lower overall cost for the guarantees. This implies lower (and thus more competitive) prices. Policyholders, however, are assumed to act financially rationally. In particular, when the value of the underlying funds is low and consequently the value of the guarantee is high, being guarantee charges fixed, policyholders will feel inclined to stay and not to lapse their contract (“in-the-money persistency”). Conversely, if the value of the underlying funds is high (in relation to the guarantees), then the value of the guarantees will be low, even vanishing in extreme cases. Policyholders will thus feel inclined to lapse and avoid the now unnecessary guarantee charges (“out-of-the-money lapsation”). Thus, the lapsation will be asset dependent. This should be reflected in the pricing assuming a certain policyholder behaviour. But this behaviour has not been explored in depth and can change over time. The resulting risk is called “policyholder behaviour risk”;

- hedging risk. It is the possibility that risk management strategies may fail. A dynamic-hedging programme requires the creation of a hedge portfolio which tightly follows the value of the guarantees. And the value of the guarantees is dependent on a whole range of parameters, for example the features of the guarantee, the value of the underlying assets, the volatility of the underlying funds, the interest rates in the policy currency, the proportion of surviving policyholders, and potentially much more. In turn, some of these parameters are dependent on more basic parameters, e.g. the value of the underlying funds is dependent on the value of the components of the funds in their denomination currency, the exchange rate between denomination currency and policy currency, the size and timing of dividends of the underlying assets and so on. The general risk of a hedging strategy is that the development of the hedging portfolio deviates from the development of the value of the guarantees. The risks involved are, among others:

- long-term volatility risk. It is the risk that the implied market volatility of
the underlying funds increases in an unforeseen way over time, resulting in losses when trying to roll over a potential hedge of the product;

– interest rate risk. It is the risk that the level of interest rates changes. It can in most cases be hedged away quite efficiently through a wide range of instruments available in the market, such as long-term swaps;

– gamma risk. Gamma measures the rate of change of Delta as the underlying moves. Delta tells us how much an option price will change given a one-point move of the underlying. But since Delta is not fixed and will increase or decrease at different rates, it needs its own measure, which is Gamma;

– foreign exchange risk. It is the risk that the guarantee is denominated in a policy currency which is different from that of the underlying funds, such that not only the asset performance of the underlying funds but also the fluctuations between the two currencies must be hedged;

– basis risk. It is the risk which emerges when there exist no hedging instruments on the underlying assets and so the solution is to map the funds to a portfolio of assets which can be hedged, typically consisting of stock market indexes, also called the “benchmark portfolio”. A special case of this risk is the “dividend risk”, which arises when the amount of dividend payments on the benchmark assets fluctuates causing hedging deviations. Another example of basis risk is the “correlation risk”. The volatility of the benchmark index, in fact, depends on the correlation between the asset indexes forming the benchmark portfolio; thus, changes in this correlation can result in hedging losses;

– funds choice risk. Usually the policyholder has the contractual right to choose the underlying funds from a prescribed list of available funds and exchange funds at market value, paying a relatively small handing fee. But some of the funds in the list could be not hedgable, and their basis risk could be so high that it exceeds the benefits of hedging;
– policyholder behaviour risk. It emerges when the asset dependent policyholder behaviour differs from that supposed when determining the hedging portfolio, e.g. assuming a certain percentage of lapses;

– liquidity risk, because of market constraints, hedging instruments which were initially liquid could become illiquid over time or even cease to exist;

– counterparty credit risk. The hedging instruments, e.g. swaps, involve counterparties which may fail to serve their obligations;

– key-person risk. It is the risk that the necessary skills and know-how required to set up a dynamic hedging programme couldn’t be retained during the whole life of the hedging operations, a time which can easily exceed 30 years;

– operational risk. It is the risk remaining after determining financing and systematic risk, and includes risks resulting from breakdowns in internal procedures, people and systems. The hedging programme typically consists of a complex series of processes, some of them IT (information technology)-related. These processes must be performed regularly in a timely manner and without mistakes. The operational risk involved is considerable and exceeds the typical level in an insurance company;

– transaction cost risk. Hedging operations require regular transactions, which are typically associated with transaction costs, e.g. in the form of bid-offer spread;

– cost of capital risk. VA products are under scrutiny from regulators, who typically require a substantial amount of risk capital. Regulations may change over time because of developments out of the control of an individual insurance company, such as political developments or insolvencies of competitors. So, risk capital requirements may increase, causing a need for liquidity and higher costs of capital for VA products;

– cost of risk-management risk;

– opaqueness premium risks. The financial community may not understand
the risk exposure of VAs or new VA features and regard the company’s financial situation as more opaque when exposed to VAs. These exposures could not be communicated adequately to investors. The result is an increase of the premium.

Note that this is only a partial overview of all possible risks involved. It is important to notice that the risks associated with a VA are typically not eliminated but only transformed. One risk is replaced with another, hopefully more easily manageable risk.

1.5 Variable annuities during the recent market Crisis

The recent severe financial crisis in 2007-2008 highlighted all the risks of variable annuities. Not only life insurers have experienced realized and unrealized losses in their general accounts from credit exposures, but their VA businesses have created exposures to equity markets that have threatened the survival of some and put pressure on the business model and balance sheet of others.

Let’s revisit briefly the fundamental changes in VA sales in the US market over the last 20 years (Chopra et al. (2009))

Before the 1990s, dividend and capital gains tax rates were higher and largely in line with marginal tax rates, so variable annuities were considered appealing because they offered policyholders the possibility to accumulate higher levels of tax-deferred savings within a life insurance policy. The 1990s saw the growth of available investment choices and enhancements of death benefits, which resulted in asset growth of 21 percent per year, with assets reaching almost $1 trillion by the end of 2000.

By 2003, affluent baby-boomers began approaching middle age with swelling 401(k) balances and limited ability to protect themselves against longevity and market risk. Changes in tax rates also weakened the traditional appeal of variable annuities as a

A 401(k) plan is a qualified (i.e., meets the standards set forth in the Internal Revenue Code (IRC) for tax-favored status) profit-sharing, stock bonus, pre-ERISA money purchase pension, or a rural cooperative plan under which an employee can elect to have the employer contribute a portion of the employee’s cash wages to the plan on a pre-tax basis. These deferred wages (elective deferrals)
By 2003, affluent baby-boomers began approaching middle age with swelling 401(k) balances and $1 trillion by the end of 2000. Living benefits provided a threshold of payments that policyholders could receive either during the accumulation or withdrawal phase (depending on the type of living benefit), regardless of their lifespan. The introduction of these guarantees set off a period of rapid development in which the market saw waves of new products with increasingly sophisticated guarantees. This innovation generated significant customer interest because it allowed clients to protect their investment from equity market declines. The introduction of the Guaranteed Withdrawal Benefit for Life in 2005 created a new standard for the industry, combining the longevity protection with the liquidity of the regular withdrawal benefit. Importantly, the introduction of these living benefits took place in the context of rising stock markets: a wave of benefits were launched beginning in 2002 and continuing to 2007, a period when the S&P 500 index grew at 9 percent per year. Hence, the period of 2003-07 was an exceptionally strong one for the life insurance industry. Figure 1.1 shows variable annuity

Figure 1.1: Variable annuities 1998-2008

<table>
<thead>
<tr>
<th>Year</th>
<th>Variable annuities sales $Billions</th>
<th>Variable annuities assets $Billions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>100</td>
<td>769</td>
</tr>
<tr>
<td>1999</td>
<td>122</td>
<td>949</td>
</tr>
<tr>
<td>2000</td>
<td>137</td>
<td>972</td>
</tr>
<tr>
<td>2001</td>
<td>111</td>
<td>904</td>
</tr>
<tr>
<td>2002</td>
<td>117</td>
<td>815</td>
</tr>
<tr>
<td>2003</td>
<td>129</td>
<td>1,004</td>
</tr>
<tr>
<td>2004</td>
<td>133</td>
<td>1,123</td>
</tr>
<tr>
<td>2005</td>
<td>137</td>
<td>1,217</td>
</tr>
<tr>
<td>2006</td>
<td>160</td>
<td>1,379</td>
</tr>
<tr>
<td>2007</td>
<td>182</td>
<td>1,491</td>
</tr>
<tr>
<td>2008</td>
<td>152</td>
<td>1,127</td>
</tr>
</tbody>
</table>

Source: LIMRA survey (1998-2007); Morningstar (2008); analyst reports

tax deferral vehicle, and growth started to flatten. These forces led to a development of the insurance industry. In the early 2000s guaranteed living benefits were introduced. Living benefits provided a threshold of payments that policyholders could receive either during the accumulation or withdrawal phase (depending on the type of living benefit), regardless of their lifespan. The introduction of these guarantees are not subject to federal income tax withholding at the time of deferral and they are not reflected as taxable income on the employee’s Individual Income Tax Return.
sales grew by around 9 percent annually over 2003-2007 (approximately $50 billion total), increasing assets to about $1.5 trillion by 2007. In some sense, VAs emerged as the natural product for affluent investors in their 50s and 60s as they transitioned from the accumulation to the decumulation stage of their investment lifecycle. With the introduction of living benefits and market performance guarantees, policyholders used variable annuities as a vehicle to invest in mutual funds. Investment choices in VAs began more and more focused on equity. Ferocious competition among VA players, intensified by the growing importance of independent sales channels, led many insurers to issue ever more generous guarantees at ever lower prices. When the crisis hit, the assumptions underlying their calculations - from equity market volatility to customer conduct - proved to be unrealistic. The sharp rise in the volatility of the stock market, in particular, led to a dramatic increase in the costs of hedging VA business. Because of significant declines in equity values, most guarantees embedded in VAs have become in-the-money, entailing losses for the issuer. This resulted in higher rider fees in new VA products. Potential policyholders, however, were not attracted by these higher prices, so sales reduced. Moreover, as fees are based on the actual account values, the considerable fall in equity prices has significantly reduced insurer’s income streams. As a result, many companies’ credit ratings were downgraded by the rating agencies.

However, the most important consequence of the market crisis is related to risk management and hedging programs. Considering that the guarantees from existing products have become more valuable and more likely to end up in-the-money, insurance companies providing them with have been forced to raise their risk-based capital requirement. As a consequence, hedging programs, which are used to counter this increase in liabilities or reinsurance arrangements for risk transfer, have gained importance. Generally, funds cannot be hedged directly, for this reason they are mapped to hedgeable indices or risk factors. This mapping is often based on simple linear relationships determined by historical data. However, during extreme market fluctuations this simple approach has often caused basis mismatches (deviations between funds and corresponding indices), which have contributed directly to hedge
ineffectiveness. Now there is an increasing tendency to include investment funds that employ different risk mitigation strategies in VA policies. Examples of these types of funds are volatility target funds and funds with a built-in downside protection. These allow VA guarantee costs to be shifted to the fund level. To avoid further guarantee costs and to benefit fully from these risk management strategies, it is essential to model these funds properly by using advanced mapping techniques and accounting for the current fund allocation.

Many insurers are still raising prices, decreasing benefits and features, discontinuing products and, in some cases, even exiting the business. Despite the somewhat steady growth of the US economy in early 2012, the global economic outlook looks less certain. During the first quarter of 2012, Hartford Life announced that it will no longer sell variable annuities. This follows the exit of Sun Life, Genworth and ING in 2011, and the scale-back by MetLife through 2011 and 2012. Consequently, distributors are now very sensitive to the possibility that a company may not offer variable annuities in the future. Product trends in 2012 continue to focus on restructuring living benefits on VA. Some companies have decreased the withdrawal percentages and bonuses, increased the charges or made the investment options more restrictive. Others have introduced new, less-competitive benefits and pulled the richer ones from the market. Insurance companies continue to look for new ways to de-risk, such as managing volatility within the funds or even buying back certain benefits. Nevertheless an increasingly positive assessment about variable annuities has been recorded in the last years. There has been a recognition that in comparison with other products, VAs are much more able to satisfy customer needs for broad coverage of biometric and capital markets related risks. This versatility gives insurers a significant competitive advantage over other financial services providers such as asset managers and additionally offers a certain protection against rapid margin erosion. Furthermore, the view that appears to be gaining the upper hand is that valuable experience was gained in the crisis that will help make VA risk management systems more robust and effective going forward.
Tables 1.4 and 1.5 show the main sellers of annuities in 2013. According to a LIMRA (Life Insurance and Market Research Association) study reported online (http://www.lifehealthpro.com), Jackson National Life Insurance Company sold $23.2 billion in annuity sales, representing the top seller of total annuities in 2013. Most of the Jackson National Life’s annuity sales were variable annuities ($20.9 billion), making it also the top seller of variable annuities in 2013. New York Life Insurance Company held the top spot for fixed annuity sales in 2013, recording $6.5 billion in fixed annuity sales. The top 5 variable annuity writers represented 50 percent of the market in 2013 - down 6 percentage points in 2012. The top 5 fixed annuity writers held 33 percent market share, which is 3 percentage points higher than in 2012. Despite extraordinary 32 percent growth in the equities market in 2013, VA sales were down 1 percent at year-end compared with 2012 and totaled $145.3 billion. Variable annuity sales marked positive growth in the fourth quarter, up 4 percent to $36.3 billion. Following a trend for the past few years, VA sales are no longer tracking with the equities markets. Companies continue to carefully manage their VA business. More emphasis on accumulation VA’s appears to be an emerging trend. In 2013, more companies introduced these types of products into their portfolios as they shift their focus to tax-deferred products with alternative investment options and indexed-linked VAs.

1.6 Variable annuities around the world

In this section we examine U.S., Japanese and European life insurance markets in order to highlight the international development of VAs and their strong potential growth (for a more detailed reading, see for example, [Ledlie et al., 2008] and [Kalberer & Ravindran, 2009]).

1.6.1 U.S.A

Variable annuities have existed in the U.S.A. since the 1950’s. In 1952 Teachers Insurance and Annuities Association (TIAA) created the College Retirement Equity Fund
<table>
<thead>
<tr>
<th>Rank</th>
<th>Company name</th>
<th>Total ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Jackson National Life</td>
<td>23,199,905</td>
</tr>
<tr>
<td>2</td>
<td>AIG Companies</td>
<td>17,502,430</td>
</tr>
<tr>
<td>3</td>
<td>Lincoln Financial Group</td>
<td>16,554,825</td>
</tr>
<tr>
<td>4</td>
<td>TIAA-CREF</td>
<td>13,929,953</td>
</tr>
<tr>
<td>5</td>
<td>MetLife</td>
<td>12,374,213</td>
</tr>
<tr>
<td>6</td>
<td>Prudential Annuities</td>
<td>12,036,730</td>
</tr>
<tr>
<td>7</td>
<td>AXA US</td>
<td>9,716,126</td>
</tr>
<tr>
<td>8</td>
<td>New York Life</td>
<td>9,672,995</td>
</tr>
<tr>
<td>9</td>
<td>Allianz Life of North America</td>
<td>9,084,876</td>
</tr>
<tr>
<td>10</td>
<td>Transamerica</td>
<td>8,556,217</td>
</tr>
<tr>
<td>11</td>
<td>Pacific Life</td>
<td>7,419,865</td>
</tr>
<tr>
<td>12</td>
<td>Nationwide Life</td>
<td>6,896,100</td>
</tr>
<tr>
<td>13</td>
<td>Security Benefit Life</td>
<td>6,361,781</td>
</tr>
<tr>
<td>14</td>
<td>RiverSource Life Insurance</td>
<td>5,480,773</td>
</tr>
<tr>
<td>15</td>
<td>American Equity Investment Life</td>
<td>4,212,355</td>
</tr>
<tr>
<td>16</td>
<td>Great American</td>
<td>3,978,280</td>
</tr>
<tr>
<td>17</td>
<td>Massachusetts Mutual Life</td>
<td>3,567,845</td>
</tr>
<tr>
<td>18</td>
<td>Thrivent Financial for Lutherans</td>
<td>3,556,494</td>
</tr>
<tr>
<td>19</td>
<td>Protective Life</td>
<td>2,563,367</td>
</tr>
<tr>
<td>20</td>
<td>Symetra Financial</td>
<td>2,539,512</td>
</tr>
<tr>
<td>Top 20</td>
<td></td>
<td>179,204,642</td>
</tr>
<tr>
<td>Total Industry</td>
<td></td>
<td>229,675,000</td>
</tr>
<tr>
<td>Top 20 share</td>
<td></td>
<td>78%</td>
</tr>
</tbody>
</table>
Table 1.5: Top 20 VA sales leaders for 2013

<table>
<thead>
<tr>
<th>Rank</th>
<th>Company name</th>
<th>Total ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Jackson National Life</td>
<td>20,941,810</td>
</tr>
<tr>
<td>2</td>
<td>Lincoln Financial Group</td>
<td>14,376,215</td>
</tr>
<tr>
<td>3</td>
<td>TIAA-CREF</td>
<td>13,929,953</td>
</tr>
<tr>
<td>4</td>
<td>AIG Companies</td>
<td>12,305,419</td>
</tr>
<tr>
<td>5</td>
<td>Prudential Annuities</td>
<td>11,427,916</td>
</tr>
<tr>
<td>6</td>
<td>MetLife</td>
<td>10,645,327</td>
</tr>
<tr>
<td>7</td>
<td>AXA US</td>
<td>9,678,056</td>
</tr>
<tr>
<td>8</td>
<td>Transamerica</td>
<td>8,406,000</td>
</tr>
<tr>
<td>9</td>
<td>Nationwide Life</td>
<td>5,741,200</td>
</tr>
<tr>
<td>10</td>
<td>RiverSource Life Insurance</td>
<td>5,230,645</td>
</tr>
<tr>
<td>11</td>
<td>Pacific Life</td>
<td>4,514,968</td>
</tr>
<tr>
<td>12</td>
<td>Thrivent Financial for Lutherans</td>
<td>3,174,818</td>
</tr>
<tr>
<td>13</td>
<td>New York Life</td>
<td>3,173,538</td>
</tr>
<tr>
<td>14</td>
<td>Allianz Life of North America</td>
<td>3,024,486</td>
</tr>
<tr>
<td>15</td>
<td>Ohio National Life Insurance Company</td>
<td>2,363,818</td>
</tr>
<tr>
<td>16</td>
<td>Fidelity Investments Life</td>
<td>2,078,599</td>
</tr>
<tr>
<td>17</td>
<td>Protective Life</td>
<td>1,868,539</td>
</tr>
<tr>
<td>18</td>
<td>Northwestern Mutual Life</td>
<td>1,631,541</td>
</tr>
<tr>
<td>19</td>
<td>Principal Financial Group</td>
<td>1,113,431</td>
</tr>
<tr>
<td>20</td>
<td>Massachusetts Mutual Life</td>
<td>907,597</td>
</tr>
</tbody>
</table>

| Top20 | 136,533,877 |
| Total Industry | 145,300,000 |
| Top 20 share | 94% |
(TIAA-CREF) to provide VA coverage within the retirement income programme of TIAA. It was not an individual annuity contract, but the development of this first VA concept paved the way for future variable contracts. For the following 40 years the growth of the VA market was gradual, mostly because of strong regulatory restrictions governing the sale of separate account-based products in many states. However, beginning in 1982, with the passage of the Tax Equity and Fiscal Responsibility Act, the Deficit Reduction Act and the Tax Reform Act of 1986, VAs began to be viewed as viable long-term retirement and savings instruments with strong tax advantages. The low-interest environment of the 1970s, followed by equity market growth starting in the late 1980s, provided additional consumer and producer incentives to look to VAs as an attractive retirement-planning tool. So, since the early 1990s, there has been an exponential growth of the market in terms of market size and complexity of the VA products, no doubt related, in addition to the previous reasons, to ageing population, consumer sophistication and availability of reinsurance covers. While in the 1980s it was common for VAs to offer only a handful of investment options (e.g. a stock fund, bond fund, fixed account or stable value fund), as the popularity of VAs grew, insurers began adding a variety of funds to their VA produce line-up. As a result, it is quite common to see current VA products with 70 or more different investment options, including diversified funds like international and emerging markets, energy, target date and “green” funds. Apart from increased investment options, a strong catalyst for VA growth has been the introduction of stronger VA savings and income guarantees. It was during the 1990s that death benefit guarantees (also known as Guaranteed Minimum Death Benefits, GMDB) were first offered to policyholders in the form of a return of premium for the purpose of estate planning. These guarantees quickly grew in complexity and severity over the years; now they include at least one of the following features: roll-ups, ratchets, resets, bonus. Policyholders can add-on these guarantees as riders to a base death benefit (which only contains the return of the premium feature) by paying an extra premium for the rider(s) chosen. The first Guaranteed Minimum Income Benefit (GMIB) followed in 1996 and it was not until the 2000s when the first Guaranteed Minimum Accumulation (GMAB)
and Withdrawal (GMWB) Benefits were introduced.

Table 1.2 summarizes the definitions and the product types that are currently offered in the US marketplace (MorganStanley 2014).

In contrast to Europe, guarantees are offered directly by the insurers, rather than by the underlying investment funds. Typical guarantees currently offered include:

- **GMDB.** It is offered as a lump sum upon death. Typically, the payout to the policyholder is a function of the higher of premium paid and account value. The typical charge level is 15-35 bp;

- **GMIB.** It is offered as a guaranteed income payment upon annuitisation. To determine the guaranteed account value to be annuitised, the initial premium minus withdrawals is accumulated at an annual rate of 5-6%, and then translated into an annual income amount. A waiting period of 5-10 years typically applies. The typical charge level is 50-75 bp;

- **GMAB.** It is offered as a one time “top up” of account value at a specified time, e.g. after 10 years. The guaranteed amount typically equals the initial premium or a roll-up thereof. The typical charge level is 25-60 bp;

- **GMWB.** It is offered as guaranteed amounts via optional annual withdrawals. These are generally limited to 5% or 6% of the initial premium or a ratchet/roll-up amount and are typically offered for life. Higher guaranteed amounts are offered if the policyholder defers the initial withdrawal. Also, guaranteed amounts may increase upon attaining certain age thresholds. The typical charge level was 60-90 bp (until 2008).

The financial market meltdown of 2008-2009 resulted in a number of significant challenges for VA providers, including significant drop-off in VA sales, write-down in deferred acquisition cost, due to reduced profitability, lower levels of hedge effectiveness, and increased losses from hedge breakage, increased levels of required VA risk-based capital and asset adequacy reserves at year-end 2008. In response, companies are currently implementing a number of product design changes: repricing
Annuities

"Bonus Share" or "X Share" annuities noted in the table below, "B Share" and...
their guarantees, charging higher fee levels and restricting certain aggressive product features or asset allocation options; introducing lower-cost fund options, like index funds; introducing simpler, low-cost products without significant guarantees. As a result of the repricing, GMWB rider fees had increased to 90-150 bp by mid-2009, with guaranteed fee levels in the 100-250 bp range. Despite the current financial crisis, the outlook for VA sales globally and in the US is still good, due to a variety of factors:

- a growing number of individuals are reaching retirement age in the US;
- there is a growing pool of retirement assets and roll-over assets;
- product charges are only rising modestly;
- only life insurers can offer lifetime guarantees; banks and mutual funds cannot provide such guarantees;
- a shift in the retirement savings responsibility from employers to employees.

