Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities.  
The complex case

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Abstract. We consider a conducting body with complex valued admittance containing a finite number of well separated thin inclusions. We derive an asymptotic formula for the boundary values of the potential in terms of the width of the inclusions.

Keywords: Thin inhomogeneities, complex coefficients.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain representing the region occupied by a conducting body and consider, at a fixed frequency $\omega$, the complex valued admittance background

$$\gamma_0 = \sigma_0 + i\omega\varepsilon_0 \text{ in } \Omega,$$

where $\sigma_0$ and $\varepsilon_0$ are real valued functions representing the electrical conductivity and permittivity, respectively.

Let $\Sigma_j \subset \Omega$, for $j = 1, \ldots, N$ be a collection of simple, regular curves and consider, for $\epsilon$ sufficiently small, a neighborhood of $\Sigma_j$ given by

$$D_j^\epsilon = \{ x \in \Omega : \text{dist}(x, \Sigma_j) < \epsilon \},$$

representing a thin inhomogeneity of admittance

$$\gamma_j = \sigma_j + i\omega\varepsilon_j.$$

Let $\psi \in H^{1/2}(\partial\Omega)$ represents a complex valued boundary current and let
\( u_0 \) be the background potential which satisfies
\[
\begin{cases}
\text{div} (\gamma_0 \nabla u_0) = 0 & \text{in } \Omega \\
\gamma_0 \frac{\partial u_0}{\partial \nu} = \psi & \text{on } \partial \Omega,
\end{cases}
\]
where \( \nu \) is the unit outer normal to \( \partial \Omega \).

Let
\[
\gamma_{\epsilon} = \gamma_0 + \sum_{j=1}^{N} (\gamma_j - \gamma_0) \chi_{D^j}
\]
and consider the perturbed complex-valued potential \( u_\epsilon \) solution to
\[
\begin{cases}
\text{div} (\gamma_\epsilon \nabla u_\epsilon) = 0 & \text{in } \Omega \\
\gamma_\epsilon \frac{\partial u_\epsilon}{\partial \nu} = \psi & \text{on } \partial \Omega.
\end{cases}
\]

The main goal of the paper is to obtain an asymptotic expansion for the boundary values \( (u_\epsilon - u_0)|_{\partial \Omega} \) as \( \epsilon \to 0 \).

The formula we derive is analogue to the one obtained in [3] in the case of constant real valued conductivities \( \sigma_0 \) and \( \sigma_1 \)
More precisely, we show that for \( y \in \partial \Omega \) and \( \epsilon \to 0 \),
\[
(u_\epsilon - u_0)(y) = \epsilon v(y) + o(\epsilon).
\]
where
\[
v(y) = \sum_{j=1}^{N} 2 \int_{\Sigma_j} (\gamma_0 - \gamma_j)(x) M^j(x) \nabla u_0(x) \nabla x N(x, y) \, d\sigma_x.
\]
Here \( N(x, y) \) is the Neumann function corresponding to the operator \( \text{div} (\gamma_0 \nabla \cdot) \)
and \( M^j \) is a two by two matrix with complex valued entries.

It is well known that this type of expansion can be used in order to solve the inverse problem of detecting the curves \( \Sigma_j, j = 1, \ldots, N \) from boundary measurements. In fact, in [1] the authors show that for the conductivity equation, it is possible to detect finitely many segments from knowledge on the boundary of the first order term \( v \) appearing in the expansion. Moreover they show the continuous dependence of the segments from the boundary measurement \( v \) is Lipschitz stable. A similar result has been obtained in the case of the system of linearized elasticity, for the case \( N = 1 \), in [2].

