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**Defined Contribution Pension Schemes: Optimal Investment Strategies**

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CHAPTER 1: INTRODUCTION

PENSION PROVISION: THE THREE PILLARS THEORY

In many developed countries, the pension provision system is often described by the so-called "three pillars theory". Following this theory, retirement income can be provided at three different levels:

1. state pension provision;
2. occupational pension schemes;
3. personal savings.

In UK, the theory requires a fourth pillar, which is the income that a pensioner can have by entering again into employment after retirement\(^1\). In other countries, like Italy, the possibility of having income from employment after retirement is not considered, and the most common theory is the one of the three pillars.

In Italy at the moment, the second pillar of the pension provision system is not yet well developed, as the first pillar has been very generous in the past, being sufficient to provide for the needs after retirement of most individuals. Due to the ageing population problem, it has been recognised that the public system will not be able in the future to provide for the retirement needs of individuals, the risk of collapsing of public system being realistic\(^2\). Therefore, recent laws\(^3\) have reformed the Italian pension system and regulated the pension funds,

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1 See, for example, Booth, P., Chadburn, R., Cooper, D., Haberman, S., James, D. (1999) Modern Actuarial Theory and Practice, Chapman & Hall/CRC.

2 The fact that PAYG systems will not be able to provide pensions at a sufficient level in most countries and that the second pillar is necessary in the pension system of any country has been pointed out by many authors recently. See for example Daykin, C.D., and Lewis, D., (1999) A Crisis of Larger Life: Reforming Pension Systems, British Actuarial Journal vol 5, pp 55-97. See also Fomero, E., (1999) L'economia dei fondi pensione: potenzialità e limiti della previdenza privata in Italia, Società Editrice Il Mulino.
which will constitute the second pillar of Italian pension provision, the so-called “previdenza integrativa” or “previdenza complementare”. The level of future pension is a serious current problem in Italy, and many individuals, worried about their future income after retirement, buy either deferred annuities or deferred capital from insurance companies in order to limit the likely negative consequences of lack of support from the State provision (insurance products can be actually regarded as the third pillar in Italy, rather than the more general description of personal savings).

It is hoped that pension funds will quickly develop in Italy, solving the retirement income problem or at least limiting the risk of increasing poverty in retirement age.

DEFINED BENEFIT AND DEFINED CONTRIBUTION PENSION SCHEMES

Pension schemes are usually divided into two groups: defined benefit and defined contribution schemes. The main difference between these two groups is the way the financial risk is treated\(^4\).

In defined contribution pension schemes, the contributions are based on a simple formula given in the scheme rules (usually but not necessarily as a percentage of the salary) and the level of the pension achieved at retirement depends on the performance of the investment returns achieved during the active membership\(^5\). That is, the member knows exactly in advance how much he/she will pay in the fund, but does not know the amount of pension he/she will

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\(^5\) An “active member” of a scheme is a member currently accruing entitlement to future benefits, while a “pensioner” is a member or a dependant currently receiving benefits and a “deferred pensioner” is a member who is currently not accruing entitlement to benefits, but has benefits preserved for him/her in the scheme which will be paid at a future date, see Khorasanee (1999).
receive at retirement. Therefore, the member bears the financial risk in defined contribution pension schemes.

In defined benefit pension schemes, the benefits are based on a simple formula given in the rules and do not depend on the investment experience of the scheme. Benefits are often proportional to the length of active membership in the scheme and may be defined as an income (i.e. a pension) and/or a cash lump sum. They are usually linked in some direct or indirect way to the final salary or to the past salaries received during service. Since benefits at retirement are guaranteed by definition and do not depend on the investment conditions during the accumulation of the contributions, it follows that the contribution rate is adjusted regularly by the actuary, depending both on the experienced returns and on the final pension that has to be paid at retirement.

Most defined benefit pension schemes are occupational and usually the member pays the same fixed percentage (e.g. 5% of the salary) while the employer pays the remaining (and obviously aleatory) part of the adjusted contribution rate. In particular, if the investment performance is poorer than expected the employer will pay a higher than expected contribution rate. If the investment performance is higher than expected, then the employer will pay a lower contribution rate or even will not pay for a while (contribution holiday). The member knows exactly in advance how much he/she will pay and also the level of pension received at retirement, whereas the employer does not know how much he/she will pay in order to match the liability. In other words, the employer bears the financial risk in defined benefit pension schemes.

Summarising, the main difference between defined benefit and defined contribution pension schemes is that in defined benefit the financial risk is borne by the employer, whereas in defined contribution it is borne by the member who
does not know in advance the amount of pension, which is strongly linked to the investment performance during membership.

CURRENT TRENDS IN ITALY AND OTHER COUNTRIES

It is worth noting that there are two possible types of pension funds in Italy: the so-called "closed" pension funds, very similar to the well known occupational pension schemes, where the membership is reserved to individuals working in a specified firm or sector or category, and the so-called "open" pension funds, where membership is not restricted to particular classes of individuals. While the latter ones can be either defined benefit or defined contribution plans, the former ones have necessarily to be defined contribution by law. This could be surprising if one thinks of the pension provision experience in UK where most occupational pension schemes are defined benefit. However, the choice adopted by the Italian legislator could be explained if we consider the gradual but continuous trend of replacement of defined benefit schemes by defined contribution schemes which is happening in most countries. The reasons for this general shift are:

1. reducing risk by employers;
2. increasing legislation;
3. surplus and overfunding;
4. high rates of return experienced in the 1980s;
5. trend towards employees' responsibility;
6. taxation structure encouraging defined contribution plans.

It seems important to underline this point as it follows that nowadays the topic of defined contribution pension schemes should be of more concern to

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6 By contribution rate we will mean (in the following) the contribution expressed as a percentage of the salary, as this is the most usual way of defining the contribution, although other definitions are theoretically possible, although very rare, as observed by Khorasanee (1999).
researchers in both academic and working environments, especially in Italy. This is one of the main reasons that led to the development of the present thesis.

**APPROACHING FINANCIAL RISK IN DEFINED CONTRIBUTION PENSION SCHEMES**

While the main advantage of defined contribution pension schemes is the removal of insolvency risk by the sponsor, the corresponding drawback is the total uncertainty about the final pension, due to the financial risk, which is controllable but cannot be eliminated.

In the thesis, the problem of financial risk in defined contribution pension schemes is considered.

In chapter 2, a panorama of the existing literature about financial risk in defined contribution pension schemes is presented. There is also a review of a number of actuarial papers, which apply control theory and dynamic programming theory to actuarial problems, with the main objectives of controlling and possibly reducing insolvency risk and contribution rate variability risk in defined benefit pension schemes. Literature has also been explored which deals with dynamic programming applied to portfolio selection (considering the major papers by Samuelson and Merton).

In chapter 3, a model is constructed which describes the dynamics over time of a defined contributions pension scheme. A problem is formulated in an attempt to control the financial risk in the considered scheme and solved using the mathematical tools of the dynamic stochastic programming.

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To go more into details, a defined contribution pension scheme is introduced in a two-asset (low-risk and high-risk), discrete-time world, the equation of the fund growth of the member is derived – allowing for lognormal returns – and the following problem is defined: what is the optimal investment allocation of the fund every year which minimises the deviations of the actual fund from some reasonably chosen final and interim targets? This problem seems to be consistent with the real investment strategy problem that the investment managers of defined contribution pension schemes have to face every year in managing the members' funds.

The problem is solved using the Bellman’s optimality principle in the stochastic case. A closed formula is derived which gives the optimal percentage of the fund to be invested every year in the high-risk asset in order to approach as much as possible the mentioned targets. The formula is analysed and a feature noticed is that the share to be invested in the high-risk asset decreases as the fund itself increases.

In chapter 4, simulations are carried out in order to understand better the behaviour of the optimal investment strategy during the membership. Four different membership lengths have been considered: 10, 20, 30 and 40 years to retirement. For each length, different rates scenarios have been made by changing the mean and standard deviation of the experienced returns. For each scenario, 1000 simulations have been made by generating each time random returns with the chosen parameters.

The resulting optimal investment strategies tend to show a decreasing trend during the years, meaning a large investment at the beginning of the membership in the high-risk asset, a gradual switching from high-risk to low-risk as time passes and a large investment (and sometimes total) in the low-risk asset in the period before retirement. This intuitive result confirms the suitability
of the so-called "lifestyle method" largely adopted in defined contribution pension schemes in the UK⁸, consisting of investing the whole fund in equities until 5-10 years to retirement, when the fund is gradually converted into index-linked, bonds and cash.

In chapter 5, the results of the simulations are analysed also from the member's point of view: the net replacement ratio achieved has been calculated every time, where by "net replacement ratio" it is meant the ratio between the pension rate and the final salary. The retiree's mortality is taken into account in the conversion of the fund into an annuity by using Italian projected mortality tables (RGS48). In order to allow also for the mortality trend, it is assumed that members with a different time to retirement have been born in different years (i.e. 1948 for members with 10 years to retirement, 1958 with 20 years, 1968 with 30 years and 1978 with 40 years). The probability of failing the final target is also calculated. Remarks about the annuity risk are made after calculating the net replacement ratio with a fixed rate and with an average of the simulated rates. In examining the resulting net replacement ratios, considerations are given to the extent of financial risk borne by the member in defined contribution pension schemes.

In chapter 6, the conclusions of the research are drawn, the limits of the model are identified, the simplifying assumptions discussed and proposals for further research outlined.

⁸ For a clear definition of "lifestyle" method see for example the comment by Sze in Knox's paper (1993).
CHAPTER 2: LITERATURE

INVESTMENT RISK AND ANNUITY RISK

The problem of financial risk in defined contribution pension schemes has been widely treated in the actuarial literature. Knox in *A critique of defined contribution plans using a simulation approach* (1993) identifies the risk borne by the member of a defined contribution plan of not achieving a sufficient income at retirement. In his paper, he analyses the distribution of retirement incomes produced from a defined contribution pension scheme, allowing for stochastic investment returns and inflation rates and changes in a number of parameters (entry and exit ages, sex, investment strategy, career paths, economic assumptions etc.). In particular, he considers three different investment strategies, the first with significant investment in equities and property, the second with significant investment in fixed interest investment, the third predominantly invested in cash and short-term stocks, the parameters of the distribution for the asset returns being chosen considering historic data in Australia. The final lump sum has been converted into a lifetime annuity and the risk faced by the member in terms of the purchased annuity has been pointed out, the level of retirement income being random and depending on the actual rates at the date of retirement. An interesting result of the paper is that a contribution rate of 9% of the salary is not enough for most employees to provide an adequate level of retirement income. The most important result of the research is the significant spread of the level of pension rate received by members who have contributed the same percentage of salary for the same number of years: Knox underlines that "in essence a system that defines a set level of contributions cannot define the level of benefits received". He stresses the fact that "a prescribed level of contributions will not provide sufficient income
for many retirees, even if, on average, it is satisfactory under certain circumstances”.

Khorasanee in *Simulation of investment returns for a money purchase fund* (1995a) also speaks about the “inequity between employees in defined contribution plans”, the level of the pension depending on whether the period of active membership of the pensioner coincided with a period of favourable or unfavourable market rates. In his paper, he considers the investment risk in money purchase pension plans and attempts to reduce it by considering different investment strategies. In modelling the rates of returns, he uses a stochastic model that considers also the correlation between dividend yields in adjacent periods and allows for long term stability in dividend yields, and, like Knox, he uses historic data for choosing the values of the parameters, but uses UK as the source.

He carries out simulations adopting different investment strategies and finds that, by investing 100% in shares, the variability in projected fund of a new entrant 40 years before retirement is high, reducing slowly as the member gets closer to retirement. This confirms the fact shown later in chapter 5 that, the longer is the duration of active membership, the higher is the volatility of the distribution of the pension achieved at retirement. The result is also consistent with Knox’s work, and, once again, highlights the importance of further research in an attempt to limit the negative outcomes that members of a defined contribution plan can experience when they retire. A second investment strategy adopted by Khorasanee consists of switching the portfolio of equities into index-linked at some time before retirement: as one would expect, the fund results tend to be lower and less volatile than by staying in equities. Some conditions are found that give an indication of the best time for switching the portfolio: the projected fund after switching will meet the member’s requirements, the equity market is overvalued relative to a historic standard, there is less than 10 years to retirement. A third investment strategy that has
been explored is a balanced strategy. This gives a reduced mean of the fund at retirement and a reduced variability, meaning a lower retirement fund for majority of members but also a reduced proportion of members who obtain a very poor pension, reducing therefore the inequity between different members. In his investigation, Khorasanee mentions the annuity risk, but does not deal with it, the results of the paper being in terms of final fund at retirement without any connection to the amount of pension received.

In a later paper, *Deterministic modelling of defined contribution pension funds* (1998), Khorasanee treats the investment risk in defined contribution plans by analysing the response of the fund to two different investment shocks that can happen and affect the pension at retirement: an instantaneous fall in the value of the fund assets and a permanent uniform reduction in the rate of earned interest. He proposes an alternative design of defined contribution plan, which incorporates a variable defined benefit scale, and analyses the response of this fund to the same investment shocks, finding that the proposed fund behaves better against a fall in asset values, but that it cannot withstand a permanent change in the force of interest earned beyond a given extreme.

Booth in *The management of investment risk for defined contribution pension schemes* (1995) tries to find optimal investment strategies for defined contribution pension schemes using different approaches to risk. He considers 500 different portfolio invested in the four asset classes of index-linked gilts, conventional gilts, cash and UK equities. He carries out 1000 simulations for each portfolio using the Wilkie stochastic investment model (1986 version) and calculates the cash sum accumulated after 35 years and after 5 years, allowing in the latter case for an initial fund accumulated in 30 years in order to allow consistent comparisons with the long investment horizon case. He calculates mean-variance and mean-semi variance efficient frontiers, considering mean and variance (or semi variance) of the accumulation of payments measured in constant purchasing power terms for both durations. He then determines the
optimal portfolio form of these frontiers in three different ways: maximisation of expected utility, allowing for a logarithmic utility function, with or without the constraint of rejecting portfolios which give the 1st or 3rd percentile below a certain level (shortfall constraints). The research finds that both with and without shortfall constraints the expected utility maximising portfolios lie on the efficient frontiers, the percentage of equities held in the portfolio reducing significantly when the constraints are applied, which is an intuitive result. Furthermore, comparing the corresponding optimal portfolios with different time horizons, it can be seen that, in each of the three cases, the proportion of equities reduces moving from the 35 to the 5 years case, which confirms again the fact that a shorter investment time horizon leads the investor to less risky investment strategies.

