General equilibrium and asset pricing: existence and speculative bubbles in pure exchange economies

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Chapter 1

Existence of solutions in intertemporal general equilibrium models

1.1 Introduction

The theory of general equilibrium is the branch of economic theory that studies the interactions between demand and supply of the different goods in the different markets in order to determine the prices of these goods (while the partial equilibrium analysis considers only the relations between demand and supply of a specific good and the price of the same good). In the study of general equilibrium some simplifications are usually introduced, in particular it is assumed that:

- markets are competitive and individuals are optimizing;
- there is no production (at least in first approximation), agents have fixed endowments of the goods and they must determine only the quantities to exchange (pure exchange economy).
One of the central features of modern economics is then the introduction of time and uncertainty, and the consequent attempt to analyse an environment characterized by the presence of these elements. The main consequence for the behaviour of individuals is that they have only a limited ability to make decisions in such an environment; with reference to the theory of general equilibrium, in particular, this implies that, when agents have limited knowledge and ability to face uncertainty, they trade sequentially and use a system of contracts which involve only limited commitments into the future.

The standard model for the analysis of general equilibrium is that developed by Arrow and Debreu; the principal objective of the Arrow-Debreu theory is to study the allocation of resources achievable through a system of markets, and the central result of this theory is that, when there are markets and associated prices for all goods and services in the economy, no externalities or public goods, and no informational asymmetries, then competitive markets allocate resources efficiently. This framework can be adapted in order to take into account the fact that economic activity (production, exchange and consumption) takes place over time and involves the presence of uncertainty; in this case the Arrow-Debreu model assumes that at the initial date there is a market for each good produced or consumed in every possible future contingency, i.e. it assumes the presence of a complete set of contingent markets. Nevertheless, this structure is an idealization of the situation we can observe in the real world, since the individuals do not have full knowledge of all possible future events and the society cannot costlessly monitor and enforce the commitments of agents. The market structure that it is possible to observe in the real world, on the contrary, takes into account the fact that agents have limited capacity to face uncertainty and that the enforcement of contracts is costly, and it consists of a sequential system of spot markets for the exchange of goods and services and of contractual (financial) markets that involve limited commitments into the future.

For this reason it is necessary to consider a general equilibrium model that constitutes an extension of the basic framework represented by the Arrow-Debreu theory. In this way it is possible, on the one hand, to maintain the simplicity and generality of the
Arrow-Debreu model, and on the other hand to consider a structure of markets that is closer to the actual structure observed in the real world. In such an extension the basic set of markets is represented by a sequence of spot markets, on which goods and services are exchanged; we then have a sequence of contractual (financial) markets to make commitments for the future, and these commitments typically involve either the promise to deliver goods and services (real contracts) or the promise to deliver a certain amount of money (nominal contracts). When there is only a limited set of such contracts, in particular, the economy is characterized by a system of incomplete markets (that is typical of the real world).

The equilibrium solution of these models (if it exists) gives the values of prices and quantities (of the goods and of the financial activities) in correspondence of which the individuals solve their optimization problem and the markets (real and financial) clear (i.e. demand equals supply on these markets). A first important problem is therefore represented by the analysis of conditions that guarantee the existence of solutions in this kind of models.

These models can then be used to analyze the issue of asset pricing, and in particular the relation between the equilibrium price of an asset and the stream of future dividends on which the asset represents a claim. What emerges is that, while in the finite-horizon case the equilibrium price equals the fundamental value of the asset (i.e. the discounted sum of future dividends), in the infinite-horizon case this is not necessarily true (in particular, it is possible for the price to be larger than the fundamental value). In this case the price of the asset is said to involve a speculative bubble. A second important question is therefore represented by the analysis of conditions that allow to exclude the presence of such bubble components, together with the study concerning the fragility of this phenomenon.

The analysis presented in these pages is divided into three parts: in the first part (Chapter 1) the problem of existence of solutions in intertemporal general equilibrium models is treated, and the main results are given. Since the literature on this argument
is very extensive, and in the light also of the results discussed in the second part, this
analysis is limited to the (relatively) simplest case, the one in which the economy is
characterized by discrete time periods and a finite number of states of nature in each
period (and in which the economy is a pure exchange one, with financial structure that
consists of real assets only). In such a framework, the case of a two-period economy is
initially studied, and the results are then extended to the case of a $T$-period economy
(with $T$ finite) and finally to the case of an infinite-horizon economy. The main conclusion
is that, even if in the passage from an economy with contingent markets to an economy
with spot and financial markets and, in this economy, from the finite-horizon case to
the infinite-horizon case, something is "lost" in the proof of existence of equilibrium,
evertheless a form of existence is always guaranteed. The model is therefore consistent,
and it can be used to explain something of the economy we are dealing with.

In particular, this kind of models can be used to investigate the problem of asset
pricing, and the relation between the price of the assets and the stream of dividends to
which they give rise. This is the question considered in the second part (Chapter 2), with
particular reference to the phenomenon of speculative bubbles that can emerge in infinite-
horizon incomplete-market economies. The analysis is inspired by a recent contribution
that tries to make order in the field and to give a definitive theoretical settlement to the
question. In this context (that is based on considerations of no-arbitrage opportunities),
in particular, the possibility of a new type of bubbles (the so-called "ambiguous bubbles")
is outlined, and the results concerning the fragility of this phenomenon are discussed.

In the last part of the analysis a different approach (based on Euler equations) is
used to study the issue of speculative bubbles, and to obtain results that, again, are in
favour of a substantial fragility of this phenomenon. In particular, then, the occurrence
of speculative bubbles as a very special circumstance is shown by means of a series of
examples (Chapter 3). In this case the use of the Euler equations' approach allows us
to obtain some new results and to show how the presence of bubbles on asset prices is
linked to the violation of specific conditions.
While, in the light of the recent results obtained, the phenomenon of speculative bubbles can be considered definitively as marginal in the kind of models analyzed, the real world is often characterized by the occurrence of "speculative episodes" in which bubbles do appear. The final considerations try to explain the mechanism that produces these results and how it is possible to reconcile the conclusions of the theory with the episodes of the reality.

1.2 Existence of equilibrium: the model with contingent markets

The analysis contained in the following pages considers the issue of existence of solutions in intertemporal general equilibrium models in the simplest case, that of a pure exchange economy; in fact, real economic activity is decomposed into production and distribution of goods, but for simplicity purposes it is possible to assume that production decisions are fixed, so that only the problem of the distribution of goods is considered. In addition, the study is limited to a situation in which the time periods are discrete and the states of the world at each date are finite (this last assumption will be removed in the approach presented at the end of Chapter 2).

The method of analysis of an equilibrium model developed in the Arrow-Debreu theory focuses on three questions: existence, optimality and determinacy of equilibrium. The question of existence is the first and most important, because it is necessary to ensure that the model has a coherent structure and a solution (otherwise it is useless!); the optimality is crucial because it allows to evaluate the efficiency of the underlying market structure as a mechanism for the allocation of resources; the determinacy (i.e. the local uniqueness of equilibria) is essential for comparative statics, that is the analysis of how an equilibrium changes when certain parameters of the model are altered. Most of the attention in this Chapter will be devoted to the question of existence; however, also the problems of optimality and of determinacy will be in some cases considered.
The starting point is represented by the simplest model that can be used to incorporate the presence of uncertainty in a general equilibrium framework, the Arrow-Debreu model with *complete contingent markets*. After a brief presentation of this contribution (in the present Section), the more realistic model that considers *spot* and *financial markets* is introduced, and in this context the distinction between a situation with finite horizon and a situation with infinite horizon is presented (in Sections 1.3 and 1.4). Finally, it is outlined as in the case of infinite horizon a phenomenon that can emerge in the problem of asset pricing is the presence of *speculative bubbles*; this is the subject of the second Chapter.

### 1.2.1 Two-period exchange economy

The standard static model of general equilibrium (Arrow (1951), Debreu (1952), Arrow-Debreu (1954)) is characterized by the presence of a market for each good and by the fact that all the goods are traded simultaneously. By suitably extending the notion of commodity traded, this standard model can be generalized to a setting with time and uncertainty (Arrow (1953), Debreu (1959)). In this case the notion of commodity used is that of *contingent commodity*, that is a contract which promises the future delivery of one unit of a particular good contingent on the occurrence of a particular state of nature. If there is a complete set of such contingent contracts (one for each good in each possible state of nature), in particular, we have the idealized situation of *complete contingent markets*. In this way the model with time and uncertainty can be reduced to the model without uncertainty (by considering the same good available in different states of nature as different goods), and in which all the decisions are taken at the initial date \(t = 0\).

The simplest case is that of a two-period exchange economy; the results can then be generalized to an economy that extends over \(T\) periods (with \(T\) finite).

In the two-period pure exchange economy model the economy consists of a finite number of agents \((i = 1, 2, ..., I)\) and a finite number of goods \((l = 1, 2, ..., L)\), and there are two time periods \((t = 0, 1)\) in which one of \(S\) states of nature \((s = 1, 2, ..., S)\) occurs.
at date 1. If we denote the date \( t = 0 \) as the state \( s = 0 \), then, there are in total \( S + 1 \) states.

Since there are \( L \) commodities available in each state, the commodity space is \( \mathbb{R}^n \) with \( n = L(S + 1) \). Each agent \( i \) has an initial endowment of the \( L \) goods in each state \( w^i = (w^i_0, w^i_1, ..., w^i_S) \) and his preference ordering is represented by a utility function:

\[
u^i : \mathbb{R}^n_+ \to \mathbb{R} \quad i = 1, 2, ..., I
\]

defined over consumption bundles \( x^i = (x^i_0, x^i_1, ..., x^i_S) \) lying in the two-period consumption set \( X^i = \mathbb{R}^n_+ \). The characteristics of agent \( i \) are therefore summarized by a utility function and an endowment vector \( (u^i, w^i) \) and we assume that they satisfy the following conditions:

**Assumption 0 (agent characteristics)**

(1) \( u^i : \mathbb{R}^n_+ \to \mathbb{R} \) is continuous on \( \mathbb{R}^n_+ \) and \( C^\infty \) on \( \mathbb{R}^n_{++} \);

(2) \( U^i(\xi) = \{ x \in \mathbb{R}^n_+ \mid u^i(x) \geq u^i(\xi) \} \subset \mathbb{R}^n_{++}, \forall \xi \in \mathbb{R}^n_{++} \);

(3) \( \forall x \in \mathbb{R}^n_+, Du^i(x) \in \mathbb{R}^n_{++} \) and \( h^T D^2 u^i(x) h < 0 \) for all \( h \neq 0 \) such that \( Du^i(x) h = 0 \);

(4) \( w^i \in \mathbb{R}^n_+ \).

These assumptions are the so called *smooth preferences*, from the fact that the function \( u^i \) is said to be smooth on \( \mathbb{R}^n_{++} \) if it is \( C^\infty \) (all its partial derivatives are continuous), and were introduced by Debreu (1972). In particular, (2) asserts that any indifference curve passing through a positive consumption bundle doesn’t intersect the boundary of the non-negative orthant, and this avoids that the solution of the agent’s maximization problem occurs at the boundary of the consumption set; (3) expresses the strong monotonicity and the fact that for each positive vector of consumption \( x \) the quadratic form of the second derivative of \( u^i \) is negative definite when restricted to the hyperplane tangent to the indifference surface through \( x \).

If we now consider \( (u, w) = (u^1, ..., u^I, w^1, ..., w^I) \), then the collection of \( I \) agents
with their characteristics \((u, w)\) constitutes the (smooth) exchange economy \(E(u, w)\) that is the basis for the analysis. An allocation of resources for the economy \(E(u, w)\) is a vector of consumption bundles \(x = (x^1, x^2, \ldots, x^I) \in \mathbb{R}_+^I\), and equilibrium theory can be considered as the study of allocations that arise when we consider different market structures in the basic exchange economy \(E(u, w)\).

In particular, in the case of a (complete) system of contingent markets, a contingent commodity for good \(l (l = 1, 2, \ldots, L)\) in state \(s (s = 0, 1, \ldots, S)\) is a contract which promises to deliver one unit of good \(l\) in state \(s\) and nothing otherwise. The price \(P_{ls}\) of this contract, payable at date \(0\) (this is essential in this model, and differentiates it from the situation with spot markets that will be introduced in Section 1.3), is measured in units of account at date \(0\), and if at this date a complete set of such contingent contracts (one for each good in each state) is available, then each agent \(i\) can sell his endowment \(w^i = (w^i_0, w^i_1, \ldots, w^i_S)\) at the prices \(P = (P_0, P_1, \ldots, P_S)\) where \(P_s = (P_{s1}, P_{s2}, \ldots, P_{sL})\) to obtain the income \(Pw^i = \sum_{s=0}^{S} P_s w^i_s\), and he can purchase any consumption vector \(x^i = (x^i_0, x^i_1, \ldots, x^i_{S})\) satisfying \(Px^i = \sum_{s=0}^{S} P_s x^i_s \leq \sum_{s=0}^{S} P_s w^i_s = Pw^i\). Since the preferences of the agents are always assumed to be monotonic, an agent will always fully spend his income, and therefore agent \(i\)'s contingent market budget set can be defined as:

\[
B(P, w^i) = \{x^i \in \mathbb{R}_+^n \mid P(x^i - w^i) = 0\}
\]

For the economy considered the following definition of equilibrium can be introduced:

**Definition 1** A contingent market equilibrium for the economy \(E(u, w)\) is a pair of allocations and prices \((\bar{x}, \bar{P})\) such that:

(i) \(\bar{x}^i = \arg \max \{w^i(x^i) \mid x^i \in B(\bar{P}, w^i)\}\) \(i = 1, 2, \ldots, I\)

(ii) \(\sum_{i=1}^{I}(\bar{x}^i - w^i) = 0\)

This means that at the equilibrium each individual solves his maximization problem and, in addition, markets clear (i.e. the total demand of each good in each state of nature is equal to the total supply). To such an economy the classical existence theorem
of general equilibrium theory can be applied; as a consequence, for all characteristics 
\((u, w)\) satisfying Assumption 0 the exchange economy \(E(u, w)\) has a contingent market 
equilibrium \((\bar{x}, \bar{P})\), and moreover the allocation \(\bar{x}\) is a Pareto optimum. We have in fact:

**Theorem 2 (existence of contingent market equilibrium)** If Assumption 0 holds, 
then the two-period exchange economy \(E(u, w)\) has at least a contingent market equilib-
rium.

**Proof.** See Debreu (1959).

We also have:

**Theorem 3 (Pareto optimality of contingent market equilibrium)** Let \(E(u, w)\) be 
an economy satisfying Assumption 0. If \((\bar{x}, \bar{P})\) is a contingent market equilibrium, then 
the allocation \(\bar{x}\) is a Pareto optimum.

**Proof.** See Debreu (1959).

With reference to the property of Pareto optimality, in particular, it is due to the fact 
that all agents face a single budget constraint induced by a common vector of prices \(\bar{P}\); 
in fact (from the first-order conditions of the maximization problem), for an allocation 
to be Pareto optimal the agents' normalized gradients must be equalized, i.e.:

\[
P^1(\bar{x}^1) = P^2(\bar{x}^2) = ... = P^I(\bar{x}^I) = \bar{P}
\]

The equality of the agents' normalized gradients can also be expressed as the condition 
that the marginal rates of substitution of all agents between all pairs of goods are equal-
ized, and if this condition is not satisfied then it is possible to find a (small) reallocation 
of the goods between the agents which is preferred by all agents (i.e. the reallocation is 
a Pareto improvement).

In conclusion, the economy considered has all the "good properties" relative to the 
equilibrium. In fact, by denoting with \(E_C(w)\) the set of contingent market equilibrium
allocations corresponding to the parameter value $w$ (where the vector of endowments $w$ lies in the open set $\Omega = \mathbb{R}^n_{++}$, that represents the endowment space), it has the following properties:

- **Existence:** $E_C(w) \neq \emptyset$ for all $w \in \Omega$;
- **Pareto optimality:** $x \in E_C(w) \implies x$ is Pareto optimal for all $w \in \Omega$;
- **Comparative statics:** generically (i.e. for values of $w$ belonging to an open set of full measure in the parameter space $\Omega$) $E_C(w)$ is a finite set (therefore the equilibrium is locally unique) and each equilibrium is locally a smooth function of the parameter $w$.

### 1.2.2 Stochastic exchange economy

The model illustrated can then be extended to the case of an economy with many time periods, $t = 0, 1, ..., T$ (where $T$ is finite), by obtaining a stochastic exchange economy. In this case the uncertainty can be modelled with an event-tree: there is a finite set of states of nature $S = \{1, 2, ..., S\}$ and a collection of partitions of $S$ given by $F = (F_0, F_1, ..., F_T)$ where $F_0 = S$, $F_T = \{\{1\}, \{2\}, ..., \{S\}\}$ and $F_t$ is finer than $F_{t-1}$ for all $t = 1, 2, ..., T$ (this fact expresses the idea that the information is revealed gradually and increases over time). In this case $F$ defines an information structure because at each date $t = 0, 1, ..., T$ exactly one of the events $\sigma \in F_t$ has occurred and this is known to each agent in the economy. If $\sigma \in F_t$ has occurred, in particular, the possible events $\sigma' \in F_{t+1}$ that can occur at $t + 1$ are those satisfying $\sigma' \subset \sigma$. We can now define an event-tree in the following way: $D = \bigcup_{t=0}^T F_t$ is the set of all nodes, and for each node $\xi \in D$ there are exactly one $t$ and one $\sigma \in F_t$ such that $\xi = (t, \sigma)$. The unique node $\xi_0 = (0, \sigma)$ is called the *initial* node. For each $\xi \in D \setminus \xi_0$ (non-initial node) with $\xi = (t, \sigma)$ there is for $t - 1$ a unique $\sigma' \in F_{t-1}$ such that $\sigma' \supset \sigma$, and the node $\xi^- = (t - 1, \sigma')$ is called the *immediate predecessor* of $\xi$. We then have that $D^- = \bigcup_{t=0}^{T-1} F_t$ denotes the set of all non-terminal nodes, and for each $\xi \in D^-$ with $\xi = (t, \sigma)$ then $\xi^+ = \{\xi' = (t + 1, \sigma') | \sigma' \subset \sigma\}$ denotes
the set of *immediate successors* of $\xi$. The number of elements in the set $\xi^+$ is called the *branching number* of the node $\xi$ and is denoted with $b(\xi)$.

Given the setting described by the event-tree $D$ and the associated commodity space $C(D, \mathbb{R}^L)$ that consists of all functions $f : D \to \mathbb{R}^L$, the economy can be described as before. In particular, each consumer $i$ has a stochastic endowment process $w^i = (w^i(\xi), \xi \in D) \in C^+$ (the strictly positive orthant of $C$) and a utility function $u^i : C_+ \to \mathbb{R}$ satisfying Assumption 0 on the commodity space $C_+$. Given the information structure $F$, the associated stochastic exchange economy is denoted by $E(u, w, F)$. By defining a contingent price process $P \in C^+$ we then have that the *contingent market budget set* of agent $i$ is defined by:

$$B(P, w^i) = \{x^i \in C_+ \mid P(x^i - w^i) = 0\}$$

and a *contingent market equilibrium* for the stochastic economy $E(u, w, F)$ is given as before by Definition 1. Existence and Pareto optimality extend to the economy $E(u, w, F)$; we have in fact:

**Theorem 4 (existence of contingent market equilibrium)** If Assumption 0 holds (with $\mathbb{R}^n$ replaced by $C$), then the stochastic exchange economy $E(u, w, F)$ has at least a contingent market equilibrium.

We also have:

**Theorem 5 (Pareto optimality of contingent market equilibrium)** If Assumption 0 holds (with $\mathbb{R}^n$ replaced by $C$), then a contingent market equilibrium allocation $\bar{x}$ for the economy $E(u, w, F)$ is a Pareto optimum.

The conclusion is that, with a system of complete contingent markets, a stochastic exchange economy behaves exactly like a two-period exchange economy, and it exhibits all the "good properties" (existence, optimality and local uniqueness) of the equilibrium.
1.3 Existence of equilibrium: the model with spot-financial markets in finite horizon

The system of contingent markets analysed in the previous Section is principally of theoretical interest and it can be viewed as an ideal system, that however is far from the sequential structure of markets that is typical of actual decentralized market economies. For this reason it is now possible to consider the model developed to represent this sequential structure (whose first formulation is due to Radner (1972)): the objective is to arrive to a model of equilibrium that is able to capture the fact that economic activity in the real world is an ongoing process that has no natural terminal horizon and in which, at every date, agents trade on markets and make limited commitments regarding their future activities. In order to obtain this result it is necessary to introduce a system of spot and financial markets (instead of contingent markets) and to describe trading on these markets as a sequential process of equilibrium over time. Like in the idealized case of contingent markets, the point of departure is represented by a two-period exchange economy; the model can then be extended to a finite horizon and finally, by taking limits of such equilibria for progressively longer horizons, it is possible to construct a model over an open-ended future.

1.3.1 Two-period exchange economy

To model the sequential structure of markets that is present in actual market economies it is possible to introduce, in the simplest case of a two-period exchange economy, a collection of real spot markets for each of the \( L \) goods at date \( t = 0 \) and in each state \( s \) at date \( t = 1 \), together with a system of financial markets. The spot markets lead to a system of \( S+1 \) budget constraints, while the financial markets provide instruments (bonds and equities, options, futures and insurance contracts, contracts between firms, between employees and firms and so on) that enable each agent to redistribute income across the states. In this case \( p = (p_0, p_1, ..., p_S) \in \mathbb{R}^n_+ \) represents the vector of spot prices, where
$p_s = (p_{s1}, p_{s2}, ..., p_{sL})$, and $p_{sl}$ denotes the price, measured in units of account, payable in state $s$ for one unit of good $l$. The essential distinction between a spot market in state $s$ and a contingent market for state $s$ is that in the former the payment is made at date 1 in state $s$ (if $s \geq 1$), while in the latter it is always made at date 0. It is this property that leads to the system of $S + 1$ budget constraints in the case of a system of spot markets and to a single budget constraint in the case of a system of contingent markets.

The financial assets considered in the economy can be of three basic types (or a combination of them): real assets (such as the equity of firms or futures contracts on real goods), nominal assets (such as bonds or financial futures) and secondary or derivative assets (such as call and put options). In the analysis that follows we will always assume that the economy is characterized by the presence of real assets (that give a dividend represented by a vector of the $L$ goods, therefore this dividend is proportional to the prices in any state, and if these prices change the dividend income that these assets generate changes in the same proportion), while we do not deal with the case in which there are nominal assets (that give a dividend represented by a certain amount of units of account, therefore in this case if prices change the dividend income that the assets generate remains unchanged and the purchasing power of the asset’s return changes; for this reason in the case of nominal assets we can have indeterminacy of equilibria - for a summary of these problems see, for instance, Magill-Shafer (1991) -). A real asset $j$ is a contract which promises to deliver a (column) vector of the $L$ goods:

$$A^j_s = (A^j_{s1}, A^j_{s2}, ..., A^j_{sL}) \in \mathbb{R}^L$$  

in each state $s$ at date 1. A real asset is therefore characterized by a date-1 commodity vector $A^j = (A^j_1, A^j_2, ..., A^j_s) \in \mathbb{R}^{LS}$ and the revenue (measured in units of account) that it gives in state $s$ is proportional to the spot price $p_s$:

$$V^j_s = p_s \cdot A^j_s$$  

$s = 1, 2, ..., S$
We assume that there are $J$ real assets in the economy, where the asset $j$ can be purchased at price $q_j$ (expressed in units of account) at date 0 and gives a random return $V^j = (V^j_1, V^j_2, ..., V^j_S)$ across the states at date 1. The (column) vectors $V^j$ can be combined to form the date-1 matrix of returns (of dimension $S \times J$):

$$V = [V^1 \ldots V^J] = \begin{bmatrix}
V^1_1 & \cdots & V^1_J \\
\vdots & \ddots & \vdots \\
V^S_1 & \cdots & V^S_J
\end{bmatrix}$$

that can also be expressed in the form:

$$V = V(p_1, A) = \begin{bmatrix}
p_1 A^1_1 & \cdots & p_1 A^1_J \\
\vdots & \ddots & \vdots \\
p_S A^1_1 & \cdots & p_S A^1_J
\end{bmatrix}$$

where $p_1 = (p_1, p_2, ..., p_S) \in \mathbb{R}^S$ is the date-1 vector of spot prices; if we then consider $p_s = (p_{s1}, p_{s2}, ..., p_{sL})$ the matrix $V$ can also be written as:

$$V(p_1, A) = \begin{bmatrix}
p_1 & 0 & \cdots & 0 \\
0 & p_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & p_S
\end{bmatrix} \begin{bmatrix}
A^1_1 & \cdots & A^1_J \\
\vdots & \ddots & \vdots \\
A^S_1 & \cdots & A^S_J
\end{bmatrix}$$

Since real assets are inflation-proof (in the sense that, for instance, doubling the spot prices in state $s$ doubles their income, therefore in an economy with only real assets price levels are unimportant) we can summarize the real asset structure with the matrix:
of dimension $LS \times J$ whose columns are the column vectors $A^j \in \mathbb{R}^{LS}$, $j = 1, 2, ..., J$. The exchange economy consisting of $I$ agents with characteristics $(u, w) = (u^1, ..., u^I, w^1, ..., w^I)$ who trade $J$ securities with date-1 payoffs given by the matrix $V$ and with real asset structure given by the matrix $A$ is denoted by $\mathcal{E}(u, w, A)$.

The matrix $V$ generates the subspace of income transfers (or market subspace) $\langle V \rangle$, i.e. the subspace of $\mathbb{R}^S$ spanned by the $J$ columns of $V$:

$$\langle V \rangle = \{\tau \in \mathbb{R}^S | \tau = Vz, z \in \mathbb{R}^J\}$$

and we have the following definition:

**Definition 6** If the subspace of income transfers satisfies $\langle V \rangle = \mathbb{R}^S$ (i.e. if this subspace has maximal dimension), then the asset structure is called complete (i.e. we have complete financial markets), otherwise the asset structure is called incomplete (i.e. we have incomplete financial markets).

This definition is fundamental for all the analysis that will be carried out in the following pages. Since $z$ represents a portfolio of the $J$ assets, the fact that $\langle V \rangle = \mathbb{R}^S$ means that every state-dependent income $\tau \in \mathbb{R}^S$ can be obtained by means of an appropriate portfolio in the economy. In this sense financial markets are complete, i.e. it is possible to construct portfolios of the $J$ assets that allow to obtain every possible state-dependent income. When such possibility does not exist (and this happens if $\langle V \rangle \neq \mathbb{R}^S$), on the contrary, financial markets are incomplete. In the reality markets are not complete, and for this reason the analysis of these Chapters will deal principally with the situation of incomplete markets.
The completeness of the asset structure requires \( \dim(V) = S \), therefore in this case the matrix \( V \) has full rank, and we must have \( J \geq S \) (with \( S \) assets that have linearly independent payoffs). As a consequence, whenever the number of the assets is less than the number of the states of nature \( (J < S) \) markets are incomplete. On the other hand, it must be observed that the definition of complete markets is relative to the system of spot prices \( p_1 = (p_1, p_2, \ldots, p_S) \). As a consequence, when \( J \geq S \) (and \( S \) assets have linearly independent payoffs, so that \( \text{rank} V = S \)) we cannot say unambiguously that markets are complete; what can be proved, anyway, is that markets are generically (or potentially) complete, i.e. for systems of spot prices belonging to an open set of full measure the asset structure is complete. This will be used in the following pages, where the issue of existence of equilibrium for an economy of this kind is considered.

To introduce this concept, at this point, we must observe that if \( z^i = (z^i_1, z^i_2, \ldots, z^i_J) \in \mathbb{R}^J \) denotes the number of units of each of the \( J \) assets purchased by agent \( i \) (where \( z^i_j < 0 \) represents a short-sale of the asset), then the \( S + 1 \) budget constraints can be written as:

\[
\begin{cases}
p_0(x_0^i - w_0^i) = -q^i \\
p_s(x_s^i - w_s^i) = V_s z^i & s = 1, 2, \ldots, S
\end{cases}
\]  

(1.1)

where \( q = (q_1, q_2, \ldots, q_J) \) and \( V_s = (V^1_s, V^2_s, \ldots, V^J_s) \) is the row \( s \) of the matrix \( V \). By defining the full matrix of returns (i.e. date-0 and date-1 returns) as:

\[
W = W(q, V) = \begin{bmatrix} -q \\ V \end{bmatrix} = \begin{bmatrix} -q_1 & \cdots & -q_J \\ V^1_1 & \cdots & V^1_J \\ \vdots & \ddots & \vdots \\ V^1_S & \cdots & V^J_S \end{bmatrix}
\]

and for \( p \in \mathbb{R}^n, x^i \in \mathbb{R}^n \) (where \( n = L(S + 1) \)) the box product:

\[
p \Box x^i = (p_0 x^i_0, p_1 x^i_1, \ldots, p_S x^i_S)
\]
we then have that the spot-financial market budget set of agent $i$ is given by:

$$B(p, q, w^i) = \{ x^i \in \mathbb{R}^+_+ \mid p \square (x^i - w^i) = W(q, V)z^i, z^i \in \mathbb{R}^J \}$$

and in this case the agent chooses a pair $(x^i, z^i)$ consisting of a vector of consumption $x^i$ and a portfolio $z^i$. It is now possible to introduce the concept of a competitive equilibrium for the $I$ agents trading on this system of $J$ financial markets, we have in fact:

**Definition 7** A spot-financial market equilibrium for the economy $E(u, w, A)$ is a pair of allocations and prices $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))$ such that:

1. $(\bar{x}^i, \bar{z}^i)$ satisfy
   $$\bar{x}^i = \arg \max \{ u^i(x^i) \mid x^i \in B(\bar{p}, \bar{q}, w^i) \} \text{ and } \bar{p} \square (\bar{x}^i - w^i) = W(q, V)\bar{z}^i$$
   $i = 1, 2, ..., I$
2. $\sum_{i=1}^I (\bar{x}^i - w^i) = 0$
3. $\sum_{i=1}^I \bar{z}^i = 0$

Again, this means that at the equilibrium individuals solve their maximization problems and markets clear, i.e. demand equals supply (in particular, here it is assumed that the total net supply of assets in the economy is zero, and for this reason the market-clearing condition on financial markets is expressed by (iii). In the second Chapter the analysis will be extended to the case of assets in positive net supply).

