## History

The journal Rendiconti dell'Istituto di Matematica dell'Università di Trieste was founded in 1969 by Arno Predonzan, with the aim of publishing original research articles in all fields of mathematics.

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In 2008 the Dipartimento di Matematica e Informatica, the owner of the journal, decided to renew it. The name of the journal however remained unchanged, but the subtitle An International Journal of Mathematics was added. The journal can be obtained by subscription, or by reciprocity with other similar journals. Currently more than 100 exchange agreements with mathematics departments and institutes around the world have been entered in.

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## Foreword

The first part of this volume is dedicated to our friend and colleague Giovanni Alessandrini, on the occasion of his sixtieth birthday. This section contains twenty-one invited papers from mathematicians who have collaborated with Giovanni, mainly in the field of partial differential equations and inverse problems. We thank all the authors for their contributions.

Giovanni Alessandrini was born in Florence on November 16, 1955. He obtained his university degree in Mathematics in 1978 at the University of Florence, with a thesis on Harmonic Functions, under the supervision of Giorgio Talenti. In the same university he worked as a researcher up to 1988 when he became associate professor at the University of Ancona. Since November 1990 he is full professor of Mathematical Analysis at the University of Trieste.
His research interests are focused on the qualitative theory of elliptic and parabolic equations and on inverse problems. In all these topics he has given several relevant scientific contributions. He is editor of important journals, in the field of ill-posed and inverse problems. During his career, Giovanni has had many students and has collaborated with mathematicians from several countries. All those who know Giovanni appreciate his deep mathematical culture and insight, as well as his kindness and generosity.

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Section 1

# Exponential decay of solutions to initial boundary value problem for anisotropic visco-elastic systems 

Gen Nakamura and Marcos Oliva<br>Dedicated to Prof. Giovanni Alessandrini on the occasion of his 60th birthday


#### Abstract

The paper concerns the asymptotic behaviour of solutions of initial boundary value problem for a general anisotropic viscoelastic system in the form of integrodifferential system of equations with homogeneous mixed boundary condition. We put a usual assumption on the relaxation tensor and assume that the inhomogeneous term of the equation and boundary data are zero. Then, by using the energy method, we show that the solutions decays exponentially with respect to time as it tends to infinity.


Keywords: anisotropic viscolastic system, decay of solutions.
MS Classification 2010: 73F15, 35B40, 35Q72.

## 1. Introduction

In this paper we will study the asymptotic behavior of solutions of the initial boundary value problem for general anisotropic viscoelastic integrodifferential system abbreviated by AVIS with homogeneous mixed type boundary condition. The main objective of this paper is to show the exponential decay of solutions with respect to time $t$ as $t \rightarrow \infty$ of solutions when the initial data are zero and the relaxation tensor $G$ satisfies a usual asymptotic behavior with respect to time as it tends to infinity. For this usual asymptotic behavior of $G$, see [1]. In many measurement devices such as a clinical diagnosing modality called the magnetic resonance elastography ([7]) and a rhelogical measurement device called the pendulum type viscoelastic spectroscopy ([8]) which use time harmonic vibrations it is important to have a very short transition time between time harmonic vibrations with different frequencies $\omega_{1}$ and $\omega_{2}$ when the frequency of vibration changes from $\omega_{1}$ to $\omega_{2}$. This can be ensured if the solutions decay exponential as $t \rightarrow \infty$ (see the argument in [4]).

In order to formulate the initial boundary value problem let $\Omega \subset \mathbb{R}^{n},(2 \leq$ $n \in \mathbb{N}$ ) be a bounded domain with $C^{1}$ smooth boundary $\partial \Omega$. Divide $\partial \Omega$ into $\partial \Omega=\overline{\Gamma_{D}} \cup \overline{\Gamma_{N}}$, where $\Gamma_{D}, \Gamma_{N} \subset \partial \Omega$ are open and assume that $\Gamma_{D} \neq \emptyset$, $\Gamma_{D} \cap \Gamma_{N}=\emptyset$ and if $n \geq 3$ then their boundaries $\partial \Gamma_{D}, \partial \Gamma_{N}$ are $C^{1}$ smooth if $n \geq 3$.

Consider the following initial boundary value problem

$$
\left\{\begin{array}{l}
\rho \partial_{t}^{2} u(\cdot, t)=\nabla \cdot\left\{C(\cdot) \nabla u(\cdot, t)-\int_{0}^{t} G(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right\}(t>0)  \tag{1}\\
u=0 \text { on } \Gamma_{D} \times(0, \infty), T u=0 \text { on } \Gamma_{N} \times(0, \infty) \\
u=0, \partial_{t} u=f \in L^{2}(\Omega) \text { on } \Omega \times\{0\}
\end{array}\right.
$$

where $\partial_{t}=\frac{\partial}{\partial t}, T u$ is the traction given by

$$
T u(\cdot, t)=\left(C(\cdot) \nabla u(\cdot, t)-\int_{0}^{t} G(\cdot, t) \nabla u(\cdot, \tau) d \tau\right) \nu
$$

with the unit outer normal vector $\nu$ of $\partial \Omega$. Here $0<\rho_{0} \leq \rho \in L^{\infty}(\Omega)$ with a positive constant $\rho_{0}, C=\left(C_{i j k \ell}\right)$ and $G=\left(G_{i j k \ell}\right)$ denote the elasticity tensor and relaxation tensor, respectively. Here we note that it is enough to consider the initial condition given above due to the Duhamel principle.

We assume the following assumptions on $C$ and $G$.
(i) $C \in L^{\infty}(\Omega)$ and $G=e^{-\kappa t} \hat{G}$ with $\hat{G}=\left(\hat{G}_{i j k \ell}\right) \in L^{\infty}(\Omega)$ and some constant $\kappa>0$.
(ii) (major symmetry) $C_{i j k \ell}=C_{k \ell i j}, G_{i j k \ell}=G_{k \ell i j}$ a.e. in $\Omega, 1 \leq i, j, k, \ell \leq n$.
(iii) (strong convexity) There exist constants $\alpha_{0}>0$ and $\beta_{0}>0$ such that for any $n \times n$ symmetric matrix $w=\left(w_{i j}\right)$

$$
\begin{equation*}
\alpha_{0}|w|^{2} \leq(C w): w \leq \beta_{0}|w|^{2}, \quad \alpha_{0}|w|^{2} \leq(\hat{G} w): w \leq \beta_{0}|w|^{2} \tag{2}
\end{equation*}
$$

where the notation ":" is defined as $(C w): w=\sum_{i, j, k, \ell=1}^{n} C_{i j k \ell} w_{i j} w_{k \ell}$.
(iv) There exists some constants $\mu_{0}>0, \nu_{0}>0$ such that for any $u(\cdot, t) \in$ $C^{0}\left([0, \infty) ; H^{1}(\Omega)\right)$,

$$
\begin{align*}
& \mu_{0} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
& \leq \int_{\Omega}\left\{\left(C(\cdot)-\int_{0}^{\infty} G(\cdot, \tau) d \tau\right) \nabla u(\cdot, t)\right\}: \nabla u(\cdot, t) d x  \tag{3}\\
& \quad \leq \nu_{0} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x,
\end{align*} \quad t \geq 0 .
$$

Remark 1.1. The last assumption can be given in the form

$$
\mu_{0} \int_{\Omega}|\nabla u|^{2} d x \leq \int_{\Omega}\left\{\left(C-\kappa^{-1} \hat{G}\right) \nabla u\right\}: \nabla u d x \leq \nu_{0} \int_{\Omega}|\nabla u|^{2} d x
$$

for any $u \in H^{1}(\Omega)$.

Our main result is as follows.

Theorem 1.2 (Main result). The solution $u \in C^{3}\left([0, \infty) ; H^{1}(\Omega)\right)$ to (1) which exists provided that the initial data $f$ satisfies the smoothness condition of order 3 will converge to zero exponentially fast in time $t$.

Remark 1.3. For the definition of the smoothness condition of order 3, see [2]. Also, the existence of $u \in C^{3}\left([0, \infty) ; H^{1}(\Omega)\right)$ easily follows from Theorem 2.2 therein.

There are several studies on the asymptotic behavior of solutions of AVIS as follows. Some abstract schemes for an integrodifferential equation were developed given in $[2,3]$ and applied showed that solutions of AVIS satisfying the Dirichlet boundary condition decay to zero as the time tends to infinity. Concerning the decay rate of the solutions, a polynomial order decay was shown in [6] by the energy method introducing an energy norm which is effective to analyze the asymptotic behavior of solutions. The first result on the exponential decay of solution was given in [5]. More precisely the author studied a special isotropic viscoelastic integrodifferential system with exponentially decaying relaxation function and gave the exponential decay of solutions satisfying the Dirichlet boundary condition. Our method is based on the aforementioned energy method given in [6] with careful estimates of constants in energy inequalities.

The rest of this paper is organized as follows. In Section 2 we introduce some notations and give the strategy of proof. Then we provide some basic identities and inequalities given in [6] in Section 3. Since we are concerned about the constants in these identities and inequalities we will give their proofs. Following the arguments in [6], we carefully carry out the strategy in Section 4. At last in Section 5, we will give some conclusion and discussion.

Here after for simplicity we will assume $\rho=1$.

## 2. Notations and strategy of proof

### 2.1. Notations

We will use the following notations

$$
\begin{gathered}
E(t, u):=\frac{1}{2}\left[\int_{\Omega}|u(\cdot, t)|^{2}+(G \square \partial u)(\cdot, t) d x\right. \\
\left.\quad+\int_{\Omega}\left\{\left(C(\cdot)-\int_{0}^{t} G(\cdot, \tau) d \tau\right) \nabla u(\cdot, t)\right\}: \nabla u(\cdot, t) d x\right], \\
G \square \partial u(\cdot, t):=\int_{0}^{t}\{G(\cdot, t-\tau) \nabla(u(\cdot, t)-u(\cdot, \tau))\}: \nabla(u(\cdot, t)-u(\cdot, \tau)) d \tau, \\
K(t, u):=\frac{1}{2} \int_{\Omega}|\ddot{u}|^{2} d x+\frac{1}{2} \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
-\int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x+\gamma \int_{\Omega}(C(\cdot) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
-\int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x \\
\operatorname{with} F(\cdot, t):=\gamma G(\cdot, t)+\dot{G}(\cdot, t), \\
I(t, u):=\int_{\Omega} \ddot{u}(\cdot, t) \dot{u}(\cdot, t) d x-\frac{1}{2} \int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x \\
\\
\quad-\frac{1}{2} \int_{\Omega}\left(\int_{0}^{t} \dot{G}(\cdot, \tau) d \tau \nabla u(\cdot, t)\right): \nabla u(\cdot, t) d x+\frac{1}{2} \int_{\Omega} \dot{G} \square \partial u(\cdot, t) d x, \\
\mathcal{L}(t, u):= \\
N_{1} E(t, u)+N_{2} E(t, \dot{u})+K(t, u)+(\gamma-c) I(t, u)+c_{p} \int_{\Omega} \dot{u} u d x
\end{gathered}
$$

where $N_{1}, N_{2}, \gamma, c, c_{p}$ are positive constants which will satisfy some condition given later in Subsection 4.4 and $c_{0}:=\left(\sup _{x \in \Omega} \int_{0}^{\infty}|G(x, t)| d t\right)^{1 / 2}$ and $c_{1}$ is the Poincaré constant of $u(\cdot, t)$.

### 2.2. Strategy of the proof

By basically following the argument in [6], we estimate $\frac{d}{d t} \mathcal{L}(t, u)$ from above by a negative constant times $\mathcal{L}(t, u)$ and $\mathcal{L}(t, u)$ from below by a positive constant times the sum $\int_{\Omega}|\nabla u(\cdot, t)|^{2} d x$ with some other positive terms depending on $u$.

By adjusting these constants $N_{1}, N_{2}, \gamma, c, c_{p}$, we have

$$
\begin{equation*}
\frac{d}{d t} \mathcal{L}(t, u) \leq-M_{1} \mathcal{L}(t, u), t>0 \tag{4}
\end{equation*}
$$

$$
\mathcal{L}(t, u) \geq M_{2} \int_{\Omega}\left\{|\ddot{u}|^{2}+|\dot{u}|^{2}+|\nabla \dot{u}|^{2}+|\nabla u|^{2}\right\} d x, t>0
$$

for some positive constants $M_{1}, M_{2}$.

## 3. Basic identities and inequalities

The key to derive estimate (4) is based on some basic identities and inequalities. In deriving these identities and inequalities, we show each step where we need the mixed type boundary condition and how constants of inequalities come in. Henceforth in this paper we assume that $u \in C^{3}\left([0, \infty) ; H^{1}(\Omega)\right)$ is the solution to (1) with the initial data $f$ satisfying the smoothness condition of order 3.

### 3.1. Basic identities

We first simply cite the following lemma given as Lemma 2.1. in [6].
Lemma 3.1. For any $v \in C^{1}\left([0, \infty) ; H^{1}(\Omega)\right)$ we have

$$
\begin{aligned}
& \int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) \nabla v(\cdot, \tau) d \tau\right): \nabla \dot{v}(\cdot, t) d x= \\
&- \frac{1}{2} \int_{\Omega}\left(\frac{d}{d t} G \square \partial v\right)(\cdot, t) d x+\frac{1}{2} \int_{\Omega}(\dot{G} \square \partial v)(\cdot, t) d x \\
&+\frac{1}{2} \int_{\Omega} \frac{d}{d t}\left(\int_{0}^{t} G(\cdot, \tau) d \tau \nabla v(\cdot, t)\right): \nabla v(\cdot, t) d x \\
& \quad-\int_{\Omega}(G(\cdot, t) \nabla v(\cdot, t)): \nabla v(\cdot, t) d x t>0 .
\end{aligned}
$$

Lemma 3.2.

$$
\begin{gathered}
\frac{d}{d t} E(t, u)=\int_{\Omega}(G(\cdot, t) \nabla u(\cdot, t)): \nabla u(\cdot, \tau) d x+\frac{1}{2} \int_{\Omega} \dot{G} \square \partial u(\cdot, x) d x \\
\frac{d}{d t} E(t, \dot{u})=\int_{\Omega}(G(\cdot, t) \nabla \dot{u}(\cdot, t)): \nabla u(\cdot, \tau) d x+\frac{1}{2} \int_{\Omega} \dot{G} \square \partial \dot{u}(\cdot, x) d x \\
\quad+\int_{\Omega}(G(\cdot, t) \nabla u(\cdot, 0)): \nabla \ddot{u}(\cdot, t) d x, t>0
\end{gathered}
$$

Proof. Let us multiply the viscoelastic equation in (1) by $\dot{u}(\cdot, x)$ to get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t}\left\{\int_{\Omega}|\dot{u}(\cdot, t)|^{2}+(C(\cdot) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x\right\} \\
& \quad=\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right) ; \nabla \dot{u}(\cdot, t) d x
\end{aligned}
$$

Using Lemma 3.1 our first assertion holds. To show the second identity, we take the time derivative of the viscoelastic equation in (1) so that

$$
\stackrel{(3)}{u}(\cdot, t)+\nabla\left\{-C(\cdot) \nabla \dot{u}(\cdot, t)+G(\cdot, 0) \nabla u(\cdot, t)+\int_{0}^{t} \dot{G}(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right\}=0
$$

where $\stackrel{(3)}{u}(\cdot, t)$ denotes the third order derivtive on $u(\cdot, t)$ with respect to $t$. Integrating by parts, this yields

$$
\stackrel{(3)}{u}(\cdot, t)+\nabla\left\{-C(\cdot) \nabla \dot{u}(\cdot, t)+\int_{0}^{t} G(\cdot, t-\tau) \nabla \dot{u}(\cdot, \tau) d \tau\right\}=-\nabla(G(\cdot, t) \nabla u(\cdot, 0)) .
$$

Finally multiplying this by $\ddot{u}(\cdot, t)$ and using again Lemma 3.1, we have the second identity.

Lemma 3.3.

$$
\begin{aligned}
& \frac{d}{d t}\{K(t, u)+(\gamma-c) I(t, u)\}= \\
& \quad-c \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+c \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot)): \nabla \dot{u}(\cdot, t) d x \\
& \quad-\int_{\Omega}(G(\cdot, 0) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x-(\gamma-c) \int_{\Omega}(\dot{G}(\cdot, t) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x \\
& \quad+\frac{1}{2}(\gamma-c) \int_{\Omega} \ddot{G} \square \partial u(\cdot, t) d x-\int_{\Omega}(F(\cdot, t) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& \quad+\int_{\Omega}\left\{\int_{0}^{t} \dot{F}(\cdot, t-\tau) \nabla(u(\cdot, t)-u(\cdot, \tau)) d \tau\right\}: \nabla \dot{u}(\cdot, t) d x, t>0
\end{aligned}
$$

Proof. First we sum $\gamma$ the viscoelastic equation and the time derivative of the viscoelastic equation in (1) to obtain

$$
\begin{align*}
\stackrel{(3)}{u}(\cdot, t) & +\gamma \ddot{u}(\cdot, t)+\nabla\{-C(\cdot) \nabla \dot{u}(\cdot, t)+G(\cdot, 0) \nabla u(\cdot, t)\} \\
& =-\nabla\left\{\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau+\gamma C(\cdot) \nabla u(\cdot, t)\right\} \tag{5}
\end{align*}
$$

Hence multiplying by $\ddot{u}(\cdot, t)$ and integrating in $\Omega$ we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} & \left\{\int_{\Omega}|\ddot{u}(\cdot, t)|^{2}+(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x\right\} \\
= & -\gamma \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+\int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \ddot{u}(\cdot, t) d x \\
& -\gamma \int_{\Omega}(C(\cdot) \nabla u(\cdot, t)): \nabla \ddot{u}(\cdot, t) d x  \tag{6}\\
& +\int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \ddot{u}(\cdot, t) d x .
\end{align*}
$$

Having in mind (6) and the following identities

$$
\begin{align*}
& \frac{d}{d t} \int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& =\int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \ddot{u}(\cdot, t) d x+\int_{\Omega}(G(\cdot, 0) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x  \tag{7}\\
& \frac{d}{d t} \int_{\Omega}(C(\cdot) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& =\gamma \int_{\Omega}(C(\cdot) \nabla u(\cdot, t)): \nabla \ddot{u}(\cdot, t) d x+\gamma \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x  \tag{8}\\
& \frac{d}{d t} \int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x \\
& = \\
& \quad \int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \ddot{u}(\cdot, t) d x  \tag{9}\\
& \quad+\int_{\Omega}\left(\int_{0}^{t} \dot{F}(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x \\
& \\
& \quad+\int_{\Omega}(F(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x
\end{align*}
$$

we obtain

$$
\begin{align*}
\frac{d}{d t} K(t, u)= & -\gamma \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x-\int_{\Omega}(G(\cdot, 0) \nabla \dot{u}(\cdot, t)): \nabla \ddot{u}(\cdot, t) d x \\
& +\gamma \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& -\int_{\Omega}(F(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x  \tag{10}\\
& -\int_{\Omega}\left(\int_{0}^{t} \dot{F}(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x
\end{align*}
$$

Now multiplying (5) with $\gamma=0$ by $\dot{u}(\cdot, t)$ and integrating in $\Omega$ we have

$$
\begin{align*}
\frac{d}{d t} \int_{\Omega} \ddot{u}(\cdot, t) \dot{u}(\cdot, t) d x= & \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x-\int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& +\int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x  \tag{11}\\
& +\int_{\Omega}\left(\int_{0}^{t} \dot{G}(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x
\end{align*}
$$

Further by the definition of $I(t, u),(11)$ and Lemma 3.1 we have

$$
\begin{align*}
\frac{d}{d t} I(t, u)= & \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x-\int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& -\int_{\Omega}(\dot{G}(\cdot, t) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x+\frac{1}{2} \int_{\Omega} \ddot{G} \square \partial u d x \tag{12}
\end{align*}
$$

Finally putting together (10) and (12) the proof is complete.

### 3.2. Basic inequalities

Lemma 3.4. For any $u, v \in C^{1}\left([0, \infty) ; H^{1}(\Omega)\right)$ we have

$$
\begin{gathered}
\left|\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau)(\nabla u(\cdot, t)-\nabla u(\cdot, \tau)) d \tau\right): \nabla v(\cdot, t) d x\right| \\
\quad \leq c_{0}\left(\int_{\Omega}(G \square \partial u)(\cdot, t) d x\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla v(\cdot, t)|^{2} d x\right)^{\frac{1}{2}} .
\end{gathered}
$$

Proof. Using Hölder's inequality we have

$$
\begin{aligned}
& \left|\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau)(\nabla u(\cdot, t)-\nabla u(\cdot, \tau)) d \tau\right): \nabla v(\cdot, t) d x\right| \\
& \quad=\left|\int_{\Omega}\left(\int_{0}^{t} G^{\frac{1}{2}}(\cdot, t-\tau) G^{\frac{1}{2}}(\cdot, t-\tau)(\nabla u(\cdot, t)-\nabla u(\cdot, \tau)) d \tau\right): \nabla v(\cdot, t) d x\right| \\
& \quad \leq c_{0} \int_{\Omega}\left((G \square \partial u)^{\frac{1}{2}}(\cdot, t)|\nabla v(\cdot, t)| d x\right) d x \\
& \quad \leq c_{0}\left(\int_{\Omega}(G \square \partial u)(\cdot, t) d x\right)^{\frac{1}{2}}\left(\int_{\Omega}|\nabla v(\cdot, t)|^{2} d x\right)^{\frac{1}{2}} .
\end{aligned}
$$

Lemma 3.5.

$$
\begin{aligned}
\frac{d}{d t} & \int_{\Omega} \dot{u}(\cdot, t) u(\cdot, t) d x \\
= & \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x-\int_{\Omega}\left\{\left(C(\cdot)-\int_{0}^{t} G(\cdot, \tau) d \tau\right) \nabla u(\cdot, t)\right\}: \nabla u(\cdot, t) d x \\
& -\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) \nabla(u(\cdot, t)-u(\cdot, \tau)) d \tau\right): \nabla u(\cdot, t) d x \\
\leq & c_{1} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x-\left(\mu_{0}+\alpha \kappa^{-1} e^{-\kappa t}\right) \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
\quad & +\frac{c_{0}}{2}\left(\varepsilon \int_{\Omega} G \square \partial u(\cdot, t) d x+\frac{1}{\varepsilon} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x\right), t>0,0 \leq \varepsilon \leq 1
\end{aligned}
$$

Proof. Using the viscoelastic equation in (1) and integrating by parts we obtain

$$
\begin{aligned}
\frac{d}{d t} \int_{\Omega} \dot{u}(\cdot, t) u(\cdot, t) d x= & \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x+\int_{\Omega} \ddot{u}(\cdot, t) u(\cdot, t) d x \\
= & \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x-\int_{\Omega}\{C(\cdot) \nabla u(\cdot, t)\}: \nabla u(\cdot, t) d x \\
& +\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla u(\cdot, t) d x
\end{aligned}
$$

Now if we add and subtract $\int_{\Omega}\left(\int_{0}^{t} G(\cdot, \tau) d \tau \nabla u(\cdot, t)\right): \nabla u(\cdot, t) d x$ we have the equality in Lemma 3.5, for the inequality we use the Poincaré inequality, (3), Lemma 3.4, and Young's inequality.

Lemma 3.6.

$$
\begin{aligned}
& \left|\int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x\right| \\
& \quad \leq \frac{1}{2} c_{0}(\gamma-\kappa)\left\{\varepsilon \int_{\Omega} G \square \partial u(\cdot, t) d x+\frac{1}{\varepsilon} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right. \\
& \left.\quad+c_{0}\left(\xi \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x+\frac{1}{\xi} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right)\right\}
\end{aligned}
$$

with $\varepsilon, \xi>0$.

Proof. By the definition of $F(\cdot, t)$ and $\dot{G}(\cdot, t)=-\kappa G(\cdot, t)$ we have

$$
\begin{aligned}
& \left|\int_{\Omega}\left(\int_{0}^{t} F(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x\right| \\
& \quad \leq(\gamma-\kappa)\left|\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau)(\nabla u(\cdot, t)-\nabla u(\cdot, \tau)) d \tau\right): \nabla \dot{u}(\cdot, t) d x\right| \\
& \quad+(\gamma-\kappa)\left|\int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) d \tau \nabla u(\cdot, t)\right): \nabla \dot{u}(\cdot, t) d x\right|
\end{aligned}
$$

Now using Lemma 3.4 and Young's inequality the proof is complete.

## 4. Estimates

4.1. $\frac{d}{d t} \mathcal{L}(t, u) \leq-M_{1} \mathcal{L}(t, u), t \geq 0$

We will bound from above $\frac{d}{d t} \mathcal{L}(t, u)$. From Lemma 3.3 and (1) we obtain:

$$
\begin{aligned}
& \frac{d}{d t} \\
& \quad \mathcal{L}(t, u)=-c \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+c \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& \quad-\int_{\Omega}(G(\cdot, 0) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x+(\gamma-c) \kappa \int_{\Omega}(G(\cdot, t) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& \quad+\frac{1}{2}(\gamma-c) \kappa^{2} \int_{\Omega} G \square \partial u(\cdot, t) d x-(\gamma-\kappa) \int_{\Omega}(G(\cdot, t) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& \quad-\kappa(\gamma-\kappa) \int_{\Omega}\left\{\int_{0}^{t} G(\cdot, t-\tau)(\nabla u(\cdot, t)-\nabla(\cdot, \tau)) d \tau\right\}: \nabla \dot{u}(\cdot, t) d x \\
& \quad-N_{1} \int_{\Omega}(G(\cdot, t) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x-\frac{1}{2} N_{1} \kappa \int_{\Omega} G \square \partial u(\cdot, t) d x \\
& \quad-N_{2} \int_{\Omega}(G(\cdot, t) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x-\frac{N_{2}}{2} \kappa \int_{\Omega} G \square \partial \dot{u}(\cdot, t) d x \\
& \quad+c_{p} \frac{d}{d t} \int_{\Omega} \dot{u}(\cdot, t) u(\cdot, t) d x .
\end{aligned}
$$

Using (2), (3), Lemma 3.4, Lemma 3.5 and Young's inequality we have

$$
\begin{aligned}
\frac{d}{d t} \mathcal{L}(t, u) \leq & -c \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+\frac{1}{2}\left\{\left(2(\gamma-c) \kappa \beta_{0}+(\gamma-\kappa) \beta_{0}-2 N_{1} \alpha_{0}\right.\right. \\
& \left.\left.-2 c_{p} \alpha_{0} \kappa^{-1}\right) e^{-\kappa t}+\frac{c_{p} c_{0}}{2 \eta}-c_{p} \mu_{0}\right\} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
+ & \frac{1}{2}\left\{\left((\gamma-\kappa) \beta_{0}-2 N_{2} \alpha_{0}\right) e^{-\kappa t}+2 c \beta_{0}+2 c_{p} c_{1}\right. \\
& \left.+\frac{c_{0}(\gamma-\kappa) \kappa}{\xi}-\alpha_{0}\right\} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x \\
+ & \frac{1}{2}\left\{(\gamma-c) \kappa^{2}+\kappa(\gamma-\kappa) c_{0} \xi+c_{p} c_{0} \eta\right. \\
& \left.\quad N_{1} \kappa\right\} \int_{\Omega} G \square \partial u(\cdot, t) d x-\frac{1}{2} N_{2} \kappa \int_{\Omega} G \square \partial \dot{u}(\cdot, t) d x .
\end{aligned}
$$

### 4.2. Estimating $-\mathcal{L}(t, u)$ from below

In this subsection we bound $-\mathcal{L}(t, u)$ from below.

$$
\begin{aligned}
- & \mathcal{L}(t, u)=-\frac{1}{2} \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x-\frac{1}{2} \int_{\Omega}(C(\cdot) \nabla \dot{u}(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& +\int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x-\gamma \int_{\Omega}(C(\cdot) \nabla u(\cdot, t)): \nabla \dot{u}(\cdot, t) d x \\
& +(\gamma-\kappa) \int_{\Omega}\left(\int_{0}^{t} G(\cdot, t-\tau) \nabla u(\cdot, \tau) d \tau\right): \nabla \dot{u}(\cdot, t) d x \\
& -(\gamma-c) \int_{\Omega} \ddot{u}(\cdot, t) \dot{u}(\cdot, t) d x+\frac{1}{2}(\gamma-c) \int_{\Omega}(G(\cdot, 0) \nabla u(\cdot, t)): \nabla u(\cdot, t) d x \\
& +\frac{\gamma-c}{2} \int_{\Omega}\left\{\left(\int_{0}^{t} \dot{G}(\cdot, t-\tau) d \tau\right) \nabla u(\cdot, t)\right\}: \nabla u(\cdot, t) d x \\
& -\frac{1}{2}(\gamma-c) \int_{\Omega} \dot{G} \square \partial u(\cdot, t) d x-\frac{1}{2} N_{1} \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x-\frac{1}{2} N_{1} \int_{\Omega} G \square \partial u(\cdot, t) d x \\
& -\frac{1}{2} N_{1} \int_{\Omega}\left\{\left(C(\cdot)-\int_{0}^{t} G(\cdot, \tau) d \tau\right) \nabla u(\cdot, t)\right\}: \nabla u(\cdot, t) d x \\
& -\frac{1}{2} N_{2} \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x-\frac{1}{2} N_{2} \int_{\Omega} G \square \partial \dot{u} d x \\
& -N_{2} \int_{\Omega}\left\{\left(C(\cdot)-\int_{0}^{t} G(\cdot, \tau) d \tau\right) \nabla \dot{u}(\cdot, t)\right\}: \nabla \dot{u}(\cdot, t) d x \\
& -c_{p} \int_{\Omega} \dot{u}(\cdot, t) u(\cdot, t) d x .
\end{aligned}
$$

Using (2), (3), Lemma 3.6, $G=e^{-\kappa t} \hat{G}$ and Young's inequality, we have:

$$
\begin{aligned}
&-\mathcal{L}(t, u) \geq- \frac{1}{2}\left\{1+N_{2}+\gamma-c\right\} \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x \\
&+ \frac{1}{2}\left\{\left(\alpha_{0}(\gamma-\kappa)-N_{1} \beta_{0} \kappa^{-1}\right) e^{-\kappa t}+N_{1} \alpha_{0} \kappa^{-1}-N_{1} \beta_{0}-c_{0}^{2}(\gamma-\kappa)\right. \\
&\left.\quad-c_{p}-(\gamma+1) \beta_{0}\right\} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
&+ \frac{1}{2}\left\{-2 N_{2} \kappa^{-1} \beta_{0} e^{-\kappa t}+2 N_{2} \kappa^{-1} \alpha_{0}-2 N_{2} \beta_{0}-\beta_{0}-c_{1}(\gamma-c)\right. \\
&\left.\quad-c_{1} N_{1}-c_{0}(\gamma-\kappa)\left(c_{0}+1\right)-c_{p}-(\gamma+1) \beta_{0}\right\} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x \\
&+\frac{1}{2}\left\{(\gamma-c) \kappa-N_{1}-c_{0}(\gamma-\kappa)\right\} \int_{\Omega} G \square \partial u(\cdot, t) d x \\
&- \frac{1}{2} N_{2} \int_{\Omega} G \square \partial \dot{u}(\cdot, t) d x .
\end{aligned}
$$

It is easy to see that if we take the constants like in Subsection 4.4 given later, all the coefficients in both bounds are less than a negative constant and are bounded. Hence there is a constant $M \geq 0$ such that $\frac{d}{d t} \mathcal{L}(t, u) \leq-M \mathcal{L}(t, u)$ for any $t \geq 0$.

### 4.3. Estimating $\mathcal{L}(t, u)$ from below

In this section we will bound $\mathcal{L}(t, u)$ from below. For this we use (3) to get the next two inequalities:
$N_{1} E(t, u) \geq \frac{1}{2} N_{1} \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x+\frac{1}{2} N_{1} \int_{\Omega} G \square \partial u(\cdot, t) d x+N_{1} \mu_{0} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x$,
$N_{2} E(t, \dot{u}) \geq \frac{1}{2} N_{2} \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+\frac{1}{2} N_{2} \int_{\Omega} G \square \partial \dot{u}(\cdot, t) d x+N_{2} \mu_{0} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x$.
We apply (4), Lemma 3.6 with $\varepsilon=\xi=1, G=e^{-\kappa t} \hat{G}$ and Young inequality with $p=q=2$ to obtain

$$
\begin{aligned}
K(t, u)+(\gamma-c) I(t, u) \geq \frac{1}{2} & {\left[\int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x+\alpha_{0} \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right.} \\
& -\beta_{0}\left(\int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right) \\
& +\gamma \alpha_{0}\left(\int_{\Omega}|\nabla u(\cdot, t)|^{2} d x+\int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right) \\
& -c_{0}(\gamma-\kappa)\left\{\int_{\Omega} G \square \partial u(\cdot, t) d x+\int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right. \\
& \left.+c_{0}\left(\int_{\Omega}|\nabla u(\cdot, t)|^{2} d x+\int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x\right)\right\} \\
& -(\gamma-c)\left\{\int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x+\int_{\Omega}|\ddot{u}(\cdot, t)|^{2}\right\} \\
& -\beta_{0}(\gamma-c) \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
& \left.-\kappa(\gamma-c) \int_{\Omega} G \square \partial u(\cdot, t) d x\right] .
\end{aligned}
$$

By using Young inequality with $p=q=2$ and Poincaré inequality, we have

$$
c_{p} \int_{\Omega} \dot{u}(\cdot, t) u(\cdot, t) d x \geq-\frac{1}{2}\left[\int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x+c_{1} \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x\right] .
$$

Putting together the last four inequalities and the definition of $\mathcal{L}(t, u)$ we obtain:

$$
\begin{aligned}
\mathcal{L}(t, u) & \geq \frac{1}{2}\left[1+N_{2}-(\gamma-c)\right] \int_{\Omega}|\ddot{u}(\cdot, t)|^{2} d x \\
& +\frac{1}{2}\left[2 N_{1} \mu_{0}+\gamma \alpha_{0}-\beta_{0}-c_{0}^{2}(\gamma-\kappa)-\beta_{0}(\gamma-c)-c_{p} c_{1}\right] \int_{\Omega}|\nabla u(\cdot, t)|^{2} d x \\
& +\frac{1}{2}\left[2 N_{2} \mu_{0}+\gamma \alpha_{0}+\alpha_{0}-\beta_{0}-c_{0}^{2}(\gamma-\kappa)-c_{0}(\gamma-\kappa)\right] \int_{\Omega}|\nabla \dot{u}(\cdot, t)|^{2} d x \\
& +\frac{1}{2}\left[N_{1}-c_{0}(\gamma-\kappa)-\kappa(\gamma-c)\right] \int_{\Omega} G \square \partial u(\cdot, t) d x \\
& +\frac{1}{2} N_{2} \int_{\Omega} G \square \partial \dot{u}(\cdot, t) d x+\frac{1}{2}\left[N_{1}-(\gamma-c)-c_{p}\right] \int_{\Omega}|\dot{u}(\cdot, t)|^{2} d x .
\end{aligned}
$$

Now if we take the constants like in Subsection 4.4 we see that exists $M_{2} \geq 0$ such that

$$
\mathcal{L}(t, u) \geq M_{2} \int_{\Omega}\left\{|\ddot{u}(\cdot, t)|^{2}+|\dot{u}(\cdot, t)|^{2}+|\nabla \dot{u}(\cdot, t)|^{2}+|\nabla u(\cdot, t)|^{2}\right\} d x, t>0
$$

### 4.4. Conditions for constants

We summarize the conditions for constants $N_{1}, N_{2}, \gamma, c, c_{p}$ as follows.

$$
\begin{aligned}
& \begin{aligned}
N_{1}>\max \{ & 2\left((\gamma-c) \kappa-c_{0}(\gamma-\kappa)\right), 2\left((\gamma-c) \kappa+c_{0}(\gamma-\kappa) \xi+c_{0} c_{p} \kappa^{-1} \eta\right), \\
& \left(\beta_{0}-\kappa^{-1} \alpha_{0}\right)^{-1}\left(\left(\alpha_{0}-c_{0}^{2}\right)(\gamma-c)-c_{0}^{2}(\gamma-\kappa)-c_{p} c_{1}-(\gamma+1) \beta_{0}\right), \\
& \frac{1}{2 \mu_{0}}\left(\beta_{0}-\gamma \alpha_{0}+c_{0}^{2}(\gamma-\kappa)+\beta_{0}(\gamma-c)+c_{p} c_{1}\right) \\
& \left.c_{0}(\gamma-\kappa)+\kappa(\gamma-c),(\gamma-c)+c_{p}\right\},
\end{aligned} \\
& N_{2}>\max \left\{(\gamma-c)-1, \frac{1}{2 \mu_{0}}\left(-\alpha_{0}+\beta_{0}-\gamma \alpha_{0}+c_{0}\left(c_{0}+1\right)(\gamma-\kappa)\right)\right\}, \\
& \xi>6 c_{0}(\gamma-\kappa) \kappa \alpha_{0}^{-1}, c<\frac{\alpha_{0}}{6 \beta_{0}}, c_{p}<\frac{\alpha_{0}}{6 c_{1}}, \eta>\frac{c_{0}}{N_{0}} \\
& \text { with } c_{0}:=\left(\sup _{x \in \Omega} \int_{0}^{\infty}|G(x, t)| d t\right)^{1 / 2} \text { and the Poincaré constant } c_{1} \text { provided } \\
& \text { that } \beta_{0}>\kappa^{-1} \alpha_{0} .
\end{aligned}
$$

It is easy to see that there are constants $N_{1}, N_{2}, \gamma, c, c_{p}$ which satisfy these conditions. Hence, from Subsections 4.1 and 4.3 , we have obtained the exponential decay of solution to (1).

## 5. Conclusion and discussion

## Conclusion

We have shown an exponential decay of solutions of (1) by applying the energy method given in [6] and carefully concerning the constants which appear in this method.

## Discussion

We will give the following list for some discussion on our result.

1. The case with density can be handled in a similar way.
2. The case with general inhomogeneous data can be handled using the Duhamel principle.
3. How the exponential decay rate depends on the assumptions and coefficients is not clear.
4. As a future work, we would like to extend this work to a more general relaxation tensor.

## 6. Acknowledgements

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# A global Riemann-Hilbert problem for two-dimensional inverse scattering at fixed energy 

Evgeny L. Lakshtanov, Roman G. Novikov and Boris R. Vainberg<br>Dedicated to Giovanni Alessandrini on the occasion of his $60^{\text {th }}$ birthday


#### Abstract

We develop the Riemann-Hilbert problem approach to inverse scattering for the two-dimensional Schrödinger equation at fixed energy. We obtain global or generic versions of the key results of this approach for the case of positive energy and compactly supported potentials. In particular, we do not assume that the potential is small or that Faddeev scattering solutions do not have singularities (i.e. we allow the Faddeev exceptional points to exist). Applications of these results to the Novikov-Veselov equation are also considered.


Keywords: two-dimensional inverse scattering, Faddeev functions, generalized Riemann-Hilbert-Manakov problem, Novikov-Veselov equation.
MS Classification 2010: 35J10, 35E25, 35P25, 35Q15, 35R30, 35Q53.

## 1. Introduction

We consider the two-dimensional Schrödinger equation

$$
\begin{equation*}
(-\Delta+v) \psi(x)=E \psi(x), x \in \mathbb{R}^{2}, \quad E>0 \tag{1}
\end{equation*}
$$

where
$v$ is a real-valued sufficiently regular function on $\mathbb{R}^{2}$
with sufficient decay at infinity.

Actually, in the present work the assumptions (2) are specified in the sense that $v$ is a real-valued, bounded, compactly supported function on $\mathbb{R}^{2}$.

For equation (1) we consider the classical scattering solutions $\psi^{+}(x, k), k \in$ $\mathbb{R}^{2}, k^{2}=E$, specified by the following asymptotics

$$
\begin{equation*}
\psi^{+}(x, k)=e^{i k x}+i \pi \sqrt{2 \pi} e^{-i \pi / 4} \frac{e^{i|k||x|}}{\sqrt{|k||x|}} f\left(k,|k| \frac{x}{|x|}\right)+o\left(\frac{1}{\sqrt{|x|}}\right) \tag{3}
\end{equation*}
$$

$$
|x| \rightarrow \infty
$$

for some a priori unknown $f$. Function $f=f(k, l)$ on

$$
\begin{equation*}
\mathcal{M}_{E}=\left\{k, l \in \mathbb{R}^{2}: k^{2}=l^{2}=E\right\} \tag{4}
\end{equation*}
$$

arising in (3) is the classical scattering amplitude for equation (1).
In order to determine $\psi^{+}$and $f$ from $v$ one can use the Lipmann-Schwinger integral equation (11) and the integral formula (12) in Section 2; see e.g. [19].

In this work we continue, in particular, studies on the following inverse scattering problem for equation (1) under assumptions (2):
Problem 1.1: Given scattering amplitude $f$ on $\mathcal{M}_{E}$ at fixed $E>0$, find the potential $v$ on $\mathbb{R}^{2}$.

When $v$ is compactly supported, that is

$$
\begin{equation*}
\operatorname{supp} v \subset D \tag{5}
\end{equation*}
$$

where $D$ is an open bounded domain in $\mathbb{R}^{2}$, we consider also the Dirichlet-toNeumann map $\Phi(E)$ for equation (1) in $D$. We recall that this map is defined via the relation

$$
\begin{equation*}
\left.\frac{\partial}{\partial \nu} \psi\right|_{\partial D}=\Phi(E)\left(\left.\psi\right|_{\partial D}\right) \tag{6}
\end{equation*}
$$

fulfilled for all sufficiently regular solutions $\psi$ of (1) in $D \cup \partial D$, where $\nu$ is the external normal vector to $\partial D$. Considering $\Phi(E)$, we assume also that

$$
\begin{equation*}
E \text { is not a Dirichlet eigenvalue for the operator }-\Delta+v \text { in } D . \tag{7}
\end{equation*}
$$

It is well known (see [18]) that, under assumptions (2), (5), Problem 1.1 is closely related with the following inverse boundary value problem for equation (1) in $D$ :
Problem 1.2: Given $\Phi(E)$ at fixed $E>0$, find $v$.
Problems 1.1, 1.2 have a long history and there are many important results on these problems; see $[6,11,19,21,23]$ and references therein in connection with Problem 1.1 and $[5,18,23,24]$ in connection with Problem 1.2.

The approach of the present work to Problems 1.1, 1.2 is based, in particular, on properties of the Faddeev exponentially increasing solutions for equation (1). We recall that the Faddeev solutions $\psi(x, k), k \in \mathbb{C}^{2} \backslash \mathbb{R}^{2}, k^{2}=E$, of equation (1) are specified by

$$
\begin{equation*}
\psi(x, k)=e^{i k x}(1+o(1)),|x| \rightarrow \infty \tag{8}
\end{equation*}
$$

see e.g. [18].
In order to determine $\psi$ from $v$ one can use the Lipmann-Schwinger-Faddeev integral equation (20) in Section 2.

In the present work, under assumptions (2), (5), we reduce Problems 1, 2 to some global generalized Riemann-Hilbert-Manakov problem for the classical scattering solutions $\psi^{+}$and the Faddeev solutions $\psi$ for equation (1); see Problem 3.5 in Section 3. A prototype of this global Riemann-Hilbert-Manakov problem for the case of equation (1) with $E<0$ was considered in Section 8 of [19].

The term "global" means, in particular, that the kernels of our Riemann-Hilbert-Manakov problem have no singularities, even if there are the Faddeev exceptional points at fixed $E$. After that we reduce our Riemann-Hilbert problem to a Fredholm linear integral equation of the second type; see Theorem 4.1 and Proposition 4.2 in Section 4.

As a result we obtain, in particular, a new generic reconstruction method for Problems 1, 2; see Proposition 4.5 and Remarks 4.6, 4.7 in Section 4.

In particular, our reconstruction from the Faddeev generalized scattering data is reduced to formulas (58), (60), (64), (65), (70), (83), (84), integral equations (67)-(69), (85), (86) and formulas (61), (75), (76), (79).

Note that the approach of the present work goes back to the soliton theory, see $[1,9,13,14,16]$. The first applications of this approach to Problems 1, 2 were given in $[12,17,18,19]$. Actually, the main result of the present work consists in a globalization of this approach to Problems 1, 2.

The reconstruction method of the present work uses properly generalized scattering data for small and large values of the complex spectral parameter at fixed energy and, therefore, is considerably more stable, generically, than the reconstruction method of [5] based exclusively on properties of some generalized scattering data for large values of complex spectral parameter. Generically, stability estimates of [22] obtained using ideas of [2], [5] can be improved using results of the present work to estimates like in [24], but without the assumptions that some norm of potential $v$ is sufficiently small in comparison with fixed $E$. This issue will be presented in detail elsewhere.

In addition, in contrasts with [5], results of the present work admit application to solving the Cauchy problem for the Novikov-Veselov equation ([15, 26])

$$
\begin{gather*}
\partial_{t} v=4 \Re\left(4 \partial_{z}^{3} v+\partial_{z}(v w)-E \partial_{z} w\right) \\
\partial_{\bar{z}} w=-3 \partial_{z} v, \quad v=\bar{v}, \quad E>0  \tag{9}\\
v=v(x, t), \quad w=w(x, t), \quad x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}, \quad t \in \mathbb{R}
\end{gather*}
$$

with compactly supported $v(x, t=0)$. Here, we used the following notations:

$$
\begin{equation*}
\partial_{t}=\frac{\partial}{\partial t}, \quad \partial_{z}=\frac{1}{2}\left(\frac{\partial}{\partial x_{1}}-i \frac{\partial}{\partial x_{2}}\right), \quad \partial_{\bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{1}}+i \frac{\partial}{\partial x_{2}}\right) . \tag{10}
\end{equation*}
$$

These applications are indicated in Section 6 of the present work and will be presented in detail elsewhere.

## 2. Preliminary results of direct scattering

### 2.1. Classical scattering functions

We recall that for the classical scattering functions $\psi^{+}$and $f$ for equation (1) the following Lipmann-Schwinger integral equation (11) and the integral formula (12) hold:

$$
\begin{gather*}
\psi^{+}(x, k)=e^{i k x}+\int_{y \in \mathbb{R}^{2}} G^{+}(x-y, \sqrt{E}) v(y) \psi^{+}(y, k) d y  \tag{11}\\
G^{+}(x, \sqrt{E})=-\frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} \frac{e^{i \xi x} d \xi}{|\xi|^{2}-E-i 0}=-\frac{i}{4} H_{0}^{1}(|x| \sqrt{E}) \\
f(k, l)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} e^{-i l y} v(y) \psi^{+}(y, k) d y \tag{12}
\end{gather*}
$$

where $x, k, l \in \mathbb{R}^{2}, k^{2}=l^{2}=E>0, H_{0}^{1}$ is the Hankel function of the first type; see e.g. [19]. In addition, it is known that equation (11) is uniquely solvable with respect to $\psi^{+}(\cdot, k) \in L^{\infty}\left(\mathbb{R}^{2}\right)$ at fixed $k$, under conditions (2) and, in particular, under the conditions that

$$
\begin{equation*}
v=\bar{v} \in L^{\infty}\left(\mathbb{R}^{2}\right), \quad \operatorname{supp} v \subset D \tag{13}
\end{equation*}
$$

where $D$ is an open bounded domain in $\mathbb{R}^{2}$; see e.g. [4] for a proof of a similar result in three dimensions.

Let

$$
\begin{gather*}
\mathbb{S}_{r}^{1}=\left\{\zeta \in \mathbb{R}^{2}: \zeta^{2}=r^{2}\right\}, r>0,  \tag{14}\\
\Sigma_{E}=\left\{\zeta \in \mathbb{C}^{2}: \zeta^{2}=E\right\}, E>0,  \tag{15}\\
\Sigma_{E, \rho}=\left\{\zeta \in \Sigma_{E}:|\Im \zeta| \geq \rho\right\}, E>0, \rho>0 \tag{16}
\end{gather*}
$$

and let

$$
\begin{equation*}
\chi_{E, \rho} \text { be the characteristic function of } \Sigma_{E, \rho} \text { in } \Sigma_{E} . \tag{17}
\end{equation*}
$$

Note that $\mathcal{M}_{E}=\mathbb{S}_{\sqrt{E}}^{1} \times \mathbb{S}_{\sqrt{E}}^{1}$, where $\mathcal{M}_{E}$ is defined by (4).
It is well known that, under conditions (2), (5),
$\psi^{+}(x, k)$ admits a holomorphic extension in $k$
$\quad$ from $\mathbb{S}_{\sqrt{E}}^{1}$ to $\Sigma_{E}$ at fixed $x$
and

$$
\begin{align*}
& f(k, l) \text { admits a holomorphic extension in }(k, l) \\
& \quad \text { from } \mathcal{M}_{E} \text { to } \Sigma_{E} \times \Sigma_{E} \tag{19}
\end{align*}
$$

with possible exponential increasing at infinity in complex domain.
As a corollary, $f$ on $\mathcal{M}_{E}$ uniquely determines $f$ on $\Sigma_{E} \times \Sigma_{E}$, under assumptions (2), (5).

### 2.2. Faddeev functions

We recall also that the Faddeev solutions $\psi(x, k)$ for (1) satisfy the following generalized Lipmann-Schwinger integral equation

$$
\begin{array}{r}
\psi(x, k)=e^{i k x}+\int_{y \in \mathbb{R}^{2}} G(x-y, k) v(y) \psi(y, k) d y \\
G(x, k)=g(x, k) e^{i k x} \\
g(x, k)=-\frac{1}{(2 \pi)^{2}} \int_{\xi \in \mathbb{R}^{2}} \frac{e^{i \xi x}}{|\xi|^{2}+2 k \xi} d \xi \tag{22}
\end{array}
$$

where $x \in \mathbb{R}^{2}, k \in \mathbb{C}^{2} \backslash \mathbb{R}^{2}, k^{2}=E>0$; see e.g. [7, 19]. In addition, we consider (20) as an equation for $\psi=e^{i k x} \mu(x, k)$, where $\mu(\cdot, k) \in L^{\infty}\left(\mathbb{R}^{2}\right)$ at fixed $k$. Note that equation (20) can be rewritten as

$$
\begin{equation*}
\mu(x, k)=1+\int_{y \in \mathbb{R}^{2}} g(x-y, k) v(y) \mu(y, k) d y \tag{23}
\end{equation*}
$$

where $x \in \mathbb{R}^{2}, k \in \mathbb{C}^{2} \backslash \mathbb{R}^{2}, k^{2}=E>0$; see e.g. [19].
Under assumptions (2) and, in particular, under assumptions (13), equations (20), (23) are uniquely solvable for $\mu(\cdot, k) \in L^{\infty}\left(\mathbb{R}^{2}\right)$ at fixed $k$ if $k \in$ $\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \backslash \mathcal{E}_{E}$, where $\mathcal{E}_{E}$ is the set of the Faddeev exceptional points on $\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}$; see e.g. [19].

Note also that, due to estimates (3.16)-(3.18) of [19], the following estimates hold for some constant $c_{0}>0$ :

$$
\begin{array}{r}
\left|G^{+}(x, \sqrt{E})\right| \leq c_{0}|x|^{-1 / 2} E^{-1 / 4} \\
|g(x, k)| \leq c_{0}|x|^{-1 / 2}|\Re k|^{-1 / 2} \tag{25}
\end{array}
$$

where $G^{+}, g$ are defined in (11), (21), $x \in \mathbb{R}^{2}, k \in \mathbb{C}^{2} \backslash \mathbb{R}^{2}, k^{2}=E>0$.
In addition, under assumptions (13), as a corollary of (24), (25), in a similar way to Proposition 4.1 in [19], we have that

$$
\begin{gather*}
\|A(k)\|_{L^{\infty}\left(\mathbb{R}^{2}\right) \rightarrow L^{\infty}\left(\mathbb{R}^{2}\right)} \leq M\left(\|v\|_{L^{\infty}(D)}, D, E, \rho\right),  \tag{26}\\
k \in \mathbb{C}^{2}, k^{2}=E>0,|\Im k|=\rho>0, \\
\Sigma_{E, \rho} \cap \mathcal{E}_{E}=\emptyset \text { if } \rho>\rho_{1}\left(\|v\|_{L^{\infty}(D)}, D, E\right), E>0, \tag{27}
\end{gather*}
$$

where $A(k)$ is the linear integral operator of equation (23),

$$
\begin{equation*}
M(q, D, E, \rho)=\frac{c_{0} q I_{1}(D)}{\left(E+\rho^{2}\right)^{1 / 4}} \tag{28}
\end{equation*}
$$

$$
\begin{array}{r}
\rho_{1}(q, D, E)=\left[\max \left(\left[c_{0} q I_{1}(D)\right]^{4}-E, 0\right)\right]^{1 / 2}, \\
q \geq 0, \quad I_{1}(D)=\max _{x \in \mathbb{R}^{2}} \int_{D} \frac{d y}{|x-y|^{1 / 2}} . \tag{29}
\end{array}
$$

In addition to $\psi$, we consider also the generalized Faddeev scattering amplitude $h(k, l)$ defined by the formula

$$
\begin{equation*}
h(k, l)=\frac{1}{(2 \pi)^{2}} \int_{\mathbb{R}^{2}} e^{-i l y} v(y) \psi(y, k) d y \tag{30}
\end{equation*}
$$

where $(k, l) \in\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \times \Sigma_{E}$; see e.g. [8, 19]. Here we assume also that $\Im k=\Im l$ if (5) is not assumed.

Note that, under assumption (13),
$h$ is (complex-valued) real-analytic on $\left(\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \backslash \mathcal{E}_{E}\right) \times \Sigma_{E}$, $h(k, \cdot)$ is holomorphic on $\Sigma_{E}$ at fixed $k$.

We say that a complex-valued function is real-analytic if its real and imaginary parts are real-analytic.

## 2.3. $\bar{\partial}$-equation on the Faddeev eigenfunctions

We recall that the following isomorphic relations are valid:

$$
\begin{equation*}
\Sigma_{E} \approx \mathbb{C} \backslash 0, \quad \mathbb{S}_{\sqrt{E}}^{1} \approx T=\{\lambda \in \mathbb{C}:|\lambda|=1\} \tag{32}
\end{equation*}
$$

More precisely:

$$
\begin{align*}
& k=\left(k_{1}, k_{2}\right) \in \Sigma_{E} \Rightarrow \lambda=\lambda(k):=\frac{k_{1}+i k_{2}}{\sqrt{E}} \in \mathbb{C} \backslash 0,  \tag{33}\\
& k=\left(k_{1}, k_{2}\right) \in \mathbb{S}_{\sqrt{E}}^{1} \Rightarrow \lambda(k) \in \mathbb{T} ; \\
& \lambda \in \mathbb{C} \backslash 0 \Rightarrow k=k(\lambda, E) \in \Sigma_{E}, \quad \lambda \in T \Rightarrow k=k(\lambda) \in \mathbb{S}_{\sqrt{E}}^{1}, \tag{34}
\end{align*}
$$

where

$$
\begin{align*}
& k(\lambda, E)=\left(k_{1}(\lambda, E), k_{2}(\lambda, E)\right), \\
& k_{1}=\left(\lambda+\frac{1}{\lambda}\right) \frac{\sqrt{E}}{2}, \quad k_{2}=\left(\frac{1}{\lambda}-\lambda\right) \frac{i \sqrt{E}}{2} . \tag{35}
\end{align*}
$$

Note also that

$$
\begin{array}{r}
|\Re k(\lambda, E)|=\frac{\sqrt{E}}{2}\left(|\lambda|+|\lambda|^{-1}\right),|\Im k(\lambda, E)|=\frac{\sqrt{E}}{2}| | \lambda\left|-|\lambda|^{-1}\right|,  \tag{36}\\
\lambda \in \mathbb{C} \backslash 0, E>0 .
\end{array}
$$

Let

$$
\begin{align*}
& L_{p, \nu}(\mathbb{C}) \text { be the function space on } \mathbb{C} \text { consisting } \\
& \text { of the functions } u \text { such that } u, u_{\nu} \in L_{p}\left(\mathcal{D}_{1}\right)  \tag{37}\\
& \text { with the norm }\|u\|_{L_{p, \nu}}=\|u\|_{L_{p}\left(\mathcal{D}_{1}\right)}+\left\|u_{\nu}\right\|_{L_{p}\left(\mathcal{D}_{1}\right)},
\end{align*}
$$

where $p \geq 1, \nu \geq 0$,

$$
\begin{gather*}
u_{\nu}(\lambda):=|\lambda|^{-\nu} u\left(\lambda^{-1}\right),  \tag{38}\\
\mathcal{D}_{1}=\{\lambda \in \mathbb{C}:|\lambda| \leq 1\} . \tag{39}
\end{gather*}
$$

It is known that the function $\psi$ of Subsection 2.2 has, in particular, the following properties, under assumptions (2) and, in particular, under assumptions (13):

$$
\begin{gather*}
\psi(x, k(\lambda))=e^{i k(\lambda) x}(1+o(1)), \text { if } \lambda \rightarrow 0 \text { or } \lambda \rightarrow \infty  \tag{40}\\
\frac{\partial}{\partial \bar{\lambda}} \psi(x, k(\lambda))=\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{\bar{\lambda}} b(k(\lambda)) \psi\left(x, k\left(-\frac{1}{\bar{\lambda}}\right)\right) \\
\psi\left(x, k\left(-\frac{1}{\bar{\lambda}}\right)\right)=\overline{\psi(x,(k(\lambda)))}, \quad k(\lambda) \in\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \backslash \mathcal{E}_{E} \tag{41}
\end{gather*}
$$

where $x \in \mathbb{R}^{2}, k(\lambda)=k(\lambda, E)$ is defined by (35),

$$
\begin{equation*}
b(k):=h(k,-\bar{k}), \tag{42}
\end{equation*}
$$

where $h$ is defined by (30); see e.g. [12, 19].
Note that $\bar{\partial}$-equations like (41) go back to $[1,3]$.

### 2.4. Some estimates related with $\bar{\partial}$-equation (41)

In particular, as a corollary of (40),

$$
\begin{array}{ll}
\psi(x,(k(\lambda))) \neq 0 & \text { if }|\lambda| \text { is sufficiently small } \\
& \text { or if }|\lambda| \text { is sufficiently large. } \tag{43}
\end{array}
$$

In addition, under assumptions (13), as a corollary of (26), we have

$$
\begin{equation*}
|\mu(x, k(\lambda))| \leq(1-M(q, D, E, \rho))^{-1} \tag{44}
\end{equation*}
$$

$$
x \in \mathbb{R}^{2}, k(\lambda)=k(\lambda, E) \in \Sigma_{E, \rho}, \rho>\rho_{1}(q, D, E),\|v\|_{L^{\infty}(D)}<q
$$

where $M$ is defined by (28), $\rho_{1}$ is defined by (29).
In connection with equation (41) we consider also

$$
\begin{equation*}
u_{E, \rho}(\lambda)=\frac{1}{\bar{\lambda}} \chi_{E, \rho}(k(\lambda)) b(k(\lambda)) \tag{45}
\end{equation*}
$$

where $\chi_{E, \rho}$ is defined by (17).
Under assumptions (13), we have:

$$
\begin{equation*}
u_{E, \rho} \in L_{p, 2}(\mathbb{C}), \quad 2<p<4, \tag{46}
\end{equation*}
$$

where $\rho>\rho_{1}\left(\|v\|_{L^{\infty}(D)}, D, E\right)$;

$$
\begin{gather*}
\left\|u_{E, \rho}\right\|_{L_{p, 2}} \leq q c_{1}(D, p, E)(1-M(q, D, E, \rho))^{-1}  \tag{47}\\
\left\|u_{E, \rho}\right\|_{L_{p, 2}}=O(q) \text { as } q \rightarrow 0 \tag{48}
\end{gather*}
$$

for fixed $E>0, \rho>\rho_{1}(q, D, E), D$ and $p$, where $\|v\|_{L^{\infty}(D)} \leq q, M$ is defined by (28), $c_{1}$ is a positive constant, $2<p<4$;

$$
\begin{equation*}
\left|\bar{\lambda} u_{E, \rho}(\lambda)\right| \leq q(2 \pi)^{-2}(1-M(q, D, E, \rho))^{-1} \int_{D} d x, \lambda \in \mathbb{C} \tag{49}
\end{equation*}
$$

where $\|v\|_{L^{\infty}(D)} \leq q, M$ is defined by $(28), \rho>\rho_{1}\left(\|v\|_{L^{\infty}(D)}, D, E\right)$; see formulas (4.4), (4.12), (4.18), (4.19) of [19]. In connection with (46)-(48) we recall that $L_{p, 2}(\mathbb{C})$ is defined in (37).

### 2.5. Final remarks

We recall also that, under the assumptions (7), (13), at fixed $E$, the scattering amplitude $f$ uniquely determines the Dirichlet-to-Neumann map $\Phi$ and vice versa; see Proposition 4 in [18].

In turn, $\Phi(E)$ uniquely determines $h$ on $\left(\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \backslash \mathcal{E}_{E}\right) \times \Sigma_{E}$; see [18].
Note also that $f$ at fixed $E$ uniquely determines $h$ on $\left(\left(\Sigma_{E} \backslash \mathbb{S}_{\sqrt{E}}^{1}\right) \backslash \mathcal{E}_{E}\right) \times$ $\Sigma_{E}$ via a two-dimensional analogue of the construction given in [20].

As a corollary, Problems 1.1, 1.2 of Section 1 are reduced to Problem 3.4 of Section 3.

## 3. Global generalized Riemann-Hilbert problem

Let
$\Lambda=\Lambda_{E, \rho}=\left\{\lambda \in \mathbb{C}: \frac{\sqrt{E}}{2}| | \lambda\left|-|\lambda|^{-1}\right|<\rho\right\}, E>0, \rho>0$,
and $\partial \Lambda=\partial \Lambda_{E, \rho}$ be the boundary of $\Lambda$ in $\mathbb{C}$ with the standard orientation.

Note that

$$
\begin{equation*}
\Sigma_{E, \rho} \approx \mathbb{C} \backslash \Lambda_{E, \rho}, \tag{51}
\end{equation*}
$$

where this isomorphism is given by formulas (33), (34).

Let

$$
\begin{align*}
& W(\lambda, \varsigma)= \frac{i}{2} \operatorname{sgn}\left(|\lambda|^{2}-1\right)\left[\frac{1}{\varsigma} \ln w_{1}(\lambda, \varsigma)+\varsigma \ln w_{2}(\lambda, \varsigma)\right] \\
&+\int_{|\eta|=1} \frac{1}{2(\varsigma-\eta)} \theta\left[\operatorname{sgn}\left(|\lambda|^{2}-1\right) i\left(\frac{|\lambda| \eta}{\lambda}-\frac{\lambda}{|\lambda| \eta}\right)\right]|d \eta|  \tag{52}\\
& \lambda, \varsigma \in \partial \Lambda
\end{align*}
$$

where

$$
w_{1}=\frac{\varsigma-\lambda}{\varsigma-\frac{\lambda}{|\lambda|}}, \quad w_{2}=\frac{\frac{-1}{\varsigma}-\bar{\lambda}}{\frac{-1}{\varsigma}-\frac{\bar{\lambda}}{|\lambda|}},
$$

and $\theta$ is the standard Heaviside step function.
Remark 3.1. Note that

$$
\begin{equation*}
\left|\arg w_{i}(\lambda, \varsigma)\right|<\pi, \quad \lambda, \varsigma \in \partial \Lambda, \quad i=1,2 \tag{53}
\end{equation*}
$$

and the logarithms in (52) are well defined by the condition $\left|\Im \ln w_{i}\right|<\pi$.
In particular, we have

$$
\begin{equation*}
W \in L_{p}(\partial \Lambda \times \partial \Lambda), p \geq 1, \partial \Lambda=\partial \Lambda_{E, \rho}, E>0, \rho>0 \tag{54}
\end{equation*}
$$

Lemma 3.2. Let $v$ satisfy (13) and let $\rho \geq \rho_{1}\left(\|v\|_{L^{\infty}(D)}, D, E\right)$, where $\rho_{1}$ is the constant in (27). Let $\psi^{+}, \psi$ be the eigenfunctions of Subsections 2.1, 2.2. Then the following relation holds:

$$
\begin{array}{r}
\psi(x, k(\lambda))=\psi^{+}(x, k(\lambda))+\int_{\partial \Lambda} W(\lambda, \varsigma) h(k(\lambda), k(\varsigma)) \psi^{+}(x, k(\varsigma)) d \varsigma  \tag{55}\\
\lambda \in \partial \Lambda .
\end{array}
$$

where $k(\lambda)=k(\lambda, E)$ is given by (35), W $(\lambda, \varsigma)=W(\lambda, \varsigma, E)$ is given by (52), $h$ is defined by (30) and the integration is taken according to the standard orientation of the $\partial \Lambda$.
Lemma 3.2 is proved in Section 6.
Note that, under assumptions (13), as a corollary of (30), (44), we have

$$
\begin{array}{r}
|h(k(\lambda), k(\varsigma))| \leq q(2 \pi)^{-2} e^{2 \rho L}(1-M(q, D, E, \rho))^{-1} \int_{D} d x  \tag{56}\\
\lambda, \varsigma \in \partial \Lambda=\partial \Lambda_{E, \rho}, \rho>\rho_{1}(q, D, E),\|v\|_{L^{\infty}(D)} \leq q
\end{array}
$$

where $M, \rho_{1}$ are defined by (28), (29),

$$
\begin{equation*}
L=\max _{x \in \partial D}|x| . \tag{57}
\end{equation*}
$$

As a corollary of properties (18), (40), (41), (55) of the functions $\psi^{+}$and $\psi$ (and using (46), (49)), we obtain the following proposition:

Proposition 3.3. Let $v$ satisfy (13) and let $\rho \geq \rho_{1}\left(\|v\|_{L^{\infty}(D)}, D, E\right)$, where $\rho_{1}$ is the constant in (27). Let $\psi, \psi^{+}$be the eigenfunctions of Subsections 2.1, 2.2. Then at fixed $x \in \mathbb{R}^{2}$ :

1. $\psi^{+}(x, k(\lambda))$ is holomorphic in $\lambda \in \Lambda$ and is continuous in $\lambda \in \Lambda \cup \partial \Lambda$;
2. $\psi(x, k(\lambda))$ has the properties (40), (41) for $\lambda \in(\mathbb{C} \backslash 0) \backslash(\Lambda \cup \partial \Lambda)$ and is continuous in $\lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda$;
3. $\psi^{+}, \psi$ are related on $\partial \Lambda$ via (55).

Now we consider the following generalized inverse scattering problem for equation (1).
Problem 3.4: Given the Faddeev functions $h$ on $\partial \Lambda \times \partial \Lambda$ and $b$ on $(\mathbb{C} \backslash 0) \backslash \Lambda$, find potential $v$ on $D$.

The approach of the present work for solving Problems 1.1, 1.2 and 3.4 is based on the reduction of Problem 3.4 to the following generalized RiemannHilbert problem.
Problem 3.5: Given functions $h$ on $\partial \Lambda \times \partial \Lambda$ and $b$ on $(\mathbb{C} \backslash 0) \backslash \Lambda$, find functions $\psi^{+}$on $\Lambda$ and $\psi$ on $(\mathbb{C} \backslash 0) \backslash \Lambda$ satisfying the properties of the items $1,2,3$ of Proposition 3.3.

Note that in Problems 3.4, 3.5 we consider $h, b$ and $\psi^{+}, \psi$ as

$$
\begin{gather*}
h=h(\lambda, \zeta, E)=h(k(\lambda), k(\zeta)), \lambda, \zeta \in \partial \Lambda,  \tag{58}\\
\psi^{+}=\psi^{+}(x, \lambda, E)=\psi^{+}(x, k(\lambda)), \lambda \in \Lambda,  \tag{59}\\
\psi=\psi(x, \lambda, E)=\psi(x, k(\lambda)), b=b(\lambda, E)=b(k(\lambda)), \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda, \tag{60}
\end{gather*}
$$

where $k(\lambda)=k(\lambda, E)$ is defined by (35), $\Lambda=\Lambda_{E, \rho}, \partial \Lambda=\partial \Lambda_{E, \rho}$ are defined in (50), $h$ is defined by (30) and $b$ is defined by (42).

In addition, if $\psi$ is the function of Subsections 2.2, 2.3, 2.4, then it determines the potential easily. Indeed, due to (1), (43), we have

$$
\begin{array}{cc}
v(x)=\frac{\left(\Delta_{x}+E\right) \psi(x, k(\lambda))}{\psi(x, k(\lambda))} & \text { on } \mathbb{R}^{2} \text { if }|\lambda| \text { is sufficiently small }  \tag{61}\\
\text { or if }|\lambda| \text { is sufficiently large. }
\end{array}
$$

Prototypes of Problems 3.4, 3.5 for the case of equation (1) with $E<0$ were considered in Section 8 of [19].

Generalized Riemann-Hilbert problems like Problem 3.5 go back to [16] and to $[9,12,13]$.

We say that Problem 3.5 is a generalized Riemann-Hilbert-Manakov problem.

We say that the results of Lemma 3.2 and Proposition 3.3 are global and that the related Problem 3.5 is global, since these results and problem are formulated
for general $v$ satisfying (13) and, in particular, without the assumption that $\mathcal{E}_{E}=\emptyset$, where $\mathcal{E}_{E}$ is the set of Faddeev exceptional points at fixed $E$. The reduction of Problem 3.4 to Problem 3.5 follows from Proposition 3.3 and, for example, from formula (61).

## 4. Integral equations for solving Problem 3.5

### 4.1. Formulas and equations

Let

$$
\begin{gather*}
\mu^{+}(\lambda):=e^{-i k(\lambda) x} \psi^{+}(x, \lambda, E), \quad \lambda \in \Lambda  \tag{62}\\
\mu(\lambda):=e^{-i k(\lambda) x} \psi(x, \lambda, E), \quad \lambda \in \mathbb{C} \backslash \Lambda  \tag{63}\\
r(x, \lambda, E)=e^{i(-k(\lambda)+k(-1 / \bar{\lambda})) x} \frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{\bar{\lambda}} \chi(\lambda) b(\lambda, E)  \tag{64}\\
=e^{-2 i \Re k(\lambda) x} u(\lambda), \quad \lambda \in \mathbb{C} \backslash 0 \\
R(x, \lambda, \zeta, E)=e^{i(k(\zeta)-k(\lambda)) x} W(\lambda, \zeta, E) h(\lambda, \zeta, E), \quad \lambda, \zeta \in \partial \Lambda \tag{65}
\end{gather*}
$$

where $\psi^{+}, \psi$ and $h=h(\lambda, \varsigma, E)=h(k(\lambda), k(\varsigma)), b=b(\lambda, E)=b(k(\lambda))$ are the functions of Problem 3.5, $\chi(\lambda)=\chi_{E, \rho}(k(\lambda))$ is defined via (17), $u(\lambda)=u_{E, \rho}(\lambda)$ is defined via (45), $k(\lambda)=k(\lambda, E)$ is defined by (35), $W$ is given by (52), $\Lambda=\Lambda_{E, \rho}$ is defined by (50).

Let

$$
\begin{equation*}
e(\lambda)=e(x, \lambda, E), \quad X_{j}(\lambda, \zeta)=X_{j}(x, \lambda, \zeta, E), j=1,2, \quad \lambda, \zeta \in \mathbb{C} \tag{66}
\end{equation*}
$$

be defined as the solutions of the following linear integral equations:

$$
\begin{gather*}
e(\lambda)=1-\frac{1}{\pi} \int_{\mathbb{C}} r(x, \zeta, E) \overline{e(\zeta)} \frac{d \Re \zeta d \Im \zeta}{\zeta-\lambda}  \tag{67}\\
X_{1}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} r(x, \eta, E) \overline{X_{1}(\eta, \zeta)} \frac{d \Re \zeta d \Im \zeta}{\eta-\lambda}=\frac{1}{2(\zeta-\lambda)}  \tag{68}\\
X_{2}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} r(x, \eta, E) \overline{X_{2}(\eta, \zeta)} \frac{d \Re \zeta d \Im \zeta}{\eta-\lambda}=\frac{1}{2 i(\zeta-\lambda)} \tag{69}
\end{gather*}
$$

In addition, we consider also

$$
\begin{equation*}
\Omega_{1}(\lambda, \zeta):=X_{1}(\lambda, \zeta)+i X_{2}(\lambda, \zeta), \quad \Omega_{2}(\lambda, \zeta):=X_{1}(\lambda, \zeta)-i X_{2}(\lambda, \zeta), \tag{70}
\end{equation*}
$$

Note that if (46) is fulfilled, then equation (67) for $e(\cdot)$ and equations (68), (69) for $X_{j}(\cdot, \zeta), j=1,2$, are uniquely solvable in $L_{q, 0}(\mathbb{C}), p /(p-1) \leq q<2$, where $L_{p, \nu}$ is defined in (37). In addition:

$$
\begin{gather*}
e(\cdot) \in C(\mathbb{C} \cup \infty), \quad e(\infty)=1,  \tag{71}\\
|e(\lambda)-1| \leq c_{2}\left(r_{0}, p\right) \\
\left|\Omega_{1}(\lambda, \zeta)-\frac{1}{\zeta-\lambda}\right|<c_{2}\left(r_{0}, p\right) \frac{1}{2|\zeta-\lambda|^{2 / p}},  \tag{72}\\
\left|\Omega_{2}(\lambda, \zeta)\right|<c_{2}\left(r_{0}, p\right) \frac{1}{2|\zeta-\lambda|^{2 / p}} \tag{73}
\end{gather*}
$$

where

$$
\begin{equation*}
r_{0}=\|r(x, \cdot, E)\|_{L_{p, 2}}, \quad \lim _{r_{0} \rightarrow 0} c_{2}\left(r_{0}, p\right)=0 \tag{74}
\end{equation*}
$$

Note that $r_{0}$ is independent of $x \in \mathbb{R}^{2}$. In connection with the functions $e, X_{1}, X_{2}, \Omega_{1}, \Omega_{2}$ and related results we refer to Chapter 3 of [25] and to Section 6 of [19].

We define

$$
\psi^{\prime}(\lambda)=\left\{\begin{array}{l}
\psi^{+}(\lambda), \lambda \in \Lambda \cup \partial \Lambda,  \tag{75}\\
\psi(\lambda), \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda,
\end{array}\right.
$$

where $\psi^{+}, \psi$ are the functions of Problem 3.5. In addition, we consider $\mu^{\prime}, \mu^{+}, \mu$, where

$$
\psi^{\prime}(\lambda)=e^{i k(\lambda) x} \mu^{\prime}(\lambda)=e^{i k(\lambda) x}\left\{\begin{array}{l}
\mu^{+}(\lambda), \lambda \in \Lambda \cup \partial \Lambda  \tag{76}\\
\mu(\lambda), \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda
\end{array}\right.
$$

Theorem 4.1. Let the data $h$ and $b$ of Problem 3.5 satisfy the following conditions:

$$
\begin{gather*}
u_{E, \rho} \in L_{p, 2}(\mathbb{C}), 2<p<4  \tag{77}\\
h(\cdot, \cdot, E) \in C(\partial \Lambda \times \partial \Lambda) \tag{78}
\end{gather*}
$$

where $u_{E, \rho}$ is defined by (45), $W$ is defined by (52), $\partial \Lambda=\partial \Lambda_{E, \rho}$ is defined in (50), $\rho>0$. Let $\psi^{\prime}$ be a solution of Problem 3.5. Then for $\mu^{\prime}$ defined by (76) the following formula holds:

$$
\mu^{\prime}(\lambda)=e(\lambda)+\frac{1}{2 \pi i} \int_{\partial \Lambda} \Omega_{1}(\lambda, \zeta) K(\zeta) d \zeta-\frac{1}{2 \pi i} \int_{\partial \Lambda} \begin{array}{r}
\Omega_{2}(\lambda, \zeta) \overline{K(\zeta)} d \bar{\zeta}  \tag{79}\\
\lambda \in \mathbb{C} \backslash \partial \Lambda
\end{array}
$$

where the integration is taken according to the standard orientation of $\partial \Lambda$,

$$
\begin{equation*}
K(\lambda):=\mu^{+}(\lambda)-\mu(\lambda), \quad \lambda \in \partial \Lambda \tag{80}
\end{equation*}
$$

In addition, this $K=K(x, \lambda, E)$ satisfies the following linear integral equation

$$
\begin{align*}
K(\lambda)+\int_{\partial \Lambda} & R\left(x, \lambda, \lambda^{\prime}, E\right)\left(e\left(\lambda^{\prime}\right)+\right. \\
& \frac{1}{2 \pi i} \int_{\partial \Lambda} \Omega_{1}\left(\lambda^{\prime}\left(1-0\left(\left|\lambda^{\prime}\right|-1\right)\right), \zeta\right) K(\zeta) d \zeta  \tag{81}\\
\quad & \left.\quad \frac{1}{2 \pi i} \int_{\partial \Lambda} \Omega_{2}\left(\lambda^{\prime}, \zeta\right) \overline{K(\zeta)} d \bar{\zeta}\right) d \lambda^{\prime}=0
\end{align*}
$$

$\lambda \in \partial \Lambda$, where $R$ is defined by (65), $\Omega_{1}, \Omega_{2}$ are defined by (70) and the integrations are taken according to the standard orientation of $\partial \Lambda$.

Note also that

$$
\begin{align*}
\int_{\partial \Lambda} & \Omega_{1}\left(\lambda^{\prime}\left(1-0\left(\left|\lambda^{\prime}\right|-1\right)\right), \zeta\right) K(\zeta) d \zeta \\
& =\lim _{0<\varepsilon \rightarrow 0} \int_{\partial \Lambda} \Omega_{1}\left(\lambda^{\prime}\left(1-\varepsilon\left(\left|\lambda^{\prime}\right|-1\right)\right), \zeta\right) K(\zeta) d \zeta, \quad \lambda^{\prime} \in \partial \Lambda \tag{82}
\end{align*}
$$

Formula (79) is similar to formula (6.7) of [19]. Equation (81) is similar to equation (6.11) of [19].

Theorem 4.1 is proved in Section 7.
Consider

$$
\begin{equation*}
I(\lambda)=I(x, \lambda, E)=-\int_{\partial \Lambda} R\left(x, \lambda, \lambda^{\prime}, E\right) e\left(\lambda^{\prime}\right) d \lambda^{\prime}, \quad \lambda \in \partial \Lambda \tag{83}
\end{equation*}
$$

$$
\begin{align*}
& A_{1}(\lambda, \zeta)=A_{1}(x, \lambda, \zeta, E) \\
& \quad=\frac{1}{2 \pi i} \int_{\partial \Lambda} R\left(x, \lambda, \lambda^{\prime}, E\right) \Omega_{1}\left(\lambda^{\prime}\left(1-0\left(\left|\lambda^{\prime}\right|-1\right)\right), \zeta\right) d \lambda^{\prime} \\
& \begin{array}{l}
A_{2}(\lambda, \zeta)=A_{2}(x, \lambda, \zeta, E) \\
\quad=\frac{-1}{2 \pi i} \int_{\partial \Lambda} R\left(x, \lambda, \lambda^{\prime}, E\right) \Omega_{2}\left(\lambda^{\prime}, \zeta\right) d \lambda^{\prime}, \quad \lambda, \zeta \in \partial \Lambda,
\end{array} \tag{84}
\end{align*}
$$

where $R, e, \Omega_{1}, \Omega_{2}$ are the functions of (65), (66), (70).
Proposition 4.2. Let the assumptions of Theorem 4.1 be fulfilled and $K$ be the function of (80), (81). Then $K, \bar{K}$ satisfy the following system of linear integral equations

$$
\begin{align*}
K(\lambda)+\int_{\partial \Lambda} A_{1}(\lambda, \zeta) K(\zeta) d \zeta+\int_{\partial \Lambda} A_{2}(\lambda, \zeta) \overline{K(\zeta)} d \bar{\zeta}=I(\lambda), & \lambda \in \partial \Lambda  \tag{85}\\
\overline{K(\lambda)}+\int_{\partial \Lambda} \overline{A_{2}(\lambda, \zeta)} K(\zeta) d \zeta+\int_{\partial \Lambda} \overline{A_{1}(\lambda, \zeta)} \overline{K(\zeta)} d \bar{\zeta}=\overline{I(\lambda)}, & \lambda \in \partial \Lambda \tag{86}
\end{align*}
$$

where $I, A_{1}, A_{2}$ are defined by (83), (84). In addition,

$$
\begin{align*}
& I \in L_{2}(\partial \Lambda), \quad A_{j} \in L_{2}(\partial \Lambda \times \partial \Lambda), j=1,2  \tag{87}\\
&\left\|A_{j}\right\|_{L_{2}} \rightarrow 0 \text { for }\|h\|_{C} \rightarrow 0, \quad r_{0} \leq r_{\text {fixed }},  \tag{88}\\
& j=1,2
\end{align*}
$$

where $|x|<c$ for fixed $c>0, r_{0}$ is defined in (74)
Proposition 4.2 is proved in Section 6.

### 4.2. Analysis of equations

Due to estimates (87), the system (85), (86) can be considered as a Fredholm linear integral equation of the second type for the vector-function $(K, \bar{K}) \in$ $L_{2}\left(\partial \Lambda, \mathbb{C}^{2}\right)$ with parameters $x \in \mathbb{R}^{2}$ and $E>0$.

The modified Fredholm determinant detA for system (85), (86) can be defined by means of the formula:

$$
\begin{equation*}
\ln \operatorname{det} \mathrm{A}=\operatorname{Tr}(\ln (\mathrm{Id}+\mathrm{A})-\mathrm{A}) \tag{89}
\end{equation*}
$$

where system (85), (86) is written as

$$
\begin{equation*}
(I d+A)\binom{K}{\bar{K}}=\binom{I}{\bar{I}} \tag{90}
\end{equation*}
$$

For the precise definition of $\operatorname{det} A$, see [10].
In addition, we have the following lemmas:
Lemma 4.3. Let $v$ satisfy (13) for fixed $D$ and $\Lambda=\Lambda_{E, \rho}$ be defined by (50) for fixed $E$ and $\rho$. Let $A_{1}, A_{2}, I$ correspond to $v$ according to formulas (20)-(23), (30), (42), (58), (60), (64), (65), (83), (84). Let $|x|<c$ for fixed $c>0$. Then:

$$
\begin{equation*}
\left\|A_{j}\right\|_{L^{2}(\partial \Lambda \times \partial \Lambda)} \rightarrow 0,\|I\|_{L^{2}(\partial \Lambda)} \rightarrow 0, \quad \text { for }\|v\|_{L^{\infty}(D)} \rightarrow 0, j=1,2 \tag{91}
\end{equation*}
$$

system (85), (86) for $(K, \bar{K}) \in L_{2}\left(\partial \Lambda, \mathbb{C}^{2}\right)$ is uniquely solvable by the method of successive approximations when $\|v\|_{L^{\infty}(D)}$ is sufficiently small (for fixed $D, E, \rho$ and c).