The choice of Booth of adopting a risk measure that takes into account only the downside part of the expected benefit like the semi variance and the shortfall constraints stresses the fact, widely debated in the literature, of the inadequacy of the volatility as risk measure. Ludvik in Investment strategy for defined contribution plans (1994) uses also such a risk measurement in presenting the results of his investigation: he considers the median of the distribution of the pension as a fraction of final salary before retirement and measures the shortfall risk borne by the member by looking at the 5th percentile of this distribution. Similarly to the work described above, Ludvik carries out simulations using an extension of the Wilkie model and investigates the results for different investment strategies, drawing conclusions similar to the ones reported above: investing more in bonds and cash than in equities gives lower shortfall values, at the cost of lower median values, the dynamic strategy being an intermediate

\[\text{It is well known that the variance as a risk measure has the fallacy of considering as "very risky" also outcomes well above the mean, which could be misleading in the case of rates of returns or pension levels in defined contribution pension schemes, as very favourable outcomes can be considered as desirable and not risky. This is not true in the case of defined benefit plans, where a high level of surplus can be not desirable for many reasons, although also in this case a very high surplus is certainly preferable to a very high deficit, see also Owadally (1998) later.}\]
situation (by dynamic strategy Ludvik means a strategy where the portfolio is switched from 100% equities to 100% bonds over the last 5 years of membership, i.e. a lifestyle strategy). He also finds out that in a dynamic strategy the switching period should be 3-5 years prior to retirement, a longer period giving inadequate protection against real risk and leading to inadequate pension provision at retirement.

The papers presented and reviewed so far are similar in that they address the investment risk problem of defined contribution pension schemes, analysing it and trying to reduce it when possible by adopting different investment strategies. They do not deal deeply with annuity risk faced by the member at retirement.

Knox mentions it, and observes how annuity rates offered at retirement can have a significant impact on the ultimate level of retirement income. He also proposes a way of overcoming this problem, which consists in offering the member a fixed period during which he/she has to convert the cash sum into annuity. In UK, this option, which is called "income drawdown", is largely adopted and allows the member to take from the accumulated capital regularly cash sums in order to supply his/her immediate needs, leaving the rest in the scheme's fund with the obligation of converting it into annuity within 10 years and not after the age of 75. It is beyond the scope of this work to investigate the best time for the member to convert the cash into annuity within the 10 years, but it is certainly worth stressing the importance of this option in reducing the annuity risk borne by the member. There is scope for work to be done in the future in analysing the potentiality of this option. An initial investigation of "drawdown" is provided by Khorasanee in A Pension Plan Incorporating both Defined Benefit and Defined Contribution Principles (1995b). Khorasanee (1995a and 1998) and Booth (1995) deliberately do not deal with the annuity risk problem, by considering the final fund at retirement only, without converting it into pension.
Ludvik (1994) avoids the problem by treating it in a deterministic way: he calculates the pension using a 15 years bond with a nominal annual coupon of 11%.

Booth & Yakoubov in *Investment policy for defined contribution pension scheme members close to retirement* (1998) address the annuity risk problem in a deeper way, reaching interesting conclusions.

In two different investigations, they consider different investment strategies, taking the observed investment returns directly from historic post war data\(^2\). This is a different approach from that adopted by other researchers: only one sample is considered (the past data) and no model has been constructed.

In the first investigation, they look at the accumulated cash fund and annuity value using 5 different investment strategies, one diversified (70% UK equities, 20% UK gilts and 10% cash), the other 4 different types of lifestyle strategies, with a more or less gradual switch from equities to cash or conventional gilts. In the second investigation, accumulated cash fund and index-linked annuities are considered and 4 investment strategies are tested, one diversified (50% UK equities, 15% US equities, 15% index-linked, 10% conventional gilts and 10% cash), the other three with a total investment in cash, conventional gilts and index-linked.

The results are presented showing the expected value, the standard deviation and the square root of semi variance of the accumulated cash fund, the fixed annuity and the index-linked annuity achieved after 10 years of accumulation. In the first investigation returns are considered from 1945 to 1988, so that the first retirement considered takes place in 1955, while in the second investigation data are considered from 1982 to 1997 inclusive, the first retirement taking

\(^2\) In particular, they use the BZW Equity-Gilt Study (1997) for the investment categories UK equities, conventional gilts, index-linked gilts and cash, whereas US equity returns were taken from S&P Composite Index converted into sterling terms.
place in 1992: this shortened period reflects the fact that index-linked are a very recent financial instrument.

The diversified strategy in each case provides the highest expected value and nearly always the highest standard deviation but also the lowest square root of semi variance, so that it is the best strategy if the semi variance is considered as a risk measure instead of the variance. This result is confirmed also from the fact that the results from this strategy stochastically dominate all the other strategies in both the investigations.

Comparing the other investment strategies, the results show that it is appropriate to invest in conventional gilts when a fixed annuity has to be bought, while index-linked are more suitable when an index-linked annuity has to be bought. This result is reasonable, as the risk that the conversion rate used to buy the annuity at retirement will be low, leading to high price of the annuity, is in some way offset by the fact that the value of the fund will be higher than expected, if the assets held in the portfolio are strongly correlated with the assets used in pricing the annuity. The annuity risk can be perfectly offset if the duration of the portfolio and the present value of the annuity have the same duration. Booth and Yakoubov comment further that cash may not be a suitable asset to match an annuity, as its capital value remains the same regardless of the movements in bond yields. Its presence can instead be justified by a duration matching argument (increasing the quota of cash in the portfolio will reduce the duration of the portfolio and this can be useful when trying to equalise the durations of asset and liabilities). They add that, if equity yields are positively correlated with bond yields (or index-linked yields), it may happen that changes in annuity rates are offset by changes in equity values, so that the volatility of the annuity bought with the proceeds of an equity investment fund reduces significantly with respect to the volatility of the corresponding final cash fund. This does not imply that equities are an appropriate match for fixed annuities, however it points out the fact that, looking at the relation between
variability of the cash fund and underlying investments, it is not sufficient to draw conclusions about the variability of the achieved pension without considering the correlation between investments and annuity rates.

In their conclusions, Booth and Yakoubov stress the fact that cash is not a suitable asset to hold in the portfolio when an annuity has to be bought, and that conventional gilts and index-linked are more appropriate for fixed and index-linked annuities respectively. They underline the fact that it is important that the duration of the investment fund should match the duration of the required annuity. They remind us that the higher expected return from equities should be considered when taking into account equity based strategies and also that the correlation between equity and yields from the appropriate asset used in matching the ultimate pension should be considered when determining the risk of the investment strategy. They conclude that there is no strong evidence from the last 10 years experience in the UK to suggest that the appropriate investment strategy should include investment in gilts. Hence, they conclude that the lifestyle method is not supported by this empirical evidence considering historic returns.

METHODOLOGY: INVESTMENT STRATEGY AS OUTPUT

The papers presented up to now are characterised by the fact that the investment and annuity risk have been analysed by use of simulations, allowing for more or less complicated stochastic models for the asset returns, often looking at historic data in order to estimate the parameters. The results of the simulations give information about the final fund accumulated at retirement and/or about the final pension perceived at retirement, giving a measure of the financial risk borne by the member of a defined contribution pension scheme.
In these studies, the investment strategy adopted is an input of the model, and a comparison between the results given by different investment strategies leads to the choice of the best investment strategy in respect to some defined objective. Cairns in *An introduction to stochastic pension fund management* (1996) underlines the importance of clearly defined objective in the decision making process of a defined contribution pension scheme. He lists of the factors that can be controlled in order to reduce the investment risk in defined benefit and defined contribution pension schemes. In defined benefit schemes the asset allocation strategy is only one of several factors which can be controlled in reducing the investment risk (the others being the method and period of amortisation, the intervaluation period, the delay in implementing a recommended contribution rate, the funding method, the valuation basis), while in defined contribution pension schemes the factors are the investment strategy and contribution rate level.

A different way of approaching the problem of reducing the investment risk in defined contribution pension schemes is to consider the investment strategy as an output of the model instead of an input. A way of doing this can be presented as follows: a model is constructed which gives a certain final outcome after a period of active membership in a scheme, typically $n$ years, the final outcome being for example the accumulated fund, or the final pension or the net replacement ratio achieved at retirement. The asset allocation is left as a parameter of the model, a defined objective is chosen and the optimal investment strategy is found in order to achieve the chosen objective. Cairns in the mentioned paper (1996) lists some possible objectives of a defined contribution pension scheme. By calling $K(n)$ the net replacement ratio achieved at retirement at time $n$ the proposed objectives are:

1. maximise $E[K(n)]$
2. maximise $E[K(n)]$ subject to $\text{Var}[K(n)] = \sigma_k^2$
3. minimise $\text{Var}[K(n)]$
4. minimise $\text{Var}[K(n)]$ subject to $E[K(n)] = \mu_k$
5. minimise $\Pr(K(n) < K_{\text{min}})$
6. maximise $E[u(K(n))]$ where $u(\cdot)$ is some utility function
7. minimise $E[(K(n) - K_{\text{est}})^2|H_s]$ where $H_s$ gives the history of the fund up until time $s$ and $K_{\text{est}}$ is the estimated future pension based on $H_s$
8. maximise $E[u(K(n))|H_s, K_{\text{est}}]$

It is proper to observe that these objectives can be used also in models that consider the investment strategy as input, considering them as decision criteria and not objectives. Booth (1995) uses the objectives 2 and 4 as criteria in finding the optimal portfolio of the efficient frontier.

Thomson in *The use of utility functions for investment channel choice in defined contribution retirement funds* (1998) sets out methods that may be used to estimate the member's utility function and, given a stochastic model of returns on alternative investment channels, to recommend a combination of these channels that will maximise the expected utility of the benefits payable on exit from a defined contribution fund. In other words, he used the sixth objective of the list by Cairns in order to find the optimal investment strategy to be adopted.

**METHODOLOGY: THE MULTIPERIOD APPROACH**

A number of other works have been developed, which recognise the multiperiod and dynamic nature of all the problems related to pension schemes and use control theory and dynamic programming theory in approaching and solving these problems. The papers here presented have been applied to defined benefit pension schemes, and applications to defined contribution schemes are at an early stage of development.

Haberman and Sung in *Dynamic approaches to pension funding* (1994) consider two types of risk of defined benefit pension schemes: the 'contribution
rate' risk, relative to the stability of the contribution rate over time, and the 'solvency risk', relative to the security of the fund. The actuary controlling the pension funding continuously tries to compromise between the conflicting interest of employers and trustees/members, the former being more concerned about the stability of the contribution rate, the latter about the security of the promised benefits. Haberman and Sung construct a discrete-time model introducing yearly targets relative to both fund and contribution rate and link the contribution risk and the solvency risk to the square of the difference between actual contribution rate or fund and the defined targets. In formulae, they introduce the quadratic performance criterion:

\[ J_T = \sum_{t=s}^{T-1} v^t [(C_t - C_{T_t})^2 + v\beta(F_{t+1} - F_{T_{t+1}})^2] \]

where the contribution risk is given by the term \((C_t - C_{T_t})^2\) and the solvency risk is given by the term \((F_{t+1} - F_{T_{t+1}})^2\), \(v\) is the discount rate and the coefficient \(\beta\) reflects the relative importance of the solvency risk in respect to the contribution risk. They apply control theory in order to find optimal contribution rates \(\{C_s, C_{s+1}, \ldots, C_{T-1}\}\) which minimise \(J_T\). In this mathematical model, the targets \(\{C_{T_t}\}\) and \(\{F_{T_t}\}\) are chosen a priori and the values of both the benefits paid from the fund and the real rate earned by the fund every year are best estimates produced by the actuary (and the trustees and/or the investment manager when appropriate), that is they are inputs to the model. The optimal contribution \(C_t^*\) at time \(t\) will depend on both the state of dynamic system at time \(t\), \(F_t\), and on the system inputs at time \(t\) \(\{C_{T_t}, F_{T_{t+1}}, i_{t+1}, B_t\}\). They apply the Bellman optimality principle in order to transform the problem from one of making \(T\) decisions simultaneously to one of making the decisions one at time but sequentially and they find closed formulae that give the value of \(C_t^*\) at any \(t\).

They define and solve the same problem also in the stochastic case, with rates of returns IID and a performance criterion that contains the expectation operator.
In their paper, Haberman and Sung consider the investment strategy as an input of the model\(^3\), the only control variable being the contribution rate. The model can be extended in order to consider also the asset allocation as a control variable. Owadally in *The Dynamics and Control of Pension Funding*, Ph.D. thesis (1998), considers an enlarged model in which both investments allocation and contribution rate are available.

The model is very similar to the one constructed by Haberman and Sung, the problems analysed involving measures of both the fund security and the contribution stability. The approach is the same as before, consisting in minimising the deviations of the actual fund and the actual contribution rate from certain targets, defined a priori. The model is in the discrete time, as before.

The innovative element in the model is the allowance for a portfolio constituted of two assets, one risk-free and one risky. The returns are assumed to be IID without any distributional assumption. The equation that gives the growth of the fund is derived and the objective is to minimise the expected discount cost from the time of joining the scheme \((t = 0)\) to retirement \((t = N)\).

In formulae, if \(c_t\) and \(y_t\) respectively are the contribution rate and the share of the fund invested in the risky asset in year \((t, t+1)\), the problem to solve is to find the set of \(\{c_t, y_t\}\) for any \(t\) \((t = 0, 1, \ldots, N-1)\) that minimise the future expected cost at time \(0\) \(C\):

\[
C = \sum_{t=1}^{N} \beta^t C_t,
\]

\(^3\) The problem is solved by modelling the investment return from the overall portfolio, without considering possible diversification of it.
where: \[ C_t = \sum_{i=1}^{N} [\theta_1 (c_i - F_i)^2 + \theta_2 (c_i - C_t)^2], \]

with the obvious notation (\( C_t \) is the contribution target at time \( t \), \( F_t \) is the fund target and \( \theta_1 \) and \( \theta_2 \) are weights given to the solvency and contribution risks, which reflect their relative importance).

The Bellman optimality principle is applied and the fact that the returns are IID (and therefore a Markov chain) is used to solve the problem and finding closed formulae for the optimal contribution rate \( C_t^* \) and \( y_t^* \) at time \( t \) (for any \( t \)).