The concept of spot-financial market equilibrium encounters difficulties in establishing the existence of equilibrium in the case in which the economy has more than two periods. For this reason it can be replaced by an equivalent concept, analytically simpler to work with, based on the idea of absence of arbitrage opportunities. With reference to this aspect, $q \in \mathbb{R}^J$ is said to be a no-arbitrage asset price if there does not exist a portfolio $z \in \mathbb{R}^J$ such that $W(q, V)z \geq 0$ (i.e. each of the $S + 1$ components of the vector $Wz$ is $\geq 0$ and at least one component is $> 0$, this means that it does not exist an investment strategy which gives a positive payoff in at least one state with non-negative payoffs in all remaining states). From the theorem of absence of arbitrage on financial markets (that constitutes the fundamental theorem on the pricing of financial securities), the
agent $i$ utility maximization problem in Definition 7 has a solution if and only if $q$ is a no-arbitrage asset price, and the absence of arbitrage opportunities in the trading of the financial assets implies the existence of a present value vector $\pi = (\pi_0, \pi_1, ..., \pi_S)$ (where the components are positive state prices, that represent the present value at date 0 of one unit of income in state $s$) such that $\pi W = 0$ (this is a well-known result, and it is a consequence of the Minkowski-Farkas lemma - see, for instance, Duffie (1988)-). This is equivalent to:

$$\pi_0 q_j = \sum_{s=1}^{S} \pi_s V_s^j \quad j = 1, 2, ..., J$$

and therefore the price of each asset equals the present value of its future income stream. From the budget constraints (1.1) the equation relative to date 0 becomes now:

$$\pi_0 p_0 (x_0^i - w_0^i) = -\pi_0 q z_i = -\sum_{s=1}^{S} \pi_s V_s z_i = -\sum_{s=1}^{S} \pi_s p_s (x_s^i - w_s^i)$$

that is:

$$\sum_{s=0}^{S} \pi_s p_s (x_s^i - w_s^i) = 0$$

and if we define the new vector of date-0 present value prices:

$$P = \pi \Box \pi$$

then the budget equation relative to date 0 reduces to:

$$P(x^i - w^i) = 0$$

i.e. to the budget constraint of the Arrow-Debreu model with contingent markets. We
then have that the budget equations relative to date 1 can be written in the form:

$$P_1 \Box (x_1^i - w_1^i) \in \langle V(P_1) \rangle$$

where $P_1 = (P_1, P_2, ..., P_S)$ is the vector of present value prices for date 1 (for details see, for instance, Magill-Shafer (1991)). As a consequence, each agent maximizes utility over the budget set:

$$P_1 \Delta (x_1^i - w_1^i) = 0$$

that is a constrained Arrow-Debreu budget set under the price system $P$. It is now necessary to consider equilibria in which the subspace of income transfers $\langle V \rangle$ is of fixed dimension $\rho$ with $0 \leq \rho \leq S$. If $G^\rho(\mathbb{R}^S)$ is the set of all linear subspaces of $\mathbb{R}^S$ of dimension $\rho$ and $\mathcal{L} \in G^\rho(\mathbb{R}^S)$ is a $\rho$-dimensional subspace of $\mathbb{R}^S$, replacing the actual subspace of income transfers $\langle V(P_1) \rangle$ by the subspace $\mathcal{L}$ the budget set becomes:

$$\mathcal{B}(P, w^i, \mathcal{L}) = \left\{ x^i \in \mathbb{R}_+^n \left| \begin{array}{l} P(x^i - w^i) = 0 \\ P_1 \Box (x_1^i - w_1^i) \in \langle V(P_1) \rangle \end{array} \right. \right\}$$

With this definition it is possible to introduce an alternative concept of equilibrium, the no-arbitrage equilibrium (that is a constrained Arrow-Debreu equilibrium); the following definition holds:

**Definition 8** A normalized no-arbitrage equilibrium of rank $\rho$ with $0 \leq \rho \leq S$ for the economy $E(u, w, A)$ is a pair $(\bar{x}, \bar{P}, \bar{\mathcal{L}})$ such that:

(i) $\bar{x}^1 = \arg \max \{ u^1(x^1) \mid x^1 \in B(\bar{P}, w^1) \}$

(ii) $\sum_{i=1}^I (\bar{x}^i - w^i) = 0$

(iii) $\langle V(\bar{P}_1) \rangle = \bar{\mathcal{L}}$

and then the following theorem (that asserts the equivalence between spot-financial
market equilibria and no-arbitrage equilibria) holds:

**Theorem 9** (i) If \(((x, z), (p, q))\) is a spot-financial market equilibrium of rank \(\rho\) (that is with rank \(V(\overline{P}_1) = \rho\)), then there exists a \(\rho\)-dimensional subspace \(\mathcal{C} \subseteq \mathcal{G}^\rho(\mathbb{R}^S)\) and a no-arbitrage vector \(\pi \in \mathbb{R}^{S+1}\) such that \((x, \pi \overline{P}, \mathcal{C})\) is a no-arbitrage equilibrium of rank \(\rho\).

(ii) If \((x, P, \mathcal{C})\) is a no-arbitrage equilibrium of rank \(\rho\), then there exist portfolios \(\overline{z} = (z^1, z^2, ..., z^\rho)\) and an asset price \(\overline{q}\) such that \(((x, \overline{z}), (P, \overline{q}))\) is a spot-financial market equilibrium of rank \(\rho\).

**Proof.** See Magill-Shafer (1985).

The importance of this theorem is due to the fact that no-arbitrage equilibria are analytically easier to handle than spot-financial market equilibria, and hence can be used to obtain the existence results in this model.

As it has been shown in the previous Section, for all characteristics \((u, w)\) that satisfy Assumption 0 the exchange economy \(\mathcal{E}(u, w)\) has a contingent market equilibrium. This is not necessarily true when we consider an economy with spot and financial markets, where the possibility of non-existence of equilibrium arises. This is due to the fact that the rank of the return matrix may change when the prices \((p, q)\) vary, and changes in the rank of this matrix may create discontinuities in the demand of the agents, that may in turn determine non-existence of equilibrium (the first to show this possibility was Hart (1975)). This is a central problem for this kind of models, and for this reason the results obtained in this framework are weaker than those obtained in an economy with contingent markets. The next pages illustrate these results, by distinguishing between the situation of generically complete markets and the situation of incomplete markets. It is worth observing that the problem of changes in the rank of the return matrix doesn't arise (apart from the situation of complete markets) when the only assets in the economy are short-lived numeraire assets (as it will be clear in Section 1.4, where these assets will be used in the infinite-horizon version of the model).
Generically complete markets

To determine the main results concerning equilibria in a two-period exchange economy with spot and financial markets we consider the exchange economy $\mathcal{E}(u, w, A)$ with financial structure $A$ and we fix the profile of utility functions $u = (u_1, u_2, ..., u^I)$ with each $u^i$ satisfying Assumption 0 and the asset structure $A \in \mathbb{R}^{LSJ}$. By letting the vector of endowments $w = (w^1, w^2, ..., w^I)$ lie in the open set $\Omega = \mathbb{R}^{nL} \_+$ (the endowment space), then, we have a parametrized family of economies $\{\mathcal{E}_A(w), w \in \Omega\}$; we now say that a property holds generically if it is true on an open set of full measure in the parameter space $\Omega$, and we introduce the following definition:

**Definition 10** Let $E_A(w)$ denote the set of spot-financial market equilibrium allocations (i.e. the vector of consumption bundles $x = (x^1, x^2, ..., x^I)$ for each spot-financial market equilibrium) for the economy $\mathcal{E}_A(w)$, and let $E_C(w)$ denote the set of contingent market equilibrium allocations for the parameter value $w$.

The properties of the set $E_C(w)$ are well known from the classical general equilibrium theory (Debreu (1959), Debreu (1970)); as it has been shown in the previous Section, they are:

- **(P1)** Existence: $E_C(w) \neq \emptyset$ for all $w \in \Omega$;
- **(P2)** Pareto optimality: $x \in E_C(w) \implies x$ is Pareto optimal for all $w \in \Omega$;
- **(P3)** Comparative statics: generically $E_C(w)$ is a finite set (therefore the equilibrium is locally unique) and each equilibrium is locally a smooth function of the parameter $w$.

The idea is now to relate these properties to the properties of the set $E_A(w)$ in order to determine results concerning equilibria in a spot-financial market economy. The techniques used to solve this problem (for instance the notion of no-arbitrage equilibrium introduced before) and the results presented here have been obtained by Magill and
Shafer (1985); these results have also been extended to the case of a stochastic exchange economy, discussed later. First of all the following definition can be introduced:

**Definition 11** The real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular if for each state of nature \( s = 1, 2, ..., S \) a row \( \tilde{a}_s \) can be selected from the \( L \times J \) matrix \( A_s = [A^1_s, A^2_s, ..., A^J_s] \) such that the collection \( (\tilde{a}_s)^S_{s=1} \) is linearly independent.

This requires \( J \geq S \), and implies the situation of generically (or potentially) complete markets (i.e. for a generic set of spot prices markets are complete). With reference to this situation several results can be proved; a first statement is the following:

**Theorem 12** If the real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular, then there exists a generic set \( \Omega' \subset \Omega \) such that

\[
E_C(w) \subset E_A(w) \quad \forall w \in \Omega'
\]

**Proof.** See Magill-Shafer (1985).  

As a consequence, using the property (P1) of the set \( E_C(w) \) we have:

**Theorem 13** (generic existence) If the real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular, then there exists a generic set \( \Omega' \subset \Omega \) such that

\[
E_A(w) \neq \emptyset \quad \forall w \in \Omega'
\]

**Proof.** See Magill-Shafer (1985).

The first theorem, together with the property (P2) of the set \( E_C(w) \), also implies that whenever \( w \in \Omega' \) there is at least one allocation \( x \in E_A(w) \) which is Pareto optimal. In the case of a spot-financial market economy there can exist equilibria that are inefficient, as pointed out by Hart (1975); in any case, equilibria of this type are exceptional, because the further following result holds:
Theorem 14 If the real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular, then there exists a generic set \( \Omega'' \subset \Omega \) such that

\[
E_A(w) \subset E_C(w) \quad \forall w \in \Omega''
\]


By using property \((P2)\) of the set \( E_C(w) \) we now obtain:

Theorem 15 (Pareto optimality) If the real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular, then there exists a generic set \( \Omega'' \subset \Omega \) such that \( x \in E_A(w) \) implies that \( x \) is Pareto optimal for all \( w \in \Omega'' \).


At this point, combining these theorems and defining \( \Omega^* = \Omega' \cap \Omega'' \) we have:

Theorem 16 (equivalence under regularity) If the real asset structure \( A \in \mathbb{R}^{LSJ} \) is regular, then there exists a generic set \( \Omega^* \subset \Omega \) such that

\[
E_A(w) = E_C(w) \quad \forall w \in \Omega^*
\]

Viceversa, if there exists a generic set \( \Omega^* \) such that \( E_A(w) = E_C(w) \) for all \( w \in \Omega^* \), then the asset structure \( A \) is regular.


As a consequence, regularity of the asset structure is a necessary and sufficient condition for the generic equivalence between the contingent market equilibrium and the spot-financial market equilibrium. This last theorem, combined with the property \((P3)\) of the set \( E_C(w) \), leads to the following further result:

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Theorem 17 (comparative statics) If the asset structure $A \in \mathbb{R}^{LSJ}$ is regular, then generically $E_A(w) \neq \emptyset$ is a finite set and each equilibrium is locally a $C^1$ function of the parameter $w$.


In conclusion, when the property of regularity of the asset structure (that implies generic completeness of markets) holds, the results of the classical general equilibrium theory (existence, optimality and local unicity of the equilibrium) extend generically (i.e. they are valid for values of the parameter $w$ belonging to an open set of full measure) to an economy with spot-financial markets.

A further important result can be obtained by considering the set of regular asset structures $\mathcal{A}_R \subset \mathbb{R}^{LSJ}$, we have in fact:

Theorem 18 (invariance of financial structure) Let $A \in \mathcal{A}_R$, then there exists a generic set $\Omega_A \subset \Omega$ such that for $w \in \Omega_A$

$$E_A'(w) = E_A(w) \quad \text{for almost all } A' \in \mathcal{A}_R$$

Furthermore $E_A(w) = E_{A+dA}(w)$ for all local changes $dA \in \mathbb{R}^{LSJ}$.


This means that under the regularity condition the equilibrium allocations of the spot-financial market model are invariant with respect to changes in the return structure of the financial assets. This is no longer true when markets are incomplete, as it will be clear in the following pages.

Incomplete markets

When $J < S$, the asset markets are incomplete; in this situation the return matrix may drop rank when prices $(p, q)$ vary, causing problems of non-existence of equilibrium. The
technique used to solve this difficulty consists in establishing the generic existence of the equilibrium: first of all the notion of pseudo-equilibrium (that is a slight weakening of the concept of no-arbitrage equilibrium) is introduced, and it is shown that a pseudo-equilibrium exists for all economies; then it is shown that for a generic set of economies the pseudo-equilibrium is also an equilibrium. While before the property of genericity has been defined with respect to the space of endowments $\Omega = \mathbb{R}^{n+}$, the parameter space is now augmented by adding the space of real asset structures $A = \mathbb{R}^{LSJ}$, and genericity is defined with respect to the parameters $(w, A) \in \Omega \times A$. We then denote with $E(w, A) = E_A(w)$ the set of spot-financial market equilibrium allocations for the economy $E(w, A) = E_A(w)$, and a first property that holds (the results presented here have been obtained by Duffie and Shafer (1985)) is the following:

**Theorem 19 (generic existence)** Let $E(u, w, A)$ be a spot-financial market exchange economy satisfying Assumption 0. If $J < S$, then there exists a generic set $\Delta' \subset \Omega \times A$ such that

$$E(w, A) \neq \emptyset \quad \forall (w, A) \in \Delta'$$

**Proof.** See Duffie-Shafer (1985). □

**Remark 20** To prove this theorem it is possible to introduce the concept of pseudo-equilibrium, defined as a normalized no-arbitrage equilibrium of maximal rank in which the condition (iii) is slightly weakened. The core of the proof consists in showing that a pseudo-equilibrium exists for all parameter values $(w, A)$; once this is established, by using a transversality argument it is possible to show that there is a generic subset of the parameters such that for all the economies in this subset every pseudo-equilibrium is a no-arbitrage equilibrium.

Another central result is the following, according to which, when markets are incomplete, equilibrium allocations are generically Pareto inefficient, and in addition agents have distinct normalized present value vectors:
Theorem 21 (Pareto inefficiency) If $J < S$, then there exists a generic set $\Delta'' \subset \Omega \times \mathcal{A}$ such that $x \in E(w, A)$ implies that $x$ is Pareto inefficient, for all $(w, A) \in \Delta''$. Furthermore the present value coefficients of the agents

$$\pi^i \in \mathbb{R}^{S+1} \quad i = 1, 2, \ldots, I$$

are distinct for each $x \in E(w, A), \forall (w, A) \in \Delta''$.


This is a fundamental difference with respect to the situation we have when markets are complete. In the case of complete markets, in fact, all agents' present value vectors coincide, i.e. $\pi^1 = \pi^2 = \ldots = \pi^I = \pi$ and there is complete agreement about the present value of a stream of date-1 income, and this property leads to the Pareto optimality of the equilibrium. When asset markets are incomplete, on the contrary, there is disagreement about the present value of a stream of date-1 income and this determines the inefficiency of the equilibrium.

With reference to the relationship between contingent market equilibrium and spot-financial market equilibrium in the case of incomplete markets, by defining the set $\Delta^* = \Delta' \cap \Delta''$ we have the following result:

Theorem 22 If $J < S$, then there exists a generic set $\Delta^* \subset \Omega \times \mathcal{A}$ such that

$$E(w, A) \cap E_C(w) = \emptyset \quad \forall (w, A) \in \Delta^*$$


As a consequence of this theorem, with incomplete markets we lose the equivalence between contingent market equilibrium and spot-financial market equilibrium. Finally, it has been shown before that in the case of generically complete markets changes in the asset structure don't alter the equilibrium allocations, and therefore with complete
markets financial changes have no real effects. This property of invariance with respect to the financial structure is no longer true when markets are incomplete; we have in fact:

**Theorem 23 (real effects of financial assets)** If \( J < S \), then there exists a generic set \( \Delta^* \subset \Omega \times \mathcal{A} \) such that for all \( (w,A) \in \Delta^* \)

\[
E(w,A) \cap E(w,A + dA) = \emptyset
\]

for almost all local changes \( dA \in \mathbb{R}^{LSJ} \).

**Proof.** See Duffie-Shafer (1985).

In this case, changes in the structure of financial assets in general alter the equilibrium allocations, and therefore with incomplete markets financial changes have real effects.

The conclusion of this analysis is that, in the two-period exchange economy with spot and financial markets, when markets are generically complete all the properties of the classical general equilibrium theory extend generically, while when markets are incomplete only generic existence is preserved.

### 1.3.2 Stochastic exchange economy

Like for the case of an economy with contingent markets, also for an economy with spot-financial markets the two-period framework can be extended to many time periods, \( t = 0,1,\ldots,T \) (with \( T \) finite). The final step, then, will be the analysis of the situation that arises when \( T \) is arbitrarily large, i.e. the analysis over an open-ended horizon.

As it has been shown above, in the case of a stochastic exchange economy the uncertainty can be modelled by means of an event-tree (denoted by \( D \)). To simplify the analysis we assume that there are \( J \) assets all initially issued at date 0 (the results obtained can however be extended to the case where assets are introduced at subsequent nodes \( \xi \neq \xi_0 \)). The real asset \( j \) is characterized by a function \( A^j : D \to \mathbb{R}^L \), and one unit of the asset held at \( \xi_0 \) promises to deliver the commodity vector \( A^j(\xi) \) at node \( \xi \). Assets
are retracted at all later dates, so that one unit of asset $j$ purchased at node $\xi$ promises the delivery of $A^j(\xi')$ for all $\xi' > \xi$. We then have that $A = (A^1, A^2, ..., A^J)$ denotes the asset structure and $A$ denotes the set of all asset structures. If $A(\xi) = [A^1(\xi), A^2(\xi), ..., A^J(\xi)]$, $\xi \in \mathbf{D}$ and $p \in C_+$ is a stochastic spot price process then:

$$V^j(\xi) = p(\xi)A^j(\xi) \quad \xi \in \mathbf{D}$$

is the dividend (in units of account) paid by asset $j$ ($j = 1, 2, ..., J$) at node $\xi$. A security price process is a function $q : \mathbf{D} \rightarrow \mathbb{R}^J$ with $q(\xi) = 0$ for $\xi \notin \mathbf{D}^-$ (the terminal value condition, according to which at the terminal node the price of the asset is 0) and $q(\xi)$ is the vector of prices of the $J$ assets at node $\xi$. Finally, the trading strategy of agent $i$ is a function $z^i : \mathbf{D} \rightarrow \mathbb{R}^J$ with $z^i(\xi) = 0$ for $\xi \notin \mathbf{D}^-$ and $z^i(\xi)$ is the portfolio of the $J$ assets purchased by agent $i$ at node $\xi$ after the previous portfolio has been liquidated.

Given these elements, agent $i$'s budget constraint is:

$$(\mathcal{B}) \begin{cases} p(\xi_0)(x^i(\xi_0) - w^i(\xi_0)) = -q(\xi_0)z^i(\xi_0) \\ p(\xi)(x^i(\xi) - w^i(\xi)) = [p(\xi)A(\xi) + q(\xi)]z^i(\xi^-) - q(\xi)z^i(\xi), \forall \xi \in \mathbf{D} \setminus \xi_0 \end{cases}$$

where it is assumed that the portfolio purchased at any node is sold at each immediate successor of the node and that a new portfolio is then purchased. The typical payoff obtained on the investment $z^i_j(\xi^-)$ in security $j$ at the predecessor $\xi^-$ of the node $\xi$ consists of two parts, the dividend $p_j(\xi)A^j(\xi)z^i_j(\xi^-)$ and the capital value $q_j(\xi)z^i_j(\xi^-)$, where the latter derives from the sale of the portfolio at node $\xi$. The capital value is the new term introduced by extending the model to the multiperiod case (it is absent at the initial and terminal dates, which are the only dates that appear in the two-period model) and it is the element that accounts for many of the differences between the behavior of the multiperiod model and of the two-period model. Given all these elements, if we denote the stochastic exchange economy considered by $\mathcal{E}(u, w, A)$, the following definition of equilibrium holds:
Definition 24  A spot-financial market equilibrium for the stochastic exchange economy $\mathcal{E}(u, w, A)$ is a pair of allocations and prices $((\overline{x}, \underline{x}), (\overline{p}, \underline{q}))$ such that:

(i) $(\overline{x}^i, \underline{x}^i)$ solves the maximization problem of agent $i$ subject to the budget constraint $(\mathcal{B})$, $i = 1, 2, ..., I$

(ii) $\sum_{i=1}^{I}(\overline{x}^i - w^i) = 0$

(iii) $\sum_{i=1}^{I} \underline{x}^i = 0$

Like in the two-period case, the asset price process $q$ in a spot-financial market equilibrium satisfies a no-arbitrage condition (this means that, given the asset structure $A$ and a spot price process $p$, there isn't a trading strategy generating a non-negative return at all nodes and a positive return for at least one node). As in the two-period economy, then, $q$ satisfies the no-arbitrage condition if and only if there exists a stochastic present value process:

$$\pi : D \rightarrow \mathbb{R}^+$$

whose components are positive state prices such that:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi') [p(\xi')A(\xi') + q(\xi')] \quad \forall \xi \in D^-$$

and therefore the present value (the value at date 0) of the asset prices at node $\xi$ is the present value of their dividends and capital values over the set of immediate successors $\xi^+$. Solving this system of equations recursively over the nodes and using the terminal condition $q(\xi') = 0, \forall \xi' \notin D^-$ leads to the equivalent expression:

$$q(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \quad \forall \xi \in D^-$$

according to which the current value of each asset at node $\xi$ is the present value of its future dividend stream over all succeeding nodes $\xi' > \xi$. This is the so-called fundamental value of an asset and, as it will be clear in the next Chapter, when the analysis is extended
to an infinite horizon it may happen that the equilibrium price of the asset is larger than its fundamental value; if this is the case, the difference represents a speculative bubble.

Also in this case, as for a two-period economy, it is possible to introduce the concept of normalized no-arbitrage equilibrium and to prove the equivalence between a spot-financial market equilibrium of rank \( p \) and a no-arbitrage equilibrium of rank \( p \) (for details see, for instance, Magill-Shafer (1991)). Again, the concept of no-arbitrage equilibrium is easier to handle and can be used to prove the generic existence of equilibrium; with reference to this aspect, also in this case we can distinguish between the situation of generically complete markets and the situation of incomplete markets.

**Generically complete markets**

In a two-period economy there are generically (or potentially) complete markets if \( J \geq S \), that is when the number of assets is sufficient to cover all the possible contingencies; in the stochastic economy the corresponding condition is \( J \geq b(\xi) \), and therefore what matters is the amount of information revealed at each node \( \xi \) measured by \( b(\xi) \). Also in this case, the notion that enables to prove the (generic) existence of the equilibrium is that of regularity of the asset structure. With reference to this aspect the following definition holds:

**Definition 25** The asset structure \( A \) in a stochastic economy is regular if for each node \( \xi \in D^- \) and each immediate successor \( \xi' \in \xi^+ \) one can choose a \( J \)-vector \( \bar{a}(\xi') \) from the rows of the collection of matrices \( (A(\xi''))_{\xi'' \geq \xi'} \) such that the collection of induced vectors over the immediate successors \( (\bar{a}(\xi'))_{\xi' \in \xi^+} \) is linearly independent.

This requires \( J \geq b(\xi) \), and therefore implies the situation of generically complete markets.

If, as before, we denote with \( E_A(w) \) the set of spot-financial market equilibrium allocations for the stochastic economy \( E_A(w) \) and with \( E_C(w) \) the set of contingent market equilibrium allocations for the parameter value \( w \), the following result can be proved:
Theorem 26 There is a generic subset $\Omega^* \subset \Omega$ such that

$$E_A(w) = E_C(w) \quad \forall w \in \Omega^*$$

if and only if the asset structure $A$ is regular.


Also in the case of a stochastic economy, therefore, it is the regularity of the asset structure that guarantees the (generic) existence of equilibrium. In addition, with generically complete markets the spot-financial market equilibria are Pareto optimal (and, again, the results of the classical general equilibrium theory extend generically to an economy with spot-financial markets).

Incomplete markets

If for some node $\xi \in D^-$ in the economy we have $J < b(\xi)$, asset markets are incomplete, and at some node agents have limited ability to redistribute their income over the immediately succeeding nodes. Again, the return matrix may drop rank, causing problems of non-existence of equilibrium, and the technique used to prove existence in this situation is the same introduced in the case of the two-period exchange economy: first it is possible to prove the existence of a pseudo-equilibrium, then to use a transversality argument in order to show that there is a generic subset of the parameter space $\Omega \times A$ for which a pseudo-equilibrium is a no-arbitrage equilibrium (for details see, for instance, Magill-Shafer (1991)). If, for a fixed information structure $F$, we denote with $E(w, A)$ the set of spot-financial market equilibrium allocations of the stochastic economy with parameters $(w, A)$ we have in fact the following result:

Theorem 27 (generic existence) If $J < b(\xi)$ for some $\xi \in D^-$, then there exists a generic set $\Delta \subset \Omega \times A$ such that $E(w, A)$ consists of a positive finite number of equilibria for each $(w, A) \in \Delta$. 31
Proof. See Duffie-Shafer (1986).

In addition, in the case of incomplete markets the equilibrium allocations are Pareto inefficient, also in the restricted sense of constrained inefficiency, i.e. they are not the best allocations that can be achieved given the restricted set of financial instruments that are available (for details on this point see Geanakoplos-Polemarchakis (1986)). The conclusion is that a stochastic exchange economy with spot-financial markets behaves exactly as a two-period exchange economy: when markets are generically complete all the properties of classical general equilibrium theory extend generically, when markets are incomplete only generic existence remains true.

This concludes the analysis of existence of the equilibrium in an exchange economy with spot-financial markets in the case of a finite horizon. Since economic activity is a process that has no natural terminal horizon, however, a realistic model of equilibrium should be able to capture this aspect, and for this reason it is possible to consider a model of equilibrium over an open-ended future. This is the object of the next Section.

1.4 Existence of equilibrium: the model with spot-financial markets in infinite horizon

In order to extend the analysis considered in the previous Section over an infinite horizon there are two possibilities: the first is to assume that the economy has a finite number of agents who are infinitely lived (infinitely lived agents models), the second is to assume that all agents are finitely lived and are succeeded by their children in an indefinite sequence of overlapping generations (overlapping generations models). The extension developed until now (for details, see Magill-Quinzii (1994, 1996)) is based on the first approach, i.e. it considers an exchange economy with a finite number of infinitely lived agents who use spot markets for the exchange of goods and a (limited) system of financial markets to redistribute their income across time periods and uncertain events.
The consideration of a sequence of markets over an infinite horizon introduces a new problem that does not appear in the case of finite horizon, and that is represented by the fact that, if agents are permitted to borrow, they may try to postpone indefinitely the repayment of their debts from one period to the next (the so-called Ponzi scheme). If this happens, there is no solution to an agent's decision problem, and hence an equilibrium cannot exist. To avoid this problem a first possibility is to impose a debt constraint (which determines a uniform bound on debt at all dates), while another possibility is to use a transversality condition (which requires that debt grow asymptotically not too fast) and this leads to the concepts of equilibrium with debt constraint and of equilibrium with transversality condition respectively.

The concept of equilibrium with transversality condition is particularly important from an abstract point of view because it permits to use the techniques and concepts of Arrow-Debreu theory in the more general setting of incomplete markets for sequence economies (in particular it leads to a proof of existence of equilibrium that uses the construction introduced by Bewley (1972)). This equilibrium, then, turns out to coincide with the equilibrium with debt constraint, and therefore it can be used to prove the existence of such an equilibrium, which is more natural from an economic point of view.

In addition, if we consider an infinite-horizon economy with a general asset structure, a further problem is represented by the possibility of non-existence of the equilibrium as a consequence of discontinuities in agents' demand created by changes in the rank of the return matrix when prices vary (a problem already discussed for the finite-horizon version of the model). To avoid such difficulties and focus on the problems created by the presence of infinite horizon, the first part of this Section considers an economy in which there are only short-lived numeraire securities (for which the problem of changes in the rank of the return matrix doesn't arise); the more complex case of a general asset structure is treated in the second part of the Section.
1.4.1 Characteristics of the economy and assumptions

To describe time, uncertainty and the revelation of information over an infinite horizon the same event-tree structure considered in the previous Sections (for a stochastic economy) can be used. In this case the set of time periods is denoted by $T = \{0, 1, 2, \ldots\}$, the set of states of nature by $S$, and again the gradual revelation of information is described by a sequence of partitions of $S$, $F = \{F_0, F_1, \ldots, F_t, \ldots\}$ where the number of subsets in $F_t$ is finite and $F_t$ is finer than $F_{t-1}$ (i.e. if $\sigma \in F_t$ and $\sigma' \in F_{t-1}$ then $\sigma \subset \sigma'$ or $\sigma \cap \sigma' = \emptyset$). The event-tree $D$ induced by $F$ is the set of all nodes $\xi = (t, \sigma)$, and the same notions of initial node, predecessor of a node and successors of a node seen above apply in this case.

The economy consists of a finite collection of infinitely lived consumers $I = \{1, 2, \ldots, I\}$ who purchase commodities on spot markets and trade securities at every node in the event-tree, and there is a set $L = \{1, 2, \ldots, L\}$ of commodities at each node. The set of all commodities over the event-tree is therefore:

$$D \times L = \{(\xi, l) | \xi \in D, l \in L\}$$

We then have that $\mathbb{R}^{D \times L}$ denotes the vector space of all functions $x : D \times L \to \mathbb{R}$ and $l_\infty(D \times L)$ denotes the subspace of $\mathbb{R}^{D \times L}$ consisting of all bounded sequences:

$$l_\infty(D \times L) = \left\{ x \in \mathbb{R}^{D \times L} | \sup_{(\xi, l) \in D \times L} |x(\xi, l)| < \infty \right\}$$

and it represents the commodity space. This is exactly the construction introduced by Bewley (1972), that allows to obtain an elegant proof of existence of equilibrium for this economy. Each agent $i \in I$ has an initial endowment process $w^i = (w^i(\xi, l), (\xi, l) \in D \times L)$ which is assumed to lie in the non-negative orthant $l_\infty^+(D \times L)$ and $w^i(\xi) = (w^i(\xi, l), l \in L)$ is the agent’s endowment of the $L$ goods at node $\xi$. He then chooses a consumption process $x^i = (x^i(\xi, l), (\xi, l) \in D \times L)$ which must lie in his consumption set $X^i = l_\infty^+(D \times L)$ and $x^i(\xi) = (x^i(\xi, l), l \in L)$ is the agent’s consumption at node $\xi$ (and this
description of the commodity space implies that each good is perfectly divisible and perishable and that the supply of goods doesn’t grow without bound). The agent’s preference among consumption processes is expressed by a preference ordering \( \succ \), and at each node there are spot markets on which the commodities are traded; in particular, \( p = (p(\xi, l), (\xi, l) \in D \times L) \) denotes the spot price process and \( p(\xi) = (p(\xi, l), l \in L) \) is the vector of spot prices for the \( L \) goods at node \( \xi \). In addition, at each node the good 1 plays the role of numeraire, so that we have \( p(\xi, 1) = 1 \forall \xi \in D \) and all payments are denominated in units of good 1.

The first part of the analysis considers the simplest class of financial assets represented by the short-lived numeraire securities, that pay dividend only at the immediate successors of their nodes of issue and whose dividends are amounts of the numeraire good (it is then possible to extend the analysis to a more general class of securities, as shown in the last part of this Chapter). In this case \( J(\xi) \) is the set of short-lived securities issued at node \( \xi \) and \( j(\xi) < \infty \) is the number of these securities. For a security \( j \in J(\xi) \) we have that \( A(\xi', j) \) with \( \xi' \in \xi^+ \) denotes the dividend (in units of good 1) at an immediate successor, while \( A(\xi') = (A(\xi', j), \xi' \in \xi^+, j \in J(\xi)) \) is the vector of dividends at \( \xi' \) of the securities issued at node \( \xi \), and \( A = (A(\xi'), \xi' \in \xi^+, \xi \in D) \) is the process of security payoffs. We then have that \( q(\xi) = (q(\xi, j), j \in J(\xi)) \) is the vector of prices of the securities issued at node \( \xi \) and \( q = (q(\xi), \xi \in D) \) is the security price process, and agent \( i \) chooses a portfolio process \( z^i = (z^i(\xi), \xi \in D) \) where \( z^i(\xi) = (z^i(\xi, j), j \in J(\xi)) \) is the vector of security holdings at node \( \xi \). By denoting with \( \succ = (\succ_1, \succ_2, ..., \succ_I) \) and \( w = (w^1, w^2, ..., w^I) \) the preference orderings and the endowments of the \( I \) agents and by \( A \) the security payoff process, then, the associated economy over the event-tree \( D \) is represented by \( E_{\infty}(D, \succ, w, A) \).

With reference to the assumptions imposed on the characteristics of this economy, the crucial element (that is required to establish the existence of an equilibrium in an infinite horizon economy) is the choice of a topology in which agents’ preferences are continuous. To this end it is possible to consider the Mackey topology, following the construction of
Bewley (1972). The following assumptions are now introduced:

**A1 (event-tree):** The branching number $b(\xi)$ is finite for each node $\xi \in D$.

**A2 (endowments):** There exist scalars $m$ and $m'$ with $0 < m < m'$ such that $\forall (\xi, l) \in D \times L, \ w^i(\xi, l) > m, \forall i \in I$ and $\sum_{i \in I} w^i(\xi, l) < m'$.

**A3 (preferences):** For $i \in I$, $\succeq_i$ is a transitive, reflexive, complete preference ordering on $X^i = l^+\infty(D \times L)$ which is convex and continuous in the Mackey topology (i.e. for all $\tilde{x}^i \in X^i$, $\{x^i \in X^i \mid x^i \succeq_i \tilde{x}^i\}$ is convex and closed in the Mackey topology and $\{x^i \in X^i \mid x^i \succ_i \tilde{x}^i\}$ is open in the Mackey topology). The preference relation $\succeq_i$ is monotone and strictly monotone in good 1, in the sense that for each $x^i \in X^i$ and for each $y \in l^+\infty(D \times L)$, $x^i + y \succeq x^i$ with strict preference if $y(\xi, l) > 0$ for some $\xi$.

