On the improved Massera’s theorem for the unique existence of the limit cycle for a Liénard equation

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Abstract. We further generalize a recent improvement obtained by G. Villari of the classical Massera’s theorem about the unique existence of the limit cycle of a Liénard equation.

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1. Background for the improved Massera’s theorem

In this paper, we consider the well-known Liénard equation

\[ \ddot{x} + f(x)\dot{x} + x = 0. \]

Throughout, we assume for the above equation that the function \( f(x) \) satisfies smoothness conditions in order to guarantee the uniqueness of solutions of initial value problems. This equation has been widely investigated in the literature (for instance see [9]). We are interested in the unique existence of the limit cycle of the equation under the following Property (A) (see [8]):

\[ f(x) \text{ is continuous and there exist } a < 0 < b \text{ such that } f(x) < 0 \text{ for } a < x < b, \]
\[ f(x) > 0 \text{ for } x \leq a \text{ or } x \geq b; \text{ moreover, } xF(x) > 0 \text{ for } |x| \text{ large, where } F(x) = \int_{0}^{x} f(t)dt. \]

Note that \( F(x) \) has three zeros at \( \alpha < 0 < \beta \) and is monotone increasing for \( x < \alpha \) and for \( x > \beta \).

It is well-known that the Liénard equation is equivalent to the Liénard system

\[ \dot{x} = y - F(x), \quad \dot{y} = -x. \] (L)

First, we recall some previous results for system (L). Levinson-Smith [3] in 1942 and Sansone [5] in 1949 (see also the paper of Villari [7] in 1985) have proved the following...
Proposition 1.1. Under the property (A) a limit cycle intersecting both the lines \( x = \alpha \) and \( x = \beta \) is at most one.


Proposition 1.2. (Massera’s Theorem) System (L) has at most one limit cycle which is stable if \( f(x) \) is monotone decreasing for \( x < 0 \) and \( f(x) \) is monotone increasing for \( x > 0 \).

We remark that the existence of a limit cycle is not guaranteed in the above theorem.

Recently Villari [8] in 2012, on these bases, has presented the following

Proposition 1.3. Under the property (A) system (L) has exactly one limit cycle, which is stable, provided that

- if \( |\alpha| > \beta \), then \( f(x) \) is monotone decreasing for \( \alpha < x < 0 \), \( f(x) \) is monotone increasing for \( 0 < x < \delta \),
- if \( |\alpha| < \beta \), then \( f(x) \) is monotone decreasing for \( \delta_1 < x < 0 \), \( f(x) \) is monotone increasing for \( 0 < x < \beta \),

where \( \delta = \sqrt{\left(1 + F(a) + \frac{\alpha^2}{2}\right)^2 + \beta^2} \) and \( \delta_1 = -\sqrt{\left(1 - F(b) + \frac{\beta^2}{2}\right)^2 + \alpha^2} \).

Our aim is to give a new criterion for the unique existence of the limit cycle of system (L) by combining Proposition 1.3 with our result [2] in 2000 below.

Proposition 1.4. Assume that \( f(x) \) is continuous, \( f(a) = f(b) = 0 \) for \( a < 0 < b \), \( f(0) < 0 \) and \( xF(x) > 0 \) for \( |x| \) large. System (L) has exactly one limit cycle, which is stable, provided that

(i) \( |\alpha| = \beta \) and \( f(x) > 0 \) for \( |x| \geq \beta \),
(ii) \( |\alpha| \leq \beta < |\alpha| \) and \( f(x) > 0 \) for \( |x| \geq \beta \),
(iii) \( b \leq |\alpha| < \beta \) and \( f(x) > 0 \) for \( |x| \geq |\alpha| \).

We produce the proof of the above proposition in the Appendix.