Hence, VAs are expected to continue to be a key product for providing retirement income. In addition, US life insurers can leverage their product expertise internationally. Already, US-style VA products are offered in most major insurance markets in Asia and Europe, generally by multinationals. Despite the current market volatility, there continues to be a significant interest in introducing VA-style products in many markets in Asia and Europe, so this global expansion of VA is likely to continue.

1.6.2 Japan

In many Asian markets, the very first VA products showed up decades ago in the form of unit-linked life contracts with implicit guarantees in case of death but without any explicit charge for those guarantees. However, “true” VAs (including explicit charges for guarantees) have a much shorter history in Asia. The first products were launched in the late 1990s, following financial deregulation within the insurance industry. At first, they had rather limited success. After the second wave of
deregulation in October 2002, which allowed banks and stockbrokers to sell variable annuities via bancassurance channels, the success story for VAs in Asia began, with a rapid growth in sales mainly in Japan. Banks became the predominant distribution channel for variable annuities with sales led by foreign insurers such as Hartford, ING, Manulife and local companies such as Mitsui Sumitomo and Sumitomo Life. Various factors contributed to this development, first of all the fact that Japan has a generally older population (the average issue age for VAs in Japan is 65). Moreover, Japanese people (in particular older generations) are in the habit of saving considerable amounts of money for their old age or to pass on to the next generation. As conservative investors, many savers make extensive use of bank deposits for some or all of their capital. However, due to the prolonged low-interest-rate environment in Japan, there is almost no return left after charges. In spite of this, there are trillion of US dollars invested in Japanese bank deposits. This underlines the higher savings rates experienced in Japan compared with the US and other wealthy countries.

Equity investments did not perform well in Japan during the past decades, so most savers are concerned about the risk of losing money. As a consequence, while savers are interested in investments with higher potential upside, they are also looking for a product with guarantees to help protect their principal. With a guarantee and exposure to upside potential, VAs appear to be a good fit for this need. Distribution through bancassurance provides a natural connection between the money held in bank deposits and the VA product. While in some cases banks may be protective of their deposits, VA sales provided some strong benefits in Japan. As well as saving solvency capital by removing deposits from their balance sheets, banks also earned some commission income in place of the small spread possible on deposits. Moreover, many banks offered investment management for the underlying funds of the VA product. This brings two benefits for the banks: not only they have new fund management income but the VA sales do not even reduce their assets under management.

Looking at all these factors, the success story for VAs in Japan seemed almost inevitable.
Between 2003 and the end of March 2008, the aggregate fund value of the variable annuity business reached ¥15.8 trillion ($154 billion) - approximately 14 times the level in 2003 - as the product benefited from rallying equity market conditions in the country. However, the financial market crisis in 2008 had a significant impact on production. It may be expected that market conditions made investment sales more difficult in general, but the VA market also suffered from some specific issue. In particular, bad news about some foreign companies raised doubts about the financial stability of their Japanese operations and discouraged clients from buying long-term investment products from them. In some cases, there was an even more direct effect on production, as providers withdrew VA products or ceased operations in Japan. Many foreign insurers, in fact, including Hartford and ING, chose to exit the market completely due to losses suffered from variable annuities. Both Hartford and ING stopped selling variable annuity products in Japan in 2009. With a fall in equity markets following the crisis, corresponding variable annuity portfolios fell significantly. Inadequate hedging programmes, over-aggressive minimum guarantees, and exotic underlyings that proved difficult to hedge have all been cited as reasons for the losses experienced by insurance companies. The planned divestment of ING’s Asian operations has also proved difficult, as the lack of buyers highlights how legacy exposures from variable annuities sold in Japan before the financial crisis continue to plague international insurers.

In 2013, ING took a €190 million ($262 million) charge on a hedge aimed at protecting capital for its Japanese variable annuity business. The business still has around 360,000 policies outstanding, with a total account value of €16 billion.

Hartford also reported a first-quarter loss in 2013, as the company realised an after-tax charge of $541 million related to an expansion of its annuities hedging programme in Japan.

Following the slump in variable annuity sales, banks turned to fixed interest rate annuity and whole-life insurance products. But despite the challenging market environment over recent years, sales in variable annuity products have picked up in the past 12 months, coinciding with the rise in the Nikkei. Total sales for variable
Annuity products in 2013 were ¥1 trillion - an increase of more than 50% on sales in 2012.

A significant de-risking by insurers, through hedging of product exposure via reinsurance and product redesign, has also played a part in the shift back to variable annuities.

Hans van Alten, regional vice-president and actuary at Aegon Asia in Tokyo, says: “In hindsight, there were mistakes made and business was priced too aggressively. If you look at some of the variable annuity products before the crisis, there were guarantees on quite a few complex funds, instead of simple index funds linked to the Topix, S&P 500 or EurostoXX 50, which are easier to hedge. Post-crisis, these complex funds or funds with too aggressive allocation to equity have been reduced”. Moreover, most products used to have fixed allocation in equities and bonds, so when markets moved, the provider could not change the allocation. That was risky and expensive. Currently, most variable annuity funds in Japan have a volatility-controlled mechanism with regular rebalancing based on the observed market volatility. Some insurers also had guarantees that were not hedged, or only partly hedged, and only later on added some hedges, but that was a little late in the game.

The introduction of a volatility control target mechanism, where funds are automatically rebalanced by moving either from equities to bonds or cash, has also meant that rather than the traditional CPPI (Constant Proportion Portfolio Insurance) products that suffer from “cash-lock” in a sharp equity downturn, investors in these products are still able to participate in a subsequent market rally. Around 95% of new variable annuity sales now use this mechanism, as opposed to the traditional CPPI mechanism for protection of principal at maturity.

Another relatively new innovation as a result of the low interest rate environment is the emergence of foreign currency denominated variable annuity products. In June 2013, Credit Agricole Life issued an AUD 10-year variable 105% annuity that has so far raised ¥37 billion, while Daichi Frontier Life issued an AUD denominated 110% variable annuity in October 2013 which has raised ¥10 billion to date. Majdi Jemel, Tokyo-based head of financial product development and sales for Japan
at BNP Paribas, says that although most distributors would have only considered Japanese yen-denominated products in the past, the perception that yen weakening will continue led clients to diversify into foreign currency products. The Credit Agricole Life and Daichi Frontier Life products are structured in a way that is similar to a CPPI product. A zero-coupon bond plus leverage on the underlying equity exposure is sold in a general account and special account format. The general account represents the fixed cash amount that will provide for capital return at maturity, while the special account provides leveraged return through the use of derivatives and a volatility target mechanism.

"Because the products are structured in this way it carries very low capital usage for the insurance company. These products also have a market value adjustment feature that allows the company to mitigate interest rate risk. The market adjustment feature allows the insurance company to reflect the mark-to-market on its investment to the clients when they decide to lapse the product. If interest rates rise, the insurer does not have to redeem 100%, but 100% minus the cost of redemption, which means you are protected against lapse risk and interest rate risk. As a result, risk and capital usage are considerably reduced in this product", says Jemel.

Rising equity markets also mean some existing or legacy policies have now come back at close to par. Policies that were issued more than five years ago, which were too expensive to hedge due to the fall in equity markets, are now less prohibitive to hedge than before.

The first generation of variable annuity products mature in the coming years. The challenge is then for insurers and banks to roll over this flow into new products.

"How insurance companies hold this money and roll it over into new products is key. The average age of previous investors who have bought variable annuities was 60 and now they are 70 years old. What insurance companies need to understand is what form of insurance they now want. We need to find new solutions and this may be in the form of a shorter term variable annuity or a regular annuity payment product. This is the conversation we are now having with insurers", says Saffon at Soci Grale.
The next step in innovation, according to BNP Paribas’ Jemel, is for the Japanese market to evolve into regular premium products, similar to what is currently seen in the Korean variable annuity market. “Once interest rates in Japan increase I think we will start to see regular premium products where young investors are looking to build a retirement tool”, he says. Maybe future products should combine features of both a fixed and variable annuity, where the accumulation period and the withdrawal period should be combined into one product. “Rather than selling a 10-year product followed by another 10-year product, issuers should look at selling a 20-year product made up of a 10-year accumulation period where you grow your investment, and 10-year withdrawal period where income is paid out to the policyholder. If you combine these two periods into one product, it makes the design much easier and the value proposition for the client much higher. This would also help address the problem of low interest rates”, he says. “If the product is properly designed with de-risking mechanisms - since we can go relatively long dated in yen interest rate hedges - insurers will be able to find appropriate hedges or reinsurance for this risk” he adds.

1.6.3 Europe

The recorded history of American-style variable annuities as a strategic innovation in Europe began in 2006, when AXA launched its first “TwinStar” product in Germany and AEGON, Royal London and Lincoln started their respective VAs in the UK (which were then quickly followed by Hartford Life and MetLife with VA launches in the UK). Before that, there had been isolated product launches that combined the benefits of unit-linked life-insurance and investment guarantees, in particular with Guaranteed Minimum Death Benefit (GMDBs) and Guaranteed Minimum Accumulation Benefit (GMABs) (such as the Generali’s “Investment Plan Plus” in Switzerland). However, such initiatives had limited visibility and were not positioned to shift the market. The initial strategic innovative push in Europe was dominated by international insurance groups, who were able to import the capabilities, infrastructure and experience required to design and manage VAs from substantial business in North America.
Furthermore, while companies in the UK stayed close to the product design known in established VA markets, AXA departed significantly from known designs by trying to replicate to a great extent the features of the traditional domestic with-profits products in Germany, by offering regular premiums and deferral periods beyond 20 years. During 2007, it was mainly AXA that pushed aggressively to expand its VAs business platform across Europe, with launches in Spain, Italy, France, Belgium and Portugal. More recently, Allianz (which has organized its European VA business in a separate new division “Allianz Global Life”), AEGON and ING all have started to enter additional markets more aggressively with a potential to catch up quickly with AXA in terms of covering of European markets. At the other end of the spectrum, a few local insurers have created their own VA products. Within the past few years VAs have become a widely spread and strategically positioned product line in Europe.

VAs in Europe don’t fill a void as they did in North America or in Japan, since in the most European markets there is still significant supply of traditional participating businesses that offer policyholders both long-term guarantees and the opportunity to participate in higher realized investment returns through profit participation. Also, many European markets are much more dominated by longer wealth-accumulation and dissaving contracts, rather than the short-term single-premium contracts typical in North America that support wealth management during the transition into retirement. In addition to this, distribution is typically through traditional channels (e.g., financial advisors and tied agents), who are often sceptical about the VA proposition and require a combination of support, incentives and direction. On the other hand, insurers have become sceptical of the economics associated with traditional with-profits business, and customers have become increasingly unhappy about the lack of transparency and choice associated with these products. Therefore, VAs appear to be a product class that meets the need of important customer segments and which can be profitably manufactured. This appears to be an important driver behind the particular activity, for instance, in Germany. One of the success factors explaining the rapid growth of VAs in Europe has been the ability to write the product in a
single legal entity and sell in different European markets under the “Freedom of Service” rule of the European Economic Area (EEA), which excludes only Switzerland as a major insurance market in Europe. By invoking this rule, insurers can choose a jurisdiction that provides suitable regulation for VAs. This is particularly important as, in some European jurisdictions, insurers cannot legally write VAs. For instance, in Germany there are strict rules around reserve accounting and the admissibility of assets that don’t allow for the investment guarantees of VAs, and recently proposed legislature that would have changed this situation failed in parliament. The jurisdiction where most VA writers are domiciliated to do pan-European business is Ireland, where the International Financial Services Centre (IFSC) has offered an attractive environment for cross-border business since 1987, and where the large companies now have their VA businesses. Luxembourg, which is considered as having particularly appealing regulation, has attracted only a few companies so far, including Swiss Life. Finally, the Principality of Liechtenstein has some relevance as a jurisdiction that uniquely offers the ability to sell into both the EEA and Switzerland, which is what attracted Swiss insurer Baloise to establish its VA business there. Apart from these pan-European domiciles, some insurers write VAs domestically (e.g., AXA in France sells VAs underwritten by the local entity and not AXA Life Europe in Dublin).

Apart from regulation, an important consideration in the selection of the domicile is the availability of the intellectual resources, expertise and experience, as well as the availability of services such as appointed actuaries, third-party administration and legal advice. Dublin, in particular, is seen as leading in these areas. The majority of the leading international groups have built their business on in-house risk management capabilities, with the ambition to set up hedging operations that would be ultimately suitable for managing all the market risks to which the insurer is exposed through the options and guarantees embedded in the products sold. In contrast, smaller, regional groups typically tend to “outsourc” the risk management by getting the support of a reinsurer or investment bank, and transacting into solutions that allow them to retain only limited market risk. However, this “outsourcing” is usually done with the intent of freeing up capacity during an initial build phase and
subsequently developing the in-house capabilities.

Despite their initial commercial success, VAs are not without problems. Because the European VA market is still young, the impact has so far been more limited than in North America and Asia. In 2007 and 2008, many VA writers blamed the market environment for the volatility in their financial statements, the need for extra capital and slippages in their hedging programmes. As a consequence, in early 2009 several insurers had to reassess their strategy (including exiting the market altogether).

As the experience of many VA writers during the months following the 2008-2009 financial crisis has confirmed, product design and risk management need to go hand in hand, as some product features such as a regular premiums or longer-duration products have proven difficult to hedge when markets become excessively volatile or dry up. In Central and Eastern European markets that are not part of the “Eurozone”, the absence of financial instruments for hedging is a limitation for the development of VAs. There will be a greater necessity to design product features and hedging strategies in synchrony, and insurers with superior access to hedging capabilities will be able to offer product features that are essential for European customers, combining elements of established products, such as participating pensions, with innovative features, such as a choice from a range of investment funds.

1.7 A brief literature review

There have been several papers devoted to the pricing and hedging of variable annuities with various forms of embedded options. The GLWB option has been launched in the market recently, therefore a detailed literature is not yet available. GMWB, which is a similar option except that it guarantees withdrawals over only a fixed number of years, has been analyzed initially by Milevsky & Salisbury (2006). The authors assume continuous withdrawals and a standard geometric Brownian motion model for the dynamics of the underlying fund. They consider two policyholder behavior strategies. Under a static withdrawal approach the contract is decomposed into a Quanto Asian Put option plus a generic term-certain annuity. Numerical
PDE methods are used to evaluate the ruin probabilities for the account process and the contract value. Considering a dynamic approach where optimal withdrawals occur, instead, an optimal stopping problem akin to pricing an American put option emerges, albeit complicated by the non-traditional payment structure. The free boundary value problem is solved numerically. The authors find fees’ values greater than those charged in the market. The optimal behavior approach has been then formalized in [Dai et al. (2008)] where a singular stochastic control problem is posed. [Chen & Forsyth (2008)] explore the effect of various modeling assumptions on the optimal withdrawal strategy of the policyholder, and examine the impact on the guarantee value under sub-optimal withdrawal behavior. The authors moreover propose numerical schemes for pricing various types of guaranteed minimum benefits in VAs using an impulse control formulation. [Bauer et al. (2008)] develop an extensive and comprehensive framework to price any of the common guarantees available with VAs. Monte-Carlo simulation is used to price the contracts assuming a deterministic behavior strategy for the policyholders. In order to price the contracts assuming an optimal withdrawal strategy, a quasi-analytic integral solution is derived and an algorithm is developed by approximating the integrals using a multidimensional discretization approach via a finite mesh. In all these papers the guarantees are priced under the assumption of constant interest rates. [Peng et al. (2012)] derive the analytic approximation solutions to the fair value of GMWB riders under both equity and interest rate risks, obtaining both the upper and the lower bound on the price process. Allowing for discrete withdrawals, [Bacinello et al. (2011)] consider a number of guarantees under a more general financial model with stochastic interest rates and stochastic volatility in addition to stochastic mortality. In particular for GMWBs, a static behavior strategy is priced using standard Monte Carlo whereas an optimal lapse approach is priced with a Least Squares Monte Carlo algorithm. The pricing models of GLWB can be considered as extensions of those concerning the GMWB guarantee together with the inclusion of mortality risk. [Shah & Bertsimas (2008)] analyze the GLWB option in a time continuous framework considering simplified assumptions on population mortality and adopting different asset pricing models. [Holz]
et al. (2012) price the contract for different product design and model parameters under the Geometric Brownian Motion dynamics of the underlying fund process. They also consider various forms of policyholder withdrawal behavior, including deterministic, probabilistic and stochastic models. Other papers investigate the impact of volatility risk, for example Kling et al. (2011). Piscopo & Haberman (2011) assess the mortality risk in GLWB but not the other risks and their interactions. A detailed analysis of the impact of systematic mortality risk on valuation and hedging, as well as its interaction with other risks underlying the GLWB is not already available. Fung et al. (2014), in particular, deal with these aspects, analyzing equity and systematic mortality risks underlying the GLWB, as well as their interactions. However, a simple Black Scholes framework is considered, thus interest rates and volatility of returns are assumed to be constant. In this thesis, we propose a generalization of the model considering more realistic assumptions.
Chapter 2

The GLWB option: the valuation model

In this chapter we will remind the main features of the GLWB option and then we will introduce the model used for its valuation.

2.1 The structure of the contract

In recent years variable annuities with a GLWB option are reaching increasingly popularity since they satisfy medium to long term investment needs providing adequate hedging against market volatility and longevity related risks. Indeed, based on an initial capital investment, the GLWB option guarantees the policyholder a stream of future payments, independently of the performance of the underlying portfolio, for his/her whole life.

In what follows we restrict our analyses to immediate variable annuities with a single premium payment up-front. Specifically, upon contract signature, the policyholder pays a sum of money that is invested in a well diversified asset portfolio. Customers can usually influence the risk-return profile of their investment by choosing from a selection of different mutual funds. Movements in the investment portfolio are recorded in an account called “Variable Annuity sub-account” (VA sub-account). The GLWB
option guarantees the policyholder a fixed or variable sum of money on set dates for his/her whole lifetime, regardless of market performance. Consequently, even if the value of the VA sub-account drops to zero while the insured is still alive, he/she can still continue to withdraw the guaranteed amount periodically until death. This sum is deducted from the VA sub-account if it has a positive balance, otherwise it is paid by the insurance company from its own capital. As we described in the first chapter, GLWB products can contain certain features that lead to an increase of the guaranteed withdrawal amount if the underlying funds perform well. Usually, on every policy anniversary, the current account value of the client is compared to a certain withdrawal benefit base. Whenever the first amount exceeds the second one, the policyholder can withdraw a higher amount than that initially guaranteed. This increase may either be permanent (withdrawal “step-up” or “ratchet”) or be effective just for the single withdrawal (“surplus distribution” or “performance bonus”). The insured can also decide not to withdraw money in the first years of the contract to take advantage of roll-up feature. In what follows we don’t consider these adjustments upwards of the benefit base, and we set it equal to the policyholder’ upfront payment. The client is allowed to surrender the contract, which is the same as withdrawing the whole account value (in this case the contract obviously terminates), or, of course, to withdraw just a portion of it. The periodic amount that may be withdrawn must not exceed some maximum value, but it’s clear that within this guarantee type, the total amount of withdrawals is free. In case the insured dies before the VA sub-account was depleted and/or the contract was surrendered, any remaining value is paid to the beneficiary as a death benefit. To cover the costs of the guarantee, the insurance company charges a fee, which is usually a pre-specified annual percentage of the account value, of course only as long as there are any assets left.

From these considerations it follows clearly that the prediction of the policyholder behaviour is one of the key element in the valuation of GLWB guarantee. For this aim, there are three alternative assumptions concerning the policyholder choices, that lead to corresponding valuation approaches, called static, dynamic and mixed.
The so-called passive or static valuation approach is characterized by the assumption that the policyholder withdraws exactly the amounts contractually specified; moreover, surrender is not allowed.

In the so-called active or dynamic valuation approach the policyholder is assumed to withdraw amounts not necessarily coinciding with those contractually specified; in particular, he/she can decide not to withdraw, or to surrender the contract. In addition, partial withdrawals or surrender decisions could be made on dates not coinciding with those contractually specified.

In the mixed approach, the policyholder is assumed to be “semiaactive”, meaning that he/she withdraws exactly when and what contractually specified but, unlike the static approach, at any time during the life of the contract he/she may decide to surrender.

From the insurer’s point of view, the dynamic approach assumes the worst case scenario since the policyholder can choose among all withdrawal strategies and, in particular, the surrender time. In the mixed approach, instead, the policyholder can choose only the surrender time, so that his/her “optimal” strategy is selected within a subset of that considered by a dynamic agent. Finally, the static approach defines a single, specific, withdrawal strategy included in the previous subset. As a consequence, the proportional fees that have to be applied to the account value to make the contract fair are ordered in the same way: they are the highest with the dynamic approach and the smallest with the static one.

In what follows we consider a static approach, in which the policyholder withdraws exactly the guaranteed amount each year. Important reasons support our choice. First of all, VA providers can influence the behavior of policyholders through imposing penalty charges on the amount of withdrawal that exceeds the guaranteed amount. In practice, additional high indirect costs in terms of taxes on the excess distributions make taking large strategic withdrawals even more unattractive. Moreover, we have to consider that these options are being introduced in pension plans, in order to ensure a constant income during retirement and provide protection
against market downside risk (Piscopo & Haberman (2011)). In addition, Holz et al. (2012) note that the value of a lifetime GMWB and so the fair guarantee fee under optimal customer behavior differs only slightly from that assuming deterministic behavior. In closing, a typical individual insured is unable to hedge risks due to his/her own longevity and less equipped than large institutions like insurance companies to hedge financial risks. Hence in our analysis we consider a typical investor with a more simplistic deterministic withdrawal behavior compared with an arbitrageur.

As seen previously, we are dealing with products that bear two main different (independent) types of risk. First of all, we can consider the financial risk (related to the market). This risk was clearly stressed during the last few years, when the major stock market indices have dropped so much. On the other hand, the insurer deals with another type of risk, let’s call it actuarial risk, related to the possibility of death for the insured (and hence the possibility for the embedded guarantee to activate). While the financial market model might be complete (any contingent claim is replicable by a trading strategy), the model that assumes both risks (financial and actuarial) is incomplete.

2.2 The valuation model

The simplest type of a GLWB attached to a VA, which will be referred to as a plain GLWB, is described in a continuous time setting. In actual practice, withdrawal of discrete amount occurs at discrete time instants during the life of the policy. Mathematically, instead, it is more convenient to construct the pricing model of the annuity policy that assumes continuous withdrawals.