**A4 (uniform lower bound on impatience):** There exists $\beta < 1$ such that, for all $i \in I$,

$$x^i \chi_{D \setminus D^+(\xi)} + \beta x^i \chi_{D^+(\xi)} + e_i^\xi \succ_i x^i \quad \forall x^i \in F, \forall \xi \in D$$

where $F$ is a bounded set and $x\chi_E = (x(\xi, l)\chi_E(\xi), (\xi, l) \in D \times L)$ with:

$$\chi_E(\xi) = \begin{cases} 1 & \text{if } \xi \in E \\ 0 & \text{if } \xi \not\in E \end{cases}$$

and $e_i^\xi$ is the process which has all components 0 except for the component of good $l$ at node $\xi$ which is 1, i.e.:

$$e_i^\xi(\xi', l') = \begin{cases} 1 & \text{if } (\xi', l') = (\xi, l) \\ 0 & \text{if } (\xi', l') \neq (\xi, l) \end{cases}$$

**A5 (securities):** Every security is a short-lived numeraire security and at each node $\xi \in D$ the number of securities $j(\xi)$ is finite.
A6 (riskless bond): For each $\xi \in D$ there exists $j_\xi \in J(\xi)$ such that:

$$A(\xi', j_\xi) = 1 \quad \forall \xi' \in \xi^+$$

In particular, A1 states that at each node there must be only a finite number of immediate successors, while A2 asserts that the aggregate endowment process $w = \sum_{i \in I} w^i$ is bounded above and hence that each individual endowment process $w^i$ is bounded above; in addition each agent has an endowment of each good which is uniformly positive across all nodes. A3, introduced by Bewley, is an abstract way of formalizing the idea that agents prefer early to distant consumption, and therefore that they are impatient. This assumption is essential for the existence of an equilibrium, and permits to approximate the infinite horizon economy by finite horizon economies, since consumption in the distant future is not important. In addition, A3 states that the good 1 is strictly desired by all agents at all nodes and therefore has a positive price at each node. With reference to A4, if we consider a feasible consumption plan for agent $i$, $x^i \in F$, and we add one unit of commodity 1 at node $\xi$ to this bundle, since commodity 1 is desired agent $i$ will strictly prefer this new consumption plan, i.e. $x^i + e^i_1 >_i x^i$. By the Mackey continuity of preferences, in addition, there exists $\beta_\xi < 1$ such that agent $i$ still prefers the new consumption plan even if it is reduced by the factor $\beta_\xi$ for all the nodes that strictly succeed $\xi$, that is $x^i \chi_{D(\xi)D^+(\xi)} + \beta_\xi x^i \chi_{D^+(\xi)} + e^i_1 >_i x^i$, and this also means that agent $i$ is ready to give up the proportion $1 - \beta_\xi > 0$ of his future consumption plan in order to have one more unit of good 1 at node $\xi$. The value $1 - \beta_\xi$ is a measure of impatience that can be called the **degree of impatience** of the agent $i$ at the node $\xi$, and the Mackey continuity of preferences implies that this degree of impatience can be bounded below by a positive number (so that it doesn't vanish asymptotically). The particular requirement present in A4 is that the degree of impatience of agent $i$ is bounded away from 0 uniformly across the nodes, so that there is a uniform lower bound on the degree of impatience of agent $i$. The last two assumptions, finally, concern the securities available to agents on the
financial markets; in particular, since both \( b(\xi) \) and \( j(\xi) \) are finite we can have either the situation in which \( j(\xi) \geq b(\xi) \) and the payoffs of \( b(\xi) \) securities are linearly independent for all \( \xi \in D \) (complete markets) or the situation in which \( j(\xi) < b(\xi) \) for some node \( \xi \in D \) (incomplete markets).

1.4.2 Debt constraints, transversality condition and equilibrium

In the case of infinite horizon two elements must be considered in the construction of the budget set of an agent; the first is the usual condition according to which the next expenditure on the spot markets must not exceed the income earned on the financial markets at each node, while the second is a new element introduced by the sequential nature of trade combined with the infinity of the horizon. If \( z^i(\xi) \) is the portfolio chosen by agent \( i \) at node \( \xi \), at the predecessor \( \xi^- \) (that is unique) he has chosen the portfolio \( z^i(\xi^-) \) which gives the payoff \( A(\xi)z^i(\xi^-) \) (by considering the normalization \( p(\xi, 1) = 1 \)). The budget constraint at node \( \xi \) is therefore (similarly to what has been shown for the finite horizon case):

\[
p(\xi)(x^i(\xi) - u^i(\xi)) = A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi)
\]

(1.3)

and the consumption-portfolio process \((x^i, z^i)\) which is chosen must satisfy this constraint at every node (we must observe that, since we are considering short-lived securities, in the right-hand side of the expression the term \( q(\xi)z^i(\xi^-) \) doesn't appear because \( q(\xi) \) in this case is equal to 0). In order to have a solution to the consumption-portfolio choice problem of the agent the prices \((p, q)\) must not offer arbitrage opportunities at any node \( \xi \in D \) and, as it has been observed in the previous Section, this is equivalent to the existence of a process \( \pi = (\pi(\xi), \xi \in D) \) of positive present value prices such that:

\[
\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')A(\xi') \quad \forall \xi \in D
\]

(1.4)
Even if the prices \((p, q)\) don't offer arbitrage opportunities, however, it is not guaranteed that the agent's choice problem has a solution if we don't impose a further restriction on the portfolio processes, because in the infinite horizon setting the agent can roll over his debt ad infinitum; it is therefore necessary to consider some form of borrowing constraint which limits the amount of debt of the agent and guarantees the existence of a solution to his consumption-portfolio choice problem.

A first possibility is to consider a debt constraint of the form:

\[
q(\xi)z^i(\xi) \geq -M \quad \forall \xi \in D
\]

for some positive number \(M\), so that an agent's debt at each date cannot exceed the bound \(M\). This leads to a budget set with explicit debt constraint:

\[
B^M_\infty (p, q, w^i, A) = \left\{ x^i \in X^i \left| \exists z^i \in Z \text{ such that } \forall \xi \in D \right. \begin{array}{l}
q(\xi)z^i(\xi) \geq -M \\
p(\xi)(x^i(\xi) - w^i(\xi)) = \\
= A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi)
\end{array}\right\}
\]

(where \(Z\) is the portfolio space) and in an economy with impatient agents the presence of this bound prevents Ponzi schemes.

Another possibility is to introduce an implicit debt constraint of the form:

\[
(qz^i) = (q(\xi)z^i(\xi), \xi \in D) \in l_\infty(D)
\]

which prevents agents from considering trading strategies that lead to debt which grows without bound (in fact the agent's debt must lie in the space of bounded sequences \(l_\infty\));
this constraint determines a budget set with implicit debt constraint:

\[
B^{DC}_\infty(p, q, w^i, A) = \left\{ x^i \in X^i \middle| \begin{array}{l}
\exists z^i \in Z \text{ with } (qz^i) \in l_\infty(D) \\
\text{such that } \forall \xi \in D \\
p(\xi)(x^i(\xi) - w^i(\xi)) = \\
A(\xi)z^i(\xi^-) - q(\xi)z^i(\xi)
\end{array} \right\}
\]

These two types of budget sets allow to introduce a first notion of equilibrium for an infinite-horizon sequence economy, we have in fact:

**Definition 28** An equilibrium with explicit (respectively implicit) debt constraint for the economy \(E_\infty(D, \succeq, w, A)\) is a pair \(((x, z), (p, q))\) such that:

(i) \((x^i, z^i)\) is \(\succeq_i\) maximal in \(B^M_\infty(p, q, w^i, A)\) (respectively in \(B^{DC}_\infty(p, q, w^i, A)\)) for each \(i \in I\)

(ii) \(\sum_{i \in I} (x^i - w^i) = 0\)

(iii) \(\sum_{i \in I} z^i = 0\)

The other approach which is used to prevent Ponzi schemes is based on the use of a transversality condition, that makes it possible to extend the techniques of the Arrow-Debreu theory to infinite horizon economies. In the case of complete markets the idea is that if \((p, q)\) is a no-arbitrage system of prices then there exists a unique vector of present value prices \(\pi\) such that the relation (1.4) is satisfied at every node. If the trading activity \((x^i, z^i)\) of an agent satisfies the budget equation (1.3) at each node, by multiplying this equation by \(\pi(\xi)\) and summing over all nodes to date \(T\) gives:

\[
\sum_{\xi' \in D^T(\xi)} \pi(\xi')p(\xi')(x^i(\xi') - w^i(\xi')) = -\sum_{\xi' \in D_T(\xi)} \pi(\xi')q(\xi')z^i(\xi')
\]

Here \(D(\xi)\) is the subtree of nodes with root \(\xi\) and \(D^T(\xi) = \{\xi' \in D(\xi) \mid t(\xi') \leq t(\xi) \leq T\} \) and \(D_T(\xi) = \{\xi' \in D(\xi) \mid t(\xi') = T\} \) (i.e. \(D^T(\xi)\) is the subset of nodes of the subtree \(D(\xi)\) between dates \(t(\xi)\) and \(T\) and \(D_T(\xi)\) is the subset of nodes of the subtree \(D(\xi)\) at date \(T\)).
If we now consider the transversality condition:

$$\lim_{T \to \infty} \sum_{\xi' \in D_T(\xi)} \pi(\xi')q(\xi')z^i(\xi') = 0 \quad (1.5)$$

(which requires that the asymptotic present value of debt be zero) then the consumption plan \(x^i\) of agent \(i\) satisfies the Arrow-Debreu budget constraint:

$$\sum_{\xi' \in D(\xi)} \pi(\xi')p(\xi')(x^i(\xi') - w^i(\xi')) = 0$$

i.e.:

$$\sum_{\xi' \in D(\xi)} P(\xi')(x^i(\xi') - w^i(\xi')) = 0$$

where \(P(\xi) = \pi(\xi)p(\xi)\) is the vector of Arrow-Debreu present value prices of the goods at node \(\xi\). At this point it is possible to consider an extension of the transversality condition (1.5) to economies with incomplete markets that can be used to place a bound on the rate at which agents accumulate debt. In this case the condition becomes:

$$\lim_{T \to \infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') = 0 \quad \forall \xi \in D \quad (1.6)$$

where the present values \(\pi^i\) used are those of each agent at the equilibrium (since with incomplete markets the no-arbitrage equation (1.4) has many solutions and therefore there is not a unique present value vector). This condition can be obtained by observing that, given an optimal consumption-portfolio plan \((x^i, z^i)\) for agent \(i\), to which is associated a present value vector \(\pi^i = (\pi^i(\xi), \xi \in D)\) that must satisfy the relation (1.4), the pair \((x^i, z^i)\) must also satisfy the condition:

$$\lim_{T \to \infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi')q(\xi')z^i(\xi') \leq 0$$
which means that an optimal portfolio doesn't leave value (i.e. it doesn't make agent $i$ a lender) at infinity (otherwise the agent could find a preferred consumption stream by decreasing his lending and increasing his earlier consumption). On the other hand, each agent knows that he cannot find lenders on the markets who finance a portfolio $z^i$ that permits him to be a borrower at infinity (because this would imply that some other traders are lenders at infinity), and therefore the relation:

$$\lim_{T \to \infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi') q(\xi') z^i(\xi') < 0$$

cannot hold. As a consequence, the transversality condition (1.6) must be satisfied.

This condition can then be used to define the budget set with transversality condition:

$$B^{TC}_\infty(p, q, \pi^i, w^i, A) = \left\{ x^i \in X^i \mid \exists z^i \in Z \text{ such that } \forall \xi \in D \right\}
\lim_{T \to \infty} \sum_{\xi' \in D_T(\xi)} \pi^i(\xi') q(\xi') z^i(\xi') = 0
p(\xi)(x^i(\xi) - w^i(\xi)) =
A(\xi)z^i(\xi^i) - q(\xi)z^i(\xi)$$

and this budget set allows to introduce a second notion of equilibrium for an infinite horizon economy, we have in fact:

**Definition 29** An equilibrium with transversality condition for the economy $E_\infty(D, \succeq, w, A)$ is a pair $((x, z), (p, q, (\pi^i)_{i \in I}))$ such that:

(i) $(x, z)$ is $\succeq$-maximal in $B^{TC}_\infty(p, q, \pi^i, w^i, A)$ for each $i \in I$;

(ii) for each $i \in I$:

(a) $\pi^i(\xi) > 0, \forall \xi \in D$ and $\overline{p}^i \in l^1(D \times L)$ where $\overline{p}^i = \left\{ \overline{p}^i(\xi), \xi \in D \right\} = (\pi^i(\xi)p(\xi), \xi \in D)$ and $l_1(D \times L)$ is the subspace of $\mathbb{R}^{D \times L}$ consisting of all summable...
sequences, i.e.:

\[ l_1(D \times L) = \left\{ P \in \mathbb{R}^{D \times L} \mid \sum_{(\xi, l) \in D \times L} P(\xi, l) |< \infty \right\} \]

(b) \( \bar{x}_i \) is \( \succeq_i \) maximal in \( B_\infty(F^i, w^i) = \left\{ x^i \in l^+_\infty(D \times L) \mid F^i(x^i - w^i) \leq 0 \right\} \)

(c) \( \bar{\pi}(\xi)(\xi, j) = \sum_{\xi' \in l^+} \bar{\pi}(\xi')A(\xi', j), \forall j \in J(\xi), \forall \xi \in D \)

(iii) \( \sum_{i \in I}(\bar{x}^i - w^i) = 0 \)

(iv) \( \sum_{i \in I} \bar{x}^i = 0 \)

In this case the condition (ii) characterizes the equilibrium present value vector \( \bar{\pi}^i \) of agent \( i \) and expresses the fact that the first-order conditions for the maximization problem of agent \( i \) are satisfied at \( (\bar{x}^i, \bar{z}) \) if \( \bar{\pi}(\xi) \) is the multiplier associated with the budget constraint at node \( \xi \), for each \( \xi \in D \).

### 1.4.3 Existence of equilibrium

Both the concepts of equilibrium introduced above are important, for different reasons. On the one hand, the notion of equilibrium with transversality condition is relevant from a theoretical point of view, because it permits to apply concepts and techniques of infinite dimensional Arrow-Debreu theory to a model of incomplete markets over infinite horizon. On the other hand, the notion of equilibrium with debt constraint is simpler and more natural from an economic point of view. A fundamental result that can be shown is that, for an economy satisfying the assumptions introduced before, these two concepts are equivalent, i.e. every equilibrium with transversality condition is an equilibrium with debt constraint and viceversa. As a consequence, if it is possible to show that the economy has an equilibrium with transversality condition, this implies that the economy has an equilibrium with debt constraint.

With reference to this aspect, the first result that can be proved is the following:
Theorem 30 (existence) Each economy $E_{\infty}(D, \succeq, w, A)$ satisfying Assumptions A1-A6 has an equilibrium with transversality condition.

Proof. See Magill-Quinzii (1994).

Remark 31 The proof of this theorem is based on the construction introduced by Bewley (1972). The idea is to start from the infinite horizon economy $E_{\infty}(D, \succeq, w, A)$ and to consider the associated $T$-truncated economy $E_T(D, \succeq, w, A)$, that is the economy with the same characteristics as $E_{\infty}$ but in which agents are constrained to stop trading at date $T$. For this economy an equilibrium exists for all the parameter values, since all securities are short-lived numeraire securities (as shown by Geanakoplos-Polemarchakis (1986)); in addition, the truncated equilibrium (the equilibrium of the truncated economy) satisfies uniform bounds, and therefore by letting $T$ tend to $\infty$ it converges. Finally, it is possible to show that the limit of the truncated equilibrium is exactly an equilibrium for the limit economy, i.e. for the infinite horizon economy $E_{\infty}$.

We then have the following further result:

Theorem 32 If $E_{\infty}(D, \succeq, w, A)$ is an economy satisfying Assumptions A1-A6, then $((x, z), (p, q))$ is an equilibrium with implicit debt constraint if and only if there exist present value vectors $(\Pi^i)_{i \in I}$ such that $((x, z), (\tilde{p}, \tilde{q}, (\Pi^i)_{i \in I}))$ is an equilibrium with transversality condition.

Proof. See Magill-Quinzii (1994).

Finally the following corollary holds:

Corollary 33 For each economy $E_{\infty}(D, \succeq, w, A)$ satisfying Assumptions A1-A6 there is a bound $M > 0$ such that the economy has an equilibrium with explicit debt constraint $M$ which is never binding.

Proof. See Magill-Quinzii (1994).
The existence of an equilibrium with implicit debt constraint, therefore, implies the existence of an equilibrium with explicit debt constraint (and the fact that this constraint is never binding at the equilibrium is important because it ensures that the debt constraint doesn’t introduce new imperfections into the model).

These are the central results for the infinite-horizon economy considered: this economy has an equilibrium with transversality condition, furthermore this equilibrium is equivalent to an equilibrium with implicit debt constraint, and this implies the existence of an equilibrium with explicit debt constraint, that is the more natural concept of equilibrium for an infinite-horizon economy from an economic point of view.

This is the analysis in the simplest case, the one in which the only type of financial assets that appear in the economy are short-lived numeraire securities; the same analysis can then be extended to a more general class of securities, the infinite-lived securities, and this is the object of the last part of this Chapter.

1.4.4 Equilibrium with infinite-lived securities

To extend the model of the previous Subsections to the case of a general asset structure we must take into account that in this situation a further problem is represented by the possibility of non-existence of equilibrium (as a consequence of changes in the rank of the return matrix when prices vary - a possibility that is excluded if the only assets in the economy are the short-lived numeraire securities previously considered, as it will be clear in the following pages -). Also in the finite-horizon case this problem arises, and the technique used to solve it consists in introducing the concept of pseudo-equilibrium; such a pseudo-equilibrium exists for all economies, and for a generic set of economies it is also an equilibrium (as it has been discussed in Section 3). The same approach can be adopted in the case of an infinite-horizon economy with long-lived assets. In this situation, first of all it is necessary to adjoin to the economy a family of potentially equivalent short-lived numeraire assets; at this point it is possible to prove that for this economy a pseudo-equilibrium exists (by taking limits of pseudo-equilibria of truncated
economies, following the technique introduced by Bewley and illustrated, for the case of a true equilibrium, in the previous Subsection); finally it can be shown that this pseudo-equilibrium is a true equilibrium for a dense set of economies (and therefore the result is weaker than for the finite-horizon model, where the same conclusion is true for a generic set of economies).

Also when we take into account a general asset structure the economy is described exactly as before, with an event-tree. The securities considered are real securities (i.e. the return of an asset at each node $\xi'$ after its node of issue $\xi$ is the value under the spot prices at node $\xi'$ of a specified bundle of the $L$ goods), and their set is denoted by $J$, while the set of securities traded at node $\xi$ is denoted by $J(\xi)$. The generic security $j$ issued at node $\xi$ promises to deliver a dividend process $(p(\xi')A(\xi',j), \xi' > \xi)$ at all nodes strictly succeeding its node of issue, and this dividend is the value of a bundle $A(\xi',j) = (A(\xi',j,l), l \in L)$ of the $L$ commodities under the spot prices $p(\xi')$. We then have that a security is short-lived if it is traded only at its node of issue and pays dividends only at the immediate successors of this node (this is the type of securities considered before), otherwise it is long-lived; if it is traded at every node after its node of issue, in particular, it is said to be infinite-lived. It is then convenient, in this case, to define the price process $q$ of each security $j$ and the portfolio process $z^i$ of each agent $i$ on the whole event-tree $D$, by setting $q(\xi,j) = 0$ if $j \notin J(\xi)$ and $z^i(\xi,j) = 0$ if $j \notin J(\xi)$, i.e. if the security is not traded at node $\xi$. Also in this case it is assumed that for each node the number of securities that are traded, $j(\xi)$, is finite. By denoting with $\succeq = (\succeq_1, \succeq_2, \ldots, \succeq_I)$ the preference orderings, with $w = (w^1, w^2, \ldots, w^I)$ the endowments and with $A$ the generic security structure, then, $E_\infty(D, \succeq, w, A)$ represents (as before) the associated economy over the event-tree $D$.

The assumptions satisfied by this economy are those introduced above (i.e. A1-A6), with the following slight modifications:

A3' (preferences): The preference ordering is strictly convex [and not only convex] in
the Mackey topology.

A5' (securities): Every security \( j \in J \) is a real security with bounded commodity payoff \( A(\cdot; j) \in l_\infty(D \times L) \) and the number of traded securities \( j(\xi) \) is finite at each node \( \xi \in D \).

A6' (short-lived numeraire bond): At each node \( \xi \in D \) a security \( j_\xi \in J(\xi) \) is issued which is traded only at this node and has a commodity payoff of one unit of good 1 at each immediate successor:

\[
A(\xi', j_\xi, l) = \begin{cases} 
1 & \text{if } \xi' \in \xi^+ \text{ and } l = 1 \\
0 & \text{otherwise}
\end{cases}
\]

These assumptions are therefore essentially the same introduced previously in the case in which the financial structure is characterized by the presence of short-lived numeraire securities only. The main difference, with a general financial structure, is the requirement of strict convexity of agents' preferred set. This is due to the fact that the proof of existence of an equilibrium for the infinite horizon economy is based on taking limits of equilibria of truncated finite horizon economies; in the case of short-lived numeraire securities the existence of an equilibrium in a finite horizon economy only requires the use of a standard Kakutani fixed-point argument, which can be applied to an economy in which agents' demands are expressed by correspondences, while for a finite horizon economy with a general security structure the existence of a (pseudo-)equilibrium requires the use of a particular theory, the so-called "degree theory", that is defined for functions, and therefore agents must have well-defined demand functions (and this is ensured by the assumption of strict convexity).

Also in this case, three concepts of equilibrium can be introduced; the simplest is that of equilibrium with debt constraint (explicit or implicit), but to obtain the existence results it is useful to introduce the more abstract concept of equilibrium with transversality condition. These equilibrium concepts differ only by the specification of the agents'
budget sets, to this end we have first of all the budget set with explicit debt constraint:

\[
B^M_{\infty}(p, q, w^i, A) = \left\{ x^i \in X^i \left| \begin{array}{l}
x^i \in Z \text{ such that } \forall \xi \in D \\
p(\xi)(x^i(\xi) - w^i(\xi)) = V(\xi)z^i(\xi^-) - q(\xi)z^i(\xi)
\end{array} \right. \right\}
\]

where \( V(\xi) = p(\xi)A(\xi) + q(\xi) \) is the vector of returns at the successor \( \xi \) of the securities traded at \( \xi^- \). The object of this constraint, as it has been outlined previously, is to eliminate Ponzi schemes (i.e. the indefinite postponement of debt), and to avoid that it introduces itself a new imperfection in the economy it must not be binding in equilibrium.

If the bound \( M \) is chosen independently of the characteristics of the economy there will always be some economies for which \( M \) is too small and is binding in equilibrium. To avoid this situation it is possible to consider a second type of budget set, the budget set with implicit debt constraint, in which the bound is left unspecified:

\[
B^{DC}_{\infty}(p, q, w^i, A) = \left\{ x^i \in X^i \left| \begin{array}{l}
x^i \in Z \text{ with } (qz^i) \in l_\infty(D) \\
\text{such that } \forall \xi \in D \\
p(\xi)(x^i(\xi) - w^i(\xi)) = V(\xi)z^i(\xi^-) - q(\xi)z^i(\xi)
\end{array} \right. \right\}
\]

where \((qz^i) = (q(\xi)z^i(\xi), \xi \in D)\). In this way, by proving that an equilibrium with implicit debt constraint exists, it is possible to prove that there is an appropriate bound \( M \) for a given economy such that there exists an equilibrium with explicit debt constraint in which the bound \( M \) is never binding. The two budget sets considered lead to the following concepts of equilibrium (the same introduced before):

**Definition 34** An equilibrium with explicit (respectively implicit) debt constraint for the economy \( E_{\infty}(D, \succeq, w, A) \) is a pair \(((\bar{x}, \bar{z}), (\bar{p}, \bar{q}))\) such that:

(i) \((\bar{x}^i, \bar{z}^i)\) is \( \succeq_i \) maximal in \( B^M_{\infty}(\bar{p}, \bar{q}, w^i, A) \) (respectively in \( B^{DC}_{\infty}(\bar{p}, \bar{q}, w^i, A) \)) for each
Finally, we also have the budget set with transversality condition, given by:

\[ B_{\infty}^{TC}(p, q, \pi^i, w^i, A) = \begin{cases} \exists z^i \in Z \text{ such that } \forall \xi \in D \\ \lim_{T \to \infty} \sum_{\xi' \in D, \xi'} p(\xi)(x^i(\xi') - w^i(\xi')) = \\ p(\xi)(x^i(\xi) - w^i(\xi)) = \\ = V(\xi)z^i(\xi^-) - q(\xi)z^i(\xi) \end{cases} \]

where \( \pi^i = (\pi^i(\xi), \xi \in D) \) is a process of present value prices. In this case the transversality condition expresses the fact that on every subtree the present value of an agent’s debt must be asymptotically zero, and again, since with incomplete markets there isn’t an objective present value vector, it is possible to use agent \( i \)'s present value vector \( \bar{\pi}^i \) to evaluate the asymptotic value of his debt. The implicit prices \( (\bar{\pi}^i)_{i \in I} \) must therefore be added to the objective market prices \( (\bar{p}, \bar{q}) \) to define an equilibrium with transversality condition, we have in fact (similarly to the case with short-lived numeraire securities):

**Definition 35** An equilibrium with transversality condition for the economy \( E_\infty(D, \succeq, w, A) \) is a pair \( ((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I})) \) such that:

(i) \( (\bar{x}, \bar{z}) \) is \( \succeq \) maximal in \( B_{\infty}^{TC}(\bar{p}, \bar{q}, \bar{\pi}^i, w^i, A) \) for each \( i \in I \)

(ii) for each \( i \in I \):

(a) \( \bar{\pi}^i(\xi) > 0, \forall \xi \in D \) and \( \bar{F}^i \in I_+^t(D \times L) \) where \( \bar{F}^i = \{ \bar{F}^i(\xi), \xi \in D \} = (\bar{\pi}^i(\xi)\bar{p}(\xi), \xi \in D) \)

(b) \( \bar{x}^i \) is \( \succeq \) maximal in \( B_{\infty}(\bar{F}^i, w^i) = \{ x^i \in I_+^t(D \times L) \mid \bar{F}^i(x^i - w^i) \leq 0 \} \)

(c) \( \bar{\pi}^i(\xi)\bar{q}(\xi, j) = \sum_{\xi' \in \xi} \bar{\pi}^i(\xi')(\bar{p}(\xi')A(\xi', j) + \bar{q}(\xi', j)), \forall j \in J(\xi), \forall \xi \in D \)

(iii) \( \sum_{i \in I}(\bar{x}^i - w^i) = 0 \)

(iv) \( \sum_{i \in I} \bar{z}^i = 0 \)
The concepts of equilibria with debt constraint and with transversality condition are used to extend the notion of equilibrium with incomplete markets for a finite horizon economy to an infinite horizon economy. When we consider a general asset structure, the problem is that, even for a finite horizon economy, such an equilibrium may not exist, because at a node $\xi$ the dimension of the subspace spanned by the columns of the return matrix (the market subspace) can change when prices $(p(\xi'), q(\xi'))_{\xi' \in \xi^+}$ vary; this can create discontinuities in agents' demands which may lead to the non-existence of an equilibrium (this is the possibility, mentioned several times in the previous pages, first shown by Hart (1975)). The technique used to solve this problem (already discussed above) consists in introducing the concept of pseudo-equilibrium and then in showing that such a pseudo-equilibrium exists for all economies and that, for "most" economies, it is an equilibrium. The difference between the finite horizon case and the infinite horizon case is that in the former the equivalence between pseudo-equilibrium and equilibrium is true for a generic set of economies (this has been shown in Section 3), while in the latter the equivalence turns out to be true only for a dense set of economies, and therefore in this case the existence result is weaker.

The idea of pseudo-equilibrium is based on the fact that in general, if at the node $\xi$, which has $b(\xi)$ successors, $j(\xi)$ securities are traded, then the rank of the $b(\xi) \times j(\xi)$ return matrix:

$$V(\xi^+) = [V(\xi', j)]_{\xi' \in \xi^+} = [p(\xi')A(\xi', j) + q(\xi', j)]_{\xi' \in \xi^+}$$

is given by:

$$a(\xi) = \min(b(\xi), j(\xi))$$

but in some cases, for some prices $(p, q)$, the rank may fall. It is precisely this fact that can create discontinuities in agents' demand and therefore non-existence of equilibrium. This problem doesn't arise in the case of short-lived numeraire securities, because in this
situation the term $q(\xi', j)$ doesn’t appear, while the price $p(\xi')$ can be normalized to 1, therefore we have:

$$\operatorname{rank} [V(\xi', j)]_{\xi' \in \xi^+} = \operatorname{rank} [A(\xi', j)]_{\xi' \in \xi^+} = j(\xi) \leq b(\xi)$$

and the matrix doesn’t drop rank (since we can assume, without loss of generality, that the returns on the securities $j \in J(\xi)$ are linearly independent). For this reason it is not necessary to introduce the notion of pseudo-equilibrium and we can consider directly the notion of equilibrium (as it has been shown in the previous Subsection). In the general case (i.e. when we have not a security structure characterized by the presence of short-lived numeraire securities only) this is not necessarily true, and we must introduce the notion of pseudo-equilibrium; in this context a pseudo-equilibrium is defined as an equilibrium of an economy in which agents are given an artificial subspace of income transfers of dimension $a(\xi)$ at node $\xi$ which contains the subspace of transfers achievable with the existing securities, but which is larger when the matrix $V(\xi^+)$ has rank less than $a(\xi)$. The artificial subspace, in particular, can be interpreted as originated by an artificial financial structure composed by $a(\xi)$ short-lived numeraire securities issued at node $\xi$, so that it is possible to use the techniques developed in the previous pages (more precisely, it is possible to show the existence of a pseudo-equilibrium for the infinite-horizon economy by taking limits of pseudo-equilibria of finite-horizon economies). The central result that can be obtained is the following:

**Theorem 36 (existence of a pseudo-equilibrium)** Each economy $\mathcal{E}_\infty(D, \succeq, w, A)$ that satisfies Assumptions A1-A6 [with A3,A5,A6 replaced by A3',A5',A6'] has a pseudo-equilibrium.

**Proof.** See Magill-Quinzii (1996). 

**Remark 37** Also in this case the idea underlying the proof of the theorem consists in taking limits of pseudo-equilibria of truncated economies in which trade stops at some finite
date. Given the infinite horizon economy $E_\infty(D, \succeq, w, A)$ the associated $T$-truncated economy $E_T(D, \succeq, w, A)$ is the economy with the same characteristics as $E_\infty$ in which agents are constrained to stop trading at date $T$. For such an economy a pseudo-equilibrium exists for every finite $T$ (from the analysis described in Section 3). In addition, this pseudo-equilibrium satisfies uniform bounds, and therefore by letting $T$ tend to $\infty$ it converges. Also in this case, finally, it is possible to show that the limit of the truncated pseudo-equilibrium is exactly a pseudo-equilibrium for the limit economy, i.e. for the infinite horizon economy $E_\infty$.

At this point, the link between a pseudo-equilibrium and an equilibrium is guaranteed by the following theorem (where $A$ is the set, opportunely defined, of admissible payoffs for the securities):

**Theorem 38 (existence of an equilibrium)** Under Assumptions A1-A6 [with A3,A5,A6 replaced by A3',A5',A6'] there exists a dense subset $A^* \subset A$ such that if $A \in A^*$ then the infinite horizon economy $E_\infty(D, \succeq, w, A)$ has an equilibrium with transversality condition.

**Proof.** See Magill-Quinzii (1996).

**Remark 39** The idea of the proof is to show that if we consider a payoff process $\bar{A} \in A$ for which a pseudo-equilibrium of the economy $E_\infty(D, \succeq, w, \bar{A})$ is not an equilibrium with transversality condition, then for all $\varepsilon > 0$ there exists a payoff process $A \in A$ in the ball of radius $\varepsilon$ around $\bar{A}$ for which a pseudo-equilibrium of the economy $E_\infty(D, \succeq, w, A)$ is an equilibrium with transversality condition.

Also in the case of infinite-lived securities, finally, it can be shown the equivalence between the different concepts of equilibrium introduced. A first result is the following:

**Theorem 40** Under Assumptions A1-A6 [with A3,A5,A6 replaced by A3',A5',A6'] $((x, z), (p, q), \pi^t)$ is an equilibrium with implicit debt constraint of $E_\infty(D, \succeq, w, A)$ if and only if there exist present value vectors $(\pi^t)_{t \in I}$ such that $((x, z), (p, q), (\pi^t)_{t \in I})$ is an equilibrium with transversality condition.