2. Main results

We show in this section that our method yields an improvement of the result of Villari [8]. Instead of the Property (A), assume the following Property (B):

\( f(x) \) is continuously differentiable and \( F(0) = F(\alpha) = F(\beta) = 0 \), \( \frac{F(x)}{x} < 0 \) for \( \alpha < 0 < \beta \), \( f(x) > 0 \) for \( x \leq p \) and \( x \geq \beta \), or \( x \leq \alpha \) and \( x \geq q \), where
\[ p = \min\{x \in (\alpha, 0) \mid F'(x) = 0, \; F''(x) \neq 0\} \]

and

\[ q = \max\{x \in (0, \beta) \mid F'(x) = 0, \; F''(x) \neq 0\}. \]

Remark that Property (B) includes Property (A). We now state our result concerning the unique existence of limit cycles of system (L).

**Theorem 2.1.** Under the property (B), if system (L) satisfies one of the conditions:

1. \(|\alpha| = \beta\) and \(f(x) > 0\) for \(|x| \geq \beta\),
2. \(|p| \leq \beta < |\alpha|\) and \(f(x) > 0\) for \(|x| \geq \beta\),
3. \(q \leq |\alpha| < \beta\) and \(f(x) > 0\) for \(|x| \geq |\alpha|\),
4. \(|\alpha| > \beta\) and \(\beta < |p|\), \(f(x) > 0\) for \(x \leq p\) and \(x \geq \beta\), \(f(x)\) is monotone decreasing for \(p \leq x < 0\), \(f(x)\) is monotone increasing for \(0 < x < \delta^*\),

where \(\delta^* = \sqrt{\left(1 + F(a^*) + \frac{p^2}{2}\right)^2 + \beta^2}\) for \(a^* = \min\{x \mid \max_{x \in (\alpha, 0)} F(x)\}\),

5. \(|\alpha| < \beta\) and \(|\alpha| < q\), \(f(x) > 0\) for \(x \leq \alpha\) and \(x \geq q\), \(f(x)\) is monotone decreasing for \(\delta_1^* < x < 0\), \(f(x)\) is monotone increasing for \(0 < x \leq q\),

where \(\delta_1^* = -\sqrt{\left(1 - F(b^*) + \frac{q^2}{2}\right)^2 + \alpha^2}\) for \(b^* = \max\{x \mid \min_{x \in (0, \beta)} F(x)\}\),

then it has a unique stable limit cycle.

**Remark 2.2.** In [8] the case of \(p = a = a^*\) or \(q = b = b^*\) is treated.

**Remark 2.3.** In Theorem 2.1 the unique limit cycle intersects the lines \(x = \pm \beta\) in the case (1) or (2). In the case (3) it intersects the lines \(x = \pm \alpha\), in the case (4) \(x = p\) and \(x = \beta\), in the case (5) \(x = \alpha\) and \(x = q\).

We now apply Theorem 2.1 to the Liénard equation with a positive parameter \(\lambda\):

\[ \ddot{x} + \lambda f(x)\dot{x} + x = 0. \]

It is equivalent to the Liénard system

\[ \dot{x} = y - \lambda F(x), \quad \dot{y} = -x. \quad (L_\lambda) \]
Theorem 2.4. Under each condition in Theorem 2.1 system $(L_\lambda)$ satisfies the following:

1. if $|\alpha| = \beta$, then it has a unique stable limit cycle intersecting the lines $x = \alpha$ and $x = \beta$, for all $\lambda > 0$,
2. if $|p| \leq \beta < |\alpha|$, then it has a unique stable limit cycle intersecting the lines $x = \pm\beta$, for all $\lambda > 0$.
3. if $q \leq |\alpha| < \beta$, then it has a unique stable limit cycle intersecting the lines $x = \pm\alpha$, for all $\lambda > 0$.
4. if $|\alpha| > \beta$ and $\beta < |p|$, then it has a unique stable limit cycle intersecting the lines $x = p$ and $x = \beta$, for all $\lambda > \tilde{\lambda}_1 = \sqrt{\frac{p^2 - \beta^2}{F^2(b^*)}}$.
5. if $|\alpha| < \beta$ and $|\alpha| < q$, then it has a unique stable limit cycle intersecting the lines $x = \alpha$ and $x = q$, for all $\lambda > \tilde{\lambda}_2 = \sqrt{\frac{q^2 - \alpha^2}{F^2(a^*)}}$.