Actually, Lemma 4.3 follows from estimates (48), (56), (71)-(73), (88).
Lemma 4.4. Let $v$ satisfy (13) for fixed $D$ and $\Lambda=\Lambda_{E, \rho}$ be defined by (50) for fixed $E$ and $\rho$, where $\rho>\rho_{1}(q, D, E),\|v\|_{L^{\infty}(D)}<q, \rho_{1}$ is defined by (29). Let $A_{1}, A_{2}, I$ correspond to sv according to formulas (20)-(23), (30), (42), (58), (60), (64), (65), (83), (84) (with sv in place of $v$ ), where $s \in]-s_{1}, s_{1}[$, where $s_{1}=q /\|v\|_{L^{\infty}(D)}$. And let $\left.\operatorname{det} \mathrm{A}=\operatorname{det} \mathrm{A}(\mathrm{x}, \mathrm{s}), \mathrm{x} \in \mathbb{R}^{2}, \mathrm{~s} \in\right]-\mathrm{s}_{1}, \mathrm{~s}_{1}[$, be the
modified Fredholm determinant of the related system (85), (86) (where $\operatorname{det} \mathrm{A}$ depends also on $v, E$ and $\rho$ ). Then:

$$
\begin{gather*}
\operatorname{det} \mathrm{A}(\mathrm{x}, 0)=1, \quad \mathrm{x} \in \mathbb{R}^{2}  \tag{92}\\
\operatorname{det} \mathrm{~A} \in \mathrm{C}\left(\mathbb{R}^{2} \times\right]-\mathrm{s}_{1}, \mathrm{~s}_{1}[, \mathbb{C})  \tag{93}\\
\operatorname{det} \mathrm{A}(\mathrm{x}, \cdot) \text { is real-analytic on }]-\mathrm{s}_{1}, \mathrm{~s}_{1}\left[\text { for fixed } \mathrm{x} \in \mathbb{R}^{2} .\right. \tag{94}
\end{gather*}
$$

Lemma 4.4 is proved in Section 8. Using Lemma 4.4 we obtain, in particular, the following result:

Proposition 4.5. Let $\Lambda=\Lambda_{E, \rho}$ be defined by (50) for fixed $E$ and $\rho$, where $\rho>\rho_{1}(q, D, E), \rho_{1}$ is defined by (29), $D$ is a fixed open bounded domain in $\mathbb{R}^{2}, q$ is a fixed positive number. Then for almost each $v$ satisfying (13) with $\|v\|_{L^{\infty}(D)} \leq q$ the system (85), (86) corresponding to $v$ (according to formulas (30), (42), (58), (60), (64), (65), (83), (84)) is uniquely solvable for almost each $x \in \mathbb{R}^{2}$.

Remark 4.6. We understand the statement of Proposition 4.5 in the sense that if $v$ satisfies (13) and $\|v\|_{L^{\infty}(D)}=q_{1}$ for fixed $q_{1}$, where $0<q_{1}<q$, then for almost each $s \in]-s_{1}, s_{1}\left[\right.$, where $s_{1}=q / q_{1}$, the system (85), (86) corresponding to $s v$ is uniquely solvable for almost each $x \in \mathbb{R}^{2}$.

Remark 4.7. If the assumptions of Proposition 4.5 are fulfilled, $\|v\|_{L^{\infty}(D)}<q$, and system (85), (86) corresponds to $v$, then, as a corollary of (93), the set of $x$, where the system (85), (86) is uniquely solvable, is an open set in $\mathbb{R}^{2}$.

Proposition 4.5 is proved in Section 8.

## 5. Applications to the Novikov-Veselov equation

In this section we suppose that $v$ and $\rho$ satisfy the assumptions of Lemma 4.4 for fixed $D, E$ and $q$.

We define

$$
\begin{array}{r}
f_{s}(k, l, t)=f_{s}(k, l) \exp \left[2 i t\left(k_{1}^{3}-3 k_{1} k_{2}^{2}-l_{1}^{3}+3 l_{1} l_{2}^{2}\right)\right],(k, l) \in \mathcal{M}_{E} \\
h_{s}(k, l, t)=h_{s}(k, l) \exp \left[2 i t\left(k_{1}^{3}-3 k_{1} k_{2}^{2}-l_{1}^{3}+3 l_{1} l_{2}^{2}\right)\right], \\
\quad(k, l) \in \partial \Sigma_{E, \rho} \times \partial \Sigma_{E, \rho}  \tag{95}\\
b_{s}(k, t)=b_{s}(k) \exp \left[2 i t\left(k_{1}^{3}+\bar{k}_{1}^{3}-3 k_{1} k_{2}^{2}-3 \bar{k}_{1} \bar{k}_{2}^{2}\right)\right], k \in \Sigma_{E, \rho}
\end{array}
$$

where $t \in \mathbb{R}, s \in]-s_{1}, s_{1}\left[, s_{1}\right.$ is defined as in Lemma 4.4 and $f_{s}, h_{s}, b_{s}$ are defined according to (11), (12), (20)-(23), (30), (42) with $s v$ in place of $v$. In
addition:

$$
\begin{array}{rlr}
h_{s}(k(\lambda), k(\varsigma), t) & =h_{s}(k(\lambda), k(\varsigma)) \exp \left[i E^{3 / 2} t\left(\lambda^{3}+\lambda^{-3}-\varsigma^{3}-\varsigma^{-3}\right)\right] \\
& =: h_{s, t}(\lambda, \varsigma, E), & (\lambda, \varsigma) \in \partial \Lambda \times \partial \Lambda, \\
b_{s}(k(\lambda), t) & =b_{s}(k(\lambda)) \exp \left[i t E^{3 / 2}\left(\lambda^{3}+\lambda^{-3}+\bar{\lambda}^{3}+\bar{\lambda}^{-3}\right)\right]  \tag{96}\\
& =: b_{s, t}(\lambda, E), & \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda,
\end{array}
$$

where $t \in \mathbb{R}, s \in]-s_{1}, s_{1}[, k(\lambda)=k(\lambda, E)$ is defined by (35), $\Lambda=\Lambda(E, \rho)$ is defined by (50).

We consider Problem 3.5 of Section 3 with $h=h_{s, t}, b=b_{s, t}, \psi^{+}=\psi_{s, t}^{+}$. As in Section 4.1, we consider the reduction of this generalized Riemann-HilbertManakov problem to formulas (75), (76), (79), (80) and the system of equations (85), (86), where $\mu^{\prime}=\mu_{s, t}^{\prime}, e=e_{s, t}, \Omega_{j}=\Omega_{j, s, t}, j=1,2, K=K_{s, t}, I=I_{s, t}$, $A_{j}=A_{j, s, t}, j=1,2$. In addition, as in Section 4.2, we consider $\operatorname{det} A(x, s, t)$ for the aforementioned system (85), (86).

We expect that using ideas of $[12,13,14,19]$ and of the present work one can obtain the following result:

Suppose that $\operatorname{det} A(x, s, t) \neq 0$ for $x \in \mathcal{X}, t \in \mathcal{T}$ at fixed $s \in]-s_{1}, s_{1}[$, where $\mathcal{X}$ is an open domain in $\mathbb{R}^{2}, \mathcal{T}$ is an open interval in $\mathbb{R}, 0 \in \mathcal{T}, s_{1}$ is defined in Lemma 4.4. Then there is a real valued $v_{s}(\cdot, t)$ such that:

$$
\begin{equation*}
v_{s}(\cdot, 0)=s v, \tag{97}
\end{equation*}
$$

where $s v$ is the potential of Lemma 4.4;

$$
\begin{align*}
& -\Delta_{x} \psi_{s, t}^{+}+v_{s}(x, t) \psi_{s, t}^{+}=E \psi_{s, t}^{+} \\
& -\Delta_{x} \psi_{s, t}+v_{s}(x, t) \psi_{s, t}=E \psi_{s, t}, \quad(x, t) \in \mathcal{X} \times \mathcal{T} \tag{98}
\end{align*}
$$

where $\psi_{s, t}^{+}=\psi_{s, t}^{+}(x, \lambda), \lambda \in \Lambda$, and $\psi_{s, t}=\psi_{s, t}(x, \lambda), \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda$, solve the aforementioned Problem 3.5; $v=v_{s}(x, t)$ solves the Novikov-Veselov equation (9) in $\mathcal{X} \times \mathcal{T}$ with appropriate $w=w_{s}(x, t)$ (and satisfies (97) on $\mathcal{X}$ ).

These studies will be given in detail elsewhere. Note that, actually, the zeroes of $\operatorname{det} A(x, s, t)$ describe the blow-up points of the potential $v_{s}(x, t)$. It remains to note that in similar way to Proposition 4.5 and Remarks 4.6, 4.7, for almost each $s \in]-s_{1}, s_{1}[$, we have that $\operatorname{det} A(x, s, t) \neq 0$ for almost each $(x, t) \in \mathbb{R}^{2} \times \mathbb{R}$; and the nonzero set of $\operatorname{det} A$ is open.

## 6. Proof of Lemma 3.2

### 6.1. Lemma for Green functions

Let

$$
\begin{equation*}
z=x_{1}+i x_{2}, \bar{z}=x_{1}-i x_{2} \text { for } x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2} \tag{99}
\end{equation*}
$$

Lemma 6.1. The following formula holds:

$$
\begin{align*}
G(x, k(\lambda))- & G^{+}(x, \sqrt{E}) \\
& =\frac{1}{(2 \pi)^{2}} \int_{\partial \Lambda} W(\lambda, \varsigma, E) e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} d \varsigma, \quad \lambda, \varsigma \in \partial \Lambda \tag{100}
\end{align*}
$$

where $G, G^{+}$are defined in (21), (11), $W$ is defined by (52), $k(\lambda)=k(\lambda, E)$ is defined in (35), $\Lambda=\Lambda_{E, \rho}$ is defined in (50).

Proof. We recall that

$$
\begin{align*}
\frac{\partial}{\partial \bar{\lambda}} G(z, k(\lambda))=\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{4 \pi \bar{\lambda}} e^{i k(-1 / \bar{\lambda}) x}, & \lambda \in(\mathbb{C} \backslash 0) \backslash T  \tag{101}\\
\frac{\partial}{\partial \lambda} G(z, k(\lambda))=\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{4 \pi \lambda} e^{i k(\lambda) x}, & \lambda \in(\mathbb{C} \backslash 0) \backslash T \tag{102}
\end{align*}
$$

where $G$ is defined by $(21),(22), k(\lambda)=k(\lambda, E)$ is defined by (35), $T$ is defined by (32); see [19].

Note that

$$
\begin{equation*}
k(-1 / \bar{\lambda}) x=-\frac{\sqrt{E}}{2}(\bar{\lambda} z+\bar{z} / \bar{\lambda}), \quad k(\lambda) x=\frac{\sqrt{E}}{2}(\lambda \bar{z}+z / \lambda) . \tag{103}
\end{equation*}
$$

Using the Cauchy formula for $e^{i k(-1 / \bar{\lambda}) x} / \bar{\lambda}$ and $e^{i k(\lambda) x} / \lambda$ we have

$$
\begin{align*}
e^{i k(-1 / \bar{\lambda}) x} / \bar{\lambda} & =\frac{1}{2 \pi i} \int_{\partial \Lambda} \frac{1}{\bar{\varsigma}} e^{-i \sqrt{E} / 2(\bar{\varsigma} z+\bar{z} / \bar{\varsigma})} \frac{d \bar{\varsigma}}{\bar{\varsigma}-\bar{\lambda}}, \quad \lambda \in \Lambda,  \tag{104}\\
e^{i k(\lambda) x} / \lambda & =\frac{1}{2 \pi i} \int_{\partial \Lambda} \frac{1}{\varsigma} e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} \frac{d \varsigma}{\varsigma-\lambda}, \quad \lambda \in \Lambda . \tag{105}
\end{align*}
$$

Due to (101), (102) and (104), (105) we have

$$
\begin{array}{r}
\frac{\partial}{\partial \bar{\lambda}} G(x, k(\lambda))=\operatorname{sgn}\left(|\lambda|^{2}-1\right) \frac{-1}{2 \pi i} \int_{\partial \Lambda} \frac{1}{4 \pi \bar{\varsigma}} e^{-i \sqrt{E} / 2(\bar{\varsigma} z+\bar{z} / \bar{\varsigma})} \frac{d \bar{\varsigma}}{\bar{\varsigma}-\bar{\lambda}} \\
\lambda \in \Lambda,|\lambda| \neq 1 \\
\frac{\partial}{\partial \lambda} G(x, k(\lambda))=\operatorname{sgn}\left(|\lambda|^{2}-1\right) \frac{1}{2 \pi i} \int_{\partial \Lambda} \frac{1}{4 \pi \varsigma} e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} \frac{d \varsigma}{\varsigma-\lambda}  \tag{107}\\
\lambda \in \Lambda, \quad|\lambda| \neq 1
\end{array}
$$

Formulas (106), (107) remain also valid with $G(x, k(\lambda))$ replaced in the left hand side by $G(x, k(\lambda))-G^{+}(x, \sqrt{E})$, where $G^{+}$is defined in (11).

Integrating the differential equation for $G-G^{+}$we obtain

$$
\begin{align*}
& G(x, k(\lambda))-G^{+}(x, \sqrt{E}) \\
& \quad=u(z, \lambda)+\left[G\left(x, k\left(\lambda_{0}\right)\right)-G^{+}(x, \sqrt{E})-u\left(z, \lambda_{0}\right)\right]  \tag{108}\\
& \quad \text { for } \lambda_{0}=\lambda_{0}(\lambda), \lambda \in \Lambda \cap \mathcal{D}_{1}, \text { or for } \lambda_{0}(\lambda), \lambda \in \Lambda \cap\left(\mathbb{C} \backslash \mathcal{D}_{1}\right),
\end{align*}
$$

where $\mathcal{D}_{1}$ is defined by (39), $\lambda_{0}=\lambda_{0}(\lambda)=\frac{\lambda}{|\lambda|}\left(1+0\left(|\lambda|^{2}-1\right)\right)$,

$$
\begin{align*}
u(z, \lambda) & =\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{2 \pi i} \int_{\partial \Lambda} \frac{1}{4 \pi \bar{\varsigma}} e^{-i \sqrt{E} / 2(\bar{\varsigma} z+\bar{z} / \bar{\varsigma})} \ln (\bar{\varsigma}-\bar{\lambda}) d \bar{\varsigma} \\
& -\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{2 \pi i} \int_{\partial \Lambda} \frac{1}{4 \pi \varsigma} e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} \ln (\varsigma-\lambda) d \varsigma, \quad \lambda \in \Lambda \backslash T, \tag{109}
\end{align*}
$$

where notation $1+0\left(|\lambda|^{2}-1\right)$ is like in (82). In the last expression logarithm is chosen such that $|\Im \ln (\cdot)|<\pi$.

We change the variable $\varsigma \rightarrow-1 / \bar{\varsigma}$ in the first integral on the right and obtain the formula

$$
\begin{align*}
& u(z, \lambda)=-\frac{\operatorname{sgn}\left(|\lambda|^{2}-1\right)}{8 \pi^{2} i} \int_{\partial \Lambda} e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)}\left[\frac{1}{\varsigma} \ln (\varsigma-\lambda)\right. \\
& \left.\quad+\varsigma \ln \left(\frac{-1}{\varsigma}-\bar{\lambda}\right)\right] d \varsigma, \quad \lambda \in \Lambda \backslash T \tag{110}
\end{align*}
$$

In the last expression logarithm is chosen such that $|\Im \ln (\cdot)|<\pi$.
We choose $\lambda_{0}$ as $\lambda_{0}=\frac{\lambda}{|\lambda|}(1 \pm 0)$ since the limiting values of $G-G^{+}$on the unit circle $T$ are given by (see [19, Section 3]):

$$
\begin{align*}
& G\left(x, k\left(\lambda_{0}\right)\right)-G^{+}(x, \sqrt{E}) \\
& \quad=\frac{\pi i}{(2 \pi)^{2}} \int_{T} e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} \times \theta\left[\operatorname{sgn}\left(|\lambda|^{2}-1\right) i\left(\frac{|\lambda| \varsigma}{\lambda}-\frac{\lambda}{|\lambda| \varsigma}\right)\right]|d \varsigma|, \tag{111}
\end{align*}
$$

where $\theta$ is the Heaviside step function.
Using the Cauchy formula for $e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)}$ in (111), we can rewrite (111) as follows:

$$
\begin{align*}
G\left(x, \lambda_{0}\right)-G^{+}(x, \sqrt{E})= & \frac{1}{8 \pi^{2}} \int_{\varsigma_{1} \in T}\left(\int_{\partial \Lambda} \frac{e^{i \sqrt{E} / 2(\varsigma \bar{z}+z / \varsigma)} d \varsigma}{\varsigma-\varsigma_{1}}\right)  \tag{112}\\
& \times \theta\left[\operatorname{sgn}\left(|\lambda|^{2}-1\right) i\left(\frac{|\lambda| \varsigma_{1}}{\lambda}-\frac{\lambda}{|\lambda| \varsigma_{1}}\right)\right]\left|d \varsigma_{1}\right| .
\end{align*}
$$

In order to complete the proof of Lemma 6.1 it remains only to put (112), (110) into (108).

In addition, to justify Remark 3.1, we need to prove (53). Assume that $\varsigma$ belongs to the part $|\varsigma|=C$ of $\partial \Lambda=\partial \Lambda_{E, \rho}$ where

$$
C=\rho / \sqrt{E}+\sqrt{(\rho / \sqrt{E})^{2}+1}
$$

Since the point $\lambda$ belongs to the disk $|\varsigma| \leq C$ and the point $\lambda_{0}$ is strictly inside of the disk, the angle $\alpha$ between vectors $\varsigma-\lambda$ and $\varsigma-\lambda_{0}$ is strictly less then $\pi$. Thus $\left|\arg w_{1}\right|=|\alpha|<\pi$ in this case. If $\varsigma$ belongs to the part $|\varsigma|=1 / C$ of the boundary of $\partial \Lambda$, then points $\lambda$ and $\lambda_{0}$ belong to the part of the ray (emitted from $\lambda=0$ ) through the point $\lambda$. This part belongs to the region $|\varsigma| \geq 1 / C$, and $\left|\arg w_{1}\right|=|\alpha|<\pi / 2$ in this case. After the estimate (53) for $w_{1}$ is proved, the estimate for $w_{2}$ becomes obvious if we replace $-1 / \varsigma$ by $\bar{\zeta}$.

### 6.2. Final part of the proof of Lemma 3.2

Let

$$
\begin{equation*}
\psi_{0}=\psi_{0}(x, k(\lambda))=e^{i k(\lambda) x}=e^{i(\sqrt{E} / 2)(\lambda \bar{z}+z / \lambda)}, \quad \lambda \in(\mathbb{C} \backslash 0) \backslash T \tag{113}
\end{equation*}
$$

where $k(\lambda)=k(\lambda, E)$ is defined by (35), $T$ is defined by (32).
We will denote by $G^{+}(\sqrt{E}), G(k)$ the convolution operators with kernels $G^{+}, G$ of $(21),(11)$, and we will denote by $G^{+}(\sqrt{E}) v, G(k) v$ the operators of multiplication by the potential $v$ followed by convolution $G^{+}(\sqrt{E})$ or $G(k)$, respectively. Then, under the assumptions of Lemma 3.2, equations (11), (20) can be considered as linear integral equations for $\psi^{+}(\cdot, k), \psi(\cdot, k) \in L^{\infty}(D)$, and can be rewritten as follows:

$$
\begin{equation*}
\psi^{+}(\cdot, k)=\left(I-G^{+}(\sqrt{E}) v\right)^{-1} \psi_{0}(\cdot, k), \quad \psi(\cdot, k)=(I-G(k) v)^{-1} \psi_{0}(\cdot, k) \tag{114}
\end{equation*}
$$

for fixed $k \in \Sigma_{E} \backslash \Sigma_{E, \rho}$, where $I$ is the identity operator.
Thus

$$
\begin{align*}
& \psi^{+}(\cdot, k)=\left(I-G^{+}(\sqrt{E}) v\right)^{-1}(I-G(k) v) \psi(\cdot, k), \\
& \psi(\cdot, k)=\left(I-G^{+}(\sqrt{E}) v\right)^{-1}\left(I-G^{+}(\sqrt{E}) v\right) \psi(\cdot, k), \quad k \in \Sigma_{E} \backslash \Sigma_{E, \rho} \tag{115}
\end{align*}
$$

Therefore,

$$
\begin{array}{r}
\psi(\cdot, k)-\psi^{+}(\cdot, k)=\left(I-G^{+}(\sqrt{E}) v\right)^{-1}\left(G(k)-G^{+}(\sqrt{E})\right) v \psi(\cdot, k)  \tag{116}\\
k \in \Sigma_{E} \backslash \Sigma_{E, \rho} .
\end{array}
$$

We take $G-G^{+}$from Lemma 6.1 and use there that $\psi_{0}(x-y, k(\lambda))=$
$\psi_{0}(x, k(\lambda)) \psi_{0}(-y, k(\lambda))$. This leads to

$$
\begin{aligned}
& \left(G(k(\lambda))-G^{+}(\sqrt{E})\right) v \psi(\cdot, k(\lambda)) \\
& \quad=\frac{1}{(2 \pi)^{2}} \int_{D} \int_{\partial \Lambda} W(\lambda, \varsigma) \psi_{0}(x, k(\varsigma)) \psi_{0}(-y, k(\varsigma)) d \varsigma v(y) \psi(y, k(\lambda)) d y \\
& \quad=\int_{\partial \Lambda} W(\lambda, \varsigma) \psi_{0}(x, k(\varsigma)) h(k(\varsigma), k(\lambda)) d \varsigma, \quad \lambda \in \partial \Lambda,
\end{aligned}
$$

where we used also (30). We plug the last relation in (116). It remains to note (see (114)) that $\left(I-G^{+}(\sqrt{E} v)\right)^{-1} \psi_{0}(\cdot, k(\varsigma))=\psi^{+}(\cdot, k(\varsigma))$.

## 7. Proofs of Theorem 4.1 and Proposition 4.2

### 7.1. Proof of Theorem 4.1

Let

$$
\begin{array}{r}
\mu_{0}^{\prime}(\lambda)=\mu^{\prime}(\lambda)-e(\lambda) \\
\mu_{0}^{+}(\lambda)=\mu^{+}(\lambda)-e(\lambda), \quad \mu_{0}(\lambda)=\mu(\lambda)-e(\lambda), \tag{117}
\end{array}
$$

where $\mu^{\prime}, \mu^{+}, \mu$ are the functions of (76), e(•) is the function of (67).
From formulas (64), (67) and from items 1 and 2 of Proposition 3.3 it follows, in particular, that

$$
\begin{array}{r}
\frac{\partial}{\partial \bar{\lambda}} e(\lambda)=r(x, \lambda, E) \overline{e(\lambda)}, \quad \lambda \in \mathbb{C}, \\
\frac{\partial}{\partial \bar{\lambda}} \mu_{0}^{\prime}(\lambda)=r(x, \lambda, E) \overline{\mu_{0}^{\prime}(\lambda)}, \quad \lambda \in \mathbb{C} \backslash \partial \Lambda,  \tag{119}\\
\mu_{0}^{\prime}(\lambda) \rightarrow 0 \text { as } \lambda \rightarrow \infty .
\end{array}
$$

Proceeding from (119) and using the generalized Cauchy formula for $\mu_{0}^{\prime}$ (see formula (10.6) of Chapter 3 of [25]) one can obtain

$$
\begin{equation*}
\mu_{0}^{\prime}(\lambda)=\frac{1}{2 \pi i} \int_{\partial \Lambda} \Omega_{1}(\lambda, \zeta) K_{0}(\zeta) d \zeta-\frac{1}{2 \pi i} \int_{\partial \Lambda} \Omega_{2}(\lambda, \zeta) \overline{K_{0}(\zeta)} d \bar{\zeta} \tag{120}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{0}(\lambda):=\mu_{0}^{+}(\lambda)-\mu_{0}(\lambda), \quad \lambda \in \partial \Lambda . \tag{121}
\end{equation*}
$$

In addition, from (80), (117) and (121) it follows that

$$
\begin{equation*}
K_{0}(\lambda)=K(\lambda), \quad \lambda \in \partial \Lambda . \tag{122}
\end{equation*}
$$

Formulas (117), (120), (122) imply formula (79).

Finally, equation (81) follows from the substitution of (79) into (55) using formulas (65), (76), (80), estimates (71)-(73) and the jump properties of the Cauchy integral.

This completes the scheme of proof of Theorem 4.1.

### 7.2. Proof of Proposition 4.2

Equation (85) follows from equation (81) and formulas (83), (84). Equation (86) follows from (85).

Estimates (87), (88) follow from formulas (64), (65), (83), (84), estimates (54), (71)-(74), (77), (78) and the estimate

$$
\begin{equation*}
\left\|\Omega_{1}^{0} u\right\|_{L_{p}(\partial \Lambda)} \leq \operatorname{const}(p, \partial \Lambda)\|u\|_{L_{p}(\partial \Lambda)}, 1<p<\infty \tag{123}
\end{equation*}
$$

where

$$
\begin{equation*}
\left(\Omega_{1}^{0} u\right)(\lambda)=\frac{1}{2 \pi i} \int_{\partial \Lambda} \frac{u(\varsigma) d \varsigma}{\varsigma-\lambda(1-0(|\lambda|-1))}, \quad \lambda \in \partial \Lambda \tag{124}
\end{equation*}
$$

$u$ is a test function on $\partial \Lambda$.

## 8. Proofs of Lemma 4.4 and Proposition 4.5

### 8.1. Proof of Lemma 4.4

Property (92) follows from (89), (91).
Property (93) follows from continuous dependence of $A_{1}, A_{2}$ with respect to $x \in \mathbb{R}^{2},|x| \leq c$, at fixed $\left.s \in\right]-s_{1}, s_{1}\left[\right.$ and continuous dependence of $A_{1}, A_{2}$ with respect to $s \in]-s_{1}, s_{1}$ [ uniformly in $x \in \mathbb{R}^{2},|x| \leq c$, in the sense of $\|\cdot\|_{L^{2}(\partial \Lambda \times \partial \Lambda)}$, for fixed $c>0$.

In turn, these continuities of $A_{1}, A_{2}$ in $x$ and in $s$ follow from formulas (72), (73), (84) and the following results:
(i) $\left.h\right|_{\partial \Lambda \times \partial \Lambda}$ depends continuously on $\left.s \in\right]-s_{1}, s_{1}\left[\right.$ in the sense of $\|\cdot\|_{C(\partial \Lambda \times \partial \Lambda)}$,
(ii) $u_{E, \rho}$ depends continuously on $\left.s \in\right]-s_{1}, s_{1}$ [ in the sense of $\|\cdot\|_{L_{p, 2}(\mathbb{C})}, 2<$ $p<4$, where $h=h(k(\lambda), k(\varsigma)), u_{E, \rho}$ correspond to $s v$ according to (20)(23), (30), (35), (42), (45);
(iii) The following estimates hold:

$$
\begin{array}{r}
\left|e^{-2 i \Re k(\lambda) x}-e^{-2 i \Re k(\lambda) x^{\prime}}\right| \leq \text { Const } \cdot\left(\sqrt{E}\left(|\lambda|+|\lambda|^{-1}\right)\left|x-x^{\prime}\right|\right)^{\alpha} \\
\lambda \in \mathbb{C} \backslash 0, x, x^{\prime} \in \mathbb{R}^{2}, 0<\alpha \leq 1 \\
\left|e^{i(k(\varsigma)-k(\lambda)) x}-e^{i(k(\varsigma)-k(\lambda)) x^{\prime}}\right| \leq 2\left(E+2 \rho^{2}\right)^{1 / 2} e^{2 \rho \max \left(|x|,\left|x^{\prime}\right|\right)}\left|x-x^{\prime}\right| \\
\varsigma, \lambda \in \partial \Lambda, x, x^{\prime} \in \mathbb{R}^{2}
\end{array}
$$

(iv) If $u \in L_{p, 2}(\mathbb{C}), 2<p<4$, then $\left(|\lambda|+\left|\lambda^{-1}\right|\right)^{\alpha} u(\lambda) \in L_{p^{\prime}, 2}(\mathbb{C})$ (as a function of $\lambda$ ), $2<p^{\prime}<p(1+\alpha p / 2)^{-1}$, where $0<\alpha<(p-2) / p$;
(v) The map (defined via (67))

$$
r \in L_{p, 2}(\mathbb{C}) \rightarrow e(\cdot) \in C(\mathbb{C})
$$

is continuous and the maps (defined via (68), (69))

$$
\begin{gathered}
r \in L_{p, 2}(\mathbb{C}) \rightarrow X_{j} \in C\left(\mathbb{C}^{2} \backslash \mathbb{C}_{\varepsilon}\right), j=1,2, \\
\mathbb{C}_{\varepsilon}=\left\{(\lambda, \varsigma) \in \mathbb{C}^{2}:|\lambda-\varsigma|<\varepsilon\right\},
\end{gathered}
$$

are continuous for any $\varepsilon>0$, where $L_{p, 2}(\mathbb{C})$ is considered with the norm of $(37), 2<p<4$, and $C(\mathbb{C}), C\left(\mathbb{C}^{2} \backslash \mathbb{C}_{\varepsilon}\right)$ are considered with the uniform norms.

In order to prove (94) we consider $s v$, where $s \in \mathbb{C}$, and we consider $h_{s}=$ $h_{s}(k(\lambda), k(\varsigma)), \lambda, \varsigma \in \partial \Lambda$, and $b_{s}=b_{s}(k(\lambda)), \lambda \in(\mathbb{C} \backslash 0) \backslash \Lambda$, where $h_{s}, b_{s}$ correspond to $s v$ according to (20)-(23), (30), (35), (42) (with $s v$ in place of $v$ ). Proceeding from these formulas and equations and from (26), (27), (51), one can show that there is an open neighbourhood $\mathcal{N}$ of the real interval ] $-s_{1}, s_{1}[$ in $\mathbb{C}$ (where $\mathcal{N}$ depends on $\left.D,\|v\|_{L^{\infty}(D)}, E, \rho, q\right)$ such that

$$
\begin{equation*}
\overline{\mathcal{N}}=\mathcal{N}, \text { i.e. } \mathcal{N} \text { is symmetric with respect to } \mathbb{R} \tag{125}
\end{equation*}
$$

$$
h_{s}(\cdot, \cdot, E) \in C(\partial \Lambda \times \partial \Lambda), u_{E, \rho, s} \in L_{p, 2}(\mathbb{C}), 2<p<4
$$

$$
\begin{equation*}
\text { with holomorphic dependence on } s \in \mathcal{N} \tag{126}
\end{equation*}
$$

where $u_{E, \rho, s}$ is defined by (45) with $b_{s}$ in place of $b$.
Next, we consider $e_{s}, X_{1, s}, X_{2, s}, \Omega_{1, s}, \Omega_{2, s}$ defined via (67), (68), (69), (70) with $r_{s}$ in place of $r$, where $r_{s}$ is defined by (64) with $b_{s}$ in place of $b$, where $s \in]-s_{1}, s_{1}\left[\right.$. And we consider $e_{s, \sigma}^{ \pm}, X_{j, s, \sigma}^{ \pm}, j=1,2$, defined via the following systems of equations:

$$
\begin{gather*}
e_{s, \sigma}^{+}(\lambda)=1-\frac{1}{\pi} \int_{\mathbb{C}} r_{s}(x, \zeta, E) e_{s, \sigma}^{-}(\zeta) \frac{d \Re \zeta d \Im \zeta}{\zeta-\lambda}, \\
e_{s, \sigma}^{-}(\lambda)=1-\frac{1}{\pi} \int_{\mathbb{C}} \overline{r_{\bar{\sigma}}(x, \zeta, E)} e_{s, \sigma}^{+}(\zeta) \frac{d \Re \zeta \Im \Im \zeta}{\bar{\zeta}-\bar{\lambda}},  \tag{127}\\
X_{1, s, \sigma}^{+}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} r_{s}(x, \eta, E) X_{1, s, \sigma}^{-}(\eta, \zeta) \frac{d \Re \zeta d \Im \zeta}{\eta-\lambda}=\frac{1}{2(\zeta-\lambda)}, \\
X_{1, s, \sigma}^{-}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} \frac{r_{\bar{\sigma}}(x, \eta, E)}{} X_{1, s, \sigma}^{+}(\eta, \zeta) \frac{d \Re \zeta d \Im \zeta}{\bar{\eta}-\bar{\lambda}}=\frac{1}{2(\bar{\zeta}-\bar{\lambda})}, \tag{128}
\end{gather*}
$$

$$
\begin{align*}
& X_{2, s, \sigma}^{+}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} r_{s}(x, \eta, E) X_{2, s, \sigma}^{-}(\eta, \zeta) \frac{d \Re \zeta d \Im \zeta}{\eta-\lambda}=\frac{1}{2 i(\zeta-\lambda)} \\
& X_{2, s, \sigma}^{-}(\lambda, \zeta)+\frac{1}{\pi} \int_{\mathbb{C}} \overline{r_{\bar{\sigma}}(x, \eta, E)} X_{2, s, \sigma}^{+}(\eta, \zeta) \frac{d \Re \zeta d \Im \zeta}{\bar{\eta}-\bar{\lambda}}=\frac{-1}{2 i(\bar{\zeta}-\bar{\lambda})} \tag{129}
\end{align*}
$$

where $s, \sigma \in \mathcal{N}, r_{s}$ is defined by (64) with $b_{s}$ in place of $b$. In addition, we consider also

$$
\begin{align*}
& \Omega_{1, s, \sigma}(\lambda, \zeta):=X_{1, s, \sigma}^{+}(\lambda, \zeta)+i X_{2, s, \sigma}^{+}(\lambda, \zeta) \\
& \Omega_{2, s, \sigma}(\lambda, \zeta):=X_{1, s, \sigma}^{+}(\lambda, \zeta)-i X_{2, s, \sigma}^{+}(\lambda, \zeta) \tag{130}
\end{align*}
$$

where $\lambda, \zeta \in \mathbb{C}, s, \sigma \in \mathcal{N}$.
Let

$$
\begin{equation*}
S:=\{(s, \sigma) \in \mathcal{N} \times \mathcal{N}: \sigma=s \in]-s_{1}, s_{1}[ \} \tag{131}
\end{equation*}
$$

Using considerations of Section 9 of Chapter 3 of [25], one can show that systems (127), (128), (129) for $e_{s, \sigma}^{ \pm}, X_{j, s, \sigma}^{ \pm}, j=1,2$, for $(s, \sigma) \in S$, are reduced to the equations for $\left.e_{s}, X_{j, s}, j=1,2, s \in\right]-s_{1}, s_{1}[$, are uniquely solvable in $L_{0}^{q}(\mathbb{C}), p /(p-1) \leq q<2$, where $p$ is the number (126). In addition:

$$
\begin{equation*}
e_{s}=e_{s, s}^{+}, \quad \overline{e_{s}}=e_{s, s}^{-}, \quad X_{j, s}=X_{j, s, s}^{+}, \quad \bar{X}_{j, s}=X_{j, s, s}^{-}, \quad \Omega_{j, s}=\Omega_{j, s, s} \tag{132}
\end{equation*}
$$

where $j=1,2, s \in]-s_{1}, s_{1}[$.
Using the definition of $r_{s}$ and holomorphic dependence of $u_{E, \rho, s}$ on $s \in \mathcal{N}$ in (126) one can show that

$$
\begin{array}{r}
r_{s}(x, \cdot, E) \in L_{p, 2}(\mathbb{C}), \quad \overline{r_{\bar{\sigma}}(x, \cdot, E)} \in L_{p, 2}(\mathbb{C}), \quad 2<p<4,  \tag{133}\\
\text { with holomorphic dependence on } s, \sigma \in \mathcal{N},
\end{array}
$$

for fixed $x \in \mathbb{R}^{2}, E>0$.
Proceeding from these results and from properties of the integral operators in (127) -(129) (presented in [25]), one can show that there is an open neighbourhood $\mathcal{S}_{x}$ of $S$ in $\mathcal{N} \times \mathcal{N}$ (where $\mathcal{S}_{x}$ depends also on $\left.v, E, \rho\right)$ such that:

> systems $(127),(128),(129)$ for $e_{s, \sigma}^{ \pm}, X_{j, s, \sigma}^{ \pm}, j=1,2$, are uniqely
> solvable in $L_{q, 0}(\mathbb{C}), p /(p-1) \leq q<2$, for $(s, \sigma) \in \mathcal{S}_{x}$

$$
\begin{array}{r}
e_{s, \sigma}^{+} \in C(\mathbb{C}), \Omega_{j, s, \sigma} \in C\left(\mathbb{C}^{2} \backslash \mathbb{C}_{\varepsilon}\right), j=1,2, \text { for any } \varepsilon>0  \tag{135}\\
\text { with holomorphic dependence on }(s, \sigma) \in \mathcal{S}_{x}
\end{array}
$$

where $\mathbb{C}_{\varepsilon}$ is defined in item (v) in the proof of property (93);

$$
\begin{equation*}
\left|\Omega_{1, s, \sigma}(\lambda, \zeta)-\frac{1}{\zeta-\lambda}\right|<\frac{c_{3}(s, \sigma, p)}{|\zeta-\lambda|^{2 / p}}, \quad\left|\Omega_{2, s, \sigma}(\lambda, \zeta)\right|<\frac{c_{3}(s, \sigma, p)}{|\zeta-\lambda|^{2 / p}} \tag{136}
\end{equation*}
$$

where $c_{3}$ depends continuously on $(s, \sigma) \in \mathcal{S}_{x}$ and depends also on $v$.
Let

$$
\begin{equation*}
\mathcal{N}_{x}:=\left\{s \in \mathcal{N}:(s, s) \in \mathcal{S}_{x}\right\}, x \in \mathbb{R}^{2} \tag{137}
\end{equation*}
$$

One can see that $\mathcal{N}_{x}$ is an open neighbourhood of the real interval ] $-s_{1}, s_{1}[$ in $\mathbb{C}$.

We consider

$$
\begin{align*}
A_{1, s}(\lambda, \zeta) & =A_{1, s}(x, \lambda, \zeta, E) \\
& =\frac{1}{2 \pi i} \int_{\partial \Lambda} R_{s}\left(x, \lambda, \lambda^{\prime}, E\right) \Omega_{1, s, s}\left(\lambda^{\prime}\left(1-0\left(\left|\lambda^{\prime}\right|-1\right)\right), \zeta\right) d \lambda^{\prime}  \tag{138}\\
A_{2, s}(\lambda, \zeta) & =A_{2, s}(x, \lambda, \zeta, E) \\
& =\frac{-1}{2 \pi i} \int_{\partial \Lambda} R_{s}\left(x, \lambda, \lambda^{\prime}, E\right) \Omega_{2, s, s}\left(\lambda^{\prime}, \zeta\right) d \lambda^{\prime}
\end{align*}
$$

$\lambda, \zeta \in \partial \Lambda$, where $R_{s}$ is defined by (65) with $h_{s}$ in place of $h, \Omega_{1, s, \sigma}, \Omega_{2, s, \sigma}$ are the functions of (130), (135), (136), $\lambda, \zeta \in \partial \Lambda, s \in \mathcal{N}_{x}$.

We consider also

$$
\begin{equation*}
\widetilde{A}_{j, s}:=\overline{A_{j, s}}, j=1,2, s \in \overline{\mathcal{N}_{x}} \tag{139}
\end{equation*}
$$

Using (126) for $h_{s}$ and (135), (136) for $\Omega_{j, s, s}, j=1,2$, we obtain

$$
\begin{align*}
& A_{j, s} \in L_{2}(\partial \Lambda \times \partial \Lambda), \quad j=1,2  \tag{140}\\
& \quad \text { with holomorphic dependence on } s \in \mathcal{N}_{x} .
\end{align*}
$$

Using (139), (140) we also obtain

$$
\begin{align*}
& \widetilde{A}_{j, s} \in L_{2}(\partial \Lambda \times \partial \Lambda), j=1,2 \\
& \quad \text { with holomorphic dependence on } s \in \overline{\mathcal{N}_{x}} . \tag{141}
\end{align*}
$$

We consider $A(x, s)$, where $s \in \mathcal{N}_{x} \cap \overline{\mathcal{N}_{x}}$, defined using (8.14), (8.15) in a similar way with $A(x, s)$ for $s \in]-s_{1}, s_{1}\left[\right.$, but with $\widetilde{A}_{j, s}$ in place of $A(x, s)$. Finally, we consider $\operatorname{det} A(x, s)$ for $s \in \mathcal{N}_{x} \cap \overline{\mathcal{N}_{x}}$.

Using (132) for $\Omega_{j, s, s},(140)$, (141), we obtain that

$$
\begin{equation*}
\operatorname{det} A(x, s) \text { is holomorphic in } s \in \mathcal{N}_{x} \cap \overline{\mathcal{N}_{x}} \text { for fixed } x \in \mathbb{R}^{2} . \tag{142}
\end{equation*}
$$

Property (142) implies property (94).

### 8.2. Proof of Proposition 4.5

Let $v$ be as in Remark 4.6 and let $\operatorname{det} A(x, s)$ be defined like in Lemma 4.4.

Let

$$
\begin{align*}
& Z:=\left\{(x, s) \in \mathbb{R}^{2} \times\right]-s_{1}, s_{1}[: \operatorname{det} A(x, s)=0\} \\
& Z_{x}:=\{s \in]-s_{1}, s_{1}[: \operatorname{det} A(x, s)=0\}, x \in \mathbb{R}^{2}  \tag{143}\\
& \left.Z_{s}:=\left\{x \in \mathbb{R}^{2}: \operatorname{det} A(x, s)=0\right\}, s \in\right]-s_{1}, s_{1}[
\end{align*}
$$

Using (92), (94), we obtain that $Z_{x}$ is a discrete set (maybe empty) without interior accumulation points in interval $]-s_{1}, s_{1}[$. Therefore, we have, in particular, that

$$
\begin{equation*}
\text { Meas } \left.\mathrm{Z}=0 \text { in } \mathbb{R}^{2} \times\right]-\mathrm{s}_{1}, \mathrm{~s}_{1}[. \tag{144}
\end{equation*}
$$

As a corollary,

$$
\begin{equation*}
\text { Meas } \left.Z_{s}=0 \text { in } \mathbb{R}^{2} \text { for almost each } \mathrm{s} \in\right]-\mathrm{s}_{1}, \mathrm{~s}_{1}[\text {. } \tag{145}
\end{equation*}
$$

Property (145) implies the result of Proposition 4.5 interpreted according to Remark 4.6.

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# On characterization of inverse data in the boundary control method 

Mikhail I. Belishev and Aleksei F. Vakulenko

Dedicated to the 60-th jubilee of Giovanni Alessandrini
Abstract. We deal with a dynamical system

$$
\begin{array}{ll}
u_{t t}-\Delta u+q u=0 & \text { in } \Omega \times(0, T) \\
\left.u\right|_{t=0}=\left.u_{t}\right|_{t=0}=0 & \text { in } \bar{\Omega} \\
\partial_{\nu} u=f & \text { in } \partial \Omega \times[0, T]
\end{array}
$$

where $\Omega \subset \mathbf{R}^{n}$ is a bounded domain, $q \in L_{\infty}(\Omega)$ is a real-valued function, $\nu$ is the outward normal to $\partial \Omega, u=u^{f}(x, t)$ is a solution. The input/output correspondence is realized by a response operator $R^{T}:\left.f \mapsto u^{f}\right|_{\partial \Omega \times[0, T]}$ and its relevant extension by hyperbolicity $R^{2 T}$. The operator $R^{2 T}$ is determined by $\left.q\right|_{\Omega^{T}}$, where $\Omega^{T}:=\{x \in$ $\Omega \mid \operatorname{dist}(x, \partial \Omega)<T\}$. The inverse problem is: Given $R^{2 T}$ to recover $q$ in $\Omega^{T}$. We solve this problem by the boundary control method and describe the necessary and sufficient conditions on $R^{2 T}$, which provide its solvability.

Keywords: determination of potential via time-domain boundary measurements, characterization of inverse data.
MS Classification 2010: 35R30, 35L05.

## 1. Introduction

## Motivation

The problem, which the paper is devoted to, was solved about 20 years ago by the BC-method, which is an approach to inverse problems (IPs) based on their relations to control and system theory $[1,3,5]$. However, in the IP-community, there are a few versions of what 'to solve an inverse problem' means. The versions may be ordered by levels as follows:

1. to establish the injectivity of the correspondence 'parameters under reconstruction $\rightarrow$ inverse data', what allows one to claim that the data determine
the parameters
2. to prove the relevant continuity of this correspondence, and thus to show that the determination is stable
3. to elaborate an efficient (preferably, realizable numerically) procedure, which determines the parameters from the data ${ }^{1}$
4. to provide a data characterization, i.e., describe the necessary and sufficient conditions on the data, which ensure solvability of the given inverse problem.
Typically, $\{i+1\}$-th level is stronger and richer in content than $i$-th one. Respectively, to reach the next level (especially, in multidimensional IPs) is more difficult. The BC-method firmly keeps level 3 (see $[3,6]$ ). In the mean time, it provides data characterization in important one-dimensional problems: see $[7,8]$.

Regarding level 4 in multidimensional IPs, there is substantial gap between the frequency-domain and time-domain problems. In the first ones, the results on the data characterization are much more promoted and successful (see [14, $17,20,21]$ and other). In time-domain problems, such results also do exist (see, e.g., [22]) but are not so deep and systematic. Our paper is an attempt to reduce the above-mentioned gap by the use of the BC-method.

## Contents and results

- We develop a general approach proposed in [2] and apply it to a concrete time-domain inverse problem for the wave equation with a potential. The approach elaborates the well-known and deep relations between inverse problems and triangular factorization of operators in the Hilbert space [1, 2, 9, 14].
- In sections 2 and 3, a forward problem is considered. With the problem one associates a relevant dynamical system. The system is endowed with standard control theory attributes: spaces and operators. In particular, a so-called extended response operator $R^{2 T}$ is introduced. It realizes the input/state correspondence and later on plays a role of the data in the inverse problem. The key property of the system is a local boundary controllability, which is relayed upon the fundamental Holmgren-John-Tataru uniqueness theorem [23]. It plays a crucial role in all versions of the BC-method.

Geometrical Optics (GO) describes propagation of wave field jumps in the system. A noticeable fact is that the GO-formulas are well interpreted in operator theory terms: they provide existence of a diagonal of the control operator and time derivative composition.

- In section 4, we present a BC-procedure, which recovers the potential from the given $R^{2 T}$. Then we prove Theorem 4.2, which is the main result. It

[^0]provides a list of necessary and sufficient conditions on an operator $\mathcal{R}^{2 T}$ to be an extended response operator.

The necessity is simple: the proof just summarizes the properties of $R^{2 T}$ stated in the forward problem. The sufficiency is richer in content. The proof is constructive: we start with an operator $\mathcal{R}^{2 T}$ obeying all the conditions, and construct a system with the response operator $R^{2 T}=\mathcal{R}^{2 T}$. In construction we follow the BC-procedure, which solves the IP.

In conclusion (section 5), a self-critical discussion of the obtained results is provided.

## 2. Geometry

All functions, function classes and spaces are real.

## Domain and subdomains

Let $\Omega \subset \mathbf{R}^{n}$ be a bounded domain with the boundary $\Gamma \in C^{\infty}$. By $\mathrm{d}(a, b)$ we denote an intrinsic distance in $\Omega$, which is defined via the length of smooth curves lying in $\bar{\Omega}$ and connecting $a$ with $b$.

For a subset $A \subset \bar{\Omega}$, we denote its metric neighborhoods by

$$
\Omega_{A}^{r}:=\{x \in \Omega \mid \mathrm{d}(x, A)<r\}, \quad r>0 .
$$

For $A=\Gamma$, we set $\Omega^{r}:=\Omega_{\Gamma}^{r}$. Later on, in dynamics, the value

$$
T_{*}:=\max _{\Omega} \tau(\cdot)=\inf \left\{r>0 \mid \Omega^{r}=\Omega\right\}
$$

is interpreted as a time needed for the waves moving from $\Gamma$ with the unit speed to fill $\Omega$.

A function $\tau(\cdot):=\mathrm{d}(\cdot, \Gamma)$ on $\bar{\Omega}$ is called an eikonal. By the definitions, we have $\Omega^{r}=\{x \in \Omega \mid \tau(x)<r\}$. In dynamics, the eikonal level sets

$$
\Gamma^{s}:=\{x \in \Omega \mid \tau(x)=s\}, \quad s \geq 0
$$

play the role of the forward fronts of waves moving from $\Gamma$.

## Semi-geodesic coordinates

- Here we introduce a separation set (cut locus) of $\Omega$ with respect to $\Gamma$ (see, e.g, [16]) and use one of its equivalent definitions.

A point in $\Omega$ is said to be multiple if it is connected with $\Gamma$ through more than one shortest geodesics (straight lines in $\mathbf{R}^{n}$ ). Denote by $c_{0}$ the set of multiple points and define

$$
c:=\bar{c}_{0} .
$$

The set $c$ is called a cut locus. It is 'small':

$$
\begin{equation*}
\operatorname{vol} c=0 \tag{1}
\end{equation*}
$$

and separated from the boundary:

$$
0<T_{c}:=\mathrm{d}(c, \Gamma) \leq T_{*}
$$

In addition, note that $\Gamma^{s} \backslash c$ is a smooth (may be, disconnected) hypersurface in $\Omega$. If $s<T_{c}$ then $\Gamma^{s}$ is smooth and diffeomorphic to $\Gamma$.

- For any $x \in \bar{\Omega} \backslash c$, there is a unique point $\gamma(x) \in \Gamma$ nearest to $x$. For such an $x$, a pair $(\gamma(x), \tau(x))$ determines its position in $\Omega$ and is said to be the semi-geodesic coordinates $(\mathrm{sgc})$. By $x(\gamma, \tau)$ we denote a point in $\bar{\Omega} \backslash c$ with the given $\operatorname{sgc}(\gamma, \tau)$.

In sgc, $\mathbf{R}^{n}$-volume element in $\Omega$ takes the well-known form

$$
\begin{equation*}
d x=\beta(\gamma, \tau) d \Gamma d \tau \tag{2}
\end{equation*}
$$

where $d \Gamma$ is Euclidean surface element on the boundary. Factor $\beta$ is a Jacobian of the passage from Cartesian coordinates to sgc.

- Denote $\Sigma^{T}:=\Gamma \times[0, T)$. A set

$$
\Theta:=\{(\gamma(x), \tau(x)) \mid x \in[\Omega \cup \Gamma] \backslash c\} \subset \Sigma^{T_{*}}
$$

is called a pattern of $\Omega$. Also, we use its parts

$$
\Theta^{T}:=\left\{(\gamma(x), \tau(x)) \mid x \in\left[\Omega^{T} \cup \Gamma\right] \backslash c\right\}=\Theta \cap \Sigma^{T}, \quad T>0 .
$$

For $T<T_{c}$, one has $\Theta^{T}=\Sigma^{T}$.

## Images

Fix a positive $T \leq T_{*}$; let $y$ be a function on $\Omega^{T} \cup \Gamma$. A function on $\Sigma^{T}$ of the form

$$
\tilde{y}^{T}(\gamma, \tau):= \begin{cases}\beta^{\frac{1}{2}}(\gamma, \tau) y(x(\gamma, \tau)), & (\gamma, \tau) \in \Theta^{T} \\ 0, & (\gamma, \tau) \in \Sigma^{T} \backslash \Theta^{T}\end{cases}
$$

is said to be an image of $y$. So, up to the factor $\beta^{\frac{1}{2}}$, image is just a function written in sgc.

An image operator $I^{T}: L_{2}\left(\Omega^{T}\right) \rightarrow L_{2}\left(\Sigma^{T}\right), I^{T} y:=\tilde{y}^{T}$ is isometric. Indeed, for $y, v \in L_{2}\left(\Omega^{T}\right)$ one has

$$
\begin{aligned}
&(y, v)_{L_{2}\left(\Omega^{T}\right)}=\int_{\Omega^{T}} y(x) v(x) d x \stackrel{(1),(2)}{=} \int_{\Theta^{T}} y(x(\gamma, \tau)) v(x(\gamma, \tau)) \beta(\gamma, \tau) d \Gamma d \tau \\
&=\left(\tilde{y}^{T}, \tilde{v}^{T}\right)_{L_{2}\left(\Sigma^{T}\right)}=\left(I^{T} y, I^{T} v\right)_{L_{2}\left(\Sigma^{T}\right)}
\end{aligned}
$$

As an isometry, $I^{T}$ obeys $\operatorname{Ran} I^{T}=\left\{g \in L_{2}\left(\Sigma^{T}\right) \mid \operatorname{supp} g \subset \overline{\Theta^{T}}\right\}$ and

$$
\begin{equation*}
\left(I^{T}\right)^{*} I^{T}=\mathbf{1}, \quad I^{T}\left(I^{T}\right)^{*}=G_{\Theta^{T}} \tag{3}
\end{equation*}
$$

where $G_{\Theta^{T}}$ cuts off functions in $\Sigma^{T}$ onto $\Theta^{T}$.

## 3. Dynamics

### 3.1. IBV problem

By $\partial_{\nu}$ we denote a derivative with respect to outward normal at the boundary Г. $H^{s}(\ldots)$ are the standard Sobolev spaces.

Consider an initial boundary-value problem

$$
\begin{array}{ll}
u_{t t}-\Delta u+q u=0 & \text { in } \Omega \times(0, T) \\
\left.u\right|_{t=0}=\left.u_{t}\right|_{t=0}=0 & \text { in } \bar{\Omega} \\
\partial_{\nu} u=f & \text { on } \overline{\Sigma^{T}} \tag{6}
\end{array}
$$

where $q \in L_{\infty}(\Omega)$ is a function (potential), $f$ is a Neumann boundary control, $u=u^{f}(x, t)$ is a solution (wave). It is a well-posed problem; its solution possesses the following properties.

- Regularity. The map $f \mapsto u^{f}$ is continuous from $L_{2}\left(\Sigma^{T}\right)$ to $C\left([0, T] ; H^{\frac{3}{5}-\varepsilon}(\Omega)\right)$, whereas $\left.f \mapsto u^{f}\right|_{\Sigma^{T}}$ acts continuously from $L_{2}\left(\Sigma^{T}\right)$ to $H^{\frac{1}{5}-2 \varepsilon}\left(\Sigma^{T}\right) \quad(\forall \varepsilon>0)$. Introduce a 'smooth' class of controls

$$
\mathcal{M}^{T}:=\left\{f \in H^{2}\left(\Sigma^{T}\right) \mid \operatorname{supp} \mathrm{f} \subset \Gamma \times(0, \mathrm{~T}]\right\}
$$

and note that each $f \in \mathcal{M}^{T}$ vanishes near $t=0$. For $f \in \mathcal{M}^{T}$ one has $u^{f} \in H^{2}(\Omega \times[0, T])$. These facts are taken from [19] (Theorem A).

- Locality. For the hyperbolic equation (4), the finiteness of the domain of influence principle holds and implies the following.

Let $\sigma \subset \Gamma$ be an open set. Take a control acting from $\sigma$, i.e., provided $\operatorname{supp} f \subset \bar{\sigma} \times[0, T]$. Then the relation

$$
\begin{equation*}
\operatorname{supp} u^{f}(\cdot, t) \subset \overline{\Omega_{\sigma}^{t}}, \quad t \geq 0 \tag{7}
\end{equation*}
$$

holds and shows that the waves propagate with the unit speed and fill the proper metric neighborhood of $\sigma$ in $\Omega$.

By the latter, solution $u^{f}$ depends on the potential locally that enables one to restate the problem (4)-(6) as follows:

$$
\begin{array}{ll}
u_{t t}-\Delta u+q u=0 & \text { in } \Omega^{T} \times(0, T) \\
\left.u\right|_{t<\tau(x)}=0 & \text { in } \overline{\Omega^{T}} \times[0, T] \\
\partial_{\nu} u=f & \text { on } \overline{\Sigma^{T}} . \tag{10}
\end{array}
$$

Such a form emphasizes that $u^{f}$ is determined by behavior of potential $q$ in $\Omega^{T}$ only (does not depend on $\left.q\right|_{\Omega \backslash \Omega^{T}}$ ) that enables one to analyze wave propagation without leaving $\Omega^{T}$.

- Steady-state property. Introduce a delay operator $\mathcal{T}_{T-\xi}^{T}$ acting on controls by the rule

$$
\left(\mathcal{T}_{T-\xi}^{T} f\right)(\cdot, t):=\left\{\begin{array}{ll}
0, & 0 \leq t<T-\xi \\
f(\cdot, t-(T-\xi)), & T-\xi \leq t \leq T
\end{array} \quad 0 \leq t \leq T\right.
$$

Since the operator $-\Delta+q$, which governs the evolution of waves, does not depend on time, one has

$$
\begin{align*}
& u^{\mathcal{T}_{T-\xi}^{T} f}(\cdot, T)=u^{f}(\cdot, \xi), \quad 0 \leq \xi \leq T \\
& u^{f_{t}}=u_{t}^{f}, u^{f_{t t}}=u_{t t}^{f} \stackrel{(4)}{=}(\Delta-q) u^{f} \quad \text { for } f \in \mathcal{M}^{T} \tag{11}
\end{align*}
$$

where the first relation implies the others.

### 3.2. System $\alpha^{T}$

Here we consider problem (8)-(10) as a dynamical system, name it by $\alpha^{T}$, and endow with standard attributes of control and system theory: spaces and operators.

## Spaces and subspaces

A space of controls $\mathcal{F}^{T}:=L_{2}\left(\Sigma^{T}\right)$ is called an outer space of the system. It contains an increasing family of subspaces, which consist of the delayed controls:

$$
\mathcal{F}^{T, \xi}:=\left\{f \in \mathcal{F}^{T} \mid \operatorname{supp} f \subset \Gamma \times[T-\xi, T]\right\}=\mathcal{T}_{T-\xi}^{T} \mathcal{F}^{T}, \quad 0 \leq \xi \leq T
$$

With an open $\sigma \subset \Gamma$ one associates the subspaces of controls

$$
\mathcal{F}_{\sigma}^{T, \xi}:=\left\{f \in \mathcal{F}^{T} \mid \operatorname{supp} f \subset \bar{\sigma} \times[T-\xi, T]\right\}, \quad 0 \leq \xi \leq T
$$

which act from $\sigma$.
A space $\mathcal{H}^{T}=L_{2}\left(\Omega^{T}\right)$ is said to be inner; waves $u^{f}(\cdot, t)$ are regarded as its elements (states) depending on time. It contains an increasing family of subspaces

$$
\mathcal{H}^{\xi}:=\left\{y \in \mathcal{H}^{T} \mid \operatorname{supp} y \subset \overline{\Omega^{T}}\right\}, \quad 0 \leq \xi \leq T
$$

Also, with $\sigma \subset \Gamma$ we associate the subspaces

$$
\mathcal{H}_{\sigma}^{\xi}:=\left\{y \in \mathcal{H}^{T} \mid \operatorname{supp} y \subset \overline{\Omega_{\sigma}^{T}}\right\}, \quad 0 \leq \xi \leq T
$$

By locality property (7) and the first relation in (11), if $f \in \mathcal{F}_{\sigma}^{T, \xi}$ then $u^{f}(\cdot, T) \in \mathcal{H}_{\sigma}^{\xi}$.

## Control operator

- In system $\alpha^{T}$, an input/state correspondence is realized by a control operator $W^{T}: \mathcal{F}^{T} \rightarrow \mathcal{H}^{T}$

$$
W^{T} f:=u^{f}(\cdot, T)
$$

By the above mentioned regularity properties of solutions to (4)-(6), it acts continuously from $\mathcal{F}^{T}$ to $H^{\frac{3}{5}-\varepsilon}(\Omega)$. Hence, for any $T>0, W^{T}$ is a compact operator.

Lemma 3.1. For $T<T_{*}$, the control operator is injective: $\operatorname{Ker} W^{T}=\{0\}$.
Proof. Let $T<T_{*}$, so that $\Omega \backslash \overline{\Omega^{T}}$ is an open set. Let $f \in \operatorname{Ker} W^{T}=\{0\}$, so that $u^{f}(\cdot, T)=0$. Define a function $U$ in $\Omega \times \mathbf{R}$ by

$$
U(\cdot, t):= \begin{cases}0, & -\infty<t<0 \\ u^{f}(\cdot, t), & 0 \leq t \leq T \\ -u^{f}(\cdot, 2 T-t), & T \leq t \leq 2 T \\ 0, & -\infty<t<0\end{cases}
$$

Owing to $u^{f}(\cdot, T)=0$, such an extension of $u^{f}$ does not violate its regularity. As a consequence, the extension satisfies

$$
U_{t t}-\Delta U+q U=0 \quad \text { in } \Omega \times \mathbf{R},\left.\quad U(\cdot, t)\right|_{\Omega \backslash \Omega^{T}}=0
$$

Applying the Fourier transform $U(\cdot, t) \mapsto \check{U}(\cdot, \omega)$, we get

$$
-\omega^{2} \check{U}-\Delta \check{U}+q \check{U}=0 \quad \text { in } \Omega,\left.\quad \check{U}(\cdot, \omega)\right|_{\Omega \backslash \Omega^{T}}=0
$$

Thus, for any $\omega \in \mathbf{R}, \check{U}(\cdot, \omega)$ satisfies an elliptic equation and vanishes on an open set. By the well-known uniqueness theorem, the latter implies $\check{U}(\cdot, \omega)=0$ everywhere in $\Omega$. Returning to the Fourier original, we get $U(\cdot, t)=0$ for all $t$ and arrive at $f=\left.\partial_{\nu} u^{f}\right|_{\Sigma^{T}}=\left.\partial_{\nu} U\right|_{\Sigma^{T}}=0$. Thus, $f \in \operatorname{Ker} W^{T}$ implies $f=0$.

- The locality property (7) and delay relation (11) lead to the embedding

$$
\begin{equation*}
W^{T} \mathcal{F}_{\sigma}^{T, \xi} \subset \mathcal{H}_{\sigma}^{\xi}, \quad 0 \leq \xi \leq T \tag{12}
\end{equation*}
$$

which is just a consequence of the finiteness of the wave propagation speed. The fact, which plays a crucial role in the BC-method, is that this embedding is dense: the relation

$$
\begin{equation*}
\overline{W^{T} \mathcal{F}_{\sigma}^{T, \xi}}=\mathcal{H}_{\sigma}^{\xi}, \quad 0 \leq \xi \leq T \tag{13}
\end{equation*}
$$

is valid for any $T>0$ and open $\sigma \subseteq \Gamma$. In control theory this fact is referred to as a local approximate boundary controllability of system $\alpha^{T}$; it is derived from the fundamental Holmgren-John-Tataru uniqueness theorem [1, 23].

- The following fact will be required in the data characterization. A multiplication of functions by a bounded $q$ is a self-adjoint bounded operator acting in $\mathcal{H}^{T}$. The last relation in (11) can be written as $\Delta W^{T} f-W^{T} f_{t t}=q W^{T} f$ that is just a form of writting the wave equation (8). Taking into account the density of $\mathcal{M}^{T}$ in $\mathcal{F}^{T}$, it is easy to conclude that a set of pairs

$$
\begin{equation*}
\left\{\left\langle\Delta W^{T} f-W^{T} f_{t t}, W^{T} f\right\rangle \mid f \in \mathcal{M}^{T}\right\} \tag{14}
\end{equation*}
$$

determines the graph of the multiplication by $q$ and, hence, determines the potential $\left.q\right|_{\Omega^{T}}$.

## Response operators

- In system $\alpha^{T}$, the input/output correspondence is realized by a response operator $R^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
R^{T} f:=\left.u^{f}\right|_{\Sigma^{T}}
$$

By the above-mentioned regularity of $u^{f}$, it acts continuously from $\mathcal{F}^{T}$ to $H^{\frac{1}{5}-2 \varepsilon}\left(\Sigma^{T}\right)$ and, hence, is a compact operator. The following is some of its basic properties. We use the auxiliary operators $Y^{T}, J^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
\left(Y^{T} f\right)(\cdot, t):=f(\cdot, T-t), \quad\left(J^{T} f\right)(\cdot, t):=\int_{0}^{t} f(\cdot, s) d s, \quad 0 \leq t \leq T
$$

Note that $\left(Y^{T}\right)^{*}=\left(Y^{T}\right)^{-1}=Y^{T}$ and $\left(Y^{T}\right)^{2}=\mathbf{1}$ holds.
Lemma 3.2. For $T>0$ and $0 \leq \xi \leq T$, the relations

$$
\begin{equation*}
R^{T} \mathcal{T}_{T-\xi}^{T}=\mathcal{T}_{T-\xi}^{T} R^{T} ; \quad R^{T} J^{T}=J^{T} R^{T} ; \quad\left(Y^{T} R^{T}\right)^{*}=Y^{T} R^{T} \tag{15}
\end{equation*}
$$

are valid.
Proof. The first relation follows from (11). The second is a simple consequence of the first. Prove the third one.

Let controls $f, g$ belong to the smooth class $\mathcal{M}^{T}$, which is dense in $\mathcal{F}^{T}$. Cauchy conditions (9) imply

$$
\left.u^{f}(\cdot, t)\right|_{t=0}=\left.u_{t}^{f}(\cdot, t)\right|_{t=0}=\left.u^{g}(\cdot, T-t)\right|_{t=T}=\left.u_{t}^{g}(\cdot, T-t)\right|_{t=T}=0
$$

Also, since each $f \in \mathcal{M}^{T}$ vanishes near $t=0$, the wave $u^{f}(\cdot, T)$ vanishes near $\Gamma^{T}$ by locality (7).

Integrating by parts, one has

$$
\begin{aligned}
& 0= \int_{\Omega^{T} \times[0, T]}\left[u_{t t}^{f}-\Delta u^{f}+q u^{f}\right](x, t) u^{g}(x, T-t) d x d t= \\
&= \int_{\Sigma^{T}}\left[u^{f}(\gamma, t) \partial_{\nu} u^{g}(\gamma, T-t)-\partial_{\nu} u^{f}(\gamma, t) u^{g}(\gamma, T-t)\right] d \Gamma d t+ \\
&+\int_{\Omega^{T} \times[0, T]} u^{f}(x, t)\left[u_{t t}^{g}-\Delta u^{g}+q u^{g}\right](x, T-t) d x d t= \\
& \stackrel{(10)}{=} \int_{\Sigma^{T}}\left[u^{f}(\gamma, t) g(\gamma, T-t)-f(\gamma, t) u^{g}(\gamma, T-t)\right] d \Gamma d t= \\
&=\left(R^{T} f, Y^{T} g\right)_{\mathcal{F}^{T}}-\left(f, Y^{T} R^{T} g\right)_{\mathcal{F}^{T}}=\left(Y^{T} R^{T} f, g\right)_{\mathcal{F}^{T}}-\left(f, Y^{T} R^{T} g\right)_{\mathcal{F}^{T}}
\end{aligned}
$$

Thus, we have $\left(Y^{T} R^{T} f, g\right)_{\mathcal{F}^{T}}=\left(f, Y^{T} R^{T} g\right)_{\mathcal{F}^{T}}$. Since $\mathcal{M}^{T}$ is dense in $\mathcal{F}^{T}$, we get the last equality in (15).

- There is one more object of system $\alpha^{T}$ related with the input/output correspondence.

Denote $D^{2 T}:=\operatorname{in}\left\{(x, t) \mid x \in \Omega^{T}, t<2 T-\tau(x)\right\}$. The problem

$$
\begin{array}{ll}
u_{t t}-\Delta u+q u=0 & \text { in } D^{2 T} \\
\left.u\right|_{t<\tau(x)}=0 & \text { in } \overline{D^{2 T}} \\
\partial_{\nu} u=f & \text { on } \overline{\Sigma^{2 T}} \tag{18}
\end{array}
$$

can be regarded as a natural extension of problem (8)-(10). Such an extension does exist and is well posed owing to the finiteness of the domains of influence (hyperbolicity). Its solution $u^{f}$ is determined by $\left.q\right|_{\Omega^{T}}$.

With problem (16)-(18) one associates an extended response operator $R^{2 T}$ : $\mathcal{F}^{2 T} \rightarrow \mathcal{F}^{2 T}$,

$$
R^{2 T} f:=\left.u^{f}\right|_{\Sigma^{2 T}}
$$

It is a compact operator with the properties quite analogous to (15):

$$
\begin{align*}
& R^{2 T} \mathcal{T}_{2 T-\xi}^{2 T}=\mathcal{T}_{2 T-\xi}^{2 T} R^{2 T}, \quad 0 \leq \xi \leq 2 T ; \quad R^{2 T} J^{2 T}=J^{2 T} R^{2 T} \\
& \left(Y^{2 T} R^{2 T}\right)^{*}=Y^{2 T} R^{2 T} \tag{19}
\end{align*}
$$

Along with the solution $u^{f}$, operator $R^{2 T}$ is determined by $\left.q\right|_{\Omega^{T}}$. By the latter, this operator must be regarded as an intrinsic object of system $\alpha^{T}$ (but not $\left.\alpha^{2 T}\right)$. Note in addition that $R^{2 T}$ is meaningful at a very general level: see [2].

## Connecting operator

- A key object of the BC-method is a connecting operator $C^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
\begin{equation*}
C^{T}:=\left(W^{T}\right)^{*} W^{T} \tag{20}
\end{equation*}
$$

By the definition, we have

$$
\left(C^{T} f, g\right)_{\mathcal{F}^{T}}=\left(W^{T} f, W^{T} g\right)_{\mathcal{H}^{T}}=\left(u^{f}(\cdot, T), u^{g}(\cdot, T)\right)_{\mathcal{H}^{T}},
$$

i.e., $C^{T}$ connects the Hilbert metrics of the outer and inner spaces. It is a compact (because $W^{T}$ is) and nonnegative operator: $\left(C^{T} f, f\right)_{\mathcal{F}^{T}} \geq 0$ holds for all $f \in \mathcal{F}^{T}$. Moreover, since $\operatorname{Ker} C^{T}=\operatorname{Ker} W^{T}$, Lemma 3.1 provides its positivity:

$$
\left(C^{T} f, f\right)_{\mathcal{F}^{T}}>0 \quad \text { for } 0 \neq f \in \mathcal{F}^{T}, T<T_{*} .
$$

- Recall that the image operator $I^{T}$ introduced in section 1 acts from $L_{2}\left(\Omega^{T}\right)$ to $L_{2}\left(\Sigma^{T}\right)$. In what follows we identify these spaces with $\mathcal{H}^{T}$ and $\mathcal{F}^{T}$ respectively, and regard $I^{T}$ as a map from $\mathcal{H}^{T}$ to $\mathcal{F}^{T}$.

The definition of images easily implies $Y^{T} I^{T} \mathcal{H}^{\xi} \subset \mathcal{F}^{T, \xi}$, whereas (12) (for $\sigma=\Gamma$ ) provides $Y^{T} I^{T} W^{T} \mathcal{F}^{T, \xi} \subset \mathcal{F}^{T, \xi}$. The latter means that an operator $Y^{T} I^{T} W^{T}$ is triangular with respect to the family of subspaces (nest) $\left\{\mathcal{F}^{T, \xi}\right\}_{0 \leq \xi \leq T}$ [13].

For the connecting operator, the relations

$$
\begin{equation*}
C^{T} \stackrel{(20)}{=}\left(W^{T}\right)^{*} W^{T} \stackrel{(3)}{=}\left(Y^{T} I^{T} W^{T}\right)^{*}\left(Y^{T} I^{T} W^{T}\right) \tag{21}
\end{equation*}
$$

hold and show that operator $Y^{T} I^{T} W^{T}$ provides a triangular factorization of the connecting operator with respect to the nest $\left\{\mathcal{F}^{T, \xi}\right\}_{0 \leq \xi \leq T}[13,15]$.

- A significant fact is that the connecting operator is determined by the extended response operator via an explicit formula:

$$
\begin{equation*}
C^{T}=-\frac{1}{2}\left(S^{T}\right)^{*} R^{2 T} J^{2 T} S^{T} \tag{22}
\end{equation*}
$$

where the map $S^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{2 T}$ extends the controls from $\Sigma^{T}$ to $\Sigma^{2 T}$ by oddness:

$$
\left(S^{T} f\right)(\cdot, t)= \begin{cases}f(\cdot, t), & 0 \leq t<T \\ -f(\cdot, 2 T-t), & T \leq t \leq 2 T\end{cases}
$$

In $[1,3]$, a relevant analog of this representation is proved for the case of the Dirichlet boundary controls. To modify the proof for obtaining (22) needs just a minor correction.

### 3.3. System $\alpha_{*}^{T}$

A dynamical system associated with the problem

$$
\begin{array}{ll}
v_{t t}-\Delta v+q v=0 & \text { in }\left\{(x, t) \mid x \in \Omega^{T}, t>\tau(x)\right\} \\
\left.v\right|_{t=T}=0,\left.\quad v_{t}\right|_{t=T}=y \in \mathcal{H}^{T} & \\
\partial_{\nu} v=0 & \text { on } \Sigma^{T}
\end{array}
$$

is denoted by $\alpha_{*}^{T}$ and said to be dual to system $\alpha^{T}$. Its solution $v=v^{y}(x, t)$ describes a wave, which is initiated by the velocity perturbation $y$ and propagates (in the reversed time) in $\Omega$. The problem is well posed owing to the finiteness of the domain of influence property.

Integration by parts provides the well-known relation

$$
\left(u^{f}(\cdot, T), y\right)_{\mathcal{H}^{T}}=\left(f, v^{y}\right)_{\mathcal{F}^{T}}, \quad f \in \mathcal{F}^{T}, y \in \mathcal{H}^{T}
$$

It is the relation, which motivates the term 'dual' $[1,3]$.
In the dual system, the state/observation correspondence is realized by an observation operator $O^{T}: \mathcal{H}^{T} \rightarrow \mathcal{F}^{T}$,

$$
O^{T} y:=\left.v^{y}\right|_{\Sigma^{T}}
$$

Being written in the form $\left(W^{T} f, y\right)_{\mathcal{H}^{T}}=\left(f, O^{T} y\right)_{\mathcal{F}^{T}}$, the duality relation leads to the equality

$$
\begin{equation*}
O^{T}=\left(W^{T}\right)^{*} \tag{26}
\end{equation*}
$$

It implies $\operatorname{Ker} O^{T}=\mathcal{H}^{T} \ominus \overline{\operatorname{Ran} W^{T}}$, whereas (13) (for $\sigma=\Gamma$ ) follows to the equality $\operatorname{Ker} O^{T}=\{0\}$. The latter is interpreted as a boundary observability of the dual system.

## 4. Visualization of waves

### 4.1. Devices

## Propagation of jumps in $\alpha_{*}^{T}$

A very general fact of the propagation of singularities theory for the hyperbolic equations is that discontinuous data produce discontinuous solutions, the discontinuities propagating along bicharacteristics and being supported on characteristic surfaces. Here we deal with the Cauchy problem (23)-(25) with a $y$ having jumps of special kind. Our goal is to describe the corresponding jumps of the image $O^{T} y$. The description is provided by the proper Geometrical Optics formulae. Since the GO-technique is rather cumbersome, we have to restrict ourselves to heuristic considerations and references to our papers [1, 5], where the rigorous analysis is developed.

We start with a simpler case $T<T_{c}$ : the simplification is that the surfaces $\Gamma^{\xi}$ are smooth as $\xi \leq T$. A characteristic function (indicator) of a set $A$ is denoted by $\chi_{A}$ :

$$
\chi_{A}(p):=\left\{\begin{array}{ll}
1, & p \in A \\
0, & p \notin A
\end{array} .\right.
$$

- Fix a $\xi$ and (small) $\Delta \xi$ provided $0<\xi<\xi+\Delta \xi<T$. A subdomain

$$
\Delta \Omega^{\xi}:=\overline{\Omega^{\xi+\Delta \xi} \backslash \Omega^{\xi}} \subset \Omega^{T}
$$

is a thin layer between the smooth surfaces $\Gamma^{\xi+\Delta \xi}$ and $\Gamma^{\xi}$.
Take a $y \in C^{\infty}\left(\overline{\Omega^{T}}\right)$. A 'slice' $\chi_{\Delta \Omega \xi} y$ is a piece-wise smooth function supported in $\overline{\Delta \Omega^{\xi}}$. Generically, it has the jumps at $\Gamma^{\xi}$ and $\Gamma^{\xi+\Delta \xi}$. In what follows, the jump at $\Gamma^{\xi}$ is of our main interest, whereas the jump at $\Gamma^{\xi+\Delta \xi}$ is introduced just for technical convenience.

Return to system (23)-(25). Putting $\left.v_{t}\right|_{t=T}=\chi_{\Delta \Omega \xi y}$ in (24), we get a Cauchy problem with discontinuous data. In particular, the data have a jump at $\Gamma^{\xi}$ :

$$
\begin{equation*}
\left.v_{t}(x(\gamma, \tau), T)\right|_{\tau=\xi-0} ^{\tau=\xi+0}=y(x(\gamma, \xi))-0=y(x(\gamma, \xi)) \tag{27}
\end{equation*}
$$

As a consequence, the solution $v^{\chi_{\Delta \Omega} \xi y}$ turns out to be non-smooth. The following is some details specific for problem (23)-(25).

- A velocity perturbation $\chi_{\Delta \Omega} \xi y$, which initiates the wave process, is separated from the boundary with the distance $\xi$. Therefore, by the finiteness of domain of influence principle, the solution $v^{\chi} \Omega^{\xi} \xi^{y}$ vanishes for $t>T-\xi-\tau(x)$, i.e., over a characteristic surface $S^{T, \xi}:=\left\{(x, t) \in \overline{\Omega^{T}} \times[0, T]\right\}$ (see Fig 4.1).


Figure 1: Propagation of jump

- Jumps of $v_{t}(\cdot, T)$ initiate jumps of the velocity $v_{t}^{\chi_{\Delta \Omega} \xi y}$. One of the velocity jumps is located at the characteristic $S^{T, \xi}{ }^{2}$. This jump propagates along the

[^1]space-time rays $r_{\gamma}^{T, \xi}$, which constitute the characteristic:
\[

$$
\begin{aligned}
r_{\gamma}^{T, \xi}: & =\left\{(x, t) \in \overline{\Omega^{T}} \times[0, T] \mid x=x(\gamma, \xi-\tau), t=T-\tau: 0 \leq \xi \leq T\right\} \\
S^{T, \xi} & =\bigcup_{\gamma \in \Gamma} r_{\gamma}^{T, \xi}
\end{aligned}
$$
\]

The jump, which moves along $r_{\gamma}^{T, \xi}$, starts from the point $a=(x(\gamma, \xi), T)$ and reaches the boundary at $b=(x(\gamma, 0), T-\xi)$. By (27), at the 'input' $a$ the value (amplitude) of the jump is $y(x(\gamma, \xi)$ ). At the endpoint $b$, its amplitude is found by the GO-technique, which provides

$$
\begin{align*}
v_{t}^{\chi_{\Delta \Omega \xi} y}\left(\left.(x(\gamma, 0), t)\right|_{t=T-\xi-0} ^{t=T-\xi+0}\right. & =0-\beta^{\frac{1}{2}}(\gamma, \xi) y(x(\gamma, \xi)) \\
& =-\beta^{\frac{1}{2}}(\gamma, \xi) y(x(\gamma, \xi)) \tag{28}
\end{align*}
$$

This relation corresponds to the well-known GO-law: the ratio of the input and output jump amplitudes is governed by the factor $\beta$, which is determined by the spreading of rays $r_{\gamma}^{T, \xi}[1,5,18]$.

- By the aforesaid, a trace $\left.v_{t}^{\chi_{\Delta \Omega \xi^{y}}}\right|_{\Sigma^{T}}$ vanishes on $\Gamma \times(T-\xi, T]$ and has a jump at the cross-section $\Sigma^{T} \cap S^{T, \xi}=\Gamma \times\{t=T-\xi\}$. In the mean time, by the regularity results, this trace is continuous as an $H^{\frac{1}{2}}(\Gamma)$-valued function of $t \in[0, T-\xi]^{3}$. The following considerations specify the behavior of $\left.v_{t}^{\chi}{ }^{\chi}{ }^{\prime \Omega \xi}\right|_{\Sigma^{T}}$ near (and below) this cross-section.

Let

$$
\Delta \Sigma^{T, \xi}:=\left\{(\gamma, t) \in \Sigma^{T} \mid \gamma \in \Gamma, T-\xi-\Delta \xi \leq t \leq T-\xi\right\}
$$

be a thin 'belt' near the cross-section (see Fig. 4.1), $\chi_{\Delta \Sigma^{T, \xi}}$ its indicator. A function on $\Sigma^{T}$ of the form $\chi_{\Delta \Sigma^{T, \xi}}\left[\left.v_{t}^{\chi_{\Delta \Omega^{\xi} \xi}}\right|_{\Sigma^{T}}\right]$ is a 'slice' of the boundary trace of the velocity. By (28), one can represented it as

$$
\begin{align*}
& \left(\chi _ { \Delta \Sigma ^ { T , \xi } } \left[v_{t}^{\left.\left.\left.\chi_{\Delta \Omega \xi}{ }_{\mid}\right|_{\Sigma^{T}}\right]\right)(\gamma, t)=} \begin{array}{ll}
-\beta^{\frac{1}{2}}(\gamma, \xi) y(x(\gamma, \xi))+w^{\xi, \Delta \xi}(\gamma, t), & (\gamma, t) \in \Delta \Sigma^{T, \xi} \\
0, & (\gamma, t) \in \Sigma^{T} \backslash \Delta \Sigma^{T, \xi},
\end{array}\right.\right.
\end{align*}
$$

where the first summand in the first line does not depend on $t$ and, hence, obeys $\left\|\beta^{\frac{1}{2}} y\right\|_{L_{2}\left(\Delta \Sigma^{T, \xi}\right)}^{2} \sim \Delta \xi$, whereas the second summand satisfies $\left\|w^{\xi, \Delta \xi}\right\|_{L_{2}\left(\Delta \Sigma^{T, \xi)}\right.}^{2} \sim o(\Delta \xi)$ uniformly with respect to $\xi \in[0, T]$ and (small enough) $\Delta \xi>0[1,5]$. So, the first summand is dominating.

[^2]
## Amplitude integral

- Choose a partition $\Xi=\left\{\xi_{i}\right\}_{i=0}^{N}: 0=\xi_{0}<\xi_{1}<\cdots<\xi_{N}=T$ of the segment $[0, T]$ and denote

$$
\begin{aligned}
& \Delta \xi_{i}=\xi_{i}-\xi_{i-1}, \quad \Delta \Sigma^{T, \xi_{i}}=\Gamma \times\left[T-\xi_{i}-\Delta \xi_{i}, T-\xi_{i}\right], \quad \Delta \Omega^{\xi_{i}}=\overline{\Omega^{\xi_{i}} \backslash \Omega^{\xi_{i-1}}} \\
& i=1,2, \ldots N \quad\left(\Omega^{0}:=\emptyset\right) ; \quad r_{\Xi}=\max _{i=1, \ldots, N} \Delta \xi_{i}
\end{aligned}
$$

Summing up the terms of the form (29) and recalling the definition of images, we get

$$
\begin{align*}
&\left(\sum_{i=1}^{N} \chi_{\Delta \Sigma^{T}, \xi_{i}}\left[\left.v_{t}^{\chi_{\Delta \Omega} \xi_{i} y}\right|_{\Sigma^{T}}\right]\right)(\gamma, T-t)= \\
&=-\left(I^{T} y\right)(\gamma, t)+\delta^{y, \Xi}(\gamma, t), \quad(\gamma, t) \in \Sigma^{T} \tag{30}
\end{align*}
$$

where $\left\|\delta^{y, \Xi}\right\|_{L_{2}\left(\Sigma^{T}\right)} \rightarrow 0$ as $r_{\Xi} \rightarrow 0$. Substituting $t$ by $T-t$, we see that, for the given smooth $y \in \mathcal{H}^{T}$, the sums converge to $-Y^{T} I^{T} y$ by the norm in $\mathcal{F}^{T}$. The smallness of $\delta^{y, \Xi}$ is justified by perfect analogy with the case of the problem with Dirichlet boundary controls $[1,5]$.