2. Main assumptions and results
For \( j = 1, \ldots, N \), let \( \Sigma_j \) be a simple, regular \( C^{2,\alpha} \) curve with \( \alpha \in (0, 1) \) and assume there exists a constant \( K > 1 \) such that, in a neighborhood of radius
\( K^{-1} \) of each point in \( \Sigma_j \), the curve is the graph of a \( C^{2,\alpha} \) function and

\[
\|\Sigma_j\|_{C^{2,\alpha}} \leq K, \quad \text{dist}(\Sigma_j, \partial\Omega) \geq K^{-1}, \\
K^{-1} \leq L(\Sigma_j) \leq K, \quad \text{dist}(\Sigma_j, \Sigma_k) \geq K^{-1} \quad \text{if } j \neq k,
\]  

(1)

where \( L \) denotes the length.

On each curve \( \Sigma_j \) we fix an continuous orthonormal system \((n_j(x), t_j(x))\) such that \( n_j(x) \) is a normal direction to \( \Sigma_j \) at its point \( x \) and \( t_j(x) \) is a tangent direction.

Assume \( \gamma_0, \gamma_j : \Omega \to \mathbb{C} \) such that \( \gamma_0 \in C^{1,\alpha}(\Omega), \gamma_j \in C^{\alpha}(\Omega) \) with

\[
\|\gamma_0\|_{C^{1,\alpha}(\Omega)}, \|\gamma_j\|_{C^{\alpha}(\Omega)} \leq K,
\]

(2)

and, furthermore, assume there exists \( c_0 > 0 \) such that

\[
\sigma_j \geq c_0, \quad \text{for } j = 0, 1, \ldots, N.
\]

(3)

Consider finally a complex valued flux \( \psi \in H^{-1/2}(\partial\Omega) \) satisfying the compatibility condition

\[
\int_{\partial\Omega} \psi = 0.
\]

(4)

Then, under the above assumptions, there exist unique weak solutions \( u_0 \) and \( u_\varepsilon \) in \( H^1(\Omega) \) to

\[
\begin{aligned}
\text{div} (\gamma_0 \nabla u_0) &= 0 \quad \text{in } \Omega, \\
\gamma_0 \frac{\partial u_0}{\partial \nu} &= \psi \quad \text{on } \partial\Omega, \\
\int_{\partial\Omega} u_0 &= 0,
\end{aligned}
\]

(5)

and

\[
\begin{aligned}
\text{div} (\gamma_\varepsilon \nabla u_\varepsilon) &= 0 \quad \text{in } \Omega, \\
\gamma_\varepsilon \frac{\partial u_\varepsilon}{\partial \nu} &= \psi \quad \text{on } \partial\Omega, \\
\int_{\partial\Omega} u_\varepsilon &= 0.
\end{aligned}
\]

(6)

We also introduce the Neumann function \( N \) solution to

\[
\begin{aligned}
\text{div} (\gamma_0 \nabla N(\cdot, y)) &= \delta_y \quad \text{in } \Omega, \\
\gamma_0 \frac{\partial N(\cdot, y)}{\partial \nu} &= \frac{1}{L(\partial\Omega)} \quad \text{on } \partial\Omega, \\
\int_{\partial\Omega} N(\cdot, y) &= 0,
\end{aligned}
\]

(7)

It is well known that under assumptions (2) and (3) there exists a unique solution to (7) (see [5]).

We are now ready to state our main result.
THEOREM 2.1. Let $\Omega \subset \mathbb{R}^2$ be bounded smooth domain and $\{\Sigma_j\}_{j=1}^N \subset \Omega$ a set of curves satisfying (1), let $\gamma_0$ and $\gamma_j$ (for $j = 1, \ldots, N$) be admittivities satisfying (2) and (3) and let $u_0$ and $u_\epsilon$ be solutions to (5) and (6), respectively. Then, for $y \in \partial \Omega$ and $\epsilon \to 0$,

$$(u_\epsilon - u_0)(y) = 2\epsilon \sum_{j=1}^N \int_{\Sigma_j} (\gamma_0(x) - \gamma_j(x)) M^j(x) \nabla u_0(x) \cdot \nabla N(x, y) \, d\sigma_x + o(\epsilon),$$

where

$$M^j(x) = \frac{\gamma_0(x)}{\gamma_j(x)} n_j(x) \otimes n_j(x) + \tau_j(x) \otimes \tau_j(x).$$

3. Proof of Theorem 2.1

We will perform the proof in the case $N = 1$. Since the curves are well separated one from each other, the same argument will work for the case of multiple inclusions.