Not surprisingly, the formula that gives \( y_t^* \) is such that the proportion invested in the risky asset decreases as the fund increases, the objective being to minimise any deviation from the target, also the deviation with a positive sign. This reflects the fact that, in defined benefit pension schemes, a surplus can be as undesirable as outcome as a deficit, and once the target has been reached (which could be for example the actuarial liability) the optimal investment strategy requires maintaining it. The same feature of the optimal investment strategy will be found in the current work, in particular many similar comments appear both in chapter 3 and 4.

Cairns in *A comparison of optimal and dynamic control strategies for continuous-time pension funds models* (1997) considers the continuous-time case. A loss function is defined which generalises in the continuous time the 'cost' incurred by the fund (square of the deviations of the actual from the target) found in Haberman & Sung and Owadally in the discrete time. The model allows for two risky assets. The reason for choosing two risky assets instead of one risky and one riskless is due to the fact that, in practice, pension funds use cash only for short term liquidity rather than an investment asset, and

\footnote{This explains the use of a quadratic performance criterion.}
least in UK. The prices of the assets are assumed to follow correlated geometric Brownian motion. The objective to minimise is the analogous version, in the stochastic case and continuous time, of the $J_T$ minimised by Haberman and Sung:

$$V(t, x)(C, p) = E\left[ \int_t^{\infty} \exp(-\beta s)L(s, C(s, X(s)), X(s))ds | X(t) = x \right]$$

where $C(t, X(t))$ is the stochastic process which gives the contribution rate and $p(t, X(t))$ is the process which gives the proportion of portfolio invested in the riskier asset.

Dynamic programming can be applied in the continuous time and stochastic case by means of the Hamilton-Jacobi-Bellman equations, which are analogous to the Bellman equations in the discrete time case. Applying the Hamilton-Jacobi-Bellman equations, Cairns finds the optimal couple $(C^*, p^*)$ that minimises $V(t, x)$, and it emerges that both the optimal contribution rate and the asset allocation are decreasing linear function of the fund size. This result is consistent with Owadally's result shown before. However, as Cairns observes, while it is intuitive that the contribution rate in a defined benefit plan is decreasing with the fund, it is not obvious and "goes against conventional actuarial wisdom" that the percentage to invest in the riskier asset has to be reduced as the fund size increases. The rationale behind this kind of optimal investment strategy is that the optimal strategy requires investing in the riskier asset at the beginning in order to approach as quickly as possible the contribution rate and the fund size desired and then stay as close as possible to

5 In particular, the loss function defined by Cairns is of the form $L(t, X(t), C(t))=(C(t)-c_m)^2+k(X(t)-X_0)^2$, with obvious notation.

6 Usually surplus arisen in a defined benefit scheme is invested in equities.
the ideal values by investing in the less risky asset. This explanation will be given again to explain some of the results of chapter 4.

DYNAMIC PROGRAMMING APPLIED TO PORTFOLIO SELECTION PROBLEMS

A review of the literature about the theme studied in the present dissertation would be incomplete without a mention of the relevant works relative to dynamic programming applied in a portfolio selection problem. We refer mainly to the major papers by Samuelson and Merton, which extend and solve by means of dynamic programming theory the classical one-period problem of portfolio selection in the discrete time (allowing for a multiperiod horizon) and continuous time cases respectively.

It is beyond the scope of the present work to discuss in detail the methodology adopted by Samuelson and Merton in their papers and only the main results will be exposed here.

Samuelson in *Lifetime portfolio selection by dynamic stochastic programming* (1969) defined the consumption and portfolio selection problem in a two-asset world (the risky and the riskfree) with a multiperiod time horizon. The classical problem consists in finding both optimal consumption and percentage of portfolio to be invested in the risky asset at time $t$ in order to maximise the expected utility of future individual's consumption. In formulae, it means find the sequence $\{C_t^*, W_t^*\}$ that maximise:

$$E \left[ \sum_{t=0}^{T} (1+p)^{-t} U(C_t) \right]$$

subject to the constraint:
where \( W_t \) is the individual's wealth at time \( t \), \( r \) is the return on riskfree asset and \( Z_t \) is the return on risky asset during year \((t, t+1)\).

He solved the problem applying the dynamic programming theory and found the optimal decision rules for consumption-savings and for portfolio selection in the form:

\[
C_t^* = f \left( W_t, Z_{t-1}, \ldots, Z_0 \right) = f_{T-t} \left( W_t \right),
\]

\[
w_t^* = g \left( W_t, Z_{t-1}, \ldots, Z_0 \right) = g_{T-t} \left( W_t \right),
\]

if the \( Z \)'s are independently distributed.

He then studied some particular cases by specifying the utility function. The main result found is that when the utility function is logarithmic or has the property of isoelastic marginal utility\(^7\), the optimal quota of the portfolio to invest in the risky asset is \( w^* \) which is independent of \( W_t \).

This interesting result implies that as the wealth increases the quota to be invested in the risky asset increases as well, its percentage of the portfolio remaining the same.

An other significant interesting result by Arrow (Essay in the Theory of Risk-Bearing, 1970) states that an individual with utility function that presents absolute risk aversion decreasing with wealth (for example, quadratic utility

\(^7\) Isoelastic marginal utility functions are utility function such that \( U'(C) = C^{\gamma-1} \) with \( \gamma < 1 \). These are the potential utility functions: \( U(C) = 1/\gamma C^\gamma \) with \( \gamma < 1 \).
function) invests a quota of portfolio in the risky asset which decreases as the wealth increases (i.e. he/she considers the risky asset an inferior good). This is indeed the case of the previous works considered in the present chapter, as a quadratic disutility function was used (the loss function, namely) which corresponds to a quadratic utility function\(^8\), and where it was found that the quota to be invested in the risky (or riskier) asset decreases as the fund size increases.

Merton in *Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case* (1969) extends the work by Samuelson, analysing the continuous time case. He examines the problem of optimal portfolio selection and consumption rules for an individual in a continuous time model with rates of return generated by a Wiener Brownian-motion process. Looking at the results relative to the optimal investment strategy, Merton confirms Samuelson's results in the discrete-time case: individuals with a utility function that presents constant relative risk aversion (i.e. logarithmic or isoelastic marginal utility) will invest in the risky asset a proportion of the portfolio independent of their wealth. Merton analyses also the case of individuals whose utility function has constant absolute risk aversion (and strictly positive, i.e. exponential utility function) and finds out that such individuals will invest in the risky asset a constant amount of their wealth, the proportion of the portfolio being therefore decreasing as the wealth increases. This result is confirmed also by Arrow (1970).

In *Optimum Consumption and Portfolio Rules in a Continuous-Time Model* (1971) Merton extends his previous paper considering the consumption-portfolio problem with more general utility functions. He derives explicit solutions for the optimal consumption and portfolio rules for utility functions of the HARA family

---

\(^8\) By operating the transformation \(x \rightarrow (-x), y \rightarrow (-y)\) a convex quadratic disutility function becomes a concave quadratic utility function.
(Hyperbolic Absolute Risk Aversion\textsuperscript{9}). In particular, he finds out that if the utility function belongs to the HARA family\textsuperscript{10} the quota to be invested in the risky asset, $w^*$, depends on the wealth, $W$, in the following way:

$$w^*W = aW + b,$$

where $a$ and $b$ are at most functions of time.

\textsuperscript{9} HARA utility functions are of the form: $U(x) = \frac{1-\gamma}{1} \left( \frac{ax}{\gamma} + b \right)$ and include the exponential, the isoelastic marginal and the quadratic utility functions.

\textsuperscript{10} To be precise, if $U(x)$ is the utility function $\exp(-pt)U(x)$ belonging to the HARA family is necessary and sufficient condition for the theorem to hold.
CHAPTER 3: THE MODEL

THE PENSION SCHEME: GENERAL ASSUMPTIONS

Consider a defined contribution pension scheme where the only decrement is retirement. The assumption of not considering other decrements is made on grounds of simplicity, however there should be a fuller discussion of its likelihood in a more realistic work. Such assumption in fact can be realistic or not depending on the age and the health of the member (which are the more relevant factors, but also other factors should be considered, as financial status, marital status, dependants etc.). It is well known for example that for very young members the withdrawals rate could be very high (also 30%), and in this case assuming only retirement as decrement could lead to distorted results, whereas for a member close to retirement the assumption can be considered realistic, and for a member not very young but still far from retirement also ill-health retirement should be considered. The death rate is usually negligible and not considering it should not lead to any distortions in the results.

In particular the member is assumed to join the scheme at year 0, contribute for N years and then retire. As N is assumed to be fixed and known in advance, also early retirement is not considered.

Taxation and commission expenses are not taken into account on grounds of simplicity. Each member contributes yearly with a percentage of his/her salary to the fund (actually the part of salary to be paid to the fund is shared between employee and employer, normally the employee pays no more than the 5% of the salary, and the employer the remaining part). The percentage of salary to be paid in the fund is fixed by construction, as a defined contribution pension scheme is considered. However, the percentage of salary could vary during the N years, due to different reasons (changes in the regulation of the scheme,
financial problems of the employer, changes in the rates of return from the financial markets, new legislation, changes in State pension provision, establishment of a defined benefit scheme in the same firm with possibility for the member to choose or switch the scheme, general advice by the actuary etc.). The assumption here made on grounds of simplicity is that the contribution rate does not change during the membership.

The fund is assumed to be invested in two different assets with two different risk levels, the high-risk and the low-risk asset. The very strong assumption here made is that the two assets remain the same for the whole membership. It may be possible that the fund is invested in 2 assets in some periods, but it is not likely that the 2 assets remain the same for long periods of time (membership to pension schemes can last 30-40 years). However, it could be argued that if the Capital Asset Pricing Model assumptions hold¹ there is only one risky portfolio in which all the traders invest their funds, the only discretion being the choice of the suitable mix between it and the riskless asset; in this case the two assets of the model would be the market portfolio (the high-risk asset) and the riskless asset (the degenerate case of a low-risk asset). This assumption is made in order to simplify the model.

Large freedom is given to the investment manager (and/or the member, depending on the scheme's rules) with regard to the level of risk of the portfolio: the proportion of the fund to be invested in the two assets is assumed to be reviewed every year, depending on the investment returns of the assets experienced and on the level of the fund compared with a specified target. This seems to be consistent with the real world, where the scheme member is informed every year about the growth of her/his position and he/she can therefore modify the investment strategy at that time, choosing the most appropriate solution in response to actual past experience. As changes in the

portfolio composition occur yearly, the model has been constructed to operate in the discrete time.

Contributions are assumed to be paid yearly in advance. The contribution is assumed to be a fixed percentage of the salary (see also above), as in most defined contribution pensions schemes. The choice of defined contribution schemes follows recent laws in Italy in the context of occupational pension schemes.

As we are dealing with a defined contribution pension scheme, the positions of the members are assumed to be independent of each other, and in the model only the growth of the fund relative to one member is considered, regardless of the size and the composition of the pension scheme. However, it should be recognised that this assumption is another simplification of the real world, as actually the whole amount of contributions from all the members is invested in the financial markets and both the size of the scheme and the average age of the members do affect the investment choices.

THE EQUATION OF THE FUND GROWTH

The level of the fund at time $t$ satisfies the following recurrence relationship:

\[(1a) \quad f_{t+1} = (f_t + cS_t)(1 - y_t)e^{r_t} + y_t e^{r_t} \]

where:

- $f_t$: fund level at time $t$
- $c$: contribution rate
$S_t$: salary at time $t$

$y_t$: proportion of fund invested in the high-risk asset during year $[t, t+1]$

$\mu_t$: real force of interest for the low-risk asset in year $[t, t+1]$, assumed to be constant over the year $[t, t+1]$

$\lambda_t$: real force of interest for the high-risk asset in year $[t, t+1]$, assumed to be constant over the year $[t, t+1]$

The word "real" here means net of price inflation. Thus, the real rates of return for the two assets on year $[t, t+1]$ are $e^{\mu_t} - 1$ and $e^{\lambda_t} - 1$.

In this model no real salary increase is considered and $S_t$ is considered equal to 1 for simplicity for all $t$, so that (1a) becomes:

$$(1b) \quad f_{t+1} = (f_t + c)[(1 - y_t)e^{\mu_t} + y_t e^{\lambda_t}]$$

It is assumed that the annual investment returns from the two assets are lognormally distributed, so that $\mu_t$ and $\lambda_t$ are normally distributed. In addition $\{\mu_t\}$ and $\{\lambda_t\}$ are assumed to be sequences of independent and identically distributed random variables, so that:

$$\mu_t \approx N(\mu, \sigma_1^2) \quad \text{and} \quad \lambda_t \approx N(\lambda, \sigma_2^2),$$

where:

$$\mu \leq \lambda \quad \text{and} \quad \sigma_1^2 \leq \sigma_2^2$$
The assumption of lognormally distributed rates of return avoids the undesired (and unrealistic) phenomenon of returns lower than \(-1\), which would lead to negative values of the fund.

**FINAL TARGET**

It is assumed that the member joins the scheme at time \(t=0\) and contributes with the same fixed contribution rate \(c\) until retirement at time \(t=N\), which is also fixed.

Since the fund and the contributions are invested partly in the low-risk asset (mean \(\mu\)) and partly in the high-risk (mean \(\lambda\)), a reasonable fund target, a priori, would be:

\[
F_N = cs_{N^*} + f_0 e^{N^*} 
\]

where

\[
r^* = \frac{1}{2} (\mu + \lambda + 0.5(\sigma_1^2 + \sigma_2^2))
\]

The term \(f_0\) which appears in (2) makes possible the allowance in the model for transferred-in members. A new member can therefore have \(f_0>0\) (\(f_0\) is actually equal to the transfer value) if he/she decides to transfer his/her position from the previous arrangement.

In the simulations which follow in chapters 4 and 5 it is assumed that the whole final fund is converted at time \(N\) into an immediate annuity: in this case the member will be more concerned about the net replacement ratio, which is the ratio between retirement income and final salary, than about the final fund. For this reason, the target \(F_N\) is converted into the actuarial expected annuity, taking
into account the retiree’s expected mortality and using the expected value of the random return of the low-risk asset as a conversion rate:

\[ B_N = F_N / \sum_{n=1}^{\Omega} nE_x = F_N / a_x \]

where \( x \) is the retirement age of the member, \( nE_x \) is calculated using appropriate mortality tables and \( v = e^{-\mu + 0.5\sigma^2} \) as discount factor, \( \Omega \) is the extreme age contemplated by the tables and \( B_N \) is the pension rate, which coincides with the net replacement ratio, since \( S_N = 1 \) (we note that we are assuming that \( S_i = 1 \) for any \( t \)).