The following corollary then holds:

**Corollary 41** Under Assumptions A1-A6 [with A3',A5',A6'] there exists a dense subset $A^* \subset A$ such that if $A \in A^*$, then the infinite horizon economy $E_\infty(D, \preceq, w, A)$ has an equilibrium with an explicit debt constraint $M$ which is never binding.


Also in this case, therefore, the different concepts of equilibrium introduced are equivalent, and this permits to retain the advantages of each of them. In particular, the notion of equilibrium with transversality condition is more natural for a mathematical analysis, while the notion of equilibrium with debt constraint is more plausible from an economic point of view. This concludes the analysis in the case of infinite-horizon economy; the central result that has been reached is that when we consider a particular asset structure, composed only by short-lived numeraire securities, then an equilibrium exists for all parameter values that characterize the economy, while when we consider a general asset structure the existence result remains true only for a dense set of economies.

The analysis presented in this Chapter has shown the main results in terms of existence of solutions in general equilibrium models with time and uncertainty. The point of departure has been represented by the standard Arrow-Debreu model, that can be extended with the introduction of the notion of contingent commodity and that leads to the concept of equilibrium with complete contingent markets, for which existence and optimality of equilibrium (for all parameter values that describe the economy) follow from the traditional general equilibrium theory. Since this model, even if particularly interesting from a theoretical point of view, is far from the situation that can be observed in actual economies, the following step has been the introduction of a model of equilibrium with spot and financial markets, that is closer to the real situation. In this case it is
no longer true that an equilibrium exists for every economy; what can be proved (both in the two-period framework and in the multi-period framework, with finite horizon) is the existence of the equilibrium for a generic set of economies. The last step has been represented by the extension of the model to an infinite horizon setting. In this case existence of equilibrium can still be proved (even if in a weaker form, with respect to the finite horizon model, when a general asset structure is considered, because in this situation the equilibrium exists only for a dense set of economies).

In conclusion, the results obtained with reference to the question of existence of equilibrium can be summarized as follows:

- **model with contingent markets**: existence of equilibrium (both in a two-period economy and in a $T$-period economy);

- **model with spot and financial markets on finite horizon**: existence of equilibrium for a generic set of economies (both in a two-period economy and in a $T$-period economy);

- **model with spot and financial markets on infinite horizon**: existence of equilibrium in the case of short-lived numeraire assets, existence of equilibrium for a dense set of economies in the case of a general financial structure.

In the transition from the model with contingent markets to the model with spot and financial markets and, in this model, from the finite horizon to the infinite horizon case (i.e. by considering situations that are more and more realistic) something is "lost" in terms of existence of solutions. Nevertheless, the existence of the equilibrium (even if in a weaker form) is preserved; the model is therefore consistent, and can be used to explain something of the economy under study. In particular, it is possible to obtain indications concerning the equilibrium prices of the assets; this is the question addressed in the second Chapter.
Chapter 2

General equilibrium models, asset pricing and speculative bubbles

2.1 Introduction

The model with spot and financial markets can be used to investigate the relation between the equilibrium price of an asset and the stream of future dividends to which this asset represents a claim. Here we recall briefly the results obtained in the first Chapter with reference to this aspect. In the finite-horizon case, the conclusion that emerges is that the equilibrium price of a security is equal to the present value of its future dividends. In fact, in a two-period economy, in which there are $S$ states of nature at date $t = 1$, the absence of arbitrage opportunities implies the existence of a vector of present values $\pi = (\pi_0, \pi_1, ..., \pi_S)$ such that:

$$\pi_0 q_j = \sum_{s=1}^{S} \pi_s V^j_s \quad j = 1, 2, ..., J$$

and, since $\pi_0$ can be normalized to 1, this in turn means that the price of each asset is equal to the present value of its future dividend stream. In the case of a $T$-period economy, similarly, the absence of arbitrage opportunities implies the existence of a present value
process \( \pi : D \rightarrow R_{++} \) such that:

\[
\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi') \left[ p(\xi') A(\xi') + q(\xi') \right] \quad \forall \xi \in D^-
\]

and solving this system of equations recursively with the terminal condition \( q(\xi) = 0, \forall \xi \notin D^- \) (where \( D^- \) is the set of non-terminal nodes) we get:

\[
q(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi') p(\xi') A(\xi') \quad \forall \xi \in D^-
\]  \hspace{1cm} (2.1)

according to which the current value of each asset at node \( \xi \) is the present value of its future dividend stream over all succeeding nodes \( \xi' > \xi \).

These are the results in the finite-horizon case. The present value of the future dividend stream deriving from an asset is the so-called fundamental value, and the conclusion obtained in the model with finite horizon is that the equilibrium price of an asset is equal to this value. When we extend the analysis to an economy defined over an infinite horizon the equality between equilibrium price and fundamental value doesn’t necessarily hold, and speculative bubbles may arise; this is precisely the question addressed in the following pages.

In particular, in the first part (Section 2.2) this question is studied by means of the infinite-horizon model introduced in the first Chapter. The issue of speculative bubbles is then considered more specifically along the lines of a recent contribution that tries to give definitive results concerning this question, and the central conclusions are derived (Section 2.3). The last part of the Chapter uses a different approach, based on Euler equations, to confirm these conclusions (Section 2.4). The important aspect is that this approach can also be used to construct and to study specific examples in which bubbles appear; this is the subject of the third Chapter.
2.2 Infinite horizon and speculative bubbles

In the model with infinite horizon an important phenomenon that can arise in the pricing of securities is represented by the so-called speculative bubbles, that can be interpreted as equilibria in which asset prices are not equal to the present values of their future dividends according to given state prices. In particular, the expression (2.1) remains true when we consider an infinite-horizon economy with finitely-lived securities, while for an infinite-lived security there is no terminal condition (i.e. the condition $q(\xi) = 0, \forall \xi \notin D^{-}$) that guarantees that the equilibrium price is equal to the fundamental value; when this relation is not satisfied the price of the asset is said to involve a bubble component. With reference to an infinite-horizon economy, the following definition can be introduced:

Definition 42 Let $((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I}))$ be an equilibrium with transversality condition (defined in Chapter 1) for the economy $E_{\infty}(D, \preceq, u, A)$. The security $j \in J$ is priced at its fundamental value if, for all agents $i \in I$, we have:

$$
\bar{q}(\xi) = \frac{1}{\bar{\pi}(\xi)} \sum_{\xi' > \xi} \bar{\pi}^i(\xi') \bar{p}(\xi') A(\xi') \quad \forall \xi \in D
$$

If this relation is not satisfied for some agent $i \in I$, the security $j$ has a speculative bubble.

In this case $\bar{\pi}^i$ denotes the present value vector of agent $i$. As it has been outlined in the first Chapter, in the case of complete markets all agents' present value vectors coincide, i.e. $\pi^1 = \pi^2 = \ldots = \pi^i = \pi$ (there is complete agreement about the present value of a stream of date-1 income), while when asset markets are incomplete the values $\pi^i$ are different. As a consequence, in the case of incomplete markets (and infinite-lived securities, because with finitely-lived securities the fundamental value is the same for all present value vectors, and it is equal to the equilibrium price of the asset) the fundamental value may be different for different values of $\pi^i$. If this happens, we may have a bubble on a security for some values $\pi^i$ but not for other values, and therefore a particular situation arises. This is the kind of bubble that, after the analysis of Santos-Woodford (1997), has
been called an *ambiguous bubble*.

With reference to the model, introduced in the last part of the first Chapter, that considers an economy extending over an infinite horizon, it is possible to obtain equilibria of this economy in which some of the securities have speculative bubbles. This depends on the type of infinite-lived securities available in the economy, in particular it is necessary to distinguish between securities in positive net supply and securities in zero net supply. For this reason the model considered, that is restricted to an economy in which securities are in zero net supply (in fact the market clearing condition relative to the assets is \( \sum_{i \in I} z^i = 0 \)), can be generalized in order to include securities in positive supply. In this case \( E_\infty(D, \geq, w, \delta, A) \) denotes an economy which is identical to that considered until now except that a subset \( J_0 \subset J(x_0) \) of the securities issued at date 0 can have positive initial supply \( \delta = (\delta_j, j \in J_0) \) where \( \delta_j = \sum_{i \in I} \delta^i_j \) and \( \delta^i_j \) is agent \( i \)'s initial holding of security \( j \). An equilibrium with transversality condition of the economy \( E_\infty(D, \geq, w, \delta, A) \) is now given by Definition 35 of Chapter 1 with the modifications represented by the fact that the new budget set \( B^{\infty}_{TC}(p, q, \pi^i, w^i, \delta^i, A) \) differs from that defined previously for the equation at the initial node, which becomes:

\[
p(x_0)(x^i_0(x_0) - w^i_0(x_0)) = q(x_0)(\delta^i - z^i(x_0))
\]

and by the fact that the market-clearing condition \((iv)\) becomes:

\[
\sum_{i \in I} z^i_0(x, j) = \delta_j \quad \forall x \in D, \forall j \in J
\]

where \( \delta_j = 0 \) if \( j \notin J_0 \). For this economy the following further assumption is introduced:

A7 If \( \delta_j > 0 \), then \( A(\cdot, j) \in l_\infty^+(D \times L) \) and \( \delta^i_j \geq 0 \) for all \( i \in I \). If \( \delta_j = 0 \), then \( \delta^i_j = 0 \) for all \( i \in I \).

This means that the securities in positive supply have non-negative payoffs and agents only inherit non-negative initial shares of such assets, while for securities in zero supply

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agents do not inherit any initial debt or credit. For an economy of this type, that admits the presence of securities in positive net supply, it is possible to prove the existence of the equilibrium when the parameters \((w, A)\) belong to a dense set (and this extends the result obtained for an economy with securities in zero net supply). With reference to the phenomenon of speculative bubbles, then, an interesting result is the following:

**Proposition 43** Under Assumptions A4 and A7, if \(((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{r}_i)_{i \in I}))\) is an equilibrium with transversality condition (defined in Chapter 1) of the economy \(E_\infty(D, \geq, w, \delta, A)\), then the price of every security in positive supply \((\delta_j > 0)\) is equal to its fundamental value.

**Proof.** See Magill-Quinzii (1996).

From this result it follows that securities in positive supply can never have (under the assumptions of the model) speculative bubbles. For infinite-lived securities in zero supply the situation is different and they admit the possibility of bubbles, we have in fact:

**Proposition 44** Let \(E_\infty(D, \geq, w, \delta, A)\) be an economy satisfying Assumption A6 with at least one infinite-lived security in zero net supply (i.e. \(\delta_j = 0\) and \(\delta'_i = 0\) \(\forall i \in I\)). The following results hold:

(i) If \(((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{r}_i)_{i \in I}))\) is an equilibrium with transversality condition of the economy and if \(\bar{q}\) is defined by:

\[
\tilde{q} (\cdot, j) = \bar{q} (\cdot, j) + \bar{p} (\cdot)
\]

\[
\tilde{q} (\cdot, j') = \bar{q} (\cdot, j') \quad j' \neq j
\]

where \(\bar{p} = (\bar{p} (\xi), \xi \in D)\) is the bubble component, then \(\tilde{q}\) is also an equilibrium security price vector, i.e. there exist portfolios \(\bar{z} = (\bar{z}^1, \bar{z}^2, ..., \bar{z}^I)\) such that \(((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{r}_i)_{i \in I}))\) is an equilibrium with transversality condition of the economy.

(ii) Conversely, if \(((\bar{x}, \bar{z}), (\bar{p}, \bar{q}, (\bar{r}_i)_{i \in I}))\) is an equilibrium with transversality condition of the economy \(E_\infty(D, \geq, w, \delta, A)\) and if the financial markets are complete even without
the infinite-lived securities whose prices exhibit a bubble, then there exists a vector of portfolios \( \bar{z} = (\bar{z}^1, \bar{z}^2, ..., \bar{z}^I) \) and a vector of security prices \( \bar{q} \), under which every security is priced at its fundamental value, such that \((\bar{z}, \bar{z}), (\bar{p}, \bar{q}, (\bar{\pi}^i)_{i \in I})\) is an equilibrium with transversality condition.

(iii) If the hypothesis in (ii) is not satisfied (i.e. if the financial markets are incomplete) then the existence of bubble components in the security prices can have real effects, i.e. there exist equilibria in which the price of some infinite-lived security has a speculative bubble, such that the same real allocation cannot be supported by a vector of security prices under which every security is priced at its fundamental value.

**Proof.** See Magill-Quinzii (1996).  ■

This proposition shows, first of all, that it is always possible to add a bubble component to the equilibrium price of an infinite-lived security in zero net supply so that the resulting price remains an equilibrium price. However, there is a difference between speculative bubbles with complete and incomplete markets. In fact, if at an equilibrium the financial markets are complete even without the securities with speculative bubbles, then the same equilibrium allocation can be supported by pricing every security at its fundamental value, and removing the bubble component doesn't affect the real equilibrium allocation. On the contrary, if markets are incomplete there exist equilibria in which infinite-lived securities have speculative bubbles such that the same equilibrium allocation cannot be obtained if securities are priced at their fundamental values, and in this case removing the bubble component affects the real equilibrium allocation.

Securities in positive supply (such as equity contracts) are introduced to model ownership rights to the income stream created by productive assets such as firms, land or other durable capital goods, and therefore an economy in which there are securities in positive initial supply serves as a model of a production economy in which all the production plans are fixed. Since equity contracts represent an important part of the capital market, the result that on such securities bubbles cannot arise reduces consistently the
role of speculation in this class of models, and this is the central conclusion emerging from the analysis of the infinite-horizon model.

The problem of bubbles on asset prices can now be studied carefully; to this end, the main reference is a recent contribution, due to Santos-Woodford (1997), that examines this problem specifically and in great depth, trying to make order in the field and to give a definitive theoretical settlement to this controversial question. It is along the lines suggested by this contribution that the following pages move.

2.3  Bubbles on asset prices: a general analysis

The analysis undertaken by Santos-Woodford provides a systematic study of rational asset pricing bubbles in an intertemporal competitive equilibrium framework. More precisely, this analysis is not concerned with the problem of existence of equilibrium (a problem that has been tackled in the first Chapter, in which it has been shown as such an equilibrium, even if with some restrictions, exists), but its goal is to give conditions under which speculative bubbles are possible or not in this kind of economies.

In particular, it is possible to study under which circumstances the so-called "fundamental theorem of asset pricing" mentioned above (according to which equilibrium asset prices are equal to the present value of the streams of future dividends to which each asset represents a claim) remains true in the case of economies with trading over an infinite horizon. The framework considered is an intertemporal general equilibrium model with spot markets for goods and markets for securities at each of a countably infinite sequence of dates (therefore the same framework for which the problem of existence of equilibrium has been studied in the last part of the first Chapter), but this model is more general than the one considered until now (in fact it considers assets with a very general structure, it covers both the situation of economies with infinitely lived agents and economies with overlapping generations, and it includes also economies that allow bubbles as an equilibrium phenomenon, like the type of monetary economy considered
by Bewley (1980), that will be treated extensively in the examples presented in the third Chapter). The central result of the analysis is that, under rather general circumstances, asset pricing bubbles do not exist in the framework considered, and the conditions under which they are possible (for example in the case of monetary equilibria) are relatively fragile.

2.3.1 Characteristics of the economy and assumptions

To investigate specifically the issue of speculative bubbles it is convenient to introduce a model that is very close to the one considered in the previous Chapter to study the problem of existence but that, in addition, is more specific relatively to the asset structure and incorporates both the situation of infinitely lived individuals and that of individuals that are finitely lived but are succeeded by their children (overlapping generations). For this reason, this model gives results that are valid for a very general class of economies. The model considers an infinite-horizon economy with sequential trading, that can be described by an event-tree $D$. The economy consists of a discrete sequence of dates $t = 0, 1, 2, \ldots$ and at each date a finite number of nodes can be reached. Each node $\xi$ (relative to the generic date $t$) has a unique immediate predecessor $\xi^-$ (relative to the date $t - 1$) and a finite number of immediate successors $\xi^+$ (relative to the date $t + 1$), and there is a unique initial node $\xi_0$ (the only one relative to date 0). Finally, in this case the notation $\xi' \geq \xi$ indicates that the node $\xi'$ belongs to the subtree whose root is $\xi$ (i.e. either $\xi' = \xi$ or $\xi'$ is a successor - not necessarily immediate - of $\xi$).

At each node $\xi$ there are spot markets for $L(\xi)$ consumption goods and $J(\xi)$ securities (both are finite numbers) and the set of agents which are able to trade in the markets at node $\xi$ is denoted by $I(\xi)$, that is a subset of the (countable) set of individuals $I$ that characterise the economy. We then denote with $D^i$ the subset of the event-tree $D$ consisting of nodes at which individual $i$ can trade, for any $i \in I$ (so that $\xi \in D^i$ if and only if $i \in I(\xi)$); with this notation the agent $i$ is infinitely lived if for any date there exists some node $\xi \in D^i$, otherwise he is finitely lived. In this way, within this framework
it becomes possible to incorporate both the case of infinitely lived agents (covered by the model introduced in the first Chapter) and the case of overlapping generations (not covered by that model). Finally, $\overline{D}^i \subset D^i$ denotes the subset of $D^i$ consisting of terminal nodes for $i$, i.e. nodes after which $i$ no longer trades, and the following hypothesis hold:

\[(i) \forall i \in I, \text{ if } \xi \in D^i \text{ and } \xi \notin \overline{D}^i \text{ then } \xi^+ \in D^i\]

\[(ii) \forall \xi \in D \text{ there exists at least one } i \in I \text{ for which } \xi \in D^i \setminus \overline{D}^i\]

The first hypothesis states that if $\xi$ is a node at which the individual $i$ trades, and it is not a terminal node for that individual, then the individual trades at the immediate successors of $\xi$ (and therefore an individual that trades at $\xi$ either trades at none of the successors of $\xi$ - if this is a terminal node for that individual - or trades at all of the immediate successors of $\xi$ - if it isn’t a terminal node -). The second hypothesis states that for every node there is at least one individual for which that node is not terminal, and this guarantees that all the economy is connected.

The securities traded are defined by a vector of prices $q(\xi)$ and the returns they give at future nodes, and in this case these returns are represented by a matrix $D(\xi)$ of dimension $L(\xi) \times J(\xi^-)$, whose components are dividends in terms of goods, and a matrix $B(\xi)$ of dimension $J(\xi) \times J(\xi^-)$, whose components are dividends in terms of other securities. As a consequence, an agent that holds a portfolio $z \in R^{J(\xi^-)}$ at the end of trading at node $\xi^-$, obtains a vector of goods dividends $D(\xi)z$ and a vector of securities $B(\xi)z$ at node $\xi$. The specific character of this framework, therefore, is represented by the fact that securities give a dividend represented not only by a vector of goods (as in the infinite-horizon model of the first Chapter), but also by a vector of other securities, and in this way it is possible to treat multi-period securities. This is the other aspect that makes this model more general with respect to that considered previously, and that allows to obtain results applicable to a wide class of economies. It is also assumed that at each node $D(\xi), B(\xi) \geq 0$, and hence the dividends in terms of goods and of future
securities to which any security represents a claim are non-negative at all nodes.

Each individual has an initial endowment of securities at the initial node \( \xi_0 \) given by \( \overline{z}(\xi_0) \), and the net supply of securities at each node can then be defined recursively in the following way:

\[
\begin{align*}
z(\xi_0) &= \sum_{i \in I(\xi_0)} \overline{z}(\xi_0) \\
z(\xi) &= B(\xi) z(\xi^-)
\end{align*}
\]

In particular, if \( I(\xi_0) \) is an infinite set, then initial securities endowments must be such that the sum is finite, so that there is a finite net supply of securities in all periods. It is also assumed that \( z(\xi_0) \geq 0 \) (even if single individuals may have negative initial endowments), and therefore \( z(\xi) \geq 0 \) at all nodes, so that all securities are in zero or positive net supply (a distinction that, as in the model considered in Section 2.2, will result important for the possibility of bubbles on their prices).

At this point it is possible to determine the streams of future dividends associated with a given security. For all \( \xi' \geq \xi \) we can define the matrix \( E(\xi') \) of dimension \( J(\xi') \times J(\xi) \) in the following way:

\[
E(\xi) = I_J(\xi) \\
E(\xi') = B(\xi') E(\xi^-) \quad \text{for all } \xi' > \xi
\]

and then we can define the matrix \( A(\xi') \) of dimension \( L(\xi') \times J(\xi) \) as:

\[
A(\xi') = D(\xi') E(\xi^-) \quad \text{for all } \xi' > \xi
\]

so that the portfolio \( z \) held at node \( \xi \) represents a claim to the stream of goods dividends \( A(\xi')z \) at each node \( \xi' > \xi \). With this notation, then, a security \( j \) traded at node \( \xi \) is of finite maturity (finitely-lived) if there exists a date such that \( E_{ij}(\xi') = 0 \) for all \( i \) and all
\( \xi' > \xi \) from that date on, otherwise the security is of \textit{infinite maturity} (infinite-lived).

At each node each individual has an endowment of consumption goods \( w^i(\xi) \in \mathbb{R}_{+}^{L(\xi)} \) and we assume that the economy has a finite aggregate endowment:

\[
\begin{align*}
  w(\xi) &= \sum_{i \in I(\xi)} w^i(\xi) \\
  \text{that is non-negative. Considering also the goods that are dividends on securities in positive net supply, then, the aggregate goods supply of the economy is given by:} \\
  \tilde{w}(\xi) &= w(\xi) + D(\xi)z(\xi^-)
\end{align*}
\]

\[
\text{that is non-negative.}
\]

Each individual has preferences described by an ordering \( \succ_i \) defined on the consumption set:

\[
X^i = \prod_{\xi \in D^i} \mathbb{R}_{+}^{L(\xi)}
\]

i.e. defined for all consumption plans that involve non-negative consumption goods at each node \( \xi \in D^i \), and the consumption sets must be bounded below.

We can now introduce the following two assumptions, that are essential for the results that will be obtained:

B1 (preferences): \textit{For each } \( i \in I \text{ the preference relation } \succ_i \text{ is non-decreasing on } X^i \text{ and strictly increasing in the consumption of some good traded at each node } \xi \in D^i.} \]

B2 (uniform impatience): \textit{For each } \( i \in I \), \textit{there exists } \( 0 \leq \beta^i < 1 \) \textit{such that:}

\[
(x^i(\xi), x^i(\xi) + \tilde{w}(\xi), \beta x^i_+(\xi)) \succ_i x^i \quad \forall \xi \in D^i
\]

\textit{for all consumption plans satisfying } \( x^i(\xi') \leq \tilde{w}(\xi') \text{ at each } \xi' \in D^i \text{ and all } \beta \geq \beta^i.} \]
In particular, B1 is very close to assumption A3 introduced in the infinite-horizon model of Chapter 1 (in this case is not required the continuity of preferences in the Mackey topology because we are not dealing with problems of existence of the equilibrium). With reference to B2, for any consumption plan $x^i \in X^i$ and any node $\xi \in D^i$ it is possible to write $x^i = (x^i_-(\xi), x^i(\xi), x^i_+(\xi))$ where $x^i_-(\xi)$ denotes the consumption at nodes other than the subtree with root $\xi$, $x^i(\xi)$ denotes the consumption at node $\xi$ and $x^i_+(\xi)$ denotes the consumption at the nodes of the subtree with root $\xi$. Assumption B2 therefore means that individuals strictly prefer to $x^i$ a consumption plan in which consumption at node $\xi$ is increased by the amount $\tilde{w}(\xi)$ and consumption at all nodes that strictly succeed $\xi$ is reduced by the factor $\beta$. This implies that they are impatient, and the value $1 - \beta$ (the proportion of their future consumption plan that they are ready to give up in order to consume more at node $\xi$) is a measure of impatience that can be called the degree of impatience. Assumption B2 implies that there is a positive lower bound on the degree of impatience uniform across the nodes, and therefore that individuals are sufficiently impatient. This has the same meaning of Assumption A4 in the infinite-horizon model of Chapter 1, and as it will be clear in the following pages, this assumption allows to strengthen the results concerning non-existence of asset pricing bubbles.

Given all these elements it is now possible to consider the choice problem faced by the agents; at each node $\xi \in D^i$ the individual $i$ chooses a vector of consumption goods $x^i(\xi)$ (of dimension $L(\xi)$) and a vector of securities $z^i(\xi)$ (of dimension $J(\xi)$) subject to the following budget constraints:

\begin{align*}
(B) \quad \begin{cases}
    p(\xi)(x^i(\xi) - w^i(\xi)) & \leq V(\xi)z^i(\xi) - q(\xi)z^i(\xi) \\
    x^i(\xi) & \geq 0 \\
    q(\xi)z^i(\xi) & \geq -M^i(\xi)
\end{cases}
\end{align*}

where $p(\xi)$ is the vector (of dimension $L(\xi)$) of goods prices in the spot market at node
\( \xi, q(\xi) \) is the vector (of dimension \( J(\xi) \)) of securities prices and:

\[
V(\xi) = p(\xi)D(\xi) + q(\xi)B(\xi)
\]

is the vector (of dimension \( J(\xi^-) \)) of one-period returns if node \( \xi \) is reached. The first constraint is the standard budget constraint for an economy with sequential trading; in particular, if agent \( i \) doesn't trade at \( \xi^- \) then the constraint has the same form but \( z^i(\xi^-) = 0 \), while at the initial date \( t = 0 \) this constraint takes the form:

\[
p(\xi_0)(z^i(\xi_0) - w^i(\xi_0)) \leq q(\xi_0)(\bar{z}^i(\xi_0) - z^i(\xi_0))
\]

The second constraint is the lower bound on the consumption set, and the third constraint represents the maximum amount that each individual can borrow at each node (in fact \( M^i(\xi) \geq 0 \) indicates the borrowing limit of agent \( i \) at node \( \xi \)). If the matrix \( V(\xi) \), which has one row corresponding to \( V(\xi^+) \) for each of the nodes \( \xi^+ \) that strictly succeed \( \xi \), has the rank equal to the number of rows (i.e. the number of immediate successor nodes, denoted in the previous Chapter by \( b(\xi) \) - the so-called branching number -) we have complete markets at node \( \xi \) (this requires that the number of securities traded at \( \xi \) be at least as large as the number of immediate successor nodes, i.e. \( J(\xi) \geq b(\xi) \)), otherwise we have incomplete markets.

Finally it is possible to introduce, for the economy described, the following definition of equilibrium:

**Definition 45** Let an economy with sequential trading be specified by an event tree \( D \), a set of agents \( I \), the participation sets \( \{D^i\} \), the securities processes \( \{B(\xi), D(\xi)\} \), the initial securities endowments \( \{\bar{z}^i\} \), the endowment processes \( \{w^i(\xi)\} \), the preferences \( \geq_i \), over consumption sets \( X^i \) and the borrowing limits \( \{M^i(\xi)\} \), given price processes \( \{p(\xi), q(\xi)\} \).

Then the processes \( \{\bar{z}^i(\xi), \bar{z}(\xi), p(\xi), q(\xi)\} \) describe an equilibrium if:

(i) for each \( i \in I \) the processes \( \{\bar{z}^i(\xi), \bar{z}(\xi)\} \) are optimal under the preferences \( \geq_i \),

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subject to the budget constraints \((B)\) and given the prices \({p(\xi), q(\xi)}\) and the borrowing limits \({M^i(\xi)}\);

(ii) for each \(\xi \in D\):

\[ p(\xi) \geq 0 \quad q(\xi) \geq 0 \]

(iii) for each \(\xi \in D\):

\[ \sum_{i \in I(\xi)} \bar{x}^i(\xi) \leq \bar{w}(\xi) \quad \sum_{i \in I(\xi)} \bar{x}^i(\xi) \leq z(\xi) \]

(iv) for each \(\xi \in D\):

\[ p(\xi) \left[ \sum_{i \in I(\xi)} x^i(\xi) - \bar{w}(\xi) \right] = 0 \quad q(\xi) \left[ \sum_{i \in I(\xi)} z^i(\xi) - z(\xi) \right] = 0 \]

Given this characterization of the economy it is now possible to focus the attention on the phenomenon of speculative bubbles on asset prices that can occur in this economy.

2.3.2 Pricing by arbitrage and the value of a dividend stream

The model introduce above can be used to determine the value of an arbitrary income stream by adopting considerations of no-arbitrage on financial markets; in this way the relation between the price of a security and the value of the stream of dividends to which it represents a claim can be determined. The existence of an equilibrium in a model of this kind requires the absence of arbitrage opportunities, and with the notation introduced in the last subsection this means that given the price processes \((p, q)\) there does not exist a portfolio \(z \in \mathbb{R}^J(\xi)\) such that:

\[ V(\xi')z \geq 0 \quad \text{for all } \xi' \in \xi^+ \]

\[ q(\xi)z \leq 0 \]
with at least one strict inequality, i.e. there isn't a trading strategy that generates a positive return in at least one node and a non-negative return in all the remaining nodes. This in turn implies the existence of positive state prices $\pi(\xi)$ and $\pi(\xi')$ for all $\xi' \in \xi^+$ such that:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')V(\xi') \forall \xi \in D^-$$  \hspace{0.5cm} (2.2)

that is the relation illustrated at the beginning of this Chapter. It is important to observe that this relation restricts only the ratios $\pi(\xi')/\pi(\xi)$, and the existence of such state prices for each node $\xi$ allows to define some state-price process \{\pi(\xi)\} for the entire event-tree such that this relation holds. In particular, it is possible to denote the set of such processes for the subtree with root $\xi$ with the symbol $\Pi(\xi)$. If at node $\xi$ there are complete markets, then the relation (2.2) uniquely determines the ratios $\pi(\xi')/\pi(\xi)$ for each $\xi' \in \xi^+$, while if there are incomplete markets there is not a unique solution to this equation (as it has been observed in the previous Section, and this is the reason that can determine the presence of a particular kind of bubbles, the so-called ambiguous bubbles).

By solving recursively the equation (2.2) it is possible to show that, in the case of infinite horizon, the price of an asset is not necessarily equal to its fundamental value; in fact, the following result holds:

**Proposition 46**  At each node $\xi \in D$, for any state-price process $\pi \in \Pi(\xi)$ the following relation is satisfied:

$$0 \leq f(\xi) \leq q(\xi)$$

where $f(\xi)$ is the vector of fundamental values for the securities traded at $\xi$. 

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Proof. The equation (2.2) can be written as:

\[ \pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')V(\xi') = \sum_{\xi' \in \xi^+} \pi(\xi') [p(\xi')D(\xi') + q(\xi')B(\xi')] = \]

\[ = \sum_{\xi' \in \xi^+} \pi(\xi')p(\xi')D(\xi') + \sum_{\xi' \in \xi^+} \pi(\xi')q(\xi')B(\xi') \]

and also:

\[ \pi(\xi)q(\xi)E(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')p(\xi')D(\xi')E(\xi) + \]

\[ + \sum_{\xi' \in \xi^+} \pi(\xi')q(\xi')B(\xi')E(\xi) \]

i.e.:

\[ \pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')p(\xi')A(\xi') + \sum_{\xi' \in \xi^+} \pi(\xi')q(\xi')E(\xi') \quad (2.3) \]

By applying repeatedly this equation we get:

\[ \pi(\xi)q(\xi) = \sum_{\xi' \in \mathcal{D}_T(\xi)} \pi(\xi')p(\xi')A(\xi') + \sum_{\xi' \in \mathcal{D}_T(\xi)} \pi(\xi')q(\xi')E(\xi') \]

where \( \mathcal{D}_T(\xi) \) denotes (as in the first Chapter) the nodes between dates \( t(\xi) \) and \( T \) that belong to the subtree \( \mathcal{D}(\xi) \) with root \( \xi \), and \( \mathcal{D}_T(\xi) \) denotes the nodes at date \( T \) that belong to the same subtree. Since the second sum in the right-hand side is non-negative we have:

\[ \pi(\xi)q(\xi) \geq \sum_{\xi' \in \mathcal{D}_T(\xi)} \pi(\xi')p(\xi')A(\xi') \]

and, since \( A(\xi') \geq 0 \) for all \( \xi' \), the right-hand side is a non-decreasing series in \( T \). This series is bounded above, and therefore it must converge to a limit no greater than the
left-hand side, i.e. for $T \to \infty$ we get:

$$\pi(\xi)q(\xi) \geq \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \geq 0$$

from which:

$$q(\xi) \geq \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \geq 0$$

where the central term is the fundamental value of the asset, and therefore:

$$0 \leq f(\xi) \leq q(\xi)$$

that is the result.