3. Proofs of theorems

Proof of Theorem 2.1. First, the cases of (1), (2) and (3) follow from [1] and [2]. So we omit the details. Next, we prove the case (4). By the Property (B), the existence of the limit cycle for system (L) is guaranteed. From [2] system (L) has at most one limit cycle intersecting the lines $x = p$ and $x = \beta$. Further it is stable. On the other hand, the limit cycle of system (L) contained in the region $D = \{(x, y) \mid p \leq x \leq \delta^*, \; y \in \mathbb{R}\}$ is at most one, by the monotonicity condition on the function $f(x)$, and is stable (see [8]). Thus we conclude from the stability of the limit cycle that system (L) has exactly one limit cycle, either intersecting the lines $x = p$ and $x = \beta$, or in $D$. Similarly, we can prove the case (5).

Proof of Theorem 2.2. The case (1)' is well-known from [1] or [8]. In the case (2)' or (3)' the result in [2] applies. So we consider the case (4)'. Any positive semitrajectory which starts from the point $(\beta, \lambda F(b^*))$ must intersect the line $x = p$ for the positive number $\lambda$ such that

\[
\sqrt{\lambda^2 F^2(b^*)} + \beta^2 \geq |p|
\]

namely, for all $\lambda > \tilde{\lambda}_1$. Then, as was mentioned in Theorem 2.1, the unique limit cycle intersecting $x = p$ and $x = \beta$ exists. Further $\delta^*$ is given by

\[
\delta^* = \sqrt{\left(1 + \lambda F(a^*) + \frac{p^2}{2}\right)^2 + \beta^2}
\]
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for each $\lambda$ satisfying $\lambda > \tilde{\lambda}_1$. Similarly, the case (5)' is discussed, where

$$\delta^*_i = -\sqrt{(1 - \lambda F(b^*) + \frac{q^2}{2})^2 + \alpha^2}$$

for all $\lambda > \tilde{\lambda}_2$. 

\[ \square \]

4. An example

We shall apply our results to some polynomial system.

**Example 4.1.** Consider the function

$$F(x) = \begin{cases} 
\frac{1}{3}x^3 + \frac{3}{2}x^2 - 4x & \text{for } x \leq -4, \ x \geq 0 \\
-\frac{1}{2}x^2 - 4x & \text{for } -4 < x < 0 
\end{cases}$$

for system (L). This system has a unique stable limit cycle. Indeed, we have $\alpha = (-9 - \sqrt{273})/4 < p(= a^*) = -4 < b = 1 < \beta = (-9 + \sqrt{273})/4$ and all conditions of the case (4) in Theorem 2.1 hold. For instance we have that $F'(x)$ is monotone decreasing for $-4 < x < 0$ and $F'(x)$ is monotone increasing for $x > 0$.

5. Appendix

We give the outline of the proof of Theorem 2 in our result in [2]. This is a special case of Theorem 1 in [2]. It is well-known from the Poincaré-Bendixson’s theorem that if System (L) satisfies the conditions that $f(0) < 0$ and $xF(x) > 0$ for $|x|$ large, then it has at least one limit cycles.

We consider the case of $|a| \leq \beta \leq |a|$ and $f(x) = F'(x) > 0$ for $|x| \geq |\beta|$. The other case can be discussed similarly. Letting $G(x) = (1/2)x^2$, there exists a negative number $-\beta \in [\alpha, 0]$ such that $G(-\beta) = G(\beta)$. Then System (L) has no limit cycles in the strip domain $\Omega = \{(x, y) \mid |x| \leq \beta, \ y \in \mathbb{R}\}$ because of $xF(x) < 0$ for $|x| < \beta$ (for instance see [1]). Thus, we know that there is a closed orbit which $C$ surrounds the origin and meets $\Omega^\varepsilon$.