A complex valued equation as

$$\text{div} \ (\gamma \nabla u) = 0$$

can be interpreted as a two by two system for real valued functions. In fact, denoting by

$$u^1 = \Re u \text{ and } u^2 = \Im u,$$

we have that the function $u = (u^1, u^2) : \Omega \to \mathbb{R}^2$ satisfies the system

$$\frac{\partial}{\partial x_k} \left( a_{ij}^k \frac{\partial u^j}{\partial x_k} \right) = 0 \text{ for } i = 1, 2$$

where, for $i, j, h, k = 1, 2$,

$$a_{ij}^k = \delta_{hk} \delta_{ij} \Re \gamma - \delta_{hk} (\delta_{i1} \delta_{j2} - \delta_{i2} \delta_{j1}) \Im \gamma.$$

If

$$\Re \gamma \geq c_0 > 0,$$

then

$$a_{ij}^k \xi^i \xi^j \geq c_0 |\xi|^2$$

which corresponds to strong ellipticity. For this reason we can apply to our equations the results that hold for strongly elliptic systems.

We first establish some key energy estimates.

**Lemma 3.1.** There exists a constant $C = C(K, c_0, \Omega)$ such that

$$\|u_\epsilon - u_0\|_{H^1(\Omega)} \leq C \|D^1_\epsilon\|^{1/2} \|\psi\|_{H^{-1/2}(\partial \Omega)}.$$
\textbf{Proof.} Since $u_0$ and $u_\epsilon$ are solutions to (5) and (6), then $w_\epsilon = u_\epsilon - u_0$ is weak solution to
\[
\begin{align*}
\div (\gamma_\epsilon \nabla w_\epsilon) &= \div ((\gamma_0 - \gamma_1) \chi_{D_1^\epsilon} \nabla u_0) \quad \text{in } \Omega, \\
\gamma_\epsilon \frac{\partial w_\epsilon}{\partial \nu} &= 0 \quad \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_\epsilon &= 0.
\end{align*}
\]
Hence, for every $\phi \in H^1(\Omega)$,
\[
\int_{\Omega} \gamma_\epsilon \nabla w_\epsilon \cdot \nabla \phi = \int_{\Omega} (\gamma_0 - \gamma_1) \chi_{D_1^\epsilon} \nabla u_0 \cdot \nabla \phi. \tag{8}
\]
By choosing $\phi = w_\epsilon \in H^1(\Omega)$, we get
\[
\int_{\Omega} \gamma_\epsilon |\nabla w_\epsilon|^2 = \int_{\Omega} (\gamma_0 - \gamma_1) \chi_{D_1^\epsilon} \nabla u_0 \cdot \nabla w_\epsilon.
\]
Now, by (3), we have
\[
||\nabla w_\epsilon||_{L^2(\Omega)} \leq C|D_1^\epsilon|^{1/2}||\gamma_0 - \gamma_1||_{L^\infty(\Omega)} \sup_{D_1^\epsilon} |\nabla u_0|.
\]
By interior regularity results (see [4, Theorem 2.1, Chapter 2]),
\[
\sup_{D_1^\epsilon} |\nabla u_0| \leq C||\psi||_{H^{-1/2}(\partial \Omega)},
\]
so that
\[
||\nabla w_\epsilon||_{L^2(\Omega)} \leq C|D_1^\epsilon|^{1/2}||\psi||_{H^{-1/2}(\partial \Omega)}
\]
where $C = C(K, c_0)$.
Finally, since
\[
\int_{\partial \Omega} w_\epsilon = 0,
\]
by Poincaré inequality we have
\[
||w_\epsilon||_{L^2(\Omega)} \leq C||\nabla w_\epsilon||_{L^2(\Omega)}
\]
with $C = C(\Omega)$ and we obtain
\[
||u_\epsilon - u_0||_{H^1(\Omega)} \leq |D_1^\epsilon|^{1/2}||\psi||_{H^{1/2}(\partial \Omega)},
\]
which ends the proof.