This conclusion shows the difference that can emerge in the infinite-horizon case with respect to the finite-horizon case. In the latter, as shown at the beginning of this Chapter, the equilibrium price of an asset must necessarily follow the relation:

$$q(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi')$$

while in the former we have:

$$q(\xi) \geq \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi')$$

and therefore the price of the security can also be larger than its fundamental value. In this case it is possible to define the corresponding vector of asset pricing bubbles as:

$$b(\xi) = q(\xi) - f(\xi)$$

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for any $\pi \in \Pi(\xi)$ for securities traded at $\xi$; from the result of the last proposition we have that this vector satisfies the bounds:

$$0 \leq b(\xi) \leq q(\xi)$$

and therefore we obtain the so-called "impossibility of negative bubbles". If we now use the fact that $q(\xi) = b(\xi) + f(\xi)$ and substitute this expression into equation (2.3) we get:

$$\pi(\xi) [b(\xi) + f(\xi)] = \sum_{\xi' \in \xi^+} \pi(\xi') p(\xi') A(\xi') + \sum_{\xi' \in \xi^+} \pi(\xi') [b(\xi') + f(\xi')] E(\xi')$$

from which:

$$\pi(\xi) b(\xi) + \pi(\xi) f(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi') p(\xi') A(\xi') + \sum_{\xi' \in \xi^+} \pi(\xi') b(\xi') E(\xi') +$$

$$+ \sum_{\xi' \in \xi^+} \pi(\xi') f(\xi') E(\xi')$$

and by substituted the definition of fundamental value:

$$\pi(\xi) b(\xi) + \pi(\xi) \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi') p(\xi') A(\xi') = \sum_{\xi' \in \xi^+} \pi(\xi') p(\xi') A(\xi') +$$

$$+ \sum_{\xi' \in \xi^+} \pi(\xi') b(\xi') E(\xi') + \sum_{\xi' \in \xi^+} \frac{1}{\pi(\xi')} \sum_{\xi'' > \xi'} \pi(\xi'') p(\xi'') A(\xi'') E(\xi')$$

that is:

$$\pi(\xi) b(\xi) + \sum_{\xi' > \xi} \pi(\xi') p(\xi') A(\xi') = \sum_{\xi' \in \xi^+} \pi(\xi') p(\xi') A(\xi') +$$

$$+ \sum_{\xi' \in \xi^+} \pi(\xi') b(\xi') E(\xi') + \sum_{\xi' \in \xi^+} \sum_{\xi'' > \xi'} \pi(\xi'') p(\xi'') A(\xi'') E(\xi')$$
\[ \pi(\xi)b(\xi) + \sum_{\xi'' > \xi'} \pi(\xi'')p(\xi'')A(\xi'') = 2 \sum_{\xi' \in \xi^+} \pi(\xi')b(\xi')E(\xi') + \sum_{\xi'' > \xi'} \pi(\xi'')p(\xi'')A(\xi'') \]

and finally:

\[ \pi(\xi)b(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi')b(\xi')E(\xi') \]

Any vector of asset pricing bubbles, therefore, must satisfy a martingale property. In particular, the last expression implies that if there exists a bubble on a security at date \( t \), there must exist a bubble at every date after \( t \), and if there exists a bubble on a security at a node \( \xi \), then there must have existed a bubble at every predecessor of the node, and in this sense a bubble can "never start".

As it has been observed, the fundamental value of the asset is defined by the expression:

\[ f(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' > \xi} \pi(\xi')p(\xi')A(\xi') \]

and a situation in which this fundamental value is unambiguously defined (even with incomplete markets) is that of finitely-lived securities; this is evident if we consider the expression, obtained before:

\[ \pi(\xi)q(\xi) = \sum_{\xi' \in D^\tau(\xi)} \pi(\xi')p(\xi')A(\xi') + \sum_{\xi' \in D_T(\xi)} \pi(\xi')q(\xi')E(\xi') \]

and we observe that in the case of a finitely-lived security there is a date \( T \) such that from that date on \( E(\xi') = 0 \) and \( A(\xi') = 0 \) (i.e. after date \( T \) the asset doesn't give any dividend, either in terms of goods or in terms of other securities); as a consequence we
get:

\[ \pi(\xi)q(\xi) = \sum_{\xi' \in D^r(\xi)} \pi(\xi')p(\xi')A(\xi') \]

and finally:

\[ q(\xi) = \frac{1}{\pi(\xi)} \sum_{\xi' \in D^r(\xi)} \pi(\xi')p(\xi')A(\xi') \]

that is:

\[ q(\xi) = f(\xi) \]

In this case, therefore, the fundamental value of a security \( j \) is the same for all state-price processes \( \pi \in \Pi(\xi) \) at it is equal to the price \( (f_j(\xi) = q_j(\xi)) \), so that there is no pricing bubble for this security.

In the case of an infinite-lived security in an economy with incompete markets, on the contrary, the fundamental value may be different for different state-price processes consistent with securities returns. In this case, anyway, it is possible to define bounds for the foundamental value; Santos and Woodford (1997) show that for any state-price process and any security \( j \) traded at node \( \xi \) the foundamental value satisfies the relation:

\[ \eta_{x^j}(\xi) \leq f_j(\xi) \leq \bar{\eta}_{x^j}(\xi) \]

Here \( \eta_{x^j}(\xi) \) is the least upper bound for the amount that can be borrowed at node \( \xi \) by an individual whose endowment is \( x^j \) at all subsequent nodes if the individual must hold non-negative wealth at all nodes after some finite date, while \( \bar{\eta}_{x^j}(\xi) \) is the greatest lower bound for the amount of wealth needed at node \( \xi \) in order to be able to purchase a consumption stream equal to the process \( x^j \) at all subsequent nodes if the individual must hold non-negative wealth at all nodes from \( \xi \) onward. These bounds represent therefore a
lower and an upper bound, respectively, for the present value of the non-negative stream of dividends \( \{x'(\xi'), \xi' > \xi \} \), and are also the tightest bounds with this property (for an analytical definition of these terms, see Santos-Woodford (1997)). In particular, if:

\[
\nu_{x'}(\xi) = \nu_{x'}(\xi)
\]

then the fundamental value of the asset is uniquely defined; this is true in the case of complete markets but can also occur with incomplete markets. On the other hand, with incomplete markets it is also possible that the two bounds do not coincide, and in this case the fundamental value may be different for different values of the state-price processes. It is in this sense that the existence of a bubble may be ambiguous, and that we may have what is called an ambiguous bubble. More precisely, there unambiguously exists no bubble if:

\[
\nu_{x'}(\xi) = \nu_{x'}(\xi) = q_j(\xi) \quad \implies \quad q_j(\xi) = f_j(\xi)
\]

and similarly there unambiguously exists a bubble if:

\[
\nu_{x'}(\xi) < q_j(\xi) \quad \implies \quad q_j(\xi) > f_j(\xi)
\]

while if:

\[
\nu_{x'}(\xi) < \nu_{x'}(\xi) = q_j(\xi)
\]

then the existence or not of a bubble depends upon the state-price process \( \pi \in \Pi(\xi) \) chosen. It is in this situation that we may have:

\[
q_j(\xi) = f_j(\xi) \quad \text{for} \quad \pi \in \Pi(\xi)
\]

\[
q_j(\xi) > f_j(\xi) \quad \text{for} \quad \pi' \in \Pi(\xi)
\]
and therefore an ambiguous bubble may arise.

2.3.3 Non-existence of asset pricing bubbles

Given the economy described at the beginning of this Section and given the characterization of the value of a certain stream of dividends introduced above, it is now possible to consider the main results concerning existence and non-existence of bubbles in this framework. In particular, these results apply to an economy in which the aggregate endowment has a finite value. A first statement is the following:

**Theorem 47** Let preferences satisfy Assumption B1 and consider an equilibrium \( \{\bar{z}, \bar{z}, p, q\} \). For any node \( \xi \in D \), suppose that the present value of the aggregate endowment of the economy is finite.

Then there exists a state-price process \( \pi \in \Pi(\xi) \) such that \( q_j(\xi') = f_j(\xi') \) for all \( \xi' \geq \xi \), for each security \( j \) traded at \( \xi' \) that is either finitely-lived or in positive net supply (i.e. for which \( z_j(\xi') > 0 \)).

**Proof.** See Santos-Woodford (1997).

As a consequence, according to this theorem there can be speculative bubbles only on securities that are both infinitely-lived and in zero net supply (that is exactly the result obtained in the infinite-horizon model considered in Section 2.2). Nevertheless, this theorem does not assert that there is an unambiguous fundamental value for any security in positive net supply, in fact it implies that \( q_j(\xi') = \bar{z}_j(\xi') \) for any such security, but it does not exclude the possibility that \( q_j(\xi') > \bar{z}_j(\xi') \), so that there also exist state-price processes \( \pi \in \Pi(\xi) \) in terms of which the market price of the security exceeds its fundamental value. In this case, an ambiguous bubble arises.

The assumption concerning the existence of a sufficient degree of impatience (i.e. the Assumption B2) can be used to strengthen the previous result (as outlined above); we have in fact:
Theorem 48 Let preferences satisfy Assumptions B1 and B2 and consider an equilibrium \( \{\xi, \tilde{z}, p, q\} \). For any node \( \xi \in D \), suppose that there exists a state-price process \( \pi \in \Pi(\xi) \) such that the present value of the aggregate endowment of the economy is finite, when it is evaluated with this state-price process.

Then \( q_j(\xi') = f_j(\xi') \) for all \( \xi' \geq \xi \), for each security \( j \) traded at \( \xi' \) that is either finitely-lived or in positive net supply (i.e. for which \( z_j(\xi') > 0 \)).


In this case, therefore, if the hypothesis are satisfied there are no bubbles regardless of the state-prices chosen (hence also ambiguous bubbles, that may arise when only Assumption B1 is considered, are now excluded); in addition, for the conclusions to be valid it is now sufficient that the present value of the aggregate endowment of the economy be finite for a state-price process, rather than for all state-price processes consistent with the values of the securities. In this sense Theorem 48 represents a strengthening of Theorem 47.

Given these results, it is worth emphasizing the role of the assumption concerning impatience of agents. In fact, in the infinite-horizon model studied in the first Chapter, a sufficient degree of impatience (Assumption A4) is crucial for the existence of the equilibrium. On the other hand, in the model presented in this Section, the corresponding hypothesis (Assumption B2) turns out to be essential in order to exclude the presence of bubbles in very general situations. This suggests the existence of a link between the two aspects (existence of equilibrium and absence of bubbles) when the agents in the economy are sufficiently impatient.

The two theorems presented above apply to a broad class of economies in which the equilibrium value of the aggregate supply of goods is bounded, and for this reason the results obtained with this analysis are very general. It is also possible to give some restrictions upon economic primitives that guarantee the applicability of these theorems; first of all the following corollary holds:
Corollary 49 Let preferences satisfy Assumption B1 and suppose that there exists a portfolio \( \hat{z} \) satisfying \( A(\xi)\hat{z} \geq w(\xi) \) for all \( \xi \geq \xi_0 \).

Then for any equilibrium \( \{\bar{x}, \bar{z}, p, q\} \) there exists a state-price process \( \pi \in \Pi(\xi_0) \) such that the conclusions of Theorem 47 hold.

If, in addition, preferences satisfy Assumption B2, then these conclusions hold for every state-price process \( \pi \in \Pi(\xi_0) \).

**Proof.** See Santos-Woodford (1997). ■

Similarly, the following further corollary holds:

Corollary 50 Let preferences satisfy Assumptions B1 and B2 and suppose that there exist an infinitely-lived agent \( i \in I \) and \( \epsilon > 0 \) such that:

(i) \( w^i(\xi) \geq \epsilon w(\xi) \) for each \( \xi \in D \) (the agent has at least a fraction \( \epsilon > 0 \) of total resources at all dates);

(ii) the borrowing limit is given by \( M^i(\xi) = \eta_{w^i}(\xi) \) (the agent is able to borrow against his future endowment income).

Then for any equilibrium \( \{\bar{x}, \bar{z}, p, q\} \) there exists a state-price process \( \pi \in \Pi(\xi_0) \) such that the conclusions of Theorem 48 hold.

**Proof.** See Santos-Woodford (1997). ■

These corollaries imply in particular that well-known examples of models that allow for speculative bubbles (for instance the overlapping generations model of Samuelson (1958), or the consumption-smoothing model of Bewley (1980) that will be considered in the third Chapter) no longer admit such bubbles if the hypothesis of the corollaries are satisfied.

In conclusion, the general analysis presented in this Section has shown that, under quite general assumptions, speculative bubbles do not exist in an infinite-horizon competitive framework (also when we introduce imperfections such as incomplete markets,
arbitrary borrowing limits and incomplete participation of agents in the sequence of markets). More specifically, if the aggregate endowment of the economy has a finite value, then bubbles do not occur for securities in positive net supply; the presence of bubbles is possible only if, in a given equilibrium, the aggregate wealth of the economy is infinite-valued. In addition, a particular situation that has been illustrated is the possibility that, with incomplete markets, the fundamental value of a security be different for different state-price processes, giving rise to ambiguous bubbles; anyway, this pathology disappears if agents are sufficiently impatient. As a consequence, known examples of pricing bubbles depend upon rather special circumstances, and therefore they are quite fragile.

The same conclusion in favour of a substantial fragility of the phenomenon of bubbles can be obtained with a different approach, based on the use of Euler equations and inequalities; this is the object of the next Section. The interest of this approach is represented mainly by the fact that, by means of it, it becomes relatively easy to construct examples of economies in which bubbles appear, and from which it emerges that the presence of such bubbles is linked to the violation of specific conditions; this last aspect will be considered in the third Chapter.

2.4 Bubbles on asset prices: the approach of Euler equations

In the last part of this Chapter the issue of bubbles on asset pricing is analysed by means of a different approach, based on the use of Euler equations and inequalities. This approach is far more limited than that considered until now (which has led to very general models, used to deal with the problem of existence of solutions and of speculative bubbles in a wide class of intertemporal economies). Nevertheless, it is of some interest because it shows how the results on fragility of bubbles obtained above can be confirmed, and especially because it provides a method that can be used to build and to study examples
in which bubbles appear.

The starting point of this part is a model with homogeneous agents; this framework can then be extended to the case of heterogeneous agents, in order to study an economy of the same type considered up to now. For this model the essential results concerning non-existence of speculative bubbles are derived in this Section. In Chapter 3, finally, a series of examples is provided, in order to show how the presence of such bubbles is related to very special conditions, therefore confirming the results obtained by means of the general analysis of the previous Section.

2.4.1 The model with homogeneous agents

The first step of the analysis based on the use of Euler equations and inequalities (for details, see Montrucchio-Privileggi (1999)) is a model "à la Lucas" (Lucas (1978)) that considers consumers which are identical in terms of utilities and endowments (hence homogeneous agents). At each trading time \((t = 0, 1, 2, \ldots)\) there are spot markets for a single non-storable consumption good and for shares in \(J\) productive assets, that produce random quantities of the good in all periods. The main difference with respect to the models introduced above is that it is no longer true that the number of states of nature at each date is finite. As a consequence, in order to describe uncertainty in this model it is necessary to introduce a probabilistic space \((\Omega, \mathcal{F}, \mu)\), where \(\Omega\) is the set of the states of the world, \(\mathcal{F}\) is a filtration of \(\sigma\)-algebras of events \(\mathcal{F} = \{\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_t \subset \ldots \subset \mathcal{F}\}\), it describes the revelation of information, that is uniform across individuals, and expresses the idea that this information is revealed gradually and increases over time), and \(\mu\) is a probability measure defined over events. Here we assume that \(\mathcal{F}_0\) is the trivial algebra; the model, however, remains true even if this is not the case (in this situation the expectation \(E\) must be replaced with \(E_0 = E(\cdot | \mathcal{F}_0)\)).

Another difference with respect to the models introduced before is that, in this case, we consider a particular type of preferences; in fact each individual has preferences given
by the separable life-time utility:

\[ \mathbb{E} \sum_{t=0}^{\infty} u_t [x_t(\omega), \omega] \]

defined over the consumption processes \( x = \{x_t(\omega)\} \). In addition, every agent receives an endowment of the consumption good described by the process \( w = \{w_t(\omega)\} \), while the process \( d = \{d_t(\omega)\} \) represents the asset dividends (where \( d_t \in \mathbb{R}_+^d \) is the amount of the consumption good yielded by one unit of each asset at each period) and \( z = \{z_t(\omega)\} \) denotes the asset holding strategy. In addition, the initial endowment of each asset is normalized to one, i.e. \( z_0 = e = (1, 1, \ldots, 1) \in \mathbb{R}_1^d \), and therefore we consider here only securities in positive net supply (and in this case bubbles on these securities can arise, contrary to the results of the models considered above, because it is violated the assumption concerning impatience of agents, on which those results are based). By denoting with \( q_t \in \mathbb{R}_+^d \) the prices of the assets and by taking the single consumption good as the numeraire (so that its price is \( p_t = 1 \)) we have that a plan \((x, z) = \{x_t(\omega), z_t(\omega)\}\) is feasible if it satisfies:

\[ x_t(\omega) + q_t(\omega)[z_{t+1}(\omega) - z_t(\omega)] \leq d_t(\omega)z_t(\omega) + w_t(\omega) \]

The assumptions on which the analysis is based are the following:

\[ C1 \ 0 < d_t(\omega) \cdot e + w_t(\omega) < +\infty \ \text{almost surely for all } t. \]

\[ C2 \ For \ each \ fixed \ \omega, \ u_t(\cdot, \omega) \ is \ concave, \ strictly \ increasing \ and \ differentiable \ over \ \mathbb{R}_{++}. \]

In particular, as a consequence of \( C2 \), also unbounded utilities (for instance, logarithmic utility) are allowed in this model. In the economy characterized by these elements (in which we also assume that short-selling is prohibited, i.e. individuals cannot borrow -
in order to avoid the so-called *Ponzi scheme* - we have a no-trade equilibrium (as a consequence of the fact that the individuals are identical in preferences and in endowments), that is an equilibrium in which agents hold their assets forever and consume all their available wealth at each date. By dropping, for simplicity, the argument \( \omega \) of the random functions considered it is now possible to characterize this equilibrium. With reference to this aspect we have (here \( X^- \) denotes the negative part of the random variable \( X \)):

**Definition 51** A no-trade equilibrium for the economy considered is a pair of allocations and prices \((\overline{x}, \overline{z}), q\) such that:

(i) the price process \( q \) satisfies:

\[
0 \leq q_t < +\infty \quad \text{almost surely for all } t
\]

(ii) the plan \( \overline{x} = \{\overline{x}_t\} = \{d_t \cdot e + w_t\}, \overline{z} = \{e\} \) is (weakly) optimal with respect to all feasible plans, i.e. it satisfies the conditions:

\[
\begin{align*}
(ii-a) \quad & \mathbb{E}[u_t(\overline{x}_t) - u_t(x_t)]^- < +\infty \quad \text{almost surely for all } t \\
(ii-b) \quad & \limsup_{N \rightarrow +\infty} \mathbb{E} \sum_{t=0}^{N-1} [u_t(\overline{x}_t) - u_t(x_t)] \geq 0 \quad \text{almost surely}
\end{align*}
\]

Given all these elements it is possible to obtain conditions that must be satisfied at the equilibrium. The first relation takes the form of a stochastic Euler inequality and represents a necessary condition of optimality in the short-run. The following result holds:

**Proposition 52** Under Assumptions C1 and C2, if \((\overline{x}, \overline{z}), q\) is an equilibrium then:

\[
u'_{t-1}(\overline{x}_{t-1})q_{t-1} \geq \mathbb{E}_{t-1}[u'_t(\overline{x}_t)(d_t + q_t)] \quad \text{for all } t \geq 1
\]

**Proof.** See Montrucchio-Privileggi (1999).  

In particular, then, the relation holds as an equality if the following two conditions
are satisfied:

\[ q_t \cdot e \leq M \bar{x}_t \]

for some scalar \( M \)

\[ \mathbb{E}_{t-1}[u_t(\bar{x}_t) - u_t(\zeta d_t \cdot e + w_t)] < +\infty \]

for some \( \zeta < 1 \)

and this is true, for instance, when the states of the world are finite at date \( t \).

The expression (2.4) can be used to introduce, also in this approach, the distinction between the fundamental value of an asset and the bubble component; starting from it we can define:

\[ u'_{t-1}(\bar{x}_{t-1})q_{t-1} = \mathbb{E}_{t-1}[u'_t(\bar{x}_t)(d_t + q_t)] + s_{t-1} \]

for \( t = 1, 2, \ldots \), where the vectors \( s_t \geq 0 \) measure the deviation from equality in the Euler equation. If we now write, for simplicity, \( u'_t(\bar{x}_t) = \pi_t \) and we iterate the last equation, we get:

\[ \pi_t q_t = \mathbb{E}_t \sum_{r=1}^{k} \pi_{t+r} d_{t+r} + \mathbb{E}_t \sum_{r=0}^{k-1} s_{t+r} + \mathbb{E}_t[\pi_{t+k} q_{t+k}] \]

and by considering the limit for \( k \) that goes to infinity:

\[ \pi_t q_t = \mathbb{E}_t \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r} + \mathbb{E}_t \sum_{r=0}^{\infty} s_{t+r} + \lim_{k \to +\infty} \mathbb{E}_t[\pi_{t+k} q_{t+k}] \quad (2.5) \]

and also:

\[ q_t = \frac{1}{\pi_t} \mathbb{E}_t \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r} + \frac{1}{\pi_t} \mathbb{E}_t \sum_{r=0}^{\infty} s_{t+r} + \frac{1}{\pi_t} \lim_{k \to +\infty} \mathbb{E}_t[\pi_{t+k} q_{t+k}] \]

The price of the asset can therefore be decomposed into three components:

\[ q_t = f_t + \bar{b}_t + \tilde{b}_t \]
where the first term of the right-hand side is the fundamental value:

$$f_t = \frac{1}{\pi_t} \sum_{r=1}^{\infty} \pi_{t+r} d_{t+r}$$

while the other two terms represent the bubble component. In particular, the second term is the bubble component due to the violation of the Euler equation:

$$\widetilde{b}_t = \frac{1}{\pi_t} \sum_{r=0}^{\infty} s_{t+r}$$

while the third term is the asymptotic bubble:

$$\bar{b}_t = \frac{1}{\pi_t} \lim_{k \to \infty} \mathbb{E}_t [\pi_{t+k} g_{t+k}]$$

It is possible to observe that the process of fundamental values $f = \{f_t(\omega)\}$ satisfies the Euler equation because we have:

$$\pi_{t-1} f_{t-1} = \mathbb{E}_{t-1} [\pi_t (d_t + f_t)]$$

while the process of asymptotic bubbles $\bar{b} = \{\bar{b}_t(\omega)\}$ follows the martingale:

$$\pi_{t-1} \bar{b}_{t-1} = \mathbb{E}_{t-1} [\pi_t \bar{b}_t]$$

and the process of the total bubbles $b = \{b_t(\omega) = \bar{b}_t(\omega) + \tilde{b}_t(\omega)\}$ follows the supermartingale:

$$\pi_{t-1} b_{t-1} \geq \mathbb{E}_{t-1} [\pi_t b_t]$$

Also in this framework we get the result, derived in the general analysis of the previous Section, according to which a bubble "never starts": if there exists a bubble on a security at date $t$, then there must have existed a bubble at date $t - 1$, and so on.
It can also be shown that this analysis is consistent with the results of the previous Section, based on considerations of no-arbitrage. At the beginning of this Chapter it has been emphasized as the absence of arbitrage opportunities implies the existence of positive state-prices $\pi$ such that the following intertemporal no-arbitrage equation is satisfied:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \pi(\xi') [p(\xi')A(\xi') + q(\xi')]$$  \hspace{1cm} (2.6)

In the model based on the use of Euler equations a similar relation can be derived. In fact, given an equilibrium in which the Euler equation is satisfied as an equality, we can write:

$$\pi_t q_t = \mathbb{E}_t[\pi_{t+1}(d_{t+1} + q_{t+1})] \hspace{1cm} \text{for all } t \geq 0$$  \hspace{1cm} (2.7)

Here the $\pi$'s are not the traditional state-prices considered above, because they are distorted by the probability law, and for this reason they can be defined pseudo-state prices. However, there is a one-to-one correspondence with the traditional state prices when the stochastic process is given through finite information nodes. In this case, in fact, the expression (2.7) becomes:

$$\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \mu(\xi' \mid \xi) \pi(\xi') [d(\xi') + q(\xi')]$$

where $\mu(\xi' \mid \xi)$ is the transition probability and $\xi, \xi'$ are adjacent nodes ($\xi' \in \xi^+$, that is $\xi'$ is an immediate successor of $\xi$). By multiplying through $\mu(\xi)$ we get:

$$\mu(\xi)\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \mu(\xi' \mid \xi)\pi(\xi') [d(\xi') + q(\xi')]$$
\[
\mu(\xi)\pi(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \mu(\xi')\pi(\xi') [d(\xi') + q(\xi')]
\]
and finally:
\[
\bar{\pi}(\xi)q(\xi) = \sum_{\xi' \in \xi^+} \bar{\pi}(\xi') [d(\xi') + q(\xi')]
\]
where \(\bar{\pi}(\xi) = \mu(\xi)\pi(\xi)\) are the traditional state-prices. The last expression is the traditional intertemporal no-arbitrage equation and corresponds to equation (2.6) (where the dividends are given by \(d(\xi') = p(\xi')A(\xi')\)). The formulation of this no-arbitrage equation in the form given by (2.7) is particularly useful when the states of the world are not necessarily finite at any date.

At this point, the final step of the analysis is represented by the determination of conditions that guarantee the existence of equilibria and that allow to rule out bubbles. A first fundamental result is the following:

**Theorem 53** Under Assumptions C1 and C2 and the condition \(E[u(t_0)(d_0^e + w_0)w_0] < +\infty\) for all \(t \geq 0\), a necessary and sufficient condition for an equilibrium to exist is that:
\[
E \sum_{t=1}^{\infty} u_t'(d_t \cdot e + w_t)d_t < +\infty \tag{2.8}
\]
In this case an equilibrium price is given by the fundamental values \(f = \{f_t(\omega)\}\).

**Proof.** See Montrucchio-Privileggi (1999). 

The next result is a sufficient condition for the uniqueness of equilibrium (due to Kamihigashi (1998)):
Theorem 54 A sufficient condition for the fundamental price $f$ to be the unique equilibrium is that for some scalar $0 < \zeta < 1$:

$$
\mathbb{E} \sum_{t=1}^{\infty} \left[ u_t(d_t \cdot e + w_t) - u_t(\zeta d_t \cdot e + w_t) \right] < +\infty
$$


In addition, a sufficient condition for this relation to hold is that:

$$
\mathbb{E} \sum_{t=1}^{\infty} u_t'(\zeta d_t \cdot e + w_t) d_t < +\infty \quad (2.9)
$$

The central result of this analysis can now be derived. It concerns the fragility of bubbles and it can be obtained starting from the observation of the similarity between the condition (2.8):

$$
\mathbb{E} \sum_{t=1}^{\infty} u_t'(d_t \cdot e + w_t) d_t < +\infty
$$

that is necessary for the existence of at least one equilibrium, and the condition (2.9):

$$
\mathbb{E} \sum_{t=1}^{\infty} u_t'(\zeta d_t \cdot e + w_t) d_t < +\infty
$$

that is sufficient for the uniqueness of the equilibrium (in which the prices of the assets are equal to their fundamental values, so that bubbles are absent). This similarity suggests that price bubbles are a fragile phenomenon and constitute a borderline event, as can be proved by considering a slight modification in the amount of assets held by the individual (or in the dividend stream). The conclusion is expressed by the following theorem:

Theorem 55 If for an initial endowment of the assets $z_0 = v \in \mathbb{R}^d_+$ a price bubble occurs, then for each initial endowment $\bar{v} \gg v$ there is only one equilibrium (without bubbles), while for each initial endowment $v \ll v$ there are no equilibria at all.
Similarly, if for a dividend sequence \( \{d_t\} \) a bubble arises, then for dividend sequences \( \{\xi d_t\} \) with \( \xi > 1 \) there is only one equilibrium (without bubbles), while for dividend sequences \( \{\zeta d_t\} \) with \( \zeta < 1 \) there are no equilibria at all.

**Proof.** We consider the case in which the initial supply of the asset is \( z_0 = v \); if there is a bubble, since condition (2.8) must hold we have:

\[
E \sum_{t=1}^{\infty} u'(d_t \cdot v + w_t) d_t < +\infty
\]

By taking a vector \( \bar{v} \) such that \( \bar{v} \gg v \), we have (since \( u' \) is decreasing):

\[
E \sum_{t=1}^{\infty} u'(d_t \cdot \bar{v} + w_t) d_t < +\infty
\]

and then there is some \( \zeta < 1 \) for which \( \zeta \bar{v} \gg v \), so that we also have:

\[
E \sum_{t=1}^{\infty} u'(\zeta d_t \cdot \bar{v} + w_t) d_t < +\infty
\]

This is the sufficient condition (2.9) for the equilibrium with initial endowment \( \bar{v} \) to be unique, hence when we consider this initial endowment the bubble component disappears.

Similarly, if we start from an initial supply of the asset \( z_0 = v \) (in correspondence of which there is a bubble) and then we take a vector \( \nu \ll v \), by assuming that equilibria do exist we must have, by condition (2.8):

\[
E \sum_{t=1}^{\infty} u'(d_t \cdot \nu + w_t) d_t < +\infty
\]

If we consider \( \xi > 1 \), then, we also have (since \( u' \) is decreasing):

\[
E \sum_{t=1}^{\infty} u'(\xi d_t \cdot \nu + w_t) d_t < +\infty
\]
and if we choose $\zeta < 1$ and $\xi > 1$ such that:

$$\nu \ll \xi \nu \ll \zeta \nu \ll \nu$$

then it must be:

$$\mathbb{E} \sum_{i=1}^{\infty} u'_t(\zeta d_i \cdot v + w_i) d_i < +\infty$$

This is the sufficient condition (2.9) for the equilibrium with initial endowment $v$ to be unique (without bubbles), but this contradicts the assumption that for $z_0 = v$ a bubble exists, and therefore we conclude that for $\nu \ll v$ there are no equilibria.

The same reasoning applies if we consider a perturbation of the dividends $d_t$, and this completes the proof. □

This theorem shows in which sense speculative bubbles are a fragile phenomenon: by starting from an equilibrium with bubbles, a slight modification of the amounts of assets (or that of dividends) has the effect that bubbles disappear. Equivalently, the set of initial endowments with bubbles has zero Lebesgue measure in $\mathbb{R}_{++}^J$.

Another fundamental result in favour of hubble fragility is related to the risk aversion of the individuals, in fact the following statement holds:

**Theorem 56** If preferences $u_t$ exhibit uniformly bounded relative risk aversion, i.e.:

$$-\frac{u''(x)x}{u'(x)} \leq R$$

for all $x \geq 0$, $t \geq 0$ and for some scalar $R$, then pricing equilibrium is uniquely determined.

**Proof.** We consider the function:

$$f(x) = u'_t(x + h)x^R \quad \text{for } x \geq 0, h \geq 0, t \geq 0$$
whose derivative is:

\[ f'(x) = u'_t(x + h)Rx^{R-1} + u''_t(x + h)x^R = \]

\[ = x^{R-1}[u'_t(x + h)R + u''_t(x + h)x] \]

and since:

\[ -\frac{u''_t(x)x}{u'_t(x)} \leq R \Rightarrow u'_t(x)R + u''_t(x)x \geq 0 \]

we also have:

\[ u'_t(x + h)R + u''_t(x + h)(x + h) \geq 0 \]

and also:

\[ u'_t(x + h)R + u''_t(x + h)x \geq 0 \]

As a consequence, \( f'(x) \geq 0 \) and the function \( f(x) \) is non-decreasing, therefore we can write, for \( \zeta \leq 1 \):

\[ u'_t(\zeta x + h)(\zeta x)^R \leq u'_t(x + h)x^R \]

i.e.:

\[ u'_t(\zeta x + h)\zeta^R \leq u'_t(x + h) \]

and finally:

\[ u'_t(\zeta x + h) \leq \zeta^{-R}u'_t(x + h) \]
At this point, if condition (2.8) (that is necessary and sufficient for the existence of an equilibrium) is satisfied, then from the last inequality we have that also condition (2.9) (that is sufficient for the uniqueness of the equilibrium) is satisfied, and therefore if an equilibrium exists it is unique and it is an equilibrium without bubbles on asset prices.