The assumption of converting the whole final fund into an immediate annuity is certainly strong and should be discussed in the context of the pension scheme rules and the general legislation. In the context of Italian occupational pension schemes, for example, the member is allowed to receive part of the final fund as a lump sum (with a “cap” of 50% of the fund) and there is no evidence to support the hypothesis that the member would decide to receive the whole benefit converted to an annuity income. However, this assumption is made in order to study the effects of the investment strategy on the net replacement ratio and furthermore the model here does not imply necessarily the conversion of the whole fund in rent and other assumptions can be made in order to analyse the suitability of the investment decisions. As in other researches, an optimal allocation of final fund in the choice between annuity and equity investment during retirement can be derived (see Kapur & Orszag, 1999) and, in the context of Italian pension funds, comparisons can be made between different uses of the final fund with regards to the trade-off between the rent and the lump sum (see, for example, Vigna, 1998), but such an extended analysis is beyond the scope of this work.
**INTERIM TARGETS**

As far as the yearly targets are concerned, only fund targets are calculated, as the fund is converted into an annuity only at retirement.

Targets have been chosen which increase linearly from:

\[ F_1 = (f_0+c) e^{r^*} \]

with \( r^* \) as above

to \( F_N \) as given by (2) above.

Linear interpolation has been chosen on grounds of simplicity.
An alternative approach could have used \( \log F_t \) as the basis of linear interpolation.

Targets could have been chosen less frequently than yearly, e.g. every 3 years (similarly to actuarial investigations in defined benefit pension schemes), every \( n > 3 \) years, or, as an extreme, every \( N \) years, i.e. considering only the final target \( F_N \).

**FORMULATION OF THE PROBLEM**

Once fixed the set of yearly targets \( \{F_t\}_{t=1,2...N} \) it is now possible to define the elements of the dynamic programming problem.

\(^2\)In the deterministic case \( v=e^{r_t} \); in the stochastic case \( v=E[e^{-\mu_t}] \), and in this case \( v=e^{-\mu + 0.5\sigma^2} \), since \( \mu_t \) is \( N(\mu, \sigma^2) \).
Following the works of Haberman & Sung (1994) and Owadally (1998) (see chapter 2) and adapting it to a defined contribution scheme, it is now possible to define the "cost" incurred by the scheme at time \( t \) for any \( t \) and at time \( N \) and choose appropriate weights (\( \theta_0 \) and \( \theta_1 \)) which give relevance to these costs:

\[
(4) \quad C(t) = \theta_1 (f_t - F_t)^2 \quad \text{for } t=1,2\ldots N-1
\]

where it is reminded that \( f_t \) is the actual fund and \( F_t \) the target defined above.

At retirement the final cost is:

\[
(5) \quad C(N) = \theta_0 (f_N - F_N)^2
\]

The difference in the coefficients \( \theta_1 \) and \( \theta_0 \) is due to the possibly lower importance that may be attributed to the achievement of interim targets compared with the achievement of the final one. It is therefore understood that:

\[
\theta_1 \leq \theta_0.
\]

However, in a more general presentation (which is beyond the scope of the current work), it is possible that \( \theta_1 > \theta_0 \), if decrements other than retirement were considered, like ill-health retirement or death, and the interim targets were linked to the benefits triggered by the cause of decrement.

The total future cost at time \( t \) is obtained by discounting the future costs until \( N \):
where $\gamma$ is the inter-temporal discount factor, which can be seen as a "psychological" discount rate, as a risk discount rate (Cairns 1997) or may also be equal to the valuation discount rate (Haberman and Sung 1994, Sung 1997).

Define $X_t$ to be the $\sigma$-field generated by all information available at time $t$:

$$X_t = \{f_0, f_1, \ldots, f_t, y_0, y_1, \ldots, y_{t-1}\}$$

The value function at time $t$ can be defined as:

$$J(X_t) = \min_{\pi_t} E[G_t | X_t]$$

where $\pi_t$ represents the set of the future feasible investment strategies, i.e.:

$$\pi_t = \{y_s = s, t, t+1, \ldots, N-1, : 0 \leq y_s \leq 1\} = \{y_t, y_{t+1}, \ldots, y_{N-1}, : 0 \leq y_s \leq 1\}$$

Thus, the objective is to choose the future investment strategies that minimise the discounted future cost.

Scope for further research is provided by the choice of the objective. Cairns (1996) proposes a set of different objectives for defined contribution pension
schemes\(^3\), and classifies them into two groups: a) those in which the scheme member is given information in advance of retirement of the likely amount of the future pension and the scheme member then expects such a target to be attained (corresponding to the formulation here); and b) those in which the scheme member is informed of the actual pension only at retirement. Another possible criterion proposed for the case considered here is the maximisation of the expected utility of the net replacement ratio at retirement, given the past history of the fund, for some utility function to be defined.

**BELLMAN’S OPTIMALITY PRINCIPLE**

Bellman’s optimality principle gives (Bellman & Kalaba 1965, pag. 43; Cugno & Montrucchio, 1998, pag. 182):

\[
J(X_t) = \min_{x_t} \mathbb{E}
\left[ \sum_{s=t}^{N} \gamma^{s-t} C(t) \mid X_t \right] = \min_{x_t} [C(t) + \gamma \mathbb{E}[J(X_{t+1}) \mid X_t]]
\]

It should be noted however that, since \(\{\mu_t\}\) and \(\{\lambda_t\}\) are assumed to be independent, \(f_t\) has the Markov property and:

\[
\Pr[f_{t+1} \mid X_t] = \Pr[f_{t+1} \mid f_t]
\]

and also:

\[
\Pr[f_{t+1}, f_{t+2}, \ldots, f_N \mid X_t] = \Pr[f_{t+1}, f_{t+2}, \ldots, f_N \mid f_t]
\]

so that:

\[
\Pr[G_t \mid X_t] = \Pr[G_t \mid f_t]
\]

\(^3\) For the complete list of objectives proposed by Cairns see chapter 2.
and:

\[ J(X_t) = \min_{X_t} E[G_t|X_t] = \min_{X_t} E[G_t|f_t] = J(f_t, t). \]

Therefore, Bellman's equation becomes:

\[ J(f_t, t) = \min_{Y_t} [C(t) + \gamma E[J(f_{t+1}, t+1)|f_t]] \]

Equation (10) will result crucial in the solution of the dynamic programming problem below outlined.

Considering the fund growth equation (1b) and the fact that \{\mu_t\} and \{\lambda_t\} are i.i.d. and normally distributed, it is easy to verify that\(^4\):

\[ \begin{align*}
E[f_{t+1}|f_t] &= (f_t+c)[(1-y_t) e^{\mu+0.5\sigma^2} + y_t e^{\lambda+0.5\sigma^2}] \\
E[f_{t+1}^2|f_t] &= (f_t+c)^2 \{(1-y_t)^2 e^{2\mu+2\sigma^2} + y_t^2 e^{2\lambda+2\sigma^2} + 2y_t(1-y_t) e^{\mu+\lambda+0.5(\sigma^2+\sigma^2)}\} \\
\text{Var}[f_{t+1}|f_t] &= (f_t+c)^2 \{(1-y_t)^2 e^{2\mu+\sigma^2} (e^\sigma - 1) + y_t^2 e^{2\lambda+\sigma^2} (e^\sigma - 1)\}
\end{align*} \]

Without loss of generality, it is assumed that \(\theta_0\) is a multiple of \(\theta_1\) and that \(\theta_1\) is equal to 1, so that:

\[ \theta_1 = 1 \quad \theta_0 = \theta, \quad \text{where it is assumed that } \theta > 1. \]

\[^4\] It follows from the well known fact that if a random variable \(X\) is \(N(\mu, \sigma^2)\), then \(E(e^{X}) = \exp(\mu t + 0.5\sigma^2 t^2)\) and from the fact that \(\text{Var}(X|Y) = E(X^2|Y) - E(X|Y)^2\), as \(\text{Var}(X|Y) = E[(X-E(X|Y))^2|Y]\). The rest is algebra.
The dynamic programming problem then becomes:

\[(DPP) \quad J(f_i, t) = \min_y [(f_i - F_t)^2 + \gamma E[J(f_{i+1}, t+1) | f_i]]\]

with boundary condition:

\[(BC) \quad J(f_N, N) = C(N) = \theta (f_N - F_N)^2\]

and \(F_N\) given by (2) above.

**SOLUTION OF THE DYNAMIC PROGRAMMING PROBLEM**

There are now all the elements to show that the dynamic programming problem DPP above outlined has the following solution:

\[(TH) \quad J(f_i, t) = P_t f_i^2 - 2Q_t f_i + R_t\]

for some sets of coefficients \(\{P_t\}, \{Q_t\}\) and \(\{R_t\}\).

**Proof.**

The proof is done by mathematical induction. (TH) is satisfied for \(t = N:\)

\[(14) \quad P_N = \theta \quad \text{and} \quad Q_N = \theta F_N\]
where the boundary condition BC has been used.

Assuming that TH is satisfied for t+1, that is:

\( (\text{IH}) \quad J(f_{t+1}, t+1) = P_{t+1} f_{t+1}^2 - 2Q_{t+1} f_{t+1} + R_{t+1}, \quad \text{inductive hypothesis} \)

it has now to be shown that it is satisfied for t.

The following results:

\[
E[ J(f_{t+1}, t+1) \mid f_t ] =
E[P_{t+1} f_{t+1}^2 - 2Q_{t+1} f_{t+1} + R_{t+1} \mid f_t ]
= P_{t+1} E[ f_{t+1}^2 \mid f_t ] - 2Q_{t+1} E[ f_{t+1} \mid f_t ] + R_{t+1}
= P_{t+1} (f_t + c)^2 [(1-y_t)^2 e^{2\mu + 2\sigma_t^2} + y_t^2 e^{2\lambda + 2\alpha_t^2} + 2y_t (1-y_t) e^{\lambda + \mu + 0.5(\sigma_t^2 + \alpha_t^2)}] - 2Q_{t+1} (f_t + c) [(1-y_t) e^{\mu + 0.5\sigma_t^2} + y_t e^{\lambda + 0.5\alpha_t^2}] + R_{t+1}
\]

where in the first equality (IH) has been used and in the third equality (11) and (12) have been used.

Reordering the long expression it comes out that:

\( (15) \quad E[ J(f_{t+1}, t+1) \mid f_t ] = L_t y_t^2 + M_t y_t + N_t = \Psi(y_t) \)

where:

\( (16) \quad L_t = P_{t+1} (f_t + c)^2 [e^{2\mu + 2\sigma_t^2} + e^{2\lambda + 2\alpha_t^2} - 2e^{\mu + \lambda + 0.5(\sigma_t^2 + \alpha_t^2)}] \)

\( (17) \quad M_t = 2P_{t+1} (f_t + c)^2 [e^{\mu + \lambda + 0.5(\sigma_t^2 + \alpha_t^2)} - e^{2\mu + 2\alpha_t^2}] - 2Q_{t+1} (f_t + c) [e^{\lambda + 0.5\sigma_t^2} - e^{\mu + 0.5\alpha_t^2}] \)
lf it can be shown that $L_1 > 0$, a unique minimum $Z^*_t$ and a corresponding optimal investment decision $y^*_t$ do exist, such that:

$$\Psi(y^*_t) = Z^*_t$$

and:

19. $$y^*_t = -\frac{M_t}{2L_t}$$

20. $$Z^*_t = N_t - \frac{M_t^2}{4L_t}$$

We assume for the moment that $L_1 > 0$. Later it will be shown that this inequality actually holds.

By substituting (16), (17) and (18) in (20), we obtain:

21. $$Z^*_t = P'_t f'_t^2 + Q'_t f_t + R'_t$$

where:

22. $$P'_t = H P_{t+1} \quad Q'_t = 2 H c P_{t+1} + 2 K Q_{t+1}$$

with:

23. $$H = \frac{1}{D} \left[ e^{2u + 2\alpha_1 + \sigma_1^2 + \sigma_2^2} \left( e^{\alpha_1^2 + \sigma_2^2} - 1 \right) \right]$$

(18) $$N_t = P_{t+1} (f_t + c)^2 e^{2u + 2\alpha_1^2} - 2Q_{t+1} (f_t + c) e^{u + 0.5\sigma_1^2} + R_{t+1}$$
Equation (10) becomes now:

\[
J(t, t) = \min_y \left[ C(t) + \gamma E[J(f_{t+1}, t+1)|f_t] \right] = \min_y \left[ (f_t - F_t)^2 + \gamma E[J(f_{t+1}, t+1)|f_t] \right] = (f_t - F_t)^2 + \gamma \min_y \left[ E[J(f_{t+1}, t+1)|f_t] \right] = (f_t - F_t)^2 + \gamma M^* = P_t f_t^2 - 2Qf_t + R_t
\]

where:

\[
P_t = 1 + \gamma P'_t \\
Q_t = F_t - 0.5\gamma Q'_t \\
R_t = F_t^2 + \gamma R'_t
\]

with \(P'_t\) and \(Q'_t\) given by (22) above, which is exactly (TH).

It remains to show that \(L_t > 0\) or, equivalently, \(P_{t+1} [...] > 0\), as \((f_t + c)^2 > 0\).

The following holds:

\[
\left[ e^{2\mu + 2\sigma_t^2} e^{2\lambda + 2\sigma_t^2} - 2e^{\lambda + 2\sigma_t^2} e^{\mu + 0.5(\sigma_t^2 + \sigma_j^2)} \right] = (e^{\mu + \sigma_t^2} - e^{\lambda + \sigma_j^2})^2 + 2e^{\mu + \lambda + \sigma_t^2 + \sigma_j^2} - 2e^{\mu + \lambda + 0.5(\sigma_t^2 + \sigma_j^2)} =
\]

\[
= (e^{\mu + \sigma_t^2} - e^{\lambda + \sigma_j^2})^2 + 2e^{\mu + \lambda + 0.5(\sigma_t^2 + \sigma_j^2)} (e^{0.5(\sigma_t^2 + \sigma_j^2)} - 1) > 0
\]

It remains to analyse the sign of \(P_{t+1}\). Using (22) and (27), we obtain:

\[
P_{t+1} = 1 + \gamma H P_{t+2}
\]
Observing that $H>0$, $\gamma>0$ and $P_N=0>0$ it follows that
\[ P_{t+1} > 0, \]
so that
\[ L_t > 0 \]
for any $t$.