We will also make use of some key regularity results for elliptic systems with discontinuous coefficients due to [6] (that extend the one established in [7] for
scalar elliptic equations). We state here a simplified version of Proposition 5.1 of [6].

Let $c$ and $M$ be two positive constants with $M > 2K$ and denote by $Q_{c,M}$ the set of points $x \in \Omega$ such that $\text{dist}(x, \partial \Omega) > M^{-1}$ and such that there is a square of size $c$ centered at $x$ that intersects $\partial D_1^1$ in at most two cartesian curves whose $C^{1,\alpha}$ norms are bounded by $M$, i.e. there exists a coordinate system at $x$ such that $\partial D_1^1 \cap [-c, c]^2$ consists in graphs of at most two functions $h_- < h_+$ with $\|h\|_{C^{1,\alpha}} \leq M$. Let us denote by $\Omega_M = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \frac{1}{2M} \}$.

**Lemma 3.2.** Let $\beta \in (0, 1/4)$ and $M > 2K$. there exists a constant $C$ depending on $\alpha$, $K$, $c_0$ and $M$ such that if $u \in H^1(\Omega)$ is a solution of

$$\text{div}(\gamma \nabla u) = 0 \text{ in } \Omega,$$

then

$$\|u_e\|_{C^{1,\beta}(Q_{c,M} \cap D_1^1)} \leq \frac{C}{c^{1+\beta}} \|u_e\|_{L^2(\Omega)}$$

(9)

and

$$\|u_e\|_{C^{1,\beta}(Q_{c,M} \setminus D_1^1)} \leq \frac{C}{c^{1+\beta}} \|u_e\|_{L^2(\Omega)}.$$  

(10)

**Lemma 3.3.** There exists $\eta > 0$ such that, if $u \in H^1(\Omega)$ is a solution to the complex valued equation

$$\text{div}(\gamma \nabla u) = f \text{ in } \Omega,$$

where $\gamma : \Omega \to \mathbb{C}$, $\gamma \in L^\infty(\Omega)$ such that

$$\Re \gamma \geq c_0 > 0$$

and $f \in H^{-1,2+\eta}(\Omega)$, then $u \in H^{1,2+\eta}(\Omega)$ and, given $B_\rho$ and $B_{2\rho}$ concentric disks contained in $\Omega$,

$$\|\nabla u\|_{L^{2+\eta}(B_{\rho})} \leq C \left( \|f\|_{H^{-1,2+\eta}(B_{2\rho})} + \rho^{\frac{2\eta}{2+\eta}} \|\nabla u\|_{L^2(B_{2\rho})} \right).$$

For the proof see [4, Chapter 2, Section 10].

**Proof of Theorem 2.1.** Take $y \in \partial \Omega$. Then, by the definition of the Neumann function, it is easy to see that

$$(u_e - u_0)(y) = \int_{D_1} (\gamma_0 - \gamma_1) \nabla u_e \cdot \nabla N(\cdot, y).$$