In the subsequent subsections, we will introduce the components of the model: the financial market and the mortality intensity. We will first describe them separately, and then successively we will combine them into the insurance market model. The valuation framework in this section follows mainly the one used in Fung et al. (2014).
2.2.1 The financial market

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a filtered probability space, where \(\mathbb{P}\) is the real world or physical probability measure and \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\) is a filtration (i.e. an increasing sequence of \(\sigma\)-algebras \(\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots\)) satisfying the usual conditions of right continuity, i.e. \(\mathcal{F}_t = \bigcap_{u \geq t} \mathcal{F}_u\), and \(\mathbb{P}\)-completeness, i.e. \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets. The filtration \(\mathcal{F}_t\) describes the total information available at time \(t\) and it has to be large enough to support the processes representing the evolution of financial variables and of mortality. We will explain later how to build it.

Consider a policyholder who invests his/her retirement savings in an immediate variable annuity with a GLWB option. Let \(P\) be the upfront single premium paid at the inception of the contract, \(t = 0\). No initial sales charge is applied, so the deposited amount is entirely used for immediate investment in the available sub accounts. Let \(x\) the age of the policyholder at time \(t = 0\), and suppose that \(\omega\) is the maximum attainable age (or limiting age), i.e. the age beyond which survival is assumed to be impossible. The limiting age \(\omega\) allows for a finite time horizon \(T = \omega - x\).

Suppose that the investment portfolio has both equity and fixed income exposure. Under the real world probability measure \(\mathbb{P}\), we assume that the riskless component (fixed income investment) is modelled by the money market account \(B(t)\) with the following ordinary differential equation:

\[
\text{dB}(t) = rB(t)dt \tag{2.1}
\]

where \(r \geq 0\) is the instantaneous interest rate. Setting \(B(0) = 1\), we have \(B(t) = e^{rt}\) for \(t \geq 0\).

The risky component is a stock (or stock index) whose price under \(\mathbb{P}\) follows the usual Geometric Brownian motion:

\[
\text{dS}(t) = \mu S(t)dt + \sigma S(t)dW_S(t), \quad S(0) > 0 \tag{2.2}
\]

where \(\mu\) is the drift of the process (whose economic interpretation is the expected return on equity, i.e. the riskless rate plus the equity risk premium), \(\sigma\) is the expected volatility of the stock price (also known as the diffusion parameter of the
process), $W_S(t)$ is a standard Wiener process. The process basically assumes that equity returns $(dS(t)/S(t))$ have a normal distribution with average $\mu dt$ and standard deviation $\sigma \sqrt{dt}$ and that returns are independent over time. It therefore models equity returns as the sum of a deterministic component proportional to the value of the parameter $\mu$ and of a random component $\sigma dW_S(t)$ which generates random increases/decreases that are, however, independent and identically distributed.

We initially assume interest rate $r$ and equity volatility $\sigma$ to be constant. In the following chapters we will discuss the cases of stochastic interest rates and/or volatility.

It is assumed that price processes $(S,B)$ are adapted, i.e. for each $t$, $S_t$ is $\mathcal{F}_t$-measurable; in particular $B$ is deterministic. This has the usual interpretation that at time $t$ we know the current outcome of the stock and the savings account.

Let $B = (B_t)_{0 \leq t \leq T}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by $(B,S)$, i.e. $B_t = \mathcal{B}_t^+ \vee \mathcal{N}$, where $\mathcal{N}$ is the $\sigma$-algebra generated by all $\mathbb{P}$-null sets and

$$\mathcal{B}_t^+ = \sigma\{(B(u), S(u)), u \leq t\} = \sigma\{S(u), u \leq t\} = \sigma\{W_S(u), u \leq t\},$$

since $W_S$ accounts for all the randomness in the model defined by (2.1) and (2.2).

Let us define the pair $\varphi(t) = (\xi(t), \eta(t))$ as the portfolio held at time $t$, where $\xi(t)$ is the number of stocks held at time $t$ and $\eta(t)$ denotes the deposit on the savings account at time $t$.

Therefore the reference investment fund $V(\cdot)$ can be written as:

$$V(t) = \xi(t)S(t) + \eta(t)B(t)$$

and so its dynamics is given by:

$$dV(t) = \xi(t)dS(t) + \eta(t)dB(t)$$

$$= \xi(t)[\mu S(t)dt + \sigma S(t)dW_S(t)] + \eta(t)[r B(t)dt]$$

$$= [\mu \xi(t)S(t) + r \eta(t)B(t)]dt + \sigma \xi(t)S(t)dW_S(t)$$

Let $\pi(t) = \frac{\xi(t)S(t)}{V(t)}$ denote the proportion of the retirement savings being invested in the equity component. Consequently, $1 - \pi(t) = \frac{\eta(t)B(t)}{V(t)}$ is that invested in the fixed income component. All the usual assumptions on the market hold: there are no
arbitrage opportunities (i.e., there is no way to make a riskless profit), it is possible to borrow and lend any amount, even fractional, of cash at the fixed riskless rate, it is possible to buy and sell any amount, even fractional, of the stock (this includes short selling) and the above transactions do not incur any fees or costs (i.e., frictionless market). However, in our model, we assume $0 \leq \pi(\cdot) \leq 1$. We can rewrite equation (2.3) as:

$$dV(t) = \left[ \mu \pi(t) + r(1 - \pi(t)) \right] V(t) dt + \sigma \pi(t) V(t) dW_S(t)$$  \hspace{1cm} (2.4)$$

Therefore the dynamics of the relative returns can be written as:

$$\frac{dV(t)}{V(t)} = \left[ \mu \pi(t) + r(1 - \pi(t)) \right] dt + \sigma \pi(t) dW_S(t)$$  \hspace{1cm} (2.5)$$

In the following we assume that $\pi(\cdot)$ is constant, that is the policyholder invests a fixed proportion of his/her retirement savings in equity and fixed income markets throughout the investment period. Clients can influence the risk-return profile of their investment by choosing from a selection of different mutual funds, from more conservative to more dynamic asset combinations. Larger values of $\pi$ correspond to a larger equity exposure, and this risk-taker could result in a higher potential growth but it will be subject to higher volatility as well. Note that from the insurer’s perspective, the buyer’s portfolio choice can have a substantial impact on the profitability of the variable annuity contract. In fact, individuals could increase risk and return in their portfolios to the point that the guarantee becomes unprofitable for the insurers. This is the reason that many actual prospectus of offered VAs restrict investment choices for their buyers.

As results from the description of the policy, the VA sub-account held by the policyholder is influenced by the variable market performance, the guarantee fees charged by the insurance company and the periodic withdrawals provided by for the contract.

Denote with $A(t)$ the VA account value at time $t$.

Since the initial premium is invested in the market, it is subject to daily fluctuations (at least considering the equity component), the size and extent of which remain a priori uncertain. Therefore, also the balance of the VA account at a given point in time $t$, $A(t)$, could be either positive or negative. Should market performance result
in low or negative returns, \( A(\cdot) \) may reduce to zero or even fall below this value.

The other two elements (fees and withdrawals) are deducted from the VA sub-account, so they reduce its value.

Let \( \alpha \) be the annual fee rate applied by the insurance company for activating the GLWB option. Fees are deducted from the account value as long as the contract is in force and the account value is positive.

Let \( \gamma(t) \) be the withdrawals made by the policyholder at time \( t \).

The above considerations imply that the dynamics of the VA sub-account can be described using the following stochastic differential equation:

\[
dA(t) = -\alpha A(t)dt - \gamma(t)dt + \frac{dV(t)}{V(t)} A(t) \tag{2.6}
\]

or equivalently, from equation (2.5) and recalling that the equity allocation \( \pi(\cdot) \) is a constant, say \( \pi \):

\[
dA(t) = -\alpha A(t)dt - \gamma(t)dt + A(t) \{ [\mu \pi + r(1 - \pi)]dt + \sigma \pi dW_S(t) \} \\
= (\mu \pi + r(1 - \pi) - \alpha) A(t)dt - \gamma(t)dt + \sigma \pi A(t)dW_S(t) \tag{2.7}
\]

This equation holds as long as \( A(\cdot) \geq 0 \). In fact, once \( A(\cdot) \) hits the zero value, it remains to be zero forever afterwards. That is, the zero value is considered to be an absorbing barrier of \( A(\cdot) \). Furthermore, being \( P \) the amount originally paid by the policyholder, we have:

\[
A(0) = P
\]

In other words, upon contract signature (at time \( t = 0 \)), the balance of the VA sub-account exactly matches the initial investment made by the policyholder.

Using \( g(t) \) to define the withdrawal rate allowed by the insurance company at time \( t \), the withdrawals \( \gamma(t) \) guaranteed at time \( t \) are given by:

\[
\gamma(t) = g(t)P
\]

It would be reasonable to assume that the withdrawals made by the policyholder at a given time \( t \) can range between a minimum value equal to zero and a maximum value equal to the value of the VA sub-account at that point in time, therefore that

\[
0 \leq \gamma(t) \leq A(t)
\]
However, our model considers a static approach for the valuation of the option: individual investors behave passively in utilizing their guarantee, in other words they always withdraw the guaranteed amount and hold the contract to maturity. From now on, we assume that the withdrawal rate does not vary over time but remains constant:

\[ g(t) = g \]

as well as withdrawals, hence we have:

\[ \gamma(t) = gP = G \]

namely, the model does not consider the possibility of increasing or reducing the amount withdrawn depending on the financial needs of the policyholder. Early lapses are not considered.

With these considerations in mind, we can write the dynamics of the VA sub-account as:

\[
\begin{aligned}
    dA(t) &= (\mu \pi + r(1 - \pi) - \alpha)A(t)dt - Gdt + \sigma \pi A(t)dW_S(t) \\
    A(0) &= P \\
    A(\cdot) &\geq 0
\end{aligned}
\]

The GLWB option is activated and has a positive value only if the process hits zero before the death date of the policyholder. If, due to declining stock markets combined with the reducing effect of fees and withdrawals, the account value of the policy becomes zero while the insured is still alive, then the GLWB guarantee becomes effective and the insured can continue to withdraw the same guaranteed amount annually until death. In this case, the account balance is not sufficient to fund the guaranteed withdrawals and intervention by the insurance company is necessary. If, on the contrary, the dynamics of the VA sub-account is such that “ruin” never occurs (or occurs after the policyholder has passed away), then the GLWB guarantee has a zero payout. Indeed, in this case, the account balance is in itself sufficient to assure the policyholder of all the withdrawals until his/her death and the guarantee therefore does not need to be activated.
2.2.2 The mortality model

An important requirement for the GLWB’s activation is the survival of the policyholder. For this reason it’s important to consider the uncertainty related to the random residual lifetime of insureds (mortality risk) in addition to that related to financial factors (financial risk). Traditionally, actuaries have been treating the demographic assumptions in a deterministic way. They have modeled mortality either adopting adjusted/projected mortality tables, either assuming suitably parametrized analytical models. The projection of mortality rates has been based on the assumption that the past represents the future and the differences between the projected rates and realized rates, the so-called mortality risk, can be diversified among individuals and/or over the time. Those are indeed very strong assumptions. Over the last century, evidences have emerged to reveal that mortality risk is neither predictable nor diversifiable. In fact, it has been shown that the mortality projections in the last fifty years have systematically underestimated the overall mortality improvement. And the consequent adverse financial impacts caused by mis-assessing mortality risk have to be carefully evaluated. As a result, the traditional deterministic actuarial approach is now seen to be inadequate for the calculation of fair values. Great efforts have been made in the past few years to explore the use of stochastic approaches to model the mortality dynamics and to evaluate the mortality-linked securities.

Traditionally, a central role in the definition of a mortality model has been played by the force of mortality (or mortality intensity), defined as the instantaneous rate of mortality at a given age $x$:

$$
\mu_x = \lim_{t \to 0} \frac{\mathbb{P}(T_x \leq t)}{t} = \lim_{t \to 0} \frac{t q_x}{t} 
$$

being $T_x$ the random variable that describes the duration of life for an $x$-years old individual and $t q_x$ the probability that he/she dies before age $x + t$ (with $x$ and $t$ real numbers). Defining the survival function as $S(x) = \mathbb{P}(T_0 > x)$ we can express the mortality intensity as the opposite of the derivative of its logarithm:

$$
\mu_x = -\frac{d}{dx} \log S(x) 
$$
Introducing the boundary condition $S(0) = 1$, it’s immediate to obtain:

$$S(x) = e^{-\int_0^x \mu_u du}$$

The force of mortality is a good tool for approximating the mortality of the individual at age $x$, since it can be shown that:

$$P(x < T_0 \leq x + \Delta x \mid T_0 > x) = \mu_x \Delta x + o(\Delta x)$$

i.e. the probability of dying in a “short” period of time after $x$, between age $x$ and age $x + \Delta x$, can be approximated by $\mu_x \Delta x$, when $\Delta x$ is small. The force of mortality is generally increasing as $x$ increases (there are some exceptions, in correspondence to very small values of $x$, due to the infant mortality, and values around 20-25, due to the young mortality hump). When allowing for mortality improvements over time, it is evident that the force of mortality has to show a dependence also on calendar year, and not only on age. Thus, the force of mortality can be described by a two variable function $\mu_x(y)$, where $y$ indicates the calendar year. As time $y$ increases and the age $x$ remains fixed, the decreasing mortality rates over time translate into a decreasing function $\mu_x(y)$. In what follows we consider a process for the mortality intensity for a particular generation and a particular initial age. Thus, the approach adopted is a “diagonal” one.

The recent literature on mortality modelling is prolific and widely inspired from credit risk theory (regarding modeling time to default of firms). Applications of this mathematical framework to dynamic mortality modeling and to insurance products pricing can be found in Biffis (2005), Milevsky & Promislow (2001), Dahl & Møller (2006), Dahl (2004), Biffis & Millossovich (2006), Ballotta & Haberman (2006), Cairns et al. (2006). The similarities between the time to default and the remaining duration of life is strong, and although the factors underlying the death of an individual and the default of a firm are obviously completely different, the mathematical tools used in the two literatures are the same. The aforementioned researchers make use of the similarities between mortality risk and interest rate risk. They suggest modifying the models arising in the interest rate sector to obtain mortality rate models. However, while mathematically similar at a certain conceptual level, mortality
rates behave very differently from interest rates. For example, the term structure of mortality rates should only be increasing to reflect the biologically reasonableness for age-specific pattern of mortality, whilst interest rates can reverse in some situation. While the mean-reverting property is a desirable property for interest rates, it is doubtful that mean-reverting is realistic for mortality dynamics.

In particular, [Cairns et al. (2006)](#) list the criteria that any “plausible” stochastic mortality model would meet:

- the model should keep the force of mortality positive;
- the model should be consistent with historical data;
- the long-term future dynamics of the model should be biologically reasonable;
- long-term deviations in mortality improvements from those anticipated should not be mean-reverting to a pre-determined target level, even if this target is time dependent and incorporates mortality improvements. In contrast, short-term deviations from the trend due to local environmental fluctuations might be mean-reverting around the stochastic long-term trend. The inclusion of mean reversion entails that if mortality improvements have been faster than anticipated in the past then the potential for further mortality improvements will be significantly reduced in the future. In extreme cases, significant past mortality improvements might be reversed if the degree of mean reversion is too strong. Such extreme mean reversion is difficult to justify on the basis of previous observed mortality changes and with reference to our perception of the timing and impact of, for example, future medical advances. Short-term trends might be detected by analysing carefully recent developments in healthcare and in the pharmaceutical industry, but even then the precise, long-term effects of such advances are difficult to judge. As we peer further into the future, it becomes even more difficult to predict what medical advances there might be, when they will happen, and what impacts they will have on survival rates. All of these uncertainties rule out strong mean reversion in a model for stochastic mortality;
the model should be comprehensive enough to deal appropriately with the
current pricing, valuation or hedging problem.

An efficient valuation model should also integrate complexity and computational
tractability of pricing and estimation. In this respect, affine diffusion processes have
shown to be useful, see for instance Biffis (2005), Dahl (2004), Dahl & Moller (2006),

Affine processes are a class of Markov processes with conditional characteristic
function of the exponential affine form. A thorough treatment of such processes is
provided in Duffie et al. (2003) and Biffis (2005). In this work, we adopt the nar-
rrower but more usual (in financial applications) perspective based on the definition
of affine processes in terms of strong solutions to specific stochastic differential equa-
tions (SDEs) in a given filtered probability space.

**Definition 1** We fix a probability space \((\Omega, \mathcal{F}, P)\) and a filtration \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) satis-
fying the usual conditions and representing the information available up to time \(t\).
An \(\mathbb{R}^n\)-valued affine diffusion \(X\) is an \(\mathcal{F}\)-Markov process specified as the strong solu-
tion to the following SDE:

\[
dX_t = \delta(t, X_t)dt + \sigma(t, X_t)dW_t
\]

where \(W\) is an \(\mathcal{F}\) standard Brownian motion in \(\mathbb{R}^n\) and where the drift \(\delta\) and the
instantaneous covariance matrix \(\sigma \sigma^T\) have affine dependence on \(X\).

The convenience of adopting an affine process in modelling the mortality intensity
lies in the fact that, under technical conditions (see Duffie & Singleton (2003)), it
yields:

\[
E[e^{\int_t^T -\mu_{x+u}(u)du}\mid \mathcal{F}_t] = e^{\alpha(T-t)+\beta(T-t)\mu_{x+t}(t)}
\]

where the coefficients \(\alpha(\cdot)\) and \(\beta(\cdot)\) satisfy generalized Riccati ordinary differential
equations (ODEs). The latter can be solved at least numerically and in some cases
analytically.
Given the criteria above and according to Fung et al. (2014) we adopt a one-factor, non mean-reverting and time homogeneous affine process for modeling mortality intensity $\mu_{x+t}(t)$ of a person aged $x$ at time $t = 0$, as follows:

$$
\begin{align*}
    d\mu_{x+t}(t) &= (a + b\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)}dW_M(t) \\
    \mu_x(0) &> 0
\end{align*}
$$

(2.9)

with $a \neq 0$, $b > 0$ and $\sigma_\mu$ being the volatility of the mortality intensity.

The values of the parameters $a$, $b$, and $\sigma_\mu$ are obtained by calibrating the survival curve implied by the mortality model to the survival curve obtained from population data as documented in the Australian Life Tables 2005-2007 (see Fung et al. (2014) for further details).

It is reasonable to assume the independence of the randomness in mortality and that in interest rates, so $W_M$ denotes a standard Brownian motion independent of $W_S$.

### 2.2.3 The combined model

Until now, we have considered a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})$ large enough to support the processes representing the evolution of financial variables and of mortality. Moreover, we have focused on a representative insured aged $x$ (=65 years old) at time $t = 0$, with random residual lifetime described by an $\mathcal{F}$-stopping time $\tau_x$. From now on we drop reference to the age, and set $\tau_x = \tau$. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ represents the flow of information available as time goes by: this includes knowledge of the evolution of all state variables up to each time $t$ and of whether the policyholder has died by then. Formally we write:

$$
\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t
$$

where $\mathcal{G}_t \vee \mathcal{H}_t$ is the $\sigma$-algebra generated by $\mathcal{G}_t \cup \mathcal{H}_t$, with

$$
\mathcal{G}_t = \sigma(W_S(s), W_M(s) : 0 \leq s \leq t)
$$

$$
\mathcal{H}_t = \sigma(1_{\{\tau \leq s\}} : 0 \leq s \leq t)
$$
Thus, $\mathcal{G}_t$ is generated by the two independent standard Brownian Motions, $W_S$ and $W_M$, which describe the uncertainties related to equity and mortality intensity, respectively, and $\mathcal{H}_t$ describes the information set that indicates if the death of the policyholder has occurred before time $t$.

It is a well-known result in asset pricing theory that, under reasonable economic assumptions, the market price of a security is given by its expected discounted cashflows. Discounting takes place at the risk-free rate and the expectation is taken with respect to a suitably risk-adjusted probability measure. The incompleteness of insurance markets implies that infinitely many such probabilities exist. We assume henceforth that the insurer has picked out a specific probability for valuation purposes, say $Q$. In particular we define $W^Q_S(t)$ and $W^Q_M(t)$ as:

$$dW^Q_S(t) = \frac{\mu - r}{\sigma}dt + dW_S(t)$$

$$dW^Q_M(t) = \lambda \sqrt{\mu_{x+t}(t)}dt + dW_M(t)$$

By the Girsanov Theorem these are standard Brownian motions under the $Q$ measure with $\frac{\mu - r}{\sigma}$ and $\lambda \sqrt{\mu_{x+t}(t)}$ representing the market price of equity risk and systematic mortality risk, respectively.

If we consider the new probability space $(\Omega, \mathcal{F}, \mathbb{F}, Q)$, the evolutions of the VA sub-account and of mortality intensity become:

$$dA(t) = (r - \alpha)A(t)dt - Gdt + \pi \sigma A(t)dW^Q_S(t)$$

$$d\mu_{x+t}(t) = (a + (b - \lambda \sigma)\mu_{x+t}(t))dt + \sigma_{\mu} \sqrt{\mu_{x+t}(t)}dW^Q_M(t)$$

### 2.2.4 The valuation formula: two valuation perspectives

There are two perspectives from which to view the GLWB rider ([Hyndman & Wenger 2014](#)). A policyholder is likely to view the VA and the GLWB rider as one combined instrument and would be interested in the total payments received over the duration of the contract. On the other hand, although the rider is embedded into the VA, the insurer might want to consider it as a separate instrument. Namely the insurer is
interested in mitigating and hedging the additional risk attributed to the rider.

The policyholder’s perspective

As extensively described previously, recall that a GLWB option offers a lifelong guarantee: the maximum amount to be periodically withdrawn is specified, but the cumulated total amount is not limited and the insured can annually request a portion of the premium paid while he or she is still alive, even if the fund value drops to zero. Moreover, any remaining account value at the time of death is paid to the beneficiary as death benefit. Therefore, from a policyholder’s perspective, the risk-neutral value at time $t$ of the GLWB is the sum of the no-arbitrage values of the living and death benefits.