**OPTIMAL INVESTMENT STRATEGY**

It is now possible to determine the optimal investment strategy by substituting (16) and (17) in (19). This leads to:

\[ y^* = \frac{Q_{t+1}V}{P_{t+1}(f_t+c)D} - \frac{W}{D} \]

where the sequences $\{P_t\}$ and $\{Q_t\}$ are given recursively by:

\[ P_t = 1 + \gamma HP_{t+1} \quad Q_t = F_t - \gamma cHP_{t+1} - \gamma KQ_{t+1} \]

with $P_N$ and $Q_N$ given by (14), $H$, $K$ and $D$ given by (23), (24) and (25) above and $V$ and $W$ given by:

\[ V = e^{1+0.5\sigma^2} - e^{\mu+0.5\sigma^2} \]
\[ W = e^{\mu+\lambda+0.5(\sigma^2+\sigma_f^2)} - e^{2\mu+2\sigma^2} \]
SOME COMMENTS ABOUT $Y^*_t$

It is possible to rewrite (28) schematically as:

\[(28b) \quad y^*_t(t) = \frac{a}{b(t + c)} - \frac{e}{d}\]

where $a$, $b$, $c$, $d$, and $e$ do not depend on $f_t$.

Looking at the sign of the coefficients $a$ and $b$ it can be seen that $b > 0$ by construction, as $D > 0$ and $P_t > 0$ for every $t$ (it was shown in the proof that $L_t > 0$, and we note in fact that $L_t = P_{t+1} D (f_t+c)^2$). Instead, the sign of $a$ is not so certain, as it depends on the sign of $Q_t$ (noting that $V > 0$). However, in all the simulations carried out in this work the sign of the resulting $Q_t$ is positive for every $t$.

Thus, differentiating $y^*_t$ with respect to $f_t$, assuming that $Q_t > 0$ we obtain:

\[(32) \quad \frac{\partial y^*_t}{\partial f_t} < 0\]

Therefore, looking at the optimal investment strategy it follows that for a fixed point of time and everything else being equal, the percentage of the fund to be invested in the high-risk asset decreases as the fund itself increases. This could appear surprising if one considers low values of $t$, looking at the beginning of the membership, when the general criterion should be to increase as much as possible the fund. However, this result is reasonable and intuitive when one considers the choice of the "cost" functions $C(t)$ that are to be minimised:
\[(f_t - F_t)^2 \quad \text{for } t = 1, 2, \ldots, N-1\]

and

\[\theta(f_N - F_N)^2 \quad \text{for } t = N.\]

This means that once the interim target has been reached the investment strategy requires that one maintains it and this implies that the share to be invested in the high-risk asset decreases as \(f_t\) increases. This feature has also been reported by Cairns (1997) and Owadally (1998) in the context of a defined benefit pension scheme (see chapter 2).

In order to generalise this result and hence find a general feature for the optimal investment strategy, it should be investigated under what assumptions the coefficient \(\theta\) is positive.

As special case, we consider the last investment decision, \(y^*_{N-1}\):

\[(33)\]

\[y^*_{N-1} = \frac{V}{D} \left[ \frac{F_N}{(f_{N-1} + c)} - \frac{W}{V} \right]\]

where the boundary condition (14) has been implemented. For the particular case of the risk-free asset \((\sigma_1 = 0)\), one notes that:

\[y^*_{N-1} = 0 \quad \text{if} \quad F_N = (f_{N-1} + c) e^\mu.\]

This is an intuitive result, since it would be sensible to stop completely investing in the high-risk asset when investing in the risk-free one would lead exactly to
the final target. Again, a corresponding result has been reported by Owadally (1998).
CHAPTER 4:

OPTIMAL INVESTMENT STRATEGIES

FRAMEWORK: PARAMETERS VALUES

The derived formulae have been implemented by carrying out many simulations, in order to understand what could be the resulting optimal investment strategy in a real context. Another important issue is the effect of implementing the optimal investment decisions on the net replacement ratio achieved by the member at retirement. Both of these points will be dealt with after discussing the simulations.

The model contains many parameters. In this investigation, a choice among the parameters has been made in order to separate the more significant from the lesser ones: these latter have been fixed in the simulation whereas the more relevant ones have been varied in order to test the sensitivity of the model and to study the behaviour of results in different situations.

FIXED PARAMETERS

$f_0$: initial value paid in the fund.

On grounds of simplicity it has been assumed that the member joins the scheme without a transfer value, so that $f_0 = 0$ in the formulae. The assumption is obviously true only in the case $N = 40$, as it is very unlikely that the member has past service elsewhere. In the other cases, and especially with a high value of $N$, the assumption could not be considered realistic, as the member could have joined other pension schemes in the past. However, the problem of transferring the past position may have been solved in some way other than by
transfer value, e.g. by leaving the accumulated contributions in the previous scheme or by receiving a lump sum. Furthermore, it has to be noticed that in some countries like Italy occupational pension schemes have not been completely developed yet\(^1\) and when they will start to be operative a situation like the one here assumed will be the normal one at the beginning, the transfer value being zero for any joining member, regardless of the age.

**c: contribution rate.**

The value of the contribution rate has been chosen equal to 12\(.\) There are two main reasons that have led to this value. Firstly, this value has been chosen in other investigations like Knox, 1993 and Booth & Yakoubov, 1998. Secondly, the recent laws in Italy which regulate the development of occupational pension schemes fix at 11.43\%\) the maximum contribution rate that allows taxation relief for both the employer and the employee\(^2\). Thus, the chosen value of 12\%\) could be seen as a maximum value for the contribution rate in the Italian case. Furthermore, as it will shown later, this value leads to results that are too high in the case of very long future membership (40 years to retirement) and rates of the return that are not too low, as it leads potentially to an excessively high net replacement ratio achieved by the member at retirement, i.e. a net replacement ratio of over 90\%. It will be shown that in certain cases the likely net replacement ratio achieved is over 100\%, which is an unacceptable result and means that the contribution rate of 12\%\) assumed was too high.

**\(\theta : \text{weight given to the final target.}\)**

The value of the parameter \(\theta\) has been chosen equal to 2, while we note that the weight for the interim targets is 1. In order to verify the robustness of the results, other simulations have been carried out with other values of \(\theta\) (\(\theta = 1, 3\))

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\(^1\) For the reasons why pension funds in Italy have many difficulties in starting see for example Fornero, 1998.

\(^2\) In particular, there is taxation relief both for the employee and the employer up to 2\% of salary (which implies that employer and employee pay in the fund up to 4\% of the salary), and, in addition, a quote of up to 7.43\% of the salary should be paid in the fund from the so-called "TFR", see Italian laws (note 3 of chapter 1).
and the results (which will be presented in the results, see Figures 8 and 9) lead to very similar conclusions.

\( \gamma \): inter-temporal discount factor.
The value of the inter-temporal discount factor has been chosen equal to 0.95. For the same reasons as before other values have been tested (\( \gamma = 0.8, 1 \)) and the results appear to be very similar (see Figures 10 and 11).

FUTURE SERVICE AND RATES OF RETURN: CONSTRUCTION OF DIFFERENT SCENARIOS

Once some of the parameters have been fixed, like \( f_0, c, \theta, \) and \( \gamma \), the sensitivity of the results has been investigated by carrying out the simulations choosing different values for the other parameters of the model. These other parameters are the length of future service and the parameters of the distribution of rates of return. By combining different values of \( N, \mu, \lambda, \sigma_1 \) and \( \sigma_2 \) different scenarios have been constructed, as is described below.

\( N \): length of future membership.
The length of future membership has been chosen equal to 10, 20, 30 and 40 years. By choosing different values for \( N \) it is possible to compare different generations of employees, the time to retirement being obviously linked to the member's age. As will be explained in chapter 5, not only a comparison between different generations has been done, but also an investigation into the mortality trend has been carried out. In fact, it has been assumed that the member joins the scheme in the year 2000 and that people with 10 years to retirement were born in 1948, with 20 years in 1958, with 30 years in 1968 and with 40 years in 1978 (allowing for a Normal Retirement Age of 62).

\( \mu, \lambda, \sigma_1, \sigma_2 \): mean and standard deviation of the rates of return.
We recall from the specification of the model (see chapter 3) that the low-risk asset return is assumed to be normal with a mean of \( \mu \) and a standard deviation of \( \sigma_1 \), while the high-risk asset return is normal with mean \( \lambda \) and standard deviation \( \sigma_2 \).

A “base” scenario has been chosen with \( \mu = 4\% \), \( \lambda = 6\% \), \( \sigma_1 = 5\% \) and \( \sigma_2 = 15\% \). The choice of the values for the mean has been made according to other works and in particular the recent authoritative work by Thornton & Wilson (1992) which fixes between 4% and 6% the “best estimate” of real rates of return recommended to actuaries in valuing pension schemes in the UK. The choice of the values for the standard deviation has been made taking into account data used in other researches (see for example Knox, 1993, Khorasanee, 1995, Cairns, 1996, and Luenberger, 1998), who consider the historic volatility of equities in UK and Australia. It has been tempted to average the different values corresponding to UK and Australia, the volatility being historically higher in UK than in Australia (and in most other countries).

The base scenario can be called either “normal volatility” or “normal mean”, depending from which point of view it is considered. The volatility and the mean have been changed and the following scenarios have been created (where the normal case is reported on grounds of completeness of exposition):

1. **Normal volatility (normal mean)**, with \( \mu = 4\% \), \( \lambda = 6\% \), \( \sigma_1 = 5\% \) and \( \sigma_2 = 15\% \)
2. **Low volatility**, with \( \mu = 4\% \), \( \lambda = 6\% \), \( \sigma_1 = 2.5\% \) and \( \sigma_2 = 10\% \)
3. **High volatility**, with \( \mu = 4\% \), \( \lambda = 6\% \), \( \sigma_1 = 10\% \) and \( \sigma_2 = 20\% \)
4. **Mixed volatility**, with \( \mu = 4\% \), \( \lambda = 6\% \), \( \sigma_1 = 2.5\% \) and \( \sigma_2 = 20\% \)
5. **Low mean**, with \( \mu = 2\% \), \( \lambda = 4\% \), \( \sigma_1 = 5\% \) and \( \sigma_2 = 15\% \)
6. **Mixed mean**, with \( \mu = 2\% \), \( \lambda = 8\% \), \( \sigma_1 = 5\% \) and \( \sigma_2 = 15\% \)
7. **Riskless & risky**, with \( \mu = 2\% \), \( \lambda = 6\% \), \( \sigma_1 = 0\% \) and \( \sigma_2 = 15\% \)
It should be noted that the volatility and the mean have been chosen in such a way that it is possible to isolate the effect of changing the risk level or the expected return of the assets. In fact, it is possible, by comparing scenarios 2 and 4, to study the effect of changing the volatility of the high-risk asset, everything else being the same. Similarly, it is possible to study the effect of changing the volatility of the low-risk asset by comparing 3 and 4, and the effect of changing the mean of the high-risk asset by comparing 5 and 6.

The 7 scenarios above described have been applied to the 4 different membership lengths (N=10, 20, 30, 40) and for each of the 28 cases considered 1000 simulations have been carried out, by generating for N years normally distributed random rates of returns with the corresponding parameters $\mu$, $\lambda$, $\sigma_1$, and $\sigma_2$.

**ANALYSIS OF RESULTS: TWO DIFFERENT POINTS OF VIEW**

The results of the simulations can give different types of information, which can be of interest to different kinds of stakeholders.

On the one hand, it is possible to study the trend of the $y^*_1$ over the N years of membership, which tells what should be the optimal investment strategy to adopt over time and specifically every year. This issue is of concern to the investment manager, and also to the member in case he/she is given the possibility to chose his/her risk profile.

On the other hand the member of the scheme will be more concerned about the final fund obtained when retires, and about the corresponding pension rate received by converting this fund in annuity, finally about the net replacement ratio achieved (where by "net replacement ratio" it is meant the ratio between the initial pension rate and the final salary). The trustees will be also interested
in the net replacement ratio promised (but not guaranteed) and in that achieved and in their comparison, the reasons being the credibility of the scheme to the members, the competition with other pension schemes, the choice of the investment manager and also the evaluation of the accuracy of the model. The net replacement ratio is therefore another important piece of information to extract from the simulations.

INVESTMENT STRATEGY: THE TREND OF THE Y^t

For each of the 7 scenarios above described and for each membership length N we present two graphs that illustrate the behaviour of y^t over time. The first one reports the minimum, the maximum and different percentiles (5th, 25th, 50th, 75th, and 95th) of y^t for any t, the second one reports the mean and the standard deviation of y^t over the 1000 simulations. The graphs are divided into scenarios: in Fig. 1 there is the normal volatility (or normal mean) case, Fig. 2 reports the low volatility case, Fig. 3 the high volatility case, Fig. 4 the mixed volatility case, Fig. 5 the low mean case, Fig. 6 the mixed mean case, while Fig. 7 the riskless & risky case.

It is now timely to note that the curves that appear in the graphs are not real investment strategies, as each single point of the curve is a percentile of the distribution of the y^t over the 1000 simulations and it may well be that percentiles with different t belong to different simulated strategies. Therefore, it is imprecise to consider the moments and percentiles illustrated as they were of the distributions of the strategies, where by strategy we mean the resulting set of {y^t}_{t=0,...,N-1} from a single simulation. However, if the y^t decrease over time, so does on average the single strategy in its whole, and if the value of the y^t presents a high volatility, so does the value of the single strategy and in the following we will speak about strategies instead of single yearly investment
decision, as the imprecision is largely compensated by the intuition and understanding of the results.

The main result found is that \( y^*_t \) is always decreasing over time, for every \( N \) and in every scenario. This means that the resulting optimal investment strategy implies investing the portfolio mainly in the high-risk asset at the beginning, switching more or less gradually from the high-risk to the low-risk one after some years and investing mainly in the low-risk in the years before retirement. This result, as we will stress later, confirms the validity of the “lifestyle method”, largely adopted by the defined contribution pension schemes in the UK. This approach consists of investing the whole fund in equities at the beginning of membership and switching the portfolio from equities into bonds and cash in the last 5-10 years before retirement.

Two criteria have been then used in analysing the trend of the strategies and their dependence on the parameters, which are \( N, \mu, \lambda, \sigma_1 \) and \( \sigma_2 \):

1) aggressiveness of the strategy: a strategy is aggressive if the share to be invested in the high-risk asset is close to 1 at the beginning (close to retirement that share will be low, due to the decreasing trend of \( y^*_t \)), meaning a high level of the \( y^*_t \) in the first years of membership;

2) diversification of the strategy between the 2 assets: a strategy is well diversified if the curve decreases gradually towards zero, as in this case the \( y^*_t \) lies mainly between 0 and 1, implying a portfolio invested in both of the assets; a less diversified strategy implies that the curve decreases steeply from 1 to 0, meaning that the portfolio is invested entirely in the high-risk asset at the beginning, diversified between the 2 assets while decreasing and entirely invested in the low-risk asset at the end: the steeper the slope the lower the number of years in which the portfolio is diversified.