We prove the theorem in the more interesting and complicated case when $\Sigma_1$ is an open curve. In fact, in this case, $\partial D_1^1$ has derivatives (near the endpoints of $\Sigma_1$) that degenerate as $\epsilon$ tends to zero. This implies that the regularity estimates of Lemma 3.2 cannot be applied near the endpoints of $\Sigma_1$. 
Let \( P_1 \) and \( Q_1 \) be the endpoints of \( \Sigma_1 \), let \( \theta \in (0, 1) \) to be chosen later, and define
\[
D_{1,\theta} = \left\{ x + \mu n(x) : x \in \Sigma_1, \text{dist}(x, P_1 \cup Q_1) > \epsilon^\theta, \mu \in (\epsilon, \epsilon) \right\}.
\]
It is easy to see that there exists a constant \( M > 2K \) and depending only on \( K \), such that
\[
D_{1,\theta} \subset Q_{\frac{\epsilon^\theta}{M}} \cap D_{1,\theta}.
\]
An application of Lemma 3.2 thus gives
\[
\| \nabla u_c \|_{C_{\tilde{\gamma}(D_{1,\theta})}} \leq C \epsilon^{-\theta(1+\beta)} \| u_c \|_{L^2(\Omega)} \leq C \epsilon^{-\theta(1+\beta)} \| \psi \|_{H^{-1/2}(\partial\Omega)}
\]
where \( C \) does not depend on \( \epsilon \).

Then,
\[
\int_{D_{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_c(x) \cdot \nabla N(x, y) \, dx
= \int_{D_{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_c(x) \cdot \nabla N(x, y) \, dx
+ \int_{D_{1,\theta} \setminus D_{1,\theta}} (\gamma_0 - \gamma_1)(x) \nabla u_c(x) \cdot \nabla N(x, y) \, dx := I_1 + I_2.
\]

Let us estimate \( I_2 \) first.
\[
|I_2| \leq \left| \int_{D_{1,\theta} \setminus D_{1,\theta}} (\gamma_0 - \gamma_1) \nabla (u_c - u_0) \cdot \nabla N(\cdot, y) \right|
+ \left| \int_{D_{1,\theta} \setminus D_{1,\theta}} (\gamma_0 - \gamma_1) \nabla u_0 \cdot \nabla N(\cdot, y) \right|.
\]

Observe that, since \( \gamma_0 \in C^{1,\alpha}(\Omega) \), by interior regularity results ([4, Theorem 2.1, Chapter 2], [5] and by (2) we get
\[
\| \nabla u_0 \|_{L^\infty(D_1)} \leq C \| \psi \|_{H^{-1/2}(\partial\Omega)}, \tag{12}
\]
\[
|\nabla N(x, y)| \leq \frac{C}{|x - y|},
\]
and, since \( y \in \partial\Omega \)
\[
\| \nabla N(\cdot, y) \|_{L^\infty(D_1)} \leq C. \tag{13}
\]