- Here we interpret (30) in operator terms.

Let $X^{T, \xi}$ be a projection in $\mathcal{F}^{T}$ onto $\mathcal{F}^{T, \xi}$, which cuts off controls onto $\Gamma \times[T-\xi, T]$. The difference $\Delta X^{T, \xi_{i}}=X^{T, \xi_{i}}-X^{T, \xi_{i-1}}$ is also the projection cutting off controls onto the belt $\Delta \Sigma^{\xi_{i}, T}: \Delta X^{T, \xi_{i}} f=\chi_{\Delta \Sigma^{T, \xi_{i}}} f$.

By $G^{\xi}$ we denote a projection in $\mathcal{H}^{T}$ onto $\mathcal{H}^{\xi}$, which cuts off functions onto $\Omega^{\xi}$. The difference $\Delta G^{\xi_{i}}=G^{\xi_{i}}-G^{\xi_{i-1}}$ cuts off functions onto the layer $\Delta \Omega^{\xi_{i}}: \Delta G^{\xi_{i}} y=\chi_{\Delta \Omega^{\xi_{i}}} y$.

Recalling the definition of the observation operator, one can represent the summands in (30) as

$$
\chi_{\Delta \Sigma^{T, \xi_{i}}}\left[\left.v_{t}^{\chi_{\Delta \Omega} \xi_{i} y}\right|_{\Sigma^{T}}\right]=\Delta X^{T, \xi_{i}} \partial_{t} O^{T} \Delta G^{\xi_{i}} y
$$

and then write (30) in the form

$$
\begin{equation*}
\lim _{r_{\Xi \rightarrow 0}}\left[\sum_{i=1}^{N} \Delta X^{T, \xi_{i}} \partial_{t} O^{T} \Delta G^{\xi_{i}}\right] y=:\left[\int_{[0, T]} d X^{T, \xi} \partial_{t} O^{T} d G^{\xi}\right] y=Y^{T} I^{T} y \tag{31}
\end{equation*}
$$

An operator construction in the square brackets is said to be an amplitude inte$\operatorname{gral}(\mathrm{AI})$. It represents the image of $y$ as a collection of the wave jumps, which pass through $\Omega^{T}$ and are detected by the external observer at the boundary.

- Recall that (31) is derived under the assumption $T<T_{c}$. The case $T>T_{c}$ is more complicated since the equidistant surfaces $\Gamma^{\xi}$ can be non-smooth and
disconnected. However, a remarkable fact is that representation (31) does survive: it is valid for any $T<T_{*}$. For the system $\alpha^{T}$ with Dirichlet boundary controls, this result is stated in $[1,5]$. To modify it for the case of Neumann controls requires just a minor technical changes. So, the following does occur.
Proposition 4.1. For any positive $T<T_{*}$, the sums in (31) converge to the limit

$$
\begin{equation*}
\lim _{r \equiv \rightarrow 0} \sum_{i=1}^{N} \Delta X^{T, \xi_{i}} \partial_{t} O^{T} \Delta G^{\xi_{i}}=: \int_{[0, T]} d X^{T, \xi} \partial_{t} O^{T} d G^{\xi}=Y^{T} I^{T} \tag{32}
\end{equation*}
$$

in the weak operator topology.

## $W^{T}$ via amplitude integral

- Multiplying (32) by $W^{T}$ from the right, we get an operator $V^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
\begin{equation*}
V^{T}:=Y^{T} I^{T} W^{T}=\left[\int_{[0, T]} d X^{T, \xi} \partial_{t} O^{T} d G^{\xi}\right] W^{T} \tag{33}
\end{equation*}
$$

which satisfies

$$
\begin{equation*}
V^{T} \mathcal{F}^{T, \xi} \subset \mathcal{F}^{T, \xi}, \quad\left(V^{T}\right)^{*} V^{T} \stackrel{(21)}{=} C^{T} \tag{34}
\end{equation*}
$$

Thus, $V^{T}$ provides triangular factorization of the connecting operator with respect to the nest $\left\{\mathcal{F}^{T, \xi}\right\}_{0 \leq \xi \leq T}$.

- Any densely defined closable linear operator acting from a Hilbert space to a Hilbert space can be represented in the form of a polar decomposition (see, e.g., [10]). For the control operator, such a decomposition is

$$
\begin{equation*}
W^{T}=U^{T}\left|W^{T}\right|:=U^{T}\left[\left(W^{T}\right)^{*} W^{T}\right]^{\frac{1}{2}} \stackrel{(21)}{=} U^{T}\left[C^{T}\right]^{\frac{1}{2}} \tag{35}
\end{equation*}
$$

where $\left|W^{T}\right|: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$ is a modulo of $W^{T}$, and $U^{T}: \mathcal{F}^{T} \rightarrow \mathcal{H}^{T}$ is an isometry, which maps $\operatorname{Ran}\left|W^{T}\right| \subset \mathcal{F}^{T}$ onto $\operatorname{Ran} W^{T} \subset \mathcal{H}^{T}$ by the rule

$$
\begin{equation*}
U^{T}\left|W^{T}\right| f=W^{T} f, \quad f \in \mathcal{F}^{T} \tag{36}
\end{equation*}
$$

By (13) with $\sigma=\Gamma$, for any $T>0$ one has $\overline{\operatorname{Ran} W^{T}}=\mathcal{H}^{T}$. In the mean time, for $T<T_{*}$, we have

$$
\overline{\operatorname{Ran}\left|W^{T}\right|}=\mathcal{F}^{T} \ominus \operatorname{Ker}\left|W^{T}\right|=\mathcal{F}^{T} \ominus \operatorname{Ker} W^{T} \stackrel{\operatorname{Lemma} 3.1}{=} \mathcal{F}^{T}
$$

As a result, if $T<T_{*}$ then $U^{T}$ can be extended by continuity from $\operatorname{Ran}\left|W^{T}\right|$ to $\mathcal{F}^{T}$, the extension being a unitary operator, which maps $\mathcal{F}^{T}$ onto $\mathcal{H}^{T}$. In what follows, we assume that such an extension is done; it satisfies

$$
\begin{equation*}
\left(U^{T}\right)^{*} U^{T}=\mathbf{1}_{\mathcal{F}^{T}}, \quad U^{T}\left(U^{T}\right)^{*}=\mathbf{1}_{\mathcal{H}^{T}} \tag{37}
\end{equation*}
$$

- Recall that $G^{\xi}$ projects in $\mathcal{H}^{T}$ onto $\mathcal{H}^{\xi}$. We say a projection $P^{\xi}$ in $\mathcal{H}^{T}$ onto the subspace $\overline{W^{T} \mathcal{F}^{T}, \xi}$ (formed by waves) to be a wave projection. A crucial point of our approach is the equality

$$
\begin{equation*}
P^{\xi} \stackrel{(13)}{=} G^{\xi}, \quad 0 \leq \xi \leq T \tag{38}
\end{equation*}
$$

which corresponds to the controllability of system $\alpha^{T}$.
Let $\tilde{P}^{T, \xi}$ be a projection in $\mathcal{F}^{T}$ onto the subspace $\overline{\left|W^{T}\right| \mathcal{F}^{T, \xi}}$. By (36), one has

$$
\begin{equation*}
U^{T} \tilde{P}^{T, \xi}=P^{\xi} U^{T}, \quad 0 \leq \xi \leq T \tag{39}
\end{equation*}
$$

that implies

$$
\begin{align*}
& O^{T} G^{\xi} W^{T} \stackrel{(26),(38)}{=}\left(W^{T}\right)^{*} P^{\xi} W^{T} \stackrel{(35)}{=}\left|W^{T}\right|\left(U^{T}\right)^{*} P^{\xi} U^{T}\left|W^{T}\right|= \\
& \stackrel{(39)}{=}\left|W^{T}\right| \tilde{P}^{T, \xi}\left|W^{T}\right| \tag{40}
\end{align*}
$$

for $0 \leq \xi \leq T$.

- Multiplying equality (33) by the isometry $\left(I^{T}\right)^{*} Y^{T}$ from the left, and taking into account (40), we get

$$
\begin{equation*}
W^{T}=U^{T}\left|W^{T}\right|, \quad U^{T}=\left(I^{T}\right)^{*} Y^{T}\left[\int_{[0, T]} d X^{T, \xi} \partial_{t}\left|W^{T}\right| d \tilde{P}^{T, \xi}\right] \tag{41}
\end{equation*}
$$

Here the operators $I^{T}, Y^{T}, X^{T, \xi}$ are standard (do not depend on potential $q$ ), whereas projections $\tilde{P}^{T, \xi}$ are obviously determined by $\left|W^{T}\right|$. Operator $W^{T}$ is triangular with respect to the pair of the nests $\left\{\mathcal{F}^{T, \xi}\right\}$ and $\left\{\mathcal{H}^{\xi}\right\}$ that means $W^{T} \mathcal{F}^{T, \xi} \subset \mathcal{H}^{\xi}, 0 \leq \xi \leq T$ (see (13)). From the operator theory viewpoint, representation (41) enables one to recover a triangular operator $W^{T}$ via its modulo $\left|W^{T}\right|$, the 'phase' part $U^{T}$ being expressed via a relevant operator integral. The integral into the square brackets is referred to as a diagonal of operator $\partial_{t} W^{T}$ with respect to the nests $\left\{\mathcal{F}^{T, \xi}\right\}$ and $\left\{\mathcal{H}^{\xi}\right\}[9,13]$.

- Introduce an operator $A^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$ by

$$
\begin{equation*}
A^{T}:=Y^{T} \int_{[0, T]} d X^{T, \xi} \partial_{t}\left[C^{T}\right]^{\frac{1}{2}} d \tilde{P}^{T, \xi} \tag{42}
\end{equation*}
$$

With regard to (38) and (39), one can write (32) in the form $A^{T}\left(U^{T}\right)^{*}=I^{T}$ that enables one to represent the phase operator in the form

$$
U^{T} \stackrel{(41)}{=}\left(I^{T}\right)^{*} A^{T} .
$$

By (37) and (3), this representation implies

$$
\begin{equation*}
\left(A^{T}\right)^{*} A^{T}=\mathbf{1}, \quad A^{T}\left(A^{T}\right)^{*}=G_{\Theta^{T}} . \tag{43}
\end{equation*}
$$

Now, writing (41) in the form

$$
\begin{equation*}
W^{T}=\left(I^{T}\right)^{*} A^{T}\left[C^{T}\right]^{\frac{1}{2}} \tag{44}
\end{equation*}
$$

we obtain the representation of the control operator, which plays a basic role in solving inverse problems. The reason is the following.

Operator $R^{2 T}$ formalizes information, which the external observer gets from measurements at the boundary $\Gamma$. The waves $u^{f}$ propagate into $\Omega$ and are invisible for him. However, the observer can determine $C^{T}$ via (22), find $\left[C^{T}\right]^{\frac{1}{2}}$, construct the integral (42), determine $W^{T}$ via (44), and eventually recover invisible waves $u^{f}(\cdot, T)=W^{T} f$. In the BC-method, such a remarkable option is referred to as a visualization of waves.

### 4.2. Solving the inverse problem

## Setup

As is mentioned in section 3.2, the extended response operator $R^{2 T}$ depends on the potential locally: it is determined by $\left.q\right|_{\Omega^{T}}$. Such a locality motivates the following setup of the inverse problem.
(IP) Given operator $R^{2 T}$, to recover potential $q$ in the subdomain $\Omega^{T}$.
The IP will be solved for an arbitrary fixed $T<T_{*}$. Surely, such an option enables one to determine $q$ in the whole $\Omega$ if $R^{2 T}$ is given for a $T \geq T_{*}$.

## Procedure

Preparatory to solving the IP, recall that geometry of the wave propagation in system $\alpha^{T}$ is governed by the leading part $\partial_{t}^{2}-\Delta$ of the wave equation (4). Since this part does not depend on the potential, the geometry is Euclidean [18]. Therefore, we have the right to regard all the geometric objects and parameters ( $\Omega^{\xi}$, sgc, $\Theta^{T}, \beta, T_{*}$, etc) as known and use them for determination of $q$. In particular, we can use the image operator $I^{T}$.

Let $T<T_{*}$ be fixed. Given $R^{2 T}$ one can recover $q$ in $\Omega^{T}$ by the following procedure.
Step 1. Find $C^{T}$ by (22). Determine $\left[C^{T}\right]^{\frac{1}{2}}$.
Step 2. Determine the subspaces $\left[C^{T}\right]^{\frac{1}{2}} \mathcal{F}^{T, \xi}$ and the corresponding projections $\tilde{P}^{T, \xi}$ for $0 \leq \xi \leq T$.
Step 3. Construct the integral (42) and, then, recover $W^{T}$ via (44).
Step 4. Determine $\left.q\right|_{\Omega^{T}}$ from the graph (14).
The IP is solved.

### 4.3. Characterization of data

## Main result

In addition to the procedure, which solves the IP, we provide the necessary and sufficient conditions for its solvability.
Theorem 4.2. Let $0<T<T_{*}$. An operator $\mathcal{R}^{2 T}: \mathcal{F}^{2 T} \rightarrow \mathcal{F}^{2 T}$ is the extended response operator of a system $\alpha^{T}$ with potential of the class $L_{\infty}\left(\Omega^{T}\right)$ if and only if it satisfies the following conditions:

1. $\mathcal{R}^{2 T}$ is a compact operator obeying

$$
\begin{equation*}
Y^{2 T} \mathcal{R}^{2 T}=\left(\mathcal{R}^{2 T} Y^{2 T}\right)^{*} ; \quad \mathcal{R}^{2 T} \mathcal{T}_{2 T-\xi}^{2 T}=\mathcal{T}_{2 T-\xi}^{2 T} \mathcal{R}^{2 T}, \quad 0 \leq \xi \leq 2 T \tag{45}
\end{equation*}
$$

2. An operator $\mathcal{C}^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
\begin{equation*}
\mathcal{C}^{T}:=-\frac{1}{2}\left(S^{T}\right)^{*} \mathcal{R}^{2 T} J^{2 T} S^{T} \tag{46}
\end{equation*}
$$

is symmetric and positive: $\left(\mathcal{C}^{T} f, f\right)_{\mathcal{F}^{T}}>0$ for $0 \neq f \in \mathcal{F}^{T}$.
3. Let $\tilde{\mathcal{P}}^{T, \xi}$ be a projection in $\mathcal{F}^{T}$ onto $\overline{\left[\mathcal{C}^{T}\right]^{\frac{1}{2}} \mathcal{F}^{T, \xi}}$. An operator integral $\mathcal{A}^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{T}$,

$$
\begin{equation*}
\mathcal{A}^{T}:=Y^{T} \int_{[0, T]} d X^{T, \xi} \partial_{t}\left[\mathcal{C}^{T}\right]^{\frac{1}{2}} d \tilde{\mathcal{P}}^{T, \xi} \tag{47}
\end{equation*}
$$

converges in the weak operator topology to an isometry, which satisfies

$$
\begin{equation*}
\left(\mathcal{A}^{T}\right)^{*} \mathcal{A}^{T}=1, \quad \mathcal{A}^{T}\left(\mathcal{A}^{T}\right)^{*}=G_{\Theta^{T}} \tag{48}
\end{equation*}
$$

4. An operator $\mathcal{W}^{T}: \mathcal{F}^{T} \rightarrow \mathcal{H}^{T}$

$$
\begin{equation*}
\mathcal{W}^{T}:=\left(I^{T}\right)^{*} \mathcal{A}^{T}\left[\mathcal{C}^{T}\right]^{\frac{1}{2}} \tag{49}
\end{equation*}
$$

satisfies $\mathcal{W}^{T} \mathcal{M}^{T} \subset H^{2}\left(\Omega^{T}\right)$.
5. The relation

$$
\begin{equation*}
\left.\partial_{\nu} \mathcal{W}^{T} f\right|_{\Gamma}=f(\cdot, T), \quad f \in \mathcal{M}^{T} \tag{50}
\end{equation*}
$$

is valid.
6. The relation

$$
\begin{equation*}
\overline{\mathcal{W}^{T} \mathcal{F}_{\sigma}^{T, \xi}}=\mathcal{H}_{\sigma}^{\xi}, \quad 0 \leq \xi \leq T \tag{51}
\end{equation*}
$$

holds for any open $\sigma \subseteq \Gamma$.
7. The relation

$$
\begin{equation*}
\sup _{0 \neq f \in \mathcal{M}^{T}} \frac{\left\|\Delta \mathcal{W}^{T} f-\mathcal{W}^{T} f_{t t}\right\|_{\mathcal{H}^{T}}}{\left\|\mathcal{W}^{T} f\right\|_{\mathcal{H}^{T}}}<\infty \tag{52}
\end{equation*}
$$

holds.
The proof consists of two parts.

## Part I (necessity)

Proof. Let $\mathcal{R}^{2 T}=R^{2 T}$, where $R^{2 T}$ is the extended response operator of a system $\alpha^{T}$ with potential $q \in L_{\infty}\left(\Omega^{T}\right)$. The system possesses the connecting, control, and phase operators $C^{T}, W^{T}$ and $U^{T}$ respectively.

1. Relations (45) hold by (19).
2. In view of (22), operator $\mathcal{C}^{T}$ defined by (46) coincides with $C^{T}$, which is a compact positive operator.
3. The equality $\mathcal{C}^{T}=C^{T}$ implies $\tilde{\mathcal{P}}^{T, \xi}=\tilde{P}^{T, \xi}$. Comparing (42) with (47), we conclude that $\mathcal{A}^{T}=A^{T}$. Hence, (48) follows from (43).
4. Comparing (49) with (44), we see that $\mathcal{W}^{T}$ coincides with $W^{T}$. Hence, $\mathcal{W}^{T} \mathcal{M}^{T} \subset H^{2}\left(\Omega^{T}\right)$ holds by the regularity results on the problem (4)-(6) (see section 3.1).
5. Since $\mathcal{W}^{T}=W^{T}$, the equality (50) is just a form of writing (10).
6. (51) holds by (13).
7. Since $\mathcal{W}^{T} f=W^{T} f=u^{f}(\cdot, T)$, we have

$$
\begin{aligned}
-\Delta \mathcal{W}^{T} f+\mathcal{W}^{T} f_{t t}=-\Delta u^{f}(\cdot, T)+ & u^{f_{t t}}(\cdot, T) \stackrel{(11)}{=} \\
& =-\Delta u^{f}(\cdot, T)+u_{t t}^{f}(\cdot, T) \stackrel{(8)}{=} q u^{f}(\cdot, T)
\end{aligned}
$$

The inequality (52) is a consequence of $q \in L_{\infty}\left(\Omega^{T}\right)$.

## Part II (sufficiency)

The proof of sufficiency is constructive: given $\mathcal{R}^{2 T}$ we provide a system $\alpha^{T}$ with the response operator $R^{2 T}=\mathcal{R}^{2 T}$. In fact, the construction follows the procedure Step 1-4, which solves the IP.

Proof. Assume that $\mathcal{R}^{2 T}$ obeys 1-5.

- Determine operator $\mathcal{C}^{T}$ by (46) and find $\left[\mathcal{C}^{T}\right]^{\frac{1}{2}}$. The latter is also positive and injective.

Construct the operator integral in (47) and get operator $\mathcal{A}^{T}$. By (48), $\mathcal{A}^{T}$ is an isometry in $\mathcal{F}^{T}$ with the range $G_{\Theta^{T}} \mathcal{F}^{T}$. Hence, it satisfies $G_{\Theta^{T}} \mathcal{A}^{T}=\mathcal{A}^{T}$.

Introduce operator $\mathcal{W}^{T}: \mathcal{F}^{T} \rightarrow \mathcal{H}^{T}$ in accordance with (49). Obviously, it is injective. By (51) (for $\xi=T$ and $\sigma=\Gamma$ ), its range $\mathcal{W}^{T} \mathcal{F}^{T}$ is dense in $\mathcal{H}^{T}$. Also, it satisfies

$$
\begin{align*}
&\left(\mathcal{W}^{T}\right)^{*} \mathcal{W}^{T}=\left[\mathcal{C}^{T}\right]^{\frac{1}{2}}\left(\mathcal{A}^{T}\right)^{*} I^{T}\left(I^{T}\right)^{*} \mathcal{A}^{T}\left[\mathcal{C}^{T}\right]^{\frac{1}{2}} \stackrel{(3)}{=}\left[\mathcal{C}^{T}\right]^{\frac{1}{2}}\left(\mathcal{A}^{T}\right)^{*} G_{\Theta^{T}} \mathcal{A}^{T}\left[\mathcal{C}^{T}\right]^{\frac{1}{2}}= \\
&=\left[\mathcal{C}^{T}\right]^{\frac{1}{2}}\left(\mathcal{A}^{T}\right)^{*} \mathcal{A}^{T}\left[\mathcal{C}^{T}\right]^{\frac{1}{2} \stackrel{(48)}{=}} \mathcal{C}^{T} \tag{53}
\end{align*}
$$

- Since $\mathcal{W}^{T}$ is injective, the set of pairs $\left\{\left\langle\mathcal{W}^{T} f, \mathcal{W}^{T} f_{t t}\right\rangle \mid f \in \mathcal{M}^{T}\right\}$ constitutes the graph of a linear operator acting in $\mathcal{H}^{T}$. This operator is denoted by $L^{T}: \mathcal{W}^{T} f \mapsto \mathcal{W}^{T} f_{t t}$. It acts in $\mathcal{H}^{T}$ and is densely defined (on $\mathcal{W}^{T} \mathcal{F}^{T}$ ).

Recall that the class of smooth controls $\mathcal{M}^{T}$ is dense in $\mathcal{F}^{T}$, its elements vanishing near $t=0$. The subclass

$$
\mathcal{M}_{0}^{T}:=\left\{f \in \mathcal{M}^{T} \mid f \text { vanishes near } t=T\right\}
$$

is also dense in $\mathcal{F}^{T}$. Hence, $\mathcal{W}^{T} \mathcal{M}_{0}^{T}$ is dense in $\mathcal{H}^{T}$ by (51) for $\sigma=\Gamma, \xi=T$. As a result, an operator $L_{0}^{T}:=\left.L^{T}\right|_{\mathcal{W}^{T} \mathcal{M}_{0}^{T}}$ is densely defined in $\mathcal{H}^{T}$. Show that it is symmetric.

Take $f, g \in \mathcal{M}_{0}^{T}$. Note that $S^{T} f$ and $S^{T} g$ are twice differentiable with respect to $t$ and vanish near $t=0$ and $t=2 T$. Also, note that the second relation in (45) implies the commutation $\mathcal{R}^{2 T} \partial_{t}^{2}=\partial_{t}^{2} \mathcal{R}^{2 T}$. Then, we have

$$
\begin{aligned}
& \left(L_{0}^{T} \mathcal{W}^{T} f, \mathcal{W}^{T} g\right)_{\mathcal{H}^{T}}=\left(L^{T} \mathcal{W}^{T} f, \mathcal{W}^{T} g\right)_{\mathcal{H}^{T}}=\left(\mathcal{W}^{T} f_{t t}, \mathcal{W}^{T} g\right)_{\mathcal{H}^{T}} \stackrel{(53)}{=} \\
& \quad=\left(\mathcal{C}^{T} f_{t t}, g\right)_{\mathcal{F}^{T}} \stackrel{(46)}{=}-\frac{1}{2}\left(\left[\mathcal{R}^{2 T} J^{2 T} S^{T}\right] f_{t t}, S^{T} g\right)_{\mathcal{F}^{2 T}}= \\
& \quad=-\frac{1}{2}\left(\left[\mathcal{R}^{2 T} J^{2 T} S^{T} f\right]_{t t}, S^{T} g\right)_{\mathcal{F}^{2 T}} \stackrel{\star}{=}-\frac{1}{2}\left(\mathcal{R}^{2 T} J^{2 T} S^{T} f,\left[S^{T} g\right]_{t t}\right)_{\mathcal{F}^{2 T}}= \\
& \quad=-\frac{1}{2}\left(\mathcal{R}^{2 T} J^{2 T} S^{T} f, S^{T} g_{t t}\right)_{\mathcal{F}^{2 T}}=-\frac{1}{2}\left(\left(S^{T}\right)^{*} \mathcal{R}^{2 T} J^{2 T} S^{T} f, g_{t t}\right)_{\mathcal{F}^{T}}= \\
& \quad=\left(\mathcal{C}^{T} f, g_{t t}\right)_{\mathcal{F}^{T}} \stackrel{(53)}{=}\left(\mathcal{W}^{T} f, \mathcal{W}^{T} g_{t t}\right)_{\mathcal{H}^{T}}=\left(\mathcal{W}^{T} f, L^{T} \mathcal{W}^{T} g\right)_{\mathcal{H}^{T}}= \\
& \quad=\left(\mathcal{W}^{T} f, L_{0}^{T} \mathcal{W}^{T} g\right)_{\mathcal{H}^{T}} .
\end{aligned}
$$

In $(\star)$ we integrate by part with respect to time in $\mathcal{F}^{T}=L_{2}\left(\Sigma^{T}\right)$. So, $L_{0}^{T}$ is symmetric.

- Owing to (52), operator $Q^{T}:=\Delta-L^{T}$ defined on the dense set $\mathcal{W}^{T} \mathcal{F}^{T} \subset$ $\mathcal{H}^{T}$, is bounded. By this, we assume that $Q^{T}$ is extended to $\mathcal{H}^{T}$ by continuity.

Operator $Q^{T}$ is self-adjoint. Indeed, in view of (50), for $f \in \mathcal{M}_{0}^{T}$ one has $\left.\partial_{\nu} \mathcal{W}^{T} f\right|_{\Gamma}=\left.f\right|_{t=T}=0$, i.e., elements of $\mathcal{W}^{T} \mathcal{M}_{0}^{T}$ satisfy the homogeneous Neumann boundary condition on $\Gamma$. By the latter, the Laplacian $\Delta$ is symmetric on $\mathcal{W}^{T} \mathcal{M}_{0}^{T}$. Hence, $\left.Q^{T}\right|_{\mathcal{W}^{T} \mathcal{M}_{0}^{T}}=\left.\Delta\right|_{\mathcal{W}^{T} \mathcal{M}_{0}^{T}}-L_{0}^{T}$ is symmetric on a dense set. Since it is bounded, we conclude that $\left(Q^{T}\right)^{*}=Q^{T}$.

- For $f \in \mathcal{M}^{T} \subset \mathcal{F}^{T}$, define a function

$$
\begin{equation*}
u^{f}(x, t):=\left(\mathcal{W}^{T} \mathcal{T}_{T-t}^{T} f\right)(x) \quad \text { in } \overline{\Omega^{T}} \times[0, T] \tag{54}
\end{equation*}
$$

The definitions of the operators imply

$$
\begin{aligned}
{\left[\Delta-Q^{T}\right] u^{f}(\cdot, t)=L^{T} u^{f}(\cdot, t)=L^{T} \mathcal{W}^{T} \mathcal{T}_{T-t}^{T} f } & =\mathcal{W}^{T}\left[\mathcal{T}_{T-t}^{T} f\right]_{t t}= \\
& =\left[\mathcal{W}^{T} \mathcal{T}_{T-t}^{T} f\right]_{t t}=u_{t t}^{f}(\cdot, t)
\end{aligned}
$$

Thus, $u^{f}$ satisfies the equation

$$
\begin{equation*}
u_{t t}-\Delta u+Q^{T} u=0 \quad \text { in } \Omega^{T} \times(0, T) \tag{55}
\end{equation*}
$$

By (51) for $\sigma=\Gamma$, we have supp $u^{\mathrm{f}}(\cdot, \mathrm{t}) \subset \overline{\Omega^{\mathrm{t}}}$, i.e., $u^{f}$ satisfies the Cauchy condition

$$
\begin{equation*}
\left.u\right|_{t<\tau(x)}=0 \quad \text { in } \overline{\Omega^{T}} \times[0, T] \tag{56}
\end{equation*}
$$

In the mean time, (50) easily implies that $u^{f}$ obeys

$$
\begin{equation*}
\partial_{\nu} u=f \quad \text { on } \overline{\Sigma^{T}} \tag{57}
\end{equation*}
$$

- Show that $Q^{T}$ is a multiplication by a bounded function. The proof follows the idea of [4].
Lemma 4.3. There is a (real) function $q \in L_{\infty}\left(\Omega^{T}\right)$ such that $Q^{T} y=q y$ holds for $y \in \mathcal{H}^{T}$.

Proof. 1. Choose a $\sigma \subset \Gamma$ and $f \in \mathcal{F}_{\sigma}^{T, \xi} \cap \mathcal{M}^{T}$. By condition 4 and (51), we have $u^{f}(\cdot, T) \in \mathcal{H}_{\sigma}^{\xi} \cap H^{2}\left(\Omega^{T}\right)$. Hence, $\Delta u^{f}(\cdot, T) \in \mathcal{H}_{\sigma}^{\xi}$. In the mean time, we have $f_{t t} \in \mathcal{F}_{\sigma}^{T, \xi} \cap \mathcal{M}^{T}$ that implies $u_{t t}^{f}=L^{T} u^{f}(\cdot, T)=\mathcal{W}^{T} f_{t t} \stackrel{(51)}{\in} \mathcal{H}_{\sigma}^{\xi}$. Therefore, $Q^{T} u^{f}(\cdot, T) \stackrel{(55)}{=} \Delta u^{f}(\cdot, T)-u_{t t}^{f} \in \mathcal{H}_{\sigma}^{\xi}$. Thus, $Q^{T} \mathcal{W}^{T} \mathcal{F}_{\sigma}^{T, \xi} \subset \mathcal{H}_{\sigma}^{\xi}$ holds. Since $\mathcal{W}^{T} \mathcal{F}_{\sigma}^{T, \xi}$ is dense in $\mathcal{H}_{\sigma}^{\xi}($ see $(51))$, we conclude that $Q^{\sigma} \mathcal{H}_{\sigma}^{\xi} \subset \mathcal{H}_{\sigma}^{\xi}$. The latter leads to $Q^{T}\left[\mathcal{H}^{T} \ominus \mathcal{H}_{\sigma}^{\xi}\right] \subset\left[\mathcal{H}^{T} \ominus \mathcal{H}_{\sigma}^{\xi}\right]$ by virtue of the symmetry $\left(Q^{T}\right)^{*}=Q^{T}$. Hence, the subspaces $\mathcal{H}_{\sigma}^{\xi}$ reduce $Q^{T}$ that is equivalent to the commutation

$$
\begin{equation*}
Q^{T} G_{\sigma}^{\xi}=G_{\sigma}^{\xi} Q^{T}, \quad \sigma \subset \Gamma, \quad 0 \leq \xi \leq T \tag{58}
\end{equation*}
$$

where $G_{\sigma}^{\xi}$ projects in $\mathcal{H}^{T}$ onto $\mathcal{H}_{\sigma}^{\xi}$, i.e., cuts off functions on $\Omega_{\sigma}^{\xi}$.
2. As is easy to verify, an operator $\tau_{\sigma}^{T}: \mathcal{H}^{T} \rightarrow \mathcal{H}^{T}$,

$$
\begin{equation*}
\tau_{\sigma}^{T} y:=\left[\int_{[0, T]} \xi d G_{\sigma}^{\xi}\right] y=\left[\lim _{r \Xi \rightarrow 0} \sum_{i=1}^{N} \xi_{i}\left[G_{\sigma}^{\xi_{i}}-G_{\sigma}^{\xi_{i-1}}\right]\right] y \tag{59}
\end{equation*}
$$

(the sums converge by the operator norm) acts by the rule

$$
\tau_{\sigma}^{T} y= \begin{cases}\mathrm{d}(\cdot, \sigma) y & \text { in } \Omega_{\sigma}^{T} \\ 0 & \text { in } \Omega^{T} \backslash \Omega_{\sigma}^{T}\end{cases}
$$

i.e., multiplies functions by the distance to $\sigma$ and, then, cuts off on $\Omega_{\sigma}^{T}$ [4]. As a consequence, an operator

$$
\hat{\tau}_{\sigma}^{T}:=\tau_{\sigma}^{T} y+T\left(\mathbf{1}_{\mathcal{H}^{T}}-G_{\sigma}^{T}\right) y
$$

multiplies functions by the continuous function $\mathrm{d}_{\sigma}^{T}(\cdot):=\max \{\mathrm{d}(\cdot, \sigma), T\}$. In the mean time, (58) implies

$$
\begin{equation*}
Q^{T} \hat{\tau}_{\sigma}^{T}=\hat{\tau}_{\sigma}^{T} Q^{T}, \quad \sigma \subset \Gamma, \quad 0 \leq \xi \leq T \tag{60}
\end{equation*}
$$

because the sums in (59) do commute with all $G_{\sigma}^{\xi}$.
3. Fix a (small) $\delta>0$. A simple geometric fact is that the functions $\left\{\mathrm{d}_{\sigma}^{T} \mid \sigma \subset \Gamma\right\}$ separate points in $\Omega^{T-\delta}$ and vanish simultaneously in no point $x_{0} \in \overline{\Omega^{T-\delta}}$. Hence, a family $\left\{\mathrm{d}_{\sigma}^{T} \mid \sigma \subset \Gamma, 0 \leq \xi \leq T\right\}$ generates the continuous function algebra $\mathbf{C}\left(\overline{\Omega^{T-\delta}}\right)$ [4].

Correspondingly, an operator family $\left\{\hat{\tau}_{\sigma}^{T} \mid \sigma \subset \Gamma, 0 \leq \xi \leq T\right\}$ generates the operator (sub)algebra $\mathbf{C}\left(\overline{\Omega^{T-\delta}}\right) \subset \mathbf{B}\left(\mathcal{H}^{T}\right)$ of multiplications by continuous functions. As a consequence of (60), we have $Q^{T} \mathbf{C}\left(\overline{\Omega^{T-\delta}}\right)=\mathbf{C}\left(\overline{\Omega^{T-\delta}}\right) Q^{T}$ that is possible if and only if $Q^{T}$ is also a multiplication by a function $q$.

Since $Q^{T}$ is bounded, we have $q \in L_{\infty}\left(\Omega^{T-\delta}\right)$. By arbitrariness of $\delta$, we get $q \in L_{\infty}\left(\Omega^{T}\right)$.

- With the above determined function $q$ one associates the system $\alpha^{T}$ of the form (8)-(10). Such a system possesses its own operators $W^{T}$ and $C^{T}$. Show that $W^{T}=\mathcal{W}^{T}$ and $C^{T}=\mathcal{C}^{T}$.

Since the problems (8)-(10) and (55)-(57) (with $Q^{T}=q$ ) are identical and uniquely solvable, their solutions (for the same $f$ 's) coincide. Writing the first relation of (11) in the form $u^{f}(\cdot, t)=W^{T} \mathcal{T}_{T-t}^{T} f$ and comparing with (54), we see that $W^{T}=\mathcal{W}^{T}$ holds.

By the latter equality and (53), we have

$$
\begin{equation*}
\mathcal{C}^{T}=\left(\mathcal{W}^{T}\right)^{*} \mathcal{W}^{T}=\left(W^{T}\right)^{*} W^{T}=C^{T} \tag{61}
\end{equation*}
$$

- System (55)-(57) (with $Q^{T}=q$ ) possesses the extended response operator $R^{2 T}$. Here we prove the equality $R^{2 T}=\mathcal{R}^{2 T}$ that completes the proof of the Theorem.

Begin with two lemmas of general character. The lemmas deal with a Hilbert space $\mathcal{F}=L_{2}([0,2 T] ; \mathcal{E})$ (with the Lebesgue measure $d t$ ), where $\mathcal{E}$ is an auxiliary Hilbert space. By $\mathcal{F}_{ \pm}$we denote the subspaces of functions, which are even and odd with respect to $t=T$. So, the decompositions $\mathcal{F}=\mathcal{F}_{+} \oplus \mathcal{F}_{-}$ holds. Let

$$
\mathcal{F}^{[a, b]}:=\{f \in \mathcal{F} \mid \operatorname{supp} f \subset[a, b]\}, \quad 0 \leq a<b \leq 2 T
$$

Lemma 4.4. If a bounded operator $N: \mathcal{F} \rightarrow \mathcal{F}$ satisfies

$$
\begin{equation*}
N \mathcal{F}_{ \pm} \subset \mathcal{F}_{ \pm} ; \quad N \mathcal{F}^{[a, 2 T]} \subset \mathcal{F}^{[a, 2 T]}, \quad 0 \leq a \leq 2 T \tag{62}
\end{equation*}
$$

then it is local, i.e., preserves the support of functions:

$$
\begin{equation*}
N \mathcal{F}^{[a, b]} \subset \mathcal{F}^{[a, b]}, \quad 0 \leq a<b \leq 2 T \tag{63}
\end{equation*}
$$

Proof. 1. Representing $\mathcal{F}=\mathcal{F}^{[0, T]} \oplus \mathcal{F}^{[T, 2 T]}$ and $f=f_{1}+f_{2}$ with $f_{1} \in$ $\mathcal{F}^{[0, T]}, f_{2} \in \mathcal{F}^{[T, 2 T]}$, we identify $f \equiv\left\langle f_{1}, f_{2}\right\rangle$.

Introduce an isometry $Y: \mathcal{F}^{[0, T]} \rightarrow \mathcal{F}^{[T, 2 T]}$ by

$$
(Y f)(t):=f(2 T-t), \quad T \leq t \leq 2 T
$$

Obviously, one has $\left.\mathcal{F}_{ \pm}=\{\langle f, \pm Y f\rangle\} \mid f \in \mathcal{F}^{[0, T]}\right\}$. Since $N$ preserves the evenness/oddness, there are two operators $k, l: \mathcal{F}^{[0, T]} \rightarrow \mathcal{F}^{[0, T]}$ such that

$$
\begin{equation*}
N\langle f, Y f\rangle=\langle k f, Y k f\rangle \quad \text { and } \quad N\langle f,-Y f\rangle=\langle l f,-Y l f\rangle \tag{64}
\end{equation*}
$$

Show that $k=l$. For a $g \in \mathcal{F}^{[0, T]}$, one has

$$
\begin{array}{r}
2 N\langle 0, Y g\rangle=N[\langle g, Y g\rangle-\langle g,-Y g\rangle] \stackrel{(64)}{=}\langle k g, Y k g\rangle-\langle l g,-Y l g\rangle= \\
=\langle[k-l] g, Y[k+l] g\rangle . \tag{65}
\end{array}
$$

In the mean time, we have $\langle 0, Y g\rangle \in \mathcal{F}^{[T, 2 T]}$ and, hence, $N\langle 0, Y g\rangle \in \mathcal{F}^{[T, 2 T]}$ holds by (62). By the latter, $2 N\langle 0, Y g\rangle$ must be of the form $\langle 0, \ldots\rangle$, i.e., $[k-l] g=$ 0 is valid and implies $k=l=: m$.
2. Putting $g=Y^{-1} h$ in (65), we get

$$
\begin{equation*}
N\langle 0, h\rangle=\left\langle 0, Y m Y^{-1} h\right\rangle . \tag{66}
\end{equation*}
$$

In the mean time, we have

$$
2 N\langle g, 0\rangle=N[\langle g, Y g\rangle+\langle g,-Y g\rangle] \stackrel{(64)}{=}\langle m g, Y g\rangle+\langle m g,-Y m g\rangle=2\langle m g, 0\rangle .
$$

Combining the latter with (66), we arrive at the representation

$$
\begin{equation*}
N\langle g, h\rangle=\left\langle m g, Y m Y^{-1} h\right\rangle . \tag{67}
\end{equation*}
$$

3. Such a representation easily provides the following fact: operator $N$ acts locally in $[0,2 T]$ if and only if operator $m$ is local in $[0, T]$. Show that the latter does occur.

Let supp $f \subset[a, b] \subset[0, T]$, so that $\left.f\right|_{0 \leq t<a}=0 \quad$ and $\left.\quad f\right|_{b<t \leq 2 T}=0$ holds. The first equality means that $f \in \mathcal{F}^{[a, 2 T]}$, implies $N f \in \mathcal{F}^{[a, 2 T]}$ by (62) and, thus, provides $\left.N f\right|_{0 \leq t<a}=0$. Hence, with regard to $f \equiv\langle f, 0\rangle$, we have

$$
0=\left.\left.\left.\left.N f\right|_{0 \leq t<a} \equiv[N\langle f, 0\rangle]\right|_{0 \leq t<a} \stackrel{(67)}{=}\langle m f, 0\rangle\right|_{0 \leq t<a} \equiv m f\right|_{0 \leq t<a}
$$

i.e., $m$ does not extend support to the left.

By the choice of $f$, one has $\operatorname{supp} Y f \subset[2 T-b, 2 T-a]$, so that $Y f \in$ $\mathcal{F}^{[2 T-b, 2 T]}$. The latter implies $N Y f \in \mathcal{F}^{[2 T-b, 2 T]}$ in accordance with (62). Hence, we have

$$
\begin{array}{r}
0=\left.\left.\left.N Y f\right|_{0 \leq t<2 T-b} \equiv[N\langle 0, Y f\rangle]\right|_{0 \leq t<2 T-b} \stackrel{(67)}{=}\langle 0, Y m f\rangle\right|_{0 \leq t<2 T-b} \equiv \\
\left.\equiv Y m f\right|_{0 \leq t<2 T-b}
\end{array}
$$

Therefore, $\left.m f\right|_{t>2 T-b}=0$, i.e., $m$ does not extend support to the right. Thus, $m$ acts locally and, eventually, $N$ is local.

In fact, the boundedness of $N$ is not substantial and the proof (mutatis mutandis) is available for a wider class of operators.

Lemma 4.5. If an operator $N$ satisfies (62) and is compact then $N=\mathbf{0}$.
Proof. A projection $X^{[a, b]}$ in $\mathcal{F}$ onto $\mathcal{F}^{[a, b]}$ cuts off functions on $[a, b]$. The complement projection $X_{\perp}^{[a, b]}=\mathbf{1}-X^{[a, b]}$ cuts off on $[0, a] \cup[b, 2 T]$. By Lemma 4.4, we have

$$
N X^{[a, b]}=X^{[a, b]} N X^{[a, b]} \quad \text { and } \quad N X_{\perp}^{[a, b]}=X_{\perp}^{[a, b]} N X_{\perp}^{[a, b]}
$$

Summing up, we get $N=X^{[a, b]} N X^{[a, b]}+X_{\perp}^{[a, b]} N X_{\perp}^{[a, b]}$ that leads to

$$
N X^{[a, b]}=X^{[a, b]} N, \quad N^{*} X^{[a, b]}=X^{[a, b]} N^{*}
$$

and, eventually, implies

$$
\begin{equation*}
N^{*} N X^{[a, b]}=X^{[a, b]} N^{*} N \tag{68}
\end{equation*}
$$

In the mean time, operator $N^{*} N$ is self-adjoint and compact. Let $\lambda \in \mathbf{R}$ be its eigenvalue, $\mathcal{D}_{\lambda}$ the corresponding eigensubspace. By (68), we have $X^{[a, b]} \mathcal{D}_{\lambda} \subset \mathcal{D}_{\lambda}$ that leads to $\operatorname{dim} \mathcal{D}_{\lambda}=\infty$. The latter is possible only for $\mathcal{D}_{0}=\operatorname{Ker} N^{*} N$. Thus, the spectrum of $N^{*} N$ is exhausted by $\lambda=0$. Hence, $N^{*} N=\mathbf{0}$. Therefore, $N=\mathbf{0}$.

- Now, we are ready to complete the proof of Theorem 4.2. Return to our system (55)-(57) (with $Q^{T}=q$ ). Recall that $S^{T}: \mathcal{F}^{T} \rightarrow \mathcal{F}^{2 T}$ extends controls from $[0, T]$ to $[0,2 T]$ by oddness with respect to $t=T$. We regard $\mathcal{F}^{2 T}=$ $L_{2}\left(\Sigma^{2 T}\right)$ as the space $L_{2}([0,2 T] ; \mathcal{E})$ with $\mathcal{E}=L_{2}(\Gamma)$. Let $\mathcal{F}_{ \pm}^{2 T}$ be the subspaces of the even and odd functions, so that the decomposition

$$
\mathcal{F}^{2 T}=\mathcal{F}_{+}^{2 T} \oplus \mathcal{F}_{-}^{2 T}
$$

occurs. The embedding $J^{2 T} \mathcal{F}_{-}^{2 T} \subset \mathcal{F}_{+}^{2 T}$ holds and is dense. Also, one has $Y^{2 T} \mathcal{F}_{ \pm}^{2 T}=\mathcal{F}_{ \pm}^{2 T}$.

Denote $N:=R^{2 T}-\mathcal{R}^{2 T}$ With regard to (22) and (46), the equality (61) leads to

$$
\left(N J^{2 T} S^{T} f, S^{T} g\right)_{\mathcal{F}^{2 T}}=0
$$

for all $f, g \in \mathcal{F}^{T}$. It shows that the embedding

$$
N \mathcal{F}_{+}^{2 T} \subset \mathcal{F}_{+}^{2 T}
$$

holds and evidently implies $Y^{2 T} N \mathcal{F}_{+}^{2 T} \subset \mathcal{F}_{+}^{2 T}$. In the mean time, operator $Y^{2 T} N$ is self-adjoint: see (19) and (45). Therefore, it is reduced by the even/odd subspaces: $Y^{2 T} N \mathcal{F}_{ \pm}^{2 T} \subset \mathcal{F}_{ \pm}^{2 T}$. The latter leads to

$$
\begin{equation*}
N \mathcal{F}_{ \pm}^{2 T} \subset \mathcal{F}_{ \pm}^{2 T} \tag{69}
\end{equation*}
$$

On the other hand, the shift invariance (19) and (45) implies

$$
N \mathcal{F}^{2 T, \xi} \subset \mathcal{F}^{2 T, \xi}, \quad 0 \leq \xi \leq 2 T
$$

Joining the latter relation with (69) and applying Lemma 4.5, we arrive at $N=\mathbf{O}$ that is $R^{2 T}=\mathcal{R}^{2 T}$. Theorem 4.2 is proved.

## 5. Comments, doubts, philosophy

- A characterization of data for an inverse problem is a list of conditions providing its solvability. The reasonable requirement to any characterization is to be checkable and possibly simple. As we guess, the only reasonable understanding of 'a condition is checkable' is that it can be verified before (without) solving the inverse problem. Formally, the conditions 1-7 of Theorem 4.2 satisfy such a requirement because they do not use the knowledge of the potential $q$. However, comparing these conditions with the procedure Step 1-4, it is easy to recognize that to check $1-7$ is almost the same as to recover $q$. Conditions 1-7 just provide the procedure to be realizable. In such a situation, can one claim that $1-7$ is an efficient characterization?

And what is 'efficient'? For instance, the key step of the procedure, as well as the characterization, is constructing the operator integral (47). If it is at our disposal, we get $W^{T}$, recover the waves $u^{f}$, and are able to check 5-7. In the mean time, having $u^{f}$ one doesn't need to check anything more but can just determine $q$ from the wave equation. So, can one regard the required in 3 convergence as an efficiently checkable condition? We don't have a convincible answer.

Also, can one avoid so long list of conditions and invent something simpler and better? ${ }^{4}$ We are rather sceptical and the following is some reasons for scepticism.

[^3]- The evolution of system (8)-(10) is governed by the operator $L_{q}=-\Delta+q$ and Neumann controls $f=\left.\partial_{\nu} u\right|_{\Sigma^{T}}$. Both of them are of very specific type. We mean, replacing them by $L_{Q}=-\sum_{i, j} \partial_{x_{i}} a^{i j} \partial_{x^{j}}+Q$ (with possibly nonlocal and time dependent $Q$ ) and, let say, $f=\left.\left[\partial_{\nu} u+\kappa u\right]\right|_{\Sigma^{T}}$, we'd got a system with the data $R_{Q}^{2 T}$ of the properties quite analogous to $R_{q}^{2 T}$. Therefore, the data characterization has to select $R_{q}^{2 T}$ from a large reserve of the response operators $R_{Q}^{2 T}$. It is such a selection, which the conditions $1-7$ do implement. Namely, the selection works as follows.
* Conditions 1, 2 appear at very general level of an abstract dynamical system with boundary control (DSBC) associated with a time-independent boundary triple [2]. Such a system necessarily satisfies (45) and (46).
$\star$ In 3, convergence of the operator integral to an isometric operator is a specific feature of hyperbolic DSBC's obeying the finiteness of domain of influence principle. System $\alpha^{T}$, which we deal with, is hyperbolic, and the characterization must provide such a property.

Also, as was noticed in sections 3.2, 4.1 (see (21), (33)), the amplitude integral is connected with a triangular factorization. One of the form of the classical factorization problem is to recover a triangular operator via its imaginary (anti-Hermitian) part. It is solved by the use of the so-called triangular truncation transformer [15], which is a kind of an operator integral. Its convergence provides a solvability criterium to the factorization problem for a class of Fredholm operators [15].

So, imposing condition 3, we follow the classicists. By the way, our construction (32) is available for a wider class of operators [9].
$\star$ The characterization should specify a regularity class of potentials, which we deal with. Condition 4, roughly speaking, rejects strongly singular potentials.
$\star$ Condition 5 excludes another types of boundary conditions like $f=\left[\partial_{\nu} u+\right.$ $\kappa u]\left.\right|_{\Sigma^{T}}$. The Neumann condition is rather specific. In contrast to the Dirichlet condition, which is connected with a Friedrichs operator extension, the Neumann one is not of invariant meaning. The characterization has to take this fact into account. Perhaps, one can specify the boundary condition right from $R^{2 T}$, without constructing $W^{T}$. It would be welcome.
$\star$ A discussable question is whether condition 6 may be efficiently checked. However, (51) is also unavoidable: it is the condition, which provides a locality of the potential.
$\star$ Assume for a while that $q \in L_{2}(\Omega) \backslash L_{\infty}(\Omega)$, so that the multiplication by $q$ is an unbounded operator. However, system $\alpha^{T}$ with such a potential does possess all the properties specified by conditions $1-6$. In the mean time, the characterization must reject such a case. We see no option to do it except of imposing (52).

So, all the conditions $1-7$ are independent and, therefore, unavoidable. We are forced to accept so long list of conditions just because we deal with a very specific class of dynamical systems. The more specific is the class, the more words is required for its description. The converse is also true: to be the response operator of an abstract DSBC, it suffices for $\mathcal{R}^{2 T}$ to satisfy nothing but (45) and (46) [2].

- A determination of $q$ from $R^{2 T}$ is conventionally regarded as an overdetermined problem. The reason is the following. One can represent

$$
\left(R^{2 T} f\right)(\gamma, t)=\int_{\Sigma^{t}} r\left(t-s, \gamma, \gamma^{\prime}\right) f\left(\gamma^{\prime}, s\right) d \Gamma_{\gamma^{\prime}} d s
$$

with a (generalized) kernel $r\left(t, \gamma, \gamma^{\prime}\right)$. The convolution form with respect to time is a consequence of the shift invariance (19). Bearing in mind that $\gamma=\left\{\gamma^{1}, \gamma^{2}, \ldots, \gamma^{n-1}\right\}$, one regards $r$ as a function of $1+2(n-1)=2 n-1$ variables, whereas a local potential $q=q\left(x^{1}, x^{2}, \ldots, x^{n}\right)$ depends on $n$ variables only. Thus, for $n \geq 2$ the data array is of higher dimension than the array of parameters under determination 'that is not natural' ${ }^{5}$.

Actually, on our opinion, in multidimensional problems such a counting parameters is not quite reasonable and reliable. Indeed, for instance, how to count the parameters if we need to recover from $R^{2 T}$ not a function (potential) but a Riemannian manifold, as in [3]? Nevertheless, the question arises: Does the characterization $1-7$ 'kill' unnecessary parameters and, if yes, in which way? The possible answer is the following.

There is a sharp necessary condition related with a locality of potential. Let $\tilde{\mathcal{P}}_{\sigma}^{T, \xi}$ be the projection in $\mathcal{F}^{T}$ onto the subspace $\left[\mathcal{C}^{T}\right]^{\frac{1}{2}} \mathcal{F}_{\sigma}^{T, \xi}$. Such a projection is unitarily equivalent (via the isometry $\left(I^{T}\right)^{*} \mathcal{A}^{T}$ : see (49)) to the projection onto $\overline{\mathcal{W}^{T} \mathcal{F}_{\sigma}^{T, \xi}}$. By (51), the latter projection coincides with the 'geometric' projection $G_{\sigma}^{\xi}$, which cuts off functions onto $\Omega_{\sigma}^{\xi}$. The geometric projections for all $\sigma$ and $\xi$ commute. As a result, we arrive at the following condition: the projection family $\left\{\tilde{\mathcal{P}}_{\sigma}^{T, \xi} \mid \sigma \subset \Gamma, 0 \leq \xi \leq T\right\}$ must be commutative. Analyzing the proof of Theorem 4.2, we see that it is the condition, which forces the 'potential' $Q$ to be a multiplication by $q$ and, thus, rejects unnecessary variables. However, the rejection mechanism is not well understood yet and we hope to clarify it in future.

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# Inverse problems for $p$-Laplace type equations under monotonicity assumptions 

Chang-Yu Guo, Manas Kar and Mikko Salo<br>Dedicated to Giovanni Alessandrini on the occasion of his 60th birthday


#### Abstract

We consider inverse problems for p-Laplace type equations under monotonicity assumptions. In two dimensions, we show that any two conductivities satisfying $\sigma_{1} \geq \sigma_{2}$ and having the same nonlinear Dirichlet-to-Neumann map must be identical. The proof is based on a monotonicity inequality and the unique continuation principle for p-Laplace type equations. In higher dimensions, where unique continuation is not known, we obtain a similar result for conductivities close to constant.


Keywords: inverse problems, p-Laplace equation, Dirichlet-to-Neumann map, unique continuation principle.
MS Classification 2010: 35J92, 35R30.

## 1. Introduction

The inverse conductivity problem posed by Calderón asks if the electrical conductivity of a medium can be determined by voltage and current measurements on its boundary. If $\Omega \subset \mathbb{R}^{n}$ is a bounded open set representing the medium, and if $\sigma \in L_{+}^{\infty}(\Omega)$ is a function representing the electrical conductivity, then Ohm's and Kirchhoff's laws imply that given a boundary voltage $f$, the electrical potential $u$ in $\Omega$ will solve the conductivity equation

$$
\left\{\begin{aligned}
\operatorname{div}(\sigma \nabla u) & =0 \text { in } \Omega \\
u & =f \text { on } \partial \Omega
\end{aligned}\right.
$$

Here and below we write

$$
L_{+}^{\infty}(\Omega)=\left\{\sigma \in L^{\infty}(\Omega) ; \sigma \geq c_{0}>0 \text { a.e. in } \Omega \text { for some } c_{0}>0\right\} .
$$

If $X(\Omega)$ is a function space (such as the space $W^{1, \infty}(\Omega)$ of Lipschitz functions), we also write

$$
X_{+}(\Omega)=X(\Omega) \cap L_{+}^{\infty}(\Omega)
$$

The boundary measurements are encoded by the Dirichlet-to-Neumann map (DN map)

$$
\Lambda_{\sigma}:\left.f \mapsto \sigma \partial_{\nu} u\right|_{\partial \Omega}
$$

where $\left.\sigma \partial_{\nu} u\right|_{\partial \Omega}$ is the electrical current flowing through the boundary, and the normal derivative $\partial_{\nu}$ is defined in a suitable weak sense. The inverse problem is to determine the conductivity $\sigma$ from knowledge of the DN map $\Lambda_{\sigma}$. This question, known as the Calderón problem, is a fundamental inverse problem with applications in industrial and medical imaging and having connections to many other inverse problems. Both the theoretical and applied aspects of the Calderón problem have been studied intensively in the last 35 years. We refer to the survey [35] for more information.

In this paper we consider a nonlinear variant of the Calderón problem. Here the standard Ohm's law $j=-\sigma \nabla u$ relating the current $j$ and potential $u$ is replaced by the nonlinear law

$$
j=-\sigma|\nabla u|^{p-2} \nabla u
$$

where $p$ is a real number with $1<p<\infty$. Combining this with Kirchhoff's law stating that $j$ is divergence free, we obtain the boundary value problem

$$
\left\{\begin{aligned}
\operatorname{div}\left(\sigma|\nabla u|^{p-2} \nabla u\right) & =0 \text { in } \Omega, \\
u & =f \text { on } \partial \Omega .
\end{aligned}\right.
$$

The boundary measurements are encoded by the nonlinear DN map

$$
\Lambda_{\sigma}:\left.f \mapsto \sigma|\nabla u|^{p-2} \partial_{\nu} u\right|_{\partial \Omega}
$$

defined in a suitable weak sense. The inverse problem is to determine the conductivity $\sigma$ from knowledge of the nonlinear map $\Lambda_{\sigma}$.

The equation $\operatorname{div}\left(\sigma|\nabla u|^{p-2} \nabla u\right)=0$ is called the weighted $p$-Laplace equation (with weight given by the positive function $\sigma$ ), and it is the Euler-Lagrange equation related to minimizing the $p$-Dirichlet energy $E(u)=\int_{\Omega} \sigma|\nabla u|^{p} d x$. The case $p=2$ is just the linear conductivity equation, but if $p \neq 2$ this is a quasilinear degenerate elliptic equation. The $p$-Laplace equation appears as a model for nonlinear dielectrics, plastic moulding, electro-rheological and thermo-rheological fluids, fluids governed by a power law, viscous flows in glaciology, or plasticity. The limiting cases $p=0$ and $p=1$ also arise in hybrid imaging inverse problems such as ultrasound modulated electrical impedance tomography (UMEIT) and current density imaging (CDI). See the references in [12] for further information. The $p$-Laplace equation is of considerable mathematical interest as well, the case $p=n$ is useful in conformal geometry [29] and also the limiting cases $p=0,1, \infty$ are relevant. We refer to $[16,23,30]$ for further details on $p$-Laplace type equations.

The inverse problem of determining $\sigma$ from the nonlinear DN map $\Lambda_{\sigma}$ was introduced in [32] as a nonlinear variant of the Calderón problem. There are several previous works related to Calderón type problems for nonlinear equations (see the references of [32]), where the inverse problem is solved by linearizing the nonlinear DN map. However, the $p$-Laplace type model has the new feature that linearizations at constant boundary values do not give any new information, and thus genuinely nonlinear methods are required to treat the inverse problem. The work [32] suggested a nonlinear version of the method of complex geometrical optics solutions that has been widely used in the original Calderón problem (see the survey [35]). The nonlinear version of this method was based on $p$-harmonic functions introduced by Wolff [36].

We are aware of the following results on the inverse problem for $p$-Laplace type equations:

- (Boundary uniqueness [32]) $\Lambda_{\sigma}$ determines $\left.\sigma\right|_{\partial \Omega}$.
- (Uniqueness for normal derivative [10]) $\Lambda_{\sigma}$ determines $\left.\partial_{\nu} \sigma\right|_{\partial \Omega}$.
- (Inclusion detection [12]) If $\sigma=1$ in $\Omega \backslash \bar{D}$ and $\sigma \geq 1+\varepsilon>1$ in $D$ where $D \subset \Omega$ is an obstacle, then $\Lambda_{\sigma}$ determines the convex hull of $D$. Further results are given in [11].

The first two results were based on Wolff type solutions and boundary determination arguments following Brown [13]. We remark that it would be interesting to see if also the boundary determination method based on singular solutions due to Alessandrini [3] applies to $p$-Laplace type equations. The third result above extends the enclosure method for inclusion detection introduced by Ikehata [26] to the $p$-Laplace case. The main new ingredient in [12] was a monotonicity inequality, which allows to estimate the difference of DN maps, $\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}$, under the condition $\sigma_{1} \geq \sigma_{2}$.

In this paper we continue the study of inverse problems for $p$-Laplace type equations. The main point is that a monotonicity assumption $\sigma_{1} \geq \sigma_{2}$, together with the monotonicity inequality and the unique continuation principle for $p$ Laplace type equations in the plane $[1,6,8,31]$, allows to establish interior uniqueness for the conductivities.

Theorem 1.1. Let $\Omega \subset \mathbb{R}^{2}$ be a bounded open set and let $1<p<\infty$. If $\sigma_{1}, \sigma_{2} \in W_{+}^{1, \infty}(\Omega)$ satisfy $\sigma_{1} \geq \sigma_{2}$ in $\Omega$, then $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$ implies $\sigma_{1}=\sigma_{2}$ in $\Omega$.

In three and higher dimensions, the unique continuation principle even for the standard $p$-Laplace equation remains an important open question (see for instance [18]). We obtain the following partial result under the additional assumption that one of the conductivities is close to a constant.

TheOrem 1.2. Let $\Omega \subset \mathbb{R}^{n}$ with $n \geq 2$ be a bounded open set with $C^{1, \alpha}$ boundary where $0<\alpha<1$, let $1<p<\infty$, and let $M>0$. There is a constant $\varepsilon=\varepsilon(n, p, \alpha, \Omega, M)>0$ such that if $\sigma_{1}, \sigma_{2} \in C^{\alpha}(\bar{\Omega})$ satisfy $\sigma_{1} \geq \sigma_{2}$ in $\Omega$ and

$$
1 / M \leq \sigma_{1} \leq M, \quad\left\|\sigma_{1}\right\|_{C^{\alpha}(\bar{\Omega})} \leq M, \quad\left\|\sigma_{2}-1\right\|_{C^{\alpha}(\bar{\Omega})} \leq \varepsilon
$$

then $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$ implies $\sigma_{1}=\sigma_{2}$ in $\Omega$.
Both of the above theorems are based on the monotonicity inequality, Lemma 2.2, and the existence of solutions whose gradient is nonvanishing in suitable sets. More precisely, Lemma 2.2 implies that for any $u \in W^{1, p}(\Omega)$ solving $\operatorname{div}\left(\sigma_{2}|\nabla u|^{p-2} \nabla u\right)=0$ in $\Omega$, one has

$$
(p-1) \int_{\Omega} \frac{\sigma_{2}}{\sigma_{1}^{1 /(p-1)}}\left(\sigma_{1}^{\frac{1}{p-1}}-\sigma_{2}^{\frac{1}{p-1}}\right)|\nabla u|^{p} d x \leq\left\langle\left(\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}\right)\left(\left.u\right|_{\partial \Omega}\right),\left.u\right|_{\partial \Omega}\right\rangle .
$$

Thus if $\sigma_{1}, \sigma_{2} \in C_{+}(\bar{\Omega})$ satisfy $\sigma_{1} \geq \sigma_{2}$ and $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$, it follows that

$$
|\nabla u|^{p}=0 \text { a.e. in } E
$$

for any solution $u$, where $E=\left\{x \in \Omega ; \sigma_{1}(x)>\sigma_{2}(x)\right\}$. We would like to show that $\sigma_{1}=\sigma_{2}$, or that $E$ is empty. But if $E$ would be nonempty, then all solutions $u$ would satisfy $\nabla u=0$ in the open set $E$. It is thus enough to exhibit one solution $u$ with $\nabla u \neq 0$ somewhere in $E$.

If $\sigma \in C_{+}^{\alpha}(\bar{\Omega})$ for some $\alpha>0$, we define the set of weak solutions

$$
S_{\sigma}=\left\{u \in W^{1, p}(\Omega) ; \operatorname{div}\left(\sigma|\nabla u|^{p-2} \nabla u\right)=0 \text { in } \Omega\right\} .
$$

Each $u \in S_{\sigma}$ is locally $C^{1}$ (see e.g. [28]), and we let $\mathcal{C}(u)$ be the critical set of $u$,

$$
\mathcal{C}(u)=\{x \in \Omega ;|\nabla u|(x)=0\} .
$$

The study of critical sets of solutions is of independent interest, and there are a number of results in the case $p=2$ and also in the two-dimensional case when $p \neq 2$ (see $[6,14]$ and references therein). The following question is relevant in our context, and further answers to this question would imply improvements in the above theorems when $n \geq 3$ :

Question. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded connected open set and let $1<p<\infty$. Given $\sigma \in C_{+}^{\alpha}(\bar{\Omega})$, consider the following statements:
(a) There is $u \in S_{\sigma}$ such that $\mathcal{C}(u)$ has Lebesgue measure zero.
(b) For any set $E \subset \Omega$ of positive Lebesgue measure there is $u \in S_{\sigma}$ such that $\left.\nabla u\right|_{E}$ is not zero a.e. in $E$.
(c) For any open set $U \subset \Omega$ there is $u \in S_{\sigma}$ such that $\left.\nabla u\right|_{U}$ is not zero a.e. in $U$.

For which $\sigma \in C_{+}^{\alpha}(\bar{\Omega})$ does (a), (b), or (c) hold?
Clearly $(\mathrm{a}) \Longrightarrow(\mathrm{b}) \Longrightarrow(\mathrm{c})$. We note that (a) holds for constant conductivities, or for conductivities only depending on $n-1$ variables (in these cases there is a linear function which is a solution with nonvanishing gradient). Also, (a) holds in two dimensions at least for Lipschitz $\sigma$, since $\mathcal{C}(u)$ for nonconstant $u \in S_{\sigma}$ is the set of zeros of a quasiregular map and hence has measure zero (see [6] or Appendix 4). Finally, the weak unique continuation principle would imply (c) since then $\mathcal{C}(u)$ has empty interior for any nonconstant $u \in S_{\sigma}$.

We remark that in the linear case $p=2$, uniqueness results for the inverse problem even without monotonicity assumptions have been known for a long time (see the survey [35]). Monotonicity arguments go back to [2, 4, 5, 24, 25, 27], and recently they have been combined with the method of localized potentials introduced in [17] to obtain reconstruction algorithms [20, 21]. However, the unique continuation principle and the Runge approximation property play a role in these arguments, and these facts are not known for $p$-Laplace type equations in dimensions $n \geq 3$.

This paper is organized as follows. Section 1 is the introduction. In Section 2 we establish the monotonicity inequality and the two-dimensional result, Theorem 1.1. Section 3 proves Theorem 1.2 which is valid in any dimension, by a perturbation argument around constant conductivities. We will do the proofs for the slightly more general equation

$$
\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0 \quad \text { in } \Omega
$$

where $\sigma$ is a positive scalar function and $A$ is a positive definite matrix function. Finally, Appendix 4 contains some interpolation and unique continuation results required in the proofs.

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## 2. Interior uniqueness in the plane

Given a bounded open set $\Omega \subset \mathbb{R}^{2}$ and a conductivity $\sigma \in L_{+}^{\infty}(\Omega)$, we consider the Dirichlet problem for the following $p$-Laplace type equation where $1<p<$ $\infty$,

$$
\left\{\begin{array}{l}
\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0 \text { in } \Omega,  \tag{1}\\
u=f \text { on } \partial \Omega,
\end{array}\right.
$$

where $A \in L_{+}^{\infty}\left(\Omega, \mathbb{R}^{n \times n}\right)$, meaning that $A=\left(a_{j k}\right)$ where $a_{j k} \in L^{\infty}(\Omega), a_{j k}=$ $a_{k j}$, and for some $c_{0}>0$ one has $\sum_{j, k=1}^{n} a_{j k}(x) \xi_{j} \xi_{k} \geq c_{0}|\xi|^{2}$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{n}$.

The problem (1) is well posed in $W^{1, p}(\Omega)$ for a given Dirichlet boundary data $f \in W^{1, p}(\Omega)$ (the boundary values are understood so that $u-f \in$ $\left.W_{0}^{1, p}(\Omega)\right)$, see for instance $[15,23,30,32]$. The solution $u$ minimizes the $p$ Dirichlet energy

$$
E_{p}(v)=\int_{\Omega} \sigma|A \nabla v \cdot \nabla v|^{p / 2} d x
$$

over all $v \in W^{1, p}(\Omega)$ with $v-f \in W_{0}^{1, p}(\Omega)$.
We formally define the nonlinear DN map by

$$
\Lambda_{\sigma}:\left.f \mapsto \sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u \cdot \nu\right|_{\partial \Omega},
$$

where $u \in W^{1, p}(\Omega)$ satisfies (1). More precisely, $\Lambda_{\sigma}$ is a nonlinear map $X \rightarrow X^{\prime}$ where $X$ is the abstract trace space $X=W^{1, p}(\Omega) / W_{0}^{1, p}(\Omega)$ and $X^{\prime}$ denotes the dual of $X$, and $\Lambda_{\sigma}$ is defined by the relation

$$
\begin{equation*}
\left\langle\Lambda_{\sigma}(f), g\right\rangle=\int_{\Omega} \sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u \cdot \nabla v d x, \quad f, g \in X \tag{2}
\end{equation*}
$$

where $u \in W^{1, p}(\Omega)$ is the unique solution of $\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0$ in $\Omega$ with $\left.u\right|_{\partial \Omega}=f$, and $v$ is any function in $W^{1, p}(\Omega)$ with $\left.v\right|_{\partial \Omega}=g$. Here $\langle\cdot, \cdot\rangle$ is the duality between $X^{\prime}$ and $X$. If $\partial \Omega$ has Lipschitz boundary, the trace space $X$ can be identified with the Besov space $B_{p p}^{1-1 / p}(\partial \Omega)$. Physically $\Lambda_{\sigma}(f)$ is the current flux density caused by the boundary potential $f$. See [32, Appendix] and [22] for further properties of the DN map.

The following is the main result of this section.
Theorem 2.1. Let $\Omega \subset \mathbb{R}^{2}$ be a bounded open set, let $A \in W^{1, \infty}\left(\Omega, \mathbb{R}^{n \times n}\right)$ be a symmetric positive definite matrix function, and let $\sigma_{1}, \sigma_{2} \in W_{+}^{1, \infty}(\Omega)$ be two conductivities such that $\sigma_{1} \geq \sigma_{2}$ in $\Omega$. If $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$, then $\sigma_{1}=\sigma_{2}$ in $\Omega$.

The proof is based on a monotonicity inequality and the unique continuation principle for solutions of (1). Let us first consider the monotonicity inequality, which holds true in any dimension $n \geq 2$. In the linear case $p=2$, the following
inequality is well known (see references in the introduction). For $p \neq 2$ this inequality was proved in [12] in the case $A=I$. The proof for general $A$ is almost identical, but we give it here for completeness.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded open set where $n \geq 2$, let $\sigma_{1}, \sigma_{2} \in$ $L_{+}^{\infty}(\Omega)$, let $A \in C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ be a symmetric positive definite matrix function, and let $1<p<\infty$. If $f \in W^{1, p}(\Omega)$, then

$$
\begin{aligned}
& (p-1) \int_{\Omega} \frac{\sigma_{2}}{\sigma_{1}^{1 /(p-1)}}\left(\sigma_{1}^{\frac{1}{p-1}}-\sigma_{2}^{\frac{1}{p-1}}\right)\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} d x \\
& \quad \leq\left\langle\left(\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}\right) f, f\right\rangle \leq \int_{\Omega}\left(\sigma_{1}-\sigma_{2}\right)\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} d x
\end{aligned}
$$

where $u_{2} \in W^{1, p}(\Omega)$ solves $\operatorname{div}\left(\sigma_{2}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{(p-2) / 2} A \nabla u_{2}\right)=0$ in $\Omega$ with $\left.u_{2}\right|_{\partial \Omega}=f$.

We emphasize that if $\sigma_{1} \geq \sigma_{2}$, then all the terms in the inequality are nonnegative, while if $\sigma_{1} \leq \sigma_{2}$, then they are nonpositive.

Proof. Let $u_{1}, u_{2} \in W^{1, p}(\Omega)$ be the solutions of the Dirichlet problem for the $p$-Laplace type equation,

$$
\left\{\begin{array}{l}
\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0 \text { in } \Omega  \tag{3}\\
u=f \text { on } \partial \Omega
\end{array}\right.
$$

corresponding to the conductivities $\sigma=\sigma_{1}$ and $\sigma=\sigma_{2}$ respectively.
Note that the solution of (3) can be characterized as the unique minimizer of the energy functional

$$
E_{p}(v)=\int_{\Omega} \sigma|A \nabla v \cdot \nabla v|^{p / 2} d x
$$

over the set $\left\{v \in W^{1, p}(\Omega) ; v-f \in W_{0}^{1, p}(\Omega)\right\}$ (see [32, Appendix]). Therefore, we obtain the following one sided inequality for the difference of DN maps:

$$
\begin{aligned}
\left\langle\left(\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}\right) f, f\right\rangle & =\int_{\Omega} \sigma_{1}\left|A \nabla u_{1} \cdot \nabla u_{1}\right|^{p / 2} d x-\int_{\Omega} \sigma_{2}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} d x \\
& \leq \int_{\Omega}\left(\sigma_{1}-\sigma_{2}\right)\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} d x
\end{aligned}
$$

Since $A$ is symmetric positive definite, $A=B^{\top} B$ for some symmetric matrix function $B \in C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$. The existence of such matrix $B$ is due to the Lemma 4.1 in the Appendix. To obtain the other side of the inequality, note that

$$
A \nabla u_{1} \cdot \nabla u_{2}=B \nabla u_{1} \cdot B \nabla u_{2}
$$

Let $\beta>0$ be a real number (whose value will be fixed later). Using (2) several times together with the fact that $\left.u_{1}\right|_{\partial \Omega}=\left.u_{2}\right|_{\partial \Omega}$, we may rewrite the difference of DN maps as follows:

$$
\begin{aligned}
& \left\langle\left(\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}\right) f, f\right\rangle \\
& =\int_{\Omega} \beta \sigma_{2}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} \\
& \quad-\left((1+\beta) \sigma_{2}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{\frac{p-2}{2}} A \nabla u_{2} \cdot \nabla u_{2}-\sigma_{1}\left|A \nabla u_{1} \cdot \nabla u_{1}\right|^{p / 2}\right) \mathrm{d} x \\
& =\int_{\Omega} \beta \sigma_{2}\left|B \nabla u_{2}\right|^{p}-\left((1+\beta) \sigma_{2}\left|B \nabla u_{2}\right|^{p-2} B \nabla u_{2} \cdot B \nabla u_{1}-\sigma_{1}\left|B \nabla u_{1}\right|^{p}\right) \mathrm{d} x
\end{aligned}
$$

Now, by applying Young's inequality $|a b| \leq \frac{|a|^{p}}{p}+\frac{|b|^{p^{\prime}}}{p^{\prime}}$ where $1 / p+1 / p^{\prime}=1$, we have

$$
\begin{aligned}
& (1+\beta) \sigma_{2}\left|B \nabla u_{2}\right|^{p-2} B \nabla u_{2} \cdot B \nabla u_{1}-\sigma_{1}\left|B \nabla u_{1}\right|^{p} \\
& =\frac{1+\beta}{p^{1 / p}} \frac{\sigma_{2}}{\sigma_{1}^{1 / p}}\left|B \nabla u_{2}\right|^{p-2} B \nabla u_{2} \cdot p^{1 / p} \sigma_{1}^{1 / p} B \nabla u_{1}-\sigma_{1}\left|B \nabla u_{1}\right|^{p} \\
& \leq \frac{1}{p^{\prime}}\left(\frac{1+\beta}{p^{1 / p}}\right)^{p^{\prime}} \frac{\sigma_{2}^{p^{\prime}}}{\sigma_{1}^{1 /(p-1)}}\left|B \nabla u_{2}\right|^{p}+\sigma_{1}\left|B \nabla u_{1}\right|^{p}-\sigma_{1}\left|B \nabla u_{1}\right|^{p} \\
& =\frac{1}{p^{\prime}}(1+\beta)^{p^{\prime}} \frac{1}{p^{1 /(p-1)}} \frac{\sigma_{2}^{p^{\prime}}}{\sigma_{1}^{1 /(p-1)}}\left|B \nabla u_{2}\right|^{p} .
\end{aligned}
$$

Therefore

$$
\begin{align*}
& \left\langle\left(\Lambda_{\sigma_{1}}-\Lambda_{\sigma_{2}}\right) f, f\right\rangle \\
& \geq \int_{\Omega}\left(\beta \sigma_{2}-\frac{1}{p^{\prime}}(1+\beta)^{p^{\prime}} \frac{1}{p^{1 /(p-1)}} \frac{\sigma_{2}^{p^{\prime}}}{\sigma_{1}^{1 /(p-1)}}\right)\left|B \nabla u_{2}\right|^{p} d x \\
& =\int_{\Omega} \frac{\beta \sigma_{2}}{\sigma_{1}^{1 /(p-1)}}\left(\sigma_{1}^{\frac{1}{p-1}}-\frac{1}{p^{\prime}} \frac{(1+\beta)^{p^{\prime}}}{\beta}\left(\frac{1}{p}\right)^{\frac{1}{p-1}} \sigma_{2}^{\frac{1}{p-1}}\right)\left|B \nabla u_{2}\right|^{p} d x \tag{4}
\end{align*}
$$

Note that $\frac{(1+\beta)^{p^{\prime^{\prime}}}}{\beta} \rightarrow \infty$ as $\beta \rightarrow \infty$ or $\beta \rightarrow 0$. So, the function $\beta \rightarrow \frac{(1+\beta)^{p^{\prime}}}{\beta}$ attains its minimum at $\beta=p-1$. Thus, we choose $\beta=p-1$ so that from (4), we obtain the required inequality.

Next we consider the unique continuation principle for solutions of the $p$ Laplace type equation

$$
\begin{equation*}
\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0 \tag{5}
\end{equation*}
$$

The case when $\sigma$ is constant and $A=I$ is well-known due to the work of Alessandrini [1], Bojarski-Iwaniec [8] and Manfredi [31], namely, if $u$ is a solution of the $p$-Laplace equation

$$
\operatorname{div}\left(|\nabla u \cdot \nabla u|^{(p-2) / 2} \nabla u\right)=0
$$

in a planar domain $\Omega \subset \mathbb{R}^{2}$ and if $u$ is constant in an open subset of $\Omega$, then it is actually constant in the whole domain $\Omega$. The proof of Alessandrini involves a linear equation for $\log |\nabla u|$, whereas the proof of Bojarski-Iwaniec (see also [7, Chapter 16] for a presentation) uses that complex gradients of solutions of the $p$-Laplace equation are quasiregular mappings, and that non-constant quasiregular mappings are discrete and open.

The unique continuation principle holds for solutions of (5) as well at least when the coefficients are Lipschitz, see [6, Proposition 3.3].
ThEOREM 2.3. If $\Omega$ is a domain in $\mathbb{R}^{2}, A \in W^{1, \infty}\left(\Omega, \mathbb{R}^{n \times n}\right)$ is a symmetric positive definite matrix function and $\sigma \in W_{+}^{1, \infty}(\Omega)$, and if $u \in W_{\mathrm{loc}}^{1, p}(\Omega)$ is a solution of (5) which is constant in an open nonempty subset of $\Omega$, then $u$ is identically constant in $\Omega$.

In the appendix, for possible later purposes we sketch an alternative proof of Theorem 2.3 for $A=I$ and $\sigma$ Lipschitz continuous, based on the theory of Beltrami equations, following the approach introduced by Bojarski and Iwaniec [8].

Proof of Theorem 2.1. We argue by contradiction and suppose that $\sigma_{1}\left(x_{0}\right)>$ $\sigma_{2}\left(x_{0}\right)$ for some $x_{0} \in \bar{\Omega}$. Since $\sigma_{1}$ and $\sigma_{2}$ are continuous, there exists some open ball $D \subset \Omega$ so that $\sigma_{1}-\sigma_{2}>0$ in $D$.

Let $u_{2} \in W^{1, p}(\Omega)$ be a solution of

$$
\operatorname{div}\left(\sigma_{2}|A \nabla u \cdot \nabla u|^{(p-2) / 2} A \nabla u\right)=0 \text { in } \Omega,
$$

with non-constant Dirichlet data $f \in W^{1, p}(\Omega)$ (i.e., $f-C \notin W_{0}^{1, p}(\Omega)$ for any constant $C$ ). Using the left hand side of the monotonicity inequality (Lemma 2.2), the assumptions that $\sigma_{1} \geq \sigma_{2}$ and $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$, and the fact that $\sigma_{1}^{\frac{1}{p-1}}-$ $\sigma_{2}^{\frac{1}{p-1}} \geq c_{0}>0$ in $D$, we deduce that

$$
\begin{equation*}
\int_{D}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2} d x \leq 0 \tag{6}
\end{equation*}
$$

Then $\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{p / 2}=0$ a.e. in $D$, and the uniform ellipticity condition for $A$ implies that $\nabla u_{2}=0$ a.e. in $D$, i.e., $u_{2}$ is constant on $D$. By the unique continuation principle (Theorem 2.3), we know that $u_{2}$ is constant on $\Omega$. This contradicts the fact that $u_{2}$ had non-constant Dirichlet data $f$.

Remark 2.4. Theorem 2.1 would remain valid in higher dimensions if the unique continuation principle would hold for solutions of (5).

## 3. Interior uniqueness in higher dimensions

In this section, we will consider interior uniqueness for $p$-Laplace type inverse problems in dimensions $n \geq 3$ (the method also works when $n=2$ ). We will show that two conductivities $\sigma_{1}, \sigma_{2}$ that satisfy $\sigma_{1} \geq \sigma_{2}$ in $\Omega$ and $\Lambda_{\sigma_{1}}=$ $\Lambda_{\sigma_{2}}$ must be identical in $\Omega$, under the additional assumption that one of the conductivities (as well as the matrix $A$ ) is close to constant.

Our main result reads as follows:
Theorem 3.1. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded open set with $C^{1, \alpha}$ boundary where $0<\alpha<1$, let $1<p<\infty$, and let $M>0$. There exists $\varepsilon=\varepsilon(n, p, \alpha, \Omega, M)>0$ such that for any $\sigma_{1}, \sigma_{2} \in C^{\alpha}(\bar{\Omega})$ and for any symmetric positive definite $A \in C^{\alpha}\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ satisfying

$$
\begin{gathered}
1 / M \leq \sigma_{1} \leq M \quad \text { in } \Omega \\
\left\|\sigma_{j}\right\|_{C^{\alpha}(\bar{\Omega})}+\|A\|_{C^{\alpha}(\bar{\Omega})} \leq M \\
\left\|\sigma_{2}-1\right\|_{L^{\infty}(\Omega)}+\|A-I\|_{L^{\infty}(\Omega)} \leq \varepsilon
\end{gathered}
$$

the conditions $\sigma_{1} \geq \sigma_{2}$ in $\Omega$ and $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$ imply that

$$
\sigma_{1}=\sigma_{2} \text { in } \Omega
$$

The proof is again based on the monotonicity inequality and the fact that one can find solutions whose critical sets are small (in fact empty). However, since it is not known if the unique continuation principle holds for our equations in dimensions $n \geq 3$, we will construct solutions with nonvanishing gradient by perturbing a linear function $u_{0}(x)=x_{1}$ which solves the constant coefficient $p$-Laplace equation

$$
\operatorname{div}\left(\left|\nabla u_{0}\right|^{p-2} \nabla u_{0}\right)=0 \text { in } \mathbb{R}^{n} .
$$

Alternatively, one could also perturb the complex geometrical optics or Wolff type solutions of the $p$-Laplace equation considered in [32] which also have nonvanishing gradient.