Results: aggressiveness.

The results are summarised and commented as follows:
• the strategies are more aggressive as $N$ increases.
Looking at the 40 years case it is possible to notice that in all scenarios 5% of the strategies have $y^*_t = 1$ for any $t$. Looking at the 10 years case, in all scenarios but one (Fig. 6), in 75% of the strategies $y^*_t$ is well below 80% from the 3rd year and from the 1st year onwards it rapidly reduces towards 0.

This result is reasonable, as the closer to retirement the member is, the shorter the time to recover from adverse outcomes in the experienced rates of return, therefore the less risky the strategy to be adopted, and vice versa.

• the strategies are more aggressive as the volatility of both assets decreases.
Looking at the graphs reporting the mean of the $\{y^*_t\}$ it is possible to see that:
- with $N = 40$ the mean of $y^*_t$ is 1 for the first 3 years in the high volatility case (look at Fig. 3), it is 1 for the first 7 years in the normal and mixed volatility case (Figs. 1 and 4), it is 1 for the first 11 years in the low volatility case (Fig. 2);
- with $N = 30$, similarly, the mean is 1 for the first 2 years with high volatility, for the first 4 years with normal and mixed volatility, for the first 6 years with low volatility;
- with $N = 20$ the mean is 1 for the first year with high volatility, for the first 2 years with normal and mixed volatility, for the first 3 years with low volatility;
- with $N = 10$ the mean for the first year is below the level of 45% with high volatility, is below 70% with normal and mixed volatility, whereas it is 1 in the first year with low volatility.

This result is reasonable, as the higher the volatility of the high-risk asset, the less confident will be the investment manager in investing the whole portfolio in that asset, and vice versa. However, the normal volatility and the mixed volatility

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3 The reader should be aware of the imprecision of this statement and of the followings ones, as explained at the beginning of this paragraph.
case present the same degree of aggressiveness, intermediate between the high and the low volatility case. This means that the aggressiveness of the portfolio is linked to the volatility of both assets, thus to the volatility of the portfolio: the more volatile the portfolio the less aggressive the strategy.

A fuller discussion about the high-volatility strategies will be provided later, when the aspect of portfolio diversification will be considered.

- **the strategies are more aggressive as the mean of both assets increases.**

  Similarly to the analysis above, it is possible by comparing Figures 1 and 5 to see that:
  - with N = 40 the mean of y*; is 1 for the first 7 years in the normal mean case, it is 1 for the first 6 years in the low mean case;
  - with N = 30 the mean is 1 for the first 4 years in the normal mean case, it is 1 for the first 3 years in the low mean case;
  - with N = 20 the mean is 1 for the first 2 years in the normal mean case, it is 1 for the first year in the low mean case;
  - with N = 10 the mean is below 70% in the normal mean case, it is below 60% in the low mean case.

The result is the converse of the results concerning volatility, as lowering the mean of both assets leaving their volatility unchanged is equivalent to raising the volatility of both assets leaving their mean unchanged, in terms of the relation between mean and standard deviation of the assets returns. Thus, the comparison between Figures 1 and 3 is equivalent to the comparison between Figures 1 and 5 and the same comments as above apply.

- **the strategies are more aggressive as the mean of the low-risk asset decreases and the mean of the high-risk asset increases, that is when the gap between \( \mu \) and \( \lambda \) increases.**
It is possible to see that the mixed mean scenario (Fig 6), with \( \mu = 4\% \) and \( \lambda = 6\% \), is the most aggressive of all scenarios and the riskless & risky scenario (Fig. 7), with \( \mu = 2\% \) and \( \lambda = 6\% \), is the second most aggressive one. This feature is particularly evident in the 10 years case, where at least 25% of the \( y^*_t \) are above 60% still in the 5th year (i.e. 5 years before retirement) in the mixed mean case and above 60% in the 4th year in the riskless & risky case, whereas in the normal case the 95% of the \( y^*_t \) are below 50% already from the 2nd year.

The explanation of this result is clear as far as the mixed mean case is concerned. The share to be invested in the high-risk asset obviously increases if its expected return increases, leaving the volatility the same, and also the share to be invested in the low-risk asset decreases if its mean decreases, with the same volatility. The good degree of aggressiveness manifested by the riskless & risky case is due to the gap between \( \mu \) and \( \lambda \), even though the low-risk asset has become a riskless one.

Results: diversification.
The results are summarised and commented on as follows:

- the strategies are more diversified as the volatility of both assets increases.

The slope of the curves is less steep in the high volatility case (Fig. 3) than in all other cases: the strategies seem to stabilise after few years around a certain percentage (the number of years depending on the time to retirement \( N \)), in the range 25%-30%. That means that on average the share to be invested in the high-risk asset is \( \frac{1}{4} \) - \( \frac{1}{3} \) of the portfolio until retirement. The slope of the curves is very steep in the low volatility case (Fig. 2), the normal and mixed case (Figs. 1 and 4) being intermediate situations. The riskless & risky scenario (Fig. 7) presents the less diversified strategy, with slope of the curve nearly vertical.
This feature can be explained as follows. In the case of low volatility of both assets, the optimal strategy is to invest the whole fund immediately in the high-risk asset in order to approach the target. Once the target has been reached (which should not be difficult because of the relatively low volatility of the asset) it is then better to maintain the fund close to the target using the low-risk asset, which offers lower volatility allowing a more defensive policy. This phenomenon is particularly evident in the riskless & risky case.

In the case of high volatility of both assets, it is not cautious to invest the whole portfolio in one asset only. Firstly, it is more difficult to reach the target using the high-risk asset due to its relatively high volatility, secondly, the maintenance of the position is not guaranteed by investing the fund in the low-risk asset due again to its high volatility. Furthermore, a diversification strategy is also suitable in order to lower the high volatility of the portfolio.

- the strategies are more diversified as N increases.

The slope of the curves is less steep with N = 10, 20 than with N = 30, 40 in the same scenario. This is a phenomenon which happens in each scenario.

This result is reasonable, as the shorter the time to retirement, the lower portfolio volatility is desirable and by diversifying it is possible to lower the portfolio variance, as explained above.

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4 It is reminded that the "diversification effect" is such that by investing a portfolio properly in 2 assets with return volatility $\sigma_1, \sigma_2$ it is possible to have the portfolio volatility lower than $\sigma_1$, provided that the linear correlation coefficient $\rho$ is such that $\rho \leq \sigma_1/\sigma_2$. In this model the assets are independent, therefore $\rho = 0$ satisfies that condition and it possible with an appropriate mix between the asset to lower the volatility of the portfolio. It is beyond the scope of this work investigate what should be the mix which gives the minimum variance portfolio of this portfolio selection problem, but this could be a proposal for further research.
SENSITIVITY ANALYSIS OF THE FIXED PARAMETERS

At the beginning of this chapter, consideration was given to fixing some of the parameters of the model. A sensitivity analysis has been carried out changing the values of some of these parameters in order to test the appropriateness of the previous choices.

It is reminded that the fixed parameters have the following values:
\[ f_0 = 0 \]
\[ c = 12\% \]
\[ \theta = 2 \]
\[ \gamma = 0.95 \]

The sensitivity analysis has been done by changing the values of \( \theta \) and \( \gamma \), in particular choosing \( \theta = 1, 3 \) and \( \gamma = 0.8, 1 \). The simulations have been carried out considering the base scenario only. The results are reported in graphs constructed in the same way as before. Fig. 8 reports the graphs relative to the case \( \theta = 1 \), Fig. 9 the case \( \theta = 3 \), Fig. 10 the case \( \gamma = 0.8 \), Fig. 11 the case \( \gamma = 1 \).

The graphs show trends very similar to the normal volatility case (in Figure 1), suggesting that the choice of the parameters \( \theta \) and \( \gamma \) does not affect materially the resulting optimal investment strategies.

It could be of interest to test the response of the model to changes also in the value of the transferred value \( f_0 \), which would apply to the case of members transferring in from other schemes.
FIG. 1 - NORMAL VOLATILITY: INVESTMENT STRATEGIES

Y*(t), normal volatility, 10 years

Y*(t), normal volatility, 10 years

Y*(t), normal volatility, 20 years

Y*(t), normal volatility, 20 years

Y*(t), normal volatility, 30 years

Y*(t), normal volatility, 30 years

Y*(t), normal volatility, 40 years

Y*(t), normal volatility, 40 years

Y*(t), normal volatility, 1 year

Y*(t), normal volatility, 2 years

Y*(t), normal volatility, 5 years

Y*(t), normal volatility, 10 years
FIG 2 - LOW VOLATILITY: INVESTMENT STRATEGIES

Y*(t), low volatility, 10 years

Y*(t), low volatility, 20 years

Y*(t), low volatility, 30 years

Y*(t), low volatility, 40 years

Y*(t), low volatility, 10 years

Y*(t), low volatility, 20 years

Y*(t), low volatility, 30 years

Y*(t), low volatility, 40 years
FIG 3 - HIGH VOLATILITY: INVESTMENT STRATEGIES

Y*(t), high volatility, 10 years

Y*(t), high volatility, 10 years

Y*(t), high volatility, 20 years

Y*(t), high volatility, 20 years

Y*(t), high volatility, 30 years

Y*(t), high volatility, 30 years

Y*(t), high volatility, 40 years

Y*(t), high volatility, 40 years
FIG 4 - MIXED VOLATILITY: INVESTMENT STRATEGIES
FIG 5 - LOW MEAN: INVESTMENT STRATEGIES

Y*(t), low mean, 10 years

Y*(t), low mean, 20 years

Y*(t), low mean, 30 years

Y*(t), low mean, 40 years
FIG 6 - MIXED MEAN: INVESTMENT STRATEGIES

Y*(t), mixed mean, 10 years

Y*(t), mixed mean, 20 years

Y*(t), mixed mean, 30 years

Y*(t), mixed mean, 40 years
FIG 7 - RISKLESS & RISKY: INVESTMENT STRATEGIES

Y*(t), riskless & risky, 10 years

Y*(t), riskless & risky, 10 years

Y*(t), riskless & risky, 20 years

Y*(t), riskless & risky, 20 years

Y*(t), riskless & risky, 30 years

Y*(t), riskless & risky, 30 years

Y*(t), riskless & risky, 40 years

Y*(t), riskless & risky, 40 years
FIG. 8 - SENSITIVITY ANALYSIS: THETA = 1

Y*(t), theta = 1, 10 years

Y*(t), theta = 1, 20 years

Y*(t), theta = 1, 30 years

Y*(t), theta = 1, 40 years

Y*(t), theta = 1, 10 years

Y*(t), theta = 1, 20 years

Y*(t), theta = 1, 30 years

Y*(t), theta = 1, 40 years

Legend:
- 5th
- 25th
- 50th
- 75th
- 95th
- min
- max

Legend:
- mean
- st.dev.
FIG. 9 - SENSITIVITY ANALYSIS: THETA = 3

Y*(t), theta = 3, 10 years

Y*(t), theta = 3, 20 years

Y*(t), theta = 3, 30 years

Y*(t), theta = 3, 40 years
FIG. 10 - SENSITIVITY ANALYSIS: GAMMA = 0.8
FIG. 11 - SENSITIVITY ANALYSIS: GAMMA = 1

Y*(t), gamma = 1, 10 years

Y*(t), gamma = 1, 20 years

Y*(t), gamma = 1, 30 years

Y*(t), gamma = 1, 40 years
CHAPTER 5: NET REPLACEMENT RATIO

INVESTMENT RISK AND ANNUITY RISK: 2 DIFFERENT WAYS OF COMPARING ACTUAL NET REPLACEMENT RATIO AND TARGET

Once the member of the defined contribution pension scheme reaches retirement, the fund is converted into an immediate annuity, which is the resulting pension. It is well known (see for example Knox, 1993, Khorasanee, 1998, Booth & Yakoubov, 1998) that the member, who has already borne the investment risk during the accumulation period, which is the risk that the returns experienced during his/her membership have been too low leading to a low final fund (see also chapter 1), now bears the annuity risk. This is the risk that the rate used in the conversion of the capital in annuity is too low, leading to a low pension rate. The actual conversion rate used to calculate the annuity is directly linked to the current market yields, and so the perceived pension will strongly depend on the level of the markets rates at retirement. In this work we tempt to separate the investment risk from the annuity risk.

To estimate the degree to which the investment risk has affected the results it is sufficient to compare the values of $f_N$ and $F_N$, i.e. the actual final fund with the target. In the case of unfavourable returns, the comparison will give $f_N < F_N$, whereas the target will be reached and even exceeded in the case of normal or favourable market returns. Although the comparison between $f_N$ and $F_N$ is certainly the most correct one, it is recognised that the member will be more concerned about the comparison between the actual net replacement ratio achieved and the one promised $N$ years before. Therefore it seems convenient

---

1 In order to avoid confusion, it seems important to specify that in the insurance context the term annuity risk is also referred to the risk that the insurer bears in paying the annuity to the policyholder until he/she survives. We do not use the term "annuity risk" with this meaning but in the sense explained above.
to transform both the target and the actual fund in the corresponding net replacement ratio, as already done in chapter 3. It is reminded that the net replacement ratio target is:

$$B_N = F_N / \sum_{n=1}^{\infty} nE_x = F_N / a_x,$$

where $x$ is the retirement age of the member, $nE_x$ is calculated using appropriate mortality tables and $v = e^{-\mu + 0.5\sigma^2}$ as a discount factor, $\Omega$ is the extreme age contemplated by the tables and $B_N$ is the pension rate, which coincides with the net replacement ratio, since we have assumed that $S_N = 1$.

As the investment risk affects $f_N$ only, it is possible to study the effect of the investment risk by converting the actual final fund into annuity in the same way:

$$b_N = f_N / \sum_{n=1}^{\infty} nE_x = f_N / a_x$$

$b_N$ is called "expected net replacement ratio", as the annuity has been calculated using the expected rate of return from the low-risk asset.

The comparison between $b_N$ and $B_N$ gives an indication of the investment risk, as it reduces to the comparison between $f_N$ and $F_N$, everything else being equal.

As explained above the real annuity will be calculated by the insurer using the current market investment returns. Therefore, a more realistic way of calculating the annuity will take into account the simulated rates of return of the low-risk asset in the years preceding retirement.