Living benefits are represented by static withdrawals made by the policyholder during the lifetime of the contract while he/she is alive. The income from these withdrawals can be regarded as an immediate life annuity, whose no-arbitrage value at time $t$ is equal to:

$$LB^P(t) = I\{\tau > t\} G \int_{0}^{\omega - x - t} s P_{x+t} e^{-rs} ds$$  \hspace{1cm} (2.14)

where $0 \leq t \leq \omega - x$, $I\{\tau > t\}$ is an indicator function taking value of one if the individual is still alive at time $t$, and zero otherwise and $P_{x+t}$ is the $Q$-survival probability at time $t + s$ of an individual alive and aged $x + t$ at time $t$. Death benefits can be calculated considering the payoff that the beneficiary will receive at the random time of policyholder’s death, $\tau$. Therefore we can write\footnote{Recall that $A(t) \geq 0 \ \forall t$ because, once the account process hits the zero value, it remains to be zero forever afterwards. That is, the zero value is an absorbing barrier of $A(\cdot)$ . Hence, we don’t need to take its positive part.}

$$DB^P(\tau) = A(\tau)$$  \hspace{1cm} (2.15)

The market value at time $t$ of the death benefit is given by:

$$DBV^P(t) = I\{\tau > t\} \int_{0}^{\omega - x - t} f_{x+t}(s) E^{Q}_{t}(e^{-rs}(A(t + s))) ds$$  \hspace{1cm} (2.16)

where $f_{x+t}(s) = -\frac{d}{ds}(s P_{x+t})$ is the density function under $Q$ of the remaining lifetime of an individual aged $x + t$ at time $t$ and $E^{Q}_{t}$ denotes conditional expectation.
Denote by $V^P(t)$ the value at time $t$ for the complete contract (VA plus GLWB rider). Both $LB^P(t)$ and $DBV^P(t)$ are cash inflows, while the amount in the investment account $A(t)$ is viewed as a cash outflow to the VA provider. The risk-neutral value of the withdrawals and any terminal account value at time $t$ is therefore:

$$V^P(t) = LB^P(t) + DBV^P(t) - \mathbb{1}_{\{\tau > t\}}A(t) \quad (2.17)$$

The value at time $t = 0$ is:

$$V^P(0) = LB^P(0) + DBV^P(0) - A(0) \quad (2.18)$$

In particular, we have:

$$sP_x = \mathbb{E}^Q[e^{-\int_0^t \mu_x(u)du}]$$

therefore:

$$f_x(s) = -\frac{d}{ds}sP_x = \mathbb{E}^Q[e^{-\int_0^s \mu_x(u)du} \mu_{x+s}(s)]$$

The contract value at time $t = 0$ is given by:

$$V^P(0) = G \int_0^{\omega-x} sP_x e^{-rs} ds + \int_0^{\omega-x} \mathbb{E}^Q[e^{-\int_0^s \mu_x(u)du} \mu_{x+s}(s)]E^Q(e^{-rs}A(s)) ds - A(0)$$

The independence between $W^Q_S$ and $W^Q_M$ implies that:

$$V^P(0) = G \int_0^{\omega-x} sP_x e^{-rs} ds + \int_0^{\omega-x} \mathbb{E}^Q[e^{-\int_0^s \mu_x(u)du} \mu_{x+s}(s)e^{-rs}A(s)] ds - A(0)$$

equivalently:

$$V^P(0) = \mathbb{E}^Q \left[ \int_0^{\omega-x} \left( Ge^{-rs} e^{-\int_0^s \mu_x(u)du} + A(s)e^{-rs} e^{-\int_0^s \mu_x(u)du} \mu_{x+s}(s) \right) ds \right] - A(0)$$

or, in more compact terms:

$$V^P(0) = \mathbb{E}^Q \left[ \int_0^{\omega-x} e^{-\int_0^s \mu_x(u)du} e^{-rs} \left( G + A(s)\mu_{x+s}(s) \right) ds \right] - A(0) \quad (2.19)$$

The guarantee is considered fair to both, policyholder and insurer, at time $t = 0$, if it holds:

$$V^P(0) = 0 \quad (2.20)$$
As a consequence, the fair fee rate is defined as the rate $\alpha^* \geq 0$ that solves (2.20):

$$\alpha^* : V^P(0; \alpha^*) = 0$$

This equation does not have a closed form solution and numerical methods must be used to find $\alpha^*$.

**The random time of death**

It is possible to obtain the risk neutral value of the contract also in terms of the policyholder’s random time of death. Recall that we are modeling the policyholder’s random residual lifetime as an $\mathbb{F}$-stopping time $\tau$ admitting a random intensity $\mu_x$. Specifically, we regard $\tau_x$ as the first jump-time of a nonexplosive $\mathbb{F}$-counting process $N$ recording at each time $t \geq 0$ whether the individual has died ($N_t \neq 0$) or not ($N_t = 0$) (Biffis (2005)). To improve analytical tractability, we further assume that $N$ is a doubly stochastic (or Cox) process driven by a subfiltration $\mathbb{G}$ of $\mathbb{F}$, with $\mathbb{G}$-predictable intensity $\mu$.

We assume that the nonnegative predictable process $\mu$ satisfies $\int_0^t \mu_s ds < \infty$ a.s. for all $t > 0$. We then fix an exponential random variable $\Phi$ with parameter 1, independent of $\mathbb{G}_\infty$. Under these assumptions, Biffis (2005) defines the random time of death $\tau$ as the first time when the process $\int_0^\tau \mu_{x+s}(s) ds$ is above the random level $\Phi$, so we set:

$$\tau = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \mu_{x+s}(s) ds > \Phi \right\}$$

With these considerations, we can express the risk-neutral value of the GLWB option as:

$$V^P(0) = E^\mathbb{Q} \left[ gA(0) \int_0^\tau e^{-rs} ds + e^{-r\tau} A(\tau) - A(0) \right]$$

and consequently, the fair fee rate as:

$$\alpha^* : E^\mathbb{Q} \left[ gA(0) \int_0^\tau e^{-rs} ds + e^{-r\tau} A(\tau) - A(0) \right] = 0$$

**The insurer’s perspective**

The alternative valuation prospective, concerning the insurer, considers the GLWB rider as a standalone product.
Recall that the trigger time defined by Milevsky & Salisbury (2006) is the first passage time of the process $A(t)$ hitting the zero value, that is
\[
\zeta = \inf\{t \geq 0 : A(t) = 0\} \quad (2.25)
\]

Once $A(t)$ hits the zero value, it remains to be zero forever afterwards. That is, the zero value is considered to be an absorbing barrier of $A(t)$ as we have already explained earlier. We use the convention $\inf(\emptyset) = \infty$. If $\zeta \leq T$ we say that the option is triggered (or exercised) at trigger time $\zeta$. Therefore, under $\mathbb{Q}$, the value process of the VA sub-account is given by:
\[
\begin{cases}
  dA(t) = (r - \alpha)A(t)dt - Gdt + \pi \sigma A(t) dW^Q_S(t) \\
  A(0) = P 
\end{cases}
\quad \text{for } 0 \leq t < \zeta \quad (2.26)
\]

and
\[
A(t) = 0 \quad \text{for } t \geq \zeta
\]

Under this approach, the rider value process can be defined as the risk-neutral expected discounted difference between future rider payouts and future fee revenues, or the expected discounted benefits minus the expected discounted premiums.

At time $\zeta$, if the policyholder is still alive, the rider guarantee entitles the policyholder to receive an annual payment of $G$ until his/her death. The expected discounted benefits are therefore calculated as
\[
B^I(t) = \mathbb{I}_{\{\tau > t\}} \int_0^{\omega - x - t} f_{x+t}(s) E_t^Q \left( \int_{t+\zeta}^{t+s} gA(0)e^{-r(v-t)} \mathbb{I}_{\{s > \zeta\}} dv \right) ds \quad (2.27)
\]
\[
= \mathbb{I}_{\{\tau > t\}} \int_0^{\omega - x - t} f_{x+t}(s) \left( \frac{gA(0)}{r} \right) E_t^Q ((e^{-r\zeta} - e^{-rs})^+) ds \quad (2.28)
\]

Fee revenue is received up to the depleting time of the account value, of course if the policyholder is alive. In other terms, the insurer charges a certain percentage of the account value up to the earliest between policyholder’s death and VA account value’s depleting. Hence, the expected discounted premiums are:
\[
P^I(t) = \mathbb{I}_{\{\tau > t\}} \int_0^{\omega - x - t} f_{x+t}(s) E_t^Q \left( \int_t^{t+(\zeta \wedge s)} e^{-r(v-t)\alpha A(v)} dv \right) ds \quad (2.29)
\]
where \( x_1 \land x_2 = \min\{x_1, x_2\} \). Denote by \( V^I(t) \) the value at time \( t \) of the GLWB contract. It is defined as:

\[
V^I(t) = B^I(t) - P^I(t)
\]

The fair guarantee fee rate can be calculated, again, as:

\[
\alpha^* : V^I(0; \alpha^*) = 0
\]

\cite{Fung et al. 2014} show the equivalence of the two approaches. While the first one is computationally more efficient, the second approach highlights the theoretical result that the market reserve of a payment process is defined as the expected discounted benefits minus the expected discounted premiums under a risk-adjusted measure \cite{Dahl & Moller 2006}.

In the implementation of the valuation model we will refer to the policyholder’s approach.
Chapter 3

The deterministic model:
numerical results

As part of our analysis, we will proceed to implement the theoretical model proposed in the previous chapter. For this purpose, we have created ad hoc codes based on the programming language MATLAB. Our numerical experiments use a Monte Carlo approach: random variables have been simulated by MATLAB high level random number generators, while for the approximation of expected values, scenario-based averages have been evaluated by exploiting MATLAB fast matrix-computation facilities. These two MATLAB specific properties have allowed to break down computational costs, in terms of complexity and time.

3.1 Numerical results

Since for SDEs involved in the valuation model described in the previous chapter there are no explicit solutions, numerical methods have to be used. Among the numerical approaches proposed in literature we have chosen the Euler-Maruyama (Euler for short) method (the interested reader can refer to Appendix).

When we consider a numerical solution of a SDE, we have to restrict our attention to a finite subinterval \([0, T]\) of the time interval \([0, +\infty]\) and it is necessary to choose
an appropriate discretization $t_0 < t_1 < \cdots < t_n < \cdots < t_N = T$ of $[0, T]$ because of computer limitations. In particular, we set an equally spaced discretization, i.e. $t_n - t_{n-1} = T/N = \Delta t$, $n = 1, \ldots, N$, where $\Delta t$ is the integration step-size.

In order to write a MATLAB code for the fair fee valuation formula, we first have to simulate Brownian Motion paths. The independent random increments of the Wiener process $\{W(t), 0 \leq t \leq T\}$ are given by:

$$W(t_n) - W(t_{n-1}) \sim N(0, \Delta t)$$  \hspace{1cm} (3.1)

or also:

$$W(t_n) - W(t_{n-1}) = \epsilon_n \sqrt{\Delta t}$$  \hspace{1cm} (3.2)

with $\epsilon_n \sim N(0,1)$ and $n = 1, \ldots, N$. In our experiments, the random number generator `randn` is used; each `randn` call produces an independent “pseudorandom” number from the $N(0,1)$ distribution. In order to make experiments repeatable, MATLAB allows the initial state of the random number generator to be set. In this way, subsequent runs of the same code would produce the same output (in our experiments we have chosen the state 12345). Different simulations can be performed by resetting the state.

In particular, our model considers a representative individual aged 65 at the inception of the contract, $t = 0$, and whose limiting age (the age beyond which survival is assumed to be impossible) is set to be 120. Therefore, we focus on the time interval $[0, 55]$. We require to use a number of samples sufficiently large and a time step sufficiently small to make numerical results more accurate. Thus, we have chosen to simulate 100000 trajectories of the Wiener process using a step-size $\Delta t = 0.02$, so 2750 points for the discretization of the interval $[0, 55]$.

With these considerations in mind, we can now show how to simulate the dynamics of the VA sub-account and of the mortality intensity.

**VA sub-account** Recall that the dynamics of the VA sub-account, under the risk-
neutral measure $Q$, can be described using the following SDE:

$$
\begin{cases}
    dA(t) = (r - \alpha)A(t)dt - Gdt + \pi \sigma A(t)dW^Q_S(t) \\
    A(0) = P \\
    A(\cdot) \geq 0
\end{cases}
$$

(3.3)

The resultant Euler scheme reads:

$$
\begin{cases}
    A(t + \Delta t) - A(t) = (r - \alpha)A(t)\Delta t - G\Delta t + \pi \sigma A(t)(W^Q_S(t + \Delta t) - W^Q_S(t)) \\
    A(0) = P \\
    A(\cdot) \geq 0
\end{cases}
$$

(3.4)

where $W^Q_S(t + \Delta t) - W^Q_S(t) = \epsilon \sqrt{\Delta t}$. Focusing on the distribution, we have not indexed the random variables $\epsilon$’s that of course change at any step.

By attributing realistic values to the parameters $r$, $\alpha$, $g$, $\pi$ and $\sigma$, we can generate several possible scenarios for the VA sub-account process, some of which are shown in Figure 3.1 and Figure 3.2. In particular, as a base case, we consider a contract which offers a withdrawal rate $g = 5\%$. Assuming an initial investment $A(0) = \euro 100$, the policyholder is then guaranteed the ability to withdraw $G = \euro 5$ until he/she is alive, independently from the market performance. In the examples presented, we show that the guarantee offered by the GLWB option is not always activated (Figure 3.1 (a)), and when it does occur, activation can take place at different times during the life of the contract. Here, we suppose that the policyholder is still alive when (and if) the account value is depleted, so we consider only the market performance contribution. In particular, in Figure 3.1 (a) we observe how the recorded performance of the assets in the underlying portfolio, after the fees charged by the insurance company have been subtracted, is sufficient to guarantee the periodic withdrawals by the policyholder for the entire duration of the contract. In Figure 3.1 (b), however, the balance of the VA sub-account is reduced to zero at around 20 years after contract signature and therefore the guarantee is activated. Figure 3.2 shows that the GLWB option can be activated in different moments during the life of the contract, with different consequences for the insurance company. In fact, the later the guarantee will be activated, the smaller will be the expenses for the insurance company.
Table 3.1: Parameters for the financial model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>100</td>
</tr>
<tr>
<td>r</td>
<td>4%</td>
</tr>
<tr>
<td>σ</td>
<td>25%</td>
</tr>
<tr>
<td>π</td>
<td>0.70</td>
</tr>
<tr>
<td>g</td>
<td>5%</td>
</tr>
</tbody>
</table>

Table 3.1 summarizes the parameters’ values for the financial component of the model used in our simulation as a base case, so if unless stated otherwise.

**Mortality intensity** It is clear from the description of the policy that an important requirement for the GLWB’s activation is the survival of the policyholder. The model chosen for the mortality intensity process is:

\[
\begin{align*}
\begin{cases}
    d\mu_{x+t}(t) = (a + (b - \lambda\sigma_{\mu})\mu_{x+t}(t))dt + \sigma_{\mu}\sqrt{\mu_{x+t}(t)}dW_{M}^{Q}(t) \\
    \mu_{x}(0) > 0
\end{cases}
\end{align*}
\]

The Euler discretization has the form:

\[
\begin{align*}
\begin{cases}
    \mu_{x+t+\Delta t}(t + \Delta t) - \mu_{x+t}(t) = (a + (b - \lambda\sigma_{\mu})\mu_{x+t}(t))\Delta t + \sigma_{\mu}\sqrt{\mu_{x+t}(t)}(W_{M}^{Q}(t + \Delta t) - W_{M}^{Q}(t)) \\
    \mu_{x}(0) > 0
\end{cases}
\end{align*}
\]

again being \(W_{M}^{Q}(t + \Delta t) - W_{M}^{Q}(t) = \epsilon \sqrt{\Delta t}\).

The values for the parameters \(a\), \(b\), and \(\sigma_{\mu}\) are obtained by calibrating the survival
A possible trajectory of the mortality intensity process is shown in Figure 3.3. The valuation formula As we described in the previous chapter, considering the policyholder’s perspective, there are two possible valuation formulae for calculating the fair fee rate. In what follows we show how to implement them. The first one consists in searching \( \alpha^* \) such that it holds:

\[
E^Q \left[ \int_0^{\omega-x} e^{-\int_0^u \mu_{x+u}(u) du} e^{-rs} \left( G + A(s) \mu_{x+s}(s) \right) ds - A(0) \right] = 0 \quad (3.7)
\]

The MATLAB code created at this aim follows the procedure described in Algorithm 1. The second valuation formula is expressed as a function of the policyholder ran-
dom time of death \( \tau \):

\[
\alpha^* : E^Q \left[ gA(0) \int_0^\tau e^{-rs} ds + e^{-r\tau} \max(A(\tau), 0) - A(0) \right] = 0
\] (3.8)

The computational procedure is described in Algorithm 2.

The MATLAB function “fzero” is used to solve our root-finding problem. This algorithm, created by T. Dekker, uses a combination of bisection, secant, and inverse quadratic interpolation methods. Moreover, we have approximated the integrals in the formula linearly through the Trapezoid Rule.

We have computed the fair fee rates using both Algorithm 1 and Algorithm 2 using the values reported in Tables 3.2 and 3.1. Results are illustrated in Table 3.4. Then we have modified each time some of these parameters in order to conduct sensitivity analyses. In particular, we have investigated the relationship between the fair fee rate and important financial and demographic factors, such as interest rates, the volatility of the reference fund, the market price of mortality risk and the volatility of the mortality intensity. Moreover, each experiment has been fulfilled considering the effect of varying guaranteed withdrawal rates. The two valuation formulae have been proved to be equivalent theoretically taking advantage of the Cox processes’ properties (see Biffis (2005)). We have tested also the computational comparability of the two Algorithms and we have proved that the numerical results
Table 3.3: Profile summary: a time comparison (seconds) between Algorithm 1 and Algorithm 2

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Algorithm 1</th>
<th>Algorithm 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.0050$</td>
<td>360.3</td>
<td>122.1</td>
</tr>
<tr>
<td>$(g, r)$</td>
<td>7323.9</td>
<td>545.6</td>
</tr>
<tr>
<td>$(g, \pi\sigma)$</td>
<td>7068.2</td>
<td>643.5</td>
</tr>
<tr>
<td>$(g, \lambda)$</td>
<td>6285.1</td>
<td>2150.4</td>
</tr>
<tr>
<td>$(g, \sigma_\mu)$</td>
<td>4353.9</td>
<td>2157.4</td>
</tr>
</tbody>
</table>

are almost the same. The reader can note the negligible gap between the fair fee rates computed through the two Algorithms.

Since Algorithm 2 is a lot more efficient in terms of computing time, all the next experiments will be carried out with this procedure. Table 3.3 summarizes the average time for each simulation required by the two Algorithms. In particular, the first line reports the time used by each code to calculate the GLWB contract value; the other lines, instead, show the average time necessary to obtain the fair fee rate.

We now analyze the impact on the fair fee rate of varying some of the parameters of the model.

**Withdrawal rate.** When the withdrawal rate $g$ increases, there are two possible effects: on the one hand, the periodic amount withdrawn ($G = gA(0)$) increases and consequently also the value of the living benefits increases; on the other hand, just because the policyholder can withdraw a greater amount, the VA sub account value decreases; thus, the value of the death benefit decreases. Overall, the relationship between $g$ and the value of the living benefit prevails (being a guaranteed amount) so that the contract becomes more valuable as $g$ increases. Figure 3.4 shows the curve representing the initial contract value (net of the initial investment value) as a decreasing function of $\alpha$. When $g$ goes up, this curve shifts to the right. Fees charged to make the contract value fair are graphically obtained through the intersection between the curve and the horizontal line corresponding to the initial premium’s
value. Therefore, as the withdrawal rate increases, also fair fee rates will be greater. We can note the positive effect of the guaranteed withdrawal rate on the GLWB value (and consequently on the fair fee rates) in all the following analyses.

**Interest rate.** As the interest rate $r$ increases, the discounted value of each withdrawal decreases; so the value of the living benefit decreases at each time point. Instead, concerning the value of the death benefit there are opposite effects. In fact, on the one hand, a greater risk-free rate increases the account value since $r$ enters its drift; on the other hand, however, the discounting takes place at a higher rate, so the discounted value of the death benefit decreases. Overall, these two effects balance out, so the contribution of death benefits disappears. A higher interest rate, therefore, results in a translation on the left of the curve reported in figure 3.4; consequently fair fee rates will be lower. The negative relationship between the interest rate level $r$ and $\alpha^*$ is reported in figure 3.5.

A remark beyond the model, in economic terms, is also possible. Recall that the GLWB option allows the policyholder to withdraw a periodic amount independently from the market performance. Therefore, other things being equal, when the interest rate level is high, policyholders will prefer more profitable investments. In this case, to attract sales leads, GLWB providers will charge lower fee rates and will suffer a
challenging situation. On the contrary, a low interest rate level will encourage clients to invest in these contracts; consequently their demand will increase and so will do the required fee rates.

**Volatility of the reference fund.** Figure 3.6 shows the sensitivity of the fair fee rate with respect to the volatility of the investment account, \( \pi \cdot \sigma \). We keep \( \sigma \) constant at the level of 25% and set \( \pi \in \{0, 0.3, 0.5, 0.7, 1\} \), so that Figure 3.6 shows also the sensitivity of the fair fee rate with respect to the equity exposure \( \pi \). As the volatility increases, the value of the living benefit does not change because the withdrawals are constant over time and do not depend on the account value, while the value of the death benefit increases. In fact, the higher is the volatility \( \pi \sigma \) the higher is the VA account value. The positive relationship between \( \alpha^* \) and \( \pi \cdot \sigma \) can be explained with financial theory: options are more expensive when volatility is high. Recall that at inception of the contract (for some products also during the term of the contract) the insured has the possibility to influence the volatility by choosing
the underlying fund from a selection of mutual funds. Since for some products offered in the market the fees do not depend on the fund choice, this possibility presents another valuable option for the policyholder. Thus, an important risk management tool for insurers offering VA guarantees is the strict limitation and control of the types of underlying funds offered within these products.

**Market price of mortality risk.** Figure 3.7 shows the impact of the market price coefficient of the systematic mortality risk $\lambda$ on the fair fee rate. One can note that, when $\lambda$ is positive and increases, the effect on the mortality intensity $\mu$ is negative; so it will be an improvement in survival probability. Higher life expectancy, so also higher probabilities of GLWB option activation, lead insurance company to increase the charged fees. Therefore, the relation between $\lambda$ and $\alpha^*$ is positive.