The first step is to show that if $u_{0}$ solves $\operatorname{div}\left(\sigma_{0}\left|A_{0} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} \nabla u_{0}\right)=$ 0 in $\Omega$, and if one perturbs $\sigma_{0}$ and $A_{0}$ slightly, then the solution $u_{1}$ of the perturbed equation will stay close to $u_{0}$ in $W^{1, p}$ norm if $\left.u_{1}\right|_{\partial \Omega}=\left.u_{0}\right|_{\partial \Omega}$.
Lemma 3.2. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded open set, let $1<p<\infty$, and let $M>0$. There is $C=C(n, p, \Omega, M)$ such that for any $\sigma_{0}, \sigma_{1} \in L_{+}^{\infty}(\Omega)$ and $A_{0}, A_{1} \in L_{+}^{\infty}\left(\Omega, \mathbb{R}^{n \times n}\right)$ satisfying

$$
1 / M \leq \sigma_{j} \leq M, \quad 1 / M \leq A_{j} \leq M \quad \text { a.e. in } \Omega,
$$

one has

$$
\begin{aligned}
& \left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}(\Omega)} \\
& \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}(\Omega)}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}(\Omega)}\right)^{\min \left\{\frac{1}{p-1}, 1\right\}}\left\|\nabla u_{0}\right\|_{L^{p}(\Omega)}
\end{aligned}
$$

whenever $u_{0}, u_{1} \in W^{1, p}(\Omega)$ solve

$$
\operatorname{div}\left(\sigma_{j}\left|A_{j} \nabla u_{j} \cdot \nabla u_{j}\right|^{\frac{p-2}{2}} A_{j} \nabla u_{j}\right)=0 \quad \text { in } \Omega
$$

and satisfy $u_{1}-u_{0} \in W_{0}^{1, p}(\Omega)$.
Proof. Consider the expression

$$
I:=\int_{\Omega}\left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{p-2}\left|\nabla u_{1}-\nabla u_{0}\right|^{2} d x .
$$

We will prove the estimate

$$
\begin{equation*}
I \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}}\right)\left\|\nabla u_{0}\right\|_{L^{p}}^{p-1}\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}} . \tag{7}
\end{equation*}
$$

This implies the statement in the lemma: if $p \geq 2$ the triangle inequality gives

$$
\int_{\Omega}\left|\nabla u_{1}-\nabla u_{0}\right|^{p} d x \leq \int_{\Omega}\left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{p-2}\left|\nabla u_{1}-\nabla u_{0}\right|^{2} d x=I
$$

and (7) yields

$$
\begin{equation*}
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}} \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}}\right)^{\frac{1}{p-1}}\left\|\nabla u_{0}\right\|_{L^{p}} \tag{8}
\end{equation*}
$$

On the other hand, if $1<p<2$ we write

$$
\begin{align*}
& \int_{\Omega}\left|\nabla u_{1}-\nabla u_{0}\right|^{p} d x \\
& =\int_{\Omega}\left[\left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{p-2}\left|\nabla u_{1}-\nabla u_{0}\right|^{2}\right]^{p / 2}\left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{\frac{p(2-p)}{2}} d x \tag{9}
\end{align*}
$$

and use Hölder's inequality with exponents $q=2 / p$ and $q^{\prime}=2 /(2-p)$ to get

$$
\begin{align*}
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}}^{p} & \leq I^{p / 2}\left(\int_{\Omega}\left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{p} d x\right)^{\frac{2-p}{2}} \\
& \leq C I^{p / 2}\left(\left\|\nabla u_{1}\right\|_{L^{p}}^{p}+\left\|\nabla u_{0}\right\|_{L^{p}}^{p}\right)^{\frac{2-p}{2}} . \tag{10}
\end{align*}
$$

One also has $\left\|\nabla u_{1}\right\|_{L^{p}} \leq C\left\|\nabla u_{0}\right\|_{L^{p}}$ (this can be seen by integrating the equation for $u_{1}$ against the test function $\left.u_{1}-u_{0} \in W_{0}^{1, p}(\Omega)\right)$. Using (7) yields

$$
\begin{equation*}
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}} \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}}\right)\left\|\nabla u_{0}\right\|_{L^{p}} \tag{11}
\end{equation*}
$$

The lemma follows by combining (8) (when $p \geq 2$ ) and (11) (when $1<p<2$ ).
It remains to show (7). For any fixed $x$ (outside a set of measure zero), we may factorize $A_{j}=B_{j}^{t} B_{j}$ so that one has $\left|B_{j} \xi\right|^{2}=A_{j} \xi \cdot \xi$ and

$$
\frac{1}{M}|\xi|^{2} \leq\left|B_{j} \xi\right|^{2} \leq M|\xi|^{2}, \quad \xi \in \mathbb{R}^{n}
$$

Using a basic inequality (see e.g. [32, equation (A.4)]) we have, for some $C=$ $C(n, p, M)$ which may change from line to line and for a.e. $x$,

$$
\begin{aligned}
& \left(\left|\nabla u_{1}\right|+\left|\nabla u_{0}\right|\right)^{p-2}\left|\nabla u_{1}-\nabla u_{0}\right|^{2} \\
& \leq C\left(\left|B_{1} \nabla u_{1}\right|+\left|B_{1} \nabla u_{0}\right|\right)^{p-2}\left|B_{1} \nabla u_{1}-B_{1} \nabla u_{0}\right|^{2} \\
& \leq C\left(\left|B_{1} \nabla u_{1}\right|^{p-2} B_{1} \nabla u_{1}-\left|B_{1} \nabla u_{0}\right|^{p-2} B_{1} \nabla u_{0}\right) \cdot\left(B_{1} \nabla u_{1}-B_{1} \nabla u_{0}\right) \\
& \leq C \sigma_{1}\left(\left|A_{1} \nabla u_{1} \cdot \nabla u_{1}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{1}-\left|A_{1} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{0}\right) \cdot\left(\nabla u_{1}-\nabla u_{0}\right) .
\end{aligned}
$$

Using that $u_{j}$ are solutions and $u_{1}-u_{0} \in W_{0}^{1, p}(\Omega)$, it follows that

$$
\begin{aligned}
& I \leq C \int_{\Omega} \sigma_{1}\left(\left|A_{1} \nabla u_{1} \cdot \nabla u_{1}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{1}-\left|A_{1} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{0}\right) \\
& \cdot\left(\nabla u_{1}-\nabla u_{0}\right) d x \\
&=-C \int_{\Omega} \sigma_{1}\left|A_{1} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{0} \cdot\left(\nabla u_{1}-\nabla u_{0}\right) d x \\
&\left(\sigma_{1}\left|A_{1} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{0}-\sigma_{0}\left|A_{0} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{0} \nabla u_{0}\right) \\
& \cdot\left(\nabla u_{1}-\nabla u_{0}\right) d x
\end{aligned}
$$

Writing $\sigma_{1}=\left(\sigma_{1}-\sigma_{0}\right)+\sigma_{0}$ and using the Hölder inequality, we get

$$
\begin{aligned}
I \leq & C\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}}\left\|\nabla u_{0}\right\|_{L^{p}}^{p-1}\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}} \\
& +C\left\|\left|A_{1} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{1} \nabla u_{0}-\left|A_{0} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{0} \nabla u_{0}\right\|_{L^{p /(p-1)}} \\
& \times\left[\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}}\right] .
\end{aligned}
$$

Writing $A_{1} \nabla u_{0}=\left(A_{1} \nabla u_{0}-A_{0} \nabla u_{0}\right)+A_{0} \nabla u_{0}$ and using that $\left|a_{1}^{\frac{p-2}{2}}-a_{0}^{\frac{p-2}{2}}\right| \leq$ $C\left|a_{1}-a_{0}\right|$ when $1 / M \leq a_{j} \leq M$ (choosing $a_{j}=A_{j} \frac{\nabla u_{0}}{\left|\nabla u_{0}\right|} \cdot \frac{\nabla u_{0}}{\left|\nabla u_{0}\right|}$ ), we obtain (7).

We now interpolate the $W^{1, p}$ control of $u_{1}-u_{0}$ in Lemma 3.2 with the uniform bounds obtained from the $C^{1, \beta}$ regularity theory of $p$-Laplace type equations to show that $\nabla u_{1}$ is actually close to $\nabla u_{0}$ in $L^{\infty}$.

Lemma 3.3. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded open set with $C^{1, \alpha}$ boundary where $0<\alpha<1$, let $1<p<\infty$, and let $M>0$. There exist $C>0$ and $\gamma>0$, depending on $n, p, \alpha, \Omega, M$, such that for any $\sigma_{0}, \sigma_{1} \in C^{\alpha}(\bar{\Omega})$ and $A_{0}, A_{1} \in L_{+}^{\infty}\left(\Omega, \mathbb{R}^{n \times n}\right)$ satisfying

$$
\begin{gathered}
1 / M \leq \sigma_{j}, A_{j} \leq M \quad \text { in } \Omega \\
\left\|\sigma_{j}\right\|_{C^{\alpha}(\bar{\Omega})}+\left\|A_{j}\right\|_{C^{\alpha}(\bar{\Omega})} \leq M
\end{gathered}
$$

and for any $f \in C^{1, \alpha}(\bar{\Omega})$ satisfying

$$
\|f\|_{C^{1, \alpha}(\bar{\Omega})} \leq M
$$

one has

$$
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{\infty}(\Omega)} \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}(\Omega)}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}(\Omega)}\right)^{\gamma}
$$

whenever $u_{0}, u_{1} \in W^{1, p}(\Omega)$ solve

$$
\operatorname{div}\left(\sigma_{j}\left|A_{j} \nabla u_{j} \cdot \nabla u_{j}\right|^{\frac{p-2}{2}} A_{j} \nabla u_{j}\right)=0 \quad \text { in } \Omega
$$

and satisfy $\left.u_{1}\right|_{\partial \Omega}=\left.u_{0}\right|_{\partial \Omega}=\left.f\right|_{\partial \Omega}$.
Proof. Under the stated assumptions, the weak solutions $u_{0}$ and $u_{1}$ are $C^{1, \beta}$ regular up to the boundary, see for instance [28]. More precisely, there exists $\beta=\beta(n, p, \alpha, \Omega, M)$ with $0<\beta<1$ so that $u_{0}$ and $u_{1}$ satisfy

$$
\begin{equation*}
\left\|u_{j}\right\|_{C^{1, \beta}(\bar{\Omega})} \leq C \tag{12}
\end{equation*}
$$

where $C=C(n, p, \alpha, \Omega, M)$ may change from line to line. Clearly also

$$
\left\|u_{j}\right\|_{W^{1, p}(\Omega)} \leq C
$$

It follows from Lemma 3.2 that

$$
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{L^{p}(\Omega)} \leq C\left(\left\|\sigma_{1}-\sigma_{0}\right\|_{L^{\infty}(\Omega)}+\left\|A_{1}-A_{0}\right\|_{L^{\infty}(\Omega)}\right)^{\min \left\{\frac{1}{p-1}, 1\right\}}
$$

On the other hand, (12) implies

$$
\left\|\nabla u_{1}-\nabla u_{0}\right\|_{C^{\beta}(\bar{\Omega})} \leq C
$$

The lemma follows by interpolating the last two estimates by using Lemma 4.2 in the appendix.

Now we show that the linear function $u_{0}(x)=x_{1}$ solving

$$
\operatorname{div}\left(\left|\nabla u_{0}\right|^{p-2} \nabla u_{0}\right)=0 \text { in } \Omega
$$

can be perturbed into a solution of $\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{\frac{p-2}{2}} A \nabla u\right)=0$ having nonvanishing gradient, if $\sigma$ and $A$ are sufficiently close to constant.

Lemma 3.4. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded open set with $C^{1, \alpha}$ boundary where $0<\alpha<1$, let $1<p<\infty$, and let $M>0$. There exists
$\varepsilon=\varepsilon(n, p, \alpha, \Omega, M)>0$ such that for any $\sigma \in C^{\alpha}(\bar{\Omega})$ and for any symmetric positive definite $A \in C^{\alpha}\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ satisfying

$$
\begin{gathered}
\|\sigma\|_{C^{\alpha}(\bar{\Omega})}+\|A\|_{C^{\alpha}(\bar{\Omega})} \leq M \\
\|\sigma-1\|_{L^{\infty}(\Omega)}+\|A-I\|_{L^{\infty}(\Omega)} \leq \varepsilon
\end{gathered}
$$

there exists a solution $u \in C^{1}(\bar{\Omega})$ of

$$
\operatorname{div}\left(\sigma|A \nabla u \cdot \nabla u|^{\frac{p-2}{2}} A \nabla u\right)=0 \text { in } \Omega
$$

satisfying $|\nabla u| \geq 1 / 2$ in $\Omega$.
Proof. Note that by taking $\varepsilon$ small enough, one has

$$
1 / 2 \leq \sigma \leq 2, \quad 1 / 2 \leq A \leq 2 \text { in } \Omega
$$

Choose $\sigma_{1}=\sigma, A_{1}=A$ and $\sigma_{0}=1, A_{0}=I$, and observe that the linear function $u_{0}(x)=x_{1}$ solves the $p$-Laplace equation

$$
\operatorname{div}\left(\sigma_{0}\left|A_{0} \nabla u_{0} \cdot \nabla u_{0}\right|^{\frac{p-2}{2}} A_{0} \nabla u_{0}\right)=0 \text { in } \Omega .
$$

By Lemma 3.3, there are $C>0$ and $\gamma>0$ so that the solution $u=u_{1}$ of

$$
\operatorname{div}\left(\sigma_{1}\left|A_{1} \nabla u \cdot \nabla u\right|^{\frac{p-2}{2}} A_{1} \nabla u\right)=0 \text { in } \Omega,\left.\quad u\right|_{\partial \Omega}=\left.u_{0}\right|_{\partial \Omega}
$$

satisfies

$$
\left\|\nabla u-\nabla u_{0}\right\|_{L^{\infty}(\Omega)} \leq C\left(\|\sigma-1\|_{L^{\infty}(\Omega)}+\|A-I\|_{L^{\infty}(\Omega)}\right)^{\gamma} .
$$

If $\varepsilon$ is chosen so that $C(2 \varepsilon)^{\gamma} \leq 1 / 2$, we have

$$
|\nabla u| \geq\left|\nabla u_{0}\right|-\left|\nabla u-\nabla u_{0}\right| \geq 1 / 2 \text { in } \Omega .
$$

Proof of Theorem 3.1. First choose $\varepsilon$ as in Lemma 3.4, and choose $u \in W^{1, p}(\Omega)$ so that $u$ solves

$$
\operatorname{div}\left(\sigma_{2}|A \nabla u \cdot \nabla u|^{\frac{p-2}{2}} A \nabla u\right)=0 \text { in } \Omega
$$

and satisfies $|\nabla u| \geq 1 / 2$ in $\Omega$. We now use a similar argument as in the proof of Theorem 2.1. The conditions $\sigma_{1} \geq \sigma_{2}$ in $\Omega$ and $\Lambda_{\sigma_{1}}=\Lambda_{\sigma_{2}}$ together with the monotonicity inequality, Lemma 2.2, imply that

$$
\left|\nabla u_{2}\right|^{p}=0 \text { a.e. in } E
$$

for any $u_{2}$ solving $\operatorname{div}\left(\sigma_{2}\left|A \nabla u_{2} \cdot \nabla u_{2}\right|^{(p-2) / 2} A \nabla u_{2}\right)=0$, where $E=\{x \in$ $\left.\Omega ; \sigma_{1}(x)>\sigma_{2}(x)\right\}$. If the open set $E$ were nonempty, one could choose $u_{2}=u$ to obtain a contradiction. Thus $E$ must be empty and we have $\sigma_{1}=\sigma_{2}$ in $\Omega$.

## 4. Appendix. Auxiliary results

In this appendix, we first prove a result related to the decomposition of a positive definite matrix having continuous entries and then state an interpolation result. Finally we finish this section by recalling a proof of the unique continuation principle for the weighted $p$-Laplace equation in the plane.

### 4.1. Matrix decomposition

Lemma 4.1. Let $A \in C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ be an $n \times n$ symmetric positive definite matrix function. Then there exists a matrix function $B \in C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ such that $A=$ $B^{\top} B$.

Proof. Consider the following inner product and norm on $\mathbb{R}^{n}$ defined for $x \in \Omega$,

$$
\langle v, w\rangle_{A(x)}=A(x) v \cdot w, \quad|v|_{A(x)}=(A(x) v \cdot v)^{1 / 2}, \quad v, w \in \mathbb{R}^{n}
$$

We apply the Gram-Schmidt orthonormalization procedure to the standard basis $\left\{e_{1}, \ldots, e_{n}\right\}$ of $\mathbb{R}^{n}$ with respect to this inner product. Define

$$
w_{1}(x)=e_{1}, \quad v_{1}(x)=w_{1} /\left|w_{1}\right|_{A(x)}
$$

and if $k \geq 2$ define inductively

$$
\begin{aligned}
w_{k}(x) & =e_{k}-\left\langle e_{k}, v_{1}(x)\right\rangle_{A(x)} v_{1}(x)-\ldots-\left\langle e_{k}, v_{k-1}(x)\right\rangle_{A(x)} v_{k-1}(x) \\
v_{k}(x) & =w_{k}(x) /\left|w_{k}(x)\right|_{A(x)}
\end{aligned}
$$

Now $v_{1}(x)=e_{1} / \sqrt{a_{11}(x)}$ is a continuous vector function in $x$, and inductively one sees that each $v_{k}(x)$ is also continuous in $x$. Here it is crucial that $A(x)$ is positive definite and that $\left\{e_{1}, \ldots, e_{n}\right\}$ are linearly independent, so the denominators $\left|w_{k}(x)\right|_{A(x)}$ are positive and continuous.

The above process leads to a basis $\left\{v_{1}(x), \ldots, v_{n}(x)\right\}$ of $\mathbb{R}^{n}$ which is orthonormal in the $A(x)$ inner product, i.e., $A(x) v_{j}(x) \cdot v_{k}(x)=\delta_{j k}$. Then $V(x)=\left(\begin{array}{lll}v_{1}(x) & \ldots & v_{n}(x)\end{array}\right)$ is a matrix function in $C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ and satisfies

$$
V(x)^{\top} A(x) V(x)=I
$$

Linear independence implies that $\operatorname{det}(V(x)) \neq 0$ for all $x \in \Omega$. It follows that the matrix $B(x)=V(x)^{-1}$ is in $C\left(\bar{\Omega}, \mathbb{R}^{n \times n}\right)$ and satisfies $B(x)^{\top} B(x)=A(x)$ in $\Omega$.

### 4.2. Interpolation

Lemma 4.2. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^{n}, n \geq 2$, let $0<\beta<1$, and let $1<p<\infty$. For any $\theta \in\left(\frac{n / p}{\beta+n / p}, 1\right]$ there is $C>0$ such that whenever
$f \in C^{\beta}(\bar{\Omega})$ satisfies

$$
\begin{aligned}
\|f\|_{L^{p}(\Omega)} & \leq M_{0} \\
\|f\|_{C^{\beta}(\bar{\Omega})} & \leq M_{1},
\end{aligned}
$$

then

$$
\|f\|_{L^{\infty}(\Omega)} \leq C M_{0}^{1-\theta} M_{1}^{\theta}
$$

Proof. Recall that, for $0<\beta<1$, the Hölder space $C^{\beta}(\bar{\Omega})$ is precisely the Besov space $B_{\infty \infty}^{\beta}(\Omega)$. We also denote by $W^{s, p}(\Omega)$ the $L^{p}$ Sobolev space with smoothness index $s$.

By the results in [33, Section 23], whenever $\varepsilon>0$ and $2 \leq q<\infty$ one has the continuous embeddings (which in fact hold for any bounded domain $\Omega$, not necessarily Lipschitz):

$$
B_{\infty \infty}^{\beta}(\Omega) \subset B_{q 2}^{\beta-\varepsilon}(\Omega) \subset F_{q 2}^{\beta-\varepsilon}(\Omega)=W^{\beta-\varepsilon, q}(\Omega)
$$

Thus we see that

$$
\begin{aligned}
\|f\|_{W^{0, p}(\Omega)} & \leq M_{0} \\
\|f\|_{W^{\beta-\varepsilon, q}(\Omega)} & \leq C M_{1}
\end{aligned}
$$

By complex interpolation [34, Theorem 2.13], we obtain for any $0 \leq t \leq 1$ that

$$
\|f\|_{W^{s_{t}, r_{t}}(\Omega)} \leq M_{0}^{1-t}\left(C M_{1}\right)^{t}
$$

where $s_{t}=t(\beta-\varepsilon)$ and $\frac{1}{r_{t}}=(1-t) \frac{1}{p}+t \frac{1}{q}$.
Now fix $\theta$ with $\frac{n / p}{\beta+n / p}<\theta \leq 1$, and fix $\varepsilon>0$ and $2 \leq q<\infty$ so that $\theta(\beta-\varepsilon)>\frac{n}{r}$ where $\frac{1}{r}=(1-\theta) \frac{1}{p}+\theta \frac{1}{q}$ (this condition is equivalent with $\theta\left(\beta-\varepsilon+\frac{n}{p}-\frac{n}{q}\right)>\frac{n}{p}$, and such $\varepsilon$ and $q$ exist since $\left.\theta>\frac{n / p}{\beta+n / p}\right)$. Choosing $t=\theta$ above and using the continuous embedding $W^{s_{\theta}, r_{\theta}}(\Omega) \subset L^{\infty}(\Omega)$, which follows from [33, Section 23] since $s_{\theta}>n / r_{\theta}$, we get

$$
\|f\|_{L^{\infty}(\Omega)} \leq C M_{0}^{1-\theta}\left(C M_{1}\right)^{\theta}
$$

as required. (After the initial embeddings, one could also use the universal extension operator for Lipschitz domains [34, Theorem 2.11] and work in $\mathbb{R}^{n}$.)

### 4.3. Unique continuation principle for weighted $p$-harmonic functions in the plane

In this subsection, we sketch a proof of the unique continuation principle for the weighted $p$-Laplace equation

$$
\begin{equation*}
\operatorname{div}\left(\sigma|\nabla u|^{p-2} \nabla u\right)=0 \tag{13}
\end{equation*}
$$

in a bounded domain $\Omega \subset \mathbb{R}^{2}$ following [8]. This result is also a special case of [6, Proposition 3.3]. Indeed, according to a very recent work of [19], even the strong unique continuation principle holds for the solutions of (13). Assume that $\sigma$ is positive and Lipschitz continuous in $\Omega$.

We first consider the case $p \geq 2$. Let us define a vector field $F: \Omega \rightarrow \mathbb{R}^{2}$ by

$$
F(x)=\sigma^{p / 2}|\nabla u(x)|^{(p-2) / 2} \nabla u(x)
$$

where $u$ satisfies (13). Then it follows from [8, Proof of Proposition 2] that $F \in W_{l o c}^{1,2}\left(\Omega, \mathbb{R}^{2}\right)$.

We write $f=\sigma u_{x}-i \sigma u_{y}$ for the complex gradient of $u$ in the complex plane, where $z=x+i y$, and define the nonlinear counterparts of the complex gradient $f$ by $F_{a}=|f|^{a} f, a>-1$. In the following computations, we will derive the nonlinear first order elliptic system for $F$, that is $F_{a}$ with $a=(p-2) / 2$.

From the definition of $f$ and $F_{a}$, we have

$$
2 u_{x}=\frac{1}{\sigma}\left|F_{a}\right|^{-\frac{a}{a+1}}\left(F_{a}+\overline{F_{a}}\right)
$$

and

$$
2 u_{y}=i \frac{1}{\sigma}\left|F_{a}\right|^{-\frac{a}{a+1}}\left(F_{a}-\overline{F_{a}}\right) .
$$

Therefore from the above equalities we have,

$$
\frac{\partial}{\partial y}\left[\frac{1}{\sigma}\left|F_{a}\right|^{-\frac{a}{a+1}}\left(F_{a}+\overline{F_{a}}\right)\right]=i \frac{\partial}{\partial x}\left[\frac{1}{\sigma}\left|F_{a}\right|^{-\frac{a}{a+1}}\left(F_{a}-\overline{F_{a}}\right)\right]
$$

This is equivalent to

$$
\begin{equation*}
\operatorname{Im} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma}\left|F_{a}\right|^{-\frac{a}{a+1}} F_{a}\right)=0 \tag{14}
\end{equation*}
$$

Note that, for $a=(p-2) / 2, F_{a}$ is differentiable almost everywhere and so (14) reduces to the complex equation

$$
\begin{align*}
\frac{\partial}{\partial \bar{z}} F_{a}-\overline{\frac{\partial}{\partial \bar{z}} F_{a}}= & -\frac{a}{a+2}\left[\frac{\bar{F}_{a}}{F_{a}} \frac{\partial}{\partial z} F_{a}-\frac{F_{a}}{\bar{F}_{a}} \overline{\frac{\partial}{\partial z} F_{a}}\right] \\
& +\sigma \frac{2 a+2}{a+2}\left[\bar{F}_{a} \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)-F_{a} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma}\right)\right] \tag{15}
\end{align*}
$$

On the other hand, since $u$ satisfies weighted $p$-Laplacian equation

$$
\nabla \cdot\left(\sigma|\nabla u|^{p-2} \nabla u\right)=0
$$

we have

$$
\nabla \cdot\left[\frac{1}{\sigma^{p-2}}\left|F_{a}\right|^{\frac{p-2-a}{a+1}}\left(F_{a}+\overline{F_{a}}, i\left(F_{a}-\overline{F_{a}}\right)\right)\right]=0
$$

which is equivalent to the equation

$$
\frac{\partial}{\partial x}\left\{\frac{1}{\sigma^{p-2}}\left|F_{a}\right|^{\frac{p-2-a}{a+1}}\left(F_{a}+\overline{F_{a}}\right)\right\}+i \frac{\partial}{\partial y}\left\{\frac{1}{\sigma^{p-2}}\left|F_{a}\right|^{\frac{p-2-a}{a+1}}\left(F_{a}-\overline{F_{a}}\right)\right\}=0 .
$$

Using the complex notation we can write

$$
\operatorname{Re} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma^{p-2}}\left|F_{a}\right|^{\frac{p-2-a}{a+1}} F_{a}\right)=0 .
$$

For $a=(p-2) / 2, F_{a}$ is differentiable almost everywhere and so the last equation can be written as

$$
\begin{align*}
\frac{\partial}{\partial \bar{z}} F_{a}+\overline{\frac{\partial}{\partial \bar{z}} F_{a}}= & -\frac{p-2-a}{a+p}\left[\frac{\overline{F_{a}}}{F_{a}} \frac{\partial}{\partial z} F_{a}+\frac{F_{a}}{\overline{F_{a}}} \frac{\partial}{\partial z} F_{a}\right] \\
& -\sigma^{p-2} \frac{2 a+2}{a+p}\left[F_{a} \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma^{p-2}}\right)+\overline{F_{a}} \frac{\partial}{\partial z}\left(\frac{1}{\sigma^{p-2}}\right)\right] . \tag{16}
\end{align*}
$$

Adding (15) and (16), we get (with $a=(p-2) / 2)$

$$
\begin{equation*}
\frac{\partial}{\partial \bar{z}} F=q_{1} \frac{\partial}{\partial z} F+q_{2} \overline{\frac{\partial}{\partial z} F}+H(z, F) \tag{17}
\end{equation*}
$$

where

$$
\begin{aligned}
q_{1} & =-\frac{1}{2}\left(\frac{p-2}{p+2}+\frac{p-2}{3 p-2}\right) \frac{\bar{F}}{F} \\
q_{2} & =-\frac{1}{2}\left(\frac{p-2}{3 p-2}-\frac{p-2}{p+2}\right) \frac{F}{\bar{F}}
\end{aligned}
$$

and

$$
\begin{aligned}
H(z, F)= & \sigma \frac{p}{p+2}\left[\bar{F} \frac{\partial}{\partial z}\left(\frac{1}{\sigma}\right)-F \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma}\right)\right] \\
& -\sigma^{p-2} \frac{p}{3 p-2}\left[\bar{F} \frac{\partial}{\partial z}\left(\frac{1}{\sigma^{p-2}}\right)+F \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\sigma^{p-2}}\right)\right] .
\end{aligned}
$$

It is easy to check that $\left|q_{1}\right|+\left|q_{2}\right|<1$ and $|H(z, F)| \leq q_{3}(z)|F|$ with $q_{3} \in L^{\infty}$. Under these structure assumptions for $q_{1}, q_{2}$ and $q_{3}$, by [ 9 , Section 8.4], the solution of (17) can be represented as

$$
\begin{equation*}
F(z)=H(\chi(z)) e^{\varphi(z)} \tag{18}
\end{equation*}
$$

where $H$ is analytic, $\chi$ is quasiconformal and $\varphi$ is Hölder continuous in $\bar{\Omega}$ with $\varphi_{\bar{z}}, \varphi_{z} \in L^{q}(\Omega)$ for some $q>2$. Write $R(\xi)=|\xi|^{\frac{2 a+2-p}{2 p}} \xi$ and it is clear that

$$
\begin{equation*}
F_{a}(z)=R \circ F_{(p-2) / 2}=|H(\chi(z))|^{\beta} H(\chi(z)) e^{(\beta+1) \varphi(z)} \tag{19}
\end{equation*}
$$

Note that the function $G_{\beta}:=|H(\chi(z))|^{\beta} H(\chi(z))$ is quasiregular as being the composition of a quasiregular mapping with an analytic function.

Suppose now $u$ is constant on an open subset of $\Omega$, then $F_{a}$ will vanish on that open subset, which together with the observation that $e^{(\beta+1) \varphi(z)}$ is nonzero, implies that $G_{\beta}$ is zero on that open subset. Since $G_{\beta}$ is quasiregular, it is either constant or both discrete and open, and $G_{\beta}$ being zero in an open subset of $\Omega$ necessarily forces $G_{\beta}$ to be zero everywhere in $\Omega$. This implies that $F_{a} \equiv 0$ in $\Omega$ and hence $u$ is identically constant in $\Omega$.

The case $p \in(1,2)$ can be treated identically as above, provided that we are able to show $F \in W_{l o c}^{1,2}\left(\Omega, \mathbb{R}^{2}\right)$. Note that since the regularity is a local issue, we may assume that $\Omega$ is a simply connected bounded domain. As in [7, Page 426-427], this can be done by a very elegant argument involving the weighted dual $q$-harmonic equation, where $\frac{1}{p}+\frac{1}{q}=1$. Namely, there exists a weighted $q$-harmonic function $v \in W_{l o c}^{1, q}(\Omega)$ satisfying

$$
\operatorname{div}\left(\sigma^{1-q}|\nabla v|^{q-2} \nabla v\right)=0
$$

such that $v_{x}=-\sigma|\nabla u|^{p-2} u_{y}$ and $v_{y}=\sigma|\nabla u|^{p-2} u_{x}$. Since $q>2$ and $\sigma^{1-q}$ is positive and Lipschitz, it follows again from [8, Proof of Proposition 2] that $|\nabla v|^{(q-2) / 2} \nabla v \in W_{l o c}^{1,2}\left(\Omega, \mathbb{R}^{2}\right)$ and so $F \in W_{l o c}^{1,2}\left(\Omega, \mathbb{R}^{2}\right)$ as desired.

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# Stable determination of an inclusion in an inhomogeneous elastic body by boundary measurements 

Antonino Morassi and Edi Rosset<br>"Dedicato a Giovanni Alessandrini per il suo sessantesimo compleanno"


#### Abstract

In this paper we consider the stability issue for the inverse problem of determining an unknown inclusion contained in an elastic body by all the pairs of measurements of displacement and traction taken at the boundary of the body. Both the body and the inclusion are made by inhomogeneous linearly elastic isotropic material. Under mild a priori assumptions about the smoothness of the inclusion and the regularity of the coefficients, we show that the logarithmic stability estimate proved in [3] in the case of piecewise constant coefficients continues to hold in the inhomogeneous case. We introduce new arguments which allow to simplify some technical aspects of the proof given in [3].


Keywords: Inverse problems, linearized elasticity, inclusions, stability, unique continuation.
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## 1. Introduction

The inverse problem of determining unknown inclusions in continuous bodies from measurements of physical parameters taken at the boundary of the body has attracted a lot of attention in the last thirty years, see, among other contributions, the reconstruction results obtained in [12, 17, 18]. Inclusions may be due to the presence of inhomogeneities or defects inside the body, and the development of non-invasive testing approaches is of great importance in several practical contexts, ranging from medicine to engineering applications.

Inverse problems of this class are usually ill-posed according to Hadamard's definition, and one of the main issues is the uniqueness of the solution, that is the determination of the boundary measurements which ensure the unique determination of the defect. Moreover, from the point of view of practical applications, it is crucial to establish how small perturbations on the data may
affect the accuracy of the identification of the inclusion, namely, the study of the stability issue.

The prototype of these inverse problems is the determination of an inclusion inside an electric conductor from boundary measurements of electric potential and current flux. Uniqueness was first proved by Isakov in ' 88 [14]. The first stability result is due to Alessandrini and Di Cristo [2], who derived a logarithmic stability estimate of the inclusion from all possible boundary measurements, that is from the full Dirichlet-to-Neumann map. More precisely, the authors considered in [2] the case of piecewise-constant coefficients and constructed an ingenious proof which, starting from Alessandrini's identity (first derived in [1]), makes use of fundamental solutions for elliptic equations with discontinuous coefficients, and suitable quantitative forms of unique continuation for solutions to Laplacian equation. An extension of the above result to the case of variable coefficients was derived in [8]. The pioneering work [2] stimulated a subsequent line of research in which methods and results were extended to other frameworks, such as, for example, the stable identification of inclusions in thermal conductors [9, 10], which involves a parabolic equation with discontinuous coefficients.

Concerning the determination of an inclusion in an elastic body from the Dirichlet-to-Neumann map, the uniqueness was proved by Ikehata, Nakamura and Tanuma in [13]. The stability issue has been recently faced in [3]. The statical equilibrium of the defected body is governed by the following system of elliptic equations

$$
\begin{equation*}
\operatorname{div}\left(\left(\mathbb{C}+\left(\mathbb{C}^{D}-\mathbb{C}\right) \chi_{D}\right) \nabla u\right)=0, \quad \text { in } \Omega \tag{1}
\end{equation*}
$$

where $u$ is the three-dimensional displacement field inside the elastic body $\Omega$, $\chi_{D}$ is the characteristic function of the inclusion $D$, and $\mathbb{C}, \mathbb{C}^{D}$ is the elasticity tensor in the background material and inside the inclusion, respectively. Given inclusions $D_{1}, D_{2}$, let $\Lambda_{D_{i}}: H^{1 / 2}(\partial \Omega) \rightarrow H^{-1 / 2}(\partial \Omega)$ be the Dirichlet-toNeumann map which gives the traction at the boundary $\partial \Omega$ corresponding to a displacement field assigned on $\partial \Omega$, when $D=D_{i}, i=1,2$. Assuming that $\mathbb{C}, \mathbb{C}^{D_{1}}=\mathbb{C}^{D_{2}}$ are constant and of Lamé type (e.g., isotropic material), and under $C^{1, \alpha}$-regularity of the boundary of the inclusion, the authors derived the following stability result. If, for some $\epsilon, 0<\epsilon<1$,

$$
\begin{equation*}
\left\|\Lambda_{D_{1}}-\Lambda_{D_{2}}\right\|_{\mathcal{L}\left(H^{1 / 2}(\partial \Omega), H^{-1 / 2}(\partial \Omega)\right)} \leq \epsilon \tag{2}
\end{equation*}
$$

then the Hausdorff distance between the two inclusions can be controlled as

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq \frac{C}{|\log \epsilon|^{\eta}}, \tag{3}
\end{equation*}
$$

where the constants $C>0$ and $\eta, 0<\eta \leq 1$, only depend on the a-priori data.

The piecewise-constant Lamé case can be considered as a simplified mathematical model of real elastic bodies. Therefore, it is of practical interest to extend the stability estimate (3) to variable coefficients both in the background, $\mathbb{C}=\mathbb{C}(x)$, and in the inclusions, $\mathbb{C}^{D_{i}}=\mathbb{C}^{D_{i}}(x), i=1,2$. More precisely, assuming $C^{1,1}$ and $C^{\tau}$ regularity, $\tau \in(0,1)$, for $\mathbb{C}$ and $\mathbb{C}^{D_{i}}$, respectively, $i=1,2$, in this paper we show that (3) continues to hold. Let us emphasize that in order to derive our result the exact knowledge of the elasticity tensor inside the inclusion is not needed. In fact, only the strong convexity conditions (16) and the bounds (17), (20), (22) are required. Moreover, as in [3], the inclusion is allowed to share a portion of its boundary with the boundary of the body $\Omega$.

Let us briefly recall the main ideas of our approach and the new mathematical tools we used in the proof of the stability result. Let $\Gamma^{D_{i}}$ be the fundamental matrix associated to the elasticity tensor $\left(\mathbb{C}+\left(\mathbb{C}^{D_{i}}-\mathbb{C}\right) \chi_{D_{i}}\right), i=1,2$. The main idea is to obtain an upper and a lower bound for $\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)(y, w)$ for points $y$ and $w$ belonging to the connected component of $\mathbb{R}^{3} \backslash\left(\overline{D_{1} \cup D_{2}}\right)$ which contains $\mathbb{R}^{3} \backslash \bar{\Omega}$, and approaching non-tangentially a suitable point $P \in \partial D_{1} \backslash \overline{D_{2}}$ (or $\partial D_{2} \backslash \overline{D_{1}}$ ). A first crucial ingredient in determining both upper and lower bounds is the integral representation of $\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)(y, w)$ given by formula (40). Next, the upper bound follows from an application of Alessandrini's identity (suitably adapted to linear elasticity, see Lemma 6.1 in [3]) and a propagation of smallness argument based on iterated use of the three spheres inequality for solutions to the Lamé system of linear elasticity with smooth variable coefficients.

In proving the lower bound (see Section 4) we introduce new arguments which entail a simplification of the proof given for the piecewise-constant coefficient case. Indeed, a generalization of Theorem 8.1 in [3], which was a key tool in proving the lower bound, should need the derivation of an asymptotic approximation of $\Gamma^{D}$ in terms of the fundamental matrix obtained by locally flattening the boundary $\partial D$ and freezing the coefficients at a point belonging to $\partial D$, which does not appear straightforward.

Finally, let us emphasize that the statement of Theorem 8.1 in [3], besides being worth of interest from a theoretical viewpoint, may have relevant interest for its possible applications. In fact, it turned out to be a fundamental ingredient in the proof of Lipschitz stability estimates for the inverse problem of determining the Lamé moduli for a piecewise constant elasticity tensor corresponding to a known partition of the body in a finite number of subdomains having regular interfaces [6], see also [7] for the case of flat interfaces.

The plan of the paper is as follows. Notation and the a priori information are introduced in section 2 , together with the statement of the stability result (Theorem 2.2). In section 3 we recall some auxiliary results, we state the upper and lower bounds on $\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)$, Theorems 3.4 and 3.5 , and we give the proof of the main Theorem 2.2. Section 4 is devoted to the proof of Theorem 3.5.

## 2. The main result

### 2.1. Notation

Let us denote $\mathbb{R}_{+}^{3}=\left\{x \in \mathbb{R}^{3} \mid x_{3}>0\right\}$ and $\mathbb{R}_{-}^{3}=\left\{x \in \mathbb{R}^{3} \mid x_{3}<0\right\}$. Given $x \in \mathbb{R}^{3}$, we shall denote $x=\left(x^{\prime}, x_{3}\right)$, where $x^{\prime}=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}, x_{3} \in \mathbb{R}$. Given $x \in \mathbb{R}^{3}$ and $r>0$, we shall use the following notation for balls in three and two dimensions:

$$
\begin{gathered}
B_{r}(x)=\left\{y \in \mathbb{R}^{3} \quad|\quad| y-x \mid<r\right\}, \quad B_{r}=B_{r}(O), \\
B_{r}^{\prime}\left(x^{\prime}\right)=\left\{y^{\prime} \in \mathbb{R}^{2} \quad| | y^{\prime}-x^{\prime} \mid<r\right\}, \quad B_{r}^{\prime}=B_{r}^{\prime}(O) .
\end{gathered}
$$

Definition 2.1 ( $C^{k, \alpha}$ regularity). Let $E$ be a domain in $\mathbb{R}^{3}$. Given $k$, $\alpha$, $k \in \mathbb{N}, 0<\alpha \leq 1$, we say that $E$ is of class $C^{k, \alpha}$ with constants $\rho_{0}, M_{0}>0$, if, for any $P \in \partial E$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$
E \cap B_{\rho_{0}}(O)=\left\{x \in B_{\rho_{0}}(O) \quad \mid \quad x_{3}>\varphi\left(x^{\prime}\right)\right\},
$$

where $\varphi$ is a $C^{k, \alpha}$ function on $B_{\rho_{0}}^{\prime}$ satisfying

$$
\begin{gathered}
\varphi(O)=0 \\
|\nabla \varphi(O)|=0, \quad \text { when } k \geq 1 \\
\|\varphi\|_{C^{k, \alpha}\left(B_{\rho_{0}}^{\prime}(O)\right)} \leq M_{0} \rho_{0}
\end{gathered}
$$

Here and in the sequel all norms are normalized such that their terms are dimensionally homogeneous. For instance

$$
\|\varphi\|_{C^{k, \alpha}\left(B_{\rho_{0}}^{\prime}(O)\right)}=\sum_{i=0}^{k} \rho_{0}^{i}\left\|\nabla^{i} \varphi\right\|_{L^{\infty}\left(B_{\rho_{0}}^{\prime}(O)\right)}+\rho_{0}^{k+\alpha}\left|\nabla^{k} \varphi\right|_{\alpha, B_{\rho_{0}}^{\prime}(O)},
$$

where

$$
\left|\nabla^{k} \varphi\right|_{\alpha, B_{\rho_{0}}^{\prime}(O)}=\sup _{\substack{x^{\prime}, y^{\prime} \in B_{\rho_{0}}^{\prime}(O) \\ x^{\prime} \neq y^{\prime}}} \frac{\left|\nabla^{k} \varphi\left(x^{\prime}\right)-\nabla^{k} \varphi\left(y^{\prime}\right)\right|}{\left|x^{\prime}-y^{\prime}\right|^{\alpha}} .
$$

Similarly, for a vector function $u: \Omega \subset \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$, we set

$$
\|u\|_{H^{1}\left(\Omega, \mathbb{R}^{3}\right)}=\left(\int_{\Omega}|u|^{2}+\rho_{0}^{2} \int_{\Omega}|\nabla u|^{2}\right)^{\frac{1}{2}}
$$

and so on for boundary and trace norms such as $\|\cdot\|_{H^{\frac{1}{2}}\left(\partial \Omega, \mathbb{R}^{3}\right)}\|\cdot\|_{H^{-\frac{1}{2}}\left(\partial \Omega, \mathbb{R}^{3}\right)}$.

For any $U \subset \mathbb{R}^{3}$ and for any $r>0$, we denote

$$
\begin{align*}
& U_{r}=\{x \in U \mid \operatorname{dist}(x, \partial U)>r\},  \tag{4}\\
& U^{r}=\left\{x \in \mathbb{R}^{3} \mid \operatorname{dist}(x, U)<r\right\} . \tag{5}
\end{align*}
$$

We denote by $\mathbb{M}^{m \times n}$ the space of $m \times n$ real valued matrices and we also use the notation $\mathbb{M}^{n}=\mathbb{M}^{n \times n}$. Let $\mathcal{L}(X, Y)$ be the space of bounded linear operators between Banach spaces $X$ and $Y$.

For every pair of real $n$-vectors $a$ and $b$, we denote by $a \otimes b$ the $n \times n$ matrix with entries

$$
\begin{equation*}
(a \otimes b)_{i j}=a_{i} b_{j}, \quad i, j=1, \ldots, n \tag{6}
\end{equation*}
$$

For every $3 \times 3$ matrices $A, B$ and for every $\mathbb{C} \in \mathcal{L}\left(\mathbb{M}^{3}, \mathbb{M}^{3}\right)$, we use the following notation:

$$
\begin{align*}
(\mathbb{C} A)_{i j} & =\sum_{k, l=1}^{3} C_{i j k l} A_{k l}  \tag{7}\\
A \cdot B & =\sum_{i, j=1}^{3} A_{i j} B_{i j}  \tag{8}\\
|A| & =(A \cdot A)^{\frac{1}{2}} \tag{9}
\end{align*}
$$

where $C_{i j k l}, A_{i j}$ and $B_{i j}$ are the entries of $\mathbb{C}, A$ and $B$ respectively.
Finally, let us recall the definition of the Hausdorff distance $d_{H}(A, B)$ of two bounded closed sets $A, B \subset \mathbb{R}^{3}$

$$
d_{H}(A, B)=\max \left\{\max _{x \in A} d(x, B), \max _{x \in B} d(x, A)\right\}
$$

### 2.2. A-priori information and main result

We make the following a-priori assumptions. The continuous body $\Omega$ is a bounded domain in $\mathbb{R}^{3}$ such that

$$
\begin{gather*}
\mathbb{R}^{3} \backslash \bar{\Omega} \text { is connected, }  \tag{10}\\
|\Omega| \leq M_{1} \rho_{0}^{3},  \tag{11}\\
\Omega \text { is of class } C^{1, \alpha}, \text { with constants } \rho_{0}, M_{0}, \tag{12}
\end{gather*}
$$

and the inclusion $D$ is a connected subset of $\Omega$ satisfying

$$
\begin{gather*}
\mathbb{R}^{3} \backslash \bar{D} \text { is connected, }  \tag{13}\\
D \text { is of class } C^{1, \alpha}, \text { with constants } \rho_{0}, M_{0} \tag{14}
\end{gather*}
$$

where $\rho_{0}, M_{0}, M_{1}$ are given positive constants, and $0<\alpha \leq 1$.
The background material is linearly elastic isotropic, with elasticity tensor $\mathbb{C}=\mathbb{C}(x)$, which - without restriction - may be defined in the whole $\mathbb{R}^{3}$. The cartesian components of $\mathbb{C}(x)$ are

$$
\begin{equation*}
C_{i j k l}(x)=\lambda(x) \delta_{i j} \delta_{k l}+\mu(x)\left(\delta_{k i} \delta_{l j}+\delta_{l i} \delta_{k j}\right), \quad \text { for every } x \in \mathbb{R}^{3}, \tag{15}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker's delta and the Lamé moduli $\lambda=\lambda(x), \mu=\mu(x)$ satisfy the strong convexity conditions

$$
\begin{equation*}
\mu(x) \geq \alpha_{0}, \quad 2 \mu(x)+3 \lambda(x) \geq \gamma_{0}, \quad \text { for every } x \in \mathbb{R}^{3}, \tag{16}
\end{equation*}
$$

for given constants $\alpha_{0}>0, \gamma_{0}>0$. We shall also assume upper bounds

$$
\begin{equation*}
\mu(x) \leq \bar{\mu}, \quad \lambda(x) \leq \bar{\lambda}, \quad \text { for every } x \in \mathbb{R}^{3}, \tag{17}
\end{equation*}
$$

where $\bar{\mu}>0, \bar{\lambda} \in \mathbb{R}$ are given constants. Let us notice that (15) clearly implies the major and minor symmetries of $\mathbb{C}$, namely

$$
\begin{equation*}
C_{i j k l}=C_{k l i j}=C_{l k i j}, \quad i, j, k, l=1,2,3 . \tag{18}
\end{equation*}
$$

The inclusion $D$ is assumed to be made by linearly elastic isotropic material having elasticity tensor $\mathbb{C}^{D}=\mathbb{C}^{D}(x)$ with components

$$
\begin{equation*}
C_{i j k l}^{D}(x)=\lambda^{D}(x) \delta_{i j} \delta_{k l}+\mu^{D}(x)\left(\delta_{k i} \delta_{l j}+\delta_{l i} \delta_{k j}\right), \quad \text { for every } x \in \bar{\Omega}, \tag{19}
\end{equation*}
$$

where the Lamé moduli $\lambda^{D}(x), \mu^{D}(x)$ satisfy the conditions (16)-(17) and, in addition,

$$
\begin{equation*}
\left(\lambda(x)-\lambda^{D}(x)\right)^{2}+\left(\mu(x)-\mu^{D}(x)\right)^{2} \geq \eta_{0}^{2}>0, \quad \text { for every } x \in \bar{\Omega}, \tag{20}
\end{equation*}
$$

for a given constant $\eta_{0}>0$.
Finally, the elasticity tensors $\mathbb{C}$ and $\mathbb{C}^{D}$ are assumed to be of $C^{1,1}$ class in $\mathbb{R}^{3}$ and of $C^{\tau}$ class in $\bar{\Omega}, \tau \in(0,1)$, respectively, that is

$$
\begin{align*}
& \|\lambda\|_{C^{1,1}\left(\mathbb{R}^{3}\right)}+\|\mu\|_{C^{1,1}\left(\mathbb{R}^{3}\right)} \leq M  \tag{21}\\
& \left\|\lambda^{D}\right\|_{C^{\tau}(\bar{\Omega})}+\left\|\mu^{D}\right\|_{C^{\tau}(\bar{\Omega})} \leq M \tag{22}
\end{align*}
$$

for a given constant $M>0$.
For any $f \in H^{\frac{1}{2}}(\partial \Omega)$, let $u \in H^{1}(\Omega)$ be the weak solution to the Dirichlet problem

$$
\left\{\begin{array}{lr}
\operatorname{div}\left(\left(\mathbb{C}+\left(\mathbb{C}^{D}-\mathbb{C}\right) \chi_{D}\right) \nabla u\right)=0, & \text { in } \Omega,  \tag{23}\\
u=f, & \text { on } \partial \Omega,
\end{array}\right.
$$

where $\chi_{D}$ is the characteristic function of $D$. The Dirichlet-to-Neumann map $\Lambda_{D}$ associated to (23)-(24),

$$
\begin{equation*}
\Lambda_{D}: H^{1 / 2}(\partial \Omega) \rightarrow H^{-1 / 2}(\partial \Omega) \tag{25}
\end{equation*}
$$

is defined in the weak form by

$$
\begin{equation*}
<\Lambda_{D} f,\left.v\right|_{\partial \Omega}>=\int_{\Omega}\left(\mathbb{C}+\left(\mathbb{C}^{D}-\mathbb{C}\right) \chi_{D}\right) \nabla u \cdot \nabla v \tag{26}
\end{equation*}
$$

for every $v \in H^{1}(\Omega)$.
We prove the following logarithmic stability estimate for the inverse problem of recovering the inclusion $D$ from the knowledge of the map $\Lambda_{D}$.
Theorem 2.2. Let $\Omega \subset \mathbb{R}^{3}$ be a bounded domain satisfying (10)-(12) and let $D_{1}, D_{2}$ be two connected inclusions contained in $\Omega$ satisfying (13)-(14). Let $\mathbb{C}(x)$ and $\mathbb{C}^{D_{i}}(x)$ be the elasticity tensor of the material of $\Omega$ and of the inclusion $D_{i}, i=1,2$, respectively, where $\mathbb{C}(x)$ given in (15) and $\mathbb{C}^{D_{i}}(x)$ given in (19) (for $D=D_{i}$ ) satisfy (16), (17), (20), (21) and (22). If, for some $\epsilon$, $0<\epsilon<1$,

$$
\begin{equation*}
\left\|\Lambda_{D_{1}}-\Lambda_{D_{2}}\right\|_{\mathcal{L}\left(H^{1 / 2}(\partial \Omega), H^{-1 / 2}(\partial \Omega)\right)} \leq \frac{\epsilon}{\rho_{0}} \tag{27}
\end{equation*}
$$

then

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq C \rho_{0}|\log \epsilon|^{-\eta} \tag{28}
\end{equation*}
$$

where $C>0$ and $\eta, 0<\eta \leq 1$, are constants only depending on $M_{0}, \alpha, M_{1}$, $\alpha_{0}, \gamma_{0}, \bar{\mu}, \bar{\lambda}, \eta_{0}, \tau, M$.
Remark 2.3. If in Theorem 2.2 we further assume that the two inclusions are at a prescribed distance from $\partial \Omega$, then the result continues to hold even when the local Dirichlet-to-Neumann map is known. The proof can be obtained by adapting the general theory developed by Alessandrini and Kim [4].

## 3. Proof of the main result

In order to state the metric Lemma 3.1 below, we need to introduce some notation.

We denote by $\mathcal{G}$ the connected component of $\mathbb{R}^{3} \backslash\left(\overline{D_{1} \cup D_{2}}\right)$ which contains $\mathbb{R}^{3} \backslash \bar{\Omega}$.

Given $O=(0,0,0)$, a unit vector $v, h>0$ and $\vartheta \in\left(0, \frac{\pi}{2}\right)$, we denote by

$$
\begin{equation*}
C(O, v, h, \vartheta)=\left\{x \in \mathbb{R}^{3}| | x-(x \cdot v) v|\leq \sin \vartheta| x \mid, 0 \leq x \cdot v \leq h\right\} \tag{29}
\end{equation*}
$$

the closed truncated cone with vertex at $O$, axis along the direction $v$, height $h$ and aperture $2 \vartheta$. Given $R, d, 0<R<d$ and $Q=-d e_{3}$, let us consider the cone
$C\left(O,-e_{3}, \frac{d^{2}-R^{2}}{d}, \arcsin \frac{R}{d}\right)$, whose lateral boundary is tangent to the sphere $\partial B_{R}(Q)$ along the circumference of its base.

Given a point $P \in \partial D_{1} \cap \partial \mathcal{G}$, let $\nu$ be the outer unit normal to $\partial D_{1}$ at $P$ and let $d>0$ be such that the segment $[P+d \nu, P]$ is contained in $\overline{\mathcal{G}}$. For a point $P_{0} \in \overline{\mathcal{G}}$, let $\gamma$ be a path in $\overline{\mathcal{G}}$ joining $P_{0}$ to $P+d \nu$. We consider the following neighbourhood of $\gamma \cup[P+d \nu, P] \backslash\{P\}$ formed by a tubular neighbourhood of $\gamma$ attached to a cone with vertex at $P$ and axis along $\nu$

$$
\begin{equation*}
V(\gamma, d, R)=\bigcup_{S \in \gamma} B_{R}(S) \cup C\left(P, \nu, \frac{d^{2}-R^{2}}{d}, \arcsin \frac{R}{d}\right) . \tag{30}
\end{equation*}
$$

Let us also define

$$
\begin{equation*}
S_{2 \rho_{0}}=\left\{x \in \mathbb{R}^{3} \mid \rho_{0}<\operatorname{dist}(x, \Omega)<2 \rho_{0}\right\} . \tag{31}
\end{equation*}
$$

Lemma 3.1. Under the assumptions of Theorem 2.2, up to inverting the role of $D_{1}$ and $D_{2}$, there exist positive constants $\bar{d}, \bar{c}$, where $\frac{\bar{d}}{\rho_{0}}$ only depends on $M_{0}$ and $\alpha$, and $\bar{c} \geq 1$ only depends on $M_{0}, \alpha$ and $M_{1}$, and there exists a point $P \in \partial D_{1} \cap \partial \mathcal{G}$ such that

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq \bar{c} \operatorname{dist}\left(P, D_{2}\right) \tag{32}
\end{equation*}
$$

and such that, giving any point $P_{0} \in S_{2 \rho_{0}}$, there exists a path $\gamma \subset \Omega^{2 \rho_{0}} \cap \mathcal{G}$ joining $P_{0}$ to $P+\bar{d} \nu$, where $\nu$ is the unit outer normal to $D_{1}$ at $P$, such that, choosing a coordinate system with origin $O$ at $P$ and axis $e_{3}=-\nu$, we have

$$
\begin{equation*}
V(\gamma, \bar{d}, \bar{R}) \subset \mathbb{R}^{3} \cap \overline{\mathcal{G}} \tag{33}
\end{equation*}
$$

where $\frac{\bar{R}}{\rho_{0}}$ only depends on $M_{0}$ and $\alpha$.
The thesis of the above lemma is a straightforward consequence of Lemma 4.1 and Lemma 4.2 in [3], and is inspired by results obtained in [5] and [2].

Let $D$ be a domain of class $C^{1, \alpha}$ with constants $\rho_{0}, M_{0}$ and $0<\alpha \leq 1$. The elasticity tensors $\mathbb{C}$ and $\mathbb{C}^{D}$ given by (15) and (19) respectively, satisfy (16), (17), (21) and (22).

Given $y \in \mathbb{R}^{3}$ and a concentrated force $l \delta(\cdot-y)$ applied at $y$, with $l \in \mathbb{R}^{3}$, let us consider the normalized fundamental solution $u^{D} \in L_{l o c}^{1}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)$ defined by

$$
\left\{\begin{align*}
\operatorname{div}_{x}\left(\left(\mathbb{C}(x)+\left(\mathbb{C}^{D}(x)-\mathbb{C}(x)\right) \chi_{D}\right) \nabla_{x} u^{D}(x, y ; l)\right)  \tag{34}\\
=-l \delta(x-y),
\end{align*} \quad \text { in } \mathbb{R}^{3} \backslash\{y\},\right.
$$

where $\delta(\cdot-y)$ is the Dirac distribution supported at $y$. It is well-known that

$$
\begin{equation*}
u^{D}(x, y ; l)=\Gamma^{D}(x, y) l \tag{35}
\end{equation*}
$$

where $\Gamma^{D}=\Gamma^{D}(\cdot, y) \in L_{\text {loc }}^{1}\left(\mathbb{R}^{3}, \mathcal{L}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)\right)$ is the normalized fundamental matrix for the operator $\operatorname{div}_{x}\left(\left(\mathbb{C}(x)+\left(\mathbb{C}^{D}(x)-\mathbb{C}(x)\right) \chi_{D}\right) \nabla_{x}(\cdot)\right)$. Existence of $\Gamma^{D}$ and asymptotic estimates are stated in the following Proposition.

Proposition 3.2. Under the above assumptions, there exists a unique fundamental matrix $\Gamma^{D}(\cdot, y) \in C^{0}\left(\mathbb{R}^{3} \backslash\{y\}\right)$, such that

$$
\begin{gather*}
\Gamma^{D}(x, y)=\left(\Gamma^{D}(y, x)\right)^{T}, \quad \text { for every } x \in \mathbb{R}^{3}, x \neq y  \tag{36}\\
\left|\Gamma^{D}(x, y)\right| \leq C|x-y|^{-1}, \quad \text { for every } x \in \mathbb{R}^{3}, x \neq y  \tag{37}\\
\left|\nabla_{x} \Gamma^{D}(x, y)\right| \leq C|x-y|^{-2}, \quad \text { for every } x \in \mathbb{R}^{3}, x \neq y \tag{38}
\end{gather*}
$$

where the constant $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
A proof of Proposition 3.2 follows by merging the regularity results by Li and Nirenberg [15] and the analysis by Hofmann and Kim [11], see [3] for details.

Let $D_{i}, i=1,2$, be a domain of class $C^{1, \alpha}$ with constants $\rho_{0}, M_{0}$ and $0<\alpha \leq 1$, and consider the elasticity tensors

$$
\begin{equation*}
\mathbb{C}^{1}=\mathbb{C} \chi_{\mathbb{R}^{3} \backslash D_{1}}+\mathbb{C}^{D_{1}} \chi_{D_{1}}, \quad \mathbb{C}^{2}=\mathbb{C} \chi_{\mathbb{R}^{3} \backslash D_{2}}+\mathbb{C}^{D_{2}} \chi_{D_{2}} \tag{39}
\end{equation*}
$$

where $\mathbb{C}^{D_{1}}, \mathbb{C}^{D_{2}}$ given in (19) (with $D=D_{1}$ and $D=D_{2}$, respectively) satisfy (16), (17) and (22).

The following Proposition 3.3 states an integral representation involving the normalized fundamental matrices corresponding to inclusions $D_{1}$ and $D_{2}$. Similar identities will be introduced in Section 4, in order to prove Theorem 3.5. Since these integral representations are basic ingredients for our approach, we present here a proof of Proposition 3.3, which is more exhaustive with respect to that given in [3, Proof of Lemma 6.2], where some details were implied.

Proposition 3.3. Let $D_{i}$ and $\mathbb{C}^{D_{i}}, i=1,2$, satisfy the above assumptions. Then, for every $y, w \in \mathbb{R}^{3}, y \neq w$, and for every $l$, $m \in \mathbb{R}^{3}$ we have

$$
\begin{align*}
& \left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)(y, w) m \cdot l= \\
& =\int_{\Omega} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m-\int_{\Omega} \mathbb{C}^{2} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m \tag{40}
\end{align*}
$$

Proof. Formula (40) is obtained by subtracting the two following identities

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m=\Gamma^{D_{2}}(y, w) m \cdot l \tag{41}
\end{equation*}
$$

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \mathbb{C}^{2} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m=\Gamma^{D_{1}}(y, w) m \cdot l . \tag{42}
\end{equation*}
$$

To prove (41), let

$$
\begin{align*}
& \mathcal{H}=\left\{f: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3} \mid f \in C^{0}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right) \cap H^{1}\left(\mathbb{R}^{3}, \mathbb{R}^{3}\right)\right. \\
&f \text { with compact support }\} . \tag{43}
\end{align*}
$$

By the weak formulation of (34) (with $D=D_{1}$ ), we have

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \varphi=\varphi(y) \cdot l, \quad \text { for every } \varphi \in \mathcal{H} \tag{44}
\end{equation*}
$$

Let $\epsilon>0, R>0$, with $\epsilon \leq \frac{|w-y|}{2}, R \geq 2 \max \{|y|,|w|\}$, and choose $\varphi \in \mathcal{H}$ such that $\operatorname{supp}(\varphi) \subset B_{2 R}(0)$ and $\left.\varphi\right|_{B_{R}(0) \backslash B_{\epsilon}(w)} \equiv \Gamma^{D_{2}}(\cdot, w) m$. Then, (44) can be rewritten as

$$
\begin{equation*}
I_{\epsilon, R}+I_{\epsilon}+I_{R, 2 R}=\Gamma^{D_{2}}(y, w) m \cdot l, \tag{45}
\end{equation*}
$$

where

$$
\begin{gather*}
I_{\epsilon, R}=\int_{B_{R}(0) \backslash B_{\epsilon}(w)} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m  \tag{46}\\
I_{\epsilon}=\int_{B_{\epsilon}(w)} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \varphi  \tag{47}\\
I_{R, 2 R}=\int_{B_{2 R}(0) \backslash B_{R}(0)} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \varphi \tag{48}
\end{gather*}
$$

Integrating by parts on $B_{\epsilon}(w)$ and recalling that $y \in \mathbb{R}^{3} \backslash \bar{B}_{\epsilon}(w)$, we have

$$
\begin{equation*}
I_{\epsilon}=\int_{\partial B_{\epsilon}(w)}\left(\mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l\right) \nu \cdot \Gamma^{D_{2}}(\cdot, w) m \tag{49}
\end{equation*}
$$

For every $x \in \partial B_{\epsilon}(w)$ and by our choice of $\epsilon$, we have $|x-y| \geq|y-w|-|w-x| \geq$ $\frac{|y-w|}{2}$. Therefore, by (37) and (38), we have

$$
\begin{equation*}
I_{\epsilon} \leq C \int_{|x-w|=\epsilon} \frac{1}{|x-y|^{2}} \frac{1}{|x-w|} \leq \frac{C \epsilon}{|y-w|^{2}} \tag{50}
\end{equation*}
$$

where the constant $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Analogously, integrating by parts in $B_{2 R}(0) \backslash B_{R}(0)$ and recalling that $\varphi=0$ on $\partial B_{2 R}(0)$ and $y \in B_{\frac{R}{2}}(0)$, we have

$$
\begin{equation*}
I_{R, 2 R}=-\int_{\partial B_{R}(0)}\left(\mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l\right) \nu \cdot \Gamma^{D_{2}}(\cdot, w) m \tag{51}
\end{equation*}
$$

For every $x \in \partial B_{R}(0)$ and by our choice of $R$, we have $|x-w| \geq|x|-|w| \geq \frac{R}{2}$ and $|x-y| \geq \frac{R}{2}$. Therefore,

$$
\begin{equation*}
I_{R, 2 R} \leq C \int_{|x|=R} \frac{1}{|x-y|^{2}} \frac{1}{|x-w|} \leq \frac{C}{R} \tag{52}
\end{equation*}
$$

where the constant $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Using the estimates (50) and (52) in (45), and taking the limit as $\epsilon \rightarrow 0$ and $R \rightarrow \infty$, we obtain (41). Symmetrically, we obtain

$$
\begin{equation*}
\int_{\mathbb{R}^{3}} \mathbb{C}^{2} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma^{D_{2}}(\cdot, w) m=\Gamma^{D_{1}}(w, y) l \cdot m \tag{53}
\end{equation*}
$$

By using (36), we obtain (42).

Let $P, P \in \partial D_{1}$, be the point introduced in Lemma 3.1. In the following two theorems, we use a cartesian coordinate system such that $P \equiv O=(0,0,0)$ and $\nu=-e_{3}$, where $\nu$ is the unit outer normal to $D_{1}$ at $P$.

TheOrem 3.4 (Upper bound on $\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)$ ). Under the notation of Lemma 3.1, let

$$
\begin{gather*}
y_{h}=P-h e_{3}  \tag{54}\\
w_{h}=P-\lambda_{w} h e_{3}, \quad 0<\lambda_{w}<1 \tag{55}
\end{gather*}
$$

with

$$
\begin{equation*}
0<h \leq \bar{h} \rho_{0} \tag{56}
\end{equation*}
$$

where $\bar{h}$ only depends on $M_{0}$ and $\alpha$.
Then, for every $l$, $m \in \mathbb{R}^{3},|l|=|m|=1$, we have

$$
\begin{equation*}
\left|\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \leq \frac{C}{\lambda_{w} h} \epsilon^{C_{1}\left(\frac{h}{\rho_{0}}\right)^{C_{2}}} \tag{57}
\end{equation*}
$$

 $\bar{\lambda}, \bar{\mu}, \tau$ and $M$.

For the proof of the above result, we refer to [3, Section 7]. To give an idea of the role played by Proposition 3.3 in proving estimate (57), let us recall Alessandrini's identity

$$
\begin{equation*}
\int_{\Omega} \mathbb{C}^{1} \nabla u_{1} \cdot \nabla u_{2}-\int_{\Omega} \mathbb{C}^{2} \nabla u_{1} \cdot \nabla u_{2}=<\left(\Lambda_{D_{1}}-\Lambda_{D_{2}}\right) u_{2}, u_{1}> \tag{58}
\end{equation*}
$$

which holds for every pair of solutions $u_{i} \in H^{1}(\Omega)$ to (1) with $D=D_{i}, i=1,2$.
By choosing in the above identity $u_{1}(\cdot)=\Gamma^{D_{1}}(\cdot, y) l, u_{2}(\cdot)=\Gamma^{D_{2}}(\cdot, w) m$ with $y, w \in S_{2 \rho_{0}}$, the first member of (58) coincides with the second member
of (40), so that, recalling the asymptotic estimate (37) and the hypothesis (27), we obtain the following smallness estimate

$$
\begin{equation*}
\left|\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)(y, w) m \cdot l\right| \leq C \frac{\epsilon}{\rho_{0}}, \quad \text { for every } y, w \in S_{2 \rho_{0}} \tag{59}
\end{equation*}
$$

where $C>0$ only depends on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
This first smallness estimate is then propagated up to the points $y_{h}, w_{h}$, with a technical construction based on iterated application of the three spheres inequality.

Theorem 3.5 (Lower bound on the function $\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)$ ). Under the notation of Lemma 3.1, let

$$
\begin{equation*}
y_{h}=P-h e_{3} . \tag{60}
\end{equation*}
$$

For every $i=1,2,3$, there exists $\lambda_{w} \in\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ and there exists $\widetilde{h} \in\left(0, \frac{1}{2}\right)$ only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \eta_{0}, \tau, M$, such that

$$
\begin{equation*}
\left.\mid\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) e_{i} \cdot e_{i}\right) \left\lvert\, \geq \frac{C}{h}\right., \quad \text { for every } h, 0<h<\widetilde{h} \operatorname{dist}\left(P, D_{2}\right) \tag{61}
\end{equation*}
$$

where

$$
\begin{equation*}
w_{h}=P-\lambda_{w} h e_{3}, \tag{62}
\end{equation*}
$$

and $C>0$ only depends on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$ and $\eta_{0}$.
The proof of this key result will be given in Section 4.
We are now in position to prove the main result of this paper.
Proof of Theorem 2.2. By the upper bound (57), with $l=m=e_{i}$ for $i \in$ $\{1,2,3\}$, and the lower bound (61), we have

$$
\begin{equation*}
C \leq \epsilon^{C_{1}\left(\frac{h}{\rho_{0}}\right)^{C_{2}}}, \quad \text { for every } h, 0<h \leq \min \left\{\bar{h} \rho_{0}, \widetilde{h} d\left(P, D_{2}\right)\right\} \tag{63}
\end{equation*}
$$

where $C, C_{1}, C_{2}$ only depend on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$ and $\eta_{0}$. By our regularity assumptions on the domains, there exists $\widetilde{C}>0$, only depending on $M_{0}, \alpha, M_{1}$, such that

$$
\begin{equation*}
d\left(P, D_{2}\right) \leq \operatorname{diam}(\Omega) \leq \widetilde{C} \rho_{0} \tag{64}
\end{equation*}
$$

Set $h^{*}=\min \left\{\frac{\bar{h}}{\bar{C}}, \widetilde{h}\right\}$. Then inequality (63) holds for every $h$ such that $h \leq$ $h^{*} d\left(P, D_{2}\right)$, with $h^{*}$ only depending on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$ and $\eta_{0}$. Taking the logarithm in (63) and recalling that $\epsilon \in(0,1)$, we obtain

$$
\begin{equation*}
h \leq C \rho_{0}\left(\frac{1}{|\log \epsilon|}\right)^{\frac{1}{C_{2}}}, \quad \text { for every } h, 0<h \leq h^{*} d\left(P, D_{2}\right) \tag{65}
\end{equation*}
$$

In particular, choosing $h=h^{*} d\left(P, D_{2}\right)$, we have

$$
\begin{equation*}
d\left(P, D_{2}\right) \leq C \rho_{0}\left(\frac{1}{|\log \epsilon|}\right)^{\frac{1}{C_{2}}} \tag{66}
\end{equation*}
$$

The thesis follows from Lemma 3.1.

## 4. Proof of Theorem 3.5

Let us recall that we have chosen a cartesian coordinate system with origin $P \equiv O$ and $e_{3}=-\nu$, where $\nu$ is the unit outer normal to $D_{1}$ at $P$.

Let $\mathbb{C}_{0}=\mathbb{C}(O)$ be the constant Lamé tensor, having Lamé moduli $\lambda \equiv \lambda(O)$, $\mu \equiv \mu(O)$, and let $\mathbb{C}_{0}^{D_{1}}=\mathbb{C}^{D_{1}}(O)$ be the constant Lamé tensor with Lamé moduli $\lambda \equiv \lambda^{D_{1}}(O), \mu \equiv \mu^{D_{1}}(O)$. Moreover, let us introduce the elasticity tensors $\mathbb{C}_{0}^{+}=\mathbb{C}_{0} \chi_{\mathbb{R}_{-}^{3}}+\mathbb{C}_{0}^{D_{1}} \chi_{\mathbb{R}_{+}^{3}}, \mathbb{C}_{0}^{1}=\mathbb{C}_{0} \chi_{\mathbb{R}^{3} \backslash D_{1}}+\mathbb{C}_{0}^{D_{1}} \chi_{D_{1}}$.

Let $\Gamma, \Gamma_{0}, \Gamma_{0}^{+}, \Gamma_{0}^{D_{1}}$ be the fundamental matrices associated to the tensors $\mathbb{C}, \mathbb{C}_{0}, \mathbb{C}_{0}^{+}, \mathbb{C}_{0}^{1}$, respectively.

In the above notation, we may write, for every $m, l \in \mathbb{R}^{3},|l|=|m|=1$,

$$
\begin{array}{r}
\left|\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \geq\left|\left(\Gamma_{0}^{+}-\Gamma_{0}\right)\left(y_{h}, w_{h}\right) m \cdot l\right|-\left|\left(\Gamma^{D_{2}}-\Gamma\right)\left(y_{h}, w_{h}\right) m \cdot l\right|- \\
-\left|\left(\Gamma-\Gamma_{0}\right)\left(y_{h}, w_{h}\right) m \cdot l\right|-\left|\left(\Gamma_{0}^{+}-\Gamma_{0}^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right|- \\
\quad-\left|\left(\Gamma_{0}^{D_{1}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| . \tag{67}
\end{array}
$$

The following Lemma, which is a straightforward consequence of Proposition 9.3 and formula (9.11), derived in [3], gives a positive lower bound for the term $\left|\left(\Gamma_{0}^{+}-\Gamma_{0}\right)\left(y_{h}, w_{h}\right) e_{i} \cdot e_{i}\right|, i=1,2,3$, for a suitable $w_{h}$.
Lemma 4.1. For every $i=1,2,3$, there exists $\lambda_{w} \in\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ such that

$$
\begin{equation*}
\left|\left(\Gamma_{0}^{+}\left(y_{h}, w_{h}\right)-\Gamma_{0}\left(y_{h}, w_{h}\right)\right) e_{i} \cdot e_{i}\right| \geq \frac{\mathcal{C}}{h}, \quad \text { for every } h>0 \tag{68}
\end{equation*}
$$

where $\mathcal{C}>0$ only depends on $\alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \eta_{0}$.
From now on, let $\lambda_{w}$ be chosen accordingly to the above lemma and let $h \leq \frac{1}{2} \min \left\{d\left(P, D_{2}\right), \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}}\right\}$.
$\operatorname{Term} \Gamma^{D_{2}}-\Gamma$.
Let us consider the vector valued function

$$
\begin{equation*}
v(x)=\left(\Gamma^{D_{2}}-\Gamma\right)\left(x, w_{h}\right) m \tag{69}
\end{equation*}
$$

Let us set $\rho=d\left(P, D_{2}\right)$. Since $d\left(w_{h}, P\right)=\lambda_{w} h \leq h \leq \frac{\rho}{2}$, we have that $d\left(w_{h}, D_{2}\right) \geq d\left(P, D_{2}\right)-d\left(w_{h}, P\right) \geq \frac{\rho}{2}$. Therefore $v(x)$ is a solution to the Lamé system

$$
\begin{equation*}
\operatorname{div}_{x}\left(\mathbb{C} \nabla_{x} v(x)\right)=0, \quad \text { in } B_{\frac{\rho}{2}}\left(w_{h}\right) \tag{70}
\end{equation*}
$$

By the regularity estimate

$$
\begin{equation*}
\sup _{B_{\frac{\rho}{4}}\left(w_{h}\right)}|v(x)| \leq \frac{C}{\rho^{\frac{3}{2}}}\left(\int_{B_{\frac{\rho}{2}}\left(w_{h}\right)}|v(x)|^{2}\right)^{\frac{1}{2}}, \tag{71}
\end{equation*}
$$

with $C$ only depending on $\alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$, and by applying the asymptotic estimates (37) to $\Gamma^{D_{2}}$ and $\Gamma$, it follows that

$$
\begin{equation*}
\sup _{B_{\frac{\rho}{4}}\left(w_{h}\right)}|v(x)| \leq \frac{C}{\rho} \tag{72}
\end{equation*}
$$

where $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Since $d\left(y_{h}, w_{h}\right)=\left(1-\lambda_{w}\right) h \leq \frac{h}{3} \leq \frac{\rho}{6}, y_{h} \in B_{\frac{\rho}{4}}\left(w_{h}\right)$ and

$$
\begin{equation*}
\left|\left(\Gamma^{D_{2}}-\Gamma\right)\left(y_{h}, w_{h}\right) m \cdot l\right|=\left|v\left(y_{h}\right) \cdot l\right| \leq \frac{C}{\rho}=\frac{C}{d\left(P, D_{2}\right)}, \tag{73}
\end{equation*}
$$

for every $l, m \in \mathbb{R}^{3},|l|=|m|=1$, with $C$ only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}$, $\bar{\mu}, \tau, M$.
$\operatorname{Term} \Gamma_{0}^{D_{1}}-\Gamma^{D_{1}}$.
By the same arguments seen in the proof of Proposition 3.3, we have that, for every $y, w \in \mathbb{R}^{3}, y \neq w$, and for every $l, m \in \mathbb{R}^{3}$,

$$
\begin{align*}
& \int_{\mathbb{R}^{3}} \mathbb{C}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma_{0}^{D_{1}}(\cdot, w) m=\Gamma_{0}^{D_{1}}(y, w) m \cdot l,  \tag{74}\\
& \int_{\mathbb{R}^{3}} \mathbb{C}_{0}^{1} \nabla \Gamma^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma_{0}^{D_{1}}(\cdot, w) m=\Gamma^{D_{1}}(y, w) m \cdot l . \tag{75}
\end{align*}
$$

Choosing $y=y_{h}$ and $w=w_{h}$, we have

$$
\begin{equation*}
\left(\Gamma_{0}^{D_{1}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l=\int_{\mathbb{R}^{3}}\left(\mathbb{C}^{1}-\mathbb{C}_{0}^{1}\right) \nabla \Gamma^{D_{1}}\left(\cdot, y_{h}\right) l \cdot \nabla \Gamma_{0}^{D_{1}}\left(\cdot, w_{h}\right) m=J+J_{0} \tag{76}
\end{equation*}
$$

with

$$
\begin{align*}
& J=\int_{D_{1}}\left(\mathbb{C}^{D_{1}}-\mathbb{C}_{0}^{D_{1}}\right) \nabla \Gamma^{D_{1}}\left(\cdot, y_{h}\right) l \cdot \nabla \Gamma_{0}^{D_{1}}\left(\cdot, w_{h}\right) m,  \tag{77}\\
& J_{0}=\int_{\mathbb{R}^{3} \backslash D_{1}}\left(\mathbb{C}-\mathbb{C}_{0}\right) \nabla \Gamma^{D_{1}}\left(\cdot, y_{h}\right) l \cdot \nabla \Gamma_{0}^{D_{1}}\left(\cdot, w_{h}\right) m . \tag{78}
\end{align*}
$$

Let us estimate $J$. We have trivially

$$
\begin{equation*}
|J| \leq C\left(I_{1}+I_{2}\right) \tag{79}
\end{equation*}
$$

where $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$ and

$$
\begin{align*}
I_{1} & =\int_{|x| \geq \rho_{0}} \frac{\left|\left(\mathbb{C}^{D_{1}}-\mathbb{C}_{0}^{D_{1}}\right)(x)\right|}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}}  \tag{80}\\
I_{2} & =\int_{|x| \leq \rho_{0}} \frac{\left|\left(\mathbb{C}^{D_{1}}-\mathbb{C}_{0}^{D_{1}}\right)(x)\right|}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} \tag{81}
\end{align*}
$$

Let us first estimate $I_{1}$. Since $h \leq \frac{\rho_{0}}{2}$ and $|x| \geq \rho_{0}$, we have that $\left|x-y_{h}\right| \geq$ $|x|-\left|y_{h}\right|=|x|-h \geq \frac{|x|}{2}$ and similarly $\left|x-w_{h}\right| \geq \frac{|x|}{2}$, so that

$$
\begin{equation*}
I_{1} \leq C \int_{|x| \geq \rho_{0}} \frac{1}{|x|^{4}}=\frac{C}{\rho_{0}} \tag{82}
\end{equation*}
$$

with $C$ only depending on $\bar{\lambda}, \bar{\mu}$. To estimate $I_{2}$, we use the fact that

$$
\begin{equation*}
\left|\left(\mathbb{C}^{D_{1}}-\mathbb{C}_{0}^{D_{1}}\right)(x)\right|=\left|\mathbb{C}^{D_{1}}(x)-\mathbb{C}^{D_{1}}(O)\right| \leq \frac{C}{\rho_{0}^{\tau}}|x|^{\tau} \tag{83}
\end{equation*}
$$

with $C$ only depending on $M$, so that

$$
\begin{equation*}
I_{2} \leq \frac{C}{\rho_{0}^{\tau}}\left(I_{2}^{\prime}+I_{2}^{\prime \prime}\right) \tag{84}
\end{equation*}
$$

where

$$
\begin{align*}
& I_{2}^{\prime}=\int_{A} \frac{|x|^{\tau}}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}}  \tag{85}\\
& I_{2}^{\prime \prime}=\int_{B} \frac{|x|^{\tau}}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} \tag{86}
\end{align*}
$$

with $A=\left\{|x| \leq \rho_{0},|x|<6\left|y_{h}-w_{h}\right|\right\}, B=\left\{6\left|y_{h}-w_{h}\right| \leq|x| \leq \rho_{0}\right\}$.
We perform the change of variables $x=\left|y_{h}-w_{h}\right| z$ in $I_{2}^{\prime}$, obtaining

$$
\begin{equation*}
I_{2}^{\prime} \leq 6^{\tau}\left|y_{h}-w_{h}\right|^{\tau-1} \int_{|z| \leq 6}\left(z-\frac{y_{h}}{\left|y_{h}-w_{h}\right|}\right)^{-2}\left(z-\frac{w_{h}}{\left|y_{h}-w_{h}\right|}\right)^{-2} \tag{87}
\end{equation*}
$$

Since the integral on the right hand side is bounded by an absolute constant, see [16, Chapter 2, Section 11], we have that

$$
\begin{equation*}
I_{2}^{\prime} \leq C\left|y_{h}-w_{h}\right|^{\tau-1} \tag{88}
\end{equation*}
$$

with $C$ only depending on $\tau$.
For every $x \in B$, we have

$$
\begin{equation*}
|x| \geq 6\left|y_{h}-w_{h}\right|=6 h\left(1-\lambda_{w}\right) \geq \frac{6}{5} h \tag{89}
\end{equation*}
$$

so that

$$
\begin{equation*}
|x| \leq\left|x-y_{h}\right|+\left|y_{h}\right|=\left|x-y_{h}\right|+h \leq\left|x-y_{h}\right|+\frac{5}{6}|x| . \tag{90}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\frac{1}{6}|x| \leq\left|x-y_{h}\right| \tag{91}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\frac{1}{6}|x| \leq\left|x-w_{h}\right| \tag{92}
\end{equation*}
$$

By (91)-(92), we have

$$
\begin{equation*}
I_{2}^{\prime \prime} \leq 6^{4} \int_{B}|x|^{\tau-4} \leq C \int_{6\left|y_{h}-w_{h}\right|}^{\rho_{0}} r^{\tau-2} d r \leq C\left|y_{h}-w_{h}\right|^{\tau-1} \tag{93}
\end{equation*}
$$

where $C$ is an absolute constant.
From (79), (82), (84), (88), (93) and noticing that $\left|y_{h}-w_{h}\right|=h\left(1-\lambda_{w}\right) \geq \frac{h}{5}$, we have

$$
\begin{equation*}
|J| \leq \frac{C}{h}\left(\frac{h}{\rho_{0}}+\left(\frac{h}{\rho_{0}}\right)^{\tau}\right) \tag{94}
\end{equation*}
$$

where $C$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
The term $J_{0}$ is estimated analogously with $\tau$ replaced by 1 , and therefore, by (76),

$$
\begin{equation*}
\left|\left(\Gamma_{0}^{D_{1}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \leq \frac{C}{h}\left(\frac{h}{\rho_{0}}+\left(\frac{h}{\rho_{0}}\right)^{\tau}\right) \tag{95}
\end{equation*}
$$

where $C$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Term $\Gamma_{0}^{+}-\Gamma_{0}^{D_{1}}$.
Arguing similarly to the proof of Proposition 3.3, we have that, for every $y, w \in \mathbb{R}^{3}, y \neq w$, and for every $l, m \in \mathbb{R}^{3}$,

$$
\begin{align*}
&\left(\Gamma_{0}^{+}-\Gamma_{0}^{D_{1}}\right)(y, w) m \cdot l= \int_{\mathbb{R}^{3}}\left(\mathbb{C}_{0}^{D_{1}}-\mathbb{C}_{0}\right)\left(\chi_{D_{1}}-\chi_{\mathbb{R}_{+}^{3}}\right) \nabla \Gamma_{0}^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma_{0}^{+}(\cdot, w) m= \\
&=\int_{D_{1} \backslash \mathbb{R}_{+}^{3}}\left(\mathbb{C}_{0}^{D_{1}}-\mathbb{C}_{0}\right) \nabla \Gamma_{0}^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma_{0}^{+}(\cdot, w) m- \\
&-\int_{\mathbb{R}_{+}^{3} \backslash D_{1}}\left(\mathbb{C}_{0}^{D_{1}}-\mathbb{C}_{0}\right) \nabla \Gamma_{0}^{D_{1}}(\cdot, y) l \cdot \nabla \Gamma_{0}^{+}(\cdot, w) m . \quad(96) \tag{96}
\end{align*}
$$

Therefore

$$
\begin{equation*}
\left|\left(\Gamma_{0}^{+}-\Gamma_{0}^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \leq C \int_{A \cup B} \frac{1}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}}, \tag{97}
\end{equation*}
$$

where

$$
\begin{align*}
& A=\left\{x \in\left(\mathbb{R}_{+}^{3} \backslash D_{1}\right) \cup\left(D_{1} \backslash \mathbb{R}_{+}^{3}\right)| | x \left\lvert\, \geq \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}}\right.\right\}  \tag{98}\\
& B=\left\{x \in\left(\mathbb{R}_{+}^{3} \backslash D_{1}\right) \cup\left(D_{1} \backslash \mathbb{R}_{+}^{3}\right)| | x \left\lvert\, \leq \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}}\right.\right\} \tag{99}
\end{align*}
$$

and $C$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$. By our hypotheses, $h \leq$ $\frac{\rho_{0}}{2 \sqrt{1+M_{0}^{2}}}$. Hence, for every $x \in A, h \leq \frac{|x|}{2},\left|x-y_{h}\right| \geq|x|-h \geq \frac{|x|}{2}$, and similarly $\left|x-w_{h}\right| \geq \frac{|x|}{2}$, so that

$$
\begin{equation*}
\int_{A} \frac{1}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} \leq 16 \int_{|x| \geq \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}}} \frac{1}{|x|^{4}}=\frac{C}{\rho_{0}} \tag{100}
\end{equation*}
$$

with $C$ only depending on $M_{0}$.
By the local representation of the boundary of $D_{1}$ as a $C^{1, \alpha}$ graph, it follows that

$$
\begin{equation*}
B \subset\left\{\left.x \in \mathbb{R}^{3}| | x^{\prime}\left|\leq \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}},\left|x_{3}\right| \leq \frac{M_{0}}{\rho_{0}^{\alpha}}\right| x^{\prime}\right|^{1+\alpha}\right\} \tag{101}
\end{equation*}
$$

By performing the change of variables $z=\frac{x}{h}$, we have

$$
\begin{align*}
& \int_{B} \frac{1}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} \\
& \leq \int_{\left|x^{\prime}\right| \leq \frac{\rho_{0}}{\sqrt{1+M_{0}^{2}}}} d x^{\prime} \int_{-\frac{M_{0}}{\rho_{0}^{0}}\left|x^{\prime}\right|^{1+\alpha}}^{\frac{M_{0}}{\rho_{0}}\left|x^{\prime}\right|^{1+\alpha}} \frac{1}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} d x_{3} \\
& \quad=\frac{1}{h} \int_{\left|z^{\prime}\right| \leq \frac{\rho_{0}}{h \sqrt{1+M_{0}^{2}}}} d z^{\prime} \int_{-M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}}^{M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}} \frac{1}{\left|z+e_{3}\right|^{2}\left|z+\lambda_{w} e_{3}\right|^{2}} d z_{3} \\
& \quad \leq \frac{1}{h} \int_{\mathbb{R}^{2}} d z^{\prime} \int_{-M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}}^{M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}} \frac{1}{\left|z+e_{3}\right|^{2}\left|z+\lambda_{w} e_{3}\right|^{2}} d z_{3} . \tag{102}
\end{align*}
$$

Denoting

$$
\begin{equation*}
D(z)=\left(\left|z^{\prime}\right|^{2}+\left(z_{3}+1\right)^{2}\right)\left(\left|z^{\prime}\right|^{2}+\left(z_{3}+\lambda_{w}\right)^{2}\right) \tag{103}
\end{equation*}
$$

we have

$$
\begin{equation*}
\int_{B} \frac{1}{\left|x-y_{h}\right|^{2}\left|x-w_{h}\right|^{2}} \leq \frac{1}{h}\left(J_{1}+J_{2}\right) \tag{104}
\end{equation*}
$$

where

$$
\begin{align*}
& J_{1}=\int_{\left|z^{\prime}\right| \leq\left(\frac{1}{3 M_{0}}\right)^{\frac{1}{1+\alpha}}} d z^{\prime} \int_{-M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}}^{M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}} \frac{1}{D(z)} d z_{3},  \tag{105}\\
& J_{2}=\int_{\left|z^{\prime}\right| \geq\left(\frac{1}{3 M_{0}}\right)^{\frac{1}{1+\alpha}}} d z^{\prime} \int_{-M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}}^{M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha}} \frac{1}{D(z)} d z_{3} . \tag{106}
\end{align*}
$$

To estimate $J_{1}$, let us notice that, recalling $h \leq \frac{\rho_{0}}{2 \sqrt{1+M_{0}^{2}}}$,

$$
\begin{equation*}
\left|z_{3}+\lambda_{w}\right| \geq \lambda_{w}-\left|z_{3}\right| \geq \frac{2}{3}-M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha} \geq \frac{1}{3} \tag{107}
\end{equation*}
$$

and, a fortiori, $\left|z_{3}+1\right| \geq \frac{1}{3}$. Hence $D(z) \geq \frac{1}{3^{4}}$ and

$$
\begin{equation*}
J_{1} \leq 3^{4} \int_{\left|z^{\prime}\right| \leq\left(\frac{1}{3 M_{0}}\right)^{\frac{1}{1+\alpha}}} 2 M_{0}\left(\frac{h}{\rho_{0}}\right)^{\alpha}\left|z^{\prime}\right|^{1+\alpha} d z^{\prime}=C\left(\frac{h}{\rho_{0}}\right)^{\alpha} \tag{108}
\end{equation*}
$$

with $C$ only depending on $M_{0}$ and $\alpha$.
To estimate $J_{2}$ we use the trivial inequality $D(z) \geq\left|z^{\prime}\right|^{4}$ when $\alpha<1$, and $D(z) \geq C\left(M_{0}\right)\left|z^{\prime}\right|^{\frac{7}{2}}$ when $\alpha=1$, so obtaining

$$
\begin{equation*}
J_{2} \leq C\left(\frac{h}{\rho_{0}}\right)^{\alpha} \tag{109}
\end{equation*}
$$

with $C$ only depending on $M_{0}$ and $\alpha$.
By (97), (100), (104), (108), (109), we have

$$
\begin{equation*}
\left|\left(\Gamma_{0}^{+}-\Gamma_{0}^{D_{1}}\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \leq \frac{C}{h}\left(\frac{h}{\rho_{0}}+\left(\frac{h}{\rho_{0}}\right)^{\alpha}\right) \tag{110}
\end{equation*}
$$

with $C$ only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Term $\Gamma-\Gamma_{0}$.
Similarly to the proof of Proposition 3.3, we have that, for every $y, w \in \mathbb{R}^{3}$, $y \neq w$,

$$
\begin{equation*}
\left(\Gamma_{0}-\Gamma\right)(y, w) m \cdot l=\int_{\mathbb{R}^{3}}\left(\mathbb{C}-\mathbb{C}_{0}\right) \nabla \Gamma(\cdot, y) l \cdot \nabla \Gamma_{0}(\cdot, w) m . \tag{111}
\end{equation*}
$$

From this identity, the arguments of the proof are similar to those seen to estimate the addend $J_{0}$ in the expression of $\left(\Gamma_{0}^{D_{1}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) l \cdot m$ given by (76), so that

$$
\begin{equation*}
\left|\left(\Gamma_{0}-\Gamma\right)\left(y_{h}, w_{h}\right) m \cdot l\right| \leq \frac{C}{h}\left(\frac{h}{\rho_{0}}\right) \tag{112}
\end{equation*}
$$

with $C$ only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$.
Conclusion. Finally, from (67), (68), (73), (95), (110), (112), we have

$$
\begin{align*}
& \left|\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) e_{i} \cdot e_{i}\right| \geq \\
& \quad \geq \frac{\mathcal{C}}{h}\left(1-C_{1} \frac{h}{d\left(P, D_{2}\right)}-C_{2} \frac{h}{\rho_{0}}-C_{3}\left(\frac{h}{\rho_{0}}\right)^{\alpha}-C_{4}\left(\frac{h}{\rho_{0}}\right)^{\tau}\right) \tag{113}
\end{align*}
$$

with $C_{i}, i=1, \ldots, 4$, only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \tau, M$ and $\mathcal{C}$ only depending on $\alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$ and $\eta_{0}$. Let $h_{1}=\min \left\{\frac{1}{2}, \frac{1}{5 C_{1}}\right\}, h_{2}=$ $\min \left\{\frac{1}{2 \sqrt{1+M_{0}^{2}}}, \frac{1}{5 C_{2}}, \frac{1}{\left(5 C_{3}\right)^{\frac{1}{\alpha}}}, \frac{1}{\left(5 C_{4}\right)^{\frac{1}{\tau}}}\right\}$. If $h \leq \min \left\{h_{1} d\left(P, D_{2}\right), h_{2} \rho_{0}\right\}$, then

$$
\begin{equation*}
\left|\left(\Gamma^{D_{2}}-\Gamma^{D_{1}}\right)\left(y_{h}, w_{h}\right) e_{i} \cdot e_{i}\right| \geq \frac{\mathcal{C}}{5 h} \tag{114}
\end{equation*}
$$

Let $\widetilde{h}=\min \left\{h_{1}, \frac{h_{2}}{\widetilde{C}}\right\}$, where $\widetilde{C}$ has been introduced in (64). Then inequality (61) holds for every $h$ such that $h \leq \widetilde{h} d\left(P, D_{2}\right)$.