In conclusion, Theorem 55 states that, generically, either the equilibrium is unique or no equilibrium exists, and Theorem 56 states that standard preferences cannot give rise to bubbles. As a consequence, the emergence of such bubbles is possible only in a non-generic set of economies, and they represent a very negligible phenomenon.

All these results have been obtained with reference to a situation characterized by the presence of homogeneous agents. Since the general analysis illustrated in the previous Section covers the more general case of heterogeneous agents, the last step consists in extending the approach based on Euler equations and inequalities to this situation. This is the topic of the last part of this Chapter.

2.4.2 The model with heterogeneous agents

The model introduced in the previous Subsection can be generalized in order to cover the situation in which individuals are heterogeneous (since the most important examples of economies with bubbles - for instance the consumption-smoothing model of Bewley - fall in this category). In this extension (for details, see Montrucchio (1999)) we consider a finite set \( I \) of consumers, each of whom has preferences given by:

\[
E \sum_{t=0}^{\infty} u^i_t [x^i_t(\omega), \omega]
\]

defined over a consumption good \( x \), where the utility function \( u^i_t(\cdot, \omega) \) is concave, strictly increasing and differentiable, with \( i \in I \) and \( t = 0, 1, 2, \ldots \). There are also \( J \) perpetual productive assets (exactly as in the case of homogeneous agents) that give a dividend in each period represented by a certain amount of the single, non-storable consumption
good, and the stream of their dividends is denoted by \( d_i \in \mathbb{R}_+^J \) (amount of the consumption good yielded by one unit of each asset at each period), while \( z_i^t \in \mathbb{R}^J \) denotes the asset holding strategy for individual \( i \) at time \( t \).

At each trading date there are spot markets both for the consumption good and for the assets, and in addition each individual receives an endowment of the consumption good \( w_i^t \geq 0 \) and, at time \( t = 0 \), a share \( z_0^i \) of the total asset supply, that is normalized to 1 (i.e. \( \sum_{i \in I} z_0^i = e = (1, 1, \ldots, 1) \in \mathbb{R}^J \)).

By denoting with \( q_t \in \mathbb{R}_+^J \) the prices of the assets and by taking the single consumption good as the numeraire (so that its price is \( p_t = 1 \)), each agent \( i \) maximizes his utility over time facing, in each period \( t \), the traditional budget constraint:

\[
x_i^t + q_t (z_{i+1}^t - z_i^t) \leq d_i z_i^t + w_i^t
\]

according to which the expenditure in each period cannot exceed the endowment of the same period (and this constraint implies the feasibility of a consumption plan). We also have a borrowing constraint, represented by \( z_i^t \geq k_i^t \). Usually it is \( k_i^t \leq 0 \) and, since a negative value of \( z_i^t \) represents a short-selling of the assets (in order to obtain a borrowing of the corresponding amount), the constraint \( z_i^t \geq k_i^t \) can be interpreted as the fact that \( k_i^t \) represents a borrowing limit. In particular, if \( k_i^t = 0 \) short-selling is prohibited. Finally, the presence of uncertainty in the model is described through a probabilistic space \( (\Omega, F, \mu) \), exactly as in the case of homogeneous agents.

Given all these elements, the following definition of equilibrium can be introduced:

**Definition 57** An equilibrium for the economy considered is a pair of allocations and prices \( ((\bar{x}^t, \bar{z}^t), q), i \in I, t = 0, 1, 2, \ldots \) such that:

(i) the price process \( q \) satisfies:

\[
0 \leq q_t < +\infty \quad \text{almost surely for all } t
\]

(ii) for every \( i \) the consumption plan \( \bar{x}^i \) is (weakly) optimal with respect to all feasible
consumption plans, i.e. it satisfies the conditions:

\[(ii - a) \quad E[u_i'(\overline{x}_t^i) - u_i'(x_t^i)]^- < +\infty\text{ almost surely}\]

\[(ii - b) \quad \limsup_{N \to +\infty} E \sum_{t=0}^{N-1} [u_i'(\overline{x}_t^i) - u_i'(x_t^i)] \geq 0 \text{ almost surely}\]

\[(iii) \quad \sum_{i \in I} \overline{z}_t^i \leq 1\text{ and } q_t (1 - \sum_{i \in I} \overline{z}_t^i) = 0 \text{ for all } t.\]

Also in the case of heterogeneous agents it is possible to obtain conditions that must be satisfied at the equilibrium, known as stochastic Euler equations and inequalities, that represent therefore necessary conditions of optimality in the short-run. We have first of all the following result (here $Du_i^t$ denotes the derivative of the utility function of agent $i$ at time $t$):

**Proposition 58** *Under the assumption $\overline{x}_t^i > 0$ almost surely for all $t$, if $((\overline{x}_t^i, \overline{z}_t^i), q)$ is an equilibrium then:*

\[Du_{t-1}^i(\overline{x}_{t-1}^i)q_{t-1} \geq E_{t-1}[Du_t^i(\overline{x}_t^i)(d_t + q_t)] \quad \text{for all } t \geq 1 \quad (2.10)\]

**Proof.** See Montrucchio (1999). \(\blacksquare\)

It is also possible to give conditions under which the previous relation holds as an equality; in this way we obtain a first-order condition of optimality of the Kuhn-Tucker type, we have in fact:

**Proposition 59** *Let $((\overline{x}_t^i, \overline{z}_t^i), q)$ be an equilibrium with $\overline{x}_t^i > 0$ for all $t$. Under the conditions:

\[(i) \text{ for some scalar } M_t:\]

\[ (d_t + q_t) \cdot (\overline{z}_t^i - k_t^i) \leq M_t \overline{x}_t^i \]

\[(ii) \text{ for some } \zeta < 1 \text{ either:}\]

\[ E_{t-1}[Du_t^i(\zeta \overline{x}_t^i)(d_t + q_t)] < +\infty \]

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or:

\[ E_{t-1}[u'_t(\bar{x}_i) - u'_t(\zeta \bar{x}_i)] < +\infty \]

then necessarily one must have:

\[ Du^i_{t-1}(\bar{x}_{t-1})q_{t-1} \geq E_{t-1}[Du^i_t(\bar{x}_t)(d_t + q_t)] \]

\[ \{Du^i_{t-1}(\bar{x}_{t-1})q_{t-1} - E_{t-1}[Du^i_t(\bar{x}_t)(d_t + q_t)]\} \cdot (\bar{z}_i^t - k_i^t) = 0 \]

**Proof.** See Montrucchio (1999). □

In particular, a situation in which the conditions (i) and (ii) are satisfied is the one in which there are finitely many states of the world at each trading date.

The next step is the distinction between the fundamental value of an asset and the bubble component. Here we proceed exactly as for the homogeneous agents case: by considering the relationship (2.10) we can define:

\[ Du^i_{t-1}(\bar{x}_{t-1})q_{t-1} = E_{t-1}[Du^i_t(\bar{x}_t)(d_t + q_t)] + s^i_{t-1} \tag{2.11} \]

where the vectors \( s_t \geq 0 \) measure the deviation from the equality in the Euler equation.

By denoting \( Du^i_t(\bar{x}_i) = \pi^i_t \) and iterating the equation (2.11) we get:

\[ \pi^i_t q_t = E_t \sum_{r=1}^{k} \pi^i_{t+r} d_{t+r} + E_t \sum_{r=0}^{k-1} s^i_{t+r} + E_t[\pi^i_{t+k} q_{t+k}] \]

and for \( k \) that goes to infinity:

\[ \pi^i_t q_t = E_t \sum_{r=1}^{\infty} \pi^i_{t+r} d_{t+r} + E_t \sum_{r=0}^{\infty} s^i_{t+r} + \lim_{k \to +\infty} E_t[\pi^i_{t+k} q_{t+k}] \tag{2.12} \]
and also:

\[ q_t = \frac{1}{\pi_t} E_t \sum_{r=1}^{\infty} \pi_{t+r}^i d_{t+r} + \frac{1}{\pi_t} E_t \sum_{r=0}^{\infty} s_{k+r}^i + \frac{1}{\pi_t} \lim_{k \to +\infty} E_t[\pi_{t+k}^i q_{t+k}] \]

Again, the price of the asset can be decomposed into three components:

\[ q_t = f_t^i + \tilde{b}_t^i + \bar{b}_t^i \]

where the first term of the right-hand side is the fundamental value, the second is the bubble component due to the violation of the Euler equation and the third is the asymptotic bubble. Finally, the fundamental values satisfy the Euler equation and the bubble components follow martingale laws, as described for the model with homogeneous agents.

Also in the model with heterogeneous agents the final step is the determination of conditions that guarantee the existence of equilibria and that allow to rule out bubbles. A first result is an immediate consequence of expression (2.12) (where, from the definition of equilibrium, we know that \( q_t < +\infty \), therefore the left-hand side of this expression is finite, i.e. \( D u_t^i(\bar{x}_t^i)q_t < +\infty \) and hence also the right-hand side must be finite):

**Proposition 60** A necessary condition for the existence of an equilibrium in the economy considered is that the following relations hold:

\[ E \sum_{t=1}^{\infty} D u_t^i(\bar{x}_t^i)d_t < +\infty \quad E \sum_{t=0}^{\infty} s_t^i < +\infty \]

A second result is the extension to the case of heterogeneous agents of the sufficient condition for uniqueness of the equilibrium due to Kamihigashi. In this case, first of all, it is necessary to introduce the following definition:

**Definition 61** An equilibrium \( ((\bar{x}^i, \bar{x}^i), q) \) is said to be uniformly interior for agent \( i \) if
there exist two scalars \( \varepsilon^i, \eta^i > 0 \) such that:

\[
\overline{x}^i_t \geq \eta^i d_t \cdot 1 \quad \text{and} \quad \overline{z}^i_t \geq k^i_t + \varepsilon^i \cdot 1
\]

for all \( t \geq 1 \).

We then have the following theorem:

**Theorem 62** If an equilibrium is uniformly interior for agent \( i \) and the condition:

\[
E \sum_{t=1}^{\infty} \left[ u^i_t(\overline{x}^i_t) - u^i_t(\overline{z}^i_t - \varepsilon^i \cdot 1) \right] \leq +\infty
\]

is satisfied for some \( \varepsilon > 0 \), then \( q_t = f^i_t \) for \( t = 0, 1, 2, \ldots \), that is pricing bubbles cannot arise.

Furthermore, a sufficient condition for this relation to hold is that:

\[
E \sum_{t=1}^{\infty} Du^i_t(\overline{x}^i_t - \varepsilon^i \cdot 1)d_t < +\infty
\]

In particular, we can observe that in the case of homogeneous agents the equilibrium is of no-trade and thus \( \overline{z}_t = 1 \) and \( \overline{x}_t = d_t \cdot 1 + w_t \); in this case, therefore, the allocation is always uniformly interior and the last theorem reduces to Kamihigashi's sufficient condition of uniqueness introduced in the previous Subsection.

Another result can be given without resorting to uniform interiority assumptions; in this case we obtain a transversality condition at infinity, that can then be used to rule out bubbles. The following statement holds (here \( X^+ \) denotes the positive part of the random variable \( X \)):

**Theorem 63** Let \( ((\overline{x}^i, \overline{z}^i), q) \) be an equilibrium; if the following conditions are fulfilled:

(i) \( E \sum_{t=1}^{\infty} Du^i_t(\overline{x}^i_t)(\overline{x}^i_t - w^i_t)^+ < +\infty \)
(ii) agent i exhibits an uniformly bounded relative risk-aversion, i.e. there is some scalar $R$ so that:

$$-\frac{D^2 u^i_t(x)x}{Du^i_t(x)} \leq R$$

for all $x$ and all $t \geq 1$

(iii) short-selling is prohibited, i.e. $k^i_t = 0$

then we must have:

$$\lim_{t \to +\infty} E[D u^i_t(\bar{x}^i_t) q_t \cdot \bar{z}^i_{t+1}] = 0$$  (2.13)

**Proof.** See Montrucchio (1999). ■

The relation (2.13) is the transversality condition at infinity; it can be used to obtain the following further result, that allows to exclude the presence of bubbles at an equilibrium:

**Corollary 64** Under the assumptions of the previous Theorem, if there exists a sequence of times $t_n$ and some scalar $\varepsilon > 0$ such that:

$$\bar{z}^i_{t_n} \geq \varepsilon \cdot 1$$

then the asymptotic bubble $\bar{b}^i_t$ is absent.

If, in addition, we have:

(i) $\bar{z}^i_t > 0$ almost surely

(ii) $(d_t + q_t) \cdot \bar{z}^i_t \leq M_t \bar{z}^i_t$ for some sequence $M_t$

then the bubble component vanishes, and therefore $q_t = f^i_t$.

**Proof.** The first part is immediate by observing that the transversality condition,
together with the condition $\overline{x}_t^n \geq \varepsilon \cdot 1$, implies:

$$\lim_{t \to +\infty} \mathbb{E}[D_u^i(\overline{x}_t^i)q_t] = 0$$

and therefore the asymptotic bubble is:

$$\overline{b}_t^i = \frac{1}{D_u^i(\overline{x}_t^i)} \lim_{t \to +\infty} \mathbb{E}[D_u^i(\overline{x}_t^i)q_t] = 0$$

The second part is based on the fact that, under the assumptions made, Proposition 59 holds and it implies that the Euler equation is satisfied as an equality; as a consequence, also $\overline{b}_t^i = 0$, and the bubble component disappears. ■

These are the central results concerning non-existence of speculative bubbles in the framework with heterogeneous agents. In this case, an important aspect is also represented by the fact that the transversality condition (2.13) can be used in order to establish sufficient conditions of optimality. We have in fact:

**Proposition 65** Let $(\overline{x}^i, \overline{z}^i)$ be an allocation for an economy where short-selling is prohibited, and $q_t$ be a price process such that:

(i) 

$$D_u^{i-1}(\overline{x}_t^{i-1})q_{t-1} \geq \mathbb{E}_{t-1}[D_u^i(\overline{x}_t^i)(d_t + q_t)]$$

$$\{D_u^{i-1}(\overline{x}_t^{i-1})q_{t-1} - \mathbb{E}_{t-1}[D_u^i(\overline{x}_t^i)(d_t + q_t)]\} \cdot \overline{z}_t^i = 0$$

for agent $i$ and all $t$

(ii) 

$$\lim \inf_{t \to +\infty} \mathbb{E}[D_u^i(\overline{x}_t^i)q_t \cdot \overline{z}_t^{i+1}] = 0$$

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Then the allocation \((\bar{x}^i, \bar{z}^i)\) is weakly optimal.

If the stronger condition:

\[
\lim_{t \to +\infty} E[Du_t^i(\bar{x}_t^i)q_t \cdot \bar{z}_t^i] = 0
\]

is satisfied, then the allocation is strongly optimal.

**Proof.** See Montrucchio (1999). ■

In this way (as it will be clear in the examples presented in the third Chapter) it becomes relatively easy to check optimality of solutions when we have bubbles caused by the violation of Euler equations. The situation is different when bubbles are of the asymptotic kind; in this case, usually, specific arguments are needed to prove the optimality of the solutions.

This concludes the analysis of the phenomenon of speculative bubbles with the method based on Euler equations and inequalities. Even if, as outlined above, this approach is much more limited than the general analysis presented in the first part of this Chapter, it is interesting because it confirms the results of the study based on no-arbitrage considerations and it also suggests a technique that turns out to be useful in the study of specific examples.

Both the approaches presented reach the conclusion according to which bubbles are linked to very special situations, and therefore are fragile. To support this idea of marginality of bubbles the last step consists in giving a series of examples in which bubbles do appear, and from which it emerges that, in order to obtain such a result, very special conditions are needed. This is the task tackled in the last Chapter.
Chapter 3

Examples of bubbles in intertemporal economies

3.1 Introduction

In the first and second Chapter two important questions concerning sequential economies in intertemporal general equilibrium models have been addressed. The first is the issue of existence of solutions in this kind of models, and the conclusion that has been reached is that such solutions exist (even if with some restrictions, especially when one considers an economy that extends over an infinite horizon). The second is the problem represented by the implications of these models in terms of asset pricing, with particular emphasis on the appearance of speculative bubbles. In this case the conclusion is that bubbles are a fragile phenomenon, and are possible only under special circumstances.

This last Chapter presents a series of examples in order to confirm this result. In all of them bubbles on asset prices appear, and the scope is to show that they are related to very special assumptions. All these examples consider a particular asset, the so-called fiat money, whose characteristic is that it doesn’t give any dividend; as a consequence, its fundamental value is unambiguously defined and it is equal to zero, regardless of the state-price process chosen. The first example (Section 3.2) is inspired
by a monetary model proposed by Bewley (1980); for this model (that is studied in great depth) it is shown (by applying the method proposed in the last part of Chapter 2) as the presence of bubbles is linked to the violation of the Euler equation and, in addition, an interesting new result is derived, represented by the presence of a multiplicity of equilibrium solutions, all involving bubble components. The same model can then be generalized, in the deterministic setting, and this is realized in the second example proposed (Section 3.3). In this case, in particular, a new result that can be obtained is a condition that allows to have "switching" of the assets (i.e. exchange from one agent to the other) in each period. The third example (Section 3.4) is an extension of the previous ones to the stochastic setting, and shows as, under opportune conditions, an equilibrium exists involving bubble components in the prices of the assets. The economies considered in these examples can also be studied with the general method proposed in the second Chapter (based on no-arbitrage considerations). This shows further as the presence of bubbles is linked to the violation of the hypothesis required to derive the general results of this approach, and confirms the results obtained with the method based on Euler equations.

The element that is common to the first three examples is that, in all of them, the occurrence of bubbles is due to the violation of Euler equations. The last example (Section 3.5) goes beyond this framework: in this case, in fact, bubbles emerge even if Euler equations are always satisfied as equalities, and therefore they are of a different nature.

Even if the conclusions of the theory (in the light of the recent results, presented in the previous Chapters) agree on the fact that rational asset pricing bubbles are substantially a negligible phenomenon, the real world is often interested by speculative episodes, in which bubbles do appear. For this reason, the last part of the Chapter deals with this apparent contrast, trying to give an explanation of this fact that allows to reconcile the theoretical results with the real situation.
3.2 A monetary model

The first example presented in this Chapter is based on a monetary model introduced by Bewley (1980). For this model it is possible to show that an equilibrium exists involving a bubble component on the price of the asset, and furthermore that the economy has a multiplicity of equilibria, all characterized by the presence of asset pricing bubbles. The economy is deterministic and there is a unique asset, traded at each date, the so-called fiat money, which gives no dividend. There are two individuals \((i = 1, 2)\) with the same preferences given by:

\[
\sum_{t=0}^{\infty} \beta^t u(x_t^i)
\]

where \(\beta\) is the discount factor with \(0 < \beta < 1\) and \(u(x_t^i)\) is strictly increasing and strictly concave. Since in the Euler equations the functions \(u_t^i\) appear, in this situation we therefore have:

\[
u_t^i(x_t^i) = \beta^t u(x_t^i)
\]

and the problem solved by each single agent is:

\[
\max \sum_{t=0}^{\infty} \beta^t u(x_t^i)
\]

s.t. \(x_t^i + q_t(x_{t+1}^i - x_t^i) \leq d_t z_t^i + w_t^i\)

\[z_0^i = z_0^i, \quad x_t^i \geq 0, \quad z_t^i \geq k_t^i\]
The endowments of the good of the two individuals are assumed to be:

\[
\begin{cases}
    \bar{w} & \text{for } t \text{ even} \\
    w & \text{for } t \text{ odd}
\end{cases}
\]

\[
\begin{cases}
    w & \text{for } t \text{ even} \\
    \bar{w} & \text{for } t \text{ odd}
\end{cases}
\]

i.e. they are symmetric across time (with \( w < \bar{w} \)), while the initial endowments of the asset (fiat money) are:

\[
\begin{align*}
    z_0^1 &= 0 \\
    z_0^2 &= 1
\end{align*}
\]

In addition, short-selling is prohibited, so that \( k_t^1 = k_t^2 = 0 \) for all \( t \), and since there are no dividends \( d_t = 0 \) \( \forall t \).

Heuristically it is possible to show (see, for instance, Sargent (1987)) that a model of this kind has, together with the "autarchic" equilibrium where fiat money is never valued (i.e. \( q_t = 0 \)) and each agent consumes its own endowment at each trading date, another equilibrium in which fiat money has a positive price (that is a monetary equilibrium) and the consumption of each individual is of the form:

\[
\begin{cases}
    x^* & \text{for } t \text{ even} \\
    x^{**} & \text{for } t \text{ odd}
\end{cases}
\]

\[
\begin{cases}
    x^{**} & \text{for } t \text{ even} \\
    x^* & \text{for } t \text{ odd}
\end{cases}
\]

while the asset holding is of the form:

\[
\begin{cases}
    0 & \text{for } t \text{ even} \\
    m & \text{for } t \text{ odd}
\end{cases}
\]

\[
\begin{cases}
    m & \text{for } t \text{ even} \\
    0 & \text{for } t \text{ odd}
\end{cases}
\]

where \( m \) is the initial endowment of the asset.

The same result can be obtained by applying the techniques illustrated in Chapter 2. In this case, the strategy followed consists in looking for an equilibrium in which the
asset holding strategies are of a certain form. The budget constraints, then, give the corresponding consumptions values, and at this point the solution candidate to be an equilibrium can be determined by means of the Euler equations. Finally, the fact that the values found represent a true equilibrium can be checked with the sufficient condition of optimality presented at the end of Chapter 2.

In particular, in this model it is possible to look for an equilibrium in which the price of the asset is constant \( (q_t = q > 0) \) and the holding strategies of the two individuals are:

\[
\begin{align*}
\bar{z}_t^1 &= \begin{cases} 
0 & \text{for } t \text{ even} \\
1 & \text{for } t \text{ odd}
\end{cases} \\
\bar{z}_t^2 &= \begin{cases} 
1 & \text{for } t \text{ even} \\
0 & \text{for } t \text{ odd}
\end{cases}
\end{align*}
\]

To find such an equilibrium we consider the budget constraint of agent 1:

\[
x_t^1 + q(z_{t+1}^1 - z_t^1) = w_t^1
\]

from which, by substituting the corresponding values for \( z \) and \( w \), we get:

\[
\bar{x}_t^1 = \begin{cases} 
\bar{w} - q & \text{for } t \text{ even} \\
\bar{w} + q & \text{for } t \text{ odd}
\end{cases}
\]

while for agent 2 the budget constraint:

\[
x_t^2 + q(z_{t+1}^2 - z_t^2) = w_t^2
\]

leads to the consumption values:

\[
\bar{x}_t^2 = \begin{cases} 
\bar{w} + q & \text{for } t \text{ even} \\
\bar{w} - q & \text{for } t \text{ odd}
\end{cases}
\]
In conclusion, at the equilibrium we are looking for we must have:

\[
\bar{x}_{t}^{1} = \begin{cases} 
\bar{w} - q & \text{for } t \text{ even} \\
\bar{w} + q & \text{for } t \text{ odd}
\end{cases} 
\bar{z}_{t}^{1} = \begin{cases} 
0 & \text{for } t \text{ even} \\
1 & \text{for } t \text{ odd}
\end{cases}
\]

\[
\bar{x}_{t}^{2} = \begin{cases} 
\bar{w} + q & \text{for } t \text{ even} \\
\bar{w} - q & \text{for } t \text{ odd}
\end{cases} 
\bar{z}_{t}^{2} = \begin{cases} 
1 & \text{for } t \text{ even} \\
0 & \text{for } t \text{ odd}
\end{cases}
\]

The solution candidate to be an equilibrium can now be determined by means of Euler equations (at the end, by using the sufficient condition of optimality, it will be possible to check that the solution found is effectively an equilibrium solution); from Proposition 59 of Chapter 2, where obviously conditions (i) and (ii) are satisfied (because we are considering a deterministic model), we know that for agent 1 the following relations hold:

\[
u'_{t-1}(\bar{x}_{t-1}^{1})q_{t-1} \geq \nu'_{t}(\bar{x}_{t}^{1})q_{t}
\]

\[
[u'_{t-1}(\bar{x}_{t-1}^{1})q_{t-1} - u'_{t}(\bar{x}_{t}^{1})q_{t}] \cdot \bar{z}_{t}^{1} = 0
\]

and by substituting the corresponding values for \(x\) and \(z\) (together with the fact that the price must be constant, equal to \(q\)) we get (here we use the fact that \(u_{t}(\cdot) = \beta^{t}u(\cdot)\), and therefore \(u'_{t}(\cdot) = \beta^{t}u'(\cdot)\)):

for \(t\) even \(\begin{cases} 
\beta^{t-1}u'(\bar{w} + q)q \geq \beta^{t}u'(\bar{w} - q)q \\
[\beta^{t-1}u'(\bar{w} + q)q - \beta^{t}u'(\bar{w} - q)q] \cdot 0 = 0
\end{cases}\)

for \(t\) odd \(\begin{cases} 
\beta^{t-1}u'(\bar{w} - q)q \geq \beta^{t}u'(\bar{w} + q)q \\
[\beta^{t-1}u'(\bar{w} - q)q - \beta^{t}u'(\bar{w} + q)q] \cdot 1 = 0
\end{cases}\)
from which:

\[ u'(w + q) \geq \beta u'(\bar{w} - q) \quad \text{for } t \text{ even} \]
\[ u'(\bar{w} - q) = \beta u'(w + q) \quad \text{for } t \text{ odd} \]

By proceeding in the same way we get, for agent 2:

\[ u'(\bar{w} - q) = \beta u'(w + q) \quad \text{for } t \text{ even} \]
\[ u'(w + q) \geq \beta u'(\bar{w} - q) \quad \text{for } t \text{ odd} \]

In conclusion, at an equilibrium the following conditions must hold:

\[ u'(\bar{w} - q) = \beta u'(w + q) \quad \text{(3.1)} \]
\[ u'(w + q) \geq \beta u'(\bar{w} - q) \]

By substituting the first expression into the second we get:

\[ u'(w + q) \geq \beta u'(\bar{w} - q) = \beta^2 u'(w + q) \]

from which:

\[ 1 \geq \beta^2 \]

that is always true (in particular, whenever \( \beta < 1 \) we have the strict inequality). In order to determine the equilibrium the condition (3.1) must therefore hold; to check the existence of a positive price \( q \) that satisfies this relation we can consider the function:

\[ f(q) = u(w - q) + \beta u(w + q) \]

that is defined over the interval \([0, \bar{w}]\). This function is strictly concave (because \( u \) has
this property), therefore the maximum exists and is unique; its derivative is:

\[ f'(q) = -u'(w - q) + \beta u'(w + q) \]

and if the maximum is interior to \([0, w]\) it is characterized by the condition \(f'(q) = 0\), i.e.:

\[ u'(w - q) = \beta u'(w + q) \]

(that is precisely condition (3.1)), together with the conditions \(f'(0) > 0\) and \(f''(w) < 0\), that are needed to avoid that the maximum is on the boundary of the interval \([0, w]\). These conditions lead to:

\[
\begin{align*}
-u'(w) + \beta u'(w) &> 0 \\
-u'(0) + \beta u'(w + w) &< 0
\end{align*}
\]

from which:

\[
\begin{align*}
u'(w) &< \beta u'(w) \tag{3.2} \\
\beta u'(w + w) &< u'(0)
\end{align*}
\]

where the second is clearly satisfied if we assume \(u'(0) = +\infty\). As a consequence, the equilibrium value of \(q\) we are looking for is the unique value \(q^*\) that satisfies equations (3.1) and (3.2).

The fact that the value \(q^*\) is a true equilibrium, finally, can be deduced by the sufficient conditions of optimality expressed in Proposition 65 of Chapter 2; we have in fact:

\[
\lim_{t \to +\infty} \beta^t u'(\bar{x}_t^i)q^* \bar{z}_t^i = 0
\]

if \(\beta < 1\), and therefore \((\bar{x}^i, \bar{z}^i)\) is a (strongly) optimal allocation, and \(((\bar{x}^i, \bar{z}^i), q^*)\) is an
equilibrium.

The results obtained can be summarized in the following proposition:

**Proposition 66** The model "à la Bewley" considered has an equilibrium with valued fiat money (monetary equilibrium). This equilibrium is characterized by \( q_t = q^* \) for each \( t \geq 0 \), where \( q^* > 0 \) is the unique quantity such that:

\[
    u'(w - q^*) = \beta u'(w + q^*)
\]
\[
    u'(w) < \beta u'(w)
\]

The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

\[
    x^1_t = \begin{cases} 
    w - q^* & \text{for } t \text{ even} \\
    w + q^* & \text{for } t \text{ odd}
    \end{cases},
    \quad
    Z^1_t = \begin{cases} 
    0 & \text{for } t \text{ even} \\
    1 & \text{for } t \text{ odd}
    \end{cases}
\]

\[
    x^2_t = \begin{cases} 
    w + q^* & \text{for } t \text{ even} \\
    w - q^* & \text{for } t \text{ odd}
    \end{cases},
    \quad
    Z^2_t = \begin{cases} 
    1 & \text{for } t \text{ even} \\
    0 & \text{for } t \text{ odd}
    \end{cases}
\]

In this example the two individuals, at the equilibrium, exchange one another, in every period, the unit of fiat money the economy is endowed with. Each agent uses a part of his endowment of the consumption good when it is high \((w)\) to buy the unit of fiat money, and reduces consequently the consumption of that period, while he sells the unit of fiat money when his endowment of the consumption good is low \((w)\) and in this way he increases the consumption of that period. For this reason, this model is known as a *consumption smoothing* model.

In this case, then, the fundamental value of the asset is zero (because the dividend at every date is zero), and the fact that at the equilibrium the price of fiat money is positive means that this price involves a bubble component. In particular, in this situation we
don't have asymptotic bubble because (by applying the definition of asymptotic bubble given in Chapter 2):

$$\lim_{t \to +\infty} \beta^t u'(x_t^i)q^* = 0$$

and hence $\tilde{b}_t^i = 0$; as a consequence, the bubble is entirely due to the violation of the Euler equation, that is $\tilde{b}_t^i > 0$. In fact, in this model, at each date one of the two individuals, alternatively, satisfies the corresponding Euler equation as an equality, but the other individual satisfies it as an inequality; in this way there is violation of this equation. A different interpretation of the presence of bubbles in this model can be given in terms of the general analysis presented in the second Chapter. In this case what is violated is the assumption (ii) in the Corollary 50; in fact, the agents cannot borrow, i.e. $M_t^i = 0$, while the present value of their future wealth is $+\infty$ (for details on this point, see Santos-Woodford (1997)), therefore it is not true that they are able to borrow against the value of their future wealth. For this reason the Corollary mentioned does not hold, and it is not true that the price of the asset is equal to its fundamental value (in this case zero).

The analysis illustrated above has shown that the monetary model considered has, together with the "autarchic" equilibrium (in which $q_t = 0$), an equilibrium in which fiat money has a positive value ($q_t = q^* > 0$). By means of the approach based on Euler equations and inequalities it is possible to show something more, and to find a new result, according to which this model, actually, has a multiplicity of equilibria.

In order to find this result we fix a sequence of prices $q_t$ and we assume that the holding strategies of the two individuals are the same as before, i.e.:

$$\bar{z}_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases} \quad \bar{z}_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$
From the budget constraints we get the consumptions of the two agents, given by:

\[
\bar{x}^1_t = \begin{cases} 
  w - q_t & \text{for } t \text{ even} \\
  w + q_t & \text{for } t \text{ odd}
\end{cases} \quad \bar{x}^2_t = \begin{cases} 
  w + q_t & \text{for } t \text{ even} \\
  w - q_t & \text{for } t \text{ odd}
\end{cases}
\]

and, by proceeding in the same way as before, we obtain the Euler equations that must be satisfied at an equilibrium and that are:

\[
\begin{align*}
  u'(w - q_{t-1})q_{t-1} &= \beta u'(w + q_t)q_t \\
  u'(w + q_{t-1})q_{t-1} &\geq \beta u'(w - q_t)q_t
\end{align*}
\]  

(3.3)

where the first relation holds for \(i = 1\) at odd-numbered dates and for \(i = 2\) at even-numbered dates, while the second relation holds for \(i = 1\) at even-numbered dates and for \(i = 2\) at odd-numbered dates. These conditions (with the additional restriction \(q_t < w\) for all \(t\)) are also sufficient to have an equilibrium, because they satisfy all the requirements of Proposition 65 of Chapter 2. Given an initial value \(q_0\), the first relation determines a sequence of prices; this dynamics has the two fix points \(q = 0\) (corresponding to the autarchic equilibrium) and \(q = q^*\) (corresponding to the equilibrium with valued fiat money found before), that represent two possible equilibria of the model. We can now show that there exist sequences of prices which form other equilibria.