**Volatility of the mortality intensity.** The effect of the volatility parameter of the mortality intensity $\sigma_\mu$ on the fair fee rate $\alpha^*$ is similar to that of the market price of mortality risk $\lambda$. Figure 3.8, in fact, shows that an increase in $\sigma_\mu$ leads to a
Figure 3.7: Sensitivity of the fair fee rate $\alpha^*$ with respect to the market price coefficient of the systematic mortality risk $\lambda$ for policyholders aged 65

decrease in the mortality intensity $\mu$, so to an improvement in survival probability. Hence, higher volatility of mortality leads not only to higher uncertainty about the timing of death of an individual, but also to an increase in the survival probability. To face this situation, the insurance company, other things being equal, has to charge higher fees.
Table 3.4: Fair guarantee fees (%) using Algorithm 1 and Algorithm 2

<table>
<thead>
<tr>
<th></th>
<th>Algorithm 1</th>
<th></th>
<th>Algorithm 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>r</td>
<td>4.5% 5% 5.5%</td>
<td>4.5% 5% 5.5%</td>
<td>4.5% 5% 5.5%</td>
<td>4.5% 5% 5.5%</td>
</tr>
<tr>
<td>1%</td>
<td>1.7909 3.3066 7.3373</td>
<td>1.7898 3.3083 7.3825</td>
<td>1.7909 3.3066 7.3373</td>
<td>1.7898 3.3083 7.3825</td>
</tr>
<tr>
<td>2%</td>
<td>0.9593 1.6279 2.8531</td>
<td>0.9550 1.6246 2.8505</td>
<td>0.9593 1.6279 2.8531</td>
<td>0.9550 1.6246 2.8505</td>
</tr>
<tr>
<td>3%</td>
<td>0.5346 0.8833 1.4477</td>
<td>0.5270 0.8774 1.4422</td>
<td>0.5346 0.8833 1.4477</td>
<td>0.5270 0.8774 1.4422</td>
</tr>
<tr>
<td>4%</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
</tr>
<tr>
<td>5%</td>
<td>0.1686 0.2812 0.4517</td>
<td>0.1557 0.2698 0.4415</td>
<td>0.1686 0.2812 0.4517</td>
<td>0.1557 0.2698 0.4415</td>
</tr>
<tr>
<td>6%</td>
<td>0.0932 0.1584 0.2576</td>
<td>0.0781 0.1444 0.2450</td>
<td>0.0932 0.1584 0.2576</td>
<td>0.0781 0.1444 0.2450</td>
</tr>
<tr>
<td>7%</td>
<td>0.0503 0.0879 0.1458</td>
<td>0.0336 0.0718 0.1307</td>
<td>0.0503 0.0879 0.1458</td>
<td>0.0336 0.0718 0.1307</td>
</tr>
<tr>
<td>8%</td>
<td>0.0260 0.0473 0.0809</td>
<td>0.0079 0.0296 0.0638</td>
<td>0.0260 0.0473 0.0809</td>
<td>0.0079 0.0296 0.0638</td>
</tr>
<tr>
<td>σμ</td>
<td>0.2340 0.3883 0.6236</td>
<td>0.2185 0.3755 0.6121</td>
<td>0.2340 0.3883 0.6236</td>
<td>0.2185 0.3755 0.6121</td>
</tr>
<tr>
<td>0.0110</td>
<td>0.2588 0.4286 0.6881</td>
<td>0.2463 0.4183 0.6789</td>
<td>0.2588 0.4286 0.6881</td>
<td>0.2463 0.4183 0.6789</td>
</tr>
<tr>
<td>0.0210</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
</tr>
<tr>
<td>0.0310</td>
<td>0.3626 0.6008 0.9715</td>
<td>0.3565 0.5955 0.9675</td>
<td>0.3626 0.6008 0.9715</td>
<td>0.3565 0.5955 0.9675</td>
</tr>
<tr>
<td>0.0410</td>
<td>0.4296 0.7362 1.2232</td>
<td>0.4450 0.7459 1.2267</td>
<td>0.4296 0.7362 1.2232</td>
<td>0.4450 0.7459 1.2267</td>
</tr>
<tr>
<td>0.0510</td>
<td>0.4846 0.8903 1.5576</td>
<td>0.5574 0.9393 1.5800</td>
<td>0.4846 0.8903 1.5576</td>
<td>0.5574 0.9393 1.5800</td>
</tr>
<tr>
<td>λ</td>
<td>-0.4 0.2354 0.3896 0.6237</td>
<td>0.2266 0.3827 0.6181</td>
<td>-0.4 0.2354 0.3896 0.6237</td>
<td>0.2266 0.3827 0.6181</td>
</tr>
<tr>
<td>0</td>
<td>0.2662 0.4396 0.7043</td>
<td>0.2568 0.4321 0.6981</td>
<td>0.2662 0.4396 0.7043</td>
<td>0.2568 0.4321 0.6981</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
<td>0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
</tr>
<tr>
<td>0.8</td>
<td>0.3397 0.5606 0.9039</td>
<td>0.3275 0.5500 0.8948</td>
<td>0.3397 0.5606 0.9039</td>
<td>0.3275 0.5500 0.8948</td>
</tr>
<tr>
<td>1.2</td>
<td>0.3825 0.6330 1.0279</td>
<td>0.3722 0.6240 1.0196</td>
<td>0.3825 0.6330 1.0279</td>
<td>0.3722 0.6240 1.0196</td>
</tr>
<tr>
<td>1.6</td>
<td>0.4283 0.7139 1.1723</td>
<td>0.4300 0.7166 1.1752</td>
<td>0.4283 0.7139 1.1723</td>
<td>0.4300 0.7166 1.1752</td>
</tr>
<tr>
<td>π · σ</td>
<td>0% 0.0003 0.0040 0.0454</td>
<td>0.0012 0.0061 0.0492</td>
<td>0% 0.0003 0.0040 0.0454</td>
<td>0.0012 0.0061 0.0492</td>
</tr>
<tr>
<td></td>
<td>7.5% 0.0377 0.0991 0.2317</td>
<td>0.0350 0.0979 0.2326</td>
<td>7.5% 0.0377 0.0991 0.2317</td>
<td>0.0350 0.0979 0.2326</td>
</tr>
<tr>
<td></td>
<td>12.5% 0.1413 0.2691 0.4874</td>
<td>0.1350 0.2645 0.4847</td>
<td>12.5% 0.1413 0.2691 0.4874</td>
<td>0.1350 0.2645 0.4847</td>
</tr>
<tr>
<td></td>
<td>17.5% 0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
<td>17.5% 0.3009 0.4963 0.7969</td>
<td>0.2905 0.4874 0.7891</td>
</tr>
<tr>
<td></td>
<td>25% 0.6029 0.8932 1.3064</td>
<td>0.5820 0.8725 1.2849</td>
<td>25% 0.6029 0.8932 1.3064</td>
<td>0.5820 0.8725 1.2849</td>
</tr>
</tbody>
</table>
Figure 3.8: Sensitivity of the fair fee rate $\alpha^*$ with respect to volatility parameter of mortality intensity $\sigma_{\mu}$ for policyholders aged 65
Algorithm 1

Require: \( N_{\text{sim}}, N_{\text{path}}, dt \) \( \triangleright \) simulation parameters
\[ dW_A, dW_\mu \] \( \triangleright \) \( N_{\text{sim}} \times N_{\text{path}} \) independent Brownian motions
\[ \mu_0, a, b, \lambda, \sigma_\mu \] \( \triangleright \) mortality intensity process parameters
\[ A_0, r, G, \pi, \sigma, \alpha \] \( \triangleright \) VA sub-account process parameters

Set \( F = 0 \)

for \( i = 1 \) to \( N_{\text{sim}} \) do

Set \( \mu_A = \mu_0, \text{cumtrapz}_\mu A = 0, \text{cumtrapz} = 0, A = A_0 \)

for \( t = 1 \) to \( N_{\text{path}} \) do

\[ \mu_B = \mu_A + [a + (b - \lambda \sigma_\mu)]dt + \sigma_\mu \sqrt{\mu_A} dW_\mu(i, t) \]
\[ \mu_B = \max(\mu_B, 0) \]

\[ B = A + [(r - \alpha) A - G]dt + \pi \sigma A dW_A(i, t) \]
\[ B = \max(B, 0) \]

\[ \text{cumtrapz}_\mu B = \text{cumtrapz}_\mu A + (\mu_A + \mu_B)dt/2 \]
\[ \text{cumtrapz} = \text{cumtrapz} + \left[ e^{-\text{cumtrapz}_\mu A - (i-1) r dt} (gA_0 + A_{\mu_A}) + \right. \\
\left. + e^{-\text{cumtrapz}_\mu B - i r dt} (gA_0 + B_{\mu_B}) \right] dt/2 \]

\[ F = F + \text{cumtrapz} \]

Set \( \mu_A = \mu_B, A = B, \text{cumtrapz}_\mu A = \text{cumtrapz}_\mu B \)

end for

end for

\[ F = F/N_{\text{sim}} \]

return \( F \) \( \triangleright \) contract expected value
Algorithm 2

Require: \(N_{\text{sim}}, N_{\text{path}}, dt\) \(\triangleright\) simulation parameters
\[
dW_A, dW_\mu \quad \triangleright\ N_{\text{sim}} \times N_{\text{path}}\ \text{independent Brownian motions}
\]
\(\xi\) \(\triangleright\) exponential random variable with parameter 1
\(\mu_0, a, b, \lambda, \sigma_\mu\) \(\triangleright\) mortality intensity process parameters
\(A_0, r, G, \pi, \sigma, \alpha\) \(\triangleright\) VA sub-account process parameters

for \(i = 1\) to \(N_{\text{sim}}\) do
    Let \(t = 0\)
    Set \(\mu_A = \mu_0, \text{cumtrapz}_\mu = 0\)
    while \(t < N_{\text{path}}\) and \(\text{cumtrapz}_\mu \leq \xi(i)\) do
        \(\mu_B = \mu_A + [a + (b - \lambda \sigma_\mu) \mu_A]dt + \sigma_\mu \sqrt{\mu_A} dW_\mu(i, t)\)\n        \(\mu_B = \max(\mu_B, 0)\)\n        \(\text{cumtrapz}_\mu = \text{cumtrapz}_\mu + (\mu_A + \mu_B)dt/2\)\n        \(\mu_A = \mu_B\)\n        \(t = t + 1\)\n    end while
    Set \(\tau(i) = t\)\n    Let \(t = 0\)
    Set \(A(1) = A_0, \ldots, A(N_{\text{sim}}) = A_0\)
    while \(t \leq \tau(i)\) and \(A(i) > 0\) do
        \(A(i) = A(i) + [(r - \alpha)A(i) - G]dt + \pi \sigma A(i)dW_A(i, t)\)
    end while
end for

\[
F = \frac{1}{N_{\text{sim}}} \sum_{i=1}^{N_{\text{sim}}} \left\{ \frac{G}{r} + e^{-r\tau(i)} \left[ -\frac{G}{r} + \max(A(i), 0) \right] \right\}
\]
return \(F\) \(\triangleright\) contract expected value
Chapter 4

Generalization of the pricing model: stochastic interest rate

Until now, the theoretical model proposed for the pricing of the GLWB option has rested upon some assumptions that are, to some extent, “counterfactual”. The valuation has been performed in a Black and Scholes economy: the sub-account value has been assumed to follow a geometric Brownian motion, thus with a constant volatility, and the term structure of interest rates has been assumed to be constant. These hypotheses, however, do not find justification in the financial markets. For the purpose of considering a model that is closer to the market, we sought to weaken these misspecifications. In particular, in this chapter we will focus on relaxing the last assumption by allowing interest rates to vary randomly.

4.1 Stochastic interest rates models

The assumption of deterministic interest rates, which can be acceptable for short-term options, is not realistic for medium or long-term contracts such as life insurance products. GLWB contracts are investment vehicles with a long term horizon and as such they are very sensitive to interest rate movements which are by nature uncertain. A stochastic modeling of the term structure is therefore appropriate. Many models
have been developed in literature (see Shao (2012)).

One of the earliest dates back to Vasicek (1977). The model specifies that the instantaneous interest rate follows the stochastic differential equation:

\[
\begin{align*}
\frac{dr(t)}{dt} &= k(\bar{r} - r(t))dt + \sigma_r dW_r(t) \\
 r(0) &= r_0
\end{align*}
\tag{4.1}
\]

where \(k, \bar{r}, \sigma_r > 0\) and \(W_r\) is a Brownian motion. This dynamics has some interesting properties that make the model attractive. Mathematically, the equation is linear and can be solved explicitly, the distribution of the short rate is Gaussian, and both the expressions and the distributions of several useful quantities related to the interest-rate world are easily obtainable. Vasicek’s model was the first one to capture mean reversion, an essential characteristic of the interest rate. As opposed to stock prices, for instance, interest rates cannot rise indefinitely. This is because at very high levels they would hamper economic activity, prompting a decrease in interest rates. Similarly, interest rates cannot decrease below 0. As a result, interest rates move in a limited range, showing a tendency to revert to a long run value. In this sense, Vasicek’s model exhibits mean reversion. Therefore, other things being equal, if the interest rate is above the long run mean \((r > \bar{r})\), then the coefficient \(k\), being positive, tends to confirm the negativity of the drift, so that the rate will be pulled down in the direction of \(\bar{r}\). Likewise, if the rate is less than the long run mean \((r < \bar{r})\), the drift is positive and the rate will be pulled up in the direction of \(\bar{r}\).

The coefficient \(k\) is, thus, the speed of adjustment of the interest rate towards its long run normal level. This feature is particularly attractive because, without it, interest rates could drift permanently upward the way stock prices do in contrast with empirical evidence. This particular type of stochastic process is referred to as an Ornstein-Uhlenbeck (OU) process. The main disadvantage is that, under Vasicek’s model, it is theoretically possible (with positive probability) for the interest rate to become negative, and this is an undesirable feature.

The general equilibrium approach developed by Cox, Ingersoll and Ross (Cox et al. (1985)) led to the introduction of a “square-root” term in the diffusion coefficient of the instantaneous short rate dynamics. The resulting model has been a benchmark
for many years because of its analytical tractability and the fact that, contrary to the Vasicek model, the instantaneous short rate is always non-negative. Hull and White (Hull & White (1990)) generalized the Vasicek model by considering a time-varying long-run mean. Later, Black and Karasinski (Black & Karasinski (1991)) assumed that the logarithm \( \ln(r(t)) \) of the instantaneous short rate evolves according to a generalized Vasicek model with time-dependent coefficients.

Here we will refer to the Cox-Ingersoll-Ross (CIR) model. Consider that, for our pricing purposes, in what follows we will express all the dynamics directly under the \( Q \) risk neutral measure.

### 4.2 The CIR model

The Cox-Ingersoll-Ross (CIR) model is a diffusion process suitable for modeling the term structure of interest rates. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an alternative of the Vasicek model. Its simplest version describes the dynamics of the interest rate \( r(t) \) as a solution of the following stochastic differential equation:

\[
\begin{align*}
    dr(t) &= k(\bar{r} - r(t))dt + \eta \sqrt{r(t)}dW^Q_r(t) \\
    r(0) &> 0
\end{align*}
\] (4.2)

where \( k > 0 \) determines the speed of adjustment of the interest rate towards its theoretical mean, \( \bar{r} > 0 \), \( \eta > 0 \) controls the volatility of the interest rate, and \( W^Q_r \) is a standard \( Q \)-Brownian motion. This process has some appealing properties from an applied point of view; for example, the interest rate stays non-negative, and is elastically pulled towards the long-term constant value \( \bar{r} \) at a speed controlled by \( k \) (mean-reverting). Those properties are attractive in modeling real-life interest rates. In particular, the condition

\[ 2k\bar{r} \geq \eta^2 \]

would ensure that the origin is inaccessible to the process, so that we can grant that \( r(t) \) remains positive. Intuitively, when the rate is at a low level (close to zero), the standard deviation \( \eta \sqrt{r(t)} \) also becomes close to zero, which dampens the effect
of the random shock on the rate. Consequently, when the rate gets close to zero, its evolution becomes dominated by the drift factor, which pushes the rate upwards (towards equilibrium). Thus, the interest rate behavior implied by this structure has the following empirically relevant properties:

i) negative interest rates are precluded;

ii) if the interest rate reaches zero, it can subsequently become positive;

iii) the absolute variance of the interest rate increases when the interest rate itself increases;

iv) there is a steady state distribution for the interest rate.

The SDE (4.2) is not explicitly solvable, hence the tractability of the CIR model is not as good as in the Vasicek model in this regard.

Considering a CIR model for the interest rate, the new dynamics of the VA sub-account become:

\[
dA(t) = (r(t) - \alpha)A(t)dt - Gdt + \pi \sigma A(t)dW_S^Q(t) \quad (4.3)
\]

The evolution of the mortality intensity remains unchanged, being not influenced by \(r(t)\):  
\[
d\mu_{x+t}(t) = (a + (b - \lambda \sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}}dW_M^Q(t) \quad (4.4)
\]

Therefore, our model is specified through the following system of stochastic differential equations:

\[
\begin{align*}
    dA(t) &= (r(t) - \alpha)A(t)dt - Gdt + \pi \sigma A(t)dW_S^Q(t), \quad A(\cdot) \geq 0, A(0) > 0 \\
    dr(t) &= k(\bar{r} - r(t))dt + \eta \sqrt{r(t)}dW_r^Q(t), \quad r(0) \geq 0 \\
    d\mu_{x+t}(t) &= (a + (b - \lambda \sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}}dW_M^Q(t), \quad \mu(0) > 0
\end{align*}
\]

(4.5)

where \(dW_S(t)dW_r(t) = \rho_{S,r}dt\), with \(|\rho_{S,r}| \leq 1\), is the correlation between the reference fund and interest rate. \(W_S^Q\) and \(W_M^Q\) are instead considered independent, as
well as we took $W^Q_S$ and $W^Q_M$. More explicitly, we can rewrite system (4.5) as:

$$
\begin{align*}
\frac{dA(t)}{dt} &= (r(t) - \alpha)A(t)dt - Gdt + \pi \sigma A(t)\left(\rho_{S,r} dW^Q_r(t) + \sqrt{1 - \rho_{S,r}^2} d\tilde{W}^Q_r(t)\right) \\
\frac{dr(t)}{dt} &= k(r(t) - \bar{r})dt + \eta \sqrt{r(t)} dW^Q_r(t) \\
\frac{d\mu_{x+t}(t)}{dt} &= (a + (b - \lambda \sigma \mu)\mu_{x+t}(t))dt + \sigma \mu \sqrt{\mu_{x+t}(t)} dW^Q_M(t)
\end{align*}
$$

(4.6)

where $\tilde{W}_r$ and $W_r$ are independent Brownian motions, $A(0), r(0), \mu(0) > 0$ and $A(\cdot) \geq 0$.

The valuation formula, under the policyholder perspective, becomes:

$$
\alpha^* : E^Q\left[gA(0) \int_0^\tau e^{-\int_0^u r(u)du}dt + e^{-\int_0^\tau r(u)du}A(\tau) - A(0)\right] = 0
$$

(4.7)

4.3 Numerical results

In this section we will present the numerical results obtained considering the CIR model for the interest rate process.

As mentioned, a drawback of the CIR process is that the SDE (4.2) is not explicitly solvable. Our pricing approach, however, requires to solve the problem of simulating a CIR process. In general there are two ways to do it, namely, exact simulation methods and approximation schemes. There are pros and cons associated with each method. Exact simulation methods usually require more computational time than a simulation with approximation schemes. Hence it should be used to compute expectations that depend on the values of the process at just a few fixed times. On the contrary, for expectation that depends on all the path (such as integrals) discretization schemes should be preferred. On the other hand, being an approximation of continuous time processes by discrete time processes, the drawback of approximation schemes in general is the bias they introduce into the estimator. And in practice, many time steps may be necessary to reduce the bias to an acceptable level.

The first method (exact simulation method) is based on the transition probability density function of the CIR process. In Cox’s paper (Cox et al. (1985)), it was noted that the distribution of $r(t)$ given $r(u)$ for any $0 < u < t$ is a noncentral chi-squared distribution. So there is not a real problem in the simulation of the mean-reverting
Complications arise, however, when we combine the CIR process with another correlated process, in this case the VA sub-account. In fact, there is no way to simulate a non central chi-squared increment together with a correlated normal increment for the account process. Therefore, a solution is the use of an Euler discretization. However, this approach leads to some problems.

A theoretical problem with discretization schemes of CIR process concerns the square-root term. In fact, the square root is not globally Lipschitz. Therefore the usual theorems leading to strong or weak convergence \cite{kloeden1999}, which require the drift and diffusion coefficients to satisfy a linear growth condition, cannot be applied. Hence, the convergence of the Euler scheme is not guaranteed. Various methods have been proposed to solve this problem and to prove the convergence (the interested reader can refer to \cite{lord2010}).

There exists another problem of practical nature. In fact, despite the domain of the square root process being the nonnegative real line, the discretization is not guaranteed to be the same. For any choice of the time grid, indeed, the probability of the interest rate becoming negative at the next time step is strictly greater than zero. Practitioners have therefore often opted for a quick “fix” by either setting the process equal to zero whenever it attains a negative value, or by reflecting it in the origin, and continuing from there on. These fixes are often referred to as absorption or reflection \cite{lord2010}). In what follows, we use $x^+ = \max(x, 0)$ as fixing function.

The Euler scheme for equation (4.2) has the form:

$$r(t + \Delta t) - r(t) = k(\bar{r} - r(t))\Delta t + \eta \sqrt{r(t)}dW^Q_r(t)$$  \hspace{1cm} (4.8)$$

Therefore, we consider only the positive part of the process:

$$r(t + \Delta t) = [r(t) + k(\bar{r} - r(t))\Delta t + \eta \sqrt{r(t)}dW^Q_r(t)]^+$$  \hspace{1cm} (4.9)$$
The parameters of the CIR process are those reported in [Grzelak & Oosterlee (2011)] and summarized in Table 4.1. We have then proceeded to price the GLWB option using the Algorithm 2 procedure. The obtained fair fee rates are reported in Tables 4.2, 4.3, 4.4 and 4.5. As in the deterministic case, we have conducted sensitivity analyses in order to study the relationship between the fair fee rate and the same financial and demographic factors, i.e., the volatility of the investment account, the market price of mortality risk and the volatility of the mortality intensity. The impact of these factors on the price of the GLWB option is analogue to the deterministic case; so, for the same reasons, we can note a positive relation of the fair fee rate $\alpha^*$ with the market price coefficient of mortality risk $\lambda$ (see figure 4.3), with the volatility of the mortality intensity $\sigma_\mu$ (see figure 4.2) and with the volatility of the investment account $\sigma_\pi$ (see figure 4.1).

We have then analyzed the impact of varying the parameters of the interest rate model on the fair fee rate. In particular, we have modified the values of the mean reversion coefficient $k$ and of the rate of diffusion $\eta$.

An increase in the mean reversion coefficient $k$, in general, doesn’t have a clear effect on the contract fair price; its contribution, in fact, depend on the sign of the difference $\tilde{r} - r(t)$. In particular, if $\tilde{r} > r$, when $k$ increases, the CIR drift factor will be greater, pushing the interest rate upwards. This will lead to a smaller value of the GLWB contract and consequently of the fair fee rate. When $r > \tilde{r}$, the relation is inverse, even if interest rates are already high, so the impact on the contract fair price could be also in this case negative. The negative relation between the fair fee rate and the mean reversion coefficient is verified by our numerical results (see Table 4.4).

Analogue considerations hold for the diffusion coefficient $\eta$. In fact, an increase in
the volatility of interest rates \( \eta \) would amplify the effect of the random shock on the rate. Therefore, depending on the sign of the \( dW_r(t) \) term (recall that Brownian motion’s increments are normally distributed with expectation zero), its impact on the interest rate (and consequently on the fair fee rate) could be both positive and negative. This non univocal relation is confirmed by numerical results (see Table 4.5).