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# An introduction to the study of critical points of solutions of elliptic and parabolic equations 

Rolando Magnanini<br>Dedicated to Giovanni Alessandrini in the occasion of his $60^{\text {th }}$ birthday


#### Abstract

We give a survey at an introductory level of old and recent results in the study of critical points of solutions of elliptic and parabolic partial differential equations. To keep the presentation simple, we mainly consider four exemplary boundary value problems: the Dirichlet problem for the Laplace's equation; the torsional creep problem; the case of Dirichlet eigenfunctions for the Laplace's equation; the initial-boundary value problem for the heat equation. We shall mostly address three issues: the estimation of the local size of the critical set; the dependence of the number of critical points on the boundary values and the geometry of the domain; the location of critical points in the domain.


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## 1. Introduction

Let $\Omega$ be a domain in the Euclidean space $\mathbb{R}^{N}, \Gamma$ be its boundary and $u: \Omega \rightarrow \mathbb{R}$ be a differentiable function. A critical point of $u$ is a point in $\Omega$ at which the gradient $\nabla u$ of $u$ is the zero vector. The importance of critical points is evident. At an elementary level, they help us to visualize the graph of $u$, since they are some of its notable points (they are local maximum, minimum, or inflection/saddle points of $u$ ). At a more sophisticated level, if we interpret $u$ and $\nabla u$ as a gravitational, electrostatic or velocity potential and its underlying field of force or flow, the critical points are the positions of equilibrium for the field of force or stagnation points for the flow and give information on the topology of the equipotential lines or of the curves of steepest descent (or stream lines) related to $u$.

A merely differentiable function can be very complicated. For instance, Whitney [88] constructed a non-constant function of class $C^{1}$ on the plane
with a connected set of critical values (the images of critical points). If we allow enough smoothness, this is no longer possible as Morse-Sard's lemma informs us: indeed, if $u$ is at least of class $C^{N}$, the set of its critical values must have zero Lebesgue measure and hence the regular values of $u$ must be dense in the image of $u$ (see [8] for a proof).

When the function $u$ is the solution of some partial differential equation, the situation improves. In this survey, we shall consider the four archetypical equations:

$$
\Delta u=0, \quad \Delta u=-1, \quad \Delta u+\lambda u=0, \quad u_{t}=\Delta u
$$

that is the Laplace's equation, the torsional creep equation, the eigenfunction equation and the heat equation.

It should be noticed at this point some important differences between the first and the remaining three equations.

One is that the critical points of harmonic functions - the solutions of the Laplace's equation - are always "saddle points" as it is suggested by the maximum and minimum principles and the fact that $\Delta u$ is the sum of the eigenvalues of the hessian matrix $\nabla^{2} u$. The other three equations instead admit solutions with maximum or minimum points.

Also, we know that the critical points of a non-constant harmonic function $u$ on an open set of $\mathbb{R}^{2}$ are isolated and can be assigned a sort of finite multiplicity, for they are the zeroes of the holomorphic function $f=u_{x}-i u_{y}$. By means of the theory of quasi-conformal mappings and generalized analytic functions, this result can be extended to solutions of the elliptic equation

$$
\begin{equation*}
\left(a u_{x}+b u_{y}\right)_{x}+\left(b u_{x}+c u_{y}\right)_{y}+d u_{x}+e u_{y}=0 \tag{1}
\end{equation*}
$$

(with suitable smoothness assumptions on the coefficients) or even to weak solutions of the elliptic equation in divergence form,

$$
\begin{equation*}
\left(a u_{x}+b u_{y}\right)_{x}+\left(b u_{x}+c u_{y}\right)_{y}=0 \text { in } \Omega, \tag{2}
\end{equation*}
$$

even allowing discontinuous coefficients.
Instead, solutions of the other three equations can show curves of critical points in $\mathbb{R}^{2}$, as one can be persuaded by looking at the solution of the torsional creep equation in a circular annulus with zero boundary values.

These discrepancies extend to any dimension $N \geq 2$, in the sense that it has been shown that the set of the critical points of a non-constant harmonic function (or of a solution of an elliptic equation with smooth coefficients modeled on the Laplace equation) has at most locally finite $(N-2)$-dimensional Hausdorff measure, while solutions of equations fashioned on the other three equations have at most locally finite ( $N-1$ )-dimensional Hausdorff measure.

Further assumptions on solutions of a partial differential equation, such as their behaviour on the boundary and the shape of the boundary itself, can
give more detailed information on the number and location of critical points. In these notes, we shall consider the case of harmonic functions with various boundary behaviors and the solutions $\tau, \phi$ and $h$ of the following three problems:

$$
\begin{gather*}
-\Delta \tau=1 \text { in } \Omega, \quad \tau=0 \text { on } \Gamma  \tag{3}\\
\Delta \phi+\lambda \phi=0 \text { in } \Omega, \quad \phi=0 \text { on } \Gamma  \tag{4}\\
h_{t}=\Delta h \text { in } \Omega \times(0, \infty),  \tag{5}\\
h=0 \text { on } \Gamma \times(0, \infty), \quad h=\varphi \text { on } \Omega \times\{0\}, \tag{6}
\end{gather*}
$$

where $\varphi$ is a given function. We will refer to (3), (4), (5)-(6), as the torsional creep problem, the Dirichlet eigenvalue problem, and the initial-boundary value problem for the heat equation, respectively.

A typical situation is that considered in Theorem 3.2: a harmonic function $u$ on a planar domain $\Omega$ is given together with a vector field $\ell$ on $\Gamma$ of assigned topological degree $D$; the number of critical points in $\Omega$ then is bounded in terms of $D$, the Euler characteristic of $\Omega$ and the number of proper connected components of the set $\{z \in \Gamma: \ell(z) \cdot \nabla u(z)>0\}$ (see Theorem 3.2 for the exact statement). We shall also see how this type of theorem has recently been extended to obtain a bound for the number of critical points of the Li-Tam Green's function of a non-compact Riemanniann surface of finite type in terms of its genus and the number of its ends.

Owing to the theory of quasi-conformal mappings, Theorem 3.2 can be extended to solutions of quite general elliptic equations and, thanks to the work of G. Alessandrini and co-authors, has found effective applications to the study of inverse problems that have as a common denominator the reconstruction of the coefficients of an elliptic equation in a domain from measurements on the boundary of a set of its solutions.

A paradigmatic example is that of Electric Impedence Tomography (EIT) in which a conductivity $\gamma$ is reconstructed, as the coefficient of the elliptic equation

$$
\operatorname{div}(\gamma \nabla u)=0 \text { in } \Omega
$$

from the so-called Neumann-to-Dirichlet (or Dirichlet-to-Neumann) operator on $\Gamma$. In physical terms, an electrical current (represented by the co-normal derivative $\gamma u_{\nu}$ ) is applied on $\Gamma$ generating a potential $u$, that is measured on $\Gamma$ within a certain error. One wants to reconstruct the conductivity $\gamma$ from some of these measurements. Roughly speaking, one has to solve for the unknown $\gamma$ the first order differential equation

$$
\nabla u \cdot \nabla \gamma+(\Delta u) \gamma=0 \text { in } \Omega
$$

once the information about $u$ has been extended from $\Gamma$ to $\Omega$. It is clear that such an equation is singular at the critical points of $u$. Thus, it is helpful to
know a priori that $\nabla u$ does not vanish and this can be done via (appropriate generalizations of) Theorem 3.2 by choosing suitable currents on $\Gamma$.

The possible presence of maximum and/or minimum points for the solutions of (3), (4), or (5)-(6) makes the search for an estimate of the number of critical points a difficult task (even in the planar case). In fact, the mere topological information only results in an estimate of the signed sum of the critical points, the sign depending on whether the relevant critical point is an extremal or saddle point. For example, for the solution of (3) or (4), we only know that the difference between the number of its (isolated) maximum and saddle points (minimum points are not allowed) must equal $\chi(\Omega)$, the Euler characteristic of $\Omega$ - a Morse-type theorem. Thus, further assumptions, such as geometric information on $\Omega$, are needed. More information is also necessary even if we consider the case of harmonic functions in dimension $N \geq 3$.

In the author's knowledge, results on the number of critical points of solutions of (3), (4), or (5)-(6) reduce to deduction that their solutions admit a unique critical point if $\Omega$ is convex. Moreover, the proof of such results is somewhat indirect: the solution is shown to be quasi-concave - indeed, log-concave for the cases of (4) and (5)-(6), and $1 / 2$-concave for the case (3) - and then its analyticity completes the argument. Estimates of the number of critical points when the domain $\Omega$ has more complex geometries would be a significant advance. In this survey, we will propose and justify some conjectures.

The problem of locating critical points is also an interesting issue. The first work on this subject dates back to Gauss [36], who proved that the critical points of a complex polynomial are its zeroes, if they are multiple, and the equilibrium points of the gravitational field of force generated by particles placed at the zeroes and with masses proportional to the zeroes' multiplicities (see Section 4). Later refinements are due to Jensen [47] and Lucas [62], but the first treatises on this matter are Marden's book [68] and, primarily, Walsh's monograph [87] that collects most of the results on the number and location of critical points of complex polynomials and harmonic functions known at that date. In general dimension, even for harmonic functions, results are sporadic and rely on explicit formulae or symmetry arguments.

Two well known questions in this context concern the location of the hot spot in a heat conductor - a hot spot is a point of (absolute or relative) maximum temperature in the conductor. The situation described by (5)-(6) corresponds with the case of a grounded conductor. By some asymptotic analysis, under appropriate assumptions on $\varphi$, one can show that the hot spots originate from the set of maximum points of the function $d_{\Omega}(x)$ - the distance of $x \in \Omega$ from $\Gamma$ - and tend to the maximum points of the unique positive solution of (4), as $t \rightarrow \infty$. In the case $\Omega$ is convex, we have only one hot spot, as already observed. In Section 4, we will describe three techniques to locate it; some of them extend their validity to locate the maximum points of the solutions to
(3) and (4). We will also give an account of what it is known about convex conductors that admit a stationary hot spot (that is the hot spot does not move with time).

It has also been considered the case in which the homogeneous Dirichlet boundary condition in (6) is replaced by the homogeneous Neumann condition:

$$
\begin{equation*}
u_{\nu}=0 \quad \text { on } \Gamma \times(0, \infty) \tag{7}
\end{equation*}
$$

These settings describe the evolution of temperature in an insulated conductor of given constant initial temperature and has been made popular by a conjecture of J. Rauch [76] that would imply that the hot spot must tend to a boundary point. Even if we now know that it is false for a general domain, the conjecture holds true for certain planar convex domains but it is still standing for unrestrained convex domains.

The remainder of the paper is divided into three sections that reflect the aforementioned features. In Section 2, we shall describe the local properties of critical points of harmonic functions or, more generally, of solutions of elliptic equations, that lead to estimates of the size of critical sets. In Section 3, we shall focus on bounds for the number of critical points that depend on the boundary behavior of the relevant solutions and/or the geometry of $\Gamma$. Finally, in Section 4, we shall address the problem of locating the possible critical points. As customary for a survey, our presentation will stress ideas rather than proofs.

This paper is dedicated with sincere gratitude to Giovanni Alessandrini an inspiring mentor, a supportive colleague and a genuine friend - on the occasion of his $60^{\text {th }}$ birthday. Much of the material presented here was either inspired by his ideas or actually carried out in his research with the author.

## 2. The size of the critical set of a harmonic function

A harmonic function in a domain $\Omega$ is a solution of the Laplace's equation

$$
\Delta u=u_{x_{1} x_{1}}+\cdots+u_{x_{N} x_{N}}=0 \text { in } \Omega
$$

It is well known that harmonic functions are analytic, so there is no difficulty to define their critical points or the critical set

$$
\mathcal{C}(u)=\{x \in \Omega: \nabla u(x)=0\} .
$$

Before getting into the heart of the matter, we present a relevant example.

### 2.1. Harmonic polynomials

In dimension two, we have a powerful tool since we know that a harmonic function is (locally) the real or imaginary part of a holomorphic function. This
remark provides our imagination with a reach set of examples on which we can speculate. For instance, the harmonic function

$$
u=\operatorname{Re}\left(z^{n}\right)=\operatorname{Re}\left[(x+i y)^{n}\right], n \in \mathbb{N}
$$

already gives some insight on the properties of harmonic functions we are interested in. In fact, we have that

$$
u_{x}-i u_{y}=n z^{n-1}
$$

thus, $u$ has only one distinct critical point, $z=0$, but it is more convenient to say that $u$ has $n-1$ critical points at $z=0$ or that $z=0$ is a critical point with multiplicity $m$ with $m=n-1$. By virtue of this choice, we can give a topological meaning to $m$.

To see that, it is advantageous to represent $u$ in polar coordinates:

$$
u=r^{n} \cos (n \theta)
$$

here, $r=|z|$ and $\theta$ is the principal branch of $\arg z$, that is we are assuming that $-\pi \leq \theta<\pi$. Thus, the topological meaning of $m$ is manifest when we look at the level "curve" $\{z: u(z)=u(0)\}$ : it is made of $m+1=n$ straight lines passing through the critical point $z=0$, divides the plane into $2 n$ cones (angles), each of amplitude $\pi / n$ and the sign of $u$ changes across those lines (see Fig. 1). One can also show that the signed angle $\omega$ formed by $\nabla u$ and the direction of the positive real semi-axis, since it equals $-(n-1) \arg z$, increases by $2 \pi m$ while $z$ makes a complete loop clockwise around $z=0$; thus, $m$ is a sort of winding number for $\nabla u$.

The critical set of a homogeneous polynomial $P: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is a cone in $\mathbb{R}^{N}$. Moreover, if $P$ is also harmonic (and non-constant) one can show that

$$
\begin{equation*}
\text { dimension of } \mathcal{C}(u) \leq N-2 \tag{8}
\end{equation*}
$$

### 2.2. Harmonic functions

If $N=2$ and $u$ is any harmonic function, the picture is similar to that outlined in the example. In fact, we can again consider the "complex gradient" of $u$,

$$
g=u_{x}-i u_{y},
$$

and observe that $g$ is holomorphic in $\Omega$, since $\partial_{\bar{z}} g=0$, and hence analytic. Thus, the zeroes of $g$ (and hence the critical points of $u$ ) in $\Omega$ are isolated and have finite multiplicity. If $z_{0}$ is a zero with multiplicity $m$ of $g$, then we can write that

$$
g(z)=\left(z-z_{0}\right)^{m} h(z)
$$



Figure 1: Level set diagram of $u=r^{6} \cos (6 \theta)$ at the critical point $z=0 ; u$ changes sign from positive to negative at dashed lines and from negative to positive at solid lines.
where $h$ is holomorphic in $\Omega$ and $h\left(z_{0}\right) \neq 0$.
On the other hand, we also know that $u$ is locally the real part of a holomorphic function $f$ and hence, since $f^{\prime}=g$, by an obvious normalization, it is not difficult to infer that

$$
f(z)=\frac{1}{n}\left(z-z_{0}\right)^{n} k(z)
$$

where $n=m+1$ and $k$ is holomorphic and $k\left(z_{0}\right)=h\left(z_{0}\right) \neq 0$. Passing to polar coordinates by $z=z_{0}+r e^{i \theta}$ tells us that

$$
f\left(z_{0}+r e^{i \theta}\right)=\frac{\left|h\left(z_{0}\right)\right|}{n} r^{n} e^{i\left(n \theta+\theta_{0}\right)}+O\left(r^{n+1}\right) \text { as } r \rightarrow 0
$$

where $\theta_{0}=\arg h\left(z_{0}\right)$. Thus, we have that

$$
u=\frac{\left|h\left(z_{0}\right)\right|}{n} r^{n} \cos \left(n \theta+\theta_{0}\right)+O\left(r^{n+1}\right) \text { as } r \rightarrow 0
$$

and hence, modulo a rotation by the angle $\theta_{0}$, in a small neighborhood of $z_{0}$, we can say that the critical level curve $\left\{z: u(z)=u\left(z_{0}\right)\right\}$ is very similar to that described in the example with 0 replaced by $z_{0}$. In particular, it is made of $n$ simple curves passing through $z_{0}$ and any two adjacent curves meet at $z_{0}$ with an angle that equals $\pi / n$ (see Fig. 2).


Figure 2: Level set diagram of a harmonic function at a critical point with multiplicity $m=5$. The curves meet with equal angles at the critical point.

If $N \geq 3$, similarly, a harmonic function can be approximated near a point 0 at which vanishes by a homogeneous harmonic polynomial of some degree $n$ :

$$
\begin{equation*}
u(x)=P_{n}(x)+O\left(|x|^{n+1}\right) \text { as }|x| \rightarrow 0 \tag{9}
\end{equation*}
$$

However, the structure of the set $\mathcal{C}(u)$ depends on whether 0 is an isolated critical point of $P_{n}$ or not. In fact, if 0 is not isolated, then $\mathcal{C}(u)$ and $\mathcal{C}\left(P_{n}\right)$ could not be diffeomorphic in general, as shown by the harmonic function

$$
u(x, y, z)=x^{2}-y^{2}+\left(x^{2}+y^{2}\right) z-\frac{2}{3} z^{3}, \quad(x, y, z) \in \mathbb{R}^{3} .
$$

Indeed, if $P_{2}(x, y, z)=x^{2}-y^{2}, \mathcal{C}\left(P_{2}\right)$ is the $z$-axis, while $\mathcal{C}(u)$ is made of 5 isolated points ([74]).

### 2.3. Elliptic equations in the plane

These arguments can be repeated with some necessary modifications for solutions of uniformly elliptic equations of the type (1), where the variable coefficients $a, b, c$ are Lipschitz continuous and $d, e$ are bounded measurable on $\Omega$ and the uniform ellipticity is assumed to take the following form:

$$
a c-b^{2}=1 \text { in } \Omega
$$

Now, the classical theory of quasi-conformal mappings comes in our aid (see $[14,86]$ and also $[4,5])$. By the uniformization theorem (see [86]), there exists
a quasi-conformal mapping $\zeta(z)=\xi(z)+i \eta(z)$, satisfying the equation

$$
\zeta_{\bar{z}}=\kappa(z) \zeta_{z} \text { with }|\kappa(z)|=\frac{a+c-2}{a+c+2}<1
$$

such that the function $U$ defined by $U(\zeta)=u(z)$ satisfies the equation

$$
\Delta U+P U_{\xi}+Q U_{\eta}=0 \text { in } \zeta(\Omega)
$$

where $P$ and $Q$ are real-valued functions depending on the coefficients in (1) and are essentially bounded on $\zeta(\Omega)$. Notice that, since the composition of $\zeta$ with a conformal mapping is still quasi-conformal, if it is convenient, by the Riemann mapping theorem, we can choose $\zeta(\Omega)$ to be the unit disk $\mathbb{D}$.

By setting $G=U_{\xi}-i U_{\eta}$, simple computations give that

$$
G_{\bar{\zeta}}=R G+\bar{R} \bar{G} \text { in } \mathbb{D},
$$

where $R=(P+i Q) / 4$ is essentially bounded. This equation tells us that $G$ is a pseudo-analytic function for which the following similarity principle holds (see [86]): there exist two functions, $H(\zeta)$ holomorphic in $\mathbb{D}$ and $s(\zeta)$ Hölder continuous on the whole $\mathbb{C}$, such that

$$
\begin{equation*}
G(\zeta)=e^{s(\zeta)} H(\zeta) \text { for } \zeta \in \mathbb{D} \tag{10}
\end{equation*}
$$

Owing to (10), it is clear that the critical points of $u$, by means of the mapping $\zeta(z)$, correspond to the zeroes of $G(\zeta)$ or, which is the same, of $H(\zeta)$ and hence we can claim that they are isolated and have a finite multiplicity.

This analysis can be further extended if the coefficients $d$ and $e$ are zero, that is for the solutions of (2). In this case, we can even assume that the coefficients $a, b, c$ be merely essentially bounded on $\Omega$, provided that we agree that $u$ is a non-constant weak solution of (1). It is well known that, with these assumptions, solutions of (1) are in general only Hölder continuous and the usual definition of critical point is no longer possible. However, in [5] we got around this difficulty by introducing a different notion of critical point, that is still consistent with the topological structure of the level curves of $u$ at its critical values.

To see this, we look for a surrogate of the harmonic conjugate for $u$. In fact, (1) implies that the 1-form

$$
\omega=-\left(b u_{x}+c u_{y}\right) d x+\left(a u_{x}+b u_{y}\right) d y
$$

is closed (in the weak sense) in $\Omega$ and hence, thanks to the theory developed in [15], we can find a so-called stream function $v \in W^{1,2}(\Omega)$ whose differential $d v$ equals $\omega$, in analogy with the theory of gas dynamics (see [13]).


Figure 3: Level set diagram of a solution of an elliptic equation with discontinuous coefficients at a geometric critical point with multiplicity $m=5$. At that point, any two consecutive curves meet with positive angles, possibly not equal to one another.

Thus, in analogy with what we have done in Subsection 2.2, we find out that the function $f=u+i v$ satisfies the equation

$$
\begin{equation*}
f_{\bar{z}}=\mu f_{z} \tag{11}
\end{equation*}
$$

where

$$
\mu=\frac{c-a-2 i b}{2+a+c} \text { and }|\mu| \leq \frac{1-\lambda}{1+\lambda}<1 \text { in } \Omega
$$

and $\lambda>0$ is a lower bound for the smaller eigenvalue of the matrix of the coefficients:

$$
\left(\begin{array}{ll}
a & b \\
b & c
\end{array}\right)
$$

The fact that $f \in W^{1,2}(\Omega, \mathbb{C})$ implies that $f$ is a quasi-regular mapping that can be factored as

$$
f=F \circ \chi \text { in } \Omega,
$$

where $\chi: \Omega \rightarrow \mathbb{D}$ is a quasi-conformal homeomorphism and $F$ is holomorphic in $\mathbb{D}$ (see [60]). Therefore, the following representation formula holds:

$$
u=U(\chi(z)) \text { for } z \in \Omega
$$

where $U$ is the real part of $F$.
This formula informs us that the level curves of $u$ can possibly be distorted by the homeomorphism $\chi$, but preserve the topological structure of a harmonic function (see Fig. 3). This remark gives grounds to the definition introduced
in [5]: $z_{0} \in \Omega$ is a geometric critical point of $u$ if the gradient of $U$ vanishes at $\chi\left(z_{0}\right) \in \mathbb{D}$. In particular, geometric critical points are isolated and can be classified by a sort of multiplicity.

### 2.4. Quasilinear elliptic equations in the plane

A similar local analysis can be replicated when $N=2$ for quasilinear equations of type

$$
\operatorname{div}\{A(|\nabla u|) \nabla u\}=0,
$$

where $A(s)>0$ and $0<\lambda \leq 1+s A^{\prime}(s) / A(s) \leq \Lambda$ for every $s>0$ and some constants $\lambda$ and $\Lambda$.


Figure 4: Level set diagram of a solution of a degenerate quasilinear elliptic equation with $B(s)=\sqrt{1+s^{2}}$ at a critical value.

These equations can be even degenerate, such as the $p$-Laplace equation with $1<p<\infty$ (see [8]). It is worth mentioning that also the case in which $A(s)=B(s) / s$, where $B$ is increasing, with $B(0)>0$, and superlinear and growing polynomially at infinity (e.g. $B(s)=\sqrt{1+s^{2}}$ ), has been studied in [23]. In this case the function $1+s A^{\prime}(s) / A(s)$ vanishes at $s=0$ and it turns out that the critical points of a solution $u$ (if any) are never isolated (Fig. 4).

### 2.5. The case $\mathrm{N} \geq 3$

As already observed, critical points of harmonic functions in dimension $N \geq 3$ may not be isolated. Besides the example given in Section 2.2, another concrete example is given by the function

$$
u(x, y, z)=J_{0}\left(\sqrt{x^{2}+y^{2}}\right) \cosh (z), \quad(x, y, z) \in \mathbb{R}^{3}
$$

where $J_{0}$ is the first Bessel function: the gradient of $u$ vanishes at the origin and on the circles on the plane $z=0$ having radii equal to the zeroes of the second Bessel function $J_{1}$. It is clear that a region $\Omega$ can be found such that $\mathcal{C}(u) \cap \Omega$ is a bounded continuum.

Nevertheless, it can be proved that $\mathcal{C}(u)$ always has locally finite $(N-$ 2)-dimensional Hausdorff measure $\mathcal{H}^{N-2}$. A nice argument to see this was suggested to me by D. Peralta-Salas [74]. If $u$ is a non-constant harmonic function and we suppose that $\mathcal{C}(u)$ has dimension $N-1$, then the general theory of analytic sets implies that there is an open and dense subset of $\mathcal{C}(u)$ which is an analytic sub-manifold (see [59]). Since $u$ is constant on a connected component of the critical set, it is constant on $\mathcal{C}(u)$, and its gradient vanishes. Thus, by the Cauchy-Kowalewski theorem $u$ must be constant in a neighborhood of $\mathcal{C}(u)$, and hence everywhere by unique continuation. Of course, this argument would also work for solutions of an elliptic equation of type

$$
\begin{equation*}
\sum_{i, j=1}^{N} a_{i j}(x) u_{x_{i} x_{j}}+\sum_{j=1}^{N} b_{j}(x) u_{x_{j}}=0 \text { in } \Omega \tag{12}
\end{equation*}
$$

with analytic coefficients.
When the coefficients $a_{i j}, b_{j}$ in (12) are of class $C^{\infty}(\Omega)$, the result has been proved in [39] (see also [38]): if $u$ is a non-constant solution of (12), then for any compact subset $K$ of $\Omega$ it holds that

$$
\begin{equation*}
\mathcal{H}^{N-2}(\mathcal{C}(u) \cap K)<\infty . \tag{13}
\end{equation*}
$$

The proof is based on an estimate similar to (8) for the complex dimension of the singular set in $\mathbb{C}^{N}$ of the complexification of the polynomial $P_{n}$ in the approximation (9).

The same result does not hold for solutions of equation

$$
\begin{equation*}
\sum_{i, j=1}^{N} a_{i j}(x) u_{x_{i} x_{j}}+\sum_{j=1}^{N} b_{j}(x) u_{x_{j}}+c(x) u=0 \text { in } \Omega \tag{14}
\end{equation*}
$$

with $c \in C^{\infty}(\Omega)$. For instance the gradient of the first Laplace-Dirichlet eigenfunction for a spherical annulus vanishes exactly on a ( $N-1$ )-dimensional
sphere. A more general counterexample is the following (see [39, Remark p. 362]): let $v$ be of class $C^{\infty}$ and with non-vanishing gradient in the unit ball $B$ in $\mathbb{R}^{N}$; the function $u=1+v^{2}$ satisfies the equation

$$
\Delta u-c u=0 \quad \text { with } c=\frac{\Delta v^{2}}{1+v^{2}} \in C^{\infty}(B)
$$

we have that $\mathcal{C}(u)=\{x \in B: v(x)=0\}$ and it has been proved that any closed subset of $\mathbb{R}^{N}$ can be the zero set of a function of class $C^{\infty}$ (see [84]).

However, once (13) is settled, it is rather easy to show that the singular set

$$
\mathcal{S}(u)=\mathcal{C}(u) \cap u^{-1}(0)=\{x \in \Omega: u(x)=0, \nabla u(x)=0\}
$$

of a non-constant solution of (14) also has locally finite $(N-2)$-dimensional Hausdorff measure [39, Corollary 1.1]. This can be done by a trick, since around any point in $\Omega$ there always exists a positive solution $u_{0}$ of (14) and it turns out that the function $w=u / u_{0}$ is a solution of an equation like (12) and that $\mathcal{S}(u) \subseteq \mathcal{C}(w)$. In particular the set of critical points on the nodal line of an eigenfunction of the Laplace operator has locally finite $(N-2)$-dimensional Hausdorff measure.

Nevertheless, for a solution of (12) the set $\mathcal{S}(u)$ can be very complicated, as a simple example in [39, p. 361]) shows: the function $u(x, y, z)=x y+f(z)^{2}$, where $f$ is a smooth function with $\left|f f^{\prime \prime}\right|+\left(f^{\prime}\right)^{2}<1 / 4$ that vanishes exactly on an arbitrary given closed subset $K$ of $\mathbb{R}$, is a solution of

$$
u_{x x}+u_{y y}+u_{z z}-\left(f^{2}\right)^{\prime \prime}(z) u_{x y}=0 \text { and } \mathcal{S}(u)=\{(0,0)\} \times K
$$

Heuristically, as in the 2-dimensional case, the proof of (13) is essentially based on the observation that, by Taylor's expansion, a harmonic function $u$ can be approximated near any of its zeroes by a homogeneous harmonic polynomial $P_{m}\left(x_{1}, \ldots, x_{n}\right)$ of degree $m \geq 1$. Technically, the authors use the fact that the complex dimension of the critical set in $\mathbb{C}^{N}$ of the complexified polynomial $P_{m}\left(z_{1}, \ldots, z_{N}\right)$ is bounded by $N-2$. A $C^{\infty}$-perturbation argument and an inequality from geometric measure theory then yield that, near a zero of $u$, the $\mathcal{H}^{N-2}$-measure of $\mathcal{C}(u)$ can be bounded in terms of $N$ and $m$. The extension of these arguments to the case of a solution of (12) is then straightforward. Recently in [26], (13) has been extended to the case of solutions of elliptic equations of type

$$
\sum_{i, j=1}^{N}\left\{a_{i j}(x) u_{x}\right\}_{x_{j}}+\sum_{j=1}^{N} b_{j}(x) u_{x_{j}}
$$

where the coefficients $a_{i j}(x)$ and $b_{j}(x)$ are assumed to be Lipschitz continuous and essentially bounded, respectively.

## 3. The number of critical points

A more detailed description of the critical set $\mathcal{C}(u)$ of a harmonic function $u$ can be obtained if we assume to have some information on its behavior on the boundary $\Gamma$ of $\Omega$. While in Section 2 the focus was on a qualitative description of the set $\mathcal{C}(u)$, here we are concerned with establishing bounds on the number of critical points.

### 3.1. Counting the critical points of a harmonic function in the plane

An exact counting formula is given by the following result.
Theorem 3.1 ([4]). Let $\Omega$ be a bounded domain in the plane and let

$$
\Gamma=\bigcup_{j=1}^{J} \Gamma_{j}
$$

where $\Gamma_{j}, j=1, \ldots, J$ are simple closed curves of class $C^{1, \alpha}$. Consider a harmonic function $u \in C^{1}(\bar{\Omega}) \cap C^{2}(\Omega)$ that satisfies the Dirichlet boundary condition

$$
\begin{equation*}
u=a_{j} \quad \text { on } \quad \Gamma_{j}, j=1, \ldots, J \tag{15}
\end{equation*}
$$

where $a_{1}, \ldots, a_{J}$ are given real numbers, not all equal.
Then $u$ has in $\bar{\Omega}$ a finite number of critical points $z_{1}, \ldots, z_{K} ;$ if $m\left(z_{1}\right), \ldots$, $m\left(z_{K}\right)$ denote their multiplicities, then the following identity holds:

$$
\begin{equation*}
\sum_{z_{k} \in \Omega} m\left(z_{k}\right)+\frac{1}{2} \sum_{z_{k} \in \Gamma} m\left(z_{k}\right)=J-2 \tag{16}
\end{equation*}
$$

Thanks to the analysis presented in Subsection 2.3, this theorem still holds if we replace the Laplace equation in (15) by the general elliptic equation (1). In fact, modulo a suitable change of variables, we can use (10) with $\operatorname{Im}(s)=0$ on the boundary.

The function considered in Theorem 3.1 can be interpreted in physical terms as the potential in an electrical capacitor and hence its critical points are the points of equilibrium of the electrical field (Fig. 5).

The proof of Theorem 3.1 relies on the fact that the critical points of $u$ are the zeroes of the holomorphic function $f=u_{x}-i u_{y}$ and hence they can be counted with their multiplicities by applying the classical argument principle to $f$ with some necessary modifications. The important remark is that, since the boundary components are level curves for $u$, the gradient of $u$ is parallel on them to the (exterior) unit normal $\nu$ to the boundary, and hence $\arg f=-\arg \nu$.

Thus, the situation is clear if $u$ does not have critical points on $\Gamma$ : the argument principle gives at once that

$$
\begin{aligned}
& \sum_{z_{k} \in \Omega} m\left(z_{k}\right)=\frac{1}{2 \pi i} \int_{+\Gamma} \frac{f^{\prime}(z)}{f(z)} d z \\
& \quad=\frac{1}{2 \pi} \operatorname{Incr}(\arg f,+\Gamma)=\frac{1}{2 \pi} \operatorname{Incr}(-\arg \nu,+\Gamma)=-[1-(J-1)]=J-2,
\end{aligned}
$$

where by $\operatorname{Incr}(\cdot,+\gamma)$ we intend the increment of an angle on an oriented curve $+\gamma$ and by $+\Gamma$ we mean that $\Gamma$ is trodden in such a way that $\Omega$ is on the left-hand side.


Figure 5: An illustration of Theorem 3.1: the domain $\Omega$ has 3 holes; $u$ has exactly 2 critical points; dashed and dotted are the level curves at critical values.

If $\Gamma$ contains critical points, we must first prove that they are also isolated. This is done, by observing that, if $z_{0}$ is a critical point belonging to some component $\Gamma_{j}$, since $u$ is constant on $\Gamma_{j}$, by the Schwarz's reflection principle (modulo a conformal transformation of $\Omega$ ), u can be extended to a function $\widetilde{u}$ which is harmonic in a whole neighborhood of $z_{0}$. Thus, $z_{0}$ is a zero of the holomorphic function $\widetilde{f}=\widetilde{u}_{x}-i \widetilde{u}_{y}$ and hence is isolated and with finite multiplicity. Moreover, the increment of $\arg \tilde{f}$ on an oriented closed simple curve $+\gamma$ around $z_{0}$ is exactly twice as much as that of $\arg f$ on the part of $+\gamma$ inside $\Omega$. This explains the second addendum in (16).

Notice that condition (15) can be re-written as

$$
u_{\tau}=0 \quad \text { on } \Gamma,
$$

where $\tau: \Gamma \rightarrow \mathbb{S}^{1}$ is the tangential unit vector field on $\Gamma$. We cannot hope to obtain an identity as (16) if $u_{\tau}$ is not constant. However, a bound for the number of critical points of a harmonic function (or a solution of (1)) can be derived in a quite general setting.

In what follows, we assume that $\Omega$ is as in Theorem 3.1 and that $\ell: \Gamma \rightarrow \mathbb{S}^{1}$ denotes a (unitary) vector field of class $C^{1}\left(\Gamma, \mathbb{S}^{1}\right)$ of given topological degree $D$, that can be defined as

$$
\begin{equation*}
2 \pi D=\operatorname{Incr}(\arg (\ell),+\Gamma) . \tag{17}
\end{equation*}
$$

Also, we will use the following definitions:
(i) if $\left(\mathcal{J}^{+}, \mathcal{J}^{-}\right)$is a decomposition of $\Gamma$ into two disjoint subsets such that $u_{\ell} \geq 0$ on $\mathcal{J}^{+}$and $u_{\ell} \leq 0$ on $\mathcal{J}^{-}$, we denote by $M\left(\mathcal{J}^{+}\right)$the number of connected components of $\mathcal{J}^{+}$which are proper subsets of some component $\Gamma_{j}$ of $\Gamma$ and set:

$$
M=\min \left\{M\left(\mathcal{J}^{+}\right):\left(\mathcal{J}^{+}, \mathcal{J}^{+}\right) \text {decomposes } \Gamma\right\}
$$

(ii) if $\mathcal{I}^{ \pm}=\left\{z \in \Gamma: \pm u_{\ell}(z)>0\right\}$, by $M^{ \pm}$we denote the number of connected components of $\mathcal{I}^{ \pm}$which are proper subsets of some component $\Gamma_{j}$ of $\Gamma$.

Notice that in (i) the definition of $M$ does not change if we replace $\mathcal{J}^{+}$by $\mathcal{J}^{-}$.
Theorem 3.2 ([4]). Let $u \in C^{1}(\bar{\Omega}) \cap C^{2}(\Omega)$ be harmonic in $\Omega$ and denote by $m\left(z_{j}\right)$ the multiplicity of a zero $z_{j}$ of $f=u_{x}-i u_{y}$.
(a) If $M$ is finite and $u$ has no critical point in $\Gamma$, then

$$
\sum_{z_{j} \in \Omega} m\left(z_{j}\right) \leq M-D
$$

(b) if $M^{+}+M^{-}$is finite, then

$$
\sum_{z_{j} \in \Omega} m\left(z_{j}\right) \leq\left[\frac{M^{+}+M^{-}}{2}\right]-D
$$

where $[x]$ is the greatest integer $\leq x$.
This theorem is clearly less sharp than Theorem 3.1 since, in that setting, it does not give information about critical points on the boundary. However, it gives the same information on the number of interior critical points, since in the setting of Theorem 3.1 the degree of the field $\tau$ on $+\Gamma$ equals $2-J$ and $M=0$.

The possibility of choosing the vector field $\ell$ arbitrarily makes Theorem 3.2 a very flexible tool: for instance, the number of critical points in $\Omega$ can be


Figure 6: An illustration of Theorem 3.2. Here, $M=M^{+}=4 ; M^{-}=4$; $D=-2$ if $\ell=\nu$ or $\tau ; D=1$ if $\ell=z /|z|$ and the origin is in $\Omega ; D=0$ if $\ell=(1,0)$ or $(0,1)$.
estimated from information on the tangential, normal, co-normal, partial, or radial (with respect to some origin) derivatives (see Fig. 6).

As an illustration, it says that in a domain topologically equivalent to a disk, in order to have $n$ interior critical point the normal (or tangential, or co-normal) derivative of a harmonic function must change sign at least $n+1$ times and a partial derivative at least $n$ times. Thus, Theorem 3.2 helps to choose Neumann data that insures the absence of critical points in $\Omega$. For this reason, in its general form for elliptic operators, it has been useful in the study of EIT and other similar inverse problems.

We give a sketch of the proof of (a) of Theorem 3.2, that hinges on the simple fact that, if we set $\theta=\arg (\ell)$ and $\omega=\arg \left(u_{x}-i u_{y}\right)$, then

$$
u_{\ell}=\ell \cdot \nabla u=|\nabla u| \cos (\theta+\omega) .
$$

Hence, if $\left(\mathcal{J}^{+}, \mathcal{J}^{-}\right)$is a minimizing decomposition of $\Gamma$ as in (i), then

$$
|\omega+\theta| \leq \frac{\pi}{2} \quad \text { on } \quad \mathcal{J}^{+} \text {and }|\omega+\theta-\pi| \leq \frac{\pi}{2} \quad \text { on } \quad \mathcal{J}^{-} .
$$

Thus, two occurrences must be checked. If a component $\Gamma_{j}$ is contained in $\mathcal{J}^{+}$or $\mathcal{J}_{-}$, then

$$
\left|\frac{1}{2 \pi} \operatorname{Incr}\left(\omega+\theta,+\Gamma_{j}\right)\right| \leq \frac{1}{2}
$$

that implies that $\omega$ and $-\theta$ must have the same increment, being the right-hand side an integer. If $\Gamma_{j}$ contains points of both $\mathcal{J}^{+}$and $\mathcal{J}^{-}$, instead, if $\sigma^{+} \subset \mathcal{J}^{+}$ and $\sigma^{-} \subset \mathcal{J}^{-}$are two consecutive components on $\Gamma_{j}$, then

$$
\frac{1}{2 \pi} \operatorname{Incr}\left(\omega+\theta,+\left(\sigma^{+} \cap \sigma^{-}\right) \leq 1\right.
$$

Therefore, if $M_{j}$ is the number of connected components of $\mathcal{J}^{+} \cap \Gamma_{j}$ (which equals that of $\mathcal{J}^{+} \cap \Gamma_{j}$ ), then

$$
\frac{1}{2 \pi} \operatorname{Incr}\left(\omega+\theta,+\Gamma_{j}\right) \leq M_{j}
$$

and hence

$$
\begin{aligned}
\sum_{z_{k} \in \Omega} m\left(z_{k}\right)=\frac{1}{2 \pi} \operatorname{Incr} & (\omega,+\Gamma)=\frac{1}{2 \pi} \operatorname{Incr}(\omega+\theta,+\Gamma)-D \\
= & \sum_{j=1}^{J} \frac{1}{2 \pi} \operatorname{Incr}\left(\omega+\theta,+\Gamma_{j}\right) \leq \sum_{j=1}^{J} M_{j}-D=M-D
\end{aligned}
$$

The obstacle problem. An estimate similar to that of Theorem 3.2 has been obtained also for $N=2$ by Sakaguchi [78] for the obstacle problem. Let $\Omega$ be bounded and simply connected and let $\psi$ be a given function in $C^{2}(\bar{\Omega})-$ the obstacle. There exists a unique solution $u \in H_{0}^{1}(\Omega)$ such that $u \geq \psi$ in $\Omega$ of the obstacle problem

$$
\int_{\Omega} \nabla u \cdot \nabla(v-u) d x \geq 0 \text { for every } v \in H_{0}^{1}(\Omega) \text { such that } u \geq \psi
$$

It turns out that $u \in C^{1,1}(\bar{\Omega})$ and $u$ is harmonic outside of the contact set $I=\{x \in \Omega: u(x)=\psi(x)\}$. In [78] it is proved that, if the number of connected components of local maximum points of $\psi$ equals $J$, then

$$
\sum_{z_{k} \in \Omega \backslash I} m\left(z_{k}\right) \leq J-1
$$

with the usual meaning for $z_{k}$ and $m\left(z_{k}\right)$. In [78], this result is also shown to hold for a more general class of quasi-linear equations. The proof of this result is based on the analysis of the level sets of $u$ at critical values, in the wake of [1] and [40].

Topological bounds as in Theorems 3.1 or 3.2 are not possible in dimension greater than 2. We give two examples.


Figure 7: The broken doughnut in a ball: $u$ must have a critical point near the center of $B$ and one between the ends of $T$.

The broken doughnut in a ball. The first is an adaptation of one contained in [28] and reproduces the situation of Theorem 3.1 (see Fig. 7). Let $B$ be the unit ball centered at the origin in $\mathbb{R}^{3}$ and $T$ an open torus with center of symmetry at the origin and such that $\bar{T} \subset B$. We can always choose coordinate axes in such a way that the $x_{3}$-axis is the axis of revolution for $T$ and hence define the set $T_{\varepsilon}=\left\{x \in T: x_{2}<\varepsilon^{-1}\left|x_{1}\right|\right\}$. $\overline{T_{\varepsilon}}$ is simply connected and tends to $T$ as $\varepsilon \rightarrow 0^{+}$. Now, set $\Omega_{\varepsilon}=B \backslash \overline{T_{\varepsilon}}$ and consider a capacity potential for $\Omega$, that is the harmonic function in $\Omega_{\varepsilon}$ with the following boundary values

$$
u=0 \quad \text { on } \partial B, \quad u=1 \text { on } \partial T_{\varepsilon} .
$$

Since $\Omega_{\varepsilon}$ has 2 planes of symmetry (the $x_{1} x_{2}$ and $x_{2} x_{3}$ planes), the partial derivatives $u_{x_{1}}$ and $u_{x_{3}}$ must be zero on the two segments that are the intersection of $\Omega_{\varepsilon}$ with the $x_{2}$-axis. If $\sigma$ is the segment that contains the origin, the restriction of $u$ to $\bar{\sigma}$ equals 1 at the point $\bar{\sigma} \cap \partial T_{\varepsilon}$, is 0 at the point $\bar{\sigma} \cap \partial B$, is bounded at the origin by a constant $<1$ independent of $\varepsilon$, and can be made arbitrarily close to 1 between the "ends" of $T_{\varepsilon}$, when $\varepsilon \rightarrow 0^{+}$, It follows that, if $\varepsilon$ is sufficiently small, $u_{x_{2}}$ (and hence $\nabla u$ ) must vanish twice on $\sigma$.

It is clear that this argument does not depend on the size or on small deformations of $T$. Thus, we can construct in $B$ a (simply connected) "chain" $C_{\varepsilon}$ of an arbitrary number $n$ of such tori, by gluing them together: the solution in
the domain obtained by replacing $T_{\varepsilon}$ by $C_{\varepsilon}$ will then have at least $2 n$ critical points.

Circles of critical points. The second example shows that, in general dimension, a finite number of sign changes of some derivative of a harmonic function $u$ on the boundary does not even imply that $u$ has a finite number of critical points.

To see this, consider the harmonic function is Subsection 2.5:

$$
u(x, y, z)=J_{0}\left(\sqrt{x^{2}+y^{2}}\right) \cosh (z) .
$$

It is easy to see that, for instance, on any sphere centered at the origin the normal derivative $u_{\nu}$ changes its sign a finite number of times. However, if the radius of the sphere is larger than the first positive zero of $J_{1}=0$, the corresponding ball contains at least one circle of critical points.

Star-shaped annuli. Nevertheless, if some additional geometric information is added, something can be done. Suppose that $\Omega=D_{0} \backslash \overline{D_{1}}$, where $D_{0}$ and $D_{1}$ are two domains in $\mathbb{R}^{N}$, with boundaries of class $C^{1}$ and such that $\overline{D_{1}} \subset D_{0}$. Suppose that $D_{0}$ and $D_{1}$ are star-shaped with respect to the same origin $O$ placed in $D_{1}$, that is the segment $O P$ is contained in the domain for every point $P$ chosen in it. Then, the capacity potential $u$ defined as the solution of the Dirichlet problem

$$
\Delta u=0 \text { in } \Omega, \quad u=0 \text { on } \partial D_{0}, \quad u=1 \text { on } \partial D_{1},
$$

does not have critical points in $\bar{\Omega}$. This is easily proved by considering the harmonic function

$$
w(x)=x \cdot \nabla u(x), x \in \Omega .
$$

Since $D_{0}$ and $D_{1}$ are starshaped and of class $C^{1}, w \geq 0$ on $\partial \Omega$. By the strong maximum principle, then $w>0$ in $\Omega$; in particular, $\nabla u$ does not vanish in $\Omega$ and all the sets $D_{1} \cup\{x \in \bar{\Omega}: u(x)>s\}$ turn out to be star shaped too (see [33]). This theorem can be extended to the capacity potential defined in $\Omega=\mathbb{R}^{N} \backslash \overline{D_{1}}$ as the solution of

$$
\Delta u=0 \text { in } \Omega, \quad u=1 \text { on } \partial \Omega, \quad u \rightarrow 0 \text { as }|x| \rightarrow \infty .
$$

Such results have been extended in $[35,75,81]$ to a very general class of nonlinear elliptic equations.

### 3.2. Counting the critical points of Green's functions on manifolds

With suitable restrictions on the coefficients, (2) can be regarded as the LaplaceBeltrami equation on the Riemannian surface $\mathbb{R}^{2}$ equipped with the metric

$$
c(d x)^{2}-2 b(d x)(d y)+a(d y)^{2} .
$$

Theorems 3.1 and 3.2 can then be interpreted accordingly.
This point of view has been considered in a more general context in [29, 30], where the focus is on Green's functions of a 2-dimensional complete Riemannian surface ( $M, g$ ) of finite topological type (that is, the first fundamental group of $M$ is finitely generated). A Green's function is a symmetric function $\mathcal{G}(x, y)$ that satisfies in $M$ the equation

$$
\begin{equation*}
-\Delta_{g} \mathcal{G}(\cdot, y)=\delta_{y}, \tag{18}
\end{equation*}
$$

where $\Delta_{g}$ is the Laplace-Beltrami operator induced by the metric $g$ and $\delta_{y}$ is the Dirac delta centered at a point $y \in M$.

A symmetric Green's function $\mathcal{G}$ can always be constructed by an approximation argument introduced in [61]: an increasing sequence of compact subsets $\Omega_{n}$ containing $y$ and exhausting $M$ is introduced and $\mathcal{G}$ is then defined as the limit on compact subsets of $M \backslash\{y\}$ of the sequence $\mathcal{G}_{n}-a_{n}$, where $\mathcal{G}_{n}$ is the solution of (18) such that $\mathcal{G}_{n}=0$ on $\Gamma_{n}$ and $a_{n}$ is a suitable constant. A Green's function defined in this way is generally not unique, but has many properties in common with the fundamental solution for Laplace's equation in the Euclidean plane.

With these premises, in $[29,30]$ it has been proved the following notable topological bound:

$$
\text { number of critical points of } \mathcal{G} \leq 2 \mathfrak{g}+\mathfrak{e}-1
$$

where $\mathfrak{g}$ and $\mathfrak{e}$ are the genus and the number of ends of $M$; the number $2 \mathfrak{g}+\mathfrak{e}-1$ is known as the first Betti number of M. Moreover, if the Betti number is attained, then $\mathcal{G}$ is Morse, that is at its critical points the Hessian matrix is non-degenerate. In [29], it is also shown that, in dimensions greater than two, an upper bound by topological invariants is impossible.

Two different proofs are constructed in [29] and [30], respectively. Both proofs are based on the following uniformization principle: since $(M, g)$ is a smooth manifold of finite topological type, it is well known (see [54]) that there exists a compact surface $\Sigma$ endowed with a metric $g^{\prime}$ of constant curvature, a finite number $J \geq 0$ of isolated points points $p_{j} \in \Sigma$ and a finite number $K \geq 0$ of (analytic) topological disks $D_{k} \subset \Sigma$ such that $(M, g)$ is conformally isometric to the manifold $\left(M^{\prime}, g^{\prime}\right)$, where $M^{\prime}$ is interior of

$$
M^{\prime}=\Sigma \backslash\left(\bigcup_{j=1}^{J}\left\{p_{j}\right\} \cup \bigcup_{k=1}^{K} D_{k}\right)
$$

That means that there exist a diffeomorphism $\Phi: M \rightarrow M^{\prime}$ and a positive function $f$ on $M$ such that $\Phi^{*} g^{\prime}=f g$; it turns out that the genus $\mathfrak{g}$ of $\Sigma$ and the number $J+K$ - that equals the number $\mathfrak{e}$ ends of $M$ - determine $M$ up to diffeomorphisms.

The proof in [29] then proceeds by analyzing the transformed Green's function $\mathcal{G}^{\prime}=\mathcal{G} \circ \Phi^{-1}$. It is proved that $\mathcal{G}^{\prime}$ satisfies the problem
$-\Delta_{g^{\prime}} \mathcal{G}^{\prime}\left(\cdot, y^{\prime}\right)=\delta_{y^{\prime}}-\sum_{j=1}^{J} c_{j} \delta_{p_{j}} \quad$ in the interior of $M^{\prime}, \quad \mathcal{G}^{\prime}=0$ on $\bigcup_{k=1}^{K} \partial D_{k}$,
where $y^{\prime}=\Phi(y)$ and the constants $c_{j}$, possibly zero (in which case $\mathcal{G}^{\prime}$ would be $g^{\prime}$-harmonic near $p_{j}$ ), sum up to 1 . Thus, a local blow up analysis of the Hopf index $\mathfrak{I}\left(z_{n}\right), j=1, \ldots, N$, of the gradient of $\mathcal{G}^{\prime}$ at the critical points $z_{1}, \ldots, z_{N}$ (isolated and with finite multiplicity), together with the Hopf Index Theorem ( $[70,71]$ ), yield the formula

$$
\sum_{n=1}^{N} \Im\left(z_{n}\right)+\sum_{c_{j} \neq 0} \mathfrak{I}\left(p_{j}\right)=\chi\left(\Sigma^{*}\right),
$$

where $\chi\left(\Sigma^{*}\right)$ is the Euler characterstic of the manifold

$$
\Sigma^{*}=\Sigma \backslash\left(D_{y^{\prime}} \cup \bigcup_{k=1}^{K} D_{k}\right)
$$

and $D_{y^{\prime}}$ is a sufficiently small disk around $y^{\prime}$. Since $\chi\left(\Sigma^{*}\right)$ is readily computed as $1-2 \mathfrak{g}-K$ and $\mathfrak{I}\left(z_{n}\right) \leq-1$, one then obtains that

$$
\begin{aligned}
& \text { number of critical points of } \mathcal{G}^{\prime}=-\sum_{n=1}^{N} \mathfrak{I}\left(z_{n}\right)= \\
& \qquad 2 \mathfrak{g}+K-1+\sum_{c_{j} \neq 0} \mathfrak{\Im}\left(p_{j}\right) \leq 2 \mathfrak{g}+J+K-1=2 \mathfrak{g}+\mathfrak{e}-1 .
\end{aligned}
$$

Of course, the gradient of $\mathcal{G}^{*}$ vanishes if and only if that of $\mathcal{G}$ does.
The proof contained in [30] has a more geometrical flavor and focuses on the study of the integral curves of the gradient of $\mathcal{G}$. This point of view is motivated by the fact that in Euclidean space the Green's function (the fundamental solution) arises as the electric potential of a charged particle at $y$, so that its critical points correspond to equilibria and the integral curves of its gradient field are the lines of force classically studied in the XIX century. Such a description relies on techniques of dynamical systems rather than on the toolkit of partial differential equations.

We shall not get into the details of this proof, but we just mention that it gives a more satisfactory portrait of the integral curves connecting the various critical points of $\mathcal{G}$ - an issue that has rarely been studied.

### 3.3. Counting the critical points of eigenfunctions

The bounds and identities on the critical points that we considered so far are based on a crucial topological tool: the index $\mathfrak{I}\left(z_{0}\right)$ of a critical point $z_{0}$.

For a function $u \in C^{1}(\Omega)$, the integer $\mathfrak{I}\left(z_{0}\right)$ is the winding number or degree of the vector field $\nabla u$ around $z_{0}$ and is related to the portrait of the set $\mathcal{N}_{u}=\left\{z \in \mathcal{U}: u(z)=u\left(z_{0}\right)\right\}$ for a sufficiently small neighborhood $\mathcal{U}$ of $z_{0}$. As a matter of fact, if $z_{0}$ is an isolated critical point of $u$, one can distinguish two situations (see [4, 77]):
(I) if $\mathcal{U}$ is sufficiently small, $\mathcal{N}_{u}=\left\{z_{0}\right\}$ and $\mathfrak{I}\left(z_{0}\right)=1$;
(II) if $\mathcal{U}$ is sufficiently small, $\mathcal{N}_{u}$ consists of $n$ simple curves and, if $n \geq 2$, each pair of such curves crosses at $z_{0}$ only; it turns out that $\mathfrak{I}\left(z_{0}\right)=1-n$.

Critical points with index $\mathfrak{I}$ equal to 1,0 , or negative are called extremal, trivial, or saddle points, respectively (see [4]) . A saddle point is simple or Morse if the hessian matrix of $u$ at that point is not trivial.

In the cases we examined so far, we always have that $\Im\left(z_{0}\right) \leq-1$, that is $z_{0}$ is a saddle point, since (I) and (II) with $n=1$ cannot occur, by the maximum principle.

The situation considerably changes when $u$ is a solution of (3), (4), or (5). Here, we shall give an account of what can be said for solutions of (4). The same ideas can be used for solutions of the semilinear equation

$$
-\Delta u=f(u) \text { in } \Omega
$$

subject to a homogeneous Dirichlet boundary condition, where the non-linearity $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the assumptions:

$$
f(t)>0 \text { if } t>0 \quad \text { or } \quad f(t) / t>0 \text { for } t \neq 0
$$

(see [4] for details). We present here the following result that is in the spirit of Theorem 3.1.

Theorem 3.3 ([4]). Let $\Omega$ be as in Theorem 3.1 and $u \in C^{1}(\bar{\Omega}) \cap C^{2}(\Omega)$ be a solution of (4). If $z_{0} \in \bar{\Omega}$ is an isolated critical point of $u$ in $\bar{\Omega}$, then
(A) either $z_{0}$ is a nodal critical point, that is $z_{0} \in \mathcal{S}(u)$, and the function $u_{x}-i u_{y}$ is asymptotic to $c\left(z-z_{0}\right)^{m}$, as $z \rightarrow z_{0}$, for some $c \in \mathbb{C} \backslash\{0\}$ and $m \in \mathbb{N}$,
(B) or $z_{0}$ is an extremal, trivial, or simple saddle critical point.

Finally, if all the critical points of $u$ in $\bar{\Omega}$ are isolated ${ }^{1}$, the following identity holds:

$$
\begin{equation*}
\sum_{z_{k} \in \Omega} m\left(z_{k}\right)+\frac{1}{2} \sum_{z_{k} \in \Gamma} m\left(z_{k}\right)+n_{S}-n_{E}=J-2 . \tag{19}
\end{equation*}
$$

Here, $n_{S}$ and $n_{E}$ denote the number of the simple saddle and extremal points of $u$.

Thus, a bound on the number of critical points in topological terms is not possible - additional information of different nature should be added.

The proof of this theorem can be outlined as follows.
First, one observes that, at a nodal critical point $z_{0} \in \Omega, \Delta u$ vanishes, and hence the situation described in Subsection 2.2 is in order, that is $u_{x}-i u_{y}$ actually behaves as specified in (A) and the index $\mathfrak{I}\left(z_{0}\right)$ equals $-m$. If $z_{0} \in \Gamma$, a reflection argument like the one used for Theorem 3.1 can be used, so that $z_{0}$ can be treated as an interior nodal critical point of an extended function with vanishing laplacian at $z_{0}$ and (A) holds; in this case, however, as done for Theorem 3.1, the contribution of $z_{0}$ must be counted as $-m / 2$.

Secondly, one examines non-nodal critical points. At these points $\Delta u$ is either positive or negative. If, say, $\Delta u\left(z_{0}\right)<0$, then at least one eigenvalue of the hessian matrix of $u$ must be negative and the remaining eigenvalue is either positive (and hence a simple saddle point arises), negative (and hence a maximum point arises) or zero (and hence, with a little more effort, either a trivial or a simple saddle point arises). Thus, the total index of these points sums up to $n_{E}-n_{S}$.

Finally, identity (19) is obtained by applying Hopf's index theorem in a suitable manner.

### 3.4. Extra assumptions: the emergence of geometry

As emerged in the previous subsection, topology is not enough to control the number of critical points of an eigenfunction or a torsion function. Here, we will explain how some geometrical information about $\Omega$ can be helpful.

Convexity is a useful information. If the domain $\Omega \subset \mathbb{R}^{N}, N \geq 2$, is convex, one can expect that the solution $\tau$ of (3) and the only positive solution $\phi_{1}$ of (4) - it exists and, as is well known, corresponds to the first Dirichlet eigenvalue $\lambda_{1}$ - have only one critical point (the maximum point). This expectation is realistic, but a rigorous proof is not straightforward.

[^5]In fact, one has to first show that $\tau$ and $\phi_{1}$ are quasi-concave, that is one shows the convexity of the level sets

$$
\{x \in \Omega: u(x) \geq s\} \text { for every } 0 \leq s \leq \max _{\bar{\Omega}} u
$$

for $u=\tau$ or $u=\phi_{1}$. It should be noted that $\phi_{1}$ is never concave and examples of convex domains $\Omega$ can be constructed such that $\tau$ is not concave (see [57]).

The quasi-concavity of $\tau$ and $\phi_{1}$ can be proved in several different ways (see [16, 17, 21, 46, 56, 57, 82]). Here, we present the argument used in [57]. There, the desired quasi-convexity is obtained by showing that the functions $\sigma=\sqrt{\tau}$ and $\psi=\log \phi_{1}$ are concave functions ( $\tau$ and $\phi_{1}$ are then said $1 / 2$-concave and log-concave, respectively).

In fact, one shows that $\sigma$ and $\psi$ satisfy the conditions

$$
\Delta \sigma=-\frac{1+2|\nabla \sigma|^{2}}{2 \sigma} \text { in } \Omega, \quad \sigma=0 \text { on } \Gamma,
$$

and

$$
\Delta \psi=-\left(\lambda_{1}+|\nabla \psi|^{2}\right) \text { in } \Omega, \quad \psi=-\infty \text { on } \Gamma .
$$

The concavity test established by Korevaar in [57], based on a maximum principle for the so-called concavity function (see also [55]), applies to these two problems and guarantees that both $\sigma$ and $\psi$ are concave. With similar arguments, one can also prove that the solution of (5)-(6) is log-concave in $x$ for any fixed time $t$.

The obtained quasi-concavity implies in particular that, for $u=\tau$ or $\phi_{1}$, the set of critical points $\mathcal{C}(u)$, that here coincides with the set

$$
\mathcal{M}(u)=\left\{x \in \Omega: u(x)=\max _{\bar{\Omega}} u\right\}
$$

is convex. This set cannot contain more than one point, due to the analyticity of $u$. In fact, if it contained a segment, being the restriction of $u$ analytic on the chord of $\bar{\Omega}$ containing that segment, $u$ would be a positive constant on this chord and this is impossible, since $u=0$ at the endpoints of this chord.

This same argument makes sure that, if $\varphi \equiv 1$ in a convex domain $\Omega$, then for any fixed $t>0$ there is a unique point $x(t) \in \Omega$ - the so-called hot spot at which the solution of (5)-(6) attains its maximum in $\bar{\Omega}$, that is

$$
h(x(t), t)=\max _{x \in \bar{\Omega}} h(x, t) \text { for } t>0
$$

The location of $x(t)$ in $\Omega$ will be one of the issues in the next section.
A conjecture. Counting (or estimating the number of) the critical points of $\tau, \phi_{1}$, or $h$ when $\Omega$ is not convex seems a difficult task. For instance, to the
author's knowledge, it is not even known whether or not the uniqueness of the maximum point holds true if $\Omega$ is assumed to be star-shaped with respect to some origin.

We conclude this subsection by offering and justifying a conjecture on the number of hot spots in a bounded simply connected domain $\Omega$ in $\mathbb{R}^{2}$. To this aim, we define for $t>0$ the set of hot spots as

$$
\mathcal{H}(t)=\{x \in \Omega: x \text { is a local maximum point of } h(\cdot, t)\} .
$$

We shall suppose that the function $\varphi$ in (6) is continuous, non-negative and not identically equal to zero in $\Omega$, so that, by Hopf's boundary point lemma, $\mathcal{H}(t) \cap \Gamma=\varnothing$. Also, by an argument based on the analyticity of $h$ similar to that used for the uniqueness of the maximum point in a convex domain, we can be sure that $\mathcal{H}(t)$ is made of isolated points (see [4] for details). (A parabolic version of ) Theorem 3.3 then yields that

$$
n_{E}(t)-n_{S}(t)=1
$$

where $n_{E}(t)$ and $n_{S}(t)$ are the number extremal and simple saddle points of $h(\cdot, t)$; clearly $n_{E}(t)$ is the cardinality of $\mathcal{H}(t)$. An estimate on the total number of critical points of $h(\cdot, t)$ will then follow from one on $n_{E}(t)$.

Notice that, if $\lambda_{n}$ and $\phi_{n}, n \in \mathbb{N}$, are Dirichlet eigenvalues (arranged in increasing order) and eigenfunctions (normalized in $L^{2}(\Omega)$ ) of the Laplace's operator in $\Omega$, then the following spectral formula

$$
\begin{equation*}
h(x, t)=\sum_{n=1}^{\infty} \widehat{\varphi}(n) \phi_{n}(x) e^{-\lambda_{n} t} \text { holds for } x \in \bar{\Omega} \text { and } t>0 \tag{20}
\end{equation*}
$$

where $\widehat{\varphi}(n)$ is the Fourier coefficient of $\varphi$ corresponding to $\phi_{n}$. Then we can infer that $e^{\lambda_{1} t} h(x, t) \rightarrow \widehat{\varphi}(1) \phi_{1}(x)$ as $t \rightarrow \infty$, with

$$
\widehat{\varphi}(1)=\int_{\Omega} \varphi(x) \phi_{1}(x) d x>0
$$

and the convergence is uniform on $\bar{\Omega}$ under sufficient assumptions on $\varphi$ and $\Omega$. This information implies that, if $x(t) \in \mathcal{H}(t)$, then

$$
\begin{equation*}
\operatorname{dist}\left(x(t), \mathcal{H}_{\infty}\right) \rightarrow 0 \text { as } t \rightarrow \infty \tag{21}
\end{equation*}
$$

where $\mathcal{H}_{\infty}$ is the set of local maximum points of $\phi_{1}$.
Now, our conjecture concerns the influence of the shape of $\Omega$ on the number $n_{E}(t)$. To rule out the possible influence of the values of $\varphi$, we assume that $\varphi \equiv$ 1: then we know that there holds the following asymptotic formula (see [85]):

$$
\begin{equation*}
\lim _{t \rightarrow 0^{+}} 4 t \log [1-h(x, t)]=-d_{\Gamma}(x)^{2} \text { for } x \in \bar{\Omega} \tag{22}
\end{equation*}
$$



Figure 8: As time $t$ increases, $\mathcal{H}(t)$ goes from $\mathcal{H}_{0}$, the set of maximum points of $d_{\Gamma}$, to $\mathcal{H}_{\infty}$, the set of maximum points of $\phi_{1}$.
here, $d_{\Gamma}(x)$ is the distance of a point $x \in \bar{\Omega}$ from the boundary $\Gamma$. The convergence in (22) is uniform on $\bar{\Omega}$ under suitable regularity assumptions on $\Gamma$.

Now, suppose that $d_{\Gamma}$ has exactly $m$ distinct local (strict) maximum points in $\Omega$. Formula (22) suggests that, when $t$ is sufficiently small, $h(\cdot, t)$ has the same number $m$ of maximum points in $\Omega$. As time $t$ increases, one expects that the maximum points of $h(\cdot, t)$ do not increase in number. Therefore, the following bounds should hold:

$$
\begin{equation*}
n_{E}(t) \leq m \text { and hence } n_{E}(t)+n_{S}(t) \leq 2 m-1 \text { for every } t>0 \tag{23}
\end{equation*}
$$

From the asymptotic analysis performed on (20), we also derive that the total number of critical points of $\phi_{1}$ does not exceeds $2 m-1$.

We stress that (23) cannot always hold with the equality sign. In fact, if $D_{\varepsilon}^{ \pm}$denotes the unit disk centered at $( \pm \varepsilon, 0)$ and we consider the domain $\Omega_{\varepsilon}$ obtained from $D_{\varepsilon}^{+} \cup D_{\varepsilon}^{-}$by "smoothing out the corners" (see Fig. 8), we notice that $m=2$ for every $0<\varepsilon<1$, while $\Omega_{\varepsilon}$ tends to the unit ball centered at the origin and hence, if $\varepsilon$ is small enough, $\phi_{1}$ has only one critical point, being $\Omega_{\varepsilon}$ "almost convex".

Based on a similar argument, inequalities like (23) should also hold for the number of critical points of the torsion function $\tau$. In fact, if $U_{s}$ is the solution of the one-parameter family of problems

$$
-\Delta U_{s}+s U_{s}=1 \text { in } \Omega, \quad U_{s}=0 \text { on } \Gamma,
$$

where $s$ is a positive parameter, we have that

$$
\lim _{s \rightarrow 0^{+}} U_{s}=\tau \quad \text { and } \quad \lim _{s \rightarrow \infty} \frac{1}{\sqrt{s}} \log \left[1-s U_{s}\right]=-d_{\Gamma}
$$

uniformly on $\bar{\Omega}$ (see again [85]).
We finally point out that the asymptotic formulas presented here hold in any dimension; thus, the bounds in (23) may be generalized in some way.

### 3.5. A conjecture by S. T. Yau

To conclude this section about the number of critical points of solutions of partial differential equations, we cannot help mentioning a conjecture proposed in [89] (also see [32, 48, 49]). This is motivated by the study of eigenfunctions of the Laplace-Beltrami operator $\Delta_{g}$ in a compact Riemannian manifold $(M, g)$.

Let $\left\{\phi_{k}\right\}_{k \in \mathbb{N}}$ be a sequence of eigenfunctions,

$$
\Delta_{g} \phi_{k}+\lambda_{k} \phi_{k}=0 \text { in } M
$$

Let $x_{k} \in M$ be a point of maximum for $\phi_{k}$ in $M$ and $B_{k}$ a geodesic ball centered at $x_{k}$ and with radius $C / \sqrt{\lambda_{k}}$. If we blow up $B_{k}$ to the unit disk in $\mathbb{R}^{2}$ and let $u_{k} / \max \phi_{k}$ be the eigenfunction after that change of variables, then a subsequence of $\left\{u_{k}\right\}_{k \in \mathbb{N}}$ will converge to a solution $u$ of

$$
\begin{equation*}
\Delta u+u=0,|u|<1 \text { in } \mathbb{R}^{2} \tag{24}
\end{equation*}
$$

If we can prove that $u$ has infinitely many isolated critical points, then we can expect that their number be unbounded also for the sequence $\left\{\phi_{k}\right\}_{k \in \mathbb{N}}$.

A naive insight built up upon the available concrete examples of entire eigenfunctions (the separated eigenfunctions in rectangular or polar coordinates) may suggest that it would be enough to prove that any solution of (24) has infinitely many nodal domains. It turns out that this is not always true, as a clever counterexample obtained in [32, Theorem 3.2] shows: there exists a solution of (24) with exactly two nodal domains.

The counterexample is constructed by perturbing the solution of (24)

$$
f=J_{1}(r) \sin \theta
$$

where $(r, \theta)$ are the usual polar coordinates and $J_{1}$ is the second Bessel's function; $f$ has infinitely many nodal domains. The desired example is thus obtained by the perturbation $h=f+\varepsilon g$, where $g(x, y)=f\left(x-\delta_{x}, y-\delta_{y}\right)$ and $\left(\delta_{x}, \delta_{y}\right)$ is suitably chosen. As a result, if $\varepsilon$ is sufficiently small, the set $\left\{(x, y) \in \mathbb{R}^{2}: h(x, y) \neq 0\right\}$ is made of two interlocked spiral-like domains (see [32, Figure 3.1]).

A related result was proved in [31], where it is shown that there is no topological upper bound for the number of critical points of the first eigenfunction on Riemannian manifolds (possibly with boundary) of dimension larger than two. In fact, with no restriction on the topology of the manifold, it is possible to construct metrics whose first eigenfunction has as many isolated critical points as one wishes.

Recently, it has been proved in [52] that, if $(M, g)$ is a non-positively curved surface with concave boundary, the number of nodal domains of $\phi_{k}$ diverges along a subsequence of eigenvalues of density 1 (see also [53] for related results). The surface needs not have any symmetries. The number can also be shown to grow like $\log \lambda_{k}([91])$. In light of such results, Yau's conjecture was updated as follows: show that, for any (generic) $(M, g)$ there exists at least one subsequence of eigenfunctions for which the number of nodal domains (and hence of the critical points) tends to infinity ([90, 91]).

## 4. The location of critical points

### 4.1. A little history

The first result that studies the critical points of a function is probably Rolle's theorem: between two zeroes of a differentiable real-valued function there is at least one critical point. Thus, a function that has $n$ distinct zeroes also has at least $n-1$ critical points - an estimate from below - and we roughly know where they are located.

After Rolle's theorem, the first general result concerning the zeroes of the derivative of a general polynomial is Gauss's theorem: if

$$
P(z)=a_{n}\left(z-z_{1}\right)^{m_{1}} \cdots,\left(z-z_{K}\right)^{m_{K}}, \text { with } m_{1}+\cdots+m_{K}=n
$$

is a polynomial of degree $n$, then

$$
\frac{P^{\prime}(z)}{P(z)}=\frac{m_{1}}{z-z_{1}}+\cdots+\frac{m_{K}}{z-z_{K}}
$$

and hence the zeroes of $P^{\prime}(z)$ are, in addition to the multiple zeroes of $P(z)$ themselves, the roots of

$$
\frac{m_{1}}{z-z_{1}}+\cdots+\frac{m_{K}}{z-z_{K}}=0
$$

These roots can be interpreted as the equilibrium points of the gravitational field generated by the masses $m_{1}, \ldots, m_{K}$ placed at the points $z_{1}, \ldots, z_{K}$, respectively.

If the zeroes of $P(z)$ are placed on the real line then, by Rolle's theorem, it is not difficult to convince oneself that the zeroes of $P^{\prime}(z)$ lie in the smallest
interval of the real axis that contains the zeroes of $P(z)$. This simple result has a geometrically expressive generalization in Lucas's theorem: the zeroes of $P^{\prime}(z)$ lie in the convex hull $\Pi$ of the set $\left\{z_{1}, \ldots, z_{K}\right\}$ - named Lucas's polygon -and no such zero lies on $\partial \Pi$ unless is a multiple zero $z_{k}$ of $P(z)$ or all the zeroes of $P(z)$ are collinear (see Fig. 9).

In fact, it is enough to observe that, if $z \notin \Pi$ or $z \in \partial \Pi$, then all the $z_{k}$ lie in the closed half-plane $H$ containing them and the side of $\Pi$ which is the closest to $z$. Thus, if $\ell=\ell_{x}+i \ell_{y}$ is an outward direction to $\partial H$, we have that

$$
\operatorname{Re}\left[\left(\sum_{k=1}^{K} \frac{m_{k}}{z-z_{k}}\right) \ell\right]=\sum_{k=1}^{K} m_{k} \frac{\operatorname{Re}\left[\overline{\left(z-z_{k}\right)} \ell\right]}{\left|z-z_{k}\right|^{2}}>0
$$

since all the addenda are non-negative and not all equal to zero, unless the $z_{k}$ 's are collinear.