To this end it is possible to analyse, first of all, a specific case. We consider the utility function:

\[
u(x_i^t) = \log x_i^t
\]

and therefore the function \(u_i^t\) is given by:

\[
u_i^t(x_i^t) = \beta^t \log x_i^t
\]  

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The stationary price equilibrium is determined by the conditions (3.1) and (3.2):

\[ u'(\tilde{w} - q^*) = \beta u'(\tilde{w} + q^*) \]
\[ u'(\tilde{w}) < \beta u'(\tilde{w}) \]

where the second becomes:

\[ \tilde{u} < \beta \tilde{w} \]

while the first is:

\[ \frac{1}{\tilde{w} - q^*} = \frac{\beta}{\tilde{w} + q^*} \]

from which we get:

\[ q^* = \frac{\beta \tilde{w} - \tilde{u}}{1 + \beta} \]

This is the stationary price, while in the general case of a sequence of prices \( q_t \) the equations (3.3) are:

\[ \frac{1}{\tilde{w} - q_{t-1}} = \frac{\beta}{\tilde{w} + q_t} \]
\[ \frac{1}{\tilde{w} + q_{t-1}} q_{t-1} \geq \frac{\beta}{\tilde{w} - q_t} q_t \]

from which:

\[ q_t = \frac{\tilde{w} q_{t-1}}{\beta \tilde{w} - (1 + \beta) q_{t-1}} \quad (3.4) \]
\[ q_t \leq \frac{\bar{w}q_{t-1}}{\beta \bar{w} + (1 + \beta)q_{t-1}} \quad (3.5) \]

In particular, the curve with equation (3.4) is defined when:

\[ \beta \bar{w} - (1 + \beta)q_{t-1} > 0 \]

i.e.:

\[ q_{t-1} < \frac{\beta \bar{w}}{1 + \beta} \]

and it is strictly increasing and strictly convex, while the curve with equation (3.5) is strictly increasing and strictly concave. Furthermore, the derivative of (3.4) at the origin is:

\[
\left( \frac{dq_t}{dq_{t-1}} \right) = \left( \frac{\beta \bar{w} \bar{w}}{[\beta \bar{w} - (1 + \beta)q_{t-1}]^2} \right) = \frac{\beta \bar{w} \bar{w}}{\beta \bar{w}^2} = \frac{w}{\beta \bar{w}} < 1
\]

while the derivative of (3.5) at the origin is:

\[
\left( \frac{dq_t}{dq_{t-1}} \right) = \left( \frac{\beta \bar{w} \bar{w}}{[\beta \bar{w} + (1 + \beta)q_{t-1}]^2} \right) = \frac{\beta \bar{w} \bar{w}}{\beta \bar{w}^2} = \frac{w}{\beta \bar{w}} = \frac{1}{\beta^2} > 1
\]

and therefore the curve (3.4) starts below the curve (3.5) and remains below it when the condition:

\[
\frac{wq_{t-1}}{\beta \bar{w} - (1 + \beta)q_{t-1}} < \frac{\bar{w}q_{t-1}}{\beta \bar{w} + (1 + \beta)q_{t-1}}
\]
is satisfied, i.e. (as can be checked after some computations) when:

\[ q_{t-1} < \frac{\beta}{1 + \beta} (\overline{w} - \underline{w}) \]

Finally, the intersections of the first curve with the 45° line can be found by considering:

\[ q = \frac{wq}{\beta \overline{w} - (1 + \beta)q} \]

from which:

\[ q = 0 \quad q^* = \frac{\beta \overline{w} - \underline{w}}{1 + \beta} \]

that are precisely the two equilibria of the model previously determined (the autarchic one and the one with constant positive price of fiat money). This situation is represented in the following graphic:

In addition, it now turns out that for any initial price \( q_0 \in (0, q^*) \) there is a sequence of asset prices \( q_t \), decreasing to 0, which forms an equilibrium. As a consequence, in this
situation an indeterminacy of equilibria arises, as the following graphic shows:

These results can be summarized as follows:

**Proposition 67** The monetary model "à la Bewley" considered, in which the utility function takes the specific form:

\[ u(x^i_t) = \log x^i_t \]

has, together with the autarchic equilibrium \( (q_t = 0) \) and the equilibrium with constant positive price of fiat money \( (q_t = q^* > 0) \), a multiplicity of other equilibria (consisting of sequences of asset prices \( q_t \) decreasing to 0), one for each initial value \( q_0 \in (0, q^*) \).

The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

\[
\begin{align*}
\bar{x}^1_t &= \begin{cases} 
\bar{w} - q_t & \text{for } t \text{ even} \\
\bar{w} + q_t & \text{for } t \text{ odd}
\end{cases} \\
\bar{z}^1_t &= \begin{cases} 
0 & \text{for } t \text{ even} \\
1 & \text{for } t \text{ odd}
\end{cases} \\
\bar{x}^2_t &= \begin{cases} 
\bar{w} + q_t & \text{for } t \text{ even} \\
\bar{w} - q_t & \text{for } t \text{ odd}
\end{cases} \\
\bar{z}^2_t &= \begin{cases} 
1 & \text{for } t \text{ even} \\
0 & \text{for } t \text{ odd}
\end{cases}
\end{align*}
\]
This result can then be extended to the general case (at least locally, around $q = 0$), and to show this we can consider the slope of the curves represented by the equations (3.3) at the origin. For the first relation, from the implicit function theorem follows that:

$$\frac{dq_t}{dq_{t-1}} = -\frac{u'\left(\bar{w} - q_{t-1}\right) + u''\left(\bar{w} - q_{t-1}\right)q_{t-1}}{\beta \left[u'(w + q_t) + u''(w + q_t)q_t\right]}$$

and then:

$$\left(\frac{dq_t}{dq_{t-1}}\right)_0 = \frac{u'(\bar{w})}{\beta u'\left(\bar{w}\right)} < 1$$

(since $u'(\bar{w}) < \beta u'(w)$), while for the second relation we have:

$$\frac{dq_t}{dq_{t-1}} = -\frac{u'(w + q_t) - u''(w + q_t)q_t}{\beta \left[u'(w - q_t) - u''(w - q_t)q_t\right]}$$

and then:

$$\left(\frac{dq_t}{dq_{t-1}}\right)_0 = \frac{u'(w)}{\beta u'(\bar{w})} > \frac{1}{\beta^2} > 1$$

(always from the fact that $u'(\bar{w}) < \beta u'(w)$ and thus $u'(w) > \beta^{-1}u'(\bar{w})$). Hence, close to the origin the first curve lies below the $45^\circ$ line, while the second curve lies above the $45^\circ$ line. In addition, the (positive) intersection of the first curve with this line is given by the value $q^*$ such that:

$$u'(\bar{w} - q^*)q^* = \beta u'(w + q^*)q^*$$

from which:

$$u'(w + q^*)q^* = \frac{1}{\beta} u'(\bar{w} - q^*)q^* > \beta u'(\bar{w} - q^*)q^*$$

while the (positive) intersection of the second curve with the $45^\circ$ line is given by the
value $q^\ast$ such that:

$$u'(w + q^\ast)q^\ast = \beta u'(\bar{w} - q^\ast)q^\ast$$

In conclusion, at the (positive) intersection of the first curve with the 45° line we have:

$$u'(w + q^\ast) > \beta u'(\bar{w} - q^\ast)$$

while at the (positive) intersection of the second curve with the same line we have:

$$u'(w + q^{**}) = \beta u'(\bar{w} - q^{**})$$

and by comparing the last two expressions we conclude (since $u'$ is decreasing) that $q^\ast < q^{**}$ (i.e. the intersection of the first curve with the 45° line is on the left of the intersection of the second curve with the same line). From this fact, and from the behaviour of the two curves close to the origin, we deduce that the first curve is always below the second curve in the interval $[0, q^\ast]$. The situation is therefore of the same type illustrated in the logarithmic case considered above, and if the first curve is monotonically increasing we have that, also in the general case, for any $q_0 \in (0, q^\ast)$ there is a sequence of prices $q_t$, decreasing to 0, which forms an equilibrium, and a multiplicity of equilibria arises.

In the general case, however, there is also a different possibility, i.e. it is possible for the first curve to be backward bending. To see this we must consider its derivative at $q = q^\ast$, which is:

$$\left( \frac{dq_t}{dq_{t-1}} \right)_{q=\ast} = \frac{u'(\bar{w} - q^\ast) - u''(\bar{w} - q^\ast)q^\ast}{\beta [u'(\bar{w} + q^\ast) + u''(\bar{w} + q^\ast)q^\ast]}$$

and a sufficient condition for the curve to be backward bending is that:

$$\frac{u'(\bar{w} - q^\ast) - u''(\bar{w} - q^\ast)q^\ast}{\beta [u'(\bar{w} + q^\ast) + u''(\bar{w} + q^\ast)q^\ast]} < 0$$

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that means (since the numerator is always positive):

\[ u'(w + q^*) + u''(w + q^*)q^* < 0 \]

A specific case that can be used to illustrate the different possibilities is the one in which the utility function is of the form \( u(x_t^i) = \frac{(x_t^i)^\alpha}{\alpha} \) with \( \alpha < 0 \) (isoelastic utilities, for \( \alpha \to 0 \) we obtain the logarithmic case considered before). In this situation the steady state equilibrium price is determined by solving:

\[ u'(w - q^*) = \beta u'(w + q^*) \]
\[ u'(w) < \beta u'(w) \]

where the second condition leads to:

\[ w < \bar{w} \beta^{\frac{1}{1-\alpha}} \]

while the first condition is:

\[ (\bar{w} - q^*)^{\alpha - 1} = \beta (w + q^*)^{\alpha - 1} \]

from which:

\[ \frac{w + q^*}{\bar{w} - q^*} = \beta^{\frac{1}{1-\alpha}} \]

By denoting \( \beta^{\frac{1}{1-\alpha}} = \mu \) we then get:

\[ w + q^* = \bar{w} \mu - \mu q^* \]
and finally:
\[ q^* = \frac{w\mu - w}{1 + \mu} = \frac{w\beta^{\frac{1}{1-\alpha}} - w}{1 + \beta^{\frac{1}{1-\alpha}}} \]

At this point the curve with equation described by the first of the relations in (3.3) is backward bending if:
\[(w + q^*)^{\alpha - 1} + (\alpha - 1)(w + q^*)^{\alpha - 2}q^* < 0\]

that is:
\[(w + q^*)^{\alpha - 2} [w + q^* + q^*(\alpha - 1)] < 0\]

that is true when:
\[w + q^* + \alpha q^* - q^* < 0\]
i.e. when:
\[\alpha < -\frac{w}{q^*}\]

that means:
\[\alpha < -\frac{w (1 + \beta^{\frac{1}{1-\alpha}})}{w\beta^{\frac{1}{1-\alpha}} - w} \quad (3.6)\]

In this case the solutions of the model can be found also with a different approach, that consists in considering separately the two sides of the equation:
\[u'(\overline{w} - q_{t-1})q_{t-1} = \beta u'(w + q_t)q_t\]
by defining:

\[ \Psi(q_{t-1}) = u'(\bar{w} - q_{t-1})q_{t-1} \]
\[ \Phi(q_t) = \beta u'(w + q_t)q_t \]

If we now consider the explicit expression of \( u \) we get:

\[ \Psi(q_{t-1}) = \frac{q_{t-1}}{(\bar{w} - q_{t-1})^{1-\alpha}} \]

that is defined for \( q_{t-1} < \bar{w} \), and then:

\[ \Psi'(q_{t-1}) = \frac{\bar{w} - \alpha q_{t-1}}{(\bar{w} - q_{t-1})^{2-\alpha}} > 0 \quad \Psi''(q_{t-1}) = \frac{(1 - \alpha)(2\bar{w} - \alpha q_{t-1})}{(\bar{w} - q_{t-1})^{3-\alpha}} > 0 \]

so that the curve \( \Psi \) is strictly increasing and strictly convex. Similarly we have:

\[ \Phi(q_t) = \frac{\beta q_t}{(w + q_t)^{1-\alpha}} \]

for which:

\[ \Phi'(q_t) = \frac{\beta(w + \alpha q_t)}{(w + q_t)^{2-\alpha}} \quad \Phi''(q_t) = -\frac{\beta(1 - \alpha)(2w + \alpha q_t)}{(w + q_t)^{3-\alpha}} \]

and for the curve \( \Phi \) we have:

\[ \Phi'(q_t) \geq 0 \iff w + \alpha q_t \geq 0 \iff q_t \leq -\frac{w}{\alpha} \]

i.e. the curve is first increasing and then decreasing, and also:

\[ \Phi''(q_t) \leq 0 \iff -(2w + \alpha q_t) \geq 0 \iff 2w + \alpha q_t \leq 0 \iff q_t \geq -\frac{2w}{\alpha} \]

i.e. the curve is first concave and then convex (and we also have that \( \lim_{q_t \to +\infty} \Phi(q_t) = 0 \)).
The two curves $\Psi$ and $\Phi$ are represented in the following graphic:

In particular, by considering the generic expressions of the functions $\Psi$ and $\Phi$, we then have:

$$\Psi'(q_{t-1}) = u'(\bar{w} - q_{t-1}) - q_{t-1}u''(\bar{w} - q_{t-1}) \quad \Rightarrow \quad \Psi'(0) = u'(\bar{w})$$

and similarly:

$$\Phi'(q_t) = \beta u'(w + q_t) + q_t\beta u''(w + q_t) \quad \Rightarrow \quad \Phi'(0) = \beta u'(w)$$

and since $u'(\bar{w}) < \beta u'(w)$ it follows that $\Psi'(0) < \Phi'(0)$, so that the slope of the curve $\Psi$ at the origin is less than the slope of the curve $\Phi$.

The two curves can be considered together, in order to determine a relation between the variables $q_{t-1}$ and $q_t$. As it is shown in the next graphic, if the value of $\alpha$ is sufficiently small (in absolute value), the relevant part of the curve $\Phi$ is increasing (because we are interested only in values of prices less than $\bar{w}$, since the curve $\Psi$ is defined only for $q_{t-1} < \bar{w}$, and for values of $\alpha$ sufficiently small the value $-\frac{w}{\alpha}$, in correspondence of which there is a change in the monotonicity of the function, is on the right of $\bar{w}$). In this situation, given a value of $q_{t-1}$ there is a corresponding value of $q_t$ that derives from
the equality $\Psi(q_{t-1}) = \Phi(q_t)$ and, since close to the origin the slope of the curve $\Psi$ is less than the slope of the curve $\Phi$, the value of $q_t$ determined in this way is less than the value of $q_{t-1}$; by proceeding in this way we therefore obtain a decreasing sequence of prices that converges to 0 and that represents an equilibrium, exactly as it has been obtained explicitly above for the logarithmic case:

For values of $\alpha$ sufficiently large (the critical value of $\alpha$ is the one expressed by relation (3.6)), on the contrary, we have the situation illustrated in the following graphic:

In this case, given a value of $q_{t-1}$ there are two possible values of $q_t$ associated to it, and this corresponds to the situation in which the curve that represents explicitely the
relationship between $q_{t-1}$ and $q_t$ is backward bending (as shown in the last graphic). In such a situation, what is crucial is the slope of this curve at $q = q^*$.

This concludes the analysis of the monetary model "à la Bewley" presented in this Section. The results obtained have shown that this model has, apart from an "autarchic" equilibrium (that is a no-trade equilibrium, in which the price of fiat money is $q = 0$) another stationary equilibrium, in which fiat money is valued ($q_t = q^* > 0$). In this case, since the fundamental value of fiat money is always zero, this implies the existence of a speculative bubble, and as long as the discount factor $\beta$ is less than 1 this bubble is due to the violation of the Euler equation. In addition, in the special case of logarithmic utility (but also in the general case, at least locally around $q = 0$) a further result has been obtained: for every initial value $q_0 \in (0, q^*)$ the model has an equilibrium, represented by a decreasing sequence of prices $q_t$, which converges to 0. In conclusion, a multiplicity of equilibria exists, and this equilibria include a bubble component (due to the violation of the Euler equation), that then vanishes as $q_t \to 0$.

### 3.3 A general deterministic model

The second example considered is a generalization of the previous one; in this case, in particular, an interesting result can be derived, concerning the possibility for the agents
to exchange one another in every period the asset available. The economy is deterministic
and, again, there are two individuals \((i = 1, 2)\) with identical preferences:

\[
\sum_{t=0}^{\infty} \beta^t u(x_t^i)
\]

(so that, as before, \(u_i^t(x_t^i) = \beta^t u(x_t^i)\)), but in this case the endowments at the beginning
of every period are generic, and are given by \(w_t^1\) and \(w_t^2\) respectively (for instance we can
assume that \(w_t^1 + w_t^2 = 1 \forall t\)). Also in this example the asset available is \(\text{fiat money}\) (in
fixed net supply of one unit at all dates), in addition short-selling is prohibited and we
assume that the strategies of the two individuals concerning the amounts of the asset
held in equilibrium are of "bang bang" type, i.e.:

\[
\bar{z}_t^1 \in \{0, 1\} \quad \bar{z}_t^2 = 1 - \bar{z}_t^1
\]

If \(q_t\) is a sequence of equilibrium prices, the budget constraint for the first individual is:

\[
x_t^1 + q_t(z_t^1 - z_t^1) = w_t^1
\]

and the Euler equations are:

\[
u'(\bar{x}_t^1)q_t \geq \beta u'(\bar{x}_{t+1}^1)q_{t+1}
\]

\[
[u'(\bar{x}_t^1)q_t - \beta u'(\bar{x}_{t+1}^1)q_{t+1}] \cdot \bar{z}_{t+1}^1 = 0
\]

and similarly for the second individual the budget constraint is:

\[
x_t^2 + q_t(z_t^2 - z_t^2) = w_t^2
\]

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while the Euler equations that must hold at the equilibrium are:

\[ u'(x_t^2)q_t \geq \beta u'(x_{t+1}^2)q_{t+1} \]

\[ [u'(x_t^2)q_t - \beta u'(x_{t+1}^2)q_{t+1}] \cdot z_{t+1}^2 = 0 \]

Given this general structure, it is now possible to obtain an interesting result by considering a sequence of "times of switching"; first of all, the following definition holds:

**Definition 68** A sequence of times of switching is a sequence \( T = \{0 = T_0, T_1, T_2, \ldots\} \) with \( 0 < T_1 < T_2 < \ldots \) that satisfies the following conditions (for \( i = 1, 2 \)):

(i) \( z_{T_n}^i \neq z_{T_{n+1}}^i \) \( \forall n \)

(ii) \( z_t^i \) is constant \( \forall t \in [T_n, T_{n+1}) \) \( \forall n \)

It follows that the "times of switching" are the dates at which the individuals exchange one another the unit of fiat money present in the economy. Given the initial values \((z_0^1, z_0^2)\) and a "time switching" \( T \), this determines a strategy \((z^1_t, z^2_t)\). With reference to this aspect, the following result holds (here \( i \) denotes one individual and \( \widehat{i} \) the other one):

**Proposition 69** If \((z^1_t, z^2_t)\) is an optimal strategy and \( T = \{0, T_1, T_2, \ldots\} \) the corresponding time switching, then the following relation must hold:

\[ \frac{u'(w_{t+1}^i)}{u'(w_t^i)} \geq \frac{u'(w_{t+1}^{\widehat{i}})}{u'(w_t^{\widehat{i}})} \quad \forall t \in [T_n, T_{n+1} - 2] \neq \emptyset \quad \forall n \]

and furthermore:

\[ z_{t+1}^i = 1 \quad \widehat{z}_{t+1}^i = 0 \quad \forall t \in [T_n, T_{n+1} - 2] \]

**Proof.** If we consider an interval \([T_n, T_{n+1} - 2] \neq \emptyset\), on this interval the values \( z_t^i \)
and $\tilde{z}_t^2$ are constant, i.e.:

$$\tilde{z}_{t+1}^1 = \tilde{z}_t^1 \quad \tilde{z}_{t+1}^2 = \tilde{z}_t^2$$

and from the budget constraints of the two individuals we get:

$$\tilde{x}_t^1 = w_t^1 \quad \tilde{x}_t^2 = w_t^2$$

If $i$ is the individual for which $\tilde{z}_i^1 = 1$ and $\hat{i}$ the individual for which $\tilde{z}_i^2 = 0$, the corresponding Euler equations are:

$$u'(w_i^1)q_t = \beta u'(w_{i+1}^1)q_{t+1} \quad \forall t \in [T_n, T_{n+1} - 2]$$

$$u'(w_i^2)q_t \geq \beta u'(w_{i+1}^2)q_{t+1} \quad \forall t \in [T_n, T_{n+1} - 2]$$

from which (by assuming that $q_t \neq 0 \ \forall t$):

$$\frac{q_t}{q_{t+1}} = \frac{\beta u'(w_{t+1}^1)}{u'(w_t^1)} \geq \frac{\beta u'(w_{t+1}^2)}{u'(w_t^2)}$$

i.e.:

$$\frac{u'(w_{t+1}^1)}{u'(w_t^1)} \geq \frac{u'(w_{t+1}^2)}{u'(w_t^2)} \quad \forall t \in [T_n, T_{n+1} - 2]$$

and also:

$$\tilde{z}_{i+1}^1 = \tilde{z}_i^1 = 1 \quad \tilde{z}_{i+1}^2 = \tilde{z}_i^2 = 0 \quad \forall t \in [T_n, T_{n+1} - 2]$$

that completes the proof. ■

This result can be used to obtain a sufficient condition for the existence of a "switching" in every period, that means $T = \{0, 1, 2, 3, \ldots\}$, that is $T_1 = 1, T_2 = 2, T_3 = 3, \ldots$ and therefore $[T_n, T_{n+1} - 2] = \emptyset \ \forall n$. The following proposition holds:
Proposition 70 If \( z_0^1 = 1 \) and \( z_0^2 = 0 \), then a sufficient condition for a switching in every period (i.e. \( T = \{0, 1, 2, 3, \ldots\} \)) is given by the relations:

\[
\frac{u'(w_{t+1}^1)}{u'(w_t^1)} < \frac{u'(w_{t+1}^2)}{u'(w_t^2)} \quad \text{for } t \text{ even}
\]
\[
\frac{u'(w_{t+1}^2)}{u'(w_t^2)} < \frac{u'(w_{t+1}^1)}{u'(w_t^1)} \quad \text{for } t \text{ odd}
\]

**Proof.** Given \( z_0^1 = 1 \) and \( z_0^2 = 0 \), from Proposition 69 it follows that in order to have \( [0, T_1 - 2] = \emptyset \) (so that \( T_1 = 1 \)) it is sufficient that the relation stated in that proposition doesn’t hold, i.e. it is sufficient to have:

\[
\frac{u'(w_1^1)}{u'(w_0^0)} < \frac{u'(w_2^1)}{u'(w_0^0)}
\]

In this way at time \( t = 1 \) we have \( z_1^1 = 0 \) and \( z_1^2 = 1 \); if we now assume:

\[
\frac{u'(w_2^2)}{u'(w_2^0)} < \frac{u'(w_2^1)}{u'(w_1^1)}
\]

then \( [1, T_2 - 2] = \emptyset \) (so that \( T_2 = 2 \)) and there is another switching, that is at time \( t = 2 \) we have \( z_2^1 = 1 \) and \( z_2^2 = 0 \), and so on. By proceeding in this way we get \( T_1 = 1, T_2 = 2, T_3 = 3, \ldots \) so that \( T = \{0, 1, 2, 3, \ldots\} \) and the result is proved. \( \blacksquare \)

In the case of "switching" in every period, if \( z_0^1 = 1 \) and \( z_0^2 = 0 \) then the equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

\[
\bar{x}_t^1 = \begin{cases} 
    w_t^1 + q_t & \text{for } t \text{ even} \\
    w_t^1 - q_t & \text{for } t \text{ odd}
\end{cases}
\]
\[
\bar{z}_t^1 = \begin{cases} 
    1 & \text{for } t \text{ even} \\
    0 & \text{for } t \text{ odd}
\end{cases}
\]

\[
\bar{x}_t^2 = \begin{cases} 
    w_t^2 - q_t & \text{for } t \text{ even} \\
    w_t^2 + q_t & \text{for } t \text{ odd}
\end{cases}
\]
\[
\bar{z}_t^2 = \begin{cases} 
    0 & \text{for } t \text{ even} \\
    1 & \text{for } t \text{ odd}
\end{cases}
\]
and the Euler equations for agent 1 are:

\[
\begin{align*}
    u'(x_t^1)q_t & \geq \beta u'(x_{t+1}^1)q_{t+1} \quad \text{for } t \text{ even} \\
    u'(x_t^1)q_t & = \beta u'(x_{t+1}^1)q_{t+1} \quad \text{for } t \text{ odd}
\end{align*}
\]

while for agent 2 they are:

\[
\begin{align*}
    u'(x_t^2)q_t & = \beta u'(x_{t+1}^2)q_{t+1} \quad \text{for } t \text{ even} \\
    u'(x_t^2)q_t & \geq \beta u'(x_{t+1}^2)q_{t+1} \quad \text{for } t \text{ odd}
\end{align*}
\]

The results obtained in this Section can be used in a specific example (that is a variant of one of those proposed by Kocherlakota (1992)). In this example the two individuals have identical preferences expressed by the function:

\[
u(x_t^i) = \log x_t^i
\]

while their endowments are given by:

\[
\begin{align*}
    w_t^1 & = \begin{cases} 
    A \rho^t & \text{for } t \text{ even} \\
    B \rho^t & \text{for } t \text{ odd}
    \end{cases} \\
    w_t^2 & = \begin{cases} 
    B \rho^t & \text{for } t \text{ even} \\
    A \rho^t & \text{for } t \text{ odd}
    \end{cases}
\end{align*}
\]

i.e. they grow at the same average rate but fluctuate over time in a deterministic fashion. In this case \( \rho \) is the factor of growth and \( 0 < A < B \). The asset in the economy is \textit{fiat money}, therefore the dividend equals zero in every period and the fundamental value is zero. The initial quantities held by the two agents are:

\[
z_0^1 = 1 \quad z_0^2 = 0
\]
and thus the agent with the smaller endowment in period 0 (agent 1) has all of the asset. In addition, short-selling is prohibited, i.e. the individuals cannot borrow.

First of all, in this example the sufficient condition expressed in the last proposition that guarantees the existence of a "switching" in every period is satisfied; in fact, this condition becomes:

\[
\begin{align*}
\frac{w_{t}^1}{w_{t+1}^1} & < \frac{w_{t}^2}{w_{t+1}^2} \quad \text{for } t \text{ even} \\
\frac{w_{t}^2}{w_{t+1}^2} & < \frac{w_{t}^1}{w_{t+1}^1} \quad \text{for } t \text{ odd}
\end{align*}
\]

i.e., in both cases:

\[
\frac{A \rho^t}{B \rho^{t+1}} < \frac{B \rho^t}{A \rho^{t+1}}
\]

that means:

\[\frac{A}{B} < \frac{B}{A}\]

that is verified since \(A < B\).

As a consequence, in this economy there is "switching" in every period, and the portfolio and consumption allocations of equilibrium are:

\[
\begin{align*}
\bar{x}_t^1 & = \begin{cases} 1 & \text{for } t \text{ even} \\
0 & \text{for } t \text{ odd} \end{cases} & \bar{x}_t^1 & = \begin{cases} A \rho^t + q_t & \text{for } t \text{ even} \\
B \rho^t - q_t & \text{for } t \text{ odd} \end{cases} \\
\bar{x}_t^2 & = \begin{cases} 0 & \text{for } t \text{ even} \\
1 & \text{for } t \text{ odd} \end{cases} & \bar{x}_t^2 & = \begin{cases} B \rho^t - q_t & \text{for } t \text{ even} \\
A \rho^t + q_t & \text{for } t \text{ odd} \end{cases}
\end{align*}
\]

In order to determine the equilibrium price we now write the Euler equations for the two
individuals. For agent 1 they are:

\[
\frac{q_t}{x_t^1} > \frac{\beta q_{t+1}}{x_{t+1}^1} \quad \text{for } t \text{ even}
\]
\[
\frac{q_t}{x_t^1} = \frac{\beta q_{t+1}}{x_{t+1}^1} \quad \text{for } t \text{ odd}
\]

that is:

\[
\frac{q_t}{A \rho^t + q_t} \geq \frac{\beta q_{t+1}}{B \rho^{t+1} - q_{t+1}} \quad \text{for } t \text{ even}
\]
\[
\frac{q_t}{B \rho^t - q_t} = \frac{\beta q_{t+1}}{A \rho^{t+1} + q_{t+1}} \quad \text{for } t \text{ odd}
\]

and for agent 2:

\[
\frac{q_t}{x_t^2} = \frac{\beta q_{t+1}}{x_{t+1}^2} \quad \text{for } t \text{ even}
\]
\[
\frac{q_t}{x_t^2} \geq \frac{\beta q_{t+1}}{x_{t+1}^2} \quad \text{for } t \text{ odd}
\]

that is:

\[
\frac{q_t}{B \rho^t - q_t} = \frac{\beta q_{t+1}}{A \rho^{t+1} + q_{t+1}} \quad \text{for } t \text{ even}
\]
\[
\frac{q_t}{A \rho^t + q_t} \geq \frac{\beta q_{t+1}}{B \rho^{t+1} - q_{t+1}} \quad \text{for } t \text{ odd}
\]

In conclusion, for every \( t \) we must have:

\[
\frac{q_t}{B \rho^t - q_t} = \frac{\beta q_{t+1}}{A \rho^{t+1} + q_{t+1}}
\]
\[
\frac{q_t}{A \rho^t + q_t} \geq \frac{\beta q_{t+1}}{B \rho^{t+1} - q_{t+1}}
\]

To solve this system with respect to \( q_t \) we can now try for the price of the asset a solution
of the form:

\[ q_t = \frac{cp^t}{B \rho^t - c \rho^t} \]

and the first of the two Euler equations becomes:

\[
\frac{cp^t}{B \rho^t - c \rho^t} = \frac{\beta cp^{t+1}}{A \rho^{t+1} + cp^{t+1}}
\]

from which:

\[
\frac{1}{B - c} = \frac{\beta}{A + c}
\]

and finally:

\[
c = \frac{\beta B - A}{1 + \beta} \tag{3.7}
\]

The Euler inequality then becomes:

\[
\frac{cp^t}{A \rho^t + c \rho^t} \geq \frac{\beta cp^{t+1}}{B \rho^{t+1} - c \rho^{t+1}}
\]

from which:

\[
\frac{1}{A + c} \geq \frac{\beta}{B - c}
\]

and finally:

\[
c \leq \frac{B - \beta A}{1 + \beta} \tag{3.8}
\]

Since expressions (3.7) and (3.8) must hold together we must have:

\[
\frac{\beta B - A}{1 + \beta} \leq \frac{B - \beta A}{1 + \beta}
\]
i.e.:

$$\beta B - A \leq B - \beta A$$

that is:

$$\beta(A + B) \leq (A + B)$$

than means:

$$\beta \leq 1$$

and since this is always true, the expression (3.7) represents the value of $c$ that satisfies the Euler equations. The equilibrium price of the asset is therefore:

$$q_t = \frac{\beta B - A}{1 + \beta^{-\rho t}}$$

and the fact that this is a true equilibrium can be checked, also in this case, by means of the sufficient condition of optimality introduced in Chapter 2. We have in fact:

$$\lim_{t \to +\infty} \beta^t u'(\bar{x}_t^i)q_t \bar{z}_{t+1} = 0$$

if $\beta < 1$, and therefore the allocation $(\bar{x}, \bar{z})$ is (strongly) optimal and $(\bar{x}, \bar{z}), q)$ is an equilibrium.

The results obtained can be summarized in the following proposition:

**Proposition 71** The model considered has an equilibrium with valued fiat money, characterized by:

$$q_t = \frac{\beta B - A}{1 + \beta^{-\rho t}}$$
with $A < \beta B$. The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$
\bar{x}_t^1 = \begin{cases} 
\frac{\beta(A+B)}{1+\beta} \rho^t & \text{for } t \text{ even} \\
\frac{A+B}{1+\beta} \rho^t & \text{for } t \text{ odd}
\end{cases} \quad \bar{x}_t^2 = \begin{cases} 
1 & \text{for } t \text{ even} \\
0 & \text{for } t \text{ odd}
\end{cases}
$$

$$
\bar{x}_t^2 = \begin{cases} 
\frac{A+B}{1+\beta} \rho^t & \text{for } t \text{ even} \\
\frac{\beta(A+B)}{1+\beta} \rho^t & \text{for } t \text{ odd}
\end{cases} \quad \bar{x}_t^2 = \begin{cases} 
0 & \text{for } t \text{ even} \\
1 & \text{for } t \text{ odd}
\end{cases}
$$

As in the first example, also in this generalization the equilibrium price of fiat money is positive, while the fundamental value is zero, therefore we have a bubble. In terms of the analysis by means of Euler equations it results:

$$
\lim_{t \to +\infty} \beta^t u'(\bar{x}_t^1)q_t = 0
$$

i.e. the asymptotic bubble is 0; this means that the bubble component is due entirely to the violation of the Euler equation.