A new element has been introduced:
The new element $\tilde{b}_N$ is called "experienced net replacement ratio", as in its calculation the past experienced returns have been taken into account in the coefficients $\tilde{\mu}$ and $\tilde{\sigma}^2_1$. It has been decided to consider the returns experienced in the last 5 years in the calculation of the mean instead of the last year only, in order to eliminate the effect of very low simulated values. For the same reason the value of 1% or 2%, depending on the mean of the low-risk asset, has been imposed as a minimum interest rate to use in the conversion. The standard deviation has been calculated considering the standard deviation experienced over the whole working life of the member ($N$ years) rather than a shorter period in order to avoid unreasonable values.
The comparison between $\tilde{b}_N$ and $B_N$ gives a measure of both the investment and the annuity risk, as it contains the comparison between $f_N$ and $F_N$ and also between $\tilde{a}_x$ and $a_x$.

It is therefore clear that by comparing $\tilde{b}_N$ and $b_N$ it is possible to isolate the effect of the annuity risk, the difference between the two net replacement ratios being only the discount factor used in converting the final fund into annuity, $\tilde{\nu}$ in the former case (that is the simulated rate), $\nu$ in the latter one (that is the expected one).

ASSUMPTIONS ABOUT AGE, MORTALITY AND JOINING DATE OF THE MEMBER (THE ITALIAN CONTEXT)

In order to apply the work in a realistic scenario, it has been assumed in the simulations that the member joins the scheme in year 2000 and that the retirement age is 62, due to continuously developing laws in Italy, which are increasing the retirement age fixing it in the range 60 - 65. With these assumptions, it follows that the member with 10 year to retirement was born in 1948, with 20 years to retirement was born in 1958, with 30 years to retirement was born in 1968 and with 40 years to retirement was born in 1978.

The choice of the year 2000 is due also to the fact that Italian pension schemes are at the moment at an early stage of their development, as already underlined in chapter 1.

The choice of having individuals of different generations has been done also in order to allow for mortality trend in the analysis of the liabilities of the pension scheme. For this reasons the mortality table used in the conversion of the accumulated fund into annuity in the simulations is the Italian projected mortality
table (RGS48), which takes into account the projected mortality of individuals still alive.

ANALYSIS OF RESULTS: PROBABILITY OF FAILING THE TARGET

The following table reports the probability of failing in all the scenarios and in both cases of expected and experienced net replacement ratio.

**Table 1: probability of failing the target**

<table>
<thead>
<tr>
<th></th>
<th>Normal volatility</th>
<th>Low volatility</th>
<th>High volatility</th>
<th>Mixed volatility</th>
<th>Low mean</th>
<th>Mixed mean</th>
<th>Riskless &amp; risky</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Pr(\tilde{b}<em>{10} &lt; B</em>{10}) )</td>
<td>54.40%</td>
<td>55.30%</td>
<td>56.90%</td>
<td>66.50%</td>
<td>53.60%</td>
<td>55.70%</td>
<td>91.70%</td>
</tr>
<tr>
<td>( Pr(b_{10} &lt; B_{10}) )</td>
<td>64.90%</td>
<td>66.80%</td>
<td>64.40%</td>
<td>87.70%</td>
<td>66.50%</td>
<td>65.80%</td>
<td>91.70%</td>
</tr>
<tr>
<td>( Pr(\tilde{b}<em>{20} &lt; B</em>{20}) )</td>
<td>55.20%</td>
<td>51.30%</td>
<td>57.90%</td>
<td>65.20%</td>
<td>54.60%</td>
<td>55.40%</td>
<td>57.60%</td>
</tr>
<tr>
<td>( Pr(b_{20} &lt; B_{20}) )</td>
<td>59.70%</td>
<td>45.40%</td>
<td>62.80%</td>
<td>74.60%</td>
<td>55.20%</td>
<td>58.20%</td>
<td>57.60%</td>
</tr>
<tr>
<td>( Pr(\tilde{b}<em>{30} &lt; B</em>{30}) )</td>
<td>55.20%</td>
<td>47.00%</td>
<td>56.00%</td>
<td>62.90%</td>
<td>57.30%</td>
<td>50.60%</td>
<td>50.50%</td>
</tr>
<tr>
<td>( Pr(b_{30} &lt; B_{30}) )</td>
<td>44.90%</td>
<td>34.60%</td>
<td>44.90%</td>
<td>41.80%</td>
<td>63.20%</td>
<td>44.00%</td>
<td>50.50%</td>
</tr>
<tr>
<td>( Pr(\tilde{b}<em>{40} &lt; B</em>{40}) )</td>
<td>55.00%</td>
<td>43.20%</td>
<td>58.30%</td>
<td>57.50%</td>
<td>55.70%</td>
<td>47.00%</td>
<td>51.40%</td>
</tr>
<tr>
<td>( Pr(b_{40} &lt; B_{40}) )</td>
<td>49.90%</td>
<td>35.90%</td>
<td>61.30%</td>
<td>55.50%</td>
<td>59.90%</td>
<td>41.90%</td>
<td>51.40%</td>
</tr>
</tbody>
</table>

It is of interest to notice the range of the values of Table 1: except in 5 of the 56 cases the figures fall in the range 40%-70%, the exceptions taking values around 35%, 75%, and 90%. The probability of failing the target seems to be generally high, and this could be very problematic in terms of credibility of the
pension scheme, if the previously calculated targets are promised (in some sense) to the member at the joining of the scheme.

By looking more deeply at the figures it is possible to study the behaviour of the probability of failing the target when N increases, fixing the scenario (ie the columns of Table 1). The results are as follows:

- In the low volatility, mixed volatility and mixed mean scenarios, the probability of failing the target decreases as N increases, in both the cases of the expected net replacement ratio and experienced net replacement ratio.
- In the normal volatility and high volatility scenarios the probability of failing the target is stable around the percentage of 55% as N increases in the case of the experienced net replacement ratio; it is decreasing from N=10 until N=30 in the case of expected net replacement ratio, then it slightly increases as N=40.
- In the riskless & risky scenario the net replacement ratio expected and experienced are obviously coincident, as the low risk asset used as conversion rate is now riskless leading to $\tilde{v} = v$. The probability of failing the target is decreasing until N=30, then it slightly increases as N=40, as seen in the previous case.
- In the low mean scenario the probability of failing the target is stable around 55% when N increases in the case of experienced net replacement ratio, while it shows no trend with N in the case of expected net replacement ratio.

The decreasing trend of the probability of failing the target as the time to retirement increases observed in many cases is reasonable: the longer the time to retirement the easier is the achievement of the target, as there is more time to react properly to unfavourable market scenarios. The highest values of the Table 1 are to be found in fact in the 10 years case, where the figures are
spread in the range of 53.60% - 91.70%, the latter being the value of the riskless & risky scenario. Note the low value of $\mu$ in this scenario, equal to 2%.

A comparison is also possible between different scenarios, fixing $N$ (i.e. the rows of Table 1). The results are as follows:

- For all the ages except $N=10$, the lowest probability of failing the target is in the low volatility scenario, for both cases of expected and experienced net replacement ratio. In the 10 years case, the minimum value is in the low mean scenario.
- For both cases of expected net replacement ratio and experienced net replacement ratio in the 10 years case, the highest probability of failing the target is in the riskless and risky scenario (equal, as observed above, to 91.70%).
- For $N=20$ and 30, the highest probability of failing the target is to be found in the mixed volatility scenario, the only exception being for $N=30$ and expected net replacement ratio, whose maximum value is in the low mean scenario.
- For $N=40$, the maximum value is in the high volatility scenario, for both cases of expected net replacement ratio and experienced net replacement ratio.
- For all values of $N$ the values of the normal volatility scenario and the high volatility scenario are similar, and in nearly all cases the value of the low mean scenario is similar to this common value\(^2\).
- For many values of $N$ (but not all the values), the figures of the low volatility scenario are similar to those of the mixed mean scenario, especially when time to retirement is short.

\(^2\) It is reminded from chapter 4 that the low mean and the high volatility scenario are similar in terms of relations between mean and standard deviation of asset returns.
The explanation why the lowest figures are to be found in the low volatility scenario recalls the explanation given in chapter 4 when it was found that the less diversified investment strategies are in the low volatility case: with low volatility of both assets the investment manager will probably invest the whole fund immediately in the high-risk asset in order to approach the target. The achievement of the target should not be difficult because of the relatively low volatility of the high-risk asset, and then maintain it investing in the low-risk asset should also be relatively easy, due again to the very low volatility of the low-risk asset. The only exception to this explanation is when time to retirement is too short (N=10): in this case there is not enough time to develop the mentioned strategy, as observed above.

**ANALYSIS OF RESULTS: EXPECTED AND EXPERIENCED NET REPLACEMENT RATIOS AND THEIR RATIOS WITH THE TARGET**

Tables 2 and 3 that follow report the distribution of \( b_N \) and \( \tilde{b}_N \) over the 1000 simulations for any \( N \) in each of the 7 scenarios described in chapter 4. The tables report, as in the previous analysis of the investment strategies, mean, standard deviation, minimum value, maximum value and different percentiles (5\(^{th}\), 25\(^{th}\), 50\(^{th}\), 75\(^{th}\) and 95\(^{th}\)) of the observed distribution.

Not only the absolute values of \( b_N \) and \( \tilde{b}_N \) have been studied, but also the more significant ratios \( b_N / B_N \) and \( \tilde{b}_N / B_N \), which indicate the percentage of the target achieved at retirement, allowing comparison between different ages (not possible looking at the absolute values of \( b_N \) and \( \tilde{b}_N \), the targets being obviously very different one from the other).

The tables report also the targets corresponding for each \( N \) and each scenario. The values of the targets change obviously by age, but also by scenario, as both mean and standard deviation of both asset returns are used in calculating
the final target. In particular we notice that in the first 4 scenarios, where only
the volatility varies, the targets are very similar to each other for each N, while in
the last 3 scenarios, where the mean has been changed, the values are spread
across a larger range. This is due to the fact that the value of the mean largely
affects the value of the accumulation factor used in the calculation of $F_N$ and
that of the discount factor used in calculating $a_x$, while the standard deviation
affects these values very slightly.

The net replacement targets $B_N$ vary in a range of 8%-12% for the 10 years
case, in a range of 19%-33% for the 20 years case, in a range of 35%-70% for
the 30 years case, in a range of 75%-138% for the 40 years case. The minimum
values of these ranges correspond in each case to the low mean scenario
(which has an obvious explanation, because a low mean for both assets
reduces $F_N$ and increases $a_x$). The range increases significantly with N, as one
would expect, the exponential accumulation law having been used.

In particular, with $N = 40$ the values of $B_N$ are very high, exceeding 100% in 4 to
7 cases (scenarios 1, 2, 3 and 4) and reaching 96% in 1 of the remaining 3
cases (scenario 6). These values are unacceptable from a practical point of
view, as it is usual to consider 70%-80% as the highest net replacement ratio
achievable at retirement. In practice, in a situation like this, the actuary would
lower the contribution rate, reducing in this way the final target to a reasonable
figure. In this simulation, the contribution rate has not been reduced and we
present all the results for the 40 years case, aware that this is an unrealistic
situation.

The targets have been approximated by rounding down the figures for $N=20,
30, 40. For the 10 years case instead, the approximation keeps also the first
decimal figure, as the targets are very low (the range being 8%-12%) and
rounding of 1% seems to be too high and would probably strongly affect the
results. The same approximation has been used in calculating the previous probabilities of failing the target.

In Tables 2 and 3 “bNt” stays for \( \tilde{b}_N \), “bN” stays for \( b_N \), and BN stays for \( B_N \).

\[ \text{In particular a strong approximation to the nearest number below, as the one that we would have if we rounded down the figures also in the 10 years case, leads to overestimated values of the achieved net replacement ratios, or, looking the other way round, to underestimated probabilities of failing the target. Therefore one should be aware that all the results here presented are overestimated in terms of net replacement ratios, although very slightly.} \]
### TABLE 2: Scenarios 1, 2, 3, 4: Targets, Expected, Experienced Net Replacement Ratios and Their Ratios

<table>
<thead>
<tr>
<th>NORMAL VOLATILITY</th>
<th>B10 11.80%</th>
<th>B20 31.00%</th>
<th>B30 84.00%</th>
<th>B40 122.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B10 to B10</td>
<td>B10 to B20</td>
<td>B10 to B30</td>
<td>B10 to B40</td>
</tr>
<tr>
<td>mean</td>
<td>0.120</td>
<td>0.078</td>
<td>0.092</td>
<td>0.095</td>
</tr>
<tr>
<td>st. dev.</td>
<td>0.034</td>
<td>0.075</td>
<td>0.114</td>
<td>0.116</td>
</tr>
<tr>
<td>5th</td>
<td>0.005</td>
<td>0.019</td>
<td>0.027</td>
<td>0.029</td>
</tr>
<tr>
<td>95th</td>
<td>0.316</td>
<td>0.424</td>
<td>0.663</td>
<td>0.646</td>
</tr>
<tr>
<td>50th</td>
<td>0.101</td>
<td>0.162</td>
<td>0.223</td>
<td>0.229</td>
</tr>
<tr>
<td>LOW VOLATILITY</td>
<td>B10 11.70%</td>
<td>B20 30.00%</td>
<td>B30 60.00%</td>
<td>B40 113.0%</td>
</tr>
<tr>
<td></td>
<td>B10 to B10</td>
<td>B10 to B20</td>
<td>B10 to B30</td>
<td>B10 to B40</td>
</tr>
<tr>
<td>mean</td>
<td>0.116</td>
<td>0.088</td>
<td>0.096</td>
<td>0.097</td>
</tr>
<tr>
<td>st. dev.</td>
<td>0.018</td>
<td>0.055</td>
<td>0.091</td>
<td>0.088</td>
</tr>
<tr>
<td>5th</td>
<td>0.015</td>
<td>0.025</td>
<td>0.023</td>
<td>0.011</td>
</tr>
<tr>
<td>95th</td>
<td>0.295</td>
<td>0.467</td>
<td>0.688</td>
<td>0.563</td>
</tr>
<tr>
<td>50th</td>
<td>0.066</td>
<td>0.115</td>
<td>0.162</td>
<td>0.118</td>
</tr>
<tr>
<td>HIGH VOLATILITY</td>
<td>B10 11.90%</td>
<td>B20 32.00%</td>
<td>B30 69.00%</td>
<td>B40 138.0%</td>
</tr>
<tr>
<td></td>
<td>B10 to B10</td>
<td>B10 to B20</td>
<td>B10 to B30</td>
<td>B10 to B40</td>
</tr>
<tr>
<td>mean</td>
<td>0.134</td>
<td>0.073</td>
<td>0.067</td>
<td>0.063</td>
</tr>
<tr>
<td>st. dev.</td>
<td>0.057</td>
<td>0.062</td>
<td>0.068</td>
<td>0.067</td>
</tr>
<tr>
<td>5th</td>
<td>0.019</td>
<td>0.017</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>95th</td>
<td>0.283</td>
<td>0.372</td>
<td>0.461</td>
<td>0.364</td>
</tr>
<tr>
<td>50th</td>
<td>0.071</td>
<td>0.101</td>
<td>0.134</td>
<td>0.096</td>
</tr>
</tbody>
</table>