We show its uniqueness. Without loss of generality we can assume that $\tilde{C}$ is outside $C$. We define Lyapunov-type functions by

$$V(x, y, t) = \begin{cases} 
V_1(x, y) = (1/2)y^2 + G(x) & \text{if } x \geq \beta, \\
V_2(x, y, t) = (1/2)y^2 + G(x) + \gamma_1t & \text{if } |x| < \beta \text{ and } y < F(x), \\
V_3(x, y) = (1/2)(y - F(a))^2 + G(x) & \text{if } x \leq -\beta, \\
V_4(x, y, t) = (1/2)y^2 + G(x) + \gamma_2t & \text{if } |x| < \beta \text{ and } y > F(x).
\end{cases}$$
We use the same notations as in [2]. Let \((x(t), y(t))\) be a periodic solution which starts from a point on the positive half of the vertical line \(x = \beta\), \(T > 0\) be its smallest period and

\[
A = y(T_2) - y(T_3) - \delta_1 \quad \text{and} \quad \tilde{A} = \tilde{y}(\bar{T}_2) - \tilde{y}(\bar{T}_3) - \delta_2
\]

for some constants \(\delta_1\) and \(\delta_2\).

We assume \(M = (T - T_3)(\bar{T}_2 - \bar{T}_1) - (\bar{T} - \bar{T}_3)(T_2 - T_1) > 0\). Then the constants \(\gamma_1\) and \(\gamma_2\) are defined by

\[
\gamma_1 = \frac{F(a)\{(\bar{T} - \bar{T}_3)A - (T - T_3)\tilde{A}\}}{M}
\]

and

\[
\gamma_2 = \frac{F(a)\{(\bar{T}_2 - \bar{T}_1)A - (T_2 - T_1)\tilde{A}\}}{M}.
\]

Since \(\tilde{y}(\bar{T}_2) - \tilde{y}(\bar{T}_3) \leq y(T_2) - y(T_3) < 0\) and \(F(a) > 0\), we can take the numbers \(\delta_1\) and \(\delta_2\) such that \(\gamma_1 > 0\), \(\gamma_2 > 0\) and \(\delta_1 \leq \delta_2\).

Then it follows from the same calculations as in [2] that \(\bar{I}_i = \int_{C_i} dV_i > \bar{I}_i = \int_{C_i} dV_i\) for \(i = 1, \ldots, 4\). Hence we have \(\bar{I} = \sum_{i=1}^{4} \bar{I}_i > \bar{I} = \sum_{i=1}^{4} I_i\).

On the other hand, we have from the choice of \(\delta_1\) and \(\delta_2\) that

\[
I = \oint \tilde{V} dV = F(a)\{y(T_2) - y(T_3)\} + \gamma_1(T_2 - T_1) - \gamma_2(T - T_3)
= F(a)(A + \delta_1) + \gamma_1(T_2 - T_1) - \gamma_2(T - T_3) = F(a)\delta_1.
\]

Similarly we have

\[
\bar{I} = F(a)(\bar{A} + \delta_2) + \gamma_1(\bar{T}_2 - \bar{T}_1) - \gamma_2(\bar{T} - \bar{T}_3) = F(a)\delta_2.
\]

Thus we have \(I \leq \bar{I}\). This contradicts \(I > \bar{I}\).

In the case \(M < 0\), by replacing with \(V_2(x, y, t) = (1/2)y^2 + G(x) - \gamma_1 t\) and \(V_4(x, y, t) = (1/2)y^2 + G(x) - \gamma_2 t\), we can take the numbers \(\delta_1\) and \(\delta_2\) satisfying \(\gamma_1 < 0\), \(\gamma_2 < 0\) and \(\delta_1 \leq \delta_2\). In the case \(M = 0\), we have by taking \(\delta_1 = \delta_2\) that \(I = \bar{I}\) for some numbers \(\gamma_1 > 0\) and \(\gamma_2 > 0\). These contradict \(I > \bar{I}\) too.

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References


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