By using the more general approach of the previous Chapter we can conclude that, again, the origin of this bubble is represented by the short-sales constraint. In fact, in even periods agent 1 is poor because his endowment is relatively low, and he owns all of the asset at the beginning of the period, that he then sells to agent 2 (in order to smooth his consumption). The problem is represented by the fact that he would like to sell even more of the asset (and agent 2 would like to buy it), but he cannot because of the short-sales constraint; the same is true for agent 2 in odd periods. In conclusion, no agent can permanently reduce his holdings because of the short-sales constraint, and furthermore agents' endowments grow as fast as the bubble, so that the individuals can always buy the asset when they are wealthy. These elements determine an increase in the price of the asset (whose fundamental value, on the other hand, is zero) and turn out to be the bubble producing factors in this economy.
3.4 A stochastic model

The third example presented is an extension of the previous ones (that are deterministic models) to the stochastic setting, and is a variant of a monetary economy analyzed by Kehoe, Levine and Woodford (1992).

In this economy at each date $t$ there is realized a random state $s_t \in \{\xi, \eta\}$, following a Markov process with transition probabilities:

$$
\pi(s_{t+1} = \xi \mid s_t = \eta) = \pi(s_{t+1} = \eta \mid s_t = \xi) = \pi
$$

$$
\pi(s_{t+1} = \xi \mid s_t = \xi) = \pi(s_{t+1} = \eta \mid s_t = \eta) = 1 - \pi
$$

where $0 < \pi < 1$, given an initial condition $s_0 \in \{\xi, \eta\}$. In this case the state space is represented by $\Omega = \{\xi, \eta\}^N$, and if we denote with $s^t$ the generic node at time $t$, with $s^t \in F$ (where $F$ is the information structure that is used to describe the uncertainty in the model), this node can be identified with a sequence:

$$
s^t = \{s_0, s_1, ..., s_t\} \in \{\xi, \eta\}^{t+1}
$$

There is a single consumption good at each node, and a single security is traded, fiat money (in fixed net supply of one unit at all dates). The economy consists of two individuals ($i = 1, 2$) with the same preferences given by:

$$
\mathbb{E} \left[ \sum_{t=0}^{\infty} \beta^t u(x^i_t) \right]
$$

where $\beta$ is the discount factor (with $0 < \beta < 1$) and $u(x^i_t)$ is a strictly increasing and strictly concave utility function. The endowments of the two agents are:

$$
w^1(s^t) = \begin{cases} 
\bar{w} & \text{if } s_t = \xi \\
\bar{w} & \text{if } s_t = \eta
\end{cases}
$$

$$
w^2(s^t) = \begin{cases} 
\bar{w} & \text{if } s_t = \xi \\
w & \text{if } s_t = \eta
\end{cases}
$$

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where \( w < \bar{w} \), and in addition one of the two individuals is initially endowed with one unit of fiat money and the other with zero units (which of them has the asset does not matter). Finally, in this economy borrowing is not possible.

As usual, we look for an equilibrium in which the asset holding strategy of the two agents is of "bang bang" type, more precisely:

\[
\bar{z}^1(s^t) = \begin{cases} 
0 & \text{if } s_t = \xi \\
1 & \text{if } s_t = \eta 
\end{cases}, \quad \bar{z}^2(s^t) = \begin{cases} 
1 & \text{if } s_t = \xi \\
0 & \text{if } s_t = \eta 
\end{cases}
\]

In this model it is convenient to write the budget constraints in the form:

\[
x^1(s^t) + q(s^t) [\bar{z}^1(s^t) - \bar{z}^1(s^{t-1})] = w^1(s^t)
\]

\[
x^2(s^t) + q(s^t) [\bar{z}^2(s^t) - \bar{z}^2(s^{t-1})] = w^2(s^t)
\]

and with this notation the initial endowments of fiat money are denoted by \( z^1(s^{-1}) \) and \( z^2(s^{-1}) \). Furthermore, in this case we suppose that equilibrium prices are of the form:

\[
q(s^t) = \Phi(|z^1(s^t) - z^1(s^{t-1})|)
\]

and since \( |z^1(s^t) - z^1(s^{t-1})| \) can assume only the values 0 and 1 we have two possible prices:

\[
q(s^t) \left\{ \begin{array}{l}
\Phi(0) = q(s^{t-2}, \xi, \xi) = q(s^{t-2}, \eta, \eta) = q^{**} \\
\Phi(1) = q(s^{t-2}, \xi, \eta) = q(s^{t-2}, \eta, \xi) = q^*
\end{array} \right.
\]

Given all these elements we can write, as usual, the Euler equations (that in this case
are true stochastic Euler equations) for the two individuals. For agent 1 we get:

\[
\begin{align*}
u'(x^1(s^{t-1})) q(s^{t-1}) &\geq \beta \pi(s_t = \xi | s^{t-1}) u'(x^1(s^{t-1}, \xi)) q(s^{t-1}, \xi) + \\
&+ \beta \pi(s_t = \eta | s^{t-1}) u'(x^1(s^{t-1}, \eta)) q(s^{t-1}, \eta) \\
[u'(x^1(s^{t-1})) q(s^{t-1}) - \beta \pi(s_t = \xi | s^{t-1}) u'(x^1(s^{t-1}, \xi)) q(s^{t-1}, \xi) - \\
- \beta \pi(s_t = \eta | s^{t-1}) u'(x^1(s^{t-1}, \eta)) q(s^{t-1}, \eta)] \cdot z^1(s^{t-1}) &= 0
\end{align*}
\]

In particular, from the second relation we have that the first expression holds as an equality for \( z^1(s^{t-1}) = 1 \), i.e. for \( s_{t-1} = \eta \), and therefore it can be written as:

\[
\begin{align*}
u'(x^1(s^{t-2}, \eta)) q(s^{t-2}, \eta) &= \beta \pi u'(x^1(s^{t-2}, \eta, \xi)) q(s^{t-2}, \eta, \xi) + \\
&+ \beta(1 - \pi) u'(x^1(s^{t-2}, \eta, \eta)) q(s^{t-2}, \eta, \eta)
\end{align*}
\]

i.e. (by using the budget constraint to determine the values of \( x^1(\cdot) \)):

\[
u'(x^1(s^{t-2}, \eta)) q(s^{t-2}, \eta) = \beta \pi u'(w + q^*) q^* + \beta(1 - \pi) u'(\overline{w}) q^{**}
\]

from which:

\[
\begin{align*}
u'(\overline{w} - q^*) q^* &= \beta \pi u'(w + q^*) q^* + \beta(1 - \pi) u'(\overline{w}) q^{**} \quad \text{if } s_{t-2} = \xi \\
u'(\overline{w}) q^{**} &= \beta \pi u'(w + q^*) q^* + \beta(1 - \pi) u'(\overline{w}) q^{**} \quad \text{if } s_{t-2} = \eta
\end{align*}
\]

When \( z^1(s^{t-1}) = 0 \), i.e. for \( s_{t-1} = \xi \), the first expression becomes:

\[
u'(x^1(s^{t-2}, \xi)) q(s^{t-2}, \xi) \geq \beta(1 - \pi) u'(x^1(s^{t-2}, \xi, \xi)) q(s^{t-2}, \xi, \xi) + \\
+ \beta \pi u'(x^1(s^{t-2}, \xi, \eta)) q(s^{t-2}, \xi, \eta)
\]

i.e.:

\[
u'(x^1(s^{t-2}, \xi)) q(s^{t-2}, \xi) \geq \beta(1 - \pi) u'(w) q^{**} + \beta \pi u'(\overline{w} - q^*) q^*
\]
from which:

\[
\begin{align*}
u'(w)q^{**} & \geq \beta(1-\pi)u'(w)q^{**} + \beta\pi u'(w-q^*)q^* \quad & \text{if } s_{t-2} = \xi \\
u'(w+q^*)q^* & \geq \beta(1-\pi)u'(w)q^{**} + \beta\pi u'(w-q^*)q^* \quad & \text{if } s_{t-2} = \eta
\end{align*}
\]

Similarly, for agent 2 the Eulèr equations are:

\[
\begin{align*}
u'(x_2(s^{t-1}))q(s^{t-1}) & \geq \beta\pi(s_t = \xi \mid s^{t-1})\nu'(x_2(s^{t-1}, \xi))q(s^{t-1}, \xi) + \\
 & + \beta\pi(s_t = \eta \mid s^{t-1})\nu'(x_2(s^{t-1}, \eta))q(s^{t-1}, \eta) \\
\left[\nu'(x_2(s^{t-1}))q(s^{t-1}) - \beta\pi(s_t = \xi \mid s^{t-1})\nu'(x_2(s^{t-1}, \xi))q(s^{t-1}, \xi) - \\
- \beta\pi(s_t = \eta \mid s^{t-1})\nu'(x_2(s^{t-1}, \eta))q(s^{t-1}, \eta)\right] \cdot z^2(s^{t-1}) & = 0
\end{align*}
\]

In this case, from the second relation we have that the first expression holds as an equality for \(z^2(s^{t-1}) = 1\), i.e. for \(s_{t-1} = \xi\), and therefore it can be written as:

\[
\begin{align*}
u'(x_2(s^{t-2}, \xi))q(s^{t-2}, \xi) & = \beta(1-\pi)\nu'(x_2(s^{t-2}, \xi, \xi))q(s^{t-2}, \xi, \xi) + \\
 & + \beta\pi u'(x_2(s^{t-2}, \xi, \eta))q(s^{t-2}, \xi, \eta)
\end{align*}
\]

i.e.:

\[
\begin{align*}
u'(x_2(s^{t-2}, \xi))q(s^{t-2}, \xi) & = \beta(1-\pi)\nu'(\overline{w})q^{**} + \beta\pi\nu'(\overline{w}+q^*)q^*
\end{align*}
\]

from which:

\[
\begin{align*}
u'(\overline{w})q^{**} & = \beta(1-\pi)\nu'(\overline{w})q^{**} + \beta\pi\nu'(\overline{w}+q^*)q^* \quad & \text{if } s_{t-2} = \xi \\
u'(\overline{w}-q^*)q^* & = \beta(1-\pi)\nu'(\overline{w})q^{**} + \beta\pi\nu'(\overline{w}+q^*)q^* \quad & \text{if } s_{t-2} = \eta
\end{align*}
\]
When $z^2(s^{t-1}) = 0$, i.e. for $s_{t-1} = \eta$, the first expression becomes:

$$u'(x^2(s^{t-2}, \eta))q(s^{t-2}, \eta) \geq \beta \pi u'(x^2(s^{t-2}, \eta, \xi))q(s^{t-2}, \eta, \xi) +$$
$$+ \beta(1 - \pi)u'(x^2(s^{t-2}, \eta, \eta))q(s^{t-2}, \eta, \eta)$$

i.e.:

$$u'(x^2(s^{t-2}, \eta))q(s^{t-2}, \eta) \geq \beta \pi u'(-q^*)q^* + \beta(1 - \pi)u'(w)q'^*$$

from which:

$$u'(w + q^*)q^* \geq \beta \pi u'(-q^*)q^* + \beta(1 - \pi)u'(w)q'^* \quad \text{if } s_{t-2} = \xi$$

(3.12)

$$u'(w)q'^* \geq \beta \pi u'(-q^*)q^* + \beta(1 - \pi)u'(w)q'^* \quad \text{if } s_{t-2} = \eta$$

By combining the expressions (3.9) - (3.12) we obtain that in every state ($s_{t-2} = \xi$ or $s_{t-2} = \eta$) the following relations must hold:

$$u'(w - q^*)q^* = \beta \pi u'(w + q^*)q^* + \beta(1 - \pi)u'(w)q'^*$$

$$u'(w)q'^* = \beta \pi u'(w + q^*)q^* + \beta(1 - \pi)u'(w)q'^*$$

$$u'(w + q^*)q^* \geq \beta \pi u'(w - q^*)q^* + \beta(1 - \pi)u'(w)q'^*$$

$$u'(w)q'^* \geq \beta \pi u'(w - q^*)q^* + \beta(1 - \pi)u'(w)q'^*$$

First of all we solve the two equations; they can be written as:

$$\begin{cases} 
  u'(w - q^*)q^* = \beta \pi u'(w + q^*)q^* + \beta(1 - \pi)u'(w)q'^* \\
  \beta \pi u'(w + q^*)q^* + [\beta(1 - \pi) - 1]u'(w)q'^* = 0 
\end{cases}$$

and from the second one we obtain:

$$u'(w)q'^* = \frac{\beta \pi}{1 - \beta(1 - \pi)}u'(w + q^*)q^*$$

(3.13)

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and by substituting into the first one:

\[ u'(\overline{w} - q^*)q^* = \beta \pi u'(\overline{w} + q^*)q^* + \beta (1 - \pi) \frac{\beta \pi}{1 - \beta (1 - \pi)} u'(\overline{w} + q^*)q^* \]

After some computations we get:

\[
\left[ u'(\overline{w} - q^*) - \frac{\beta \pi}{1 - \beta (1 - \pi)} u'(\overline{w} + q^*) \right] q^* = 0
\]

from which, apart from the value \( q^* = 0 \), we obtain that this relation is satisfied by the value \( q^* \) such that:

\[
\frac{u'(\overline{w} - q^*)}{u'(\overline{w} + q^*)} = \frac{\beta \pi}{1 - \beta (1 - \pi)} \tag{3.14}
\]

and by substituting into (3.13) we get:

\[
q^{**} = \frac{u'(\overline{w} - q^*)}{u'(\overline{w})} q^* \tag{3.15}
\]

Now we consider the two inequalities; they can be written as:

\[
\begin{cases}
  u'(w + q^*)q^* \geq \beta \pi u'(\overline{w} - q^*)q^* + \beta (1 - \pi) u'(w)q^{**} \\
  [1 - \beta (1 - \pi)] u'(w)q^{**} \geq \beta \pi u'(\overline{w} - q^*)q^*
\end{cases}
\]

and by substituting the expression (3.15):

\[
\begin{cases}
  u'(w + q^*)q^* \geq \beta \pi u'(\overline{w} - q^*)q^* + \beta (1 - \pi) u'(w) \frac{u'(\overline{w} - q^*)}{u'(w)} q^* \\
  [1 - \beta (1 - \pi)] u'(w) \frac{u'(\overline{w} - q^*)}{u'(w)} q^* \geq \beta \pi u'(\overline{w} - q^*)q^*
\end{cases}
\]

from which:

\[
\begin{cases}
  u'(w + q^*)q^* \geq \left[ \beta \pi + \beta (1 - \pi) \frac{u'(w)}{u'(\overline{w})} \right] u'(\overline{w} - q^*)q^* \\
  [1 - \beta (1 - \pi)] \frac{u'(w)}{u'(\overline{w})} \geq \beta \pi
\end{cases}
\]

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and also:

\[
\begin{align*}
1 & \geq \left[ \beta \pi + \beta (1 - \pi) \frac{u'(w)}{u'(w)} \right] \frac{u'(w)}{u'(w)} \\
[1 - \beta (1 - \pi)] \frac{u'(w)}{u'(w)} & \geq \beta \pi
\end{align*}
\]

and by substituting the expression (3.14) in the first inequality we get:

\[
\begin{align*}
1 & \geq \left[ \beta \pi + \beta (1 - \pi) \frac{u'(w)}{u'(w)} \right] \frac{\beta \pi}{1 - \beta (1 - \pi)} \\
[1 - \beta (1 - \pi)] \frac{u'(w)}{u'(w)} & \geq \beta \pi
\end{align*}
\]

From the second inequality we now have:

\[
\frac{u'(w)}{u'(w)} \leq \frac{1 - \beta (1 - \pi)}{\beta \pi}
\]

while from the first inequality we can write:

\[
\frac{1 - \beta (1 - \pi)}{\beta \pi} \geq \beta \pi + \beta (1 - \pi) \frac{u'(w)}{u'(w)}
\]

i.e.:

\[
\frac{1 - \beta + \beta \pi - \beta^2 \pi^2}{\beta \pi} \geq \beta (1 - \pi) \frac{u'(w)}{u'(w)}
\]

and finally:

\[
\frac{u'(w)}{u'(w)} \geq \frac{\beta^2 \pi (1 - \pi)}{1 - \beta + \beta \pi (1 - \beta \pi)}
\]

In conclusion, from the two inequalities that must be satisfied in every state, we get the following limitations:

\[
\frac{\beta^2 \pi (1 - \pi)}{1 - \beta + \beta \pi (1 - \beta \pi)} \leq \frac{u'(w)}{u'(w)} \leq \frac{1 - \beta (1 - \pi)}{\beta \pi}
\]  

(3.16)
while from the two equalities that must be satisfied in every state we have that the values $q^*$ and $q^{**}$ we are looking for are those that satisfy the equations:

\[
\frac{u'(\bar{w} - q^*)}{u'(\bar{w} + q^*)} = \frac{\beta \pi}{1 - \beta(1 - \pi)}
\]

\[
q^{**} = \frac{u'(-\bar{w} - q^*)}{u'(-\bar{w})} q^*
\]

These values are unique, and to show this we can write the first expression as:

\[
u'(\bar{w} - q^*) = \delta u'(\bar{w} + q^*) \quad \text{with} \quad \delta = \frac{\beta \pi}{1 - \beta(1 - \pi)}
\]

where $0 < \delta < 1$, and then (by using the same technique applied in the first example of this Chapter) consider the function:

\[
f(q) = u(\bar{w} - q) + \delta u(\bar{w} + q)
\]

that is defined over the interval $[0, \bar{w}]$ and is strictly concave (therefore it has a unique maximum). Its derivative is:

\[
f'(q) = -u'(\bar{w} - q) + \delta u'(\bar{w} + q)
\]

and the maximum (if it is interior to $[0, \bar{w}]$) is characterized by the condition $f''(q) = 0$, i.e.:

\[
u'(\bar{w} - q) = \delta u'(\bar{w} + q)
\]

(that is exactly the condition that must be satisfied by the value $q^*$ we are looking for), together with the conditions $f'(0) > 0$ and $f'(\bar{w}) < 0$, that are necessary to avoid that
the maximum is on the boundary of the interval \([0, \bar{w}]\). These conditions lead to:

\[-u'(\bar{w}) + \delta u'(w) > 0\]
\[-u'(0) + \delta u'(w + \bar{w}) < 0\]

that is:

\[u'(\bar{w}) < \delta u'(w)\]
\[\delta u'(w + \bar{w}) < u'(0)\]

where the second condition is clearly satisfied if we assume \(u'(0) = +\infty\). In conclusion, the value we are looking for is the unique (positive) value \(q = q^*\) that satisfies the relations:

\[u'(\bar{w} - q) = \frac{\beta \pi}{1 - \beta(1 - \pi)} u'(w + q)\]

\[u'(\bar{w}) < \frac{\beta \pi}{1 - \beta(1 - \pi)} u'(w)\]  \(\text{(3.17)}\)

As a consequence, also the value \(q^{**}\) is unique, and since:

\[q^{**} = \frac{u'(\bar{w} - q^*)}{u'(\bar{w})} q^*\]

we have (as \(u'\) is decreasing) \(q^{**} > q^*\). From the expression (3.17) we also get a further limitation for the ratio \(\frac{u'(\bar{w})}{u'(w)}\), and by combining it with the relation (3.16) we have that the limitations that must be satisfied are the following:

\[\frac{\beta^2 \pi (1 - \pi)}{1 - \beta + \beta \pi (1 - \beta \pi)} \leq \frac{u'(\bar{w})}{u'(w)} < \frac{\beta \pi}{1 - \beta (1 - \pi)}\]

Finally, it is possible to determine all the equilibrium values, and the results can be
summarized in the following statement:

**Proposition 72** Under the assumption:

\[
\frac{\beta^2 \pi (1 - \pi)}{1 - \beta + \beta \pi (1 - \beta \pi)} \leq \frac{u'(\bar{w})}{u'(w)} \leq \frac{\beta \pi}{1 - \beta (1 - \pi)}
\]

the stochastic model considered has an equilibrium with valued fiat money given by the following processes:

- if \( s_{t-1} = \xi \) then \( \bar{x}^1(s^t) = w \), \( \bar{x}^1(s^t) = 0 \), \( q(s^t) = q^{**} \)
  \( s_t = \xi \)
  \( \bar{x}^2(s^t) = \bar{w} \), \( \bar{x}^2(s^t) = 1 \)

- if \( s_{t-1} = \xi \) then \( \bar{x}^1(s^t) = \bar{w} - q^{*} \), \( \bar{x}^1(s^t) = 1 \), \( q(s^t) = q^{*} \)
  \( s_t = \eta \)
  \( \bar{x}^2(s^t) = w + q^{*} \), \( \bar{x}^2(s^t) = 0 \)

- if \( s_{t-1} = \eta \) then \( \bar{x}^1(s^t) = w + q^{*} \), \( \bar{x}^1(s^t) = 0 \), \( q(s^t) = q^{*} \)
  \( s_t = \xi \)
  \( \bar{x}^2(s^t) = w - q^{*} \), \( \bar{x}^2(s^t) = 1 \)

- if \( s_{t-1} = \eta \) then \( \bar{x}^1(s^t) = \bar{w} \), \( \bar{x}^1(s^t) = 1 \), \( q(s^t) = q^{**} \)
  \( s_t = \eta \)
  \( \bar{x}^2(s^t) = w \), \( \bar{x}^2(s^t) = 0 \)

where \( q^{*} \) and \( q^{**} \) are the unique quantities satisfying the relations:

\[
q^{**} = \frac{u'(\bar{w} - q^{*})}{u'(\bar{w})} q^{*}
\]

and \( q^{**} > q^{*} \).

The fact that this is a true equilibrium, once again, can be verified by applying the
sufficient condition of optimality; we have in this situation:

\[ \lim_{t \to +\infty} \beta^t E[u'(\bar{x}^t(s^t))q(s^t)\bar{z}^t(s^t+1)] = 0 \]

and therefore the allocation found is (strongly) optimal. In this case at the equilibrium the entire money supply, at the end of trading at each node, is held by the individual that had the higher endowment \((\bar{w})\) at that node, while the individual with the lower endowment \((w)\) spends during the period any money that he holds at the beginning of the period. The exchange value of fiat money, then, is \(q^*\) if the agent that holds the entire money supply at the beginning of the period is the agent with endowment \(w\), while it is \(q^{**}\) if the agent with the money has endowment \(\bar{w}\). Again, we have a bubble, since the equilibrium price of fiat money is positive, while its fundamental value is zero. In terms of the general analysis presented in the previous Chapter, this is due to the fact that in this example the value of the aggregate endowment of the economy is infinite when it is evaluated using any state-price process satisfying the usual no-arbitrage equation. As a consequence, it is not possible to apply Theorem 48 of Chapter 2, and it is not true that the equilibrium price of fiat money is equal to its fundamental value (i.e. equal to zero). In terms of the analysis based on Euler equations, on the other hand, the asymptotic bubble is zero:

\[ \lim_{t \to +\infty} \beta^t E[u'(\bar{x}^t(s^t))q(s^t)] = 0 \]

and therefore the bubble component, when \(\beta < 1\) (as it is assumed) is due to the violation of the Euler equation. In fact, in every period, one of the two individuals, alternatively, satisfies the Euler equation as an equality, but the other individual satisfies it as an inequality, and in this way there is violation of this relation.
3.5 An example of asymptotic bubble

All the previous examples are characterized by the fact that the presence of a speculative bubble is linked to the violation of the Euler equation (alternatively, in each period, by one of the two agents, as a consequence of the symmetry of these agents that is assumed). This last example shows a different situation, in which a bubble arises even if the Euler equation is always satisfied as an equality, and hence no violation of this relation occurs.

For the case of homogeneous agents, an example of bubble even when the Euler equation is never violated is given in Montrucchio-Privileggi (1999). For the case of heterogeneous agents it is possible to consider a variant of the monetary model examined in the first example. We assume that everything is as in that example, except for the fact that the two agents don't discount the future, so that $\beta = 1$. In this situation the equilibrium is characterized by the conditions:

\[
u'(\overline{w} - q^*) = \nu'(\overline{w} + q^*)
\]
\[
u'(\overline{w}) < \nu'(w)
\]

from which:

\[
\overline{w} - q^* = \overline{w} + q^*
\]
\[
\overline{w} > w
\]

i.e.:

\[
q^* = \frac{\overline{w} - w}{2}
\]
\[
\overline{w} > w
\]
and then:

$$\bar{w} - q^* = \frac{\bar{w} + w}{2} = w + q^*$$

and therefore:

$$\bar{x}_t^1 = \bar{x}_t^2 = \frac{\bar{w} + w}{2} \quad \forall t$$

while the holding strategies are the same as before (the two agents hold alternatively the unit of the asset in even periods and in odd periods).

In this case we then have:

$$\lim \inf_{t \to +\infty} u'(\bar{x}_t^1)q^*\bar{x}_{t+1}^1 = 0$$

so that, by Proposition 65 of Chapter 2, the allocation \((\bar{x}^i, \bar{z}^i)\) is weakly optimal and the solution found is an equilibrium. In this example, anyway, these allocation is not strongly optimal because:

$$\lim \sup_{t \to +\infty} u'(\bar{x}_t^1)q^*\bar{x}_{t+1}^1 > 0$$

In this case the Euler equations are always satisfied as equalities, in fact since \(\beta = 1\) we have:

$$u'(\bar{w} - q) = u'(\bar{w} + q)$$

$$u'(\bar{w} + q) \geq u'(\bar{w} - q)$$

and this implies that the two Euler equations reduce to the same relation:

$$u'(\bar{w} - q) = u'(\bar{w} + q)$$

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and are satisfied as equalities. In this example, therefore, $\tilde{b}_t^i = 0$, i.e. the bubble component due to the violation of the Euler equation is zero, and as a consequence all the bubble is of asymptotic kind, in fact we have:

$$\bar{b}_t^i = \frac{1}{D u_t^i(x_t^i)} \lim_{k \to +\infty} D u_{i+k}^i(x_{i+k}) q^* = q^*$$

The results obtained can be summarized in the following proposition:

**Proposition 73** The monetary model "à la Bewley" in which agents do not discount the future (i.e. $\beta = 1$) has an equilibrium with valued fiat money where the price of the asset is:

$$q^* = \frac{w - \bar{w}}{2} \quad \forall t$$

with $\bar{w} > w$. The corresponding equilibrium consumption allocations and portfolio allocations of the two individuals are the following:

$$x_t^1 = \frac{\bar{w} + w}{2} \quad \forall t \quad x_t^1 = \begin{cases} 0 & \text{for } t \text{ even} \\ 1 & \text{for } t \text{ odd} \end{cases}$$

$$x_t^2 = \frac{\bar{w} + w}{2} \quad \forall t \quad x_t^2 = \begin{cases} 1 & \text{for } t \text{ even} \\ 0 & \text{for } t \text{ odd} \end{cases}$$

Furthermore, in this case Euler equations are never violated, and the bubble on the price of the asset is entirely of asymptotic kind.

The result concerning the nature of the bubble term extends to the other examples involving fiat money that have been considered in this Chapter, whenever $\beta \neq 1$. The conclusion is that in models with valued fiat money of the types examined, if $0 < \beta < 1$ the bubble component on the price of the asset is due to the violation of the Euler
equation, while if $\beta = 1$ the bubble is of asymptotic kind.

3.6 Conclusions

The examples considered in the previous Sections of this Chapter show how the presence of bubbles on asset prices is linked to particular situations. More precisely, in terms of the analysis based on Euler equations and inequalities, the appearance of bubbles is usually related to the violation of such equations. On the other hand, by considering the more general analysis introduced in Chapter 2, the appearance of bubbles is due to the violation of some of the hypothesis on which this analysis is based (in the examples introduced above, in particular, the impossibility for the individuals to borrow against the value of their future wealth - i.e. the short-sales constraint - or the fact that the aggregate endowment of the economy is infinite, are the cause of bubbles). Since these hypothesis are quite general, and are satisfied by a wide class of economies, this confirms that their violation (and the consequent bubbles) represent a rather special circumstance. This is the conclusion of the whole analysis presented in these Chapters, according to which speculative bubbles in intertemporal general equilibrium models are therefore a negligible phenomenon. As it emerges from the analysis of the models discussed, in these models it is assumed that individuals use all the available information to make their predictions (and furthermore that these predictions are based on the correct model of the economy), i.e that they have rational expectations. This assumption, hence, implies that the price of a security is equal to its fundamental value (the present value of its future dividend stream), and deviations from this value are only occasional.

Nevertheless, the observation of the real world makes it clear that there are periods in which this is not true, and the prices of some assets far exceed their fundamental values. These periods are the so-called speculative booms, in which speculative bubbles on the prices of the assets arise. Historically, the most famous examples of situations of this kind are the "Tulipmania" of the middle 1600 in Holland (when the appearance of new
varieties of tulips determined an enormous increase in their price level), the Mississippi Company and the South Sea Bubble in the first half of 1700 (when the prices of the shares of these companies rose enormously and then collapsed) and the famous stock market boom and crash of 1929 in this century. Less strong versions of such speculative manias, then, characterize capital markets intermittently.

There is thus an evident contrast between these episodes and the conclusions of the theory discussed in these Chapters; for this reason it is necessary to find an explanation of this situation, in order to be able to reconcile theoretical results and real episodes. A possible answer is represented by the fact that, during the periods of speculative boom, there is a "breakdown" of rational expectations, on which the theory is based. It is worth observing that the crucial characteristic of the asset markets is their liquidity. In fact, a long-lived security or capital good (an asset) can be purchased for two reasons: the first is the possibility to receive the future stream of dividends that it offers, the second is the possibility to resell it subsequently and to obtain a capital gain. When an agent buys an asset for the first reason, he will never accept to pay more for it than the present value of its future dividend stream, but when he buys an asset for the second reason what matters in assessing its value is what other agents will be ready to pay for it later. If agents can evaluate the future dividends of the asset with reasonable precision, and believe that all other agents can make a similar evaluation, then they have no reason to believe that the price of the asset at any future date will differ from the value of its remaining dividend stream. As a consequence, they will not accept to pay for the asset more than the present value of its future dividends, even when they buy it in order to resell it later. The element that allows to explain the appearance of speculative bubbles is the presence of some new event, for instance an innovation or a new discovery, whose consequences may be important but whose probability of success is difficult to evaluate (for instance the appearance of new varieties of tulips in the case of the "Tulipmania", or significant changes in techniques of production in other situations). In such a situation agents don’t know how to evaluate the assets, and at the same time the development of
the "fashion" draws new agents into the capital markets. At this point individuals have not experience to use in assessing future values of the assets and know that other agents face the same difficulties, therefore it is no longer rational for them to believe that the price of a security should equal its fundamental value; as a consequence, the common knowledge of rationality disappears, and the same happens to the hypothesis of rational expectations.

In this context, it often happens that at this moment professional investors enter the market (aware of the possibility of the innovation to create gains in the future), and in this way begin to give up prices; this confirms the expectations of these individuals and we have an initial phase of rising prices. When this process has continued for a while, the information concerning the possibility of consistent gains in the future spreads among a broader segment of investors, that in turn enter the market. The more investors are attracted to the market, the more the prices rise and the more expectations of rising prices become self-fulfilling. At this point agents' expectations of rising prices begin to feed on themselves and cease to be related to the rational valuation made by an agent who buys the asset in order to receive its future dividends stream. It is precisely in this moment that the assumption of common knowledge of rationality breaks down, because agents recognize that there are other agents in the market who are not pricing the assets by their fundamental values, but are basing their valuation on the upward inertia of the market. This is the phase in which speculative bubbles reach their maximum level.

Finally, after this process has continued for a certain period, an increasing number of agents begin to have doubts about the possibility of the market to continue this phase of rising prices; as a consequence, they start selling the assets, and since the number of buyers declines the process of rising prices gradually ends. As soon as there is a general perception that the market has lost its upward inertia, no agent has a reason to be a buyer, hence they try to get out of the market, the prices decrease and the bubble bursts. At the end of this process, the prices of the assets reach once again a level that corresponds approximately with agents' perceptions of their fundamental values.
Outside of the periods in which the rational expectations behaviour of the individuals breaks down, therefore, the price of an asset is equal to its fundamental value, and the results of the models considered are valid; it is in this way that the conclusions of the theory, presented in these Chapters, are consistent with the reality.
Bibliography