### MIXED VOLATILITY

<table>
<thead>
<tr>
<th>NORMAL VOLATILITY</th>
<th>B10 12.00%</th>
<th>B20 33.00%</th>
<th>B30 70.00%</th>
<th>B40 137.0%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B10 to B10</td>
<td>B10 to B20</td>
<td>B10 to B30</td>
<td>B10 to B40</td>
</tr>
<tr>
<td>mean</td>
<td>0.115</td>
<td>0.090</td>
<td>0.091</td>
<td>0.092</td>
</tr>
<tr>
<td>st. dev.</td>
<td>0.079</td>
<td>0.090</td>
<td>0.101</td>
<td>0.110</td>
</tr>
<tr>
<td>5th</td>
<td>0.007</td>
<td>0.015</td>
<td>0.019</td>
<td>0.020</td>
</tr>
<tr>
<td>95th</td>
<td>0.261</td>
<td>0.363</td>
<td>0.461</td>
<td>0.365</td>
</tr>
<tr>
<td>50th</td>
<td>0.071</td>
<td>0.094</td>
<td>0.100</td>
<td>0.093</td>
</tr>
<tr>
<td>5th</td>
<td>0.123</td>
<td>0.132</td>
<td>0.139</td>
<td>0.144</td>
</tr>
<tr>
<td>95th</td>
<td>0.090</td>
<td>0.109</td>
<td>0.117</td>
<td>0.122</td>
</tr>
<tr>
<td>min</td>
<td>0.071</td>
<td>0.080</td>
<td>0.089</td>
<td>0.092</td>
</tr>
<tr>
<td>max</td>
<td>0.183</td>
<td>0.201</td>
<td>0.218</td>
<td>0.236</td>
</tr>
</tbody>
</table>
TABLE 3: SCENARIOS 1, 5, 6, 7: TARGETS, EXPECTED, EXPERIENCED NET REPLACEMENT RATIOS AND THEIR RATIOS

<table>
<thead>
<tr>
<th>TARGETS</th>
<th>50%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>NORMAL MEAN</td>
<td>810</td>
<td>820</td>
<td>830</td>
<td>840</td>
</tr>
<tr>
<td>50%</td>
<td>0.087</td>
<td>0.094</td>
<td>0.097</td>
<td>0.099</td>
</tr>
<tr>
<td>90%</td>
<td>0.109</td>
<td>0.114</td>
<td>0.117</td>
<td>0.119</td>
</tr>
<tr>
<td>95%</td>
<td>0.129</td>
<td>0.133</td>
<td>0.137</td>
<td>0.140</td>
</tr>
<tr>
<td>99%</td>
<td>0.149</td>
<td>0.153</td>
<td>0.156</td>
<td>0.159</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>LOW MEAN</th>
<th>50%</th>
<th>90%</th>
<th>95%</th>
<th>99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td>0.090</td>
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Looking at Tables 2 and 3 we identify the following important features:

1. The standard deviation of $\tilde{b}_N$ is much higher than the standard deviation of $b_N$ for each scenario and for any value of $N$: in nearly all cases it is more than double. This variability of the results in the case of experienced net replacement ratio indicates that with a variable and past-depending conversion rate the final pension can be much higher (the 95th and 75th percentiles of the distribution of $\tilde{b}_N$ are always higher than the corresponding percentiles of $b_N$), but also much lower (the 5th and 25th percentiles of the distribution of $\tilde{b}_N$ are always lower than the corresponding of $b_N$). This highlights the effect of the annuity risk on the final pension, which results to be one of the major causes of uncertainty in retirement income for the member of defined contribution pension schemes. This is probably the main result that can be extracted from these tables.

2. The mean of the ratios $\frac{b_N}{B_N}$ and $\frac{\tilde{b}_N}{B_N}$ are very similar for $N = 10, 20, 30, 40$, which indicates that the probability of achieving a certain percentage of final target is on average approximately the same, regardless of the age and the length of future service. This consistency between different generations that be surprising. However, it can be explained by noting that the formula for the final targets takes into account the duration of future service.

3. However, looking at the standard deviation of these ratios it is possible to see that this increases as $N$ increases. In terms of percentiles this issue leads to 5th and 25th percentiles generally decreasing and 75th and 95th percentiles generally increasing as $N$ increases. This means that the results are more spread out as the time to retirement increases, which is reasonable, as the number of possible events and outcomes increases exponentially with $N$. 
4. The standard deviation of these ratios increases also as the volatility of both assets increases and especially as the volatility of the low-risk asset increases. The volatility of the high-risk asset affects only the volatility of the actual final fund $f_N$ while the volatility of the low-risk asset affects also the volatility of the annuity $a_x$. Thus, the lowest standard deviation of the ratios $\frac{b_{N}}{B_{N}}$ and $\frac{\tilde{b}_{N}}{B_{N}}$ are to be found in the riskless & risky scenario for all $N$, followed by the low and mixed volatility scenarios, while the highest is in the high volatility scenario.

By looking at the 5th and 25th percentiles of the reported distributions it is possible to observe the magnitude of the unfavourable outcomes. The discussion that follows will report the results relative to the experienced net replacement ratio $\tilde{b}_{N}$, although a similar discussion may be done for the expected net replacement ratio $b_{N}$. The results are the following:

5. As one would expect (see discussion above), the most adverse results are to be found in the high volatility scenario. The 5th percentile of the distribution of the ratio $\frac{\tilde{b}_{N}}{B_{N}}$ varies between 0.37 ($N=40$) and 0.56 ($N=10$), meaning that in the 5% of the simulated cases the achieved net replacement ratio does not achieve the 40% of the target in the 40 years case and the 60% of the target in the 10 years case (the other ages being intermediate situations). The 25th percentile varies between 63% and 70%. It seems more intuitive to illustrate the results in terms of absolute values.

- In the 10 years case, the n.r.r.$^4$ target being 11.80%, the results are that in 5% of the cases the n.r.r. achieved is less than 6.70%, and in 25% of the cases it is less than 8.3%$^5$.

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$^4$ By "n.r.r." we mean net replacement ratio.

$^5$
- In the 20 years case, the n.r.r. target being 32%, the results are that in 5% of the cases the n.r.r. achieved is less than 16%, and in 25% of the cases it is less than 22%;

- In the 30 years case, the n.r.r. target being 69%, it results that in 5% of the cases the n.r.r. achieved is less than 29%, and in 25% of the cases it is less than 44%;

- In the 40 years case, the n.r.r. target being 138%, it results that in 5% of the cases the n.r.r. achieved is less than 51%, and in 25% of the cases it is less than 87%, although we remind that these last figures are only indicative, this high net replacement ratio target being unrealistic (see discussion above).

6. Adverse results are to be found also in the normal volatility scenario. The 5th percentile of the distribution of the ratio $\frac{\tilde{B}_N}{B_N}$ varies between 0.49 (N=40) and 0.66 (N=10). The same analysis above reported which compares the n.r.r. target and the achieved one can be applied also in this case.

7. The results improve for the mixed volatility case and are acceptable in the low volatility scenarios. In the latter case the 5th percentile varies between 66% and 75%, meaning that only in the 5% of the cases the n.r.r. achieved is less than 66% (or 75%) of n.r.r. target. It is of interest to note that these scenarios are characterised by low volatility of the low-risk asset. This is reasonable as the conversion of the fund into annuity has been done using the return from the low-risk asset and a low volatility of the conversion rate leads to lower volatility of the final outcome (see also comment in point n. 4 above).

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5 It seems proper to state precisely that in the 5% or 25% of the simulated cases the results are the ones above exposed, the word "simulated" has been omitted in order to have a shorter exposition.
CHAPTER 6: MAIN RESULTS AND CONCLUDING COMMENTS

MAIN RESULTS

As it has widely been commented, the main criticism of defined contribution pension schemes is the uncertainty of retirement income, in comparison to defined benefit schemes. This uncertainty is due to the financial risk borne by the member, which is composed of investment risk, experienced in the accumulation of contributions, and annuity risk, experienced at retirement when the pension is bought.

In this research the financial risk in defined contribution pension schemes has been investigated in both its components.

A model has been constructed which gives an optimal investment strategy with the objective of reducing the investment risk. The mathematical tools of dynamic programming theory, used in many actuarial papers with the objective of controlling insolvency risk and instability of contribution rate in defined benefit schemes, have been used in a defined contribution scheme with the objective of controlling investment risk. A closed formula has been found and implemented in realistic scenarios and the main result is that the optimal investment strategy requires investing at the beginning of the period of active membership in the high-risk asset, shifting towards the low-risk asset as time passes and retirement approaches. This is consistent with the lifestyle policy adopted in defined contribution pension schemes in UK, which seems to be the appropriate investment strategy in this kind of scheme. Therefore, this research seems to confirm through a scientific and rigorous approach what investment managers and actuaries have already discovered working on experience and intuition.
The annuity risk has been also investigated, by looking at the difference between the pension achieved at retirement using a fixed conversion factor (therefore removing the annuity risk) and that achieved by using a variable conversion factor, depending on the experienced returns prior to retirement. The main result is the large variability of the level of pension achieved at retirement in the case of the variable conversion rate in comparison with the fixed rate case, stressing the impact of the annuity risk on the final benefit.

This last issue underlines the fact that controlling only the investment risk is necessary but not sufficient to guarantee an adequate pension level to the member of the scheme, the annuity risk has to be controlled as well if very unfavourable outcomes have to be avoided. The present research does not make any proposal in reducing annuity risk in defined contribution pension schemes and it is intention of the author to continue in further research in order to approach this relevant problem. The author believes that defined contribution pension schemes will be central in the pension system of most countries in the future, hence the importance of continuing with research on this topic, in order to limit the negative consequences of this kind of scheme on the member’s expected pensions.

CONCLUDING COMMENTS

This research is done with many simplifying assumptions, in order to make the model easy to treat mathematically. Therefore, the model presented has a number of limitations and can still be improved. In this final section, we want to show the limitations of the model and give suggestion for further research.

In the model the parameter values of the distribution of asset returns have been chosen without direct reference to any historic observations, and a sensitivity
analysis of these parameters has been done after having chosen some "normal" values. The research could be improved by taking into account historic data, chosen for example from the returns actually experienced in the country where the defined contribution scheme works, as other researchers have done (see for example Knox, 1993, Ludvik, 1994, Khorasanee, 1995, Booth ad Yakoubov, 1998).

The returns on assets are supposed to be temporally independent and different categories of assets are suppose to be uncorrelated, while, in the real world, returns on the same assets in different period are not independent (see for example Khorasanee, 1995 and Booth and Yakoubov, 1998). Autoregressive models can be also used in modelling the dynamics of assets returns\(^1\). Also the hypothesis of identical distribution can be criticised, as well as the lognormal distribution assumption.

A simplifying assumption is the coincidence of asset allocation decisions and targets (both yearly). This assumption could be relaxed by choosing interim targets less frequent than yearly, as every three years or even less frequently until the extreme position with one target only (it could be worthwhile to analyse the resulting optimal investment strategies, which could present a different behaviour, taking into account the fact that only the final capital has to be maximised).

The definition of the target \(F_t\) can be different, including for example also other constraints, like the guarantee of a minimum benefit for the member, as it would be normal in defined contribution pension schemes in Italy.

A more general model can be made also by varying the level of the contribution rate during the membership, considering for example the contribution rate as a

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\(^1\) For an exhaustive description of autoregressive models see for example Daykin, Pentikainen and Pesonen (1994): *Practical Risk Theory for Actuaries.*
step-function \( c(t) \) such that \( c(t) = c_j \) for \( l_j \leq t \leq l_{j+1} \), where the intervals \([l_j, l_{j+1}]\) (\( j=0,\ldots,K-1 \), with \( l_0 = 0 \) and \( l_K = N \)) are a partition of the interval \([0, N] \).

Other limitations are the absence of expenses and other loadings in the pricing of annuities, and the failure to consider decrements other than retirement. The choice of using the whole final fund to buy an annuity at retirement has also to be mentioned, whereas a more general model would include other uses of that fund, eg a lump sum.

The assumption of having 2 assets for the whole working life period seems to be very strong. The different scenarios of mean and volatility may well represent different classes of assets. It may be reasonable to assume that for a certain period of time the fund can be invested in 2 assets only, diversifying between 2 different risky assets. It is certainly a simplification to assume that the 2 assets, in which the fund are invested, should remain the same for the whole period of active membership. Therefore, it would also be of interest to investigate the optimal investment strategy in an n-asset model.

An important assumption in this thesis is the choice of the objective. As already observed in this thesis (chapter 2), in a defined benefit pension scheme, a surplus is as undesirable as the deficit (with the obvious difference that a big deficit is much more undesirable than a big surplus), and so the variance is an appropriate risk measure in a defined benefit scheme. Therefore a loss function that penalises both surplus and deficit is appropriate in a defined benefit scheme.

This is less true in a defined contribution pension scheme, where the main objective should be to increase as much as possible the accumulated fund and hence the level of benefits. The loss function chosen in this work penalises the
fund accumulated above the target, leading to possible distortions in the realistic objectives of the trustees and drastically limiting the real potentiality of defined contribution pension scheme, which consists in giving the member the possibility of gaining on higher than expected market returns. However, it may be a sensible strategy for a scheme member to be investing in order to achieve a formula-based target with the aim of reducing the risk in the resulting investment strategy. Furthermore, the problem underlined above can be easily solved if we consider an unrealistically high final target (like a final pension equal to 400% of the final salary), the objective becoming in practice the maximisation of the final fund.

Another way of overcoming this problem is to modify the loss function in such a way that the increase in the fund above a certain benchmark is not penalised: the author believes that this approach could be more realistic in treating problems related to defined contribution pension schemes, and that further research should be developed in this direction.

2 About the possibility of drawing a part of the benefits as lump sum in countries like Italy it has been commented elsewhere in this thesis (chapter 3): we refer to this chapter and to other references for more details.
REFERENCES