Figure 9: Lucas's theorem: the zeroes of $P^{\prime}(z)$ must fall in the convex envelope of those of $P(z)$.

If $P(z)$ has real coefficients, we know that its non-real zeroes occur in conjugate pairs. Using the circle whose diameter is the segment joining such a pair - this is called a Jensen's circle of $P(z)$ - one can obtain a sharper estimate of the location of the zeroes of $P^{\prime}(z)$ : each non-real zero of $P^{\prime}(z)$ lies on or within a Jensen's circle of $P(z)$. This result goes under the name of Jensen's theorem (see [87] for a proof).

All these results can be found in Walsh's treatise [87], that contains many other results about zeroes of complex polynomials or rational functions and their extensions to critical points of harmonic functions: among them restricted
versions of Theorem 3.1 give information (i) on the critical points of the Green's function of an infinite region delimited by a finite collection of simple closed curves and (ii) of harmonic measures generated by collections of Jordan arcs. Besides the argument's principle already presented in these notes, a useful ingredient used in those extensions is a Hurwitz's theorem (based on the classical Rouché's theorem): if $f_{n}(z)$ and $f(z)$ are holomorphic in a domain $\Omega$, continuous on $\bar{\Omega}, f(z)$ is non-zero on $\Gamma$ and $f_{n}(z)$ converges uniformly to $f(z)$ on $\bar{\Omega}$, then there is a $n_{0} \in \mathbb{N}$ such that, for $n>n_{0}, f_{n}(z)$ and $f(z)$ have the same number of zeroes in $\Omega$.

### 4.2. Location of critical points of harmonic functions in space

The following result is somewhat an analog of Lucas's theorem and is related to [87, Theorem 1, p. 249], which holds in the plane.

Theorem 4.1 ([28]). Let $D_{1}, \ldots, D_{J}$ be bounded domains in $\mathbb{R}^{N}, N \geq 3$, with boundaries of class $C^{1, \alpha}$ and with mutually disjoint closures, and set

$$
\Omega=\mathbb{R}^{N} \backslash \bigcup_{j=1}^{J} \overline{D_{j}}
$$

Let $u \in C^{0}(\bar{\Omega}) \cap C^{2}(\Omega)$ be the solution of the boundary value problem

$$
\begin{equation*}
\Delta u=0 \quad \text { in } \Omega, \quad u=1 \quad \text { on } \Gamma, \quad u(x) \rightarrow 0 \quad \text { as }|x| \rightarrow \infty . \tag{25}
\end{equation*}
$$

If $\mathcal{K}$ denotes the convex hull of

$$
\bigcup_{j=1}^{J} D_{j}
$$

then $u$ does not have critical points in $\overline{\mathbb{R}^{N} \backslash \mathcal{K}}$ (sse Fig. 10).
This theorem admits at least two proofs and it is worth to present both of them. The former is somewhat reminiscent of Lucas's proof and is based on an explicit formula for $u$,

$$
u(x)=\frac{1}{(N-2) \omega_{N}} \int_{\Gamma} \frac{u_{\nu}(y)}{|x-y|^{N-2}} d S_{y}, x \in \Omega
$$

that can be derived as a consequence of Stokes's formula. Here, $\omega_{N}$ is the surface area of a unit sphere in $\mathbb{R}^{N}, d S_{y}$ denotes the $(N-1)$-dimensional surface measure, and $u_{\nu}$ is the (outward) normal derivative of $u$.

By the Hopf's boundary point lemma, $u_{\nu}>0$ on $\Gamma$. Also, if $x \in \overline{\mathbb{R}^{N} \backslash \mathcal{K}}$, we can choose a hyperplane $\pi$ passing through $x$ and supporting $\mathcal{K}$ (at some point). If $\ell$ is the unit vector orthogonal to $\pi$ at $x$ and pointing into the halfspace containing $\mathcal{K}$, we have that $(x-y) \cdot \ell$ is non-negative and is not identically zero for $y \in \Gamma$. Therefore,

$$
u_{\ell}(x)=-\frac{1}{\omega_{N}} \int_{\Gamma} \frac{u_{\nu}(y)(x-y) \cdot \ell}{|x-y|^{N}} d S_{y}<0
$$

which means that $\nabla u(x) \neq 0$.


Figure 10: No critical points outside of the convex envelope.
The latter proof is based on a symmetry argument ([79]) and, as it will be clear, can also be extended to more general non-linear equations. Let $\pi$ be any hyperplane contained in $\bar{\Omega}$ and let $H$ be the open half-space containing $\mathcal{K}$ and such that $\partial H=\pi$. Let $x^{\prime}$ be the mirror reflection in $\pi$ of any point $x \in H \cap \Omega$. Then the function defined by

$$
u^{\prime}(x)=u\left(x^{\prime}\right) \text { for } x \in H \cap \Omega
$$

is harmonic in $H \cap \Omega$, tends to 0 as $|x| \rightarrow \infty$ and

$$
u^{\prime}<u \text { in } H \cap \Omega, \quad u^{\prime}=u \text { on } \pi \backslash \Gamma
$$

Therefore, by the Hopf's boundary point lemma, $u_{\ell}(x) \neq 0$ at any $x \in \pi \backslash \Gamma$ for any direction $\ell$ not parallel to $\pi$. Of course, if $x \in \Gamma \cap \pi$, we obtain that $u_{\nu}(x)>0$ by directly using the Hopf's boundary point lemma.

Generalizations of Lucas's theorem hold for other problems. Here, we mention the well known result of Chavel and Karp [25] for the minimal solution of the Cauchy problem for the heat equation in a Riemannian manifold $(M, g)$ :

$$
\begin{equation*}
u_{t}=\Delta_{g} u \text { in } M \times(0, \infty), \quad u=\varphi \text { on } M \times\{0\} \tag{26}
\end{equation*}
$$

where $\varphi$ is a bounded initial data with compact support in $M$. In [23], it is shown that, if $M$ is complete, simply connected and of constant curvature, then the set of the hot spots of $u$,

$$
\mathcal{H}(t)=\left\{x \in M: u(x, t)=\max _{y \in M} u(y, t)\right\}
$$

is contained in the convex hull of the support of $\varphi$. The proof is based on an explicit formula for $u$ in terms of the initial values $\varphi$. For instance, when $M=\mathbb{R}^{N}$, we have the formula

$$
u(x, t)=(4 \pi t)^{-N / 2} \int_{\mathbb{R}^{N}} e^{-|x-y|^{2}} \varphi(y) d y \text { for }(x, t) \in \mathbb{R}^{N} \times(0, \infty)
$$

With this formula in hand, by looking at the second derivatives of $u$, one can also prove that there is a time $T>0$ such that, for $t>T, \mathcal{H}(t)$ reduces to the single point

$$
\frac{\int_{\mathbb{R}^{N}} y \varphi(y) d y}{\int_{\mathbb{R}^{N}} \varphi(y) d y}
$$

which is the center of mass of the measure space $\left(\mathbb{R}^{N}, \varphi(y) d y\right)$ (see [51]).
We also mention here the work of Ishige and Kabeya ([43, 44, 45]) on the large time behavior of hot spots for solutions of the heat equation with a rapidly decaying potential and for the Schrödinger equation.

### 4.3. Hot spots in a grounded conductor

From a physical point of view, the solution (26) describes the evolution of the temperature of $M$ when its initial value distribution is known on $M$. The situation is more difficult if $\partial M$ is not empty. We shall consider here the case of a grounded heat conductor, that is we will study the solution $h$ of the Cauchy-Dirichlet problem (5)-(6).

Bounded conductor. As already seen, if $\varphi \geq 0$, (20) implies (21). For an arbitrary continuous function $\varphi$, from (20) we can infer that, if $m$ is the first integer such that $\widehat{\varphi}(n) \neq 0$ and $m+1, \ldots, m+k-1$ are all the integers such that $\lambda_{m}=\lambda_{m+1}=\cdots=\lambda_{m+k-1}$, then

$$
e^{\lambda_{m} t} h(x, t) \rightarrow \sum_{n=m}^{m+k-1} \widehat{\varphi}(n) \phi_{n}(x) \text { if } t \rightarrow \infty .
$$

Also, when $\varphi \equiv 1$, (22) holds and hence

$$
\begin{equation*}
\operatorname{dist}\left(x(t), \mathcal{H}_{0}\right) \rightarrow 0 \text { as } t \rightarrow 0 \tag{27}
\end{equation*}
$$

where $\mathcal{H}_{0}$ is the set of local (strict) maximum points of $d_{\Gamma}$. These informations give a rough picture of the set of trajectories of the hot spots:

$$
\mathcal{T}=\bigcup_{t>0} \mathcal{H}(t)
$$

Notice in passing that, if $\Omega$ is convex and has $N$ distinct hyperplanes of symmetry, it is clear that $\mathcal{T}$ is made of the same single point - the intersection of the hyperplanes - that is the hot spot does not move or is stationary. Also, it is not difficult to show (see [24]) that the hot spot does not move if $\Omega$ is invariant under an essential group $G$ of orthogonal transformations (that is for every $x \neq 0$ there is $A \in G$ such that $A x \neq 0)$. Characterizing the class $\mathcal{P}$ of convex domains that admit a stationary hot spot seems to be a difficult task: some partial results about convex polygons can be found in [64, 65] (see also [63]). There it is proved that: (i) the equilateral triangle and the parallelogram are the only polygons with 3 or 4 sides in $\mathcal{P}$; (ii) the equilateral pentagon and the hexagons invariant under rotations of angles $\pi / 3,2 \pi / 3$, or $\pi$ are the only polygons with 5 or 6 sides all touching the inscribed circle centered at the hot spot.

The analysis of the behavior of $\mathcal{H}(t)$ for $t \rightarrow 0^{+}$and $t \rightarrow \infty$ helps us to show that hot spots do move in general.


Figure 11: The reflected $D^{*}$ is contained in $D^{+}$, hence $h^{\prime}$ can be defined in $D^{*}$.

To see this, it is enough to consider the half-disk (see Fig. 11)

$$
D^{+}=\left\{(x, y) \in \mathbb{R}^{2}:|x|<1, x_{1}>0\right\}
$$

being $D^{+}$convex, for each $t>0$, there is a unique hot spot that, as $t \rightarrow 0^{+}$, tends to the maximum point $x_{0}=(1 / 2,0)$ of $d_{\Gamma}$. Thus, it is enough to show that $x_{0}$ is not a spatial critical point of $h(x, t)$ for some $t>0$ or, if you like, for $\phi_{1}$.

This is readily seen by Alexandrov's reflection principle. Let $D^{*}=\{x \in$ $\left.D^{+}: x_{1}>1 / 2\right\}$ and define

$$
h^{\prime}\left(x_{1}, x_{2}, t\right)=h\left(1-x_{1}, x_{2}, t\right) \text { for }\left(x_{1}, x_{2}, t\right) \in \overline{D^{*}} \times(0, \infty) ;
$$

$h^{\prime}$ is the reflection of $h$ in the line $x_{1}=1 / 2$. We clearly have that

$$
\begin{gathered}
\left(h^{\prime}-h\right)_{t}=\Delta\left(h-h^{\prime}\right) \text { in } D^{*} \times(0, \infty), \quad h^{\prime}-h=0 \text { on } D^{*} \times\{0\}, \\
h^{\prime}-h>0 \text { on }\left(\partial D^{*} \cap \partial^{+}\right) \times(0, \infty), \quad h^{\prime}-h=0 \quad \text { on }\left(\partial D^{*} \cap D^{+}\right) \times(0, \infty) .
\end{gathered}
$$

Thus, the strong maximum principle and the Hopf's boundary point lemma imply that

$$
-2 h_{x_{1}}\left(1 / 2, x_{2}, t\right)=h_{x_{1}}^{\prime}\left(1 / 2, x_{2}, t\right)-h_{x_{1}}\left(1 / 2, x_{2}, t\right)>0
$$

for $\left(1 / 2, x_{2}, t\right) \in\left(\partial D^{*} \cap D^{+}\right) \times(0, \infty)$, and hence $x_{0}$ cannot be a critical point of $h$.

The Alexandrov's principle just mentioned can also be employed to estimate the location of a hot spot. In fact, as shown in [18], by the same arguments one can prove that hot spots must belong to the subset $\triangle(\Omega)$ of $\Omega$ defined as follows. Let $\pi_{\omega}$ be a hyperplane orthogonal to the direction $\omega \in \mathbb{S}^{N-1}$ and let $H_{\omega}^{+}$and $H_{\omega}^{-}$be the two half-spaces defined by $\pi_{\omega}$; let $\mathcal{R}_{\omega}(x)$ denote the mirror reflection of a point $x$ in $\pi_{\omega}$. Then, the heart ${ }^{2}$ of $\Omega$ is defined by

$$
\checkmark(\Omega)=\bigcap_{\omega \in \mathbb{S}^{N-1}}\left\{H_{\omega}^{-} \cap \Omega: \mathcal{R}_{\omega}\left(H_{\omega}^{+} \cap \Omega\right) \subset \Omega\right\}
$$

When $\Omega$ is convex, then $\Omega(\Omega)$ is also convex and, if $\Gamma$ is of class $C^{1}$, we are sure that its distance from $\Gamma$ is positive (see [34]). Also, we know that $\mathcal{H}(t)$ is made of only one point $x(t)$, so that

$$
\operatorname{dist}(x(t), \Gamma) \geq \operatorname{dist}(\bigcirc(\Omega), \Gamma)
$$

The set $\Theta(\Omega)$ contains many notable geometric points of the set $\Omega$, such as the center of mass, the incenter, the circumcenter, and others; see [19], where

[^6]further properties of the heart of a convex body are presented. See also [80] for related research on this issue.

As clear from [18], the estimate just presented is of purely geometric nature, that is it only depends on the lack of symmetry of $\Omega$ and does not depend on the particular equation we are considering in $\Omega$, as long as the equation is invariant by reflections.

A different way to estimate the location of the hot spot of a grounded convex heat conductor or the maximum point of the solution of certain elliptic equations is based on ideas related to Alexandrov-Bakelman-Pucci's maximum principle and does take into account the information that comes from the relevant equation. For instance, in [18] it is proved that the maximum point $x_{\infty}$ of $\phi_{1}$ in $\bar{\Omega}$ is such that

$$
\begin{equation*}
\operatorname{dist}\left(x_{\infty}, \Gamma\right) \geq C_{N} r_{\Omega}\left(\frac{r_{\Omega}}{\operatorname{diam}(\Omega)}\right)^{N^{2}-1} \tag{28}
\end{equation*}
$$

where $C_{N}$ is a constant only depending on $N, r_{\Omega}$ is the inradius of $\Omega$ (the radius of a largest ball contained in $\Omega$ ) and $\operatorname{diam}(\Omega)$ is the diameter of $\Omega$.

The idea of the proof of (28) is to compare the concave envelope $f$ of $\phi_{1}-$ the smallest concave function above $\phi_{1}$ - and the function $g$ whose graph is the surface of the (truncated) cone based on $\Omega$ and having its tip at the point $\left(x_{\infty}, \phi\left(x_{\infty}\right)\right)$ (see Fig. 12).

Since $f \geq g$ and $f\left(x_{\infty}\right)=g\left(x_{\infty}\right)$, we can compare their respective subdifferential images:

$$
\begin{aligned}
& \partial f(\Omega)=\bigcup_{x \in \bar{\Omega}}\left\{p \in \mathbb{R}^{N}: f(x)+p \cdot(y-x) \geq f(y) \text { for } y \in \bar{\Omega}\right\} \\
& \partial g(\Omega)=\bigcup_{x \in \bar{\Omega}}\left\{p \in \mathbb{R}^{N}: g(x)+p \cdot(y-x) \geq g(y) \text { for } y \in \bar{\Omega}\right\}
\end{aligned}
$$

in fact, it holds that $\partial g(\Omega) \subseteq \partial f(\Omega)$.
Now, $\partial g(\Omega)$ has a precise geometrical meaning: it is the set $\phi_{1}\left(x_{\infty}\right) \Omega^{*}$, that is a multiple of the polar set of $\Omega$ with respect to $x_{\infty}$ defined by

$$
\Omega^{*}=\left\{y \in \mathbb{R}^{N}:\left(x-x_{\infty}\right) \cdot\left(y-x_{\infty}\right) \leq 1 \text { for every } x \in \bar{\Omega}\right\}
$$

The volume $|\partial f(\Omega)|$ can be estimated by the formula of change of variables to obtain:

$$
\phi_{1}\left(x_{\infty}\right)^{N}\left|\Omega^{*}\right|=|\partial g(\Omega)| \leq|\partial f(\Omega)| \leq \int_{C}\left|\operatorname{det} D^{2} f\right| d x=\int_{C}\left|\operatorname{det} D^{2} \phi_{1}\right| d x
$$

where $C=\left\{x \in \bar{\Omega}: f(x)=\phi_{1}(x)\right\}$ is the contact set. Since the determinant and the trace of a matrix are the product and the sum of the eigenvalues of the
matrix, by the arithmetic-geometric mean inequality, we have that $\left|\operatorname{det} D^{2} \phi_{1}\right| \leq$ $\left(-\Delta \phi_{1} / N\right)^{N}$, and hence we can infer that

$$
\left|\Omega^{*}\right| \leq \int_{C}\left[\frac{-\Delta \phi_{1}}{N \phi_{1}\left(x_{\infty}\right)}\right]^{N} d x=\int_{C}\left[\frac{\lambda_{1}(\Omega) \phi_{1}}{N \phi_{1}\left(x_{\infty}\right)}\right]^{N} d x \leq\left[\frac{\lambda_{1}(\Omega)}{N}\right]^{N}|\Omega|
$$

being $\phi_{1} \leq \phi_{1}\left(x_{\infty}\right)$ in $\bar{\Omega}$. Finally, in order to get (28) explicitly, one has to bound $\left|\Omega^{*}\right|$ from below by the volume of the polar set of a suitable half-ball containing $\Omega$, and $\lambda_{1}(\Omega)$ from above by the isodiametric inequality (see [18] for details).


Figure 12: The concave envelope of $\phi_{1}$ and the cone $g$. The dashed cap is the image $f(C)=\phi_{1}(C)$ of the contact set $C$.

The two methods we have seen so far, give estimates of how far the hot spot must be from the boundary. We now present a method, due to Grieser and Jerison [37], that gives an estimate of how far the hot spot can be from a specific point in the domain. The idea is to adapt the classical method of separation of variables to construct a suitable approximation $u$ of the first Dirichlet eigenfunction $\phi_{1}$ in a planar convex domain. Clearly, if $\Omega$ were a rectangle, say $[a, b] \times[0,1]$, then that approximation would be exact: in fact

$$
u(x, y)=\phi_{1}(x, y)=\sin [\pi(x-a) /(b-a)] .
$$

If $\Omega$ is not a rectangle, after some manipulations, we can suppose that

$$
\Omega=\left\{(x, y): a<x<b, f_{1}(x)<y<f_{2}(x)\right\}
$$

where, in $[a, b], f_{1}$ is convex, $f_{2}$ is concave and

$$
0 \leq f_{1} \leq f_{2} \leq 1 \text { and } \min _{[a, b]} f_{1}=0, \max _{[a, b]} f_{2}=1
$$

(see Fig. 13).
The geometry of $\Omega$ does not allow to find a solution by separation of variables as in the case of the rectangle. However, one can operate "as if" that


Figure 13: Estimating the hot spot in the "long" convex set $\Omega$.
separation were possible. To understand that, consider the length of the section of foot $x$, parallel to the $y$-axis, by

$$
h(x)=f_{2}(x)-f_{1}(x) \text { for } a \leq x \leq b,
$$

and notice that, if we set

$$
\alpha(x, y)=\pi \frac{y-f_{1}(x)}{h(x)}
$$

the function

$$
e(x, y)=\sqrt{2 / h(x)} \sin \alpha(x, y) \text { for } f_{1}(x) \leq y \leq f_{2}(x)
$$

satisfies for fixed $x$ the problem

$$
e_{y y}+\pi^{2} e=0 \text { in }\left(f_{1}(x), f_{2}(x)\right), \quad e\left(x, f_{1}(x)\right)=e\left(x, f_{2}(x)\right)=0
$$

- thus, it is the first Dirichlet eigenfunction in the interval $\left(f_{1}(x), f_{2}(x)\right)$, normalized in the space $L^{2}\left(\left[f_{1}(x), f_{2}(x)\right]\right)$. The basic idea is then that $\phi_{1}(x, y)$ should be (and in fact it is) well approximated by its lowest Fourier mode in the $y$-direction, computed for each fixed $x$, that is by the projection of $\phi_{1}$ along $e$ :

$$
\psi(x) e(x, y) \text { where } \psi(x)=\int_{f_{1}(x)}^{f_{2}(x)} \phi_{1}(x, \eta) e(x, \eta) d \eta
$$

To simplify matters, a further approximation is needed: it turns out that $\psi$ and its first derivative can be well approximated by $\phi / \sqrt{2}$ and its derivative, where $\phi$ is the first eigenfunction of the problem

$$
\phi^{\prime \prime}(x)+\left[\mu-\frac{\pi^{2}}{h(x)^{2}}\right] \phi(x)=0 \text { for } a<x<b, \quad \phi(a)=\phi(b)=0 .
$$

Since near the maximum point $x_{1}$ of $\phi,\left|\phi^{\prime}(x)\right|$ can be bounded from below by a constant times $\left|x-x_{1}\right|$, the constructed chain of approximations gives that, if $\left(x_{0}, y_{0}\right)$ is the maximum point of $\phi_{1}$ on $\bar{\Omega}$, then there is an absolute constant $C$ such that

$$
\left|x_{1}-x_{0}\right| \leq C
$$

$C$ is independent of $\Omega$, but the result has clearly no content unless $b-a>C$.
Unbounded conductor. If $\Omega$ is unbounded, by working with suitable barriers, one can still prove formula (22) when $\varphi \equiv 1$ (see $[66,67]$ ), the convergence holding uniformly on compact subsets of $\Omega$. Thus, any hot spot $x(t)$ will again satisfy (27).

To the author's knowledge, [51] is the only reference in which the behavior of hot spots for large times has been studied for some grounded unbounded conductors. There, the cases of a half-space $\mathbb{R}_{+}^{N}=\left\{x \in \mathbb{R}^{N}: x_{N}>0\right\}$ and the exterior of a ball $B^{c}=\left\{x \in \mathbb{R}^{N}:|x|>1\right\}$ are considered. It is shown that there is a time $T>0$ such that for $t>T$ the set $\mathcal{H}(t)$ is made of only one hot spot $x(t)=\left(x_{1}(t), \ldots, x_{N}(t)\right)$ and

$$
x_{j}(t) \rightarrow \frac{\int_{\mathbb{R}^{N-1}} y_{j} y_{N} \varphi\left(y^{\prime}\right) d y^{\prime}}{\int_{\mathbb{R}^{N-1}} y_{N} \varphi\left(y^{\prime}\right) d y^{\prime}}, 1 \leq j \leq N-1, \quad \frac{x_{N}(t)}{\sqrt{2 t}} \rightarrow 1 \quad \text { as } t \rightarrow \infty
$$

if $\Omega=\mathbb{R}_{+}^{N}$, while for $\Omega=B^{c}$, if $\varphi$ is radially symmetric, then there is a time $T>0$ such that $\mathcal{H}(t)=\left\{x \in \mathbb{R}^{N}:|x|=r(t)\right\}$, for $t>T$, where $r(t)$ is some smooth function of $t$ such that

$$
\limsup _{t \rightarrow \infty} r(t)=\infty
$$

Upper bounds for $\mathcal{H}(t)$ are also given in [51] for the case of the exterior of a smooth bounded domain.

### 4.4. Hot spots in an insulated conductor

We conclude this survey by giving an account on the so-called hot spot conjecture by J. Rauch [76]. This is related to the asymptotic behavior of hot spots in a perfectly insulated heat conductor modeled by the following initial-boundary value problem:

$$
\begin{equation*}
h_{t}=\Delta h \text { in } \Omega \times(0, \infty), \quad h=\varphi \text { on } \Omega \times\{0\}, \quad \partial_{\nu} u=0 \text { on } \Gamma \times(0, \infty) \tag{29}
\end{equation*}
$$

Observe that, similarly to (20), a spectral formula also holds for the solution of (29):

$$
\begin{equation*}
h(x, t)=\sum_{n=1}^{\infty} \widehat{\varphi}(n) \psi_{n}(x) e^{-\mu_{n} t}, \text { for } x \in \bar{\Omega} \text { and } t>0 \tag{30}
\end{equation*}
$$

Here $\left\{\mu_{n}\right\}_{n \in \mathbb{N}}$ is the increasing sequence of Neumann eigenvalues and $\left\{\psi_{n}\right\}_{n \in \mathbb{N}}$ is a complete orthonormal system in $L^{2}(\Omega)$ of eigenfunctions corresponding to the $\mu_{n}$ 's, that is $\psi_{n}$ is a non-zero solution of

$$
\begin{equation*}
\Delta \psi+\mu \psi=0 \text { in } \Omega, \quad \partial_{\nu} \psi=0 \text { on } \Gamma, \tag{31}
\end{equation*}
$$

with $\mu=\mu_{n}$. The numbers $\widehat{\varphi}(n)$ are the Fourier coefficients of $\varphi$ corresponding to $\psi_{n}$, that is

$$
\widehat{\varphi}(n)=\int_{\Omega} \varphi(x) \psi_{n}(x) d x, n \in \mathbb{N}
$$

Since $\mu_{1}=0$ and $\psi_{1}=1 / \sqrt{|\Omega|}$, we can infer that

$$
\begin{equation*}
e^{\mu_{m} t}\left[h(x, t)-\frac{1}{\sqrt{|\Omega|}} \int_{\Omega} \varphi d x\right] \rightarrow \sum_{n=m}^{m+k-1} \widehat{\varphi}(n) \psi_{n}(x) \text { as } t \rightarrow \infty \tag{32}
\end{equation*}
$$

where $m$ is the first integer such that $\widehat{\varphi}(n) \neq 0$ and $m+1, \ldots, m+k-1$ are all the integers such that $\mu_{m}=\mu_{m+1}=\cdots=\mu_{m+k-1}$. Thus, similarly to what happens for the case of a grounded conductor, as $t \rightarrow \infty$, a hot spot $x(t)$ of $h$ tends to a maximum point of the function at the right-hand side of (32).

Now, roughly speaking, the conjecture states that, for "most" initial conditions $\varphi$, the distance from $\Gamma$ of any hot and cold spot of $h$ must tend to zero as $t \rightarrow \infty$, and hence it amounts to prove that the right-hand side of (32) attains its maximum and minimum at points in $\Gamma$.

It should be noticed now that the quotes around the word most are justified by the fact that the conjecture does not hold for all initial conditions. In fact, as shown in [10], if $\Omega=(0,2 \pi) \times(0,2 \pi) \subset \mathbb{R}^{2}$, the function defined by

$$
h\left(x_{1}, x_{2}, t\right)=-e^{-t}\left(\cos x_{1}+\cos x_{2}\right),\left(x_{1}, x_{2}\right) \in \Omega, t>0
$$

is a solution of $(29)$ - with $\varphi\left(x_{1}, x_{2}\right)=-\left(\cos x_{1}+\cos x_{2}\right)$ - that attains its maximum at $(-\pi, \pi)$ for any $t>0$. However, it turns out that in this case $h\left(x_{1}, x_{2}, t\right)=-e^{-\mu_{4} t} \psi_{4}\left(x_{1}, x_{2}\right)$. Thus, it is wiser to rephrase the conjecture by asking whether or not the hot and cold spots tend to $\Gamma$ if the coefficient $\widehat{\varphi}(2)$ of the first non-constant eigenfunction $\psi_{2}$ is not zero or, which is the same, whether or not maximum and minimum points of $\psi_{2}$ in $\bar{\Omega}$ are attained only on $\Gamma$.

In [55], a weaker version of this last statement is proved to hold for domains of the form $D \times(0, a)$, where $D \subset \mathbb{R}^{N-1}$ has a boundary of class $C^{0,1}$. In [55], the conjecture has also been reformulated for convex domains. Indeed, we now know that it is false for fairly general domains: in [20] a planar domain with two holes is constructed, having a simple second eigenvalue and such that the corresponding eigenfunction attains its strict maximum at an interior point of the domain. It turns out that in that example the minimum point is on
the boundary. Nevertheless, in [12] it is given an example of a domain whose second Neumann eigenfunction attains both its maximum and minimum points at interior points. In both examples the conclusion is obtained by probabilistic methods.

Besides in [55], positive results on this conjecture can be found in [9, 10, 11, $27,50,69,73,83]$. In [10], the conjecture is proved for planar convex domains $\Omega$ with two orthogonal axis of symmetry and such that

$$
\frac{\operatorname{diam}(\Omega)}{\operatorname{width}(\Omega)}>1.54
$$

This restriction is removed in [50]. In [73], $\Omega$ is assumed to have only one axis of symmetry, but $\psi_{2}$ is assumed anti-symmetric in that axis. A more general result is contained in [9]: the conjecture holds true for domains of the type

$$
\Omega=\left\{\left(x_{1}, x_{2}\right): f_{1}\left(x_{1}\right)<x_{2}<f_{2}\left(x_{1}\right)\right\}
$$

where $f_{1}$ and $f_{2}$ have unitary Lipschitz constant. In [27], a modified version is considered: it holds true for general domains, if vigorous maxima are considered (see [27] for the definition). If no symmetry is assumed for a convex domain $\Omega$, Y. Miyamoto [69] has verified the conjecture when

$$
\frac{\operatorname{diam}(\Omega)^{2}}{|\Omega|}<1.378
$$

(for a disk, this ratio is about 1.273).
For unbounded domains, the situation changes. For the half-space, Jimbo and Sakaguchi proved in [51] that there is a time $T$ after which the hot spot equals a point on the boundary that depends on $\varphi$. In [51], the case of the exterior $\Omega$ of a ball $\overline{B_{R}}$ is also considered for a radially symmetric $\varphi$. For a suitably general $\varphi$, Ishige [41] has proved that the behavior of the hot spot is governed by the point

$$
A_{\varphi}=\frac{\int_{\Omega} x\left(1+\frac{R^{N}}{N-1}|x|^{-N}\right) \varphi(x) d x}{\int_{\Omega} \varphi(x) d x}
$$

If $A_{\varphi} \in B_{R}$, then $\mathcal{H}(t)$ tends to the boundary point $R A_{\varphi} /\left|A_{\varphi}\right|$, while if $A_{\varphi} \notin$ $B_{R}$, then $\mathcal{H}(t)$ tends to $A_{\varphi}$ itself.

Results concerning the behavior of hot spots for parabolic equations with a rapidly decaying potential can be found in [43, 44].

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# Two-phase heat conductors with a stationary isothermic surface 

Shigeru Sakaguchi

Dedicated to Professor Giovanni Alessandrini on his sixtieth birthday


#### Abstract

We consider a two-phase heat conductor in $\mathbb{R}^{N}$ with $N \geq 2$ consisting of a core and a shell with different constant conductivities. Suppose that, initially, the conductor has temperature 0 and, at all times, its boundary is kept at temperature 1. It is shown that, if there is a stationary isothermic surface in the shell near the boundary, then the structure of the conductor must be spherical. Also, when the medium outside the two-phase conductor has a possibly different conductivity, we consider the Cauchy problem with $N \geq 3$ and the initial condition where the conductor has temperature 0 and the outside medium has temperature 1. Then we show that almost the same proposition holds true.


Keywords: heat equation, diffusion equation, two-phase heat conductor, transmission condition, initial-boundary value problem, Cauchy problem, stationary isothermic surface, symmetry.
MS Classification 2010: 35K05, 35K10, 35B40, 35K15, 35K20.

## 1. Introduction

Let $\Omega$ be a bounded $C^{2}$ domain in $\mathbb{R}^{N}(N \geq 2)$ with boundary $\partial \Omega$, and let $D$ be a bounded $C^{2}$ open set in $\mathbb{R}^{N}$ which may have finitely many connected components. Assume that $\Omega \backslash \bar{D}$ is connected and $\bar{D} \subset \Omega$. Denote by $\sigma=$ $\sigma(x)\left(x \in \mathbb{R}^{N}\right)$ the conductivity distribution of the medium given by

$$
\sigma= \begin{cases}\sigma_{c} & \text { in } D \\ \sigma_{s} & \text { in } \Omega \backslash D \\ \sigma_{m} & \text { in } \mathbb{R}^{N} \backslash \Omega\end{cases}
$$

where $\sigma_{c}, \sigma_{s}, \sigma_{m}$ are positive constants and $\sigma_{c} \neq \sigma_{s}$. This kind of three-phase electrical conductor has been dealt with in [7] in the study of neutrally coated inclusions.

In the present paper we consider the heat diffusion over two-phase or threephase heat conductors. Let $u=u(x, t)$ be the unique bounded solution of either the initial-boundary value problem for the diffusion equation:

$$
\begin{array}{ll}
u_{t}=\operatorname{div}(\sigma \nabla u) & \text { in } \Omega \times(0,+\infty) \\
u=1 & \text { on } \partial \Omega \times(0,+\infty) \\
u=0 & \text { on } \Omega \times\{0\} \tag{3}
\end{array}
$$

or the Cauchy problem for the diffusion equation:

$$
\begin{equation*}
u_{t}=\operatorname{div}(\sigma \nabla u) \quad \text { in } \mathbb{R}^{N} \times(0,+\infty) \text { and } u=\mathcal{X}_{\Omega^{c}} \text { on } \mathbb{R}^{N} \times\{0\} \tag{4}
\end{equation*}
$$

where $\mathcal{X}_{\Omega^{c}}$ denotes the characteristic function of the set $\Omega^{c}=\mathbb{R}^{N} \backslash \Omega$. Consider a bounded domain $G$ in $\mathbb{R}^{N}$ satisfying

$$
\begin{equation*}
\bar{D} \subset G \subset \bar{G} \subset \Omega \text { and } \operatorname{dist}(x, \partial \Omega) \leq \operatorname{dist}(x, \bar{D}) \text { for every } x \in \partial G \tag{5}
\end{equation*}
$$

The purpose of the present paper is to show the following theorems.
THEOREM 1.1. Let $u$ be the solution of problem (1)-(3) for $N \geq 2$, and let $\Gamma$ be a connected component of $\partial G$ satisfying

$$
\begin{equation*}
\operatorname{dist}(\Gamma, \partial \Omega)=\operatorname{dist}(\partial G, \partial \Omega) \tag{6}
\end{equation*}
$$

If there exists a function $a:(0,+\infty) \rightarrow(0,+\infty)$ satisfying

$$
\begin{equation*}
u(x, t)=a(t) \text { for every }(x, t) \in \Gamma \times(0,+\infty) \tag{7}
\end{equation*}
$$

then $\Omega$ and $D$ must be concentric balls.
Corollary 1.2. Let $u$ be the solution of problem (1)-(3) for $N \geq 2$. If there exists a function $a:(0,+\infty) \rightarrow(0,+\infty)$ satisfying

$$
\begin{equation*}
u(x, t)=a(t) \text { for every }(x, t) \in \partial G \times(0,+\infty) \tag{8}
\end{equation*}
$$

then $\Omega$ and $D$ must be concentric balls.
Theorem 1.3. Let $u$ be the solution of problem (4) for $N \geq 3$. Then the following assertions hold:
(a) If there exists a function $a:(0,+\infty) \rightarrow(0,+\infty)$ satisfying (8), then $\Omega$ and $D$ must be concentric balls.
(b) If $\sigma_{s}=\sigma_{m}$ and there exists a function a: $(0,+\infty) \rightarrow(0,+\infty)$ satisfying (7) for a connected component $\Gamma$ of $\partial G$ with (6), then $\Omega$ and $D$ must be concentric balls.

Corollary 1.2 is just an easy by-product of Theorem 1.1. Theorem 1.3 is limited to the case where $N \geq 3$, which is not natural; that is required for technical reasons in the use of the auxiliary functions $U, V, W$ given in section 4. We conjecture that Theorem 1.3 holds true also for $N=2$.

The condition (7) means that $\Gamma$ is an isothermic surface of the normalized temperature $u$ at every time, and hence $\Gamma$ is called a stationary isothermic surface of $u$. When $D=\emptyset$ and $\sigma$ is constant on $\mathbb{R}^{N}$, a symmetry theorem similar to Theorem 1.1 or Theorem 1.3 has been proved in [13, Theorem 1.2, p. 2024] provided the conclusion is replaced by that $\partial \Omega$ must be either a sphere or the union of two concentric spheres, and a symmetry theorem similar to Corollary 1.2 has also been proved in [10, Theorem 1.1, p. 932]. The present paper gives a generalization of the previous results to multi-phase heat conductors.

We note that the study of the relationship between the stationary isothermic surfaces and the symmetry of the problems has been initiated by Alessandrini $[2,3]$. Indeed, when $D=\emptyset$ and $\sigma$ is constant on $\mathbb{R}^{N}$, he considered the problem where the initial data in (3) is replaced by the general data $u_{0}$ in problem (1)-(3). Then he proved that if all the spatial isothermic surfaces of $u$ are stationary, then either $u_{0}-1$ is an eigenfunction of the Laplacian or $\Omega$ is a ball where $u_{0}$ is radially symmetric. See also $[8,14]$ for this direction.

The following sections are organized as follows. In section 2, we give four preliminaries where the balance laws given in [9, 10] play a key role on behalf of Varadhan's formula (see (12)) given in [15]. Section 3 is devoted to the proof of Theorem 1.1. Auxiliary functions $U, V$ given in section 3 play a key role. If $D$ is not a ball, we use the transmission condition (35) on $\partial D$ to get a contradiction to Hopf's boundary point lemma. In section 4, we prove Theorem 1.3 by following the proof of Theorem 1.1. Auxiliary functions $U, V, W$ given in section 4 play a key role. We notice that almost the same arguments work as in the proof of Theorem 1.1.

## 2. Preliminaries for $N \geq 2$

Concerning the behavior of the solutions of problem (1)-(3) and problem (4), we start with the following lemma.
Lemma 2.1. Let $u$ be the solution of either problem (1)-(3) or problem (4). We have the following assertions:
(a) For every compact set $K \subset \Omega$, there exist two positive constants $B$ and $b$ satisfying

$$
0<u(x, t)<B e^{-\frac{b}{t}} \quad \text { for every }(x, t) \in K \times(0,1]
$$

(b) There exists a constant $M>0$ satisfying

$$
0 \leq 1-u(x, t) \leq \min \left\{1, M t^{-\frac{N}{2}}|\Omega|\right\}
$$

for every $(x, t) \in \Omega \times(0,+\infty)$ or $\in \mathbb{R}^{N} \times(0, \infty)$, where $|\Omega|$ denotes the Lebesgue measure of the set $\Omega$.
(c) For the solution $u$ of problem (1)-(3), there exist two positive constants $C$ and $\lambda$ satisfying

$$
0 \leq 1-u(x, t) \leq C e^{-\lambda t} \quad \text { for every }(x, t) \in \Omega \times(0,+\infty)
$$

(d) For the solution $u$ of problem (4) where $N \geq 3$, there exist two positive constants $\beta$ and $L$ satisfying

$$
\beta^{-1}|x|^{2-N} \leq \int_{0}^{\infty}(1-u(x, t)) d t \leq \beta|x|^{2-N} \quad \text { if } \quad|x| \geq L
$$

$$
\text { where } \bar{\Omega} \subset B_{L}(0)=\left\{x \in \mathbb{R}^{N}:|x|<L\right\} .
$$

Proof. We make use of the Gaussian bounds for the fundamental solutions of parabolic equations due to Aronson [4, Theorem 1, p. 891] (see also [5, p. 328]). Let $g=g(x, t ; \xi, \tau)$ be the fundamental solution of $u_{t}=\operatorname{div}(\sigma \nabla u)$. Then there exist two positive constants $\alpha$ and $M$ such that

$$
\begin{equation*}
M^{-1}(t-\tau)^{-\frac{N}{2}} e^{-\frac{\alpha|x-\xi|^{2}}{t-\tau}} \leq g(x, t ; \xi, \tau) \leq M(t-\tau)^{-\frac{N}{2}} e^{-\frac{|x-\xi|^{2}}{\alpha(t-\tau)}} \tag{9}
\end{equation*}
$$

for all $(x, t),(\xi, \tau) \in \mathbb{R}^{N} \times(0,+\infty)$ with $t>\tau$.
For the solution $u$ of problem (4), $1-u$ is regarded as the unique bounded solution of the Cauchy problem for the diffusion equation with initial data $\mathcal{X}_{\Omega}$ which is greater than or equal to the corresponding solution of the initialboundary value problem for the diffusion equation under the homogeneous Dirichlet boundary condition by the comparison principle. Hence we have from (9)

$$
1-u(x, t)=\int_{\mathbb{R}^{N}} g(x, t ; \xi, 0) \mathcal{X}_{\Omega}(\xi) d \xi \leq M t^{-\frac{N}{2}}|\Omega|
$$

The inequalities $0 \leq 1-u \leq 1$ follow from the comparison principle. This completes the proof of (b). Moreover, (d) follows from (9) as is noted in [4, 5. Remark, pp. 895-896].

For (a), let $K$ be a compact set contained in $\Omega$. We set

$$
\mathcal{N}_{\rho}=\left\{x \in \mathbb{R}^{N}: \operatorname{dist}(x, \partial \Omega)<\rho\right\}
$$

where $\rho=\frac{1}{2} \operatorname{dist}(K, \partial \Omega)(>0)$. Define $v=v(x, t)$ by

$$
v(x, t)=\lambda \int_{\mathcal{N}_{\rho}} g(x, t ; \xi, 0) d \xi \quad \text { for every }(x, t) \in \mathbb{R}^{N} \times(0,+\infty)
$$

where a number $\lambda>0$ will be determined later. Then it follows from (9) that

$$
v(x, t) \geq \lambda M^{-1} t^{-\frac{N}{2}} \int_{\mathcal{N}_{\rho}} e^{-\frac{\alpha|x-\xi|^{2}}{t}} d \xi \quad \text { for }(x, t) \in \mathbb{R}^{N} \times(0,+\infty)
$$

and hence we can choose $\lambda>0$ satisfying

$$
v \geq 1 \text { on } \partial \Omega \times(0,1]
$$

Thus the comparison principle yields that

$$
\begin{equation*}
u \leq v \text { in } \Omega \times(0,1] \tag{10}
\end{equation*}
$$

On the other hand, it follows from (9) that

$$
v(x, t) \leq \lambda M t^{-\frac{N}{2}} \int_{\mathcal{N}_{\rho}} e^{-\frac{|x-\xi|^{2}}{\alpha t}} d \xi \quad \text { for }(x, t) \in \mathbb{R}^{N} \times(0,+\infty)
$$

Since $|x-\xi| \geq \rho$ for every $x \in K$ and $\xi \in \mathcal{N} \rho$, we observe that

$$
v(x, t) \leq \lambda M t^{-\frac{N}{2}} e^{-\frac{\rho^{2}}{\alpha t}}\left|\mathcal{N}_{\rho}\right| \quad \text { for every }(x, t) \in K \times(0,+\infty)
$$

where $\left|\mathcal{N}_{\rho}\right|$ denotes the Lebesgue measure of the set $\mathcal{N}_{\rho}$. Therefore (10) gives (a).
For (c), for instance choose a large ball $B$ with $\bar{\Omega} \subset B$ and let $\varphi=\varphi(x)$ be the first positive eigenfunction of the problem

$$
-\operatorname{div}(\sigma \nabla \varphi)=\lambda \varphi \text { in } B \text { and } \varphi=0 \text { on } \partial B
$$

with $\sup _{B} \varphi=1$. Choose $C>0$ sufficiently large to have

$$
1 \leq C \varphi \text { in } \bar{\Omega} .
$$

Then it follows from the comparison principle that

$$
1-u(x, t) \leq C e^{-\lambda t} \varphi(x) \quad \text { for every }(x, t) \in \Omega \times(0,+\infty)
$$

which gives (c).
The following asymptotic formula of the heat content of a ball touching at $\partial \Omega$ at only one point tells us about the interaction between the initial behavior of solutions and geometry of domain.

Proposition 2.2. Let $u$ be the solution of either problem (1)-(3) or problem (4). Let $x \in \Omega$ and assume that the open ball $B_{r}(x)$ with radius $r>0$
centered at $x$ is contained in $\Omega$ and such that $\overline{B_{r}(x)} \cap \partial \Omega=\{y\}$ for some $y \in \partial \Omega$. Then we have:

$$
\begin{equation*}
\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(x)} u(z, t) d z=C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{r}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} \tag{11}
\end{equation*}
$$

Here, $\kappa_{1}(y), \ldots, \kappa_{N-1}(y)$ denote the principal curvatures of $\partial \Omega$ at $y$ with respect to the inward normal direction to $\partial \Omega$ and $C(N, \sigma)$ is a positive constant given by

$$
C(N, \sigma)=\left\{\begin{aligned}
2 \sigma_{s}^{\frac{N+1}{4}} c(N) & \text { for problem }(1)-(3) \\
\frac{2 \sqrt{\sigma_{m}}}{\sqrt{\sigma_{s}}+\sqrt{\sigma_{m}}} \sigma_{s}^{\frac{N+1}{4}} c(N) & \text { for problem }(4)
\end{aligned}\right.
$$

where $c(N)$ is a positive constant depending only on $N$. (Notice that if $\sigma_{s}=\sigma_{m}$ then $C(N, \sigma)=\sigma_{s}^{\frac{N+1}{4}} c(N)$ for problem (4), that is, just half of the constant for problem (1)-(3).)

When $\kappa_{j}(y)=1 / r$ for some $j \in\{1, \cdots, N-1\}$, (11) holds by setting the right-hand side to $+\infty$ (notice that $\kappa_{j}(y) \leq 1 / r$ always holds for all $j$ 's).

Proof. For the one-phase problem, that is, for the heat equation $u_{t}=\Delta u$, this lemma has been proved in [12, Theorem 1.1, p. 238] or in [13, Theorem B, pp. 2024-2025 and Appendix, pp. 2029-2032]. The proof in [13] was carried out by constructing appropriate super- and subsolutions in a neighborhood of $\partial \Omega$ in a short time with the aid of the initial behavior [13, Lemma B.2, p. 2030] obtained by Varadhan's formula [15] for the heat equation $u_{t}=\Delta u$

$$
\begin{equation*}
-4 t \log u(x, t) \rightarrow \operatorname{dist}(x, \partial \Omega)^{2} \text { as } t \rightarrow+0 \tag{12}
\end{equation*}
$$

uniformly on every compact set in $\Omega$. (See also [13, Theorem A, p. 2024] for the formula.) Here, with no need of Varadhan's formula, (a) of Lemma 2.1 gives sufficient information on the initial behavior [13, Lemma B.2, p. 2030]. We remark that since problem (1)-(3) is one-phase with conductivity $\sigma_{s}$ near $\partial \Omega$, we can obtain formula (11) for problem (1)-(3) only by scaling in $t$. On the other hand, problem (4) is two-phase with conductivities $\sigma_{m}, \sigma_{s}$ near $\partial \Omega$ if $\sigma_{m} \neq \sigma_{s}$. Therefore, it is enough for us to prove formula (11) for problem (4) where $\sigma_{m} \neq \sigma_{s}$.

Let $u$ be the solution of problem (4) where $\sigma_{m} \neq \sigma_{s}$, and let us prove this lemma by modifying the proof of Theorem B in [13, Appendix, pp. 2029-2032].

Let us consider the signed distance function $d^{*}=d^{*}(x)$ of $x \in \mathbb{R}^{N}$ to the boundary $\partial \Omega$ defined by

$$
d^{*}(x)=\left\{\begin{align*}
\operatorname{dist}(x, \partial \Omega) & \text { if } x \in \Omega  \tag{13}\\
-\operatorname{dist}(x, \partial \Omega) & \text { if } x \notin \Omega
\end{align*}\right.
$$

Since $\partial \Omega$ is bounded and of class $C^{2}$, there exists a number $\rho_{0}>0$ such that $d^{*}(x)$ is $C^{2}$-smooth on a compact neighborhood $\mathcal{N}$ of the boundary $\partial \Omega$ given by

$$
\begin{equation*}
\mathcal{N}=\left\{x \in \mathbb{R}^{N}:-\rho_{0} \leq d^{*}(x) \leq \rho_{0}\right\} \tag{14}
\end{equation*}
$$

We make $\mathcal{N}$ satisfy $\mathcal{N} \cap \bar{D}=\emptyset$. Introduce a function $F=F(\xi)$ for $\xi \in \mathbb{R}$ by

$$
F(\xi)=\frac{1}{2 \sqrt{\pi}} \int_{\xi}^{\infty} e^{-s^{2} / 4} d s
$$

Then $F$ satisfies

$$
\begin{aligned}
& F^{\prime \prime}+\frac{1}{2} \xi F^{\prime}=0 \text { and } F^{\prime}<0 \text { in } \mathbb{R} \\
& F(-\infty)=1, F(0)=\frac{1}{2}, \quad \text { and } F(+\infty)=0
\end{aligned}
$$

For each $\varepsilon \in(0,1 / 4)$, we define two functions $F_{ \pm}=F_{ \pm}(\xi)$ for $\xi \in \mathbb{R}$ by

$$
F_{ \pm}(\xi)=F(\xi \mp 2 \varepsilon)
$$

Then $F_{ \pm}$satisfies

$$
\begin{aligned}
& F_{ \pm}^{\prime \prime}+\frac{1}{2} \xi F_{ \pm}^{\prime}= \pm \varepsilon F_{ \pm}^{\prime}, F_{ \pm}^{\prime}<0 \text { and } F_{-}<F<F_{+} \text {in } \mathbb{R} \\
& F_{ \pm}(-\infty)=1, F_{ \pm}(0) \gtrless \frac{1}{2}, \text { and } F_{ \pm}(+\infty)=0 .
\end{aligned}
$$

By setting $\eta=t^{-\frac{1}{2}} d^{*}(x), \mu=\sqrt{\sigma_{m}} / \sqrt{\sigma_{s}}$ and $\theta_{ \pm}=1+(\mu-1) F_{ \pm}(0)(>0)$, we introduce two functions $v_{ \pm}=v_{ \pm}(x, t)$ by

$$
v_{ \pm}(x, t)=\left\{\begin{align*}
\frac{\mu}{\theta_{ \pm}} F_{ \pm}\left(\sigma_{s}^{-\frac{1}{2}} \eta\right) & \text { for }(x, t) \in \Omega \times(0,+\infty)  \tag{15}\\
\frac{1}{\theta_{ \pm}}\left[F_{ \pm}\left(\sigma_{m}^{-\frac{1}{2}} \eta\right)+\theta_{ \pm}-1\right] & \text { for }(x, t) \in \Omega^{c} \times(0,+\infty)
\end{align*}\right.
$$

Then $v_{ \pm}$satisfies the transmission conditions

$$
\begin{equation*}
\left.v_{ \pm}\right|_{+}=\left.v_{ \pm}\right|_{-} \quad \text { and }\left.\sigma_{m} \frac{\partial v_{ \pm}}{\partial \nu}\right|_{+}=\left.\sigma_{s} \frac{\partial v_{ \pm}}{\partial \nu}\right|_{-} \quad \text { on } \partial \Omega \times(0,+\infty) \tag{16}
\end{equation*}
$$

where + denotes the limit from outside and - that from inside of $\Omega$ and $\nu=$ $\nu(x)$ denotes the outward unit normal vector to $\partial \Omega$ at $x \in \partial \Omega$, since $\nu=-\nabla d^{*}$ on $\partial \Omega$. Moreover we observe that for each $\varepsilon \in(0,1 / 4)$, there exists $t_{1, \varepsilon} \in(0,1]$ satisfying

$$
\begin{equation*}
( \pm 1)\left\{\left(v_{ \pm}\right)_{t}-\sigma \Delta v_{ \pm}\right\}>0 \quad \text { in } \quad(\mathcal{N} \backslash \partial \Omega) \times\left(0, t_{1, \varepsilon}\right] \tag{17}
\end{equation*}
$$

In fact, a straightforward computation gives

$$
\left(v_{ \pm}\right)_{t}-\sigma \Delta v_{ \pm}=\left\{\begin{aligned}
&-\frac{\mu}{t \theta_{ \pm}}\left( \pm \varepsilon+\sqrt{\sigma_{s} t} \Delta d^{*}\right) F_{ \pm}^{\prime} \\
&-\frac{1}{t \theta_{ \pm}}\left( \pm \varepsilon+\sqrt{\sigma_{m} t} \Delta d^{*}\right) F_{ \pm}^{\prime} \\
& \text { in }(\mathcal{N} \cap \Omega) \times(0,+\infty)
\end{aligned}\right.
$$

Then, for each $\varepsilon \in(0,1 / 4)$, by setting $t_{1, \varepsilon}=\frac{1}{\max \left\{\sigma_{s}, \sigma_{m}\right\}}\left(\frac{\varepsilon}{2 M}\right)^{2}$, where $M=$ $\max _{x \in \mathcal{N}}\left|\Delta d^{*}(x)\right|$, we obtain (17).

Then, in view of (a) of Lemma 2.1 and the definition (15) of $v_{ \pm}$, we see that there exist two positive constants $E_{1}$ and $E_{2}$ satisfying

$$
\begin{equation*}
\max \left\{\left|v_{+}\right|,\left|v_{-}\right|,|u|\right\} \leq E_{1} e^{-\frac{E_{2}}{t}} \text { in } \overline{\Omega \backslash \mathcal{N}} \times(0,1] \tag{18}
\end{equation*}
$$

By setting, for $(x, t) \in \mathbb{R}^{N} \times(0,+\infty)$,

$$
\begin{equation*}
w_{ \pm}(x, t)=(1 \pm \varepsilon) v_{ \pm}(x, t) \pm 2 E_{1} e^{-\frac{E_{2}}{t}} \tag{19}
\end{equation*}
$$

since $v_{ \pm}$and $u$ are all nonnegative, we obtain from (18) that

$$
\begin{equation*}
w_{-} \leq u \leq w_{+} \text {in } \overline{\Omega \backslash \mathcal{N}} \times(0,1] \tag{20}
\end{equation*}
$$

Moreover, in view of the facts that $F_{ \pm}(-\infty)=1$ and $F_{ \pm}(+\infty)=0$, we see that there exists $t_{\varepsilon} \in\left(0, t_{1, \varepsilon}\right]$ satisfying

$$
\begin{equation*}
w_{-} \leq u \leq w_{+} \quad \text { on }\left((\partial \mathcal{N} \backslash \Omega) \times\left(0, t_{\varepsilon}\right]\right) \cup(\mathcal{N} \times\{0\}) \tag{21}
\end{equation*}
$$

Then, in view of (16), (17), (20), (21) and the definition (19) of $w_{ \pm}$, we have from the comparison principle over $\mathcal{N}$ that

$$
\begin{equation*}
w_{-} \leq u \leq w_{+} \quad \text { in } \quad(\overline{\mathcal{N}} \cup \Omega) \times\left(0, t_{\varepsilon}\right] \tag{22}
\end{equation*}
$$

By writing

$$
\Gamma_{s}=\left\{x \in \Omega: d^{*}(x)=s\right\} \text { for } s>0
$$

let us quote a geometric lemma from [11] adjusted to our situation.
Lemma 2.3. ([11, Lemma 2.1, p. 376]) If $\max _{1 \leq j \leq N-1} \kappa_{j}(y)<\frac{1}{r}$, then we have:

$$
\lim _{s \rightarrow 0^{+}} s^{-\frac{N-1}{2}} \mathcal{H}^{N-1}\left(\Gamma_{s} \cap B_{r}(x)\right)=2^{\frac{N-1}{2}} \omega_{N-1}\left\{\prod_{j=1}^{N-1}\left(\frac{1}{r}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}}
$$

where $\mathcal{H}^{N-1}$ is the standard ( $N-1$ )-dimensional Hausdorff measure, and $\omega_{N-1}$ is the volume of the unit ball in $\mathbb{R}^{N-1}$.

Let us consider the case where $\max _{1 \leq j \leq N-1} \kappa_{j}(y)<\frac{1}{r}$. Then it follows from (22) that for every $t \in\left(0, t_{\varepsilon}\right]$

$$
\begin{equation*}
t^{-\frac{N+1}{4}} \int_{B_{r}(x)} w_{-} d z \leq t^{-\frac{N+1}{4}} \int_{B_{r}(x)} u d z \leq t^{-\frac{N+1}{4}} \int_{B_{r}(x)} w_{+} d z . \tag{23}
\end{equation*}
$$

On the other hand, with the aid of the co-area formula, we have

$$
\begin{aligned}
& \int_{B_{r}(x)} v_{ \pm} d z= \\
& \frac{\mu}{\theta_{ \pm}}\left(\sigma_{s} t\right)^{\frac{N+1}{4}} \int_{0}^{2 r\left(\sigma_{s} t\right)^{-\frac{1}{2}}} F_{ \pm}(\xi) \xi^{\frac{N-1}{2}}\left(\left(\sigma_{s} t\right)^{\frac{1}{2}} \xi\right)^{-\frac{N-1}{2}} \mathcal{H}^{N-1}\left(\Gamma_{\left(\sigma_{s} t\right)^{\frac{1}{2}} \xi^{n}} \cap B_{r}(x)\right) d \xi,
\end{aligned}
$$

where $v_{ \pm}$is defined by (15). Thus, by Lebesgue's dominated convergence theorem and Lemma 2.3, we get

$$
\begin{aligned}
& \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(x)} w_{ \pm} d x= \\
& \frac{\mu}{\theta_{ \pm}}\left(\sigma_{s}\right)^{\frac{N+1}{4}} 2^{\frac{N-1}{2}} \omega_{N-1}\left\{\prod_{j=1}^{N-1}\left(\frac{1}{r}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} \int_{0}^{\infty} F_{ \pm}(\xi) \xi^{\frac{N-1}{2}} d \xi .
\end{aligned}
$$

Moreover, again by Lebesgue's dominated convergence theorem, since

$$
\lim _{\varepsilon \rightarrow 0} \theta_{ \pm}=1+(\mu-1) F(0)=\frac{\mu+1}{2} \text { and } \mu=\sqrt{\sigma_{m}} / \sqrt{\sigma_{s}},
$$

we see that

$$
\begin{aligned}
& \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(x)} w_{ \pm} d x= \\
& \frac{2 \sqrt{\sigma_{m}}}{\sqrt{\sigma_{s}}+\sqrt{\sigma_{m}}}\left(\sigma_{s}\right)^{\frac{N+1}{4}} 2^{\frac{N-1}{2}} \omega_{N-1}\left\{\prod_{j=1}^{N-1}\left(\frac{1}{r}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} \int_{0}^{\infty} F(\xi) \xi^{\frac{N-1}{2}} d \xi .
\end{aligned}
$$

Therefore (23) gives formula (11) provided $\max _{1 \leq j \leq N-1} \kappa_{j}(y)<\frac{1}{r}$.
Once this is proved, the case where $\kappa_{j}(y)=1 / r$ for some $j \in\{1, \cdots, N-1\}$ can be dealt with as in [12, p. 248] by choosing a sequence of balls $\left\{B_{r_{k}}\left(x_{k}\right)\right\}_{k=1}^{\infty}$ satisfying:
$r_{k}<r, y \in \partial B_{r_{k}}\left(x_{k}\right)$, and $B_{r_{k}}\left(x_{k}\right) \subset B_{r}(x)$ for every $k \geq 1$, and $\lim _{k \rightarrow \infty} r_{k}=r$.

Then, because of $\max _{1 \leq j \leq N-1} \kappa_{j}(y) \leq \frac{1}{r}<\frac{1}{r_{k}}$, applying formula (11) to each ball $B_{r_{k}}\left(x_{k}\right)$ yields that

$$
\liminf _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(x)} u(z, t) d z=+\infty
$$

This completes the proof of Proposition 2.2.
In order to determine the symmetry of $\Omega$, we employ the following lemma.
Lemma 2.4. Let $u$ be the solution of either problem (1)-(3) or problem (4). Under the assumption (7) of Theorem 1.1 and Theorem 1.3, the following assertions hold:
(a) There exists a number $R>0$ such that

$$
\operatorname{dist}(x, \partial \Omega)=R \text { for every } x \in \Gamma ;
$$

(b) $\Gamma$ is a real analytic hypersurface;
(c) there exists a connected component $\gamma$ of $\partial \Omega$, that is also a real analytic hypersurface, such that the mapping $\gamma \ni y \mapsto x(y) \equiv y-R \nu(y) \in \Gamma$, where $\nu(y)$ is the outward unit normal vector to $\partial \Omega$ at $y \in \gamma$, is a diffeomorphism; in particular $\gamma$ and $\Gamma$ are parallel hypersurfaces at distance $R$;
(d) it holds that

$$
\begin{equation*}
\max _{1 \leq j \leq N-1} \kappa_{j}(y)<\frac{1}{R} \text { for every } y \in \gamma \tag{24}
\end{equation*}
$$

where $\kappa_{1}(y), \cdots, \kappa_{N-1}(y)$ are the principal curvatures of $\partial \Omega$ at $y \in \gamma$ with respect to the inward unit normal vector $-\nu(y)$ to $\partial \Omega$;
(e) there exists a number $c>0$ such that

$$
\begin{equation*}
\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}(y)\right)=c \quad \text { for every } y \in \gamma \tag{25}
\end{equation*}
$$

Proof. First it follows from the assumption (5) that

$$
B_{r}(x) \subset \Omega \backslash \bar{D} \text { for every } x \in \partial G \text { with } 0<r \leq \operatorname{dist}(x, \partial \Omega)
$$

Therefore, since $\sigma=\sigma_{s}$ in $\Omega \backslash \bar{D}$, we can use a balance law (see [10, Theorem 2.1, pp. 934-935] or [9, Theorem 4, p. 704]) to obtain from (7) that

$$
\begin{equation*}
\int_{B_{r}(p)} u(z, t) d z=\int_{B_{r}(q)} u(z, t) d z \text { for every } p, q \in \Gamma \text { and } t>0 \tag{26}
\end{equation*}
$$

provided $0<r \leq \min \{\operatorname{dist}(p, \partial \Omega), \operatorname{dist}(q, \partial \Omega)\}$. Let us show assertion (a). Suppose that there exist a pair of points $p$ and $q$ satisfying

$$
\operatorname{dist}(p, \partial \Omega)<\operatorname{dist}(q, \partial \Omega)
$$

Set $r=\operatorname{dist}(p, \partial \Omega)$. Then there exists a point $y \in \partial \Omega$ such that $y \in \overline{B_{r}(p)} \cap \partial \Omega$. Choose a smaller ball $B_{\hat{r}}(x) \subset B_{r}(p)$ with $0<\hat{r}<r$ and $\overline{B_{\hat{r}}(x)} \cap \partial B_{r}(p)=\{y\}$. Since $\max _{1 \leq j \leq N-1} \kappa_{j}(y) \leq \frac{1}{r}<\frac{1}{\hat{r}}$, by applying Proposition 2.2 to the ball $B_{\hat{r}}(x)$, we get

$$
\liminf _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(p)} u(z, t) d z \geq \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{\hat{r}}(x)} u(z, t) d z>0
$$

On the other hand, since $\overline{B_{r}(q)} \subset \Omega$, it follows from (a) of Lemma 2.1 that

$$
\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{r}(q)} u(z, t) d z=0
$$

which contradicts (26), and hence assertion (a) holds true.
We can find a point $x_{*} \in \Gamma$ and a ball $B_{\rho}\left(z_{*}\right)$ such that $B_{\rho}\left(z_{*}\right) \subset G$ and $x_{*} \in \partial B_{\rho}\left(z_{*}\right)$. Since $\Gamma$ satisfies (6), assertion (a) yields that there exists a point $y_{*} \in \partial \Omega$ satisfying

$$
B_{R+\rho}\left(z_{*}\right) \subset \Omega, y_{*} \in \overline{B_{R+\rho}\left(z_{*}\right)} \cap \partial \Omega, \text { and } \overline{B_{R}\left(x_{*}\right)} \cap \partial \Omega=\left\{y_{*}\right\}
$$

Observe that

$$
\max _{1 \leq j \leq N-1} \kappa_{j}\left(y_{*}\right) \leq \frac{1}{R+\rho}<\frac{1}{R} \text { and } x_{*}=y_{*}-R \nu\left(y_{*}\right) \equiv x\left(y_{*}\right)
$$

Define $\gamma \subset \partial \Omega$ by

$$
\begin{gathered}
\gamma=\left\{y \in \partial \Omega: \overline{B_{R}(x)} \cap \partial \Omega=\{y\} \text { for } x=y-R \nu(y) \in \Gamma\right. \\
\text { and } \left.\max _{1 \leq j \leq N-1} \kappa_{j}(y)<\frac{1}{R}\right\} .
\end{gathered}
$$

Hence $y_{*} \in \gamma$ and $\gamma \neq \emptyset$. By Proposition 2.2 we have that for every $y \in \gamma$ and $x=x(y)(=y-R \nu(y))$

$$
\begin{equation*}
\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{R}(x)} u(z, t) d z=C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} \tag{27}
\end{equation*}
$$

Here let us show that, if $y \in \gamma$ and $x=x(y)$, then $\nabla u(x, t) \neq 0$ for some $t>0$, which guarantees that in a neighborhood of $x, \Gamma$ is a part of a real analytic
hypersurface properly embedded in $\mathbb{R}^{N}$ because of (7), real analyticity of $u$ with respect to the space variables, and the implicit function theorem. Moreover, this together with the implicit function theorem guarantees that $\gamma$ is open in $\partial \Omega$ and the mapping $\gamma \ni y \mapsto x(y) \in \Gamma$ is a local diffeomorphism, which is also real analytic. If we can prove additionally that $\gamma$ is closed in $\partial \Omega$, then the mapping $\gamma \ni y \mapsto x(y) \in \Gamma$ is a diffeomorphism and $\gamma$ is a connected component of $\partial \Omega$ since $\Gamma$ is a connected component of $\partial G$, and hence all the remaining assertions (b) - (e) follow from (26), (27) and the definition of $\gamma$. We shall prove this later in the end of the proof of Lemma 2.4.

Before this we show that, if $y \in \gamma$ and $x=x(y)$, then $\nabla u(x, t) \neq 0$ for some $t>0$. Suppose that $\nabla u(x, t)=0$ for every $t>0$. Then we use another balance law (see [10, Corollary 2.2, pp. 935-936]) to obtain that

$$
\begin{equation*}
\int_{B_{R}(x)}(z-x) u(z, t) d z=0 \text { for every } t>0 \tag{28}
\end{equation*}
$$

On the other hand, (a) of Lemma 2.1 yields that

$$
\begin{equation*}
\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{K} u(z, t) d z=0 \text { for every compact set } K \subset \Omega \tag{29}
\end{equation*}
$$

and hence by (27) it follows that for every $\varepsilon>0$

$$
\begin{equation*}
\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{R}(x) \cap B_{\varepsilon}(y)} u(z, t) d z=C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} \tag{30}
\end{equation*}
$$

This implies that

$$
\begin{aligned}
& \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{R}(x)}(z-x) u(z, t) d z= \\
& C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}}(y-x) \neq 0
\end{aligned}
$$

which contradicts (28).
It remains to show that $\gamma$ is closed in $\partial \Omega$. Let $\left\{y^{n}\right\}$ be a sequence of points in $\gamma$ with $\lim _{n \rightarrow \infty} y^{n}=y^{\infty} \in \partial \Omega$, and let us prove that $y^{\infty} \in \gamma$. By combining (26) with (27), we see that there exists a positive number $c$ satisfying assertion (e) and hence by continuity

$$
\begin{equation*}
\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}\left(y^{\infty}\right)\right)=c>0 \text { and } \max _{1 \leq j \leq N-1} \kappa_{j}\left(y^{\infty}\right) \leq \frac{1}{R} \tag{31}
\end{equation*}
$$

since $y^{j} \in \gamma$ for every $j$. Thus $\max _{1 \leq j \leq N-1} \kappa_{j}\left(y^{\infty}\right)<\frac{1}{R}$. Let $x^{\infty}=y^{\infty}-$ $R \nu\left(y^{\infty}\right)\left(=x\left(y^{\infty}\right)\right)$. It suffices to show that $\overline{B_{R}\left(x^{\infty}\right)} \cap \partial \Omega=\left\{y^{\infty}\right\}$. Suppose that there exists another point $y \in \overline{B_{R}\left(x^{\infty}\right)} \cap \partial \Omega$. Then for every $\hat{R} \in(0, R)$ we can find two points $p^{\infty}$ and $p$ in $B_{R}\left(x^{\infty}\right)$ such that

$$
B_{\hat{R}}\left(p^{\infty}\right) \cup B_{\hat{R}}(p) \subset B_{R}\left(x^{\infty}\right), \overline{B_{\hat{R}}\left(p^{\infty}\right)} \cap \partial \Omega=\left\{y^{\infty}\right\}, \text { and } \overline{B_{\hat{R}}(p)} \cap \partial \Omega=\{y\}
$$

Hence by Proposition 2.2 we have

$$
\begin{aligned}
& \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{\hat{R}}\left(p^{\infty}\right)} u(z, t) d z=C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{\hat{R}}-\kappa_{j}\left(y^{\infty}\right)\right)\right\}^{-\frac{1}{2}}, \\
& \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{\hat{R}}(p)} u(z, t) d z=C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{\hat{R}}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}} .
\end{aligned}
$$

Thus, with the same reasoning as in (30) by choosing small $\varepsilon>0$, we have from (31), (26), (27) and assertion (e) that for every $x \in \gamma$

$$
\begin{aligned}
& C(N, \sigma)\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_{j}\left(y^{\infty}\right)\right)\right\}^{-\frac{1}{2}}=C(N, \sigma) c^{-\frac{1}{2}} \\
& =\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{R}(x)} u(z, t) d z=\lim _{t \rightarrow+0} t^{-\frac{N+1}{4}} \int_{B_{R}\left(x^{\infty}\right)} u(z, t) d z \\
& \geq \lim _{t \rightarrow+0} t^{-\frac{N+1}{4}}\left[\int_{B_{\hat{R}}\left(p^{\infty}\right) \cap B_{\varepsilon}\left(y^{\infty}\right)} u(z, t) d z+\int_{B_{\hat{R}}(p) \cap B_{\varepsilon}(y)} u(z, t) d z\right] \\
& =C(N, \sigma)\left[\left\{\prod_{j=1}^{N-1}\left(\frac{1}{\hat{R}}-\kappa_{j}\left(y^{\infty}\right)\right)\right\}^{-\frac{1}{2}}+\left\{\prod_{j=1}^{N-1}\left(\frac{1}{\hat{R}}-\kappa_{j}(y)\right)\right\}^{-\frac{1}{2}}\right] .
\end{aligned}
$$

Since $\hat{R} \in(0, R)$ is arbitrarily chosen, this gives a contradiction, and hence $\gamma$ is closed in $\partial \Omega$.

Lemma 2.5. Let $u$ be the solution of problem (4). Under the assumption (8) of Theorem 1.3, the same assertions (a)-(e) as in Lemma 2.4 hold provided $\Gamma$ and $\gamma$ are replaced by $\partial G$ and $\partial \Omega$, respectively.

Proof. By the same reasoning as in assertion (a) of Lemma 2.4 we have assertion (a) from the assumption (8). Since every component $\Gamma$ of $\partial G$ has the same distance $R$ to $\partial \Omega$, every component $\Gamma$ satisfies the assumption (6). Therefore,
we can use the same arguments as in the proof of Lemma 2.4 to prove this lemma. Here we must have

$$
\partial \Omega=\left\{x \in \mathbb{R}^{N}: \operatorname{dist}(x, \bar{G})=R\right\}
$$

## 3. Proof of Theorem 1.1

Let $u$ be the solution of problem (1)-(3) for $N \geq 2$. With the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.4 yields that $\gamma$ and $\Gamma$ are concentric spheres. Denote by $x_{0} \in \mathbb{R}^{N}$ the common center of $\gamma$ and $\Gamma$. By combining the initial and boundary conditions of problem (1)-(3) and the assumption (7) with the real analyticity in $x$ of $u$ over $\Omega \backslash \bar{D}$, we see that $u$ is radially symmetric with respect to $x_{0}$ in $x$ on $(\Omega \backslash \bar{D}) \times(0, \infty)$. Here we used the assumption that $\Omega \backslash \bar{D}$ is connected. Moreover, in view of the Dirichlet boundary condition (2), we can distinguish the following two cases:
(I) $\Omega$ is a ball;
(II) $\Omega$ is a spherical shell.

By virtue of (c) of Lemma 2.1, we can introduce the following two auxiliary functions $U=U(x), V=V(x)$ by

$$
\begin{array}{ll}
U(x)=\int_{0}^{\infty}(1-u(x, t)) d t & \text { for } x \in \Omega \backslash \bar{D} \\
V(x)=\int_{0}^{\infty}(1-u(x, t)) d t & \text { for } x \in D \tag{33}
\end{array}
$$

Then we observe that

$$
\begin{align*}
& -\Delta U=\frac{1}{\sigma_{s}} \text { in } \Omega \backslash \bar{D},-\Delta V=\frac{1}{\sigma_{c}} \text { in } D  \tag{34}\\
& U=V \text { and } \sigma_{s} \frac{\partial U}{\partial \nu}=\sigma_{c} \frac{\partial V}{\partial \nu} \text { on } \partial D  \tag{35}\\
& U=0 \text { on } \partial \Omega \tag{36}
\end{align*}
$$

where $\nu=\nu(x)$ denotes the outward unit normal vector to $\partial D$ at $x \in \partial D$ and (35) is the transmission condition. Since $U$ is radially symmetric with respect to $x_{0}$, by setting $r=\left|x-x_{0}\right|$ for $x \in \Omega \backslash \bar{D}$ we have

$$
\begin{equation*}
-\frac{\partial^{2}}{\partial r^{2}} U-\frac{N-1}{r} \frac{\partial}{\partial r} U=\frac{1}{\sigma_{s}} \text { in } \Omega \backslash \bar{D} . \tag{37}
\end{equation*}
$$

Solving this ordinary differential equation yields that

$$
U= \begin{cases}c_{1} r^{2-N}-\frac{1}{2 N \sigma_{s}} r^{2}+c_{2} & \text { if } N \geq 3  \tag{38}\\ -c_{1} \log r-\frac{1}{4 \sigma_{s}} r^{2}+c_{2} & \text { if } N=2\end{cases}
$$

where $c_{1}, c_{2}$ are some constants depending on $N$. Remark that $U$ can be extended as a radially symmetric function of $r$ in $\mathbb{R}^{N} \backslash\left\{x_{0}\right\}$.

Let us first show that case (II) does not occur. Set $\Omega=B_{\rho_{+}}\left(x_{0}\right) \backslash \overline{B_{\rho_{-}}\left(x_{0}\right)}$ for some numbers $\rho_{+}>\rho_{-}>0$. Since $\Omega \backslash \bar{D}$ is connected, (36) yields that $U\left(\rho_{+}\right)=U\left(\rho_{-}\right)=0$ and hence $c_{1}<0$. Moreover we observe that

$$
\begin{equation*}
U^{\prime \prime}<0 \text { on }\left[\rho_{-}, \rho_{+}\right] \tag{39}
\end{equation*}
$$

Recall that $D$ may have finitely many connected components. Let us take a connected component $D_{*} \subset D$. Then, since $\overline{D_{*}} \subset \Omega$, we see that there exist $\rho_{*} \in\left(\rho_{-}, \rho_{+}\right)$and $x_{*} \in \partial D_{*}$ which satisfy

$$
\begin{equation*}
U\left(\rho_{*}\right)=\min \left\{U(r): r=\left|x-x_{0}\right|, x \in \partial D_{*}\right\} \text { and } \rho_{*}=\left|x_{*}-x_{0}\right| . \tag{40}
\end{equation*}
$$

Notice that $\nu\left(x_{*}\right)$ equals either $\frac{x_{*}-x_{0}}{\rho_{*}}$ or $-\frac{x_{*}-x_{0}}{\rho_{*}}$. For $r>0$, set

$$
\begin{equation*}
\hat{U}(r)=U\left(\rho_{*}\right)+\frac{\sigma_{s}}{\sigma_{c}}\left(U(r)-U\left(\rho_{*}\right)\right) \tag{41}
\end{equation*}
$$

Since

$$
\begin{equation*}
\hat{U}(r)-U(r)=\left(\frac{\sigma_{s}}{\sigma_{c}}-1\right)\left(U(r)-U\left(\rho_{*}\right)\right) \tag{42}
\end{equation*}
$$

it follows that

$$
\hat{U}\left\{\begin{array}{ll}
\geq U & \text { if } \quad \sigma_{s}>\sigma_{c}  \tag{43}\\
\leq U & \text { if } \quad \sigma_{s}<\sigma_{c}
\end{array} \quad \text { on } \partial D_{*} .\right.
$$

Moreover, we remark that $\hat{U}$ never equals $U$ identically on $\partial D_{*}$ since $\Omega \backslash \overline{D_{*}}$ is connected and $\Omega$ is a spherical shell. Observe that

$$
\begin{equation*}
-\Delta \hat{U}=\frac{1}{\sigma_{c}} \text { and } \frac{\partial \hat{U}}{\partial r}=\frac{\sigma_{s}}{\sigma_{c}} \frac{\partial U}{\partial r} \text { in } \overline{D_{*}} \tag{44}
\end{equation*}
$$

On the other hand, we have

$$
\begin{equation*}
-\Delta V=\frac{1}{\sigma_{c}} \text { in } D_{*} \text { and } V=U \text { on } \partial D_{*} \tag{45}
\end{equation*}
$$

Then it follows from (43) and the strong comparison principle that

$$
\hat{U}\left\{\begin{array}{ll}
>V & \text { if } \sigma_{s}>\sigma_{c}  \tag{46}\\
<V & \text { if } \sigma_{s}<\sigma_{c}
\end{array} \text { in } D_{*}\right.
$$

since $\hat{U}$ never equals $U$ identically on $\partial D_{*}$. The transmission condition (35) with the definition of $\hat{U}$ tells us that

$$
\begin{equation*}
\hat{U}=V \quad \text { and } \quad \frac{\partial \hat{U}}{\partial \nu}=\frac{\partial V}{\partial \nu} \text { at } x=x_{*} \in \partial D_{*} \tag{47}
\end{equation*}
$$

since $\nu\left(x_{*}\right)$ equals either $\frac{x_{*}-x_{0}}{\rho_{*}}$ or $-\frac{x_{*}-x_{0}}{\rho_{*}}$. Therefore applying Hopf's boundary point lemma to the harmonic function $\hat{U}-V$ gives a contradiction to (47), and hence case (II) never occurs. (See [6, Lemma 3.4, p. 34] for Hopf's boundary point lemma.)

Let us consider case (I). Set $\Omega=B_{\rho}\left(x_{0}\right)$ for some number $\rho>0$. We distinguish the following three cases:
(i) $c_{1}=0$;
(ii) $c_{1}>0$;
(iii) $c_{1}<0$.

We shall show that only case (i) occurs. Let us consider case (i) first. Note that

$$
\begin{equation*}
U^{\prime}(r)<0 \text { if } r>0, \text { and } U^{\prime}(0)=0 . \tag{48}
\end{equation*}
$$

Take an arbitrary component $D_{*} \subset D$. Then, since $\overline{D_{*}} \subset \Omega=B_{\rho}\left(x_{0}\right)$, we see that there exist $\rho_{*} \in(0, \rho)$ and $x_{*} \in \partial D_{*}$ which also satisfy (40). Notice that $\nu\left(x_{*}\right)$ equals $\frac{x_{*}-x_{0}}{\rho_{*}}$. For $r \geq 0$, define $\hat{U}=\hat{U}(r)$ by (41). Then, by (42) we also have (43). Observe that both (44) and (45) also hold true. Then it follows from (43) and the comparison principle that

$$
\hat{U}\left\{\begin{array}{ll}
\geq V & \text { if } \sigma_{s}>\sigma_{c}  \tag{49}\\
\leq V & \text { if } \sigma_{s}<\sigma_{c}
\end{array} \quad \text { in } D_{*}\right.
$$

The transmission condition (35) with the definition of $\hat{U}$ also yields (47) since $\nu\left(x_{*}\right)$ equals $\frac{x_{*}-x_{0}}{\rho_{*}}$. Therefore, by applying Hopf's boundary point lemma to the harmonic function $\hat{U}-V$, we conclude from (47) that

$$
\hat{U} \equiv V \quad \text { in } D_{*}
$$

and hence $D_{*}$ must be a ball centered at $x_{0}$. In conclusion, $D$ itself is connected and must be a ball centered at $x_{0}$, since $D_{*}$ is an arbitrary component of $D$.

Next, let us show that case (ii) does not occur. In case (ii) we have

$$
\begin{equation*}
U^{\prime}(r)<0 \text { if } r>0, \lim _{r \rightarrow 0} U(r)=+\infty, \text { and } x_{0} \in D \tag{50}
\end{equation*}
$$

Let us choose the connected component $D_{*}$ of $D$ satisfying $x_{0} \in D_{*}$. Then, since $\overline{D_{*}} \subset \Omega=B_{\rho}\left(x_{0}\right)$, we see that there exist $\rho_{* 1}, \rho_{* 2} \in(0, \rho)$ and $x_{* 1}, x_{* 2} \in$ $\partial D_{*}$ which satisfy that $\rho_{* 1} \leq \rho_{* 2}$ and

$$
\begin{align*}
& U\left(\rho_{* 1}\right)=\max \left\{U(r): r=\left|x-x_{0}\right|, x \in \partial D_{*}\right\} \text { and } \rho_{* 1}=\left|x_{* 1}-x_{0}\right|  \tag{51}\\
& U\left(\rho_{* 2}\right)=\min \left\{U(r): r=\left|x-x_{0}\right|, x \in \partial D_{*}\right\} \text { and } \rho_{* 2}=\left|x_{* 2}-x_{0}\right| \tag{52}
\end{align*}
$$

Notice that $\nu\left(x_{* i}\right)$ equals $\frac{x_{* i}-x_{0}}{\rho_{* i}}$ for $i=1,2$. Also, the case where $\rho_{* 1}=\rho_{* 2}$ may occur for instance if $D_{*}$ is a ball centered at $x_{0}$. For $r>0$, we set

$$
\hat{U}(r)= \begin{cases}U\left(\rho_{* 2}\right)+\frac{\sigma_{s}}{\sigma_{c}}\left(U(r)-U\left(\rho_{* 2}\right)\right) & \text { if } \sigma_{s}>\sigma_{c}  \tag{53}\\ U\left(\rho_{* 1}\right)+\frac{\sigma_{s}}{\sigma_{c}}\left(U(r)-U\left(\rho_{* 1}\right)\right) & \text { if } \sigma_{s}<\sigma_{c}\end{cases}
$$

Then, as in (43), it follows that

$$
\begin{equation*}
\hat{U} \geq U \text { on } \partial D_{*} \tag{54}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
-\Delta \hat{U}=\frac{1}{\sigma_{c}} \text { and } \frac{\partial \hat{U}}{\partial r}=\frac{\sigma_{s}}{\sigma_{c}} \frac{\partial U}{\partial r} \text { in } \overline{D_{*}} \backslash\left\{x_{0}\right\}, \quad \text { and } \quad \lim _{x \rightarrow x_{0}} \hat{U}=+\infty \tag{55}
\end{equation*}
$$

Therefore, since we also have (45), it follows from (54) and the strong comparison principle that

$$
\begin{equation*}
\hat{U}>V \text { in } D_{*} \backslash\left\{x_{0}\right\} \tag{56}
\end{equation*}
$$

The transmission condition (35) with the definition of $\hat{U}$ tells us that

$$
\begin{equation*}
\hat{U}=V \quad \text { and } \quad \frac{\partial \hat{U}}{\partial \nu}=\frac{\partial V}{\partial \nu} \quad \text { at } x=x_{* i} \in \partial D_{*}, \tag{57}
\end{equation*}
$$

since $\nu\left(x_{* i}\right)$ equals $\frac{x_{* i}-x_{0}}{\rho_{* i}}$ for $i=1,2$. Therefore applying Hopf's boundary point lemma to the harmonic function $\hat{U}-V$ gives a contradiction to (57), and hence case (ii) never occurs.