Hence, by (12), (13) and Lemma 3.1,
\[
|I_2| \leq C \left| D_{1,\theta} \setminus D_{1,\theta} \right| \| \psi \|_{H^{-1/2}(\partial\Omega)} \leq C \epsilon^{1+\theta} \| \psi \|_{H^{-1/2}(\partial\Omega)}.
\]
We now define
\[ \Sigma_0^\theta = \{ x + \eta u_1(x) : x \in \Sigma_1, \text{dist}(x, P_1 \cup Q_1) > \epsilon^\theta \} . \]
Due to the regularity of \( \Sigma_1 \), if we denote by \( d\sigma^\eta_x \) the arclength measure on \( \Sigma_0^\theta \) and by \( d\sigma_x \) the arclength measure on \( \Sigma_1 \), we have
\[ d\sigma^\eta_x = (1 + O(\eta))d\sigma_x. \]
For every point \( x + \eta u(x) \in D_1^1 \), let \( x_\epsilon = x + \epsilon n(x) \). By (11),
\[ |\nabla u_\epsilon(x + \eta u(x)) - \nabla u_\epsilon(x_\epsilon)| \leq C|x + \eta u(x) - x_\epsilon|^{\beta}e^{-\theta(1+\beta)}\|\psi\|_{H^{-1/2}(\partial\Omega)} \]
and
\[ |(\gamma_1 - \gamma_0)(x + \eta u(x)) - (\gamma_1 - \gamma_0)(x_\epsilon)| \leq C\epsilon^\alpha \]
so that
\[ \int_{D_1^1 \setminus \Omega} (\gamma_0 - \gamma_1)\nabla u_\epsilon \nabla x N = 2\epsilon \int_{\Sigma_0^\theta} (\gamma_0 - \gamma_1)\nabla u_\epsilon^\eta \nabla x N + o(\epsilon), \]
where we set
\[ u_\epsilon^i = u_\epsilon|_{D_1^1 \setminus \Omega}, \quad u_\epsilon^\eta = u_\epsilon|_{\Omega \setminus \Sigma_0^\theta}. \]
We now use the transmission conditions
\[ u_\epsilon^i = u_\epsilon^\eta, \quad \frac{\partial u_\epsilon^i}{\partial n} = \frac{\partial u_\epsilon^\eta}{\partial n}, \]
that are satisfied on \( \partial D_1^1 \setminus \Omega \) pointwise, in order to obtain, finally,
\[ \int_{D_1^1 \setminus \Omega} (\gamma_0 - \gamma_1)\nabla u_\epsilon \nabla N \]
\[ = 2\epsilon \int_{\Sigma_0^\theta} (\gamma_0 - \gamma_1) \left\{ \frac{\gamma_0}{\gamma_1} \frac{\partial u_\epsilon^\eta}{\partial n} \frac{\partial N}{\partial x} + \frac{\partial u_\epsilon^\eta}{\partial n} \frac{\partial N}{\partial r_x} \right\} + o(\epsilon). \]
Assume that
\[ \|\nabla u_\epsilon^\eta - \nabla u_\epsilon^\eta\|_{L^\infty(\Sigma_0^\theta)} \leq C\epsilon^{\beta_1} \|\psi\|_{H^{-1/2}(\partial\Omega)} \]
for some \( \theta_1 > 0 \). Then
\[ \int_{D_1^1 \setminus \Omega} (\gamma_0 - \gamma_1)(x)\nabla u_\epsilon (x) \nabla N(x, y)dx \]
\[ = 2\epsilon \int_{\Sigma_0^\theta} (\gamma_0 - \gamma_1) \left\{ \frac{\gamma_0}{\gamma_1} \frac{\partial u_\epsilon^\eta}{\partial n} \frac{\partial N}{\partial n_1} + \frac{\partial u_\epsilon^\eta}{\partial n_1} \frac{\partial N}{\partial r_1} \right\} d\sigma_x + o(\epsilon) \]
\[ = 2\epsilon \int_{\Sigma_1} (\gamma_0 - \gamma_1)\nabla u_\epsilon (x) \nabla N(x, y) d\sigma_x + o(\epsilon), \]
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which concludes the proof.

So, we are left with the proof of (14). Let \( 2 \varepsilon < d < \frac{1}{2d} \) and let

\[ \Omega_d' = \{ x \in \Omega : \text{dist}(x, \partial(\Omega \setminus D_d^1)) > d \}. \]

Since \( u_\varepsilon - u_0 \) is solution of

\[ \nabla (\gamma_0 \nabla (u_\varepsilon - u_0)) = 0 \text{ in } \Omega \setminus D_d^1, \]

the regularity assumption on \( \gamma_0 \) implies that \( u_\varepsilon - u_0 \in H^2_{\text{loc}}(\Omega \setminus D_d^1) \) (see [4, Theorem 2.1, Chapter 2]).

Consider \( \Phi^k = \frac{\partial}{\partial x_k} (u_\varepsilon - u_0) \) for \( k = 1, 2 \).

The function \( \Phi^k \) satisfies in \( \Omega \setminus D_d^1 \)

\[ \nabla (\gamma_0 \nabla \Phi^k) = -\nabla (\partial \gamma_0 \nabla (u_\varepsilon - u_0)) = F. \]

By Caccioppoli inequality and by Lemma 8 we have that

\[
\| \nabla \Phi^k \|_{L^2(\Omega_{d/4}')} \leq C \frac{1}{d^2} \| \Phi^k \|_{L^2(\Omega_{d/4}')} + \| F \|_{H^{-1}(\Omega_{d/4})} \\
\leq C \left( \frac{1}{d^2} + 1 \right) \| \nabla (u_\varepsilon - u_0) \|_{L^2(\Omega')} \\
\leq C \left( \frac{1}{d^2} + 1 \right) \| \psi \|_{H^{-1/2}(\partial \Omega)}. 
\]