In closing we have studied the impact on the fair fee rate of the long-run mean of \( r \), \( \bar{r} \), and of its initial value, \( r(0) \). Intuitively, when the rate is at a low level (close to zero), the diffusion term \( \eta \sqrt{r(t)} \) also becomes close to zero, and this dampens the effect of the random shock on the rate. Consequently, when the rate decreases, its evolution becomes dominated by the drift factor, which pushes the rate upwards. The value of the GLWB is consequently smaller and the same holds for the fair fee rate. Similarly, when the long-run mean \( \bar{r} \) increases, the drift term will be greater, while the diffusion one will be unaffected. Overall, the effect on the interest rate evolution is positive, and therefore the impact on the fair fee rate is negative. In particular, if \( k = 0 \), so when only the diffusion of the CIR process is present, Table 4.2 confirms that the fair contract price is not dependent on the long-run mean of \( r \). Moreover, if \( \eta = 0 \), so if the random shock on the rate is zero, we are in the case of deterministic interest rates, and more precisely, if in addition \( \bar{r} = r \) (the diagonal of the first table in Table 4.3), we are considering constant interest rates.
Figure 4.1: Sensitivity of the fair fee rate $\alpha^*$ with respect to the volatility of the investment account $\pi\sigma$ for policyholders aged 65.

Figure 4.2: Sensitivity of the fair fee rate $\alpha^*$ with respect to the volatility parameter of mortality intensity $\sigma_\mu$ for policyholders aged 65.
Figure 4.3: Sensitivity of the fair fee rate $\alpha^*$ with respect to the market price coefficient of systematic mortality risk $\lambda$ for policyholders aged 65

Table 4.2: Sensitivity of the fair fee rate $\alpha^*$ with respect to the long-run mean $\bar{r}$ and to the initial value $r(0)$, with different values for the mean reversion coefficient $k$

<table>
<thead>
<tr>
<th>$k = 0.00$</th>
<th>$r(0)$</th>
<th>$k = 0.01$</th>
<th>$r(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>3.3178</td>
<td>1.6570</td>
<td>0.5244</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.3178</td>
<td>1.6570</td>
<td>0.5244</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.3178</td>
<td>1.6570</td>
<td>0.5244</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$k = 0.50$</th>
<th>$r(0)$</th>
<th>$k = 1.00$</th>
<th>$r(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>3.2854</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.7880</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.6461</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

92
Table 4.3: Sensitivity of the fair fee rate $\alpha^*$ with respect to the long-run mean $\bar{r}$ and to the initial value $r(0)$, with different values for the diffusion rate $\eta$

<table>
<thead>
<tr>
<th>$\eta = 0.000$</th>
<th>$r(0)$</th>
<th>$\eta = 0.005$</th>
<th>$r(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.01</td>
<td>0.02</td>
<td>0.04</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>0.01</td>
<td>3.2809</td>
<td>1.7078</td>
</tr>
<tr>
<td></td>
<td>0.02</td>
<td>3.0729</td>
<td>1.6190</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>2.7104</td>
<td>1.4583</td>
</tr>
<tr>
<td>$\eta = 0.010$</td>
<td>$r(0)$</td>
<td>$\eta = 0.020$</td>
<td>$r(0)$</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>3.6268</td>
<td>2.0990</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>0.02</td>
<td>3.4304</td>
<td>2.0052</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>3.0778</td>
<td>1.8318</td>
</tr>
<tr>
<td>$\eta = 0.050$</td>
<td>$r(0)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>3.3760</td>
<td>1.8052</td>
</tr>
<tr>
<td>$\bar{r}$</td>
<td>0.02</td>
<td>3.1668</td>
<td>1.7142</td>
</tr>
<tr>
<td></td>
<td>0.04</td>
<td>2.8019</td>
<td>1.5493</td>
</tr>
</tbody>
</table>
Table 4.4: Fair fees (%) varying the mean reversion coefficient $k$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\pi\sigma$</th>
<th>$\sigma_\mu$</th>
<th>$0$</th>
<th>$0.01$</th>
<th>$0.5$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>1.1957</td>
<td>1.1933</td>
<td>1.1724</td>
<td>1.1704</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.3982</td>
<td>1.3952</td>
<td>1.3708</td>
<td>1.3687</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>1.6570</td>
<td>1.6531</td>
<td>1.6237</td>
<td>1.6214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>1.9817</td>
<td>1.9764</td>
<td>1.9406</td>
<td>1.9380</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>2.4250</td>
<td>2.4177</td>
<td>2.3725</td>
<td>2.3696</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>3.0584</td>
<td>3.0474</td>
<td>2.9868</td>
<td>2.9835</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.5727</td>
<td>0.5696</td>
<td>0.5524</td>
<td>0.5519</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7.5%</td>
<td>0.8610</td>
<td>0.8571</td>
<td>0.8294</td>
<td>0.8275</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12.5%</td>
<td>1.2315</td>
<td>1.2276</td>
<td>1.1985</td>
<td>1.1963</td>
<td></td>
<td></td>
</tr>
<tr>
<td>17.5%</td>
<td>1.6570</td>
<td>1.6531</td>
<td>1.6237</td>
<td>1.6214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25%</td>
<td>2.3244</td>
<td>2.3205</td>
<td>2.2915</td>
<td>2.2892</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.1844</td>
<td>1.1821</td>
<td>1.1611</td>
<td>1.1592</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.011</td>
<td>1.3483</td>
<td>1.3455</td>
<td>1.3217</td>
<td>1.3196</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.021</td>
<td>1.6570</td>
<td>1.6531</td>
<td>1.6237</td>
<td>1.6214</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.031</td>
<td>2.2486</td>
<td>2.2417</td>
<td>2.1999</td>
<td>2.1972</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.041</td>
<td>3.4358</td>
<td>3.4203</td>
<td>3.3459</td>
<td>3.3424</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.051</td>
<td>6.0376</td>
<td>5.9913</td>
<td>5.8115</td>
<td>5.8060</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 4.5: Fair fees (%) varying the diffusion rate $\eta$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\pi\sigma$</th>
<th>0.051</th>
<th>0.041</th>
<th>0.031</th>
<th>0.021</th>
<th>0.011</th>
<th>0.005</th>
<th>0.01</th>
<th>0.02</th>
<th>0.05</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.4</td>
<td>0</td>
<td>0.856</td>
<td>1.2193</td>
<td>0.9218</td>
<td>0.6219</td>
<td>0.8256</td>
<td>0.5696</td>
<td>0.5563</td>
<td>0.5517</td>
<td>1.1683</td>
</tr>
<tr>
<td>0</td>
<td>0.4</td>
<td>1.190</td>
<td>1.6190</td>
<td>1.3781</td>
<td>1.3664</td>
<td>1.6190</td>
<td>1.6190</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.3664</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>1.511</td>
<td>1.9353</td>
<td>1.9764</td>
<td>1.9511</td>
<td>1.9353</td>
<td>1.9353</td>
<td>1.9511</td>
<td>1.9511</td>
<td>1.9353</td>
</tr>
<tr>
<td>1.2</td>
<td>1.2</td>
<td>2.300</td>
<td>2.3666</td>
<td>2.4177</td>
<td>2.3855</td>
<td>2.3855</td>
<td>2.3855</td>
<td>2.3855</td>
<td>2.3855</td>
<td>2.3666</td>
</tr>
<tr>
<td>1.6</td>
<td>1.6</td>
<td>2.634</td>
<td>2.9802</td>
<td>3.0474</td>
<td>3.0037</td>
<td>3.0037</td>
<td>3.0037</td>
<td>3.0037</td>
<td>3.0037</td>
<td>2.9802</td>
</tr>
<tr>
<td>7.5%</td>
<td>12.5%</td>
<td>1.207</td>
<td>1.940</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.6325</td>
<td>1.6325</td>
</tr>
<tr>
<td>17.5%</td>
<td>25%</td>
<td>2.300</td>
<td>2.2867</td>
<td>2.3205</td>
<td>2.3205</td>
<td>2.3205</td>
<td>2.3205</td>
<td>2.3205</td>
<td>2.3205</td>
<td>2.2867</td>
</tr>
<tr>
<td>25%</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ \sigma_{\mu} \]

\[ \sigma_{\varphi} \]
Chapter 5

Generalization of the pricing model: stochastic volatility

In this chapter we will focus on relaxing some of the assumptions of the theoretical model proposed for the pricing of the GLWB option.

It is widely recognized that financial models which consider a constant volatility parameter (such as the Black-Scholes one) are no longer sufficient to capture modern market phenomena, especially since the 1987 crash. Empirical studies of stock price returns, in fact, show that volatility exhibits “random” characteristics. The natural extension of these models that has been pursued in the literature and in practice, suggests to modify the specification of volatility to make it a stochastic process. What makes this approach particularly challenging is first that volatility is a hidden process: it drives prices and yet cannot be directly observed. Second, volatility tends to fluctuate at a high level for a while, then at a low level for a similar period, then high again, and so on. It “mean reverts” many times during the life of a derivative contract.

In this chapter we will show how the assumption of stochastic volatility could impact on the pricing of the GLWB option.
5.1 Stochastic volatility models

Stochastic volatility models predict that volatility itself follows a stochastic process
(Fouque et al. (2000)):

\[ \{ \sigma(t), t \geq 0 \} \quad \text{(5.1)} \]

with

\[ \sigma(t) = f(V(t)) \quad \text{(5.2)} \]

where \( f(v) > 0 \ \forall v \in \mathbb{R} \) and \( \{ V(t), t \geq 0 \} \) represents a stochastic process.

One feature that most models seem to appraise is mean reversion. The term “mean reverting” refers to the characteristic (typical) time it takes for a process to get back to the mean level of its invariant distribution (the long-run distribution of the process). From a financial modeling perspective, mean reverting refers to a linear pull-back term in the drift of the volatility process itself, or in the drift of some (underlying) process of which volatility is a function. This property means that \( V(t) \) satisfies a stochastic differential equation of the following type:

\[ dV(t) = a(m - V(t))dt + \beta(t)dW_V(t) \quad \text{(5.3)} \]

where \( a \) is the mean reversion coefficient, \( m \) denotes the long-run mean of \( V \), the process \( \beta(t) \) is the volatility of \( V(t) \) and \( W_V(t) \) is a Brownian motion correlated with the Wiener process \( W_S(t) \) which appears in the VA sub-account process according to a correlation coefficient \( \rho_{S,V} \in [-1, 1] \). The drift term pulls \( V \) towards \( m \), so we would expect that \( \sigma(t) \) is pulled towards the mean value of \( f(V) \) with respect to the long-run distribution of \( V \). The correlation coefficient \( \rho_{S,V} \) is defined by

\[ d(W_S(t), W_V(t)) = \rho_{S,V}dt \quad \text{(5.4)} \]

or also

\[ W_V(t) = \rho W_S + \sqrt{1 - \rho^2} \tilde{W}_V(t) \quad \text{(5.5)} \]

where \( \tilde{W}_V(t) \) is a standard Brownian motion independent of \( W_S(t) \). From the empirical data, we deduce that \( \rho_{S,V} < 0 \) and there are also economic arguments for a negative correlation or leverage effect between financial asset price and volatility.
shocks. Indeed, empirical studies show that asset prices tend to decrease when volatility increases. In general terms, the correlation may be time dependent, therefore it would be more correct to write $\rho_{S,V}(t) \in [-1, 1]$, however it is typically assumed to be constant, both to simplify the notation and because this assumption is the most widely used in most practical situations.

Various alternative models have been proposed in literature, differentiated for the driving process $V(t)$ and for the function $f$.

Some common driving processes $V(t)$ are:

- Lognormal,

$$dV(t) = c_1 V(t)dt + c_2 V(t)dW_V(t)$$

with $c_1 \in \mathbb{R}$ and $c_2 > 0$;

- Ornstein-Uhlenbeck (OU),

$$dV(t) = a(m - V(t))dt + \beta dW_V(t)$$

with $a, m, \beta > 0$

- Feller or Cox-Ingersoll-Ross (CIR),

$$dV(t) = \theta(\bar{V} - V(t))dt + \gamma \sqrt{V(t)}dW_V(t)$$

with $\theta, \bar{V}, \gamma > 0$

Also for the function $f$ we have a wide choice, among which we can cite the exponential function, the square root one, that considering the absolute value, etc. Table 5.1 summarizes some of the main models studied in the literature.

Among those suggested, we have chosen to consider the Heston model. It dated back to 1993 and is described by a square-root CIR process. Recall that for our pricing purposes, we express all the dynamics directly under the $Q$ risk neutral measure. Considering the reference fund volatility not constant anymore, the new dynamics
Table 5.1: Models for the volatility

<table>
<thead>
<tr>
<th>Authors</th>
<th>Correlation</th>
<th>$f(v)$</th>
<th>$V$ process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hull-White</td>
<td>$\rho = 0$</td>
<td>$f(v) = \sqrt{v}$</td>
<td>Lognormal</td>
</tr>
<tr>
<td>Scott</td>
<td>$\rho = 0$</td>
<td>$f(v) = e^v$</td>
<td>Mean-reverting OU</td>
</tr>
<tr>
<td>Stein-Stein</td>
<td>$\rho = 0$</td>
<td>$f(v) =</td>
<td>v</td>
</tr>
<tr>
<td>Ball-Roma</td>
<td>$\rho = 0$</td>
<td>$f(v) = \sqrt{v}$</td>
<td>CIR</td>
</tr>
<tr>
<td>Heston</td>
<td>$\rho \neq 0$</td>
<td>$f(v) = \sqrt{v}$</td>
<td>CIR</td>
</tr>
</tbody>
</table>

of the VA sub-account becomes:

$$dA(t) = (r - \alpha)A(t)dt - Gdt + \pi \sigma(t)A(t)dW^Q_S(t)$$  (5.6)

where $\sigma(t) = \sqrt{V(t)}$ and

$$dV(t) = \theta(\bar{V} - V(t))dt + \gamma \sqrt{V(t)}dW_V(t)$$  (5.7)

The evolution of the mortality intensity remains unchanged, being not influenced by $\sigma(t)$:

$$d\mu_{x+t}(t) = (a + (b - \lambda \sigma_\mu)\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)}dW^Q_M(t)$$  (5.8)

### 5.2 Numerical results

In this section we will present the numerical results obtained considering the Heston model for the volatility process.

As for the experiments described in the previous chapter, we first have to discretize the involved processes.

In particular, the Euler scheme for equation [5.7] has the form:

$$V(t + \Delta t) - V(t) = \theta(\bar{V} - V(t))\Delta t + \gamma \sqrt{V(t)}dW_V(t)$$  (5.9)

As described in the previous chapter for the interest rates CIR process, also in this case using an Euler discretization can give rise to a problem of practical nature. In
Table 5.2: Calibrated parameters for the volatility process

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.3</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.6</td>
</tr>
<tr>
<td>$V(0)$</td>
<td>$\bar{V}$</td>
</tr>
<tr>
<td>$\rho_{S,V}$</td>
<td>$-0.3$</td>
</tr>
</tbody>
</table>

fact, it is not guaranteed the positivity of the domain of the square root process. So, as in the interest rate case, in what follows, we use $x^+ = \max(x, 0)$ as fixing function:

$$V(t + \Delta t) = [V(t) + \theta(\bar{V} - V(t))\Delta t + \gamma\sqrt{V(t)} dW_V(t)]^+$$

The parameters of the volatility process are those reported in Grzelak & Oosterlee (2011) and summarized in Table 5.2.

We have then proceeded to price the GLWB option using the Algorithm 2 procedure. The fair fee rates obtained are reported in Tables 5.3 and 5.4. As in the deterministic case, we have conducted sensitivity analyses in order to study the relationship between the fair fee rate and the same financial and demographic factors, i.e. interest rates, the equity exposure, the market price of mortality risk and the volatility of the mortality intensity. The impact of these factors on the price of the GLWB option is analogue to the determinist case; so, for the same reasons, we can note a negative relationship between the interest rate level $r$ and $\alpha^*$ and a positive relation with the market price coefficient of mortality risk $\lambda$, the volatility of the mortality intensity $\sigma_{\mu}$ and the equity exposure $\pi$. At the same time, we have analyzed the sensitivity of the fair contract price to the parameters of the volatility model. In particular, we have changed the values of the mean reversion coefficient $\theta$ and of the rate of diffusion $\gamma$. Numerical results are reported in figures 5.1 and 5.2. Considerations are analogue to those for the CIR interest rates process described in the previous chapter.
Figure 5.1: Sensitivity of the fair fee rate $\alpha^*$ with respect to $\sigma_{\mu}$, $\lambda$, $\pi$ and $r$ varying the mean reversion coefficient $\theta$ for policyholders aged 65
Figure 5.2: Sensitivity of the fair fee rate $\alpha^*$ with respect to $\sigma_\mu$, $\lambda$, $\pi$ and $r$ varying the rate of diffusion $\gamma$ for policyholders aged 65
Table 5.3: Fair guarantee fees (%) varying the mean reversion coefficient $\theta$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>0</th>
<th>0.3</th>
<th>0.5</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>2.5925</td>
<td>3.1683</td>
<td>3.1535</td>
<td>3.1298</td>
</tr>
<tr>
<td>2%</td>
<td>0.9708</td>
<td>1.5395</td>
<td>1.5298</td>
<td>1.5059</td>
</tr>
<tr>
<td>3%</td>
<td>0.4043</td>
<td>0.8550</td>
<td>0.8428</td>
<td>0.8160</td>
</tr>
<tr>
<td>4%</td>
<td>0.2142</td>
<td>0.5113</td>
<td>0.4962</td>
<td>0.4672</td>
</tr>
<tr>
<td>5%</td>
<td>0.1488</td>
<td>0.3227</td>
<td>0.3052</td>
<td>0.2758</td>
</tr>
<tr>
<td>6%</td>
<td>0.1128</td>
<td>0.2118</td>
<td>0.1936</td>
<td>0.1660</td>
</tr>
<tr>
<td>7%</td>
<td>0.0876</td>
<td>0.1425</td>
<td>0.1254</td>
<td>0.1013</td>
</tr>
<tr>
<td>8%</td>
<td>0.0685</td>
<td>0.0969</td>
<td>0.0819</td>
<td>0.0615</td>
</tr>
<tr>
<td>$\lambda$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.4</td>
<td>0.1861</td>
<td>0.4119</td>
<td>0.3994</td>
<td>0.3717</td>
</tr>
<tr>
<td>0</td>
<td>0.1989</td>
<td>0.4615</td>
<td>0.4461</td>
<td>0.4161</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2142</td>
<td>0.5113</td>
<td>0.4962</td>
<td>0.4672</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2300</td>
<td>0.5632</td>
<td>0.5478</td>
<td>0.5185</td>
</tr>
<tr>
<td>1.2</td>
<td>0.2532</td>
<td>0.6281</td>
<td>0.6110</td>
<td>0.5813</td>
</tr>
<tr>
<td>1.6</td>
<td>0.2847</td>
<td>0.6955</td>
<td>0.6819</td>
<td>0.6534</td>
</tr>
<tr>
<td>$\pi$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.0060</td>
<td>0.0060</td>
<td>0.0060</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0813</td>
<td>0.1298</td>
<td>0.1190</td>
<td>0.1043</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1371</td>
<td>0.3031</td>
<td>0.2879</td>
<td>0.2637</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2142</td>
<td>0.5113</td>
<td>0.4962</td>
<td>0.4672</td>
</tr>
<tr>
<td>1</td>
<td>0.3117</td>
<td>0.8512</td>
<td>0.8388</td>
<td>0.8119</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0.1862</td>
<td>0.4196</td>
<td>0.4019</td>
<td>0.3715</td>
</tr>
<tr>
<td>0.011</td>
<td>0.1939</td>
<td>0.4474</td>
<td>0.4332</td>
<td>0.4030</td>
</tr>
<tr>
<td>0.021</td>
<td>0.2142</td>
<td>0.5113</td>
<td>0.4962</td>
<td>0.4672</td>
</tr>
<tr>
<td>0.031</td>
<td>0.2554</td>
<td>0.6041</td>
<td>0.5890</td>
<td>0.5609</td>
</tr>
<tr>
<td>0.041</td>
<td>0.3272</td>
<td>0.7426</td>
<td>0.7282</td>
<td>0.6994</td>
</tr>
<tr>
<td>0.051</td>
<td>0.4377</td>
<td>0.9126</td>
<td>0.9028</td>
<td>0.8779</td>
</tr>
</tbody>
</table>
Table 5.4: Fair guarantee fees (%) varying the rate of diffusion $\gamma$

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>0</th>
<th>0.3</th>
<th>0.6</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>3.1092</td>
<td>3.0021</td>
<td>3.1683</td>
<td>3.5182</td>
</tr>
<tr>
<td>2%</td>
<td>1.4748</td>
<td>1.4009</td>
<td>1.5395</td>
<td>1.8248</td>
</tr>
<tr>
<td>3%</td>
<td>0.7715</td>
<td>0.7431</td>
<td>0.8550</td>
<td>1.0791</td>
</tr>
<tr>
<td>4%</td>
<td>0.4170</td>
<td>0.4224</td>
<td>0.5113</td>
<td>0.6856</td>
</tr>
<tr>
<td>5%</td>
<td>0.2274</td>
<td>0.2519</td>
<td>0.3227</td>
<td>0.4554</td>
</tr>
<tr>
<td>6%</td>
<td>0.1236</td>
<td>0.1555</td>
<td>0.2118</td>
<td>0.3123</td>
</tr>
<tr>
<td>7%</td>
<td>0.0667</td>
<td>0.0987</td>
<td>0.1425</td>
<td>0.2182</td>
</tr>
<tr>
<td>8%</td>
<td>0.0357</td>
<td>0.0635</td>
<td>0.0969</td>
<td>0.1535</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.4</td>
<td>0.3216</td>
<td>0.3362</td>
<td>0.4119</td>
</tr>
<tr>
<td>0</td>
<td>0.3657</td>
<td>0.3768</td>
<td>0.4615</td>
<td>0.6334</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4170</td>
<td>0.4224</td>
<td>0.5113</td>
<td>0.6856</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4672</td>
<td>0.4697</td>
<td>0.5632</td>
<td>0.7416</td>
</tr>
<tr>
<td>1.2</td>
<td>0.5358</td>
<td>0.5292</td>
<td>0.6281</td>
<td>0.8239</td>
</tr>
<tr>
<td>1.6</td>
<td>0.6087</td>
<td>0.5957</td>
<td>0.6955</td>
<td>0.8893</td>
</tr>
<tr>
<td>$\pi$</td>
<td>0.0060</td>
<td>0.0060</td>
<td>0.0060</td>
<td>0.0060</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0848</td>
<td>0.0977</td>
<td>0.1298</td>
<td>0.1932</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2252</td>
<td>0.2415</td>
<td>0.3031</td>
<td>0.4269</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4170</td>
<td>0.4224</td>
<td>0.5113</td>
<td>0.6856</td>
</tr>
<tr>
<td>1</td>
<td>0.7638</td>
<td>0.7268</td>
<td>0.8512</td>
<td>1.0553</td>
</tr>
<tr>
<td>$\sigma_i^\mu$</td>
<td>0</td>
<td>0.3165</td>
<td>0.3345</td>
<td>0.4196</td>
</tr>
<tr>
<td>0.011</td>
<td>0.3513</td>
<td>0.3622</td>
<td>0.4474</td>
<td>0.6214</td>
</tr>
<tr>
<td>0.021</td>
<td>0.4170</td>
<td>0.4224</td>
<td>0.5113</td>
<td>0.6856</td>
</tr>
<tr>
<td>0.031</td>
<td>0.5158</td>
<td>0.5110</td>
<td>0.6041</td>
<td>0.7814</td>
</tr>
<tr>
<td>0.041</td>
<td>0.6536</td>
<td>0.6402</td>
<td>0.7426</td>
<td>0.9347</td>
</tr>
<tr>
<td>0.051</td>
<td>0.8417</td>
<td>0.8069</td>
<td>0.9126</td>
<td>1.1163</td>
</tr>
</tbody>
</table>
Chapter 6

The multi-factor model

6.1 Multi-factor models

In this chapter we will price the GLWB option considering a stochastic process for both the interest rate and the volatility. Derivatives that depend on a variety of factors, in fact, can be modeled through the specification of a system of stochastic differential equations, that correspond to the involved state variables (in our specific case reference fund, interest rate and volatility). By correlating the SDEs from the different asset classes one can define so-called hybrid models. Even if each of these SDEs yields a closed form solution, a non-zero correlation structure between the processes may cause difficulties for modelling and product pricing. Typically, a closed form solution of the hybrid models is not known, and numerical approximation has to be employed for model evaluation.