It remains to show that case (iii) does not occur. In case (iii), since $c_{1}<0$, there exists a unique critical point $r=\rho_{c}$ of $U(r)$ such that

$$
\begin{align*}
& U\left(\rho_{c}\right)=\max \{U(r): r>0\}>0 \text { and } 0<\rho_{c}<\rho  \tag{58}\\
& U^{\prime}(r)<0 \text { if } r>\rho_{c} \text { and } U^{\prime}(r)>0 \text { if } 0<r<\rho_{c}  \tag{59}\\
& \lim _{r \rightarrow 0} U(r)=-\infty \text { and } x_{0} \in D \tag{60}
\end{align*}
$$

Let us choose the connected component $D_{*}$ of $D$ satisfying $x_{0} \in D_{*}$. Then, since $\overline{D_{*}} \subset \Omega=B_{\rho}\left(x_{0}\right)$, as in case (ii), we see that there exist $\rho_{* 1}, \rho_{* 2} \in(0, \rho)$ and $x_{* 1}, x_{* 2} \in \partial D_{*}$ which satisfy (51) and (52). In view of the shape of the graph of $U$, we have from the transmission condition (35) that at $x_{* i} \in \partial D_{*}, i=$ 1,2 ,

$$
\frac{\partial V}{\partial \nu}=\frac{\sigma_{s}}{\sigma_{c}} \frac{\partial U}{\partial \nu}=\left\{\begin{align*}
0 & \text { if } \rho_{* i}=\rho_{c}  \tag{61}\\
\frac{\sigma_{s}}{\sigma_{c}} U^{\prime} & \text { if } \rho_{* i} \neq \rho_{c}
\end{align*}\right.
$$

where, in order to see that $\nu\left(x_{* i}\right)$ equals $\frac{x_{* i}-x_{0}}{\rho_{* i}}$ if $\rho_{* i} \neq \rho_{c}$, we used the fact that both $D_{*}$ and $B_{\rho}\left(x_{0}\right) \backslash \overline{D_{*}}$ are connected and $x_{0} \in D_{*}$. Also, the case where $\rho_{* 1}=\rho_{* 2}$ may occur for instance if $D_{*}$ is a ball centered at $x_{0}$. For $r>0$, we define $\hat{U}=\hat{U}(r)$ by

$$
\hat{U}(r)= \begin{cases}U\left(\rho_{* 1}\right)+\frac{\sigma_{s}}{\sigma_{c}}\left(U(r)-U\left(\rho_{* 1}\right)\right) & \text { if } \sigma_{s}>\sigma_{c}  \tag{62}\\ U\left(\rho_{* 2}\right)+\frac{\sigma_{s}}{\sigma_{c}}\left(U(r)-U\left(\rho_{* 2}\right)\right) & \text { if } \sigma_{s}<\sigma_{c}\end{cases}
$$

Remark that (62) is opposite to (53). Then, as in (54), it follows that

$$
\begin{equation*}
\hat{U} \leq U \text { on } \partial D_{*} . \tag{63}
\end{equation*}
$$

Hence, by proceeding with the strong comparison principle as in case (ii), we conclude that

$$
\begin{equation*}
\hat{U}<V \text { in } D_{*} \backslash\left\{x_{0}\right\} \tag{64}
\end{equation*}
$$

Then, it follows from the definition of $\hat{U}$ and (61) that (57) also holds true. In conclusion, applying Hopf's boundary point lemma to the harmonic function $\hat{U}-V$ gives a contradiction to (57), and hence case (iii) never occurs.

## 4. Proof of Theorem 1.3

Let $u$ be the solution of problem (4) for $N \geq 3$. For assertion (b) of Theorem 1.3, with the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.4 yields that $\gamma$ and $\Gamma$ are concentric spheres. Denote by $x_{0} \in \mathbb{R}^{N}$ the common center of $\gamma$ and $\Gamma$. By combining the initial condition of problem (4) and the assumption (7) with the real analyticity in $x$ of $u$ over $\mathbb{R}^{N} \backslash \bar{D}$ coming from $\sigma_{s}=\sigma_{m}$, we see that $u$ is radially symmetric with respect to $x_{0}$ in $x$ on $\left(\mathbb{R}^{N} \backslash \bar{D}\right) \times(0, \infty)$. Here we used the assumption that $\Omega \backslash \bar{D}$ is connected. Moreover, in view of the initial condition of problem (4), we can distinguish the following two cases as in section 3 :

$$
\begin{array}{ll}
\text { (I) } \Omega \text { is a ball; } & \text { (II) } \Omega \text { is a spherical shell. }
\end{array}
$$

For assertion (a) of Theorem 1.3, with the aid of Aleksandrov's sphere theorem [1, p. 412], Lemma 2.5 yields that $\partial G$ and $\partial \Omega$ are concentric spheres, since every component of $\partial \Omega$ is a sphere with the same curvature. Therefore, only the case (I) remains for assertion (a) of Theorem 1.3. Also, denoting by $x_{0} \in \mathbb{R}^{N}$ the common center of $\partial G$ and $\partial \Omega$ and combining the initial condition of problem (4) and the assumption (8) with the real analyticity in $x$ of $u$ over $\Omega \backslash \bar{D}$ yield that $u$ is radially symmetric with respect to $x_{0}$ in $x$ on $\left(\mathbb{R}^{N} \backslash \bar{D}\right) \times(0, \infty)$.

By virtue of (b) of Lemma 2.1, since $N \geq 3$, we can introduce the following three auxiliary functions $U=U(x), V=V(x)$ and $W=W(x)$ by

$$
\begin{array}{ll}
U(x)=\int_{0}^{\infty}(1-u(x, t)) d t & \text { for } x \in \Omega \backslash \bar{D} \\
V(x)=\int_{0}^{\infty}(1-u(x, t)) d t & \text { for } x \in D \\
W(x)=\int_{0}^{\infty}(1-u(x, t)) d t & \text { for } x \in \mathbb{R}^{N} \backslash \bar{\Omega} \tag{67}
\end{array}
$$

Then we observe that

$$
\begin{align*}
& -\Delta U=\frac{1}{\sigma_{s}} \text { in } \Omega \backslash \bar{D},-\Delta V=\frac{1}{\sigma_{c}} \text { in } D,-\Delta W=0 \text { in } \mathbb{R}^{N} \backslash \bar{\Omega},  \tag{68}\\
& U=V \text { and } \sigma_{s} \frac{\partial U}{\partial \nu}=\sigma_{c} \frac{\partial V}{\partial \nu} \text { on } \partial D  \tag{69}\\
& U=W \text { and } \sigma_{s} \frac{\partial U}{\partial \nu}=\sigma_{m} \frac{\partial W}{\partial \nu} \text { on } \partial \Omega,  \tag{70}\\
& \lim _{|x| \rightarrow \infty} W(x)=0 \tag{71}
\end{align*}
$$

where $\nu=\nu(x)$ denotes the outward unit normal vector to $\partial D$ at $x \in \partial D$ or to $\partial \Omega$ at $x \in \partial \Omega$ and (69) - (70) are the transmission conditions. Here we used (d) of Lemma 2.1 to obtain (71).

Let us follow the proof of Theorem 1.1. We first show that case (II) for assertion (b) of Theorem 1.3 does not occur. Set $\Omega=B_{\rho_{+}}\left(x_{0}\right) \backslash \overline{B_{\rho_{-}}\left(x_{0}\right)}$ for some numbers $\rho_{+}>\rho_{-}>0$. Since $u$ is radially symmetric with respect to $x_{0}$ in $x$ on $\left(\mathbb{R}^{N} \backslash \bar{D}\right) \times(0, \infty)$, we can obtain from (68)-(71) that for $r=\left|x-x_{0}\right| \geq 0$

$$
\begin{array}{cl}
U=c_{1} r^{2-N}-\frac{1}{2 N \sigma_{s}} r^{2}+c_{2} & \text { for } \rho_{-} \leq r \leq \rho_{+} \\
W=c_{3} r^{2-N} & \text { for } r \geq \rho_{+} \\
W=c_{4} & \text { for } 0 \leq r \leq \rho_{-}
\end{array}
$$

where $c_{1}, \ldots, c_{4}$ are some constants, since $\Omega \backslash \bar{D}$ is connected. Remark that $U$ can be extended as a radially symmetric function of $r$ in $\mathbb{R}^{N} \backslash\left\{x_{0}\right\}$. We observe that $c_{4}>0$ and $c_{3}>0$. Also it follows from (70) that $U^{\prime}\left(\rho_{-}\right)=0$ and $U^{\prime}\left(\rho_{+}\right)<0$, and hence

$$
c_{1}<0 \text { and } U^{\prime}<0 \text { on }\left(\rho_{-}, \rho_{+}\right] .
$$

Then the same argument as in the corresponding case in the proof of Theorem 1.1 works and a contradiction to the transmission condition (69) can be obtained. Thus case (II) for assertion (b) of Theorem 1.3 never occurs.

Let us proceed to case (I). Set $\Omega=B_{\rho}\left(x_{0}\right)$ for some number $\rho>0$. Since $u$ is radially symmetric with respect to $x_{0}$ in $x$ on $\left(\mathbb{R}^{N} \backslash \bar{D}\right) \times(0, \infty)$, we can obtain from (68)-(71) that for $r=\left|x-x_{0}\right| \geq 0$

$$
\begin{array}{cl}
U=c_{1} r^{2-N}-\frac{1}{2 N \sigma_{s}} r^{2}+c_{2} & \text { for } x \in \bar{\Omega} \backslash D, \\
W=c_{3} r^{2-N} & \text { for } r \geq \rho,
\end{array}
$$

where $c_{1}, c_{2}, c_{3}$ are some constants, since $\Omega \backslash \bar{D}$ is connected. Remark that $U$ can be extended as a radially symmetric function of $r$ in $\mathbb{R}^{N} \backslash\left\{x_{0}\right\}$. Therefore it follows from (70) that $U^{\prime}(\rho)<0$. As in the proof of Theorem 1.1, We distinguish the following three cases:
(i) $c_{1}=0$;
(ii) $c_{1}>0$;
(iii) $c_{1}<0$.

Because of the fact that $U^{\prime}(\rho)<0$, the same arguments as in the proof of Theorem 1.1 works to conclude that only case (i) occurs and $D$ must be a ball centered at $x_{0}$.

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# Quantitative uniqueness for zero-order perturbations of generalized Baouendi-Grushin operators 

Agnid Banerjee and Nicola Garofalo<br>Dedicated to Giovanni Alessandrini, on his 60-th birthday, with great affection and admiration


#### Abstract

Based on a variant of the frequency function approach of Almgren, we establish an optimal bound on the vanishing order of solutions to stationary Schrödinger equations associated to a class of subelliptic equations with variable coefficients whose model is the socalled Baouendi-Grushin operator. Such bound provides a quantitative form of strong unique continuation that can be thought of as an analogue of the recent results of Bakri and Zhu for the standard Laplacian.


Keywords: quantitative uniqueness, Baouendi-Grushin operators.
MS Classification 2010: 35, 35K.

## 1. Introduction

In this note we study quantitative uniqueness for zero-order perturbations of variable coefficient subelliptic equations whose "constant coefficient" model is the so called Baouendi-Grushin operator. Precisely, in $\mathbb{R}^{N}$, with $N=m+k$, we analyze equations of the form

$$
\begin{equation*}
\sum_{i=1}^{N} X_{i}\left(a_{i j}(z, t) X_{j} u\right)=V(z, t) u \tag{1}
\end{equation*}
$$

where $z \in \mathbb{R}^{m}, t \in \mathbb{R}^{k}$, and the vector fields $X_{1}, \ldots, X_{N}$ are given by

$$
\begin{equation*}
X_{i}=\partial_{z_{i}}, \quad i=1, \ldots m, \quad X_{m+j}=|z|^{\beta} \partial_{t_{j}}, \quad j=1, \ldots k, \quad \beta>0 \tag{2}
\end{equation*}
$$

Besides ellipticity, the $N \times N$ matrix-valued function $A(z, t)=\left[a_{i j}(z, t)\right]$ is requested to satisfy certain structural hypothesis that will be specified in (20), (21) in Section 2 below. These assumptions reduce to the standard Lipschitz continuity when the dimension $k=0$, or the parameter $\beta \rightarrow 0$. The assumptions on the potential function $V(z, t)$ are specified in (22) below. They
represent the counterpart, with respect to the non-isotropic dilations associated with the vector fields $X_{1}, \ldots, X_{N}$, of the requirements

$$
\begin{equation*}
|V(x)| \leq M, \quad|<x, D V(x)>| \leq M \tag{3}
\end{equation*}
$$

for the classical Schrödinger equation $\Delta u=V u$ in $\mathbb{R}^{n}$. To put this paper in the proper historical perspective we recall that for this operator, and under the hypothesis (3), quantitative unique continuation results akin to our have been recently obtained in [2], by Carleman estimates, and in [18], by means of a variant of Almgren's frequency function introduced in [17]. In these papers the authors established sharp estimates on the order of vanishing of solution to Schrödinger equations which generalized those in [6] and [7] for eigenvalues of the Laplacian on a compact manifold. Our results should be seen as a generalization of those in [2] and [18] to subelliptic equations such as (1) above. As the reader will realize such generalization is made possible by the combination of several quite non-trivial geometric facts that beautifully combine. Some of these facts are based on the previous work [13]. We also mention that the frequency approach in [17] and [18] has been recently extended in [3] to obtain sharp quantitative estimates at the boundary of Dini domains for more general elliptic equations with Lipschitz principal part.

When in (1) we take $\left[a_{i j}\right]=I_{N}$, the identity matrix in $\mathbb{R}^{N}$, then the operator in the left-hand side of (1) reduces to the well known Baouendi-Grushin operator

$$
\begin{equation*}
\mathcal{B}_{\beta} u=\sum_{i=1}^{N} X_{i}^{2} u=\Delta_{z} u+|z|^{2 \beta} \Delta_{t} u \tag{4}
\end{equation*}
$$

which is degenerate elliptic along the $k$-dimensional subspace $M=\{0\} \times \mathbb{R}^{k}$. We observe that $\mathcal{B}_{\beta}$ is not translation invariant in $\mathbb{R}^{N}$. However, it is invariant with respect to the translations along $M$. When $\beta=1$ the operator $\mathcal{B}_{\beta}$ is intimately connected to the sub-Laplacians in groups of Heisenberg type. In such Lie groups, in fact, in the exponential coordinates with respect to a fixed orthonormal basis of the Lie algebra the sub-Laplacian is given by

$$
\begin{equation*}
\Delta_{H}=\Delta_{z}+\frac{|z|^{2}}{4} \Delta_{t}+\sum_{\ell=1}^{k} \partial_{t_{\ell}} \sum_{i<j} b_{i j}^{\ell}\left(z_{i} \partial_{z_{j}}-z_{j} \partial_{z_{i}}\right) \tag{5}
\end{equation*}
$$

where $b_{i j}^{\ell}$ indicate the group constants. If $u$ is a solution of $\Delta_{H}$ that further annihilates the symplectic vector field $\sum_{\ell=1}^{k} \partial_{t_{\ell}} \sum_{i<j} b_{i j}^{\ell}\left(z_{i} \partial_{z_{j}}-z_{j} \partial_{z_{i}}\right)$, then we see that, in particular, $u$ solves (up to a normalization factor of 4) the operator $\mathcal{B}_{\beta}$ obtained by letting $\beta=1$ in (4) above.

We recall that a more general class of operators modeled on $\mathcal{B}_{\beta}$ was first introduced by Baouendi, who studied the Dirichlet problem in weighted Sobolev spaces in [4]. Subsequently, Grushin in [14, 15] studied the hypoelliptcity of the
operator $\mathcal{B}_{\beta}$ when $\beta \in \mathbb{N}$, and showed that this property is drastically affected by addition of lower order terms.

In the paper [10] the second named author introduced a frequency function associated with $\mathcal{B}_{\beta}$, and proved that such frequency is monotone nondecreasing on solutions of $\mathcal{B}_{\beta} u=0$. Such result, which generalized Almgren's in [1], was used to establish the strong unique continuation property for $\mathcal{B}_{\beta}$. The results in [10] were extended to more general equations of the form (1) by the second named author and Vassilev in [13], following the circle of ideas in the works $[11,12]$. We mention that a version of the Almgren type monotonicity formula for $\mathcal{B}_{\beta}$ played an extensive role also in the recent work [5] on the obstacle problem for the fractional Laplacian. Remarkably, the operator $\mathcal{B}_{\beta}$ also played an important role in the recent work [16] on the higher regularity of the free boundary in the classical Signorini problem.

We can now state our main result.
Theorem 1.1. Let $u$ be a solution to (1) in $B_{10}$ such that ( $a_{i j}$ ) satisfy (20), (21) and $V$ satisfy (22) below. We furthermore assume that $X_{i} X_{j} u \in L_{\text {loc }}^{2}\left(B_{10}\right)$ and $|u| \leq C_{0}$. Then, there exist universal $R_{1}>0, a \in(0,1 / 3)$, depending only on $\bar{R}, \Lambda$ in (20), (21), and constants $C_{1}, C_{2}$ depending on $m, k, \beta, \lambda, \Lambda, C_{0}$ and $\int_{B_{\frac{R_{1}}{3}}} u^{2} \psi$, such that for all $0<r<a R_{1}$ one has

$$
\begin{equation*}
\|u\|_{L^{\infty}\left(B_{r}\right)} \geq C_{1}\left(\frac{r}{R_{1}}\right)^{C_{2} \sqrt{K}} \tag{6}
\end{equation*}
$$

It is worth emphasizing that, when $k=0$, we have $N=m$ and then (14) below gives $\psi \equiv 1$. In such a case the constant $K$ in (22) below can be taken to be $\|V\|_{W^{1, \infty}}+1$. We thus see that Theorem 1.1 , when $A \equiv I_{N}$, reduces to the cited Euclidean result in [2] and [18]. Therefore, Theorem 1.1 can be thought of as a subelliptic generalization of this sharp quantitative uniqueness result for the standard Laplacian. We also would like to mention that, to the best of our knowledge, Theorem 1.1 is new even for $\mathcal{B}_{\beta} u=V u$ where $\mathcal{B}_{\beta}$ is as in (4).

The present paper is organized as follows. In Section 2 we introduce the basic notations and gather some crucial preliminary results from [10] and [13]. In Section 3 we establish a monotonicity theorem for a generalized frequency Such result plays a central role in this paper. In Section 4, we finally prove our main result, Theorem 1.1 above.

## 2. Notations and preliminary results

Henceforth in this paper we follow the notations adopted in [10] and [13], with one notable proviso: the parameter $\beta>0$ in (2), (4), etc. in this paper plays
the role of $\alpha>0$ in [10] and [13]. The reason for this is that we have reserved the greek letter $\alpha$ for the powers of the weight $\left(r^{2}-\rho\right)^{\alpha}$ in definitions (30), (31) and (32) below. Let $\left\{X_{i}\right\}$ for $i=1, \ldots N$ be defined as in (2). We denote an arbitrary point in $\mathbb{R}^{N}$ as $(z, t) \in \mathbb{R}^{m} \times \mathbb{R}^{k}$. Given a function $f$, we denote

$$
\begin{equation*}
X f=\left(X_{1} f, \ldots . X_{N} f\right), \quad|X f|^{2}=\sum_{i=1}^{N}\left(X_{i} f\right)^{2} \tag{7}
\end{equation*}
$$

respectively the intrinsic gradient and the square of its length. We recall from [10] that the following family of anisotropic dilations are associated with the vector fields in (2)

$$
\begin{equation*}
\delta_{a}(z, t)=\left(a z, a^{\beta+1} t\right), \quad a>0 \tag{8}
\end{equation*}
$$

Let

$$
\begin{equation*}
Q=m+(\beta+1) k \tag{9}
\end{equation*}
$$

Since denoting by $d z d t$ Lebesgue measure in $\mathbb{R}^{N}$ we have $d\left(\delta_{a}(z, t)\right)=a^{Q} d z d t$, the number $Q$ plays the role of a dimension in the analysis of the operator $\mathcal{B}_{\beta}$. For instance, one has the following remarkable fact (see [10]) that the fundamental solution $\Gamma$ of $\mathcal{B}_{\beta}$ with pole at the origin is given by the formula

$$
\Gamma(z, t)=\frac{C}{\rho(z, t)^{Q-2}}, \quad(z, t) \neq(0,0)
$$

where $\rho$ is the pseudo-gauge

$$
\begin{equation*}
\rho(z, t)=\left(|z|^{2(\beta+1)}+(\beta+1)^{2}|t|^{2}\right)^{\frac{1}{2(\beta+1)}} . \tag{10}
\end{equation*}
$$

We respectively denote by

$$
B_{r}=\left\{(z, t) \in \mathbb{R}^{N} \mid \rho(z, t)<r\right\}, \quad S_{r}=\left\{(z, t) \in \mathbb{R}^{N} \mid \rho(z, t)=r\right\}
$$

the gauge pseudo-ball and sphere centered at 0 with radius $r$. The infinitesimal generator of the family of dilations (8) is given by the vector field

$$
\begin{equation*}
Z=\sum_{i=1}^{m} z_{i} \partial_{z_{i}}+(\beta+1) \sum_{j=1}^{k} t_{j} \partial_{y_{j}} \tag{11}
\end{equation*}
$$

We note the important facts that

$$
\begin{equation*}
\operatorname{div} Z=Q, \quad\left[X_{i}, Z\right]=X_{i}, \quad i=1, \ldots, N \tag{12}
\end{equation*}
$$

A function $v$ is $\delta_{a}$-homogeneous of degree $\kappa$ if and only if $Z v=\kappa v$. Since $\rho$ in (10) is homogeneous of degree one, we have

$$
\begin{equation*}
Z \rho=\rho \tag{13}
\end{equation*}
$$

We also need the angle function $\psi$ introduced in [10]

$$
\begin{equation*}
\psi=|X \rho|^{2}=\frac{|z|^{2 \beta}}{\rho^{2 \beta}} \tag{14}
\end{equation*}
$$

The function $\psi$ vanishes on the characteristic manifold $M=\mathbb{R}^{n} \times\{0\}$ and clearly satisfies $0 \leq \psi \leq 1$. Since $\psi$ is homogeneous of degree zero with respect to (8), one has

$$
\begin{equation*}
Z \psi=0 . \tag{15}
\end{equation*}
$$

A first basic assumption on the matrix-valued function $A=\left[a_{i j}\right]$ is that it be symmetric and uniformly elliptic. I.e., $a_{i j}=a_{j i}, i, j=1, \ldots, N$, and there exists $\lambda>0$ such that for every $(z, t) \in \mathbb{R}^{N}$ and $\eta \in \mathbb{R}^{N}$ one has

$$
\begin{equation*}
\lambda|\eta|^{2} \leq<A(z, t) \eta, \eta>\leq \lambda^{-1}|\eta|^{2} \tag{16}
\end{equation*}
$$

On the potential $V$ we preliminarily assume that $V \in L_{l o c}^{\infty}\left(\mathbb{R}^{N}\right)$. With these hypothesis in place we can introduce the notion of weak solution of (1).

Definition 2.1. A weak solution to (1) in an open set $\Omega \subset \mathbb{R}^{N}$ is a function $u \in L_{l o c}^{2}(\Omega)$ such that the distributional horizontal gradient $X u \in L_{l o c}^{2}(\Omega)$, and for which the following equality holds for all $\varphi \in C_{0}^{\infty}(\Omega)$

$$
\begin{equation*}
\int_{\Omega}<A X u, X \varphi>=\int_{\Omega} V u \varphi . \tag{17}
\end{equation*}
$$

We note that when $A \equiv I_{N}$, and for a class of vector fields which are modeled on (2) above, in the pioneering paper [9] it was proved that a weak solution $u$ to (1) is locally Hölder continuous in $\Omega$ with respect to the control metric associated with the vector fields (2). In particular, it is continuous with respect to the Euclidean topology of $\mathbb{R}^{N}$. For the general situation of (17) the local Hölder continuity of weak solutions can be proved essentially following [9], but see also [8] where such result is discussed for more general equations in the case in which $V=0$ in (17) above. In this paper, however, all we need is the local boundedness of weak solutions of (17), and we do assume it a priori in Theorem 1.1 above, so we do not need to derive it.

Throughout the paper we assume that

$$
\begin{equation*}
A(0,0)=I_{N}, \tag{18}
\end{equation*}
$$

where $I_{N}$ indicates the identity matrix in $\mathbb{R}^{N}$. In order to state our main assumptions (H) on the matrix $A$ it will be useful to represent the latter in the following block form

$$
A=\left(\begin{array}{ll}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{array}\right)
$$

Here, the entries are respectively $m \times m, m \times k, k \times m$ and $k \times k$ matrices, and we assume that $A_{12}^{t}=A_{21}$. We shall denote by $B$ the matrix

$$
B=A-I_{N},
$$

and thus

$$
\begin{equation*}
B(0,0)=O_{N}, \tag{19}
\end{equation*}
$$

thanks to (18). The proof of Theorem 1.1 relies crucially on the following assumptions on the matrix $A$. These will be our main hypothesis and, without further mention, will be assumed to hold throughout the paper.

HYPOTHESIS. There exists a positive constant $\Lambda$ such that, for some $\bar{R}>0$, one has in $B_{\bar{R}}$ the following estimates

$$
\begin{gather*}
\left|b_{i j}\right|=\left|a_{i j}-\delta_{i j}\right| \leq \begin{cases}\Lambda \rho, & \text { for } 1 \leq i, j \leq m \\
\Lambda \psi^{\frac{1}{2}+\frac{1}{2 \beta}} \rho=\Lambda \frac{|z|^{\beta+1}}{\rho^{\beta}}, & \text { otherwise }\end{cases}  \tag{20}\\
\left|X_{k} b_{i j}\right|=\left|X_{k} a_{i j}\right| \leq \begin{cases}\Lambda, & \text { for } 1 \leq k \leq m, \\
\text { and } 1 \leq i, j \leq m \\
\Lambda \psi^{\frac{1}{2}}=\Lambda \frac{|z|^{\beta}}{\rho^{\beta}}, & \text { otherwise. }\end{cases} \tag{21}
\end{gather*}
$$

Remark 2.2. We note that in the situation when $k=0$ the above hypothesis coincide with the usual Lipschitz continuity at the origin of the coefficients $a_{i j}$.

Now we assume that $V$ in (1) satisfy the following hypothesis for some $K \geq 0$

$$
\begin{equation*}
|V| \leq K \psi, \quad|Z V| \leq K \psi \tag{22}
\end{equation*}
$$

where $\psi$ indicates the function introduced in (14) above. Without loss of generality we assume henceforth that $K \geq 1$.

We next collect several preliminary results established in [13] that will be important in the proof of Theorem 1.1. We consider the quantity

$$
\begin{equation*}
\mu=<A X \rho, X \rho> \tag{23}
\end{equation*}
$$

We note that, by the uniform ellipticity (16) of $A$, the function $\mu$ is comparable to $\psi$ defined in (14), in the sense that

$$
\begin{equation*}
\lambda \psi \leq \mu \leq \lambda^{-1} \psi . \tag{24}
\end{equation*}
$$

By (24) it is clear that, similarly to $\psi$, the function $\mu$ vanishes on the characteristic manifold $M=\left\{(0, t) \in \mathbb{R}^{N} \mid t \in \mathbb{R}^{k}\right\}$. The following vector field $F$
introduced in [13] will play an important role in this paper:

$$
\begin{equation*}
F=\frac{\rho}{\mu} \sum_{i, j=1}^{N} a_{i j} X_{i} \rho X_{j} \tag{25}
\end{equation*}
$$

It is clear that $F$ is singular on $M$. However, using (29) below and the assumptions (20), (21) on the matrix $A$, it was shown in [13] that $F$ can be extended to all of $\mathbb{R}^{N}$ to a continuous vector field that, near the characteristic manifold $M$, gives a small perturbation of the Euler vector field $Z$ in (11) above, but see also the Remark 2.3 below. We note from (25) that

$$
\begin{equation*}
F \rho=\rho . \tag{26}
\end{equation*}
$$

More in general, the action of $F$ on a function $u$ is given by

$$
\begin{equation*}
F u=\frac{\rho}{\mu}<A X \rho, X u> \tag{27}
\end{equation*}
$$

We also let

$$
\begin{equation*}
\sigma=<B X \rho, X \rho>=\mu-\psi \tag{28}
\end{equation*}
$$

As in (2.13) in [13], $F$ can be represented in the following way

$$
\begin{equation*}
F=Z-\frac{\sigma}{\mu} Z+\frac{\rho}{\mu} \sum_{i, j=1}^{N} b_{i j} X_{i} \rho X_{j} \tag{29}
\end{equation*}
$$

Remark 2.3. We emphasize that when $A(z, t) \equiv I_{N}$, then $B(z, t) \equiv 0_{N}$. In such case we immediately see from (29) that $F \equiv Z$.

Henceforth, for any two vector fields $U$ and $W,[U, W]=U W-W U$ denotes their commutator. In the next theorem we collect several important estimates that have been established in [10] and [13].

Theorem 2.4. There exists a constant $C(\beta, \lambda, \Lambda, N)>0$ such that for any function u one has:
(i) $|Q-\operatorname{div} F| \leq C \rho$;
(ii) $|F \mu| \leq C \rho \psi$;
(iii) $\operatorname{div}\left(\frac{\sigma Z}{\mu}\right) \leq C \rho$;
(iv) $\left|X_{i} \rho\right| \leq \psi^{1+\frac{1}{2 \beta}}, \quad i=1, \ldots, m, \quad\left|X_{m+j} \rho\right| \leq(\beta+1) \rho^{1 / 2}, \quad j=1, \ldots, k$;
(v) $|F-Z| \leq C \rho^{2}$;
(vi) $|<F A X u, X u>|\leq C \rho| X u|^{2}$;
(vii) $\left|\left[X_{i}, F\right] u-X_{i} u\right| \leq C \rho|X u|, \quad i=1, \ldots, N$;
(viii) $|\sigma| \leq C \rho \psi^{3 / 2+\frac{1}{2 \beta}}|X \sigma| \leq C \psi^{3 / 2}$;
(ix) $\left|\frac{b_{i j} X_{j} \rho X_{i}}{\mu}\right| \leq C|z|$;
(x) $\left|X_{i} \psi\right| \leq \frac{C \beta \psi}{|z|}, i=1, \ldots, m, \quad\left|X_{n+j} \psi\right| \leq \frac{C \beta \psi}{\rho}, j=1, \ldots, k$;
(xi) $\left|\frac{\sigma}{\mu}\right| \leq C \rho \psi,|Z \sigma| \leq C \rho \psi,\left|X_{k} \sigma\right| \leq C \psi^{3 / 2}$;
(xii) $\left|\left[X_{i},-\frac{\sigma Z}{\mu}\right] u\right| \leq C \rho|X u|, \quad$ (Lemma 2.7 in [13]);
(xiii) $\left|\left[X_{\ell}, \frac{\rho}{\mu} \sum_{i, j=1}^{N} \frac{b_{i j} X_{j} \rho}{X}{ }_{i}\right] u\right| \leq C \rho|X u|, \quad \ell=1, \ldots, N$.

The properties expressed in (i) and (vii) should be compared with (12) above.

## 3. Monotonicity of a generalized frequency

Henceforth, we denote by $u$ a weak solution to (1) in $B_{10}$. For the sake of brevity in all the integrals involved we will routinely omit the variable of integration $(z, t) \in \mathbb{R}^{N}$, as well as Lebesgue measure $d z d t$. When we say that a constant is universal, we mean that it depends exclusively on $m, k, \beta$, on the ellipticity bound $\lambda$ on $A(z, t)$, see (16) above, and on the Lipschitz bound $\Lambda$ in (20), (21). Likewise, we will say that $O(1), O(r)$, etc. are universal if $|O(1)| \leq C$, $|O(r)| \leq C r$, etc., with $C \geq 0$ universal.

For $0<r<\bar{R}$, where $\bar{R}$ is as in the hypotheses (20), (21) above, we define the generalized height function of $u$ in $B_{r}$ as follows

$$
\begin{equation*}
H(r)=\int_{B_{r}} u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu, \tag{30}
\end{equation*}
$$

where $\rho$ is the pseudo-gauge in (10) above, the function $\mu$ is defined in (23), and $\alpha>-1$ is going to be fixed later (precisely, in passing from (55) to (56) below). We also introduce the generalized energy of $u$ in $B_{r}$

$$
\begin{equation*}
I(r)=\int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}+\int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1} \tag{31}
\end{equation*}
$$

where, besides (16), the $N \times N$ matrix-valued function $A(z, t)$ fulfills the requirements (20), (21) above, whereas the potential $V(z, t)$ satisfies the hypothesis (22) above. We define the generalized frequency of $u$ as follows

$$
\begin{equation*}
N(r)=\frac{I(r)}{H(r)} \tag{32}
\end{equation*}
$$

The central result of this section is the following monotonicity result for the frequency $N(r)$.

THEOREM 3.1. There exists $R_{1}>0$, depending only on $\bar{R}$ and $\Lambda$ in (20), (21), such that the function

$$
r \rightarrow e^{C_{1} r}\left(N(r)+C_{2} K r^{2}\right)
$$

is monotone non-decreasing on the interval $\left(0, R_{1}\right)$. Here, $C_{1}$ and $C_{2}$ are two universal nonnegative numbers.

The proof of Theorem 3.1 will be divided into several steps. We begin by noting that although the gauge $\rho$ in (10) above is not smooth at the origin, nevertheless all subsequent calculations can be justified by integrating over the set $B_{r}-B_{\varepsilon}$, and then let $\varepsilon \rightarrow 0$. Moreover, by standard approximation type arguments as in [13] which crucially use the estimates in Theorem 2.4, we can assume that all the computations hereafter are classical. The initial step in the proof of Theorem 3.1 is the following result that provides a crucial alternative representation of the generalized energy (31).

Lemma 3.2. For every $0<r<\bar{R}$ one has

$$
\begin{equation*}
I(r)=2(\alpha+1) \int_{B_{r}} u F u\left(r^{2}-\rho^{2}\right)^{\alpha} \mu \tag{33}
\end{equation*}
$$

Proof. Using the definition of $F$, the divergence theorem and (1), we find

$$
\begin{array}{rl}
2(\alpha+1) \int_{B_{r}} u & u\left(r^{2}-\rho^{2}\right)^{\alpha} \mu=-\int_{B_{r}} u<A X u, X\left(r^{2}-\rho^{2}\right)^{\alpha+1}> \\
=\int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}+\int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1}
\end{array}
$$

which proves (33) above.
Lemma 3.3 (First variation formula for $H(r))$. There exists a universal $O(1)$ such that for every $r \in(0, \bar{R})$ one has

$$
\begin{equation*}
H^{\prime}(r)=\frac{2 \alpha+Q}{r} H(r)+O(1) H(r)+\frac{1}{(\alpha+1) r} I(r) \tag{34}
\end{equation*}
$$

Proof. Differentiating (30), and using the fact that $\left(r^{2}-\rho^{2}\right)^{\alpha}$ vanishes on $S_{r}$, we find that

$$
H^{\prime}(r)=2 \alpha r \int_{B_{r}} u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha-1} \mu
$$

Using the identity

$$
\left(r^{2}-\rho^{2}\right)^{\alpha-1}=\frac{1}{r^{2}}\left(r^{2}-\rho^{2}\right)^{\alpha}+\frac{\rho^{2}}{r^{2}}\left(r^{2}-\rho^{2}\right)^{\alpha-1}
$$

the latter equation can be rewritten as

$$
H^{\prime}(r)=\frac{2 \alpha}{r} H(r)+\frac{2 \alpha}{r} \int_{B_{r}} u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha-1} \rho^{2} \mu
$$

Recalling (26), we have

$$
H^{\prime}(r)=\frac{2 \alpha}{r} H(r)-\frac{1}{r} \int_{B_{r}} u^{2} F\left(r^{2}-\rho^{2}\right)^{\alpha} \mu .
$$

Integrating by parts, we obtain

$$
\begin{aligned}
H^{\prime}(r) & =\frac{2 \alpha}{r} H(r)+\frac{1}{r} \int_{B_{r}} \operatorname{div}\left(\mu u^{2} F\right)\left(r^{2}-\rho^{2}\right)^{\alpha} \\
& =\frac{2 \alpha}{r} H(r)+\frac{2}{r} \int_{B_{r}} u F u\left(r^{2}-\rho^{2}\right)^{\alpha} \mu \\
& +\frac{1}{r} \int_{B_{r}} u^{2} \operatorname{div}(F)\left(r^{2}-\rho^{2}\right)^{\alpha} \mu+\frac{1}{r} \int_{B_{r}} u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} F \mu .
\end{aligned}
$$

Using (i) in Theorem 2.4 to estimate the third term in the right-hand side, and (ii) to estimate the forth one, we obtain

$$
\begin{equation*}
H^{\prime}(r)=\frac{2 \alpha+Q}{r} H(r)+O(1) H(r)+\frac{2}{r} \int_{B_{r}} u F u\left(r^{2}-\rho^{2}\right)^{\alpha} \mu . \tag{35}
\end{equation*}
$$

Using (33) in (35) we conclude that (34) holds.
Our next result is a basic first variation formula of the generalized energy $I(r)$. Its proof will be quite laborious, and it displays many of the beautiful geometric properties of the Baouendi-Grushin vector fields (2).

Lemma 3.4 (First variation formula for $I(r)$ ). There exists a universal $O(1)$ and $R_{1}$ depending on $\bar{R}, \Lambda$ as in (20), (21) such that for every $r \in\left(0, R_{1}\right)$ one has

$$
\begin{align*}
I^{\prime}(r)= & \frac{2 \alpha+Q}{r} I(r) \\
& +\frac{4(\alpha+1)}{r} \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu+O(1) I(r)+O(1) K r H(r), \tag{36}
\end{align*}
$$

where $K \geq 1$ is the constant in (22).

Proof. Differentiating the expression (31) of $I(r)$ we obtain,

$$
\begin{aligned}
I^{\prime}(r)=2(\alpha+1) r \int_{B_{r}}<A X u, X u>\left(r^{2}-\right. & \left.\rho^{2}\right)^{\alpha} \\
& +2(\alpha+1) r \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} .
\end{aligned}
$$

Using the identity

$$
\left(r^{2}-\rho^{2}\right)^{\alpha}=\frac{1}{r^{2}}\left(r^{2}-\rho^{2}\right)^{\alpha+1}+\frac{\rho^{2}}{r^{2}}\left(r^{2}-\rho^{2}\right)^{\alpha}
$$

we find

$$
\begin{align*}
I^{\prime}(r)=\frac{2(\alpha+1)}{r} \int_{B_{r}}<A X u, X u> & \left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& +\frac{2(\alpha+1)}{r} \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha} \rho^{2} \\
& +2(\alpha+1) r \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \tag{37}
\end{align*}
$$

The second term in the right-hand side of (37) is dealt with as follows

$$
\begin{align*}
\frac{2(\alpha+1)}{r} \int_{B_{r}}<A X u, X u> & \left(r^{2}-\rho^{2}\right)^{\alpha} \rho^{2} \\
& =-\frac{1}{r} \int_{B_{r}}<A X u, X u>F\left(r^{2}-\rho^{2}\right)^{\alpha+1} \tag{38}
\end{align*}
$$

To compute the integral in the right-hand side of (38) we now use the following Rellich type identity in Lemma 2.11 in [13]:

$$
\begin{align*}
& \int_{\partial B_{r}}<A X u, X u><G, \nu>=2 \int_{\partial B_{r}} a_{i j} X_{i} u<X_{j}, \nu>G u \\
&-2 \int_{B_{r}} a_{i j}\left(\operatorname{div} X_{i}\right) X_{j} u G u-2 \int_{B_{r}} a_{i j} X_{i} u\left[X_{j}, G\right] u+\int_{B_{r}} \operatorname{div} G<A X u, X u> \\
&+\int_{B_{r}}<(G A) X u, X u>-2 \int_{B_{r}} G u X_{i}\left(a_{i j} X_{j} u\right), \tag{39}
\end{align*}
$$

where $G$ is a vector field, $G A$ is the matrix with coefficients $G a_{i j}, \nu$ denotes the outer unit normal to $B_{r}$, and the summation convention over repeated indices has been adopted. Since for the vector fields $X_{1}, \ldots, X_{N}$ in (2) above we have
$\operatorname{div} X_{i}=0$, if in (39) we take a vector field such that $G \equiv 0$ on $\partial B_{r}$, we obtain

$$
\begin{align*}
& \int_{B_{r}} \operatorname{div} G<A X u, X u>=2 \int_{B_{r}} a_{i j} X_{i} u\left[X_{j}, G\right] u \\
&-\int_{B_{r}}<(G A) X u, X u>+2 \int_{B_{r}} G u X_{i}\left(a_{i j} X_{j} u\right) . \tag{40}
\end{align*}
$$

In the identity (40) we now take $G=\left(r^{2}-\rho^{2}\right)^{\alpha+1} F$. We remark that, while in our situation the vector fields $X_{i}$ and $G$ are not smooth, one can nonetheless rigorously justify the implementation of (40) as in [13] by standard approximation arguments based on the key estimates in Theorem 2.4 above. Now we look at each individual term in (40). We first note that from (1) the last integral in the right-hand side of (40) equals $-2 \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1}$. For the left-hand side of (40) we have instead

$$
\begin{align*}
\int_{B_{r}} \operatorname{div} G<A X u, X u>=\int_{B_{r}} & \operatorname{div} F<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& +\int_{B_{r}}<A X u, X u>F\left(r^{2}-\rho^{2}\right)^{\alpha+1} . \tag{41}
\end{align*}
$$

Combining (40) and (41), we reach the conclusion

$$
\begin{align*}
&-\int_{B_{r}}<A X u, X u>F\left(r^{2}-\rho^{2}\right)^{\alpha+1}=\int_{B_{r}} \operatorname{div} F<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
&+\int_{B_{r}}<(F A) X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}-2 \int_{B_{r}} a_{i j} X_{i} u\left[X_{j}, G\right] u \\
&-2 \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1} \tag{42}
\end{align*}
$$

Using (i) in Theorem 2.4 we find

$$
\begin{array}{r}
\int_{B_{r}} \operatorname{div} F<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}=Q \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
+O(r) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} . \tag{43}
\end{array}
$$

Using (vi) in Theorem 2.4 we have

$$
\begin{align*}
& \int_{B_{r}}<(F A) X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
&=O(r) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \tag{44}
\end{align*}
$$

We next keep in mind that

$$
\left[X_{j}, G\right]=-2(\alpha+1) \rho\left(r^{2}-\rho^{2}\right)^{\alpha} X_{j} \rho F+\left(r^{2}-\rho^{2}\right)^{\alpha+1}\left[X_{j}, F\right]
$$

This gives

$$
\begin{aligned}
& a_{i j} X_{i} u\left[X_{j}, G\right] u=-2( \alpha+1)\left(r^{2}-\rho^{2}\right)^{\alpha} \rho<A X \rho, X u>F u \\
& \quad+\left(r^{2}-\rho^{2}\right)^{\alpha+1} a_{i j} X_{i} u\left[X_{i}, F\right] u \\
&=-2(\alpha+1)\left(r^{2}-\rho^{2}\right)^{\alpha}(F u)^{2} \mu \\
&+\left(r^{2}-\rho^{2}\right)^{\alpha+1} a_{i j} X_{i} u\left(\left[X_{j}, F\right] u-X_{j} u\right) \\
& \quad \quad+\left(r^{2}-\rho^{2}\right)^{\alpha+1}<A X u, X u>,
\end{aligned}
$$

where we have used the fact that

$$
\rho<A X \rho, X u>=\mu F u
$$

which follows from (27) above. We thus conclude that

$$
\begin{align*}
& -2 \int_{B_{r}} a_{i j} X_{i} u\left[X_{j}, G\right] u=-2 \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}  \tag{45}\\
& +O(r) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}+4(\alpha+1) \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu
\end{align*}
$$

where we have used the crucial estimate (vii) in Theorem 2.4 to control the integral

$$
\int_{B_{r}} a_{i j} X_{i} u\left(\left[X_{j}, F\right] u-X_{j} u\right)\left(r^{2}-\rho^{2}\right)^{\alpha+1}
$$

Using (43), (44) and (45) in (42), we conclude

$$
\begin{align*}
&-\int_{B_{r}}<A X u, X u>F\left(r^{2}-\rho^{2}\right)^{\alpha+1}=(Q-2) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
&+O(r) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}+4(\alpha+1) \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu \\
&-2 \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1} \tag{46}
\end{align*}
$$

With (46) in hands we now return to (38) to find

$$
\begin{align*}
& \quad \frac{2(\alpha+1)}{r} \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha} \rho^{2} \\
& =\frac{Q-2}{r} \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1}+O(1) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& \quad+\frac{4(\alpha+1)}{r} \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu-\frac{2}{r} \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1} . \tag{47}
\end{align*}
$$

The equation (47) is the central one in the proof of the first variation of the energy. Such equation allows us to unravel the second term in the right-hand side of (38) above, to which we now return to find

$$
\begin{aligned}
& I^{\prime}(r)=\frac{2 \alpha+Q}{r} \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& +\frac{4(\alpha+1)}{r} \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu+O(1) \int_{B_{r}}<A X u, X u>\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& \quad-\frac{2}{r} \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1}+2(\alpha+1) r \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha}
\end{aligned}
$$

Recalling the definition (31) of $I(r)$ we see that we can rewrite the latter equation as follows

$$
\begin{align*}
& I^{\prime}(r)=\frac{2 \alpha+Q}{r} I(r)-\frac{2 \alpha+Q}{r} \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& +\frac{4(\alpha+1)}{r} \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu+O(1) I(r)-O(1) \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& \quad-\frac{2}{r} \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1}+2(\alpha+1) r \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} . \tag{48}
\end{align*}
$$

An integration by parts now gives

$$
\begin{aligned}
& -\frac{2}{r} \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1}=-\frac{1}{r} \int_{B_{r}} F\left(u^{2} / 2\right) V\left(r^{2}-\rho^{2}\right)^{\alpha+1} \\
& =\frac{1}{2 r} \int_{B_{r}} u^{2} \operatorname{div}\left(\left(r^{2}-\rho^{2}\right)^{\alpha+1} V F\right)=\frac{1}{2 r} \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1} \operatorname{div} F \\
& \\
& \quad+\frac{1}{2 r} \int_{B_{r}} u^{2} F V\left(r^{2}-\rho^{2}\right)^{\alpha+1}-\frac{\alpha+1}{r} \int_{B_{r}} V u^{2} \rho F \rho\left(r^{2}-\rho^{2}\right)^{\alpha} .
\end{aligned}
$$

Since one has trivially $\left(r^{2}-\rho^{2}\right)^{\alpha+1} \leq r^{2}\left(r^{2}-\rho^{2}\right)^{\alpha}$, from the assumptions (22) above, from (16) and from (i) in Theorem 2.4, we find

$$
\left|\frac{1}{2 r} \int_{B_{r}} V u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha+1} \operatorname{div} F\right| \leq C K r \int_{B_{r}} u^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu=C K r H(r),
$$

where $C=C(\beta, m, k, \lambda)>0$ is universal. Similarly, one has

$$
\left|\frac{1}{2 r} \int_{B_{r}} u^{2} F V\left(r^{2}-\rho^{2}\right)^{\alpha+1}\right| \leq C K r H(r)
$$

Finally, since by (26) we have $F \rho=\rho$, we obtain

$$
\left|-\frac{\alpha+1}{r} \int_{B_{r}} V u^{2} \rho F \rho\left(r^{2}-\rho^{2}\right)^{\alpha}\right| \leq C K r H(r)
$$

In conclusion, we have for a universal $O(1)$

$$
-\frac{2}{r} \int_{B_{r}} F u V u\left(r^{2}-\rho^{2}\right)^{\alpha+1}=O(1) K r H(r) .
$$

The other terms containing $V$ in the right-hand side of (48) are estimated similarly. We thus conclude
$I^{\prime}(r)=\frac{2 \alpha+Q}{r} I(r)+\frac{4(\alpha+1)}{r} \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu+O(1) I(r)+O(1) K r H(r)$, which is (36).

We are now in a position to provide the
Proof of Theorem 3.1. Using (32), and the equations (34) in Lemma 3.3 and (36) in Lemma 3.4, we find for some universal $C_{1}, C_{3} \geq 0$,

$$
\begin{align*}
N^{\prime}(r)= & \frac{I^{\prime}(r)}{H(r)}-\frac{H^{\prime}(r)}{H(r)} N(r)=O(1) N(r)+O(1) K r \\
& +\left(4(\alpha+1) \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu-\frac{1}{(\alpha+1)} \frac{I(r)^{2}}{H(r)}\right) \frac{1}{r H(r)} \\
& \geq-C_{1} N(r)-C_{3} K r \tag{49}
\end{align*}
$$

where in the last inequality, we have used the fact that, in view of (33) in Lemma 3.2, the Cauchy-Schwarz inequality and the definition of $H(r)$, we have

$$
\begin{aligned}
I(r)^{2} & =4(\alpha+1)^{2}\left(\int_{B_{r}} u F u\left(r^{2}-\rho^{2}\right)^{\alpha} \mu\right)^{2} \\
& \leq 4(\alpha+1)^{2} H(r) \int_{B_{r}}(F u)^{2}\left(r^{2}-\rho^{2}\right)^{\alpha} \mu
\end{aligned}
$$

The inequality (49) implies that, with $C_{2}=C_{3} / 2$, the function

$$
r \rightarrow e^{C_{1} r}\left(N(r)+C_{2} K r^{2}\right)
$$

is nondecreasing.

## 4. Proof of Theorem 1.1

This final section is devoted to proving the main result in this paper, Theorem 1.1. We start from Theorem 3.1 which implies

$$
e^{C_{1} r}\left(N(r)+C_{2} K r^{2}\right) \leq e^{C_{1} s}\left(N(s)+C_{2} K s^{2}\right), \quad \text { for } 0<r<s<R_{1}
$$

Henceforth, without loss of generality we assume that $R_{1} \leq 1$. The latter monotonicity property implies, in particular, the existence of universal constants $C_{2}>0$ and $\bar{C} \geq 1$ such that

$$
\begin{equation*}
N(r) \leq \bar{C}\left(N(s)+C_{2} K\right), \quad \text { for } 0<r<s<R_{1} \tag{50}
\end{equation*}
$$

Returning to (34) in Lemma 3.3, we rewrite it in the following form

$$
\begin{equation*}
\frac{d}{d r} \log \left(\frac{H(r)}{r^{2 \alpha+Q}}\right)=O(1)+\frac{1}{(\alpha+1) r} N(r), \quad 0<r<R_{1} \tag{51}
\end{equation*}
$$

where $|O(1)| \leq C$, with $C$ universal.
Suppose now that $0<r_{1}<r_{2}<2 r_{2}<r_{3}<R_{1}$. Integrating (51) between $r_{1}$ and $2 r_{2}$, and using (50), we find

$$
\begin{equation*}
\frac{\log \frac{H\left(2 r_{2}\right)}{H\left(r_{1}\right)}-C}{\log \left(\frac{2 r_{2}}{r_{1}}\right)}-(2 \alpha+Q) \leq \frac{\bar{C}}{\alpha+1}\left(N\left(2 r_{2}\right)+C_{2} K\right) . \tag{52}
\end{equation*}
$$

Next, we integrate (51) between $2 r_{2}$ and $r_{3}$, and again using (50) we find

$$
\begin{equation*}
\frac{\bar{C}}{\alpha+1}\left(N\left(2 r_{2}\right)-\bar{C} C_{2} K\right) \leq \bar{C}^{2}\left[\frac{\log \frac{H\left(r_{3}\right)}{H\left(2 r_{2}\right)}+C}{\log \left(\frac{r_{3}}{2 r_{2}}\right)}-(2 \alpha+Q)\right] \tag{53}
\end{equation*}
$$

Combining (52) and (53) we conclude

$$
\frac{\log \frac{H\left(2 r_{2}\right)}{H\left(r_{1}\right)}-C}{\bar{C}^{2} \log \left(\frac{2 r_{2}}{r_{1}}\right)} \leq \frac{\log \frac{H\left(r_{3}\right)}{H\left(2 r_{2}\right)}+C}{\log \left(\frac{r_{3}}{2 r_{2}}\right)}+C^{\prime} \frac{K}{\alpha+1}-\left(1-\frac{1}{\bar{C}^{2}}\right)(2 \alpha+Q),
$$

where we have let $C^{\prime}=(\bar{C}+1) / \bar{C}$. Since $\bar{C} \geq 1$, if we now set

$$
\alpha_{0}=\log \left(\frac{r_{3}}{2 r_{2}}\right), \quad \beta_{0}=\bar{C}^{2} \log \left(\frac{2 r_{2}}{r_{1}}\right),
$$

then we obtain

$$
\begin{equation*}
\alpha_{0} \log \frac{H\left(2 r_{2}\right)}{H\left(r_{1}\right)} \leq \beta_{0} \log \frac{H\left(r_{3}\right)}{H\left(2 r_{2}\right)}+C\left(\alpha_{0}+\beta_{0}\right)+C^{\prime} \frac{K}{\alpha+1} \alpha_{0} \beta_{0} . \tag{54}
\end{equation*}
$$

Dividing both sides of the latter inequality by the quantity $\alpha_{0}+\beta_{0}$, we find

$$
\log \left(\frac{H\left(2 r_{2}\right)}{H\left(r_{1}\right)}\right)^{\frac{\alpha_{0}}{\alpha_{0}+\beta_{0}}} \leq \log \left(\frac{H\left(r_{3}\right)}{H\left(2 r_{2}\right)}\right)^{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}}+C+C^{\prime} \frac{K}{\alpha+1} \frac{\alpha \beta_{0}}{\alpha_{0}+\beta_{0}}
$$

This gives

$$
\begin{equation*}
\log H\left(2 r_{2}\right) \leq \log \left[H\left(r_{3}\right)^{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}} H\left(r_{1}\right)^{\frac{\alpha_{0}}{\alpha_{0}+\beta_{0}}}\right]+C+C^{\prime} \frac{K}{\alpha+1} \alpha_{0} \tag{55}
\end{equation*}
$$

where we have used the trivial estimate $\frac{\beta_{0}}{\alpha_{0}+\beta_{0}} \leq 1$. Exponentiating both sides of (55) and choosing $\alpha=\sqrt{K}$, we conclude

$$
\begin{equation*}
H\left(2 r_{2}\right) \leq e^{C}\left(\frac{r_{3}}{2 r_{2}}\right)^{C^{\prime} \sqrt{K}} H\left(r_{3}\right)^{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}} H\left(r_{1}\right)^{\frac{\alpha_{0}}{\alpha_{0}+\beta_{0}}} \tag{56}
\end{equation*}
$$

We now consider the quantity

$$
\begin{equation*}
h(r)=\int_{B_{r}} u^{2} \mu \tag{57}
\end{equation*}
$$

The following estimates are easily verified from (30) and (57)

$$
H(r) \leq r^{2 \alpha} h(r), \quad \text { and } \quad h(r) \leq \frac{H(s)}{\left(s^{2}-r^{2}\right)^{\alpha}}, 0<r<s<R_{1}
$$

From these estimates and (56) we obtain

$$
\begin{equation*}
h\left(r_{2}\right) \leq e^{C}\left(\frac{r_{3}}{2 r_{2}}\right)^{C^{\prime \prime} \sqrt{K}} h\left(r_{3}\right)^{\frac{\beta_{0}}{\alpha_{0}+\beta_{0}}} h\left(r_{1}\right)^{\frac{\alpha_{0}}{\alpha_{0}+\beta_{0}}}, \tag{58}
\end{equation*}
$$

for $r_{1}<r_{2}<2 r_{2}<r_{3}<R_{1}$. At this point, we take $r_{2}=\frac{R_{1}}{3}, r_{3}=R_{1}$. If

$$
C_{0}=\|u\|_{L^{\infty}\left(B_{R_{1}}\right)}^{2} \int_{B_{R_{1}}} \mu>0
$$

then we clearly have $h\left(R_{1}\right) \leq C_{0}$, and we conclude from (58) that

$$
\begin{equation*}
h\left(R_{1} / 3\right)^{1+\frac{\beta_{0}}{\alpha_{0}}} \leq e^{C\left(1+\frac{\beta_{0}}{\alpha_{0}}\right)}\left(\frac{3}{2}\right)^{C^{\prime \prime}\left(1+\frac{\beta_{0}}{\alpha_{0}}\right) \sqrt{K}} C_{0}^{\frac{\beta_{0}}{\alpha_{0}}} h(r), \quad 0<r<R_{1} / 3 \tag{59}
\end{equation*}
$$

If we set $A=e^{C}$ and $\gamma=\frac{\bar{C}^{2}}{\log (3 / 2)}$, then $q=\beta_{0} / \alpha_{0}=-\log \left(r / R_{1}\right)^{\gamma}-\bar{C}^{2}$, and recalling that $\bar{C} \geq 1$ we obtain from (59) for $0<r<R_{1} / 3$

$$
h(r) \geq C_{0}\left(\frac{h\left(R_{1} / 3\right)}{A C_{0}}\right)^{1+q}\left(\frac{3}{2}\right)^{-C^{\prime \prime}(1+q) \sqrt{K}} \geq C_{0} M_{0}^{1+q}\left(\frac{r}{R_{1}}\right)^{B \sqrt{K}}
$$

where we have let $M_{0}=\frac{h\left(R_{1} / 3\right)}{A C_{0}}$, and $B=\gamma C^{\prime \prime} \log (3 / 2)$. If $M_{0} \geq 1$ this estimate implies in a trivial way for $0<r<R_{1} / 3$

$$
h(r) \geq C_{0}\left(\frac{r}{R_{1}}\right)^{B \sqrt{K}}
$$

If instead $0<M_{0} \leq 1$, keeping in mind that $\bar{C} \geq 1$, with $B^{\prime}=\max \{B$, $\left.\gamma \log \left(1 / M_{0}\right)\right\}$ we obtain for $0<r<R_{1} / 3$

$$
h(r) \geq C_{0}\left(\frac{r}{R_{1}}\right)^{B \sqrt{K}+\gamma \log \left(1 / M_{0}\right)} \geq C_{0}\left(\frac{r}{R_{1}}\right)^{B^{\prime}(1+\sqrt{K})} \geq C_{0}\left(\frac{r}{R_{1}}\right)^{2 B^{\prime} \sqrt{K}}
$$

where the last inequality follows by remembering that $K \geq 1$. In either case, the desired conclusion of Theorem 1.1 follows by noticing that $h(r) \leq$ $\|u\|_{L^{\infty}\left(B_{r}\right)}^{2} \int_{B_{r}} \mu$, and that $\int_{B_{r}} \mu \leq \lambda^{-1} \int_{B_{r}} \psi=\lambda^{-1} \omega r^{Q}$, where we have let $\omega=\int_{B_{1}} \psi$. In fact, we would find

$$
\|u\|_{L^{\infty}\left(B_{r}\right)} \geq C_{3}\left(\frac{r}{R_{1}}\right)^{C_{4} \sqrt{K}}
$$

with $C_{3}=C_{0} \sqrt{\frac{\lambda}{\omega R_{1}^{Q}}}$ and $C_{4}=2 B^{\prime}$. This finishes the proof of Theorem 1.1.

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# Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. The complex case 

Elena Beretta and Elisa Francini

Dedicated to Giovanni Alessandrini on the occasion of his 60th birthday


#### Abstract

We consider a conducting body with complex valued admittivity containing a finite number of well separated thin inclusions. We derive an asymptotic formula for the boundary values of the potential in terms of the width of the inclusions.


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## 1. Introduction

Let $\Omega \subset \mathbb{R}^{2}$ be a smooth bounded domain representing the region occupied by a conducting body and consider, at a fixed frequency $\omega$, the complex valued admittivity background

$$
\gamma_{0}=\sigma_{0}+i \omega \varepsilon_{0} \text { in } \Omega
$$

where $\sigma_{0}$ and $\varepsilon_{0}$ are real valued functions representing the electrical conductivity and permittivity, respectively.

Let $\Sigma_{j} \subset \subset \Omega$, for $j=1, \ldots, N$ be a collection of simple, regular curves and consider, for $\epsilon$ sufficiently small, a neighborhood of $\Sigma_{j}$ given by

$$
D_{\epsilon}^{j}=\left\{x \in \Omega: \operatorname{dist}\left(x, \Sigma_{j}\right)<\epsilon\right\},
$$

representing a thin inhomogeneity of admittivity

$$
\gamma_{j}=\sigma_{j}+i \omega \varepsilon_{j}
$$

Let $\psi \in H^{1 / 2}(\partial \Omega)$ represents a complex valued boundary current and let
$u_{0}$ be the background potential which satisfies

$$
\left\{\begin{array}{rll}
\operatorname{div}\left(\gamma_{0} \nabla u_{0}\right)=0 & \text { in } \quad \Omega \\
\gamma_{0} \frac{\partial u_{0}}{\partial \nu}=\psi & \text { on } \quad \partial \Omega,
\end{array}\right.
$$

where $\nu$ is the unit outer normal to $\partial \Omega$.
Let

$$
\gamma_{\epsilon}=\gamma_{0}+\sum_{j=1}^{N}\left(\gamma_{j}-\gamma_{0}\right) \chi_{D_{\epsilon}^{j}}
$$

and consider the perturbed complex-valued potential $u_{\epsilon}$ solution to

$$
\left\{\begin{array}{rll}
\operatorname{div}\left(\gamma_{\epsilon} \nabla u_{\epsilon}\right)=0 & \text { in } & \Omega \\
\gamma_{\epsilon} \frac{\partial u_{\epsilon}}{\partial \nu}=\psi & \text { on } & \partial \Omega
\end{array}\right.
$$

The main goal of the paper is to obtain an asymptotic expansion for the boundary values $\left(u_{\epsilon}-u_{0}\right)_{\mid \partial \Omega}$ as $\epsilon \rightarrow 0$.

The formula we derive is analogue to the one obtained in [3] in the case of constant real valued conductivities $\sigma_{0}$ and $\sigma_{1}(\omega=0)$.

More precisely, we show that for $y \in \partial \Omega$ and $\epsilon \rightarrow 0$,

$$
\left(u_{\epsilon}-u_{0}\right)(y)=\epsilon v(y)+o(\epsilon) .
$$

where

$$
v(y)=\sum_{j=1}^{N} 2 \int_{\Sigma_{j}}\left(\gamma_{0}-\gamma_{j}\right)(x) \mathbb{M}^{j}(x) \nabla u_{0}(x) \nabla_{x} N(x, y) d \sigma_{x}
$$

Here $N(x, y)$ is the Neumann function corresponding to the operator $\operatorname{div}\left(\gamma_{0} \nabla \cdot\right)$ and $\mathbb{M}^{j}$ is a two by two matrix with complex valued entries.

It is well known that this type of expansion can be used in order to solve the inverse problem of detecting the curves $\Sigma_{j}, j=1, \ldots, N$ from boundary measurements. In fact, in [1] the authors show that for the conductivity equation, it is possible to detect finitely many segments from knowledge on the boundary of the first order term $v$ appearing in the expansion. Moreover they show the continuous dependence of the segments from the boundary measurement $v$ is Lipschitz stable. A similar result has been obtained in the case of the system of linearized elasticity, for the case $N=1$, in [2].

## 2. Main assumptions and results

For $j=1, \ldots, N$, let $\Sigma_{j}$ be a simple, regular $C^{2, \alpha}$ curve with $\alpha \in(0,1)$ and assume there exists a contant $K>1$ such that, in a neighborhood of radius
$K^{-1}$ of each point in $\Sigma_{j}$, the curve is the graph of a $C^{2, \alpha}$ function and

$$
\begin{gather*}
\left\|\Sigma_{j}\right\|_{C^{2, \alpha}} \leq K, \quad \operatorname{dist}\left(\Sigma_{j}, \partial \Omega\right) \geq K^{-1} \\
K^{-1} \leq \mathcal{L}\left(\Sigma_{j}\right) \leq K, \quad \operatorname{dist}\left(\Sigma_{j}, \Sigma_{k}\right) \geq K^{-1} \text { if } j \neq k, \tag{1}
\end{gather*}
$$

where $\mathcal{L}$ denotes the length.
On each curve $\Sigma_{j}$ we fix an continuous orthonormal system $\left(n_{j}(x), t_{j}(x)\right)$ such that $n_{j}(x)$ is a normal direction to $\Sigma_{j}$ at its point $x$ and $\tau_{j}(x)$ is a tangent direction.

Assume $\gamma_{0}, \gamma_{j}: \Omega \rightarrow \mathbb{C}$ such that $\gamma_{0} \in C^{1, \alpha}(\Omega), \gamma_{j} \in C^{\alpha}(\Omega)$ with

$$
\begin{equation*}
\left\|\gamma_{0}\right\|_{C^{1, \alpha}(\Omega)},\left\|\gamma_{j}\right\|_{C^{\alpha}(\Omega)} \leq K \tag{2}
\end{equation*}
$$

and, furthermore, assume there exists $c_{0}>0$ such that

$$
\begin{equation*}
\sigma_{j} \geq c_{0}, \text { for } j=0,1, \ldots, N \tag{3}
\end{equation*}
$$

Consider finally a complex valued flux $\psi \in H^{-1 / 2}(\partial \Omega)$ satisfying the compatibility condition

$$
\begin{equation*}
\int_{\partial \Omega} \psi=0 . \tag{4}
\end{equation*}
$$

Then, under the above assumptions, there exist unique weak solutions $u_{0}$ and $u_{\epsilon}$ in $H^{1}(\Omega)$ to

$$
\left\{\begin{align*}
& \operatorname{div}\left(\gamma_{0} \nabla u_{0}\right)=0 \quad \text { in } \quad \Omega,  \tag{5}\\
& \gamma_{0} \frac{\partial u_{0}}{\partial \nu}=\psi \quad \text { on } \quad \partial \Omega, \\
& \int_{\partial \Omega} u_{0}=0,
\end{align*}\right.
$$

and

$$
\left\{\begin{array}{rll}
\operatorname{div}\left(\gamma_{\epsilon} \nabla u_{\epsilon}\right)=0 & \text { in } \quad \Omega,  \tag{6}\\
\gamma_{\epsilon} \frac{\partial u_{\epsilon}}{\partial \nu}=\psi & \text { on } & \partial \Omega, \\
\int_{\partial \Omega} u_{\epsilon}=0 . & &
\end{array}\right.
$$

We also introduce the Neumann function $N$ solution to

$$
\left\{\begin{array}{rll}
\operatorname{div}\left(\gamma_{0} \nabla N(\cdot, y)\right)=\delta_{y} & \text { in } & \Omega,  \tag{7}\\
\gamma_{0} \frac{\partial N(\cdot, y)}{\partial \nu}=\frac{1}{\mathcal{L}(\partial \Omega)} & \text { on } & \partial \Omega, \\
\int_{\partial \Omega} N(\cdot, y)=0, &
\end{array}\right.
$$

It is well known that under assumptions (2) and (3) there exists a unique solution to (7) (see [5]).

We are now ready to state our main result.

Theorem 2.1. Let $\Omega \subset \mathbb{R}^{2}$ be bounded smooth domain and $\{\Sigma\}_{j=1}^{N} \subset \subset \Omega$ a set of curves satisfying (1), let $\gamma_{0}$ and $\gamma_{j}($ for $j=1, \ldots, N$ ) be admittivities satisfying (2) and (3) and let $u_{0}$ and $u_{\epsilon}$ be solutions to (5) and (6), respectively. Then, for $y \in \partial \Omega$ and $\epsilon \rightarrow 0$,

$$
\left(u_{\epsilon}-u_{0}\right)(y)=2 \epsilon \sum_{j=1}^{N} \int_{\Sigma_{j}}\left(\gamma_{0}(x)-\gamma_{j}(x)\right) \mathbb{M}^{j}(x) \nabla u_{0}(x) \cdot \nabla N(x, y) d \sigma_{x}+o(\epsilon)
$$

where

$$
\mathbb{M}^{j}(x)=\frac{\gamma_{0}(x)}{\gamma_{j}(x)} n_{j}(x) \otimes n_{j}(x)+\tau_{j}(x) \otimes \tau_{j}(x)
$$

## 3. Proof of Theorem 2.1

We will perform the proof in the case $N=1$. Since the curves are well separated one from each other, the same argument will work for the case of multiple inclusions.

A complex valued equation as

$$
\operatorname{div}(\gamma \nabla u)=0
$$

can be interpreted as a two by two system for real valued functions. In fact, denoting by

$$
u^{1}=\Re u \text { and } u^{2}=\Im u
$$

we have that the function $u=\left(u^{1}, u^{2}\right): \Omega \rightarrow \mathbb{R}^{2}$ satisfies the system

$$
\frac{\partial}{\partial x_{k}}\left(a_{i j}^{h k} \frac{\partial u^{j}}{\partial x_{k}}\right)=0 \text { for } i=1,2
$$

where, for $i, j, h, k=1,2$,

$$
a_{i j}^{h k}=\delta_{h k} \delta_{i j} \Re \gamma-\delta_{h k}\left(\delta_{i 1} \delta_{j 2}-\delta_{i 2} \delta_{j 1}\right) \Im \gamma .
$$

If

$$
\Re \gamma \geq c_{0}>0
$$

then

$$
a_{i j}^{h k} \xi_{h}^{i} \xi_{k}^{j} \geq c_{0}|\xi|^{2}
$$

which corresponds to strong ellipticity. For this reason we can apply to our equations the results that hold for strongly elliptic systems.

We first establish some key energy estimates.
Lemma 3.1. There exists a constant $C=C\left(K, c_{o}, \Omega\right)$ such that

$$
\left\|u_{\epsilon}-u_{0}\right\|_{H^{1}(\Omega)} \leq C\left|D_{\epsilon}^{1}\right|^{1 / 2}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
$$

Proof. Since $u_{0}$ and $u_{\epsilon}$ are solutions to (5) and (6), then $w_{\epsilon}=u_{\epsilon}-u_{0}$ is weak solution to

$$
\left\{\begin{aligned}
\operatorname{div}\left(\gamma_{\epsilon} \nabla w_{\epsilon}\right) & =\operatorname{div}\left(\left(\gamma_{0}-\gamma_{1}\right) \chi_{D_{\epsilon}^{1}} \nabla u_{0}\right) \text { in } \Omega \\
\gamma_{\epsilon} \frac{\partial w_{\epsilon}}{\partial \nu} & =0 \text { on } \partial \Omega \\
\int_{\partial \Omega} u_{\epsilon} & =0
\end{aligned}\right.
$$

Hence, for every $\phi \in H^{1}(\Omega)$,

$$
\begin{equation*}
\int_{\Omega} \gamma_{\epsilon} \nabla w_{\epsilon} \cdot \nabla \phi=\int_{\Omega}\left(\gamma_{0}-\gamma_{1}\right) \chi_{D_{\epsilon}^{1}} \nabla u_{0} \cdot \nabla \phi . \tag{8}
\end{equation*}
$$

By choosing $\phi=\bar{w}_{\epsilon} \in H^{1}(\Omega)$, we get

$$
\int_{\Omega} \gamma_{\epsilon}\left|\nabla w_{\epsilon}\right|^{2}=\int_{\Omega}\left(\gamma_{0}-\gamma_{1}\right) \chi_{D_{\epsilon}^{1}} \nabla u_{0} \cdot \nabla \bar{w}_{\epsilon}
$$

Now, by (3), we have

$$
\left\|\nabla w_{\epsilon}\right\|_{L^{2}(\Omega)} \leq C\left|D_{\epsilon}^{1}\right|^{1 / 2}\left\|\gamma_{0}-\gamma_{1}\right\|_{L^{\infty}(\Omega)} \sup _{D_{\epsilon}^{1}}\left|\nabla u_{0}\right| .
$$

By interior regularity results (see [4, Theorem2.1, Chapter 2]),

$$
\sup _{D_{\epsilon}^{1}}\left|\nabla u_{0}\right| \leq C\|\psi\|_{H^{-1 / 2}(\partial \Omega)}
$$

so that

$$
\left\|\nabla w_{\epsilon}\right\|_{L^{2}(\Omega)} \leq C\left|D_{\epsilon}^{1}\right|^{1 / 2}\|\psi\|_{H^{-1 / 2}(\partial \Omega)}
$$

where $C=C\left(K, c_{0}\right)$.
Finally, since

$$
\int_{\partial \Omega} w_{\epsilon}=0
$$

by Poincaré inequality we have

$$
\left\|w_{\epsilon}\right\|_{L^{2}(\Omega)} \leq C\left\|\nabla w_{\epsilon}\right\|_{L^{2}(\Omega)}
$$

with $C=C(\Omega)$ and we obtain

$$
\left\|u_{\epsilon}-u_{0}\right\|_{H^{1}(\Omega)} \leq\left|D_{\epsilon}^{1}\right|^{1 / 2}\|\psi\|_{H^{1 / 2}(\partial \Omega)}
$$

which ends the proof.
We will also make use of some key regularity results for elliptic systems with discontinuous coefficients due to [6] (that extend the one established in [7] for
scalar elliptic equations). We state here a simplified version of Proposition 5.1 of [6].

Let $c$ and $M$ be two positive constants with $M>2 K$ and denote by $Q_{c, M}$ the set of points $x \in \Omega$ such that $\operatorname{dist}(x, \partial \Omega)>M^{-1}$ and such that there is a square of size $c$ centered at $x$ that intersects $\partial D_{\epsilon}^{1}$ in at most two cartesian curves whose $C^{1, \alpha}$ norms are bounded by $M$, i.e. there exists a coordinate system at $x$ such that $\partial D_{\epsilon}^{1} \cap[-c, c]^{2}$ consists in graphs of at most two functions $h_{-}<h_{+}$ with $\left\|h_{ \pm}\right\|_{C^{1, \alpha}} \leq M$. Let us denote by $\Omega_{M}=\left\{x \in \Omega: \operatorname{dist}(x, \partial \Omega)>\frac{1}{2 M}\right\}$.

Lemma 3.2. Let $\beta \in(0,1 / 4)$ and $M>2 K$. there exists a constant $C$ depending on $\alpha, K, c_{0}$ and $M$ such that if $\epsilon \in\left(0, \frac{1}{3 K}\right), 0<c<\frac{1}{6 K}$ and $u_{\epsilon} \in H^{1}(\Omega)$ is a solution of

$$
\operatorname{div}\left(\gamma_{\epsilon} \nabla u\right)=0 \text { in } \Omega,
$$

then

$$
\begin{equation*}
\left\|u_{\epsilon}\right\|_{C^{1, \beta}\left(Q_{c, M} \cap \bar{D}_{\epsilon}^{1}\right)} \leq \frac{C}{c^{1+\beta}}\left\|u_{\epsilon}\right\|_{L^{2}(\Omega)} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|u_{\epsilon}\right\|_{C^{1, \beta}\left(Q_{c, M} \backslash D_{\epsilon}^{1}\right)} \leq \frac{C}{c^{1+\beta}}\left\|u_{\epsilon}\right\|_{L^{2}(\Omega)} . \tag{10}
\end{equation*}
$$

Lemma 3.3. There exists $\eta>0$ such that, if $u \in H^{1}(\Omega)$ is a solution to the complex valued equation

$$
\operatorname{div}(\gamma \nabla u)=f \text { in } \Omega
$$

where $\gamma: \Omega \rightarrow \mathbb{C}, \gamma \in L^{\infty}(\Omega)$ such that

$$
\Re \gamma \geq c_{0}>0
$$

and $f \in H^{-1,2+\eta}(\Omega)$, then $u \in H_{l o c}^{1,2+\eta}(\Omega)$ and, given $B_{\rho}$ and $B_{2 \rho}$ concentric disks contained in $\Omega$,

$$
\|\nabla u\|_{L^{2+\eta}\left(B_{\rho}\right)} \leq C\left(\|f\|_{H^{-1,2+\eta}\left(B_{2 \rho}\right)}+\rho^{\frac{2}{2+\eta}-1}\|\nabla u\|_{L^{2}\left(B_{2 \rho}\right)}\right) .
$$

For the proof see [4, Chapter 2, Section 10].
Proof of Theorem 2.1. Take $y \in \partial \Omega$. Then, by the definition of the Neumann function, it is easy to see that

$$
\left(u_{\epsilon}-u_{0}\right)(y)=\int_{D_{\epsilon}^{1}}\left(\gamma_{0}-\gamma_{1}\right) \nabla u_{\epsilon} \cdot \nabla N(\cdot, y)
$$

We prove the theorem in the more interesting and complicated case when $\Sigma_{1}$ is an open curve. In fact, in this case, $\partial D_{\epsilon}^{1}$ has derivatives (near the endpoints of $\Sigma_{1}$ ) that degenerate as $\epsilon$ tends to zero. This implies that the regularity estimates of Lemma 3.2 cannot be applied near the endpoints of $\Sigma_{1}$.

Let $P_{1}$ and $Q_{1}$ be the endpoints of $\Sigma_{1}$, let $\theta \in(0,1)$ to be chosen later, and define

$$
D_{\epsilon}^{1, \theta}=\left\{x+\mu n(x): x \in \Sigma_{1}, \operatorname{dist}\left(x, P_{1} \cup Q_{1}\right)>\epsilon^{\theta}, \mu \in(\epsilon, \epsilon)\right\}
$$

It is easy to see that there exists a constant $M>2 K$ and depending only on $K$, such that

$$
D_{\epsilon}^{1, \theta} \subset Q_{\frac{\epsilon \beta}{4}, M} \cap D_{\epsilon}^{1}
$$

An application of Lemma 3.2 thus gives

$$
\begin{equation*}
\left\|\nabla u_{\epsilon}\right\|_{C^{\beta}\left(\overline{D_{\epsilon}^{1, \theta}}\right)} \leq C \epsilon^{-\theta(1+\beta)}\left\|u_{\epsilon}\right\|_{L^{2}(\Omega)} \leq C \epsilon^{-\theta(1+\beta)}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \tag{11}
\end{equation*}
$$

where $C$ does not depend on $\epsilon$.
Then,

$$
\begin{aligned}
& \int_{D_{\epsilon}^{1}}\left(\gamma_{0}-\gamma_{1}\right)(x) \nabla u_{\epsilon}(x) \cdot \nabla N(x, y) d x \\
& \quad=\int_{D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right)(x) \nabla u_{\epsilon}(x) \cdot \nabla N(x, y) d x \\
& \quad \quad \quad \int_{D_{\epsilon}^{1} \backslash D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right)(x) \nabla u_{\epsilon}(x) \cdot \nabla N(x, y) d x:=I_{1}+I_{2}
\end{aligned}
$$

Let us estimate $I_{2}$ first.

$$
\begin{aligned}
\left|I_{2}\right| \leq & \left|\int_{D_{\epsilon}^{1} \backslash D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right) \nabla\left(u_{\epsilon}-u_{0}\right) \cdot \nabla N(\cdot, y)\right| \\
& +\left|\int_{D_{\epsilon}^{1} \backslash D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right) \nabla u_{0} \cdot \nabla N(\cdot, y)\right|
\end{aligned}
$$

Observe that, since $\gamma_{0} \in C^{1, \alpha}(\Omega)$, by interior regularity results ([4, Theorem 2.1, Chapter 2], by [5] and by (2) we get

$$
\begin{gather*}
\left\|\nabla u_{0}\right\|_{L^{\infty}\left(D_{\epsilon}^{1}\right)} \leq C\|\psi\|_{H^{-1 / 2}(\partial \Omega)}  \tag{12}\\
|\nabla N(x, y)| \leq \frac{C}{|x-y|}
\end{gather*}
$$

and, since $y \in \partial \Omega$

$$
\begin{equation*}
\|\nabla N(\cdot, y)\|_{L^{\infty}\left(D_{\epsilon}^{1}\right)} \leq C \tag{13}
\end{equation*}
$$

Hence, by (12), (13) and Lemma 3.1,

$$
\left|I_{2}\right| \leq C\left|D_{\epsilon}^{1} \backslash D_{\epsilon}^{1, \theta}\right|\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \leq C \epsilon^{1+\theta}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
$$

We now define

$$
\Sigma_{\eta}^{\theta}=\left\{x+\eta n_{1}(x): x \in \Sigma_{1}, \operatorname{dist}\left(x, P_{1} \cup Q_{1}\right)>\epsilon^{\theta}\right\} .
$$

Due to the regularity of $\Sigma_{1}$, if we denote by $d \sigma_{x}^{\eta}$ the arclength measure on $\Sigma_{\eta}^{\theta}$ and by $d \sigma_{x}$ the arclength measure on $\Sigma_{1}$, we have

$$
d \sigma_{x}^{\eta}=(1+O(\eta)) d \sigma_{x}
$$

For every point $x+\eta n(x) \in D_{\epsilon}^{1, \theta}$, let $x_{\epsilon}=x+\epsilon n(x)$. By (11),

$$
\begin{aligned}
\left|\nabla u_{\epsilon}(x+\eta n(x))-\nabla u_{\epsilon}\left(x_{\epsilon}\right)\right| & \leq C\left|x+\eta n(x)-x_{\epsilon}\right|^{\beta} \epsilon^{-\theta(1+\beta)}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \\
& \leq C \epsilon^{\beta-\theta(1+\beta)}\|\psi\|_{H^{-1 / 2}(\partial \Omega)}
\end{aligned}
$$

and, also,

$$
\left|\left(\gamma_{1}-\gamma_{0}\right)(x+\eta n(x))-\left(\gamma_{1}-\gamma_{0}\right)\left(x_{\epsilon}\right)\right| \leq C \epsilon^{\alpha}
$$

so that

$$
\int_{D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right) \nabla u_{\epsilon} \nabla_{x} N=2 \epsilon \int_{\Sigma_{\epsilon}^{\theta}}\left(\gamma_{0}-\gamma_{1}\right) \nabla u_{\epsilon}^{i} \nabla_{x} N+o(\epsilon),
$$

where we set

$$
u_{\epsilon}^{i}=u_{\left.\epsilon\right|_{D_{\epsilon}^{1}} ^{1}}, \quad u_{\epsilon}^{e}=u_{\left.\epsilon\right|_{\Omega \backslash \bar{D}_{\epsilon}^{1}}} .
$$

We now use the transmission conditions

$$
\begin{aligned}
u_{\epsilon}^{i} & =u_{\epsilon}^{e}, \\
\gamma_{1} \frac{\partial u_{\epsilon}^{i}}{\partial n} & =\gamma_{0} \frac{\partial u_{\epsilon}^{e}}{\partial n}
\end{aligned}
$$

that are satisfied on $\partial D_{\epsilon}^{1, \theta}$ pointwise, in order to obtain, finally,

$$
\begin{aligned}
& \int_{D_{\epsilon}^{1, \theta}}\left(\gamma_{0}-\gamma_{1}\right) \nabla u_{\epsilon}^{i} \nabla N \\
& \quad=2 \epsilon \int_{\Sigma_{\epsilon}^{\theta}}\left(\gamma_{0}-\gamma_{1}\right)\left\{\frac{\gamma_{0}}{\gamma_{1}} \frac{\partial u_{\epsilon}^{e}}{\partial n} \frac{\partial N}{\partial n_{x}}+\frac{\partial u_{\epsilon}^{e}}{\partial \tau} \frac{\partial N}{\partial \tau_{x}}\right\}+o(\epsilon) .
\end{aligned}
$$

Assume that

$$
\begin{equation*}
\left\|\nabla u_{\epsilon}^{e}-\nabla u_{0}^{e}\right\|_{L^{\infty}\left(\Sigma_{\epsilon}^{\theta}\right)} \leq C \epsilon^{\theta_{1}}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \tag{14}
\end{equation*}
$$

for some $\theta_{1}>0$. Then

$$
\begin{aligned}
\int_{D_{\epsilon}^{1, \theta}} & \left(\gamma_{0}-\gamma_{1}\right)(x) \nabla u_{\epsilon}^{i}(x) \nabla N(x, y) d x \\
& =2 \epsilon \int_{\Sigma_{\epsilon}^{\theta}}\left(\gamma_{0}-\gamma_{1}\right)\left\{\frac{\gamma_{0}}{\gamma_{1}} \frac{\partial u_{0}^{e}}{\partial n_{1}} \frac{\partial N}{\partial n_{1}}+\frac{\partial u_{0}^{e}}{\partial \tau_{1}} \frac{\partial N}{\partial \tau_{1}}\right\} d \sigma_{x}^{\epsilon}+o(\epsilon) \\
& =2 \epsilon \int_{\Sigma_{1}}\left(\gamma_{0}-\gamma_{1}\right) \mathbb{M}^{1}(x) \nabla u_{0}(x) \nabla_{x} N(x, y) d \sigma_{x}+o(\epsilon),
\end{aligned}
$$

which concludes the proof.
So, we are left with the proof of (14). Let $2 \epsilon<d<\frac{1}{2 K}$ and let

$$
\Omega_{d}^{\epsilon}=\left\{x \in \Omega: \operatorname{dist}\left(x, \partial\left(\Omega \backslash D_{\epsilon}^{1}\right)\right)>d\right\} .
$$

Since $u_{\epsilon}-u_{0}$ is solution of

$$
\operatorname{div}\left(\gamma_{0} \nabla\left(u_{\epsilon}-u_{0}\right)\right)=0 \text { in } \Omega \backslash D_{\epsilon}^{1}
$$

the regularity assumption on $\gamma_{0}$ implies that $u_{\varepsilon}-u_{0} \in H_{l o c}^{2}\left(\Omega \backslash D_{\epsilon}^{1}\right)$ (see [4, Theorem 2.1, Chapter 2]).