Hence

\[ \| \Phi^k \|_{H^1(\Omega_{d/2}')} \leq C \sqrt{d} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \]

Applying Lemma 3.3 to \( \Phi^k \) gives

\[ \| \nabla \Phi^k \|_{L^{2+\eta}(\Omega_d')} \leq C \left( \| F \|_{H^{-1,2+\eta}(\Omega_{d/2})} + d^{\frac{2+\eta}{2(2+\eta)}} \| \nabla \Phi^k \|_{L^2(\Omega_{d/4})} \right). \]

Now, by

\[ \| F \|_{H^{-1,2+\eta}(\Omega_{d/2})} \leq C \| \nabla (u_\varepsilon - u_0) \|_{L^{2+\eta}(\Omega_{d/2}')}, \]

and by the interior regularity estimates and Sobolev Immersion Theorem

\[ \| \nabla (u_\varepsilon - u_0) \|_{L^{2+\eta}(\Omega_{d/2}')} \leq C \| u_\varepsilon - u_0 \|_{H^2(\Omega_{d/2}')} \]

\[ \leq C \| u_\varepsilon - u_0 \|_{H^1(\Omega_{d/2}')} \leq \| u_\varepsilon - u_0 \|_{H^1(\Omega)}. \]
and hence
\[ \| \nabla \Phi_k \|_{L^{2+\eta}(\Omega_d')} \leq C \left( \| u_\epsilon - u_0 \|_{H^1(\Omega)} + d^{\frac{2}{2+\eta}} \| \nabla \Phi_k \|_{L^2(\Omega_d')/2} \right) \]
\[ \leq C \left( 1 + d^{\frac{2}{2+\eta}} \right) \sqrt{\epsilon} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \]

Finally, since \( \frac{2}{2+\eta} - 2 < 0 \) and \( d < 1 \), from last inequality, we derive
\[ \| \nabla \Phi_k \|_{L^{2+\eta}(\Omega_d')} \leq Cd^\frac{2}{2+\eta} \sqrt{\epsilon} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \]

On the other hand, applying Lemma 3.3 to \( u_\epsilon - u_0 \) we have
\[ \| \Phi_k \|_{L^{2+\eta}(\Omega_d')} \leq Cd^\frac{2}{2+\eta} \sqrt{\epsilon} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \]

By Sobolev Imbedding Theorem we than have
\[ \left\| \frac{\partial}{\partial x_k} (u_\epsilon - u_0) \right\|_{L^\infty(\Omega'_d)} \leq Cd^\frac{2}{2+\eta} \sqrt{\epsilon} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \tag{15} \]

Now let \( y \in \Sigma'_d \) and \( y_d \) be the closest point to \( y \) in \( \Omega_d' \). By (11) we have
\[ |\nabla u_\epsilon(y) - \nabla u_\epsilon(y_d)| \leq C \frac{d^\beta}{\epsilon^{\theta(\beta+1)}} \| \psi \|_{H^{-1/2}(\partial \Omega)}. \tag{16} \]

Hence, by (15) and (16) we have
\[ |\nabla(u_\epsilon - u_0)(y)| \leq C \left( d^{\beta} \epsilon^{-\theta(\beta+1)} + d^{-2} + \frac{2}{\pi} \sqrt{\epsilon} \right) \| \psi \|_{H^{-1/2}(\partial \Omega)}. \]

Choosing \( \theta < \frac{\beta}{2(2+\frac{\beta}{2+\eta})(\beta+1)} \) we get
\[ |\nabla(u_\epsilon - u_0)(y)| \leq C \epsilon^{\theta_1} \]
with \( \theta_1 > 0 \).

\[ \Box \]

References

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