6.2 Heston-CIR hybrid pricing model

As described in the previous chapters, the hypothesis of a constant interest rate may be inappropriate for pricing interest rate sensitive products, as well as the assumption of a constant volatility is no longer able to capture modern market phenomena. For these reasons, we weakened these misspecifications by allowing interest rates and volatility to vary randomly. Until now, we analyzed separately the impact
on the pricing of the GLWB option of the stochasticity of interest rates and of volatility. In this chapter we will consider the combined effect on the contract fair fee of both stochasticities. In particular, we will introduce a hybrid model adding to the Heston model the square root Cox-Ingersoll-Ross (CIR) process, both described in the previous chapters. The generalized model, under the $\mathbb{Q}$ measure, can be expressed in the following way:

\[
\begin{align*}
\frac{dA}{dt} &= [r(t) - \alpha]A(t) - G(t)dt + \pi\sqrt{v(t)}A(t)dW^Q_S(t) \\
\frac{dV}{dt} &= \theta(\bar{V} - V(t))dt + \gamma\sqrt{V(t)}dW^Q_V(t) \\
\frac{dr}{dt} &= k(\bar{r} - r(t))dt + \eta\sqrt{r(t)}dW^Q_r(t) \\
\frac{dm_{x+t}}{dt} &= [a + (b - \lambda)\mu_{x+t}(t)]dt + \sigma_{\mu}\sqrt{\mu_{x+t}(t)}dW^Q_M(t)
\end{align*}
\]

(6.1)

with $A(0), V(0), r(0), \mu(0) > 0$ and $A(\cdot) \geq 0$. The various random factors may be independent, but more realistically, there is often correlation between them. Recall that for multifactor Wiener processes $(W_1(t), W_2(t), \ldots, W_k(t))$ the generalization of Ito's Formula requires that

\[
\begin{align*}
\frac{dt}{dt} &= 0 \\
\frac{dt}{dW_i} &= dW_i(t)dt = 0 \\
dW_i(t)dW_j(t) &= \rho_{ij}dt
\end{align*}
\]

(6.2)

where $\rho_{ij}$ represents the statistical correlation between $W_i(t)$ and $W_j(t)$. As usual, the correlation $\rho$ of two random variables $X_1$ and $X_2$ is defined as

\[
\rho(X_1, X_2) = \frac{\text{cov}(X_1, X_2)}{\sqrt{V(X_1)}\sqrt{V(X_2)}}
\]

Note that $\rho(X_1, X_1) = 1$, and $X_1$ and $X_2$ are uncorrelated if $\rho(X_1, X_2) = 0$. In our model, we have 4 Wiener processes: those related to VA-sub account, interest rate and volatility processes are all correlated each other, while we assume independence between financial and systematic mortality risk, so $\rho_{M,S} = \rho_{M,V} = \rho_{M,r} = 0$.

There isn’t a closed form solution of our hybrid model, therefore numerical approximation has to be employed.

To construct discretized correlated Wiener processes for use in SDE solvers, we begin with a desired correlation matrix that we would like to specify for the Wiener
processes $W_S, W_V, W_r$.

$$C = \begin{bmatrix}
\rho_{S,S} & \rho_{S,V} & \rho_{S,r} \\
\rho_{V,S} & \rho_{V,V} & \rho_{V,r} \\
\rho_{r,S} & \rho_{r,V} & \rho_{r,r}
\end{bmatrix}$$

$C$ is a symmetric matrix with units on the main diagonal. To simplify and lighten the notation, we set

$$\rho_1 = \rho_{S,v}, \quad \rho_2 = \rho_{S,r} \quad \rho_3 = \rho_{V,r}$$

So, we have:

$$C = \begin{bmatrix}
1 & \rho_1 & \rho_2 \\
* & 1 & \rho_3 \\
* & * & 1
\end{bmatrix}$$

Our aim is to write the system of SDEs (6.1) in terms of independent Brownian motions in order to simulate the involved processes.

We make use of the Cholesky decomposition to factorize the positive definite matrix $C$ into the product of a unique lower triangular matrix $L$ with strictly positive entries on the main diagonal and its transpose:

$$C = LL^T$$

with

$$L = \begin{bmatrix}
1 & 0 & 0 \\
\rho_1 & \sqrt{1 - \rho_1^2} & 0 \\
\rho_2 & \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} & \sqrt{1 - \rho_2^2 - \left(\frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}}\right)^2}
\end{bmatrix}$$

$L$ is called the Cholesky factor of $C$ and it can be interpreted as a generalized square root of $C$. 

$$C = LL^T = \begin{bmatrix}
1 & 0 & 0 & 0 \\
\rho_1 & \sqrt{1 - \rho_1^2} & 0 & 0 \\
\rho_2 & \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} & \sqrt{1 - \rho_2^2 - \left(\frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}}\right)^2} & 0 \\
0 & 0 & 0 & \sqrt{1 - \rho_2^2 - \left(\frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}}\right)^2}
\end{bmatrix}$$
With these considerations in mind, with the help of the upper matrix, we can rewrite the system of SDEs (6.1) as:

\[
\begin{bmatrix}
    dA(t) \\
    dV(t) \\
    dr(t)
\end{bmatrix} =
\begin{bmatrix}
    (r(t) - \alpha)A(t) - G \\
    \theta(V - \nu(t)) \\
    k(\bar{r} - r(t))
\end{bmatrix} dt + B
\begin{bmatrix}
    d\tilde{W}^{Q}_A(t) \\
    d\tilde{W}^{Q}_V(t) \\
    d\tilde{W}^{Q}_r(t)
\end{bmatrix}
\]

where

\[
B =
\begin{bmatrix}
    \pi \sqrt{v(t)} & \rho_1 \pi \sqrt{v(t)} & \rho_2 \pi \sqrt{v(t)} \\
    0 & \sqrt{1 - \rho_1^2} \gamma \sqrt{v(t)} & \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \gamma \sqrt{v(t)} \\
    0 & 0 & \sqrt{1 - \rho_2^2} - \left( \frac{\rho_3 - \rho_1 \rho_2}{\sqrt{1 - \rho_1^2}} \right)^2 \eta \sqrt{r(t)}
\end{bmatrix}
\]

and \(d\tilde{W}^{Q}_i(t) (i = A, v, r)\) are independent Brownian motions. After the Euler discretization of the involved processes, we have proceeded to price the GLWB option using Algorithm 2. We used the values reported in the previous two chapters for the parameters of the CIR and Heston processes. In addition, we set \(\rho_{r,V}\) equal to 0.15. Numerical results are reported in Table 6.1.

In particular, as until now, we have conducted sensitivity analyses in order to study the relationship between the fair fee rate and the demographic factors already examined, i.e. the market price coefficient of mortality risk and the volatility of the mortality intensity. As in the previous experiments, we can note a positive relation of the fair fee rate \(\alpha^*\) with \(\lambda\) (see figure 6.1) and with \(\sigma_{\mu}\) (see figure 6.2). In addition, as in many papers on this topics the correlation coefficient \(\rho_{r,V}\) is set equal to zero, we have considered also this hypothesis. Numerical analyses show the stability of the results: little changes in the correlation coefficient correspond to little changes in the fair fee rates.

In conclusion, we have summarized in Table 6.2 all the obtained results in order to compare them. In particular we show how the fair price of the GLWB contract changes when we consider a deterministic approach (first column), a stochastic process only for the term structure of interest rates (second column), a stochastic process only for the volatility of the reference fund (third column) or a combined stochastic...
Table 6.1: Sensitivity of the fair fee rate $\alpha^*$ with respect to $\sigma_\mu$ and $\lambda$ with different values for the correlation coefficient $\rho_{r,V}$

<table>
<thead>
<tr>
<th>$\rho_{r,V} = 0.00$</th>
<th>$\sigma_\mu$</th>
<th>g</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0.6854</td>
<td>1.1021</td>
<td>1.8295</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.7692</td>
<td>1.2520</td>
<td>2.1275</td>
</tr>
<tr>
<td></td>
<td>0.020</td>
<td>0.9210</td>
<td>1.5367</td>
<td>2.7266</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>1.2178</td>
<td>2.1167</td>
<td>4.0845</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>1.7497</td>
<td>3.2978</td>
<td>7.8394</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>2.6391</td>
<td>5.9230</td>
<td>28.6995</td>
</tr>
<tr>
<td></td>
<td>-0.4</td>
<td>0.6989</td>
<td>1.1142</td>
<td>1.8327</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.7966</td>
<td>1.2966</td>
<td>2.2051</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.9210</td>
<td>1.5367</td>
<td>2.7266</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.0830</td>
<td>1.8580</td>
<td>3.4897</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>1.2893</td>
<td>2.2977</td>
<td>4.7085</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>1.5520</td>
<td>2.9283</td>
<td>6.9568</td>
</tr>
<tr>
<td>$\rho_{r,V} = 0.15$</td>
<td>$\sigma_\mu$</td>
<td>g</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.6792</td>
<td>1.0971</td>
<td>1.8262</td>
</tr>
<tr>
<td></td>
<td>0.010</td>
<td>0.7622</td>
<td>1.2461</td>
<td>2.1220</td>
</tr>
<tr>
<td></td>
<td>0.020</td>
<td>0.9161</td>
<td>1.5317</td>
<td>2.7219</td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>1.2150</td>
<td>2.1152</td>
<td>4.0845</td>
</tr>
<tr>
<td></td>
<td>0.040</td>
<td>1.7483</td>
<td>3.2980</td>
<td>7.8472</td>
</tr>
<tr>
<td></td>
<td>0.050</td>
<td>2.6426</td>
<td>5.9275</td>
<td>28.7127</td>
</tr>
<tr>
<td></td>
<td>-0.4</td>
<td>0.6928</td>
<td>1.1080</td>
<td>1.8265</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.7902</td>
<td>1.2906</td>
<td>2.1990</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>0.9161</td>
<td>1.5317</td>
<td>2.7219</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.0792</td>
<td>1.8549</td>
<td>3.4875</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>1.2835</td>
<td>2.2932</td>
<td>4.7077</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>1.5462</td>
<td>2.9230</td>
<td>6.9673</td>
</tr>
</tbody>
</table>
Figure 6.1: Sensitivity of the fair fee rate $\alpha^*$ with respect to the market price coefficient of the systematic mortality risk $\lambda$ for policyholders aged 65

![Figure 6.1: Sensitivity of the fair fee rate $\alpha^*$ with respect to the market price coefficient of the systematic mortality risk $\lambda$ for policyholders aged 65](image)

Figure 6.2: Sensitivity of the fair fee rate $\alpha^*$ with respect to the volatility parameter of mortality intensity $\sigma_\mu$ for policyholders aged 65

![Figure 6.2: Sensitivity of the fair fee rate $\alpha^*$ with respect to the volatility parameter of mortality intensity $\sigma_\mu$ for policyholders aged 65](image)
Table 6.2: Summary comparison

<table>
<thead>
<tr>
<th></th>
<th>$\eta = 0$</th>
<th>$\eta = 0.01$</th>
<th>$\eta = 0$</th>
<th>$\eta = 0.01$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 0$</td>
<td>0.9882</td>
<td>1.0122</td>
<td>1.0796</td>
<td>1.0971</td>
</tr>
<tr>
<td></td>
<td>1.1386</td>
<td>1.1659</td>
<td>1.2256</td>
<td>1.2461</td>
</tr>
<tr>
<td></td>
<td>1.4335</td>
<td>1.4669</td>
<td>1.5054</td>
<td>1.5317</td>
</tr>
<tr>
<td>$\sigma_\mu$</td>
<td>1.9964</td>
<td>2.0417</td>
<td>2.0765</td>
<td>2.1152</td>
</tr>
<tr>
<td></td>
<td>3.1473</td>
<td>3.2212</td>
<td>3.2305</td>
<td>3.2980</td>
</tr>
<tr>
<td></td>
<td>5.6904</td>
<td>5.8595</td>
<td>5.7648</td>
<td>5.9275</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>-0.4</td>
<td>1.0044</td>
<td>1.0906</td>
<td>1.1080</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1.1914</td>
<td>1.2693</td>
<td>1.2906</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>1.4335</td>
<td>1.5054</td>
<td>1.5317</td>
</tr>
<tr>
<td></td>
<td>0.8</td>
<td>1.7492</td>
<td>1.8217</td>
<td>1.8549</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>2.1741</td>
<td>2.2507</td>
<td>2.2932</td>
</tr>
<tr>
<td></td>
<td>1.6</td>
<td>2.7910</td>
<td>2.8658</td>
<td>2.9230</td>
</tr>
</tbody>
</table>

Numerical results confirm that introducing random shocks on interest rates and/or volatility increases the value of the fees that GLWB issuers have to charge in order to price fairly the contract. A stochastic approach, especially that obtained allowing both interest rates and volatility to vary randomly, is more able to describe the real fluctuations of the market, so it is necessary in order to avoid underestimation of the policy.
Chapter 7

Conclusion

This thesis dealt with the problem of pricing a particular rider embedded in variable annuity contracts: the Guaranteed Lifetime Withdrawal Benefit (GLWB) guarantee. This option meets medium to long-term investment needs, while providing adequate hedging against market volatility and longevity-related risks. Indeed, based on an initial capital investment, it guarantees the policyholder a stream of future payments, regardless of the performance of the underlying policy, for his/her whole life. In this work, we have proposed a valuation model for the policy using tractable financial and stochastic mortality processes in a continuous time framework. We have analyzed the contract considering both the policyholder’s and the insurer’s point of view and assuming a static approach, in which clients withdraw exactly the guaranteed amount each year. In particular, we based our analyses on the model presented in the paper “Systematic mortality risk: an analysis of guaranteed lifetime withdrawal benefits in variable annuities” by M. C. Fung, K. Ignatieva and M. Sherris (2014), with the aim at proposing its generalization. The valuation, indeed, has been performed initially in a Black and Scholes economy: the sub-account value has been assumed to follow a geometric Brownian motion, thus with a constant volatility, and the term structure of interest rates has been assumed to be constant. These hypotheses, however, do not find justification in the financial markets. In order to consider a more realistic model we sought to weaken these misconceptions. Specifically we have introduced a Cox,
Ingersoll and Ross (CIR) stochastic process for the term structure of interest rates and a Heston model for the volatility of the underlying account. We have addressed these two hypotheses separately at first, and jointly thereafter. We have implemented the theoretical model using a Monte Carlo approach. To this end, we have created ad hoc codes based on the programming language MATLAB, exploiting its fast matrix-computation facilities. Sensitivity analyses have been conducted in order to investigate the relation between the fair price of the contract and important financial and demographic factors. In particular, we have observed a negative relationship between the interest rate level $r$ and the fair fee rate $\alpha^*$ and a positive one with the market price coefficient of mortality risk $\lambda$, the volatility of the mortality intensity $\sigma_\mu$ and the equity exposure $\pi$. At the same time, we have analyzed the impact on the fair contract price of two important parameters of the volatility and interest rate model: the mean reversion coefficient and the rate of diffusion. Comparing the deterministic approach with the stochastic one, we have found an increase in the value of the fees that GLWB issuers have to charge in order to price the contract fairly. The introduction of random shocks on interest rates and/or volatility, indeed, allows to consider the real fluctuations of the market. A stochastic approach, especially the one obtained allowing both interest rates and volatility to vary randomly, is therefore necessary in order to avoid an underestimation of the policy.
Appendix A

An introduction to Stochastic Calculus

The aim of this thesis is to obtain the fair value of the GLWB option embedded in a variable annuity contract. The goal is reached throughout the numerical solution of a system of stochastic differential equations. In this appendix we will introduce a quick survey of the most fundamental concepts from stochastic calculus that are needed to proceed with the description of the GLWB’s valuation model. For full details, the reader may consult, for example, Oksendal (2003) and Kloeden & Platen (1999).

A.1 Preliminary notions

About three hundred years ago, Newton and Leibniz developed the differential calculus, allowing us to model continuous time dynamical systems in most areas of science and contributing to the revolutionary developments in technology, science, and manufacturing that the world has experienced over the last two centuries. As we try to build more realistic models, stochastic effects need to be taken into account. In areas such as finance, the randomness in the system dynamics is in fact the essential phenomenon to be modeled. Continuous time stochastic dynamics appear in
many areas of application, such as biology, physics, economics, finance, insurance, chemistry, medicine, etc. Practical problems arising in these areas in the mid-1900s led to the development of a corresponding stochastic calculus.

Stochastic calculus is concerned with the study of stochastic processes, which model the uncertainties using probability models. The basic object in a probability model is a probability space, which is a triple \((\Omega, \mathcal{F}, P)\) consisting of a set \(\Omega\), usually denoted as the sample space, a \(\sigma\)-field \(\mathcal{F}\) of subsets of \(\Omega\) and a probability \(P\) defined on \(\mathcal{F}\). The set \(\Omega\) can be considered as the set of all possible outcomes \(\omega \in \Omega\) of some random experiment or phenomenon. To any event we can associated the subset \(A \subset \Omega\) consisting of all scenarios at which the event occurs. Such a subset will also be denoted as an event and \(\mathcal{F}\) is the collection of all events. \(\mathcal{F}\) is called the event space and it represents both the amount of information available as a result of the experiment conducted and the collection of all events of possible interest to us. From a mathematical point of view, it is important to consider only collections of events that have the structure of a \(\sigma\)-field.

Definition 2 A collection \(\mathcal{F}\) of subsets of a set \(\Omega\) is called a \(\sigma\)-field if

i) \(\Omega \in \mathcal{F}\);

ii) if \(A \in \mathcal{F}\) then \(A^C \in \mathcal{F}\) as well (where \(A^C = \Omega \setminus A\) is the complement of \(A\) in \(\Omega\));

iii) if \(A_i \in \mathcal{F}\) for \(i = 1, 2, \ldots\) then also \(\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}\).

A measurable space is a pair \((\Omega, \mathcal{F})\), where \(\Omega\) is a set and \(\mathcal{F}\) a \(\sigma\)-field of subsets of \(\Omega\).

Definition 3 A probability measure \(P\) defined on a \(\sigma\)-field \(\mathcal{F}\) is a map from \(\mathcal{F}\) to the interval \([0, 1]\) such that

i) \(0 \leq P(A) \leq 1\) for all \(A \in \mathcal{F}\);

ii) \(P(\Omega) = 1\);

iii) \(P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)\) for any pairwise disjoint sequence \((A_n) \subset F\). Pairwise disjoint means that \(A_i \cap A_j = \emptyset\) for \(i \neq j\).
Definition 4 A random variable is a function $X : \Omega \to \mathbb{R}$ such that $\forall \alpha \in \mathbb{R}$ the set \{ $\omega : X(\omega) \leq \alpha$ \} is in $\mathcal{F}$ (such a function is also called a $\mathcal{F}$-measurable or, simply, measurable function).

Definition 5 Given a random variable $X$ we denote by $\sigma(X)$ the smallest $\sigma$-field $\mathcal{G} \subseteq \mathcal{F}$ such that $X(\omega)$ is measurable on $(\Omega, \mathcal{G})$. One can show that $\sigma(X) = \sigma(\{ \omega : X(\omega) \leq \alpha \})$. We call $\sigma(X)$ the $\sigma$-field generated by $X$ and interchangeably use the notations $\sigma(X)$ and $\mathcal{F}_X$. Similarly, given the random variables $X_1, \ldots, X_n$ on the same measurable space $(\Omega, \mathcal{F})$, denote by $\sigma(X_k, k \leq n)$ the smallest $\sigma$-field $\mathcal{F}$ such that $X_k(\omega)$, $k = 1, \ldots, n$ are measurable on $(\Omega, \mathcal{F})$. That is, $\sigma(X_k, k \leq n)$ is the smallest $\sigma$-field containing $\sigma(X_k)$ for $k = 1, \ldots, n$.

The concept of $\sigma$-field is needed in order to produce a rigorous mathematical theory. It further has the crucial role of quantifying the amount of information we have.

A.2 Stochastic Processes

Definition 6 A stochastic process is a parametrized collection of random variables

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}^n$.

The set $T$ is called parameter space. When $T = \mathbb{N} = \{0, 1, 2, \ldots\}$, the process $\{X_t\}_{t \in T}$ is said to be a discrete parameter process. If $T$ is not countable, the process is said to have a continuous parameter. In the latter case the usual examples are $T = \mathbb{R}_+ = [0, \infty)$, or $T$ equal to an interval $[a, b] \subset \mathbb{R}$.

Note that for each $t \in T$ fixed we have a random variable

$$\omega \rightarrow X_t(\omega); \quad \omega \in \Omega.$$ 

On the other hand, fixing $\omega \in \Omega$ we can consider the function

$$t \rightarrow X_t(\omega); \quad t \in T$$
which is called a path (realization, trajectory) of $X$.

It may be useful for the intuition to think of $t$ as “time” and each $\omega$ as an individual “particle” or “experiment”. With this picture, $X_t(\omega)$ would represent the position (or result) at time $t$ of the particle (experiment) $\omega$. Sometimes it is convenient to write $X(t, \omega)$ instead of $X_t(\omega)$. Thus we may also regard the process as a function of two variables

$$(t, \omega) \rightarrow X(t, \omega)$$

from $T \times \Omega$ into $\mathbb{R}^n$. This is often a natural point of view in stochastic analysis, because it is crucial to have $X(t, \omega)$ jointly measurable in $(t, \omega)$.

It’s important to build the notion of time into our probability space. To this end, recall that we can collect more and more information as time goes on through a filtration.

**Definition 7** A filtration is a non-decreasing family of sub-$\sigma$-fields $\{F_n\}$ of a measurable space $(\Omega, \mathcal{F})$. That is, $\mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \mathcal{F}$ and $\mathcal{F}_n$ is a $\sigma$-field for each $n$. The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, P)$ is called a filtered probability space.

Given a filtration, we are interested in stochastic processes such that for each $n$ the information gathered by that time suffices for evaluating the value of the $n$-th element of the process. That is,

**Definition 8** A stochastic process $\{X_n, n = 0, 1, \ldots\}$ is adapted to a filtration $\{\mathcal{F}_n\}$ if $\omega \rightarrow X_n(\omega)$ is a random variable on $(\Omega, \mathcal{F}_n)$ for each $n$, that is, if $\sigma(X_n) \subseteq \mathcal{F}_n$ for each $n$ ($X_n$ is $\mathcal{F}_n$-measurable for every $n$). A stochastic process $\{X_n\}$ is called $\mathcal{F}_n$-predictable if $X_n$ is $\mathcal{F}_{n-1}$-measurable for every $n$.

**Definition 9** The filtration $\{\mathcal{G}_n\}$ with $\mathcal{G}_n = \sigma(X_0, X_1, \ldots, X_n)$ is the minimal filtration with respect to which $\{X_n\}$ is adapted. We therefore call it the canonical filtration for the stochastic process $\{X_n\}$.

Conversely, if we have some probability space $(\Omega, \mathcal{F}, P)$ and a sequence of random variables $\{X_n\}$, we can use this sequence to generate a filtration:
Definition 10 Let $(\Omega, \mathcal{F}, P)$ be a probability space and \{X_n\} be a stochastic process. The filtration generated by \{X_n\} is defined as $\mathcal{F}_n^X = \sigma\{X_0, \ldots, X_n\}$, and the process \{X_n\} is $\mathcal{F}_n^X$-adapted by construction.

A martingale is a very special type of stochastic process. It consists of a filtration and an adapted stochastic process which has the property of being a “fair game”, that is, the expected future reward given current information is exactly the current value of the process. We now make this into a rigorous definition.

Definition 11 A martingale is a pair $(X_n, \mathcal{F}_n)$, where \{\mathcal{F}_n\} is a filtration and $X_n$ an integrable (i.e. $E|X_n| < \infty$), stochastic process adapted to this filtration such that

$$E[X_{n+1} | \mathcal{F}_n] = X_n \quad \forall n \text{ a.s.}$$

A.2.1 The Wiener process

The notion of stochastic process is very important both in mathematical theory and its applications in science, engineering, economics, etc. It is used to model a large number of various phenomena where the quantity of interest varies discretely or continuously through time in a non-predictable fashion. Since 1827, the botanist R. Brown described the motion of a pollen particle suspended in a fluid as an irregular, random movement. The mathematic foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, and for this reason the process is also called a Wiener process.

Definition 12 A standard Brownian motion (or Wiener process) \{W_t, t \geq 0\} is a continuous-time stochastic process with the following properties:

i) $W_0 = 0$

ii) For all $0 \leq t_1 \leq \cdots \leq t_n$ the increments $W_{t_n} - W_{t_{n-1}}, \ldots, W_{t_2} - W_{t_1}, W_{t_1}$ are independent random variables

iii) If $0 \leq s < t$, the increment $W_t - W_s$ has a Normal distribution $N(0, t-s)$
iv) The process \( \{W_t\} \) has continuous trajectories

Given a Wiener process \( W_t \), we can introduce its natural filtration \( F_t^W = \sigma\{W_s : s \leq t\} \). More generally, it is sometimes convenient to speak of an \( \mathcal{F}_t \)-Wiener process when it is \( \mathcal{F}_t \)-adapted. It can be proved that an \( \mathcal{F}_t \)-Wiener process \( W_t \) is an \( \mathcal{F}_t \)-martingale. And it is also a Markov process.

**Definition 13** An \( \mathcal{F}_t \)-adapted process \( X_t \) is called an \( \mathcal{F}_t \)-Markov process if we have
\[
E(f(X_t) | \mathcal{F}_s) = E(f(X_t) | X_s)
\]
for all \( t \geq s \) and all bounded measurable functions \( f \). When the filtration is not specified, the natural filtration \( \mathcal{F}_t^X \) is implied.

### A.3 Stochastic Differential Equations

Consider a Brownian motion \( \{W_t, t \geq 0\} \) defined on a probability space \((\Omega, \mathcal{F}, P)\). Suppose that \( \{\mathcal{F}_t, t \geq 0\} \) is a filtration such that \( W_t \) is \( \mathcal{F}_t \)-adapted and for any \( 0 \leq s < t \), the increment \( W_t - W_s \) is independent of \( \mathcal{F}_s \). We aim to solve stochastic differential equations of the form
\[
dX_t = a(t, X_t)dt + b(t, X_t)dW_t
\]
with an initial condition \( X_0 \), which is a random variable independent of the Brownian motion \( W_t \). The coefficients \( a(t, x) \) and \( b(t, x) \) are called, respectively, drift and diffusion coefficient. If the diffusion coefficient vanishes, then we have that (A.1) is the ordinary differential equation:
\[
\frac{dX_t}{dt} = a(t, X_t)
\]
For instance, in the linear case \( b(t, x) = a(t)x \), the solution of this equation, with the initial condition, is
\[
X_t = X_0 e^{\int_0^t a(s)ds}
\]
The stochastic differential equation (A.1) has the following heuristic interpretation. The increment \( \Delta X_t = X_{t+\Delta t} - X_t \) can be approximatively decomposed into the sum of \( a(t, X_t)\Delta t \) plus the term \( b(t, X_t)\Delta W_t \) which is interpreted as a random impulse. The approximate distribution of this increment will be the normal distribution.
with mean \( a(t, X_t) \Delta t \) and variance \( b(t, X_t)^2 \Delta t \). Notice that the SDE (A.1) is given in differential form, unlike the derivative form of an ODE. That is because many interesting stochastic processes, like Brownian motion, are continuous but not differentiable. Therefore, a formal meaning of Equation (A.1) is obtained by rewriting it in integral form, using stochastic integrals:

\[
X_t = X_0 + \int_0^t a(s, X_s) \, ds + \int_0^t b(s, X_s) \, dW_s \tag{A.2}
\]

where the last integral is called an Itô integral. The solution will be an Itô process \( \{X_t, t \geq 0\} \). The solutions of stochastic differential equations are called diffusion processes.

The main result on the existence and uniqueness of SDEs solutions is the following.

**Theorem 1** Fix a time interval \([0, T]\). Suppose that the coefficients of Equation (A.1) satisfy the following Lipschitz and linear growth properties:

\[
|a(t, x) - a(t, y)| \leq D_1 |x - y|
|b(t, x) - b(t, y)| \leq D_2 |x - y|
|a(t, x)| \leq C_1 (1 + |x|)
|b(t, x)| \leq C_2 (1 + |x|) \tag{A.3}
\]

for all \( x, y \in \mathbb{R}, t \in [0, T] \). Suppose that \( X_0 \) is a random variable independent of the Brownian motion \( \{W_t, 0 \leq t \leq T\} \) and such that \( E(X_0^2) < \infty \). Then, there exists a unique continuous and adapted stochastic process \( \{X_t, t \in [0, T]\} \) such that

\[
E(\int_0^T |X_s|^2 \, ds) < \infty
\]

which satisfies Equation (A.2).

To solve SDEs analytically, we need to introduce the chain rule for stochastic differentials, called the Itô formula: If \( Y = f(t, X) \), then

\[
dY = \frac{\partial f}{\partial t}(t, X) dt + \frac{\partial f}{\partial x}(t, X) dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X) dx dx \tag{A.4}
\]
where the $dx dx$ term is defined using the identities

\[ dtdt = 0 \]
\[ dt dW_t = dW_t dt = 0 \]
\[ dW_t dW_t = dt \] \hspace{1cm} (A.5)

The Ito formula is the stochastic analogue to the chain rule of conventional calculus. Although it is expressed in differential form for ease of understanding, its meaning is precisely the equality of the Ito integral of both sides of the equation. It is proved under rather weak hypotheses by referring the equation back to the definition of Ito integral (Oksendal (2003)).

Unfortunately, explicitly solvable SDEs are rare in practical applications. There are, however, an increasing number of numerical methods for the solution of SDEs mentioned in the literature.

A.4 Numerical approaches to SDEs

Since analytical solutions of SDEs are rare, numerical approximations have been developed. Several different approaches have been proposed in the literature to handle SDEs numerically (see Kloeden & Platen (1999) for further details). On the very general level, there is a method due to Boyce (1978) by means of which one can investigate, at least in principle, general random systems by Monte Carlo methods. For SDEs this method is somewhat inefficient because it does not use the special structure of these equations, specifically their characterization by drift and diffusion coefficients. Kushner (1974) and Kushner & Dupuis (1992) proposed the discretization of both time and space variables, so the approximating processes are then finite state Markov chains, manageable through their transition matrices. Higher order Markov chain approximations are developed in Platen (1992). In comparison with the information encompassed succinctly in the drift and diffusion coefficients of an SDE, transition matrices contain a considerable amount of superfluous information which must be repeatedly reprocessed during computations. Consequently
such a Markov chain approach seems applicable only for low dimensional problems on bounded domains.

The most efficient and widely applicable approach to solving SDEs seems to be the simulation of sample paths of time discrete approximations. This is based on a finite discretization of the time interval \([0, T]\) under consideration, and generates approximate values of the sample paths step by step at the discretization times. The simulated sample paths can then be analysed by usual statistical methods to determine how good the approximation is and in what sense it is close to the exact solution. An advantage of considerable practical importance of this approach is that the computational costs such as time and memory required increase only polynomially with the dimension of the problem.

A.4.1 Time discrete approximations

Simulation experiments and theoretical studies have shown that not all classical or heuristic time discrete approximations of SDEs converge in a useful sense to the corresponding solution process as the step-size \(\Delta\) tends to zero. Consequently a systematic investigation of different methods is needed in order to select a sufficiently efficient and reliable numerical method for the problem at hand.

We shall consider a time discretization

\[
0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N = T
\]

doing a time interval \([0, T]\). Time discretizations could be also random, but usually a maximum step-size \(\Delta\) must be specified.

**Euler-Maruyama Method**

The simplest effective computational example of such a method is the Euler-Maruyama method \([\text{Maruyama}}, 1955]\), which is the analogue of the Euler method for ordinary differential equations. Given the SDE initial value problem

\[
\begin{align*}
\begin{cases}
dX(t) = a(X, t)dt + b(X, t)dW_t \\
X(0) = X_0
\end{cases}
\end{align*}
\]  

(A.6)
the approximation has the form:

\[ w_{i+1} = w_i + a(t_i, w_i) \Delta t_{i+1} + b(t_i, w_i) \Delta W_{i+1} \quad (A.7) \]

for \( i = 0, 1, \ldots, N - 1 \) with initial value

\[ w_0 = X_0 \]

where

\[ \Delta t_{i+1} = t_{i+1} - t_i, \]

represents the step-size, and

\[ \Delta W_{i+1} = W(t_{i+1}) - W(t_i) \quad (A.8) \]

denotes the increment of the Wiener process in the time interval \([t_i, t_{i+1}]\). The recursive scheme \((A.7)\) obviously gives values of the approximation only at the discretization times. If values are required at intermediate instants, then either piecewise constant values from the preceding discretization point or some interpolation, especially a linear interpolation, of the values of the two immediate enclosing discretization points could be used.

To simulate a realization of the Euler approximation, one needs to simulate the Brownian motion \(\Delta W_i\). The random variables \(\Delta W_i\) defined in \((A.8)\) are independent \(N(0, \Delta t_i)\), i.e. normally distributed random variables with mean zero and variance \(\Delta t_i\). If we define \(N(0, 1)\) to be the standard random variable that is normally distributed with mean 0 and standard deviation 1, each random number \(\Delta W_i\) can be computed as

\[ \Delta W_i = z_i \sqrt{\Delta t_i} \quad (A.9) \]

where \(z_i\) is generated from independent \(N(0, 1)\). Summarizing, each set of \(\{w_0, \ldots, w_N\}\) produced by the Euler-Maruyama method is an approximate realization of the solution stochastic process \(X(t)\) which depends on the random numbers \(z_i\) that were chosen. Since \(W_t\) is a stochastic process, each realization will be random and so will the approximations.
A.4.2 Convergence of SDE solvers

It is convenient to have some measure of the efficiency of a numerical scheme by identifying its order of convergence. In stochastic numerical analysis, there are two major types of convergence to be distinguished (see Platen (1999) or Sauer (2012)). These can be identified by whether one requires

a) approximations to the sample paths, or

b) approximations to the corresponding distributions.

For convenience, we choose a rather simple characterization of each of these two types of convergence for the classification of numerical algorithms, and call them the strong and the weak convergence criterion, respectively.

**Strong convergence of SDE solvers**

Tasks involving direct simulations of paths, such as the generation of a stock market price scenario, the computation of a filter estimate for some hidden unobserved variable, or the testing of a statistical estimator for parameters in some SDEs, require that the simulated sample paths be close to those of the solution of the original SDE. This implies that in these cases, among others, some strong convergence criterion should be used.

A discrete-time approximation is said to converge strongly to the solution $X(t)$ at time $T$ if

$$\lim_{\Delta t \to 0} E\{|X(T) - w_{\Delta t}(T)|\} = 0$$

where $w_{\Delta t}$ is the approximate solution computed with constant step-size $\Delta t$, and $E$ denotes expected value. For strongly convergent approximations, we further quantify the rate of convergence by the concept of order.

An SDE solver converges strongly with order $m$ if the expected value of the error is of $m$th order in the step-size, i.e. if for any time $T$,

$$E\{|X(T) - w_{\Delta t}(T)|\} = O((\Delta t)^m)$$
for sufficiently small step-size $\Delta t$. This definition generalizes the standard convergence criterion for ordinary differential equations, reducing to the usual definition when the stochastic part of the equation goes to zero.

Although the Euler method for ordinary differential equations has order 1, the strong order for the Euler-Maruyama method for stochastic differential equations is $1/2$. This result was proved in Gikhman & Skorokhod (1979) under appropriate conditions on the functions $a$ and $b$ in (A.6). In order to build a strong order 1 method for SDEs, another term in the “stochastic Taylor series” must be added to the method. Consider the stochastic differential equation (A.6); we can define the Milstein scheme as follows:

**Milstein Method**

\[
\begin{align*}
  w_0 &= X_0 \\
  w_{i+1} &= w_i + a(w_i, t_i)\Delta t_i + b(w_i, t_i)\Delta W_i \\
  &\quad + \frac{1}{2} b(w_i, t_i) \frac{\partial b}{\partial x}(w_i, t_i)(\Delta W_i^2 - \Delta t_i)
\end{align*}
\]  

(A.10)

The Milstein Method has order one. Note that it is identical to the Euler-Maruyama method if there is no $X$ term in the diffusion part $b(X, t)$ of the equation. In case there is, Milstein will in general converge to the correct stochastic solution process more quickly than Euler-Maruyama as the step size $\Delta t_i$ goes to zero. The Milstein method is a Taylor method, meaning that it is derived from a truncation of the stochastic Taylor expansion of the solution. This is in many cases a disadvantage, since the partial derivative appears in the approximation method, and must be provided explicitly by the user. This is analogous to Taylor methods for solving ordinary differential equations, which are seldom used in practice for that reason. To counter this problem, Runge-Kutta methods were developed for ODEs, which trade these extra partial derivatives in the Taylor expansion for extra function evaluations from the underlying equation.

In the stochastic differential equation context, the same trade can be made with the Milstein method, resulting in a strong order 1 method that requires evaluation of $b(X)$ at two places on each step. A heuristic derivation can be carried out by
making the replacement

\[ b_x(w_i) \approx \frac{b(w_i + b(w_i)\sqrt{\Delta t_i}) - b(w_i)}{b(w_i)\sqrt{\Delta t_i}} \]

in the Milstein formula (A.10), which leads to the following method:

**Strong Order 1.0 Runge-Kutta Method**

\[
\begin{align*}
    w_0 &= X_0 \\
    w_{i+1} &= w_i + a(w_i)\Delta t_i + b(w_i)\Delta W_i + \\
    &\quad + \frac{1}{2}[b(w_i + b(w_i)\sqrt{\Delta t_i}) - b(w_i)](\Delta W_i^2 - \Delta t_i) / \sqrt{\Delta t_i}
\end{align*}
\]

(A.11)

The orders of the methods introduced here for SDEs, 1/2 for Euler-Maruyama and 1 for Milstein and the Runge-Kutta counterpart, would be considered low by ODE standards. Higher-order methods can be developed for SDEs, but become much more complicated as the order grows. As an example, consider the following strong order 1.5 scheme for the SDE (A.6):

**Strong Order 1.5 Taylor Method**

\[
\begin{align*}
    w_0 &= X_0 \\
    w_{i+1} &= w_i + a\Delta t_i + b\Delta W_i + \frac{1}{2}bb_x(\Delta W_i^2 - \Delta t_i) + \\
    &\quad + a_x b\Delta Z_i + \frac{1}{2}(aa_x + \frac{1}{2}b^2a_{xx})\Delta t_i^2 + \\
    &\quad + (ab_x + \frac{1}{2}b^2b_{xx})(\Delta W_i\Delta t_i - \Delta Z_i) + \\
    &\quad + \frac{1}{2}b(bb_{xx} + b_x^2)(\frac{1}{3}\Delta W_i^2 - \Delta t_i)\Delta W_i
\end{align*}
\]

(A.12)

where partial derivatives are denoted by subscripts, and where the additional random variable \(\Delta Z_i\) is normally distributed with mean 0, variance \(E(\Delta Z_i^2) = \frac{1}{3}\Delta t_i^3\) and correlated with \(\Delta W_i\) with covariance \(E(\Delta Z_i\Delta W_i) = \frac{1}{2}\Delta t_i^2\). Note that \(\Delta Z_i\) can be generated as

\[
\Delta Z_i = \frac{1}{2}\Delta t_i(\Delta W_i + \Delta V_i / \sqrt{3})
\]
where $\Delta V_i$ is chosen independently from $\sqrt{\Delta t_i}N(0,1)$.

Whether higher-order methods are needed in a given application depends on how the resulting approximate solutions are to be used. In the ordinary differential equation case, the usual assumption is that the initial condition and the equation are known with accuracy. Then it makes sense to calculate the solution as closely as possible to the same accuracy, and higher-order methods are called for. In the context of stochastic differential equations, in particular if the initial conditions are chosen from a probability distribution, the advantages of higher-order solvers are often less compelling, and if they come with added computational expense, may not be warranted.

**Weak convergence of SDE solvers**

Strong convergence allows accurate approximations to be computed on an individual realization basis. For some applications such detailed pathwise information is not required. If one aims to compute, for instance, a moment of $X$, a probability related to $X$, an option price on a stock price $X$ or a general functional of the form $E(g(X_T))$, then single realizations are not of primary interest. Rather, it is sufficient to approximate adequately the probability distribution that corresponds to $X$. Weak solvers seek to fill this need. They can be simpler than corresponding strong methods, since their goal is to replicate the probability distribution only. The following additional definition is useful.

A discrete-time approximation $w_{\Delta t}$ with step-size $\Delta t$ is said to converge weakly to the solution $X(T)$ if

$$\lim_{\Delta t \to 0} E\{f(w_{\Delta t}(T))\} = E\{f(X(T))\}$$

for all polynomials $f(x)$. According to this definition, all moments converge as $\Delta t \to 0$. If the stochastic part of the equation is zero and the initial value is deterministic, the definition agrees with the strong convergence definition, and the usual ordinary differential equation definition.

Weakly convergent methods can also be assigned an order of convergence.
We say that a the solver *converges weakly with order* $m$ if the error in the moments is of $m$th order in the step-size, or

$$|E\{f(X(T))\} - E\{f(\omega \Delta t(T))\}| = O((\Delta t)^m)$$

for sufficiently small stepsize $\Delta t$.

In general, the rates of weak and strong convergence do not agree. Unlike the case of ordinary differential equations, where the Euler method has order 1, the Euler-Maruyama method for SDEs has strong order $m = 1/2$. However, Euler-Maruyama is guaranteed to converge weakly with order 1.

Higher order weak methods can be much simpler than corresponding strong methods, and are available in several different forms. The most direct approach is to exploit the Ito-Taylor expansion (Kloeden & Platen (1999)), the Ito calculus analogue of the Taylor expansion of deterministic functions. An example of SDE solver that converges weakly with order 2 is the following:

**Weak Order 2 Taylor Method**

\[
\begin{align*}
  w_0 &= X_0 \\
  w_{i+1} &= w_i + a\Delta t_i + b\Delta W_i + \frac{1}{2}bb_x(\Delta W_i^2 - \Delta t_i) \\
  &\quad + a_x b\Delta Z_i + \frac{1}{2}(aa_x + \frac{1}{2}a_{xx}b^2)\Delta t_i^2 \\
  &\quad + (ab_x + \frac{1}{2}b_{xx}b^2)(\Delta W_i\Delta t_i - \Delta Z_i)at_i)\Delta W_i
\end{align*}
\]

(A.13)

where $\Delta W_i$ is chosen from $\sqrt{\Delta t_i}N(0, 1)$ and $\Delta Z_i$ is distributed as in the above Strong Order 1.5 Method.

A second approach is to mimic the idea of Runge-Kutta solvers for ordinary differential equations. These solvers replace the explicit higher derivatives in the Ito-Taylor solvers with extra function evaluations at interior points of the current solution interval. A weak order 2 solver of Runge-Kutta type is:

**Weak Order 2 Runge-Kutta Method**

131
\[ w_0 = X_0 \]

\[ w_{i+1} = w_i + \frac{1}{2}[a(u) + a(w_i)]\Delta t_i \]

\[ + \frac{1}{4}[b(u_+) + b(u_-) + 2b(w_i)]\Delta W_i \]

\[ + \frac{1}{4}[b(u_+) + b(u_-)][(\Delta W_i^2 - \Delta t_i)/\sqrt{\Delta t_i}] \]  

(A.14)

where

\[ u = w_i + a\Delta t_i + b\Delta W_i \]

\[ u_+ = w_i + a\Delta t_i + b\sqrt{\Delta t_i} \]

\[ u_- = w_i + a\Delta t_i - b\sqrt{\Delta t_i} \]  

(A.15)

Several other higher-order weak solvers can be found in Kloeden & Platen (1999). Weak Taylor methods of any order can be constructed, as well as Runge-Kutta analogues that reduce or eliminate the derivative calculations.

This thesis would not seek to optimize the efficiency of the convergence criterion. So, since the numerical methods that can be constructed with respect to a weak convergence criterion are much easier to implement than those required by the strong convergence criterion, we will use an Euler-Maruyama approximation, so an order 1 weak Taylor scheme.
Bibliography