Consider $\Phi_{\epsilon}^{k}=\frac{\partial}{\partial x_{k}}\left(u_{\epsilon}-u_{0}\right)$ for $k=1,2$.
The function $\Phi_{\epsilon}^{k}$ satisfies in $\Omega \backslash D_{\epsilon}^{1}$

$$
\operatorname{div}\left(\gamma_{0} \nabla \Phi_{\epsilon}^{k}\right)=-\operatorname{div}\left(\frac{\partial \gamma_{0}}{\partial x_{k}} \nabla\left(u_{\epsilon}-u_{0}\right)\right)=: F
$$

By Caccioppoli inequality and by Lemma 8 we have that

$$
\begin{aligned}
\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2}\left(\Omega_{d / 2}^{\epsilon}\right)}^{2} & \leq \frac{C}{d^{2}}\left\|\Phi_{\epsilon}^{k}\right\|_{L^{2}\left(\Omega_{d / 4}^{\epsilon}\right)}^{2}+\|F\|_{H^{-1}\left(\Omega_{d / 4}^{\epsilon}\right)}^{2} \\
& \leq C\left(\frac{1}{d^{2}}\left\|\Phi_{\epsilon}^{k}\right\|_{L^{2}\left(\Omega_{d / 4}^{\epsilon}\right)}^{2}+\left\|\nabla\left(u_{\epsilon}-u_{0}\right)\right\|_{L^{2}(\Omega)}^{2}\right) \\
& \leq C\left(\frac{1}{d^{2}}+1\right)\left\|\nabla\left(u_{\epsilon}-u_{0}\right)\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C\left(\frac{1}{d^{2}}+1\right) \epsilon\|\psi\|_{H^{-1 / 2}(\partial \Omega)}^{2} .
\end{aligned}
$$

Hence

$$
\left\|\Phi_{\epsilon}^{k}\right\|_{H^{1}\left(\Omega_{d / 2}^{\epsilon}\right)} \leq C \sqrt{\epsilon} d^{-1}\|\psi\|_{H^{-1 / 2}(\partial \Omega)}
$$

Applying Lemma 3.3 to $\Phi_{\epsilon}^{k}$ gives

$$
\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2+\eta}\left(\Omega_{d}^{\epsilon}\right)} \leq C\left(\|F\|_{H^{-1,2+\eta}\left(\Omega_{d / 2}^{\epsilon}\right)}+d^{\frac{2}{2+\eta}-1}\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2}\left(\Omega_{d / 2}^{\epsilon}\right)}^{2}\right) .
$$

Now, by

$$
\|F\|_{H^{-1,2+\eta}\left(\Omega_{d / 2}^{\epsilon}\right)} \leq C\left\|\nabla\left(u_{\epsilon}-u_{0}\right)\right\|_{L^{2+\eta}\left(\Omega_{d / 2}^{\epsilon}\right)}
$$

and by the interior regularity estimates and Sobolev Immersion Theorem

$$
\begin{aligned}
\left\|\nabla\left(u_{\epsilon}-u_{0}\right)\right\|_{L^{2+\eta}\left(\Omega_{d / 2}^{\epsilon}\right)} & \leq C\left\|u_{\epsilon}-u_{0}\right\|_{H^{2}\left(\Omega_{d / 2}^{\epsilon}\right)} \\
& \leq C\left\|u_{\epsilon}-u_{0}\right\|_{H^{1}\left(\Omega_{d / 4}^{\epsilon}\right)} \leq\left\|u_{\epsilon}-u_{0}\right\|_{H^{1}(\Omega)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2+\eta}\left(\Omega_{d}^{\epsilon}\right)} & \leq C\left(\left\|u_{\epsilon}-u_{0}\right\|_{H^{1}(\Omega)}+d^{\frac{2}{2+\eta}-1}\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2}\left(\Omega_{d / 2}^{\epsilon}\right)}^{2}\right) \\
& \leq C\left(1+d^{\frac{2}{2+\eta}-2}\right) \sqrt{\epsilon}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
\end{aligned}
$$

Finally, since $\frac{2}{2+\eta}-2<0$ and $d<1$, from last inequality, we derive

$$
\left\|\nabla \Phi_{\epsilon}^{k}\right\|_{L^{2+\eta}\left(\Omega_{d}^{\epsilon}\right)} \leq C d^{\frac{2}{2+\eta}}-2 \sqrt{\epsilon}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
$$

On the other hand, applying Lemma 3.3 to $u_{\epsilon}-u_{0}$ we have

$$
\left\|\Phi_{\epsilon}^{k}\right\|_{L^{2+\eta}\left(\Omega_{d}^{\epsilon}\right)} \leq C d^{\frac{2}{2+\eta}-1} \sqrt{\epsilon}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
$$

By Sobolev Imbedding Theorem we than have

$$
\begin{equation*}
\left\|\frac{\partial}{\partial x_{k}}\left(u_{\epsilon}-u_{0}\right)\right\|_{L^{\infty}\left(\Omega_{d}^{\epsilon}\right)} \leq C d^{\frac{2}{2+\eta}-2} \sqrt{\epsilon}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \tag{15}
\end{equation*}
$$

Now let $y \in \Sigma_{\epsilon}^{\prime}$ and $y_{d}$ be the closest point to $y$ in $\Omega_{d}^{\epsilon}$. By (11) we have

$$
\begin{equation*}
\left|\nabla u_{\epsilon}(y)-\nabla u_{\epsilon}\left(y_{d}\right)\right| \leq C \frac{d^{\beta}}{\epsilon^{\theta(\beta+1)}}\|\psi\|_{H^{-1 / 2}(\partial \Omega)} \tag{16}
\end{equation*}
$$

Hence, by (15) and (16) we have

$$
\left|\nabla\left(u_{\epsilon}^{e}-u_{0}\right)(y)\right| \leq C\left(d^{\beta} \epsilon^{-\theta(\beta+1)}+d^{-2+\frac{2}{2+\eta}} \sqrt{\epsilon}\right)\|\psi\|_{H^{-1 / 2}(\partial \Omega)} .
$$

Choosing $\theta<\frac{\beta}{2\left(2-\frac{2}{2+\eta}\right)(\beta+1)}$ we get

$$
\left|\nabla\left(u_{\epsilon}^{e}-u_{0}\right)(y)\right| \leq C \epsilon^{\theta_{1}}
$$

with $\theta_{1}>0$.

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# Wave equation with Robin condition, quantitative estimates of strong unique continuation at the boundary 

Eva Sincich and Sergio Vessella<br>"Dedicated to Giovanni Alessandrini on the occasion of his 60th birthday"


#### Abstract

The main result of the present paper consists in a quantitative estimate of unique continuation at the boundary for solutions to the wave equation. Such estimate is the sharp quantitative counterpart of the following strong unique continuation property: let u be a solution to the wave equation that satisfies an homogeneous Robin condition on a portion $S$ of the boundary and the restriction of $u_{\mid S}$ on $S$ is flat on a segment $\{0\} \times J$ with $0 \in S$ then $u_{\mid S}$ vanishes in a neighbourhood of $\{0\} \times J$.


Keywords: Stability Estimates, Unique Continuation Property, Hyperbolic Equations, Robin problem.
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## 1. Introduction

The strong unique continuation properties at the boundary and the related quantitative estimates have been well understood in the context of second order elliptic equations, $[1,22]$, and in the context of second order parabolic equations $[16,17,32]$. For instance, in the framework of elliptic equations, the doubling inequality at the boundary and three sphere inequality are the typical forms in which such quantitative estimates of unique continuation occur [4]. Similar forms, like three cylinder inequality or two-sphere one cylinder inequality, occur in the parabolic case [32]. In the context of hyperbolic equation, strong properties of unique continuation at the interior and the related quantitative estimates are less studied $[6,24,25,31]$. Also, we recall here the papers [ $11,12,26]$ in which unique continuation properties are proved along and across lower dimensional manifolds for the wave equation. We refer to $[8,9,23]$ for recent result of quantitative estimate for hyperbolic equations. Such results are the quantitative counterpart of the unique continuation properties for equation with partially analytic coefficients proved in [19, 27, 30], see also [20].

Quantitative estimates of strong unique continuation at the boundary are
one of most important tool which enables to prove sharp stability estimates for inverse problems for PDE with unknown boundaries or with unknown boundary coefficients of Robin type, [3, 29] (elliptic equations), [5, 10, 14, 32] (parabolic equations), [33] (hyperbolic equations). In the context of elliptic and parabolic equations, the stability estimates that were proved are optimal $[2,13,14]$.

To the authors knowledge there exits no result in the literature concerning quantitative estimates of strong unique continuation at the boundary for hyperbolic equations.

In order to make clear what we mean, we illustrate our result in a particular and meaningful case. Let $A(x)$ be a real-valued symmetric $n \times n, n \geq 2$, matrix whose entries are functions of Lipschitz class satisfying a uniform ellipticity condition. Let $u$ be a solution to

$$
\begin{equation*}
\partial_{t}^{2} u-\operatorname{div}\left(A(x) \nabla_{x} u\right)=0, \quad \text { in } B_{1}^{+} \times J, \tag{1}
\end{equation*}
$$

where $B_{1}^{+}=\left\{x=\left(x^{\prime}, x_{n}\right) \in \mathbb{R}^{n}:|x|<1, x_{n}>0\right\}$ and $J=(-T, T)$ is an interval of $\mathbb{R}$. Assume that $u$ satisfies the following Robin condition

$$
\begin{equation*}
A\left(x^{\prime}, 0\right) \nabla_{x} u\left(x^{\prime}, 0, t\right) \cdot \nu+\gamma\left(x^{\prime}\right) u\left(x^{\prime}, 0, t\right)=0, \quad \text { in } B_{1}^{\prime} \times J \tag{2}
\end{equation*}
$$

where $B_{1}^{\prime}$ is the $\mathbb{R}^{n-1}$ ball of radius 1 centred at $0, \nu$ denotes the outer unit normal to $B_{1}^{\prime}$ and $\gamma$, the Robin coefficient, is of Lipschitz class. The quantitative estimate of strong unique continuation that we provide here may be briefly described as follows. Let $r \in(0,1)$ and assume that

$$
\begin{equation*}
\sup _{t \in J}\|u(\cdot, 0, t)\|_{L^{2}\left(B_{r}^{\prime}\right)} \leq \varepsilon \quad \text { and } \quad\|u(\cdot, 0)\|_{H^{2}\left(B_{1}^{+}\right)} \leq 1 \tag{3}
\end{equation*}
$$

where $\varepsilon<1$. Then

$$
\begin{equation*}
\|u(\cdot, 0,0)\|_{L^{2}\left(B_{s_{0}}^{\prime}\right)} \leq C\left|\log \left(\varepsilon^{\theta}\right)\right|^{-\alpha} \tag{4}
\end{equation*}
$$

where $s_{0} \in(0,1), C \geq 1, \alpha>0$ are constants independent of $u$ and $r$ and

$$
\begin{equation*}
\theta=|\log r|^{-1} \tag{5}
\end{equation*}
$$

For the precise statement of our result we refer to Theorem 2.1. Roughly speaking, in such a Theorem the half ball $B_{1}^{+}$is replaced by the region $\left\{\left(x^{\prime}, x_{n}\right) \in B_{1}\right.$ : $\left.x_{n}>\phi\left(x^{\prime}\right)\right\}$ where $\phi \in C^{1,1}\left(B_{1}^{\prime}\right)$ satisfies $\phi(0)=\left|\nabla_{x^{\prime}} \phi(0)\right|=0$. In addition, $u$ satisfies the Robin condition (2) on $S_{1} \times J$ where $S_{1}=\left\{\left(x^{\prime}, \phi\left(x^{\prime}\right)\right): x^{\prime} \in B_{1}^{\prime}\right\}$.

The estimate (4) is a sharp estimate from two points of view:
(i) The logarithmic character of the estimate cannot be improved as it is shown by a well-known counterexample of John for the wave equation, [21];
(ii) The sharp dependence of $\theta$ by $r$. Indeed it is easy to check that the estimate (4) implies that the following strong unique continuation property at the boundary holds true. Let $u$ satisfy (1) and (2) and assume that

$$
\sup _{t \in J}\|u(\cdot, 0, t)\|_{L^{2}\left(B_{r}^{\prime}\right)}=\mathcal{O}\left(r^{N}\right), \forall N \in \mathbb{N}, \text { as } r \rightarrow 0
$$

then we have

$$
u\left(x^{\prime}, 0, t\right)=0 \quad \text { for every }\left(x^{\prime}, t\right) \in \mathcal{U}
$$

where $\mathcal{U}$ is a neighbourhood of $\{0\} \times J$.
In order to prove the quantitative estimate (4), we have mainly refined the strategy developed in [31] in which the author, among various results, proved that if

$$
\sup _{t \in J}\|u(\cdot, t)\|_{L^{2}\left(B_{r}^{+}\right)} \leq \varepsilon \quad \text { and } \quad\|u(\cdot, 0)\|_{H^{2}\left(B_{1}^{+}\right)} \leq 1
$$

then

$$
\begin{equation*}
\|u(\cdot, 0)\|_{L^{2}\left(B_{s_{0}}^{+}\right)} \leq C\left|\log \left(\varepsilon^{\theta}\right)\right|^{-1 / 6} \tag{6}
\end{equation*}
$$

where $\theta=|\log r|^{-1}, s_{0} \in(0,1), C \geq 1$ are constants independent of $u$ and $r$ and an homogeneous Neumann boundary condition applies instead of (2). To carry out our proof, we first adapt an argument used in [28] in the elliptic context which enable to reduce the Robin boundary condition into a Neumann boundary one. Subsequently we need a careful refinement of some arguments used in [31]. Actually, to fulfil our proof it is not sufficient to apply the above estimate (6). In order to illustrate this point, a comparison with the analogue elliptic context (i.e. $u$ is time independent) could be useful. In such an elliptic context [28] instead of (3) we would have

$$
\|u(\cdot, 0)\|_{L^{2}\left(B_{r}^{\prime}\right)} \leq \varepsilon \quad \text { and } \quad\|u\|_{H^{2}\left(B_{1}^{+}\right)} \leq 1
$$

Thus, from stability estimates for the Cauchy problem [4] and regularity result we would obtain the following Holder estimate

$$
\|u\|_{L^{2}\left(B_{\frac{+}{2}}^{+}\right)} \leq C \varepsilon^{\beta}
$$

where $C$ and $\beta \in(0,1)$ are independent on $u$ and $r$. By using the above estimate, the three sphere inequality at the boundary and standard regularity results we would have

$$
\|u\|_{H^{1}\left(B_{\rho}^{+}\right)} \leq C \varepsilon^{\vartheta}
$$

where $0<\rho<1$ and $\vartheta \sim|\log r|^{-1}$ as $r \rightarrow 0$. Finally, by trace inequality we would obtain

$$
\|u\|_{L^{2}\left(B_{\rho / 2}^{\prime}\right)} \leq C \varepsilon^{\vartheta}
$$

The application of the same argument in the hyperbolic case would lead to a loglog type estimate instead of the desired single log one (4). In fact, opposite to the elliptic case, in the hyperbolic context the dependence of the interior values of the solution upon the Cauchy data is logarithmic. As a consequence, by combining such a log dependence with the logarithmic estimate in (6) we would obtain a loglog type estimate for $\|u(\cdot, 0,0)\|_{L^{2}\left(B_{s_{0}}^{\prime}\right)}$.

The plan of the paper is as follows. In Section 2 we state the main result of this paper. In Section 3 we prove our main theorem, in Section 4 we discuss some auxiliary results and in Section 5 we conclude by summarizing the main steps of our proof.

## 2. The main result

### 2.1. Notation and Definition

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^{n}$ will be adopted. For $n \geq 2$, a point $x \in \mathbb{R}^{n}$ will be denoted by $x=\left(x^{\prime}, x_{n}\right)$, where $x^{\prime} \in \mathbb{R}^{n-1}$ and $x_{n} \in \mathbb{R}$. Moreover, given $r>0$, we will denote by $B_{r}, B_{r}^{\prime}, \widetilde{B}_{r}$ the ball of $\mathbb{R}^{n}, \mathbb{R}^{n-1}$ and $\mathbb{R}^{n+1}$ of radius $r$ centred at 0 . For any open set $\Omega \subset \mathbb{R}^{n}$ and any function (smooth enough) $u$ we denote by $\nabla_{x} u=\left(\partial_{x_{1}} u, \cdots, \partial_{x_{n}} u\right)$ the gradient of $u$. Also, for the gradient of $u$ we use the notation $D_{x} u$. If $j=0,1,2$ we denote by $D_{x}^{j} u$ the set of the derivatives of $u$ of order $j$, so $D_{x}^{0} u=u, D_{x}^{1} u=\nabla_{x} u$ and $D_{x}^{2} u$ is the Hessian matrix $\left\{\partial_{x_{i} x_{j}} u\right\}_{i, j=1}^{n}$. Similar notation are used whenever other variables occur and $\Omega$ is an open subset of $\mathbb{R}^{n-1}$ or a subset of $\mathbb{R}^{n+1}$. By $H^{\ell}(\Omega), \ell=0,1,2$ we denote the usual Sobolev spaces of order $\ell$, in particular we have $H^{0}(\Omega)=L^{2}(\Omega)$.

For any interval $J \subset \mathbb{R}$ and $\Omega$ as above we denote

$$
\mathcal{W}(J ; \Omega)=\left\{u \in C^{0}\left(J ; H^{2}(\Omega)\right): \partial_{t}^{\ell} u \in C^{0}\left(J ; H^{2-\ell}(\Omega)\right), \ell=1,2\right\} .
$$

We shall use the letters $C, C_{0}, C_{1}, \cdots$ to denote constants. The value of the constants may change from line to line, but we shall specified their dependence everywhere they appear.

### 2.2. Statements of the main results

Let $A(x)=\left\{a^{i j}(x)\right\}_{i, j=1}^{n}$ be a real-valued symmetric $n \times n$ matrix whose entries are measurable functions and they satisfy the following conditions for given constants $\rho_{0}>0, \lambda \in(0,1]$ and $\Lambda>0$,

$$
\begin{equation*}
\lambda|\xi|^{2} \leq A(x) \xi \cdot \xi \leq \lambda^{-1}|\xi|^{2}, \quad \text { for every } x, \xi \in \mathbb{R}^{n} \tag{7a}
\end{equation*}
$$

$$
\begin{equation*}
|A(x)-A(y)| \leq \frac{\Lambda}{\rho_{0}}|x-y|, \quad \text { for every } x, y \in \mathbb{R}^{n} \tag{7b}
\end{equation*}
$$

Let $\phi$ be a function belonging to $C^{1,1}\left(B_{\rho_{0}}^{\prime}\right)$ that satisfies

$$
\begin{gather*}
\phi(0)=\left|\nabla_{x^{\prime}} \phi(0)\right|=0  \tag{8a}\\
\|\phi\|_{C^{1,1}\left(B_{\rho_{0}}^{\prime}\right)} \leq E \rho_{0} \tag{8b}
\end{gather*}
$$

where

$$
\|\phi\|_{C^{1,1}\left(B_{\rho_{0}}^{\prime}\right)}=\|\phi\|_{L^{\infty}\left(B_{\rho_{0}}^{\prime}\right)}+\rho_{0}\left\|\nabla_{x^{\prime}} \phi\right\|_{L^{\infty}\left(B_{\rho_{0}}^{\prime}\right)}+\rho_{0}^{2}\left\|D_{x^{\prime}}^{2} \phi\right\|_{L^{\infty}\left(B_{\rho_{0}}^{\prime}\right)} .
$$

For any $r \in\left(0, \rho_{0}\right]$ denote by

$$
K_{r}:=\left\{\left(x^{\prime}, x_{n}\right) \in B_{r}: x_{n}>\phi\left(x^{\prime}\right)\right\}
$$

and

$$
S_{r}:=\left\{\left(x^{\prime}, \phi\left(x^{\prime}\right)\right): x^{\prime} \in B_{r}^{\prime}\right\} .
$$

We assume that the Robin coefficient $\gamma$ belongs to $C^{0,1}\left(S_{\rho_{0}}\right)$ and for a given $\bar{\gamma}>0$ is such that

$$
\begin{equation*}
\|\gamma\|_{C^{0,1}\left(S_{\rho_{0}}\right)} \leq \bar{\gamma} \tag{9}
\end{equation*}
$$

Let $U \in \mathcal{W}\left(\left[-\lambda \rho_{0}, \lambda \rho_{0}\right] ; K_{\rho_{0}}\right)$ be a solution to

$$
\begin{equation*}
\partial_{t}^{2} U-\operatorname{div}\left(A(x) \nabla_{x} U\right)=0, \quad \text { in } K_{\rho_{0}} \times\left(-\lambda \rho_{0}, \lambda \rho_{0}\right) \tag{10}
\end{equation*}
$$

satisfying the following Robin condition

$$
\begin{equation*}
A \nabla_{x} U \cdot \nu+\gamma U=0, \quad \text { on } S_{\rho_{0}} \times\left(-\lambda \rho_{0}, \lambda \rho_{0}\right) \tag{11}
\end{equation*}
$$

where $\nu$ denotes the outer unit normal to $S_{\rho_{0}}$.
Let $r_{0} \in\left(0, \rho_{0}\right]$ and denote

$$
\begin{equation*}
\varepsilon=\sup _{t \in\left(-\lambda \rho_{0}, \lambda \rho_{0}\right)}\left(\rho_{0}^{-n+1} \int_{S_{r_{0}}} U^{2}(\sigma, t) d \sigma\right)^{1 / 2} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
H=\left(\sum_{j=0}^{2} \rho_{0}^{j-n} \int_{K_{\rho_{0}}}\left|D_{x}^{j} U(x, 0)\right|^{2} d x\right)^{1 / 2} \tag{13}
\end{equation*}
$$

Theorem 2.1. Let (7) be satisfied. Let $U \in \mathcal{W}\left(\left[-\lambda \rho_{0}, \lambda \rho_{0}\right] ; K_{\rho_{0}}\right)$ be a solution to (10) satisfying (12) and (13). Assume that u satisfies (11). There exist constants $\bar{s}_{0} \in(0,1)$ and $C \geq 1$ depending on $\lambda, \Lambda$ and $E$ only such that for every $0<r_{0} \leq \rho \leq \bar{s}_{0} \rho_{0}$ the following inequality holds true

$$
\begin{equation*}
\|U(\cdot, 0)\|_{L^{2}\left(S_{\rho}\right)} \leq \frac{C\left(\rho_{0} \rho^{-1}\right)^{C}(H+e \varepsilon)}{\left(\widetilde{\theta} \log \left(\frac{H+e \varepsilon}{\varepsilon}\right)\right)^{1 / 6}} \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{\theta}=\frac{\log \left(\rho_{0} / C \rho\right)}{\log \left(\rho_{0} / r_{0}\right)} \tag{15}
\end{equation*}
$$

From now on we shall refer to the a priori bounds as the following set of quantities: $\lambda, \Lambda, \rho_{0}, E, \bar{\gamma}$.

## 3. Proof of Theorem 2.1

In what follows we use the following
Proposition 3.1. There exists a radius $r_{1}>0$ depending on the a priori data only, such that the problem

$$
\begin{cases}\operatorname{div}(A \nabla \psi)=0, & \text { in } K_{r_{1}}  \tag{16}\\ A \nabla \psi \cdot \nu+\gamma \psi=0, & \text { in } S_{r_{1}},\end{cases}
$$

admits a solution $\psi \in H^{1}\left(K_{r_{1}}\right)$ satisfying

$$
\begin{equation*}
\psi(x) \geq 1 \quad \text { for every } x \in K_{r_{1}} \tag{17}
\end{equation*}
$$

Moreover, there exists a constant $\bar{\psi}>0$ depending on the a priori data only, such that

$$
\begin{equation*}
\|\psi\|_{C^{1}\left(K_{r_{1}}\right)} \leq \bar{\psi} \tag{18}
\end{equation*}
$$

Proof. See Section 4
Let $r_{1}$ and $\psi$ be the radius and the function introduced in Proposition 3.1. Denoting with

$$
\begin{equation*}
u^{\star}=\frac{U}{\psi} \tag{19}
\end{equation*}
$$

it follows that $u^{\star} \in \mathcal{W}\left(\left[-\lambda r_{1}, \lambda r_{1}\right] ; K_{r_{1}}\right)$ is a solution to

$$
\begin{equation*}
\psi^{2}(x) \partial_{t}^{2} u^{\star}-\operatorname{div}\left(A^{\star}(x) \nabla_{x} u^{\star}\right)=0, \quad \text { in } K_{r_{1}} \times\left(-\lambda r_{1}, \lambda r_{1}\right), \tag{20}
\end{equation*}
$$

satisfying the following Neumann condition

$$
\begin{equation*}
A^{\star} \nabla_{x} u^{\star} \cdot \nu=0, \quad \text { on } S_{r_{1}} \times\left(-\lambda r_{1}, \lambda r_{1}\right), \tag{21}
\end{equation*}
$$

where $\nu$ denotes the outer unit normal to $S_{r_{1}}$ and $A^{\star}(x)=\psi^{2}(x) A(x)$. Repeating the arguments in [31, Subsection 3.2] (partly based on the techniques introduced in [1]), we can assume with no loss of generality that $A^{\star}(0)=I$ with $I$ identity matrix $n \times n$ and we infer that there exist $\rho_{1}, \rho_{2}$ and a function $\phi \in C^{1,1}\left(\bar{B}_{\rho_{2}}, \mathbb{R}^{n}\right)$ such that

$$
\begin{gather*}
\Phi\left(B_{\rho_{2}}\right) \subset B_{\rho_{1}}  \tag{22a}\\
\Phi(y, 0)=\left(y^{\prime}, \phi\left(y^{\prime}\right)\right)  \tag{22b}\\
C^{-1} \leq|\operatorname{det} D \Phi(y)| \leq C, \quad \text { for every } y \in B_{\rho_{2}} \tag{22c}
\end{gather*}
$$

Let us define the matrix $\bar{A}(y)=\{\bar{a}(y)\}_{i, j=1}^{n}$ as follows (below $\left(D \Phi^{-1}\right)^{\operatorname{tr}}$ denotes the transposed matrix of $\left.\left(D \Phi^{-1}\right)\right)$

$$
\begin{gather*}
\bar{A}(y)=|\operatorname{det} D \Phi(y)|\left(D \Phi^{-1}\right)(\Phi(y)) A^{\star}(\Phi(y))\left(D \Phi^{-1}\right)^{t r}(\Phi(y)) \\
z(y, t)=u^{\star}(\Phi(y), t)  \tag{23}\\
u(y, t)=z\left(y^{\prime},\left|y_{n}\right|, t\right) \tag{24}
\end{gather*}
$$

and hence we get that $u$ is a solution to

$$
\begin{equation*}
q(y) \partial_{t}^{2} u-\operatorname{div}(\tilde{A}(y) \nabla u)=0, \quad \text { in } B_{\rho_{2}} \times\left(-\lambda \rho_{2}, \lambda \rho_{2}\right) \tag{25}
\end{equation*}
$$

where for every $y \in B_{\rho_{2}}$ we denote

$$
q(y)=\left|\operatorname{det} D \Phi\left(y^{\prime},\left|y_{n}\right|\right)\right| \psi^{2}\left(y^{\prime},\left|y_{n}\right|\right)
$$

and $\tilde{A}(y)=\left\{\tilde{a}_{i j}(y)\right\}_{i, j=1}^{n}$ is the matrix whose entries are given by

$$
\begin{gather*}
\tilde{a}_{i j}\left(y^{\prime}, y_{n}\right)=\bar{a}_{i j}\left(y^{\prime},\left|y_{n}\right|\right), \text { if either } i, j \in\{1, \ldots, n-1\}, \text { or } i=j=n,  \tag{26a}\\
\tilde{a}_{n j}\left(y^{\prime}, y_{n}\right)=\tilde{a}_{j n}\left(y^{\prime}, y_{n}\right)=\operatorname{sgn}\left(y_{n}\right) \bar{a}^{n j}\left(y^{\prime},\left|y_{n}\right|\right), \text { if } 1 \leq j \leq n-1 \tag{26b}
\end{gather*}
$$

From (7a), (7b), (22c), (17) and (18) there exist constants $\tilde{\Lambda}, \tilde{\lambda}>0$ depending on the a priori data only such that

$$
\begin{align*}
& \tilde{\lambda}|\xi|^{2} \leq \tilde{A}(y) \xi \cdot \xi \leq \tilde{\lambda}^{-1}|\xi|^{2}, \quad \text { for every } y \in B_{\rho_{2}}, \xi \in \mathbb{R}^{n}  \tag{27a}\\
& \left|\tilde{A}\left(y_{1}\right)-\tilde{A}\left(y_{2}\right)\right| \leq \frac{\tilde{\Lambda}}{\rho_{0}}\left|y_{1}-y_{2}\right|, \quad \text { for every } y_{1}, y_{2} \in B_{\rho_{2}} \tag{27b}
\end{align*}
$$

and

$$
\begin{equation*}
\tilde{\lambda} \leq q(y) \leq \tilde{\lambda}^{-1}, \quad \text { for every } y \in B_{\rho_{2}} \tag{28a}
\end{equation*}
$$

$$
\begin{equation*}
\left|q\left(y_{1}\right)-q\left(y_{2}\right)\right| \leq \frac{\tilde{\Lambda}}{\rho_{0}}\left|y_{1}-y_{2}\right|, \quad \text { for every } y_{1}, y_{2} \in B_{\rho_{2}} \tag{28b}
\end{equation*}
$$

Let us recall that, by construction, the function $u$ in (24) is even w.r.t. the variable $y_{n}$ and moreover with no loss of generality we may assume that $u$ (up to replacing it with its even part w.r.t the variable $t$ as in [31]) is even w.r.t. $t$ also. From now for the sake of simplicity we shall assume that $\rho_{2}=1$.

By (12) and by (13) we have that there exist $C_{1}, C_{2}>0$ constants depending on the a priori data only such that

$$
\begin{align*}
& \epsilon=\sup _{t \in(-\lambda, \lambda)}\left(\int_{B_{r_{0}}^{\prime}} u^{2}\left(y^{\prime}, 0, t\right) d y^{\prime}\right)^{1 / 2} \leq C_{1} \varepsilon  \tag{29}\\
& H_{1}=\left(\sum_{j=0}^{2} \int_{B_{1}}\left|D_{x}^{j} u(y, 0)\right|^{2} d y\right)^{1 / 2} \leq C_{2} H . \tag{30}
\end{align*}
$$

As in [31], let $\widetilde{u}_{0}$ be an even extension w.r.t. $y_{n}$ of the function $u_{0}:=u(\cdot, 0)$ such that $\widetilde{u}_{0} \in H^{2}\left(B_{2}\right) \cap H_{0}^{1}\left(B_{2}\right)$ and

$$
\begin{equation*}
\left\|\widetilde{u}_{0}\right\|_{H^{2}\left(B_{2}\right)} \leq C H_{1}, \tag{31}
\end{equation*}
$$

where $C$ is an absolute constant.
Let us denote by $\lambda_{j}$, with $0<\lambda_{1} \leq \lambda_{2} \leq \cdots \leq \lambda_{j} \leq \cdots$ the eigenvalues associated to the Dirichlet problem

$$
\left\{\begin{array}{l}
\operatorname{div}\left(\tilde{A}(y) \nabla_{y} v\right)+\omega q(y) v=0, \quad \text { in } B_{2}  \tag{32}\\
v \in H_{0}^{1}\left(B_{2}\right)
\end{array}\right.
$$

and by $e_{j}(\cdot)$ the corresponding eigenfunctions normalized by

$$
\begin{equation*}
\int_{B_{2}} e_{j}^{2}(y) q(y) d y=1 \tag{33}
\end{equation*}
$$

Let us stress that we may choose the eigenfunctions $e_{j}$ to be even w.r.t $y_{n}$ (see Remark 4.1 in Section 4). By (7a), (28) and Poincaré inequality we have for every $j \in \mathbb{N}$

$$
\begin{equation*}
\lambda_{j}=\int_{B_{2}} \tilde{A}(y) \nabla_{x} e_{j}(y) \cdot \nabla_{y} e_{j}(y) d y \geq c \lambda^{2} \int_{B_{2}} e_{j}^{2}(y) q(y) d y=c \lambda^{2} \tag{34}
\end{equation*}
$$

where $c$ is an absolute constant. Denote by

$$
\begin{equation*}
\alpha_{j}:=\int_{B_{2}} \widetilde{u}_{0}(y) e_{j}(y) q(y) d y, \tag{35}
\end{equation*}
$$

and let

$$
\begin{equation*}
\widetilde{u}(y, t):=\sum_{j=1}^{\infty} \alpha_{j} e_{j}(y) \cos \sqrt{\lambda_{j}} t \tag{36}
\end{equation*}
$$

By Proposition 3.3 in [31] we have that

$$
\begin{equation*}
\sum_{j=1}^{\infty}\left(1+\lambda_{j}\right)^{2} \alpha_{j}^{2} \leq C H_{1}^{2} \tag{37}
\end{equation*}
$$

where $C>0$ depends on $\tilde{\lambda}$ and $\tilde{\Lambda}$ only.
Moreover, as a consequence of the uniqueness for the Cauchy problem for the equation (25) (see (3.9) in [31] for a detailed discussion) we have that

$$
\begin{equation*}
\tilde{u}(y, t)=u(y, t) \quad \text { for } \quad|y|+\tilde{\lambda}^{-1}|t|<1 \tag{38}
\end{equation*}
$$

We define for any $\mu \in(0,1]$ and for any $k \in \mathbb{N}$ the following mollified form of the Boman transformation of $\widetilde{u}(y, \cdot)[7]$

$$
\begin{equation*}
\widetilde{u}_{\mu, k}(x)=\int_{\mathbb{R}} \widetilde{u}(x, t) \varphi_{\mu, k}(t) d t, \text { for } x \in B_{2} \tag{39}
\end{equation*}
$$

where $\left\{\varphi_{\mu, k}\right\}_{k=1}^{\infty}$ is a suitable sequence of mollifiers, [31, Section 3.1], such that $\operatorname{supp} \varphi_{\mu, k} \subset\left[-\frac{\lambda(\mu+1)}{4}, \frac{\lambda(\mu+1)}{4}\right], \varphi_{\mu, k} \geq 0, \varphi_{\mu, k}$ even function and such that $\int_{\mathbb{R}} \varphi_{\mu, k}(t) d t=1$.

From now on we fix $\bar{\mu}:=k^{-\frac{1}{6}}$ for $k \geq 1$ and we denote

$$
\begin{equation*}
\widetilde{u}_{k}:=\widetilde{u}_{\bar{\mu}, k} \tag{40}
\end{equation*}
$$

By Proposition 3.3 im [31], it follows that

$$
\begin{equation*}
\left\|u(\cdot, 0)-\widetilde{u}_{\mu, k}\right\|_{L^{2}\left(B_{1}\right)} \leq C H k^{-1 / 6} \tag{41}
\end{equation*}
$$

Let

$$
\widehat{\varphi}_{\bar{\mu}, k}(\tau)=\int_{\mathbb{R}} \varphi_{\bar{\mu}, k}(t) e^{-i \tau t} d t=\int_{\mathbb{R}} \varphi_{\bar{\mu}, k}(t) \cos \tau t d t, \tau \in \mathbb{R} .
$$

Let us introduce now, for every $k \in \mathbb{N}$ an even function $g_{k} \in C^{1,1}(\mathbb{R})$ such that if $|z| \leq k$ then we have $g_{k}(z)=\cosh z$, if $|z| \geq 2 k$ then we have $g_{k}(z)=\cosh 2 k$ and such that it satisfies the condition

$$
\begin{equation*}
\left|g_{k}(z)\right|+\left|g_{k}^{\prime}(z)\right|+\left|g_{k}^{\prime \prime}(z)\right| \leq c e^{2 k}, \text { for every } z \in \mathbb{R} \tag{42}
\end{equation*}
$$

where $c$ is an absolute constant.

Let us introduce the following quantities

$$
\begin{gather*}
h_{k}(z)=e^{2 k} \min \left\{1,\left(4 \pi \lambda^{-1}|z|\right)^{2 k}\right\}, \quad z \in \mathbb{R},  \tag{43a}\\
f_{k}(y, z)=\sum_{j=1}^{\infty} \lambda_{j} \alpha_{j} \widehat{\varphi_{\bar{\mu}}, k}\left(\sqrt{\lambda_{j}}\right)\left(g_{k}^{\prime \prime}\left(z \sqrt{\lambda_{j}}\right)-g_{k}\left(z \sqrt{\lambda_{j}}\right)\right) e_{j}(y) \\
y \in B_{2}, z \in \mathbb{R}  \tag{43b}\\
F_{k}(y, t, z)=\sum_{j=1}^{\infty} \alpha_{j} \sqrt{\lambda_{j}} \gamma_{k}\left(z \sqrt{\lambda_{j}}\right) \sin \left(\sqrt{\lambda_{j}} t\right) e_{j}(y), y \in B_{2}, t, z \in \mathbb{R}  \tag{43c}\\
\gamma_{k}\left(z \sqrt{\lambda_{j}}\right)=g_{k}^{\prime \prime}\left(z \sqrt{\lambda_{j}}\right)-g_{k}\left(z \sqrt{\lambda_{j}}\right), \quad z \in \mathbb{R} . \tag{43d}
\end{gather*}
$$

Proposition 3.2. Let

$$
\begin{equation*}
v_{k}(y, z):=\sum_{j=1}^{\infty} \alpha_{j} \widehat{\varphi}_{\bar{\mu}, k}\left(\sqrt{\lambda_{j}}\right) g_{k}\left(y \sqrt{\lambda_{j}}\right) e_{j}(z), \text { for }(y, z) \in B_{2} \times \mathbb{R} \tag{44}
\end{equation*}
$$

We have that $v_{k}(\cdot, z)$ belongs to $H^{2}\left(B_{2}\right) \cap H_{0}^{1}\left(B_{2}\right)$ for every $y \in \mathbb{R}, v_{k}(y, z)$ is an even function with respect to $z$ and it satisfies

$$
\left\{\begin{array}{l}
q(y) \partial_{z}^{2} v_{k}+\operatorname{div}\left(\tilde{A}(y) \nabla_{x} v_{k}\right)=f_{k}(y, z), \quad \text { in } B_{2} \times \mathbb{R}  \tag{45}\\
v_{k}(\cdot, 0)=\widetilde{u}_{k}, \quad \text { in } B_{2}
\end{array}\right.
$$

Moreover we have

$$
\begin{gather*}
\sum_{j=0}^{2}\left\|\partial_{y}^{j} v_{k}(\cdot, z)\right\|_{H^{2-j}\left(B_{2}\right)} \leq C H e^{2 k}, \text { for every } z \in \mathbb{R},  \tag{46}\\
\left\|f_{k}(\cdot, z)\right\|_{L^{2}\left(B_{2}\right)} \leq C H e^{2 k} \min \left\{1,\left(4 \pi \lambda^{-1}|z|\right)^{2 k}\right\}, \text { for every } z \in \mathbb{R},  \tag{47}\\
\left\|F_{k}(\cdot, 0, t, z)\right\|_{H^{\frac{1}{2}}\left(B_{1}^{\prime}\right)} \leq C H_{1} h_{k}(z), \text { for every } t, z \in \mathbb{R}, \tag{48}
\end{gather*}
$$

where $C$ depends on $\tilde{\lambda}$ and $\Lambda$ only.
Proof. Except for the inequality (48) which is discussed below, the proofs of the remaining results follow along the lines of Proposition 3.4 in [31]. From the arguments in Proposition 3.4 in [31] we deduce that

$$
\begin{equation*}
\left|\gamma_{k}\left(z \sqrt{\lambda_{j}}\right)\right| \leq \operatorname{ch}_{k}(z), \tag{49}
\end{equation*}
$$

where $c>0$ is an absolute constant constant, which in turn implies that

$$
\begin{equation*}
\left\|F_{k}(\cdot, 0, t, z)\right\|_{L^{2}\left(B_{2}\right)} \leq c h_{k}^{2} \sum_{j=1}^{\infty} \alpha_{j}^{2} \lambda_{j} \leq C H_{1}^{2} h_{k}^{2}(z) \tag{50}
\end{equation*}
$$

with $C>0$ constant depending on $\tilde{\lambda}$.
From (27a) we have

$$
\begin{align*}
& \tilde{\lambda} \int_{B_{2}}\left|\nabla_{y} F_{k}(y, t, z)\right|^{2} \mathrm{~d} y \leq \int_{B_{2}} \tilde{A}(y) \nabla_{y} F_{k}(y, t, z) \cdot \nabla_{y} F_{k}(y, t, z) \mathrm{d} y  \tag{51}\\
& \quad=\sum_{j=1}^{\infty} \alpha_{j} \sqrt{\lambda_{j}} \sin \left(\sqrt{\lambda_{j}} t\right) \gamma_{k}\left(z \sqrt{\lambda_{j}}\right) \int_{B_{2}} \tilde{A}(y) \nabla_{y} e_{j}(y) \cdot \nabla_{y} F_{k}(y, t, z) \mathrm{d} y \\
& \quad=\sum_{j=1}^{\infty} \alpha_{j} \sqrt{\lambda_{j}} \sin \left(\sqrt{\lambda_{j}} t\right) \gamma_{k}\left(z \sqrt{\lambda_{j}}\right) \int_{B_{2}} \lambda_{j} q(y) e_{j}(y) F_{k}(y, t, z) \mathrm{d} y \\
& \quad=\sum_{j=1}^{\infty} \alpha_{j}^{2} \lambda_{j}^{2}\left(\sin \left(\sqrt{\lambda_{j}} t\right) \gamma_{k}\left(z \sqrt{\lambda_{j}}\right)\right)^{2} \leq \sum_{j=1}^{\infty} \alpha_{j}^{2} \lambda_{j}^{2}\left(c h_{k}(z)\right)^{2} \leq C H_{1}^{2} h_{k}^{2}(z),
\end{align*}
$$

where $C>0$ is a constant depending on $\tilde{\lambda}$ and $\tilde{\Lambda}$ only.
Combining (50) and (51) we get

$$
\begin{equation*}
\left\|F_{k}(\cdot, t, z)\right\|_{H^{1}\left(B_{2}\right)} \leq C H_{1} h_{k}(z) \tag{52}
\end{equation*}
$$

which in view of standard trace estimates leads to

$$
\begin{equation*}
\left\|F_{k}(\cdot, 0, t, z)\right\|_{H^{\frac{1}{2}}\left(B_{1}^{\prime}\right)} \leq C H_{1} h_{k}(z) . \tag{53}
\end{equation*}
$$

Let us now consider a function $\Phi \in L^{2}\left(B_{r_{0}}^{\prime}\right)$ and let us define for any $(t, z) \in R=\left\{(t, z) \in \mathbb{R}^{2}:|t|<\tilde{\lambda},|z|<1\right\}$

$$
\begin{equation*}
w_{k}(t, z)=\int_{B_{r_{0}}^{\prime}} W_{k}\left(y^{\prime}, 0, t, z\right) \Phi\left(y^{\prime}\right) \mathrm{d} y^{\prime} \tag{54}
\end{equation*}
$$

where

$$
\begin{equation*}
W_{k}(y, t, z)=\sum_{j=1}^{\infty} \alpha_{j} \cos \left(\sqrt{\lambda_{j}} t\right) g_{k}\left(z \sqrt{\lambda_{j}}\right) e_{j}(y) \tag{55}
\end{equation*}
$$

Note that from (44) we have

$$
\begin{equation*}
v_{k}(y, z)=\int_{\mathbb{R}} \varphi_{\bar{\mu}, k}(t) W_{k}(y, t, z) \mathrm{d} t \tag{56}
\end{equation*}
$$

Proposition 3.3. We have that $w_{k}(\cdot, \cdot)$ belongs to $H^{1}(R)$ is a weak solution to

$$
\begin{equation*}
\Delta_{t, z} w_{k}(t, z)=-\partial_{t} \tilde{F}_{k}(t, z) \tag{57}
\end{equation*}
$$

satisfying

$$
\begin{gather*}
\left|w_{k}(t, 0)\right| \leq \epsilon\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)}  \tag{58a}\\
\partial_{z} w_{k}(t, 0)=0 \tag{58b}
\end{gather*}
$$

where

$$
\begin{equation*}
\tilde{F}_{k}(t, z)=\int_{B_{r_{0}}^{\prime}} F_{k}\left(y^{\prime}, 0, t, z\right) \Phi\left(y^{\prime}\right) d y^{\prime} \tag{59}
\end{equation*}
$$

Moreover, for any $(t, z) \in R$ we have that

$$
\begin{gather*}
\left|w_{k}(t, z)\right| \leq C H_{1} e^{2 k}\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)},  \tag{60a}\\
\left|\tilde{F}_{k}(t, z)\right| \leq C H_{1} h_{k}(z)\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)} \tag{60b}
\end{gather*}
$$

where $C>0$ is a constant depending on $\tilde{\lambda}$ and $\tilde{\Lambda}$ only.
Proof. We start by proving (57). To this aim we consider a test function $\phi \in H_{0}^{1}(R)$ and by integration by parts we get

$$
\begin{align*}
& \int_{R} \nabla_{t, z} w_{k} \cdot \nabla \phi \mathrm{~d} t \mathrm{~d} y  \tag{61}\\
& =\sum_{j=1}^{\infty} \int_{R} \lambda_{j} \alpha_{j}<e_{j}, \Phi>\left(g_{k}\left(z \sqrt{\lambda_{j}}\right)-g_{k}^{\prime \prime}\left(z \sqrt{\lambda_{j}}\right)\right) \cos \left(\sqrt{\lambda_{j}} t\right) \phi(t, z) \mathrm{d} t \mathrm{~d} z \\
& =\sum_{j=1}^{\infty}-\int_{R} \partial_{t}\left(\sqrt{\lambda_{j}} \alpha_{j}<e_{j}, \Phi>\gamma_{k}\left(z \sqrt{\lambda_{j}}\right) \sin \left(\sqrt{\lambda_{j}} t\right)\right) \phi(t, z) \mathrm{d} t \mathrm{~d} z
\end{align*}
$$

where we mean $\left.<e_{j}, \Phi>=\int_{B_{r_{0}}^{\prime}} e_{j}\left(y^{\prime}, 0\right) \Phi\left(y^{\prime}\right)\right) \mathrm{d} y^{\prime}$. Again by integration by parts with respect to the variable $t$ we get

$$
\begin{equation*}
\int_{R} \nabla_{t, z} w_{k} \cdot \nabla \phi \mathrm{~d} t \mathrm{~d} y=\int_{R}\left(\int_{B_{r_{0}}^{\prime}} F_{k}\left(y^{\prime}, 0, t, z\right) \Phi\left(y^{\prime}\right) \mathrm{d} y^{\prime}\right) \partial_{t} \phi \mathrm{~d} t \mathrm{~d} z \tag{62}
\end{equation*}
$$

and hence (57) follows.
Let us now prove (58a) and (58b). We have that by (36)

$$
\begin{equation*}
w_{k}(t, 0)=\int_{B_{r_{0}}^{\prime}} \tilde{u}\left(y^{\prime}, 0, t\right) \varphi\left(y^{\prime}\right) \mathrm{d} y^{\prime} \tag{63}
\end{equation*}
$$

Hence by (38) and (29) we have that

$$
\begin{equation*}
\left|w_{k}(t, 0)\right| \leq\left(\int_{B_{r_{0}}^{\prime}}\left|\tilde{u}\left(y^{\prime}, 0, t\right)\right|^{2} \mathrm{~d} y^{\prime}\right)^{\frac{1}{2}}\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)} \leq \epsilon\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)} \tag{64}
\end{equation*}
$$

By (55) we also get that

$$
\begin{equation*}
\partial_{z} w_{k}(t, 0)=\left.\int_{B_{r_{0}}^{\prime}} W_{k}\left(y^{\prime}, 0, t, z\right)\right|_{z=0} \Phi\left(y^{\prime}\right) \mathrm{d} y^{\prime}=0 \tag{65}
\end{equation*}
$$

Let us now prove (60a). By a standard trace inequality, by (37) and by (42) we have

$$
\begin{align*}
& \left|w_{k}(t, z)\right| \leq\left\|W_{k}\right\|_{H^{1}\left(B_{2}\right)}\|\Phi\|_{L^{2}\left(B^{\prime} r_{0}\right)} \\
& \quad \leq C e^{2 k}\left(\sum_{j=1}^{\infty}\left(1+\lambda_{j}\right) \alpha_{j}^{2}\right)^{\frac{1}{2}}\|\Phi\|_{L^{2}\left(B^{\prime} r_{0}\right)} \leq C H_{1} e^{2 k}\|\Phi\|_{L^{2}\left(B^{\prime} r_{0}\right)} . \tag{66}
\end{align*}
$$

Finally (60b) follows from (48).
Proposition 3.4. Let $w_{k}$ be the function introduced in (54), then we have that

$$
\begin{equation*}
\left|w_{k}(t, z)\right| \leq C r_{0}^{\frac{1}{2}} \sigma_{k}\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)} \quad \text { for any }|t| \leq \frac{\tilde{\lambda}}{2},|z| \leq \frac{r_{0}}{8} \tag{67}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{k}=\left(\epsilon+H_{1}\left(C r_{0}\right)^{2 k}\right)^{\beta}\left(H_{1}\left(C r_{0}\right)^{2 k}+H_{1} e^{2 k}\right)^{1-\beta} \tag{68}
\end{equation*}
$$

Proof. We notice that by (57) and by a standard local boundedness estimate it follows that for any $t_{0} \in\left(-\frac{\tilde{\lambda}}{2}, \frac{\tilde{\lambda}}{2}\right)$ we have

$$
\begin{equation*}
\left\|w_{k}\right\|_{L^{\infty}\left(B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right)\right)} \leq \frac{1}{r_{0}}\left\|w_{k}\right\|_{L^{2}\left(B_{\frac{r 0}{4}}^{4}\left(t_{0}, 0\right)\right)} \tag{69}
\end{equation*}
$$

where we denote $B_{r}^{(2)}\left(t_{0}, 0\right)=\left\{(t, z) \in \mathbb{R}^{2}:\left|t-t_{0}\right|^{2}+|z|^{2} \leq r^{2}\right\}$ for any $r>0$.
Let $\tilde{w}_{k} \in H^{1}\left(B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right)\right)$ be the solution to the following Dirichlet problem

$$
\left\{\begin{array}{l}
\Delta_{t, z} \tilde{w}_{k}=-\partial_{t} \tilde{F}_{k}(t, z) \quad \text { in } B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right),  \tag{70}\\
\tilde{w_{k}}=0 \quad \text { on } \partial B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right) .
\end{array}\right.
$$

We observe that being $\partial_{t} \tilde{F}_{k}(t, z)$ odd with respect the variable $z$, we have that $\tilde{w}_{k}$ is odd with respect the variable $z$ as well. Moreover, we have that
$\partial_{z} \tilde{w}_{k}(t, z)=0$ on $B_{\frac{r_{0}}{8}}^{(1)}$ where we denote $B_{r}^{(1)}=\left(t_{0}-r, t_{0}+r\right) \times\{0\}$ for any $r>0$.

Now denoting

$$
\begin{equation*}
\hat{w}_{k}=w_{k}-\tilde{w}_{k} \tag{71}
\end{equation*}
$$

we have that

$$
\left\{\begin{array}{l}
\Delta_{t, z} \hat{w}_{k}=0 \quad \text { in } B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right),  \tag{72}\\
\hat{w}_{k}=0 \text { on } B_{\frac{r_{0}}{8}}^{(1)} .
\end{array}\right.
$$

By the argument in Proposition 3.5 of [31], which in turn are based on wellknown stability estimates for the Cauchy problem (see for instance [4]), it follows that

$$
\begin{equation*}
\int_{B_{\frac{r_{0}}{32}}^{(2)}\left(t_{0}, 0\right)}\left|\hat{w}_{k}\right|^{2} \leq C\left(\int_{B_{\frac{r_{0}}{8}}^{(2)}\left(t_{0}, 0\right)}\left|\hat{w}_{k}\right|^{2}\right)^{1-\beta}\left(\int_{B_{\frac{r_{0}}{16}}^{16}\left(t_{0}, 0\right)}\left|\hat{w}_{k}\right|^{2}\right)^{\beta} \tag{73}
\end{equation*}
$$

Furthermore we have that by (58a), (60b) and (60a)

$$
\begin{gather*}
\left\|\hat{w}_{k}\right\|_{L^{2}\left(B_{\frac{r_{0}}{16}}^{(1)}\left(t_{0}, 0\right)\right)} \leq C\left(\epsilon+H_{1}\left(C r_{0}\right)^{2 k}\right)\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)}  \tag{74a}\\
\left\|\hat{w}_{k}\right\|_{L^{2}\left(B_{\frac{r 0}{8}}^{8}\left(t_{0}, 0\right)\right)} \leq C\left(H_{1} e^{2 k}+H_{1}\left(C r_{0}\right)^{2 k}\right)\|\Phi\|_{L^{2}\left(B_{r_{0}}^{\prime}\right)} \tag{74b}
\end{gather*}
$$

where $C>0$ is a constant depending on the a priori data only. Inserting (74a) and (74b) in (73) we get the thesis.

Proposition 3.5. Let $v_{k}$ be defined in (44), then we have

$$
\begin{equation*}
\int_{B_{r_{0}}^{\prime}}\left|v_{k}\left(y^{\prime}, 0, z\right)\right|^{2} d y^{\prime} \leq\left(C r_{0}^{-\frac{1}{2}} \sigma_{k}\right)^{2} \tag{75}
\end{equation*}
$$

where $C>0$ depends on $\tilde{\lambda}$ and $\tilde{\Lambda}$ only.
Proof. From (54), (67) and the dual characterization of the norm, we have that

$$
\begin{equation*}
\int_{B_{r_{0}}^{\prime}}\left|W_{k}\left(y^{\prime}, 0, t, z\right)\right|^{2} \mathrm{~d} y^{\prime} \leq\left(C r_{0}^{-\frac{1}{2}} \sigma_{k}\right)^{2} \tag{76}
\end{equation*}
$$

for $|t| \leq \frac{\tilde{\lambda}}{2},|z| \leq \frac{r_{0}}{8}$. On the other hand by using equality (56), we have that

$$
\begin{align*}
&\left|v_{k}\left(y^{\prime}, 0, z\right)\right|^{2} \leq\left|\int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \varphi_{\bar{\mu}, k}(t) W_{k}\left(y^{\prime}, 0, t, z\right) \mathrm{d} t\right|^{2} \\
& \leq\left(\int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \varphi_{\bar{\mu}, k}(t) \mathrm{d} t\right)\left(\int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \varphi_{\bar{\mu}, k}(t)\left|W_{k}\left(y^{\prime}, 0, t, z\right)\right|^{2} \mathrm{~d} t\right) \\
&=\left(\int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \varphi_{\bar{\mu}, k}(t)\left|W_{k}\left(y^{\prime}, 0, t, z\right)\right|^{2} \mathrm{~d} t\right) . \tag{77}
\end{align*}
$$

Hence from (76) we have

$$
\begin{array}{r}
\int_{B_{r_{0}}^{\prime}}\left|v_{k}\left(y^{\prime}, 0, z\right)\right|^{2} \mathrm{~d} y^{\prime} \leq \int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \mathrm{~d} t\left(\varphi_{\bar{\mu}, k}(t) \int_{B_{r_{0}}^{\prime}}\left|W_{k}\left(y^{\prime}, 0, t, z\right)\right|^{2} \mathrm{~d} y^{\prime}\right) \\
\leq\left(\int_{\frac{-\tilde{\lambda}(\bar{\mu}+1)}{4}}^{\frac{\tilde{\lambda}(\bar{\mu}+1)}{4}} \varphi_{\bar{\mu}, k}(t) \mathrm{d} t\right)\left(C r_{0}^{-\frac{1}{2}} \sigma_{k}\right)^{2} \leq\left(C r_{0}^{-\frac{1}{2}} \sigma_{k}\right)^{2} \tag{78}
\end{array}
$$

We are now in position to conclude the proof of Theorem 2.1. We observe that since the eigenfunctions $e_{j}$ introduced in (33) are even with respect $y_{n}$ and since by (26b) we have

$$
\begin{equation*}
\tilde{a}_{i, n}\left(y^{\prime}, 0\right)=0 \text { for } 1 \leq i \leq n-1 \tag{79}
\end{equation*}
$$

it follows that for any $\left|y^{\prime}\right| \leq 2$

$$
\begin{align*}
\tilde{A}\left(y^{\prime}, 0\right) \nabla v_{k} & \cdot \nu \\
& =-\tilde{a}_{n, n}\left(y^{\prime}, 0\right) \sum_{j=1}^{\infty} \alpha_{j} \hat{\varphi}_{\bar{\mu}, k}\left(\sqrt{\lambda_{j}}\right) g_{k}\left(z \sqrt{\lambda_{j}}\right) \partial_{y_{n}} e_{j}\left(y^{\prime}, 0\right)=0 \tag{80}
\end{align*}
$$

where $\nu=(0, \ldots, 0,-1)$. Hence by (45), (75) and (80)

$$
\begin{cases}q(y) \partial_{z}^{2} v_{k}+\operatorname{div}\left(\tilde{A}(y) \nabla_{x} v_{k}\right)=f_{k}(y, z), & |y| \leq r_{0},|z| \leq \frac{r_{0}}{8}  \tag{81}\\ \left\|v_{k}(\cdot, 0, z)\right\|_{L^{2}\left(B_{r_{0}}^{\prime}\right) \leq C r_{0}^{-\frac{1}{2}} \sigma_{k},} & |z| \leq \frac{r_{0}}{8} \\ \tilde{A}\left(y^{\prime}, 0\right) \nabla v_{k} \cdot \nu=0, & \left|y^{\prime}\right| \leq r_{0}, \quad|z| \leq \frac{r_{0}}{8}\end{cases}
$$

Finally combining (46), (47), quantitative estimates for the Cauchy problem (81) (see Theorems 3.5 and 3.6 in [31]), we obtain the following

$$
\begin{equation*}
\left\|v_{k}\right\|_{L^{2}\left(\tilde{B}_{\frac{r_{0}}{32}}\right.} \leq C\left(\epsilon+H_{1}\left(C r_{0}\right)^{2 k}\right)^{\beta^{2}}\left(H_{1} e^{2 k}+H_{1}\left(C r_{0}\right)^{2 k}\right)^{1-\beta^{2}} \tag{82}
\end{equation*}
$$

where $C>0$ depends on $\tilde{\lambda}$ and $\tilde{\Lambda}$.
Let us observe that the above inequality replace Theorem 3.6 in [31]. The same arguments discussed in [31] from Theorem 3.7 and on go through for the present case and lead to the desired estimate (14).

## 4. Auxiliary results

Proof of Proposition 3.1. Let $\Psi \in C^{1,1}\left(B_{\rho_{0}}\right)$ be the map defined as

$$
\begin{equation*}
\Psi\left(y^{\prime}, y_{n}\right)=\left(y^{\prime}, y_{n}+\phi\left(y^{\prime}\right)\right) . \tag{83}
\end{equation*}
$$

For any $r \in\left(0, \frac{\rho_{0}}{\sqrt{2}(C+1)}\right)$ we have that

$$
\begin{equation*}
K_{\frac{r}{\sqrt{2}(E+1)}} \subset \Psi\left(B_{r}^{-}\right) \subset K_{\sqrt{2}(E+1) r} \tag{84}
\end{equation*}
$$

where $B_{r}^{-}=\left\{y \in \mathbb{R}^{n}:\left|y^{\prime}\right|<r, y_{n}<0\right\}$ and furthermore we get

$$
\begin{equation*}
|\operatorname{det} D \Psi|=1 \tag{85}
\end{equation*}
$$

Denoting by

$$
\begin{align*}
& \sigma(y)=\left(D \Psi^{-1}\right)(\Psi(y)) A(\Psi(y))\left(D \Psi^{-1}\right)^{T}(\Psi(y))  \tag{86}\\
& \gamma^{\prime}(y)=\gamma(\Psi(y))  \tag{87}\\
& \gamma_{0}^{\prime}=\gamma^{\prime}(0) \tag{88}
\end{align*}
$$

it follows that

$$
\begin{align*}
& \sigma(0)=A(0)  \tag{89}\\
& \left\|\sigma_{i, j}\right\|_{C^{0,1}\left(B^{+} \frac{\rho_{0}}{\sqrt{2}(C+1)}\right)} \leq \Sigma, \text { for } i, j=1, \ldots, n,  \tag{90}\\
& \left\|\gamma_{i, j}^{\prime}\right\|_{C^{0,1}\left(B^{\prime} \frac{\rho_{0}}{\sqrt{2}(C+1)}(0)\right)} \leq \Lambda^{\prime} \tag{91}
\end{align*}
$$

where $\Sigma, \Lambda^{\prime}$ are positive constants depending on $E, \Lambda, \rho_{0}$ only.
Dealing as in Proposition 4.3 in [28] we look for a solution to (16) of the form

$$
\begin{equation*}
\psi\left(x^{\prime}, x_{n}\right)=\psi^{\prime}\left(\Psi^{-1}\left(x^{\prime}, x_{n}\right)\right) \tag{92}
\end{equation*}
$$

where $\psi^{\prime}$ is a solution to

$$
\begin{cases}\operatorname{div}\left(\sigma(y) \nabla \psi^{\prime}\right)=0, & \text { in } B_{r_{2}}^{-}  \tag{93}\\ \sigma \nabla \psi^{\prime} \cdot \nu^{\prime}+\gamma^{\prime} \psi^{\prime}=0, & \text { on } B_{r_{2}}^{\prime}\end{cases}
$$

with $r_{2}=\min \left\{\rho_{0}, \frac{\lambda n^{-n / 2}}{12 \bar{\gamma}}\right\}$.
And in turn, as in Claim 4.4 of [28], we search for a solution $\psi^{\prime}$ to (93) such that $\psi^{\prime}=\psi_{0}-s$, where $\psi_{0}$ is a solution to

$$
\begin{cases}\operatorname{div}\left(A(0) \nabla \psi_{0}\right)=0, & \text { in } B_{r_{2}}^{-}  \tag{94}\\ A(0) \nabla \psi_{0} \cdot \nu^{\prime}+\gamma_{0}^{\prime} \psi_{0}=0, & \text { on } B_{r_{2}}^{\prime}\end{cases}
$$

satisfying $\psi_{0} \geq 2$ in $B_{r_{2}}^{-}$and where $s \in H^{1}\left(B_{r_{2}}^{-}\right)$is a weak solution to the problem

$$
\left\{\begin{array}{lc}
\operatorname{div}(\sigma \nabla s)=-\operatorname{div}\left((\sigma-A(0)) \nabla \psi_{0}\right), & \text { in } B_{r_{2}}^{-}  \tag{95}\\
\sigma \nabla s \cdot \nu^{\prime}+\gamma^{\prime} s=(\sigma-A(0)) \nabla \psi_{0} \cdot \nu^{\prime}+\left(\gamma^{\prime}-\gamma_{0}\right) \psi_{0}, & \text { on } B_{r_{2}}^{\prime} \\
s=0, & \text { on }|y|=r_{2}
\end{array}\right.
$$

such that $s(y)=\mathcal{O}\left(|y|^{2}\right)$ near the origin. The proof of the latter relies on a slight adaptation of the arguments in Claim 4.4 of [28].

In order to construct $\psi_{0}$, we introduce the following linear change of variable $L=\left(l_{i, j}\right)_{i, j=1, \ldots, n}$ (see also [18])

$$
\begin{align*}
L: & \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}  \tag{96}\\
& \xi \mapsto L \xi=R \sqrt{A^{-1}(0)} \xi \tag{97}
\end{align*}
$$

where $R$ is the planar rotation in $\mathbb{R}^{n}$ that rotates the unit vector $\frac{v}{\|v\|}$, where $v=\sqrt{A(0)} e_{n}$ to the $n$th standard unit vector $e_{n}$, and such that

$$
\left.\left.R\right|_{(\pi)^{\perp}} \equiv I d\right|_{(\pi)^{\perp}}
$$

where $\pi$ is the plane in $\mathbb{R}^{n}$ generated by $e_{n}, v$ and $(\pi)^{\perp}$ denotes the orthogonal complement of $\pi$ in $\mathbb{R}^{n}$. For this choice of $L$ we have
i) $A(0)=L^{-1} \cdot\left(L^{-1}\right)^{T}$,
ii) $(L \xi) \cdot e_{n}=\frac{1}{\|v\|} \xi \cdot e_{n}$.
which means that $L^{-1}: x \mapsto \xi$ is the linear change of variables that maps $I$ into $A(0)$.

By defining $\tilde{L}$ as the $(n-1) \times(n-1)$ matrix such that $\tilde{L}=(l)_{i, j=1 \cdots, n-1}$ we have that the function

$$
\begin{equation*}
\bar{\psi}(\xi)=8 e^{-|\operatorname{det} L||\operatorname{det} \tilde{L}|^{-1} \gamma_{0}^{\prime} \xi_{n}} \cos \left(|\operatorname{det} L||\operatorname{det} \tilde{L}|^{-1} \xi_{1} \gamma_{0}^{\prime}\right) \tag{98}
\end{equation*}
$$

is a solution to

$$
\begin{cases}\Delta \bar{\psi}=0, & \text { in } B_{r_{3}}^{-}  \tag{99}\\ \nabla \bar{\psi} \cdot \nu^{\prime}+|\operatorname{det} L||\operatorname{det} \tilde{L}|^{-1} \gamma_{0}^{\prime} \bar{\psi}=0, & \text { on } B_{r_{3}}^{\prime}\end{cases}
$$

where $r_{3}=\frac{1}{2} \frac{\Lambda^{\frac{1}{2}}}{\rho_{0}} r_{2}$.
Finally we observe that by setting

$$
\begin{equation*}
\psi_{0}(y)=\bar{\psi}(L y) \tag{100}
\end{equation*}
$$

we end up with a weak solution to (94) such that

$$
\begin{equation*}
\left|\psi_{0}\right|>2 \text { in } B_{r_{2}}^{-}(0) . \tag{101}
\end{equation*}
$$

Hence the thesis follows by choosing $r_{1}=\frac{r_{2}}{\sqrt{2}(E+1)} \psi\left(x^{\prime}, x_{n}\right)=\psi^{\prime}\left(\phi^{-1}\left(x^{\prime}, x_{n}\right)\right)$ and $\psi^{\prime}=\psi_{0}-s$.

Proposition 4.1. There exists a complete orthonormal system of eigenfunctions $e_{j}$ in $L_{+}^{2}\left(B_{2}, q d y\right)=\left\{f \in L^{2}\left(B_{2}, q d y\right)\right.$ s.t. $\left.f\left(y^{\prime}, y_{n}\right)=f\left(y^{\prime},-y_{n}\right)\right\}$ associated to the Dirichlet problem (31).

Proof. Let us start by observing that from (26) and since

$$
\begin{gather*}
\tilde{a}_{n i}\left(y^{\prime}, 0\right)=\bar{a}_{i n}\left(y^{\prime}, 0\right)=0, \quad \text { for } i \in\{1, \ldots, n-1\},  \tag{102a}\\
\tilde{a}_{n n}(0)=1, \tag{102b}
\end{gather*}
$$

it follows that

$$
\begin{equation*}
\operatorname{div}\left(\tilde{A}(y) \nabla_{y}\left(u\left(y^{\prime},-y_{n}\right)\right)\right)=\left.\operatorname{div}\left(\tilde{A}(z) \nabla_{z}(u(z))\right)\right|_{z=\left(y^{\prime},-y_{n}\right)} \tag{103}
\end{equation*}
$$

for any smooth function $u$.
We set

$$
\begin{equation*}
u^{+}(y)=\frac{u\left(y^{\prime}, y_{n}\right)+u\left(y^{\prime},-y_{n}\right)}{2} \tag{104}
\end{equation*}
$$

and we observe that being $q$ even with respect to $y_{n}$ then we have that if $u$ is a solution to (32) then $u^{+}$is a solution to (32) as well.

Let us denote by $\lambda_{j}$, with $0<\lambda_{1} \leq \lambda_{2} \leq \ldots \lambda_{j} \leq \ldots$ the eigenvalues associated to the Dirichlet problem (32) and let $\left\{\mathrm{S}_{1}, \mathrm{~S}_{2}, \ldots, \mathrm{~S}_{\mathrm{j}}, \ldots\right\}$ be a complete orthonormal system of eigenfunctions in $L^{2}\left(B_{2}, q \mathrm{~d} y\right)$.

Let us now fix $j \in \mathbb{N}$ and let $\left\{\mathrm{S}_{\mathrm{j}_{1}}, \mathrm{~S}_{\mathrm{j}_{2}}, \ldots, \mathrm{~S}_{\mathrm{j}_{\mathrm{k}}}\right\}$ be such that they span the eigenspace corresponding to the eigenvalue $\lambda_{j}$. We restrict our attention to the non trivial functions $\mathrm{S}_{\mathrm{j}_{1}}^{+}, \mathrm{S}_{\mathrm{j}_{2}}^{+}, \ldots, \mathrm{S}_{\mathrm{j}_{\mathrm{h}_{\mathrm{j}}}}^{+}$among $\mathrm{S}_{\mathrm{j}_{1}}^{+}, \mathrm{S}_{\mathrm{j}_{2}}^{+}, \ldots, \mathrm{S}_{\mathrm{j}_{\mathrm{k}_{\mathrm{j}}}}^{+}$with $h_{j} \leq k_{j}$.

Using a Gram-Schmidt orthogonalization procedure in the Hilbert space $L_{+}^{2}\left(B_{2}, q \mathrm{~d} y\right)$ we may find our desired eigenfunctions $e_{j_{1}}, \ldots, e_{j_{h_{j}}}$ such that

$$
\begin{equation*}
\left(e_{j_{l}}, e_{j_{k}}\right)=\int_{B_{2}} q(y) e_{j_{l}}(y) e_{j_{k}}(y) \mathrm{d} y=\delta_{j_{l} j_{k}} \tag{105}
\end{equation*}
$$

and $e_{j_{l}}$ are even in $y_{n}$ for $l=1, \ldots, h_{j}$.
It turns out that the system of eigenfunctions

$$
\begin{equation*}
\mathcal{S}=\left\{e_{1_{1}}, \ldots, e_{1_{h_{1}}}, e_{2_{1}}, \ldots, e_{2_{h_{2}}}, \ldots, e_{j_{1}}, \ldots, e_{j_{h_{j}}}, \ldots\right\} \tag{106}
\end{equation*}
$$

is an orthonormal system by construction. Finally we wish to prove that $\mathcal{S}$ is complete in $L_{+}^{2}\left(B_{2}, q \mathrm{~d} y\right)$. To this end we assume that $f \in L_{+}^{2}\left(B_{2}, q \mathrm{~d} y\right)$ is such that

$$
\begin{equation*}
\int_{B_{2}} f(y) e(y) q(y) \mathrm{d} y=0 \quad \forall e \in \mathcal{S} \tag{107}
\end{equation*}
$$

and we claim that $f \equiv 0$.
In order to prove the claim above, we observe that by (107) we have that for any $j \in \mathbb{N}$ the function $f$ in (107) is orthogonal with respect the $L_{+}^{2}\left(B_{2}, q \mathrm{~d} y\right)$ scalar product to the $\operatorname{span}\left\{e_{j_{1}}, \ldots, e_{j_{h_{j}}}\right\}$ and as a consequence to the $\operatorname{span}\left\{\mathrm{S}_{\mathrm{j}_{1}}^{+}, \ldots, \mathrm{S}_{\mathrm{j}_{\mathrm{k}_{\mathrm{j}}}}^{+}\right\}$as well. In particular the following holds

$$
\begin{equation*}
\int_{B_{2}} f(y) q(y) \mathrm{S}_{\mathrm{j}_{\mathrm{i}}}^{+}(y) \mathrm{d} y=0 \quad, \quad j=1, \ldots, k_{j} \tag{108}
\end{equation*}
$$

On the other hand since $q$ and $f$ are even w.r.t. $y_{n}$ we have that

$$
\begin{equation*}
\int_{B_{2}} f(y) q(y) \mathrm{S}_{\mathrm{j}_{\mathrm{i}}}^{+}(y) \mathrm{d} y=\int_{B_{2}} f(y) q(y) \mathrm{S}_{\mathrm{j}_{\mathrm{i}}}(y) \mathrm{d} y \quad, \quad j=1, \ldots, k_{j} . \tag{109}
\end{equation*}
$$

Finally we observe that being the system $\left\{\mathrm{S}_{1}, \mathrm{~S}_{2}, \ldots, \mathrm{~S}_{\mathrm{j}}, \ldots\right\}$ complete in $L^{2}\left(B_{2}, q \mathrm{~d} y\right)$ then $f \equiv 0$ as claimed above.

## 5. Conclusions

Let us conclude by summarizing the main steps of our strategy.

- We first introduce in Proposition 3.1 a strictly positive solution $\psi$ to the elliptic problem (16) such that by the change of variable

$$
\begin{equation*}
u^{\star}=\frac{U}{\psi} \tag{110}
\end{equation*}
$$

we reformulate our original problem for a Robin boundary condition (10)(11) in terms of a new one (20)-(21) where a Neumann condition arises instead.

- Second, in (39) we take advantage of the Boman transform [7] in order to perform a suitable transformation of the wave equation in a nonhomogeneous second order elliptic equation (45). Furthermore, we observe that the solution $v_{k}$ to (45) may be represented as

$$
\begin{equation*}
v_{k}(y, z)=\int_{\mathbb{R}} \varphi_{\bar{\mu}, k}(t) W_{k}(y, t, z) \mathrm{d} t \tag{111}
\end{equation*}
$$

where $\varphi_{\bar{\mu}, k}$ is a suitable sequence of mollifiers and $W_{k}\left(y^{\prime}, 0, \cdot, \cdot\right)$ is a solution to the following two dimensional Cauchy problem for a nonhomogeneous elliptic equation

$$
\left\{\begin{array}{l}
\Delta_{t, z} W_{k}\left(y^{\prime}, 0, t, z\right)=\partial_{t} F_{k}\left(y^{\prime} 0, t, z\right)  \tag{112}\\
W_{k}\left(y^{\prime}, 0, t, 0\right)=\sum_{j=1}^{\infty} \alpha_{j} \cos \left(\sqrt{\lambda_{j}} t\right) e_{j}\left(y^{\prime}, 0\right)=\tilde{u}\left(y^{\prime}, 0, t\right) \\
\partial_{z} W_{k}\left(y^{\prime}, 0, t, 0\right)=0
\end{array}\right.
$$

for any $y \in B_{2}$.
We furthermore, observe that the Dirichlet datum of the above problem can be controlled from above by $\epsilon$ in view of (38) and (29), whereas the Neumann datum vanishes in view of the specific choice discussed in Proposition 4.1 for the eigenfunctions $e_{j}$. The right hand side of the elliptic equation in (112), although is in divergence form, it can be handled as well by gathering a refinements of the arguments in Proposition 3.6 of [31] and in Theorem 1.7 of [4], in order to get the following estimate

$$
\begin{equation*}
\int_{B_{r_{0}}^{\prime}}\left|W_{k}\left(y^{\prime}, 0, t, z\right)\right| d y^{\prime} \leq\left(C r_{0}^{\frac{1}{2}} \sigma_{k}\right)^{2} \tag{113}
\end{equation*}
$$

- Finally, by combining the latter with (56) and again the special choice for the eigenfunctions $e_{j}$ we end up with the Cauchy problem (81) which in turn leads to the desired estimate (82).


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# Some properties of De Giorgi classes 

Emmanuele DiBenedetto and Ugo Gianazza

Dedicated to Giovanni Alessandrini for his 60th Birthday


#### Abstract

The De Giorgi classes $[D G]_{p}(E ; \gamma)$, defined in $(1)_{ \pm}$below encompass, solutions of quasilinear elliptic equations with measurable coefficients as well as minima and Q-minima of variational integrals. For these classes we present some new results (§ 2 and § 3.1), and some known facts scattered in the literature (§3-§5), and formulate some open issues (§ 6).

Keywords: De Giorgi classes, Hölder continuity, Harnack inequality, higher integrability, boundary behavior, decay estimates. MS Classification 2010: 35J15, 49N60, 35J92.


## 1. Introduction

Let $E$ be open subset of $\mathbb{R}^{N}$ and for $y \in \mathbb{R}^{N}$, let $K_{\rho}(y)$ denote a cube of edge $2 \rho$ centered at $y$. The De Giorgi classes $[D G]_{p}^{ \pm}(E ; \gamma)$ in $E$ are the collection of functions $u \in W_{l o c}^{1, p}(E)$, for some $p>1$, satisfying

$$
\begin{equation*}
\int_{K_{\rho}(y)}\left|D(u-k)_{ \pm}\right|^{p} d x \leq \frac{\gamma}{(R-\rho)^{p}} \int_{K_{R}(y)}\left|(u-k)_{ \pm}\right|^{p} d x \tag{1}
\end{equation*}
$$

for all cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$, and all $k \in \mathbb{R}$, for a given positive constant $\gamma$. We further define

$$
\begin{equation*}
[D G]_{p}(E ; \gamma)=[D G]_{p}^{+}(E ; \gamma) \cap[D G]_{p}^{-}(E ; \gamma) \tag{2}
\end{equation*}
$$

A celebrated theorem of De Giorgi [2] states that functions $u \in[D G]_{p}(E ; \gamma)$ are locally bounded and locally Hölder continuous in $E$. Moreover, non-negative functions $u \in[D G]_{p}(E ; \gamma)$ satisfy the Harnack inequality [7].

Local sub(super)-solutions, in $W_{\text {loc }}^{1, p}(E)$, of quasi-linear elliptic equations in divergence form belong to $[D G]_{p}^{+(-)}(E ; \gamma)([12])$, with $\gamma$ proportional to the ratio of upper and lower modulus of ellipticity. Local minima and/or $Q$ minima of variational integrals with $p$-growth with respect to $|D u|$ belong to these classes ([10]). Thus the $[D G]_{p}$-classes include local solutions of elliptic equations with merely bounded and measurable coefficients, only subject to
some upper and lower ellipticity condition. They also include local minima or $Q$-minima of rather general functionals, even if not admitting a Euler equation.

The interest in the De Giorgi classes stems from the large class of, seemingly unrelated functions they encompass, and from properties, such as local Hölder continuity ([2]), and the Harnack inequality ([7]), typically regarded as properties of solutions of elliptic partial differential equations ( $[12,14]$ ).

The purpose of this note is to present some new results on De Giorgi classes (§ 2 and $\S 3.1$ ), as well as collecting some known facts scattered in the literature (§ 3-§5), and formulate some open issues (§6) to serve as a basis for further investigations.

## 2. De Giorgi Classes and Sub(Super)-Harmonic Functions

The generalized De Giorgi classes $[G D G]_{p}^{ \pm}(E ; \gamma)$, are the collection of functions $u \in W_{\text {loc }}^{1, p}(E)$, for some $p>1$, satisfying

$$
\begin{equation*}
\int_{K_{\rho}(y)}\left|D(u-k)_{ \pm}\right|^{p} d x \leq \frac{\gamma}{(R-\rho)^{p}}\left(\frac{R}{R-\rho}\right)^{N p} \int_{K_{R}(y)}\left|(u-k)_{ \pm}\right|^{p} d x \tag{3}
\end{equation*}
$$

for all cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$, and all $k \in \mathbb{R}$, for a given positive constant $\gamma$. Convex, monotone, non-decreasing functions of sub-harmonic functions are sub-harmonic. Similarly, concave, non-decreasing, functions of super-harmonic functions are super-harmonic. Similar statements hold for weak, sub(super)solutions of linear elliptic equations with measurable coefficients ([14]). The next lemma establishes analogous properties for functions $u \in[D G]_{p}^{ \pm}(E ; \gamma)$. Given any such class, we refer to the set of parameters $\{p, \gamma, N\}$ as the data and say that a constant $C=C$ (data) depends only on the data if it can be quantitatively determined a-priori only in terms of the indicated set of parameters.

Lemma 2.1. Let $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ be convex and non-decreasing, and let $u \in$ $[D G]_{p}^{+}(E ; \gamma)$. There exists a positive constant $\bar{\gamma}$ depending only on the data, and independent of $u$, such that $\varphi(u) \in[G D G]_{p}^{+}(E ; \bar{\gamma})$.

Likewise let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be concave and non-decreasing, and let $u \in$ $[D G]_{p}^{-}(E ; \gamma)$. There exist a positive constant $\bar{\gamma}$ depending only on the data, and independent of $u$, such that $\psi(u) \in[G D G]_{p}^{-}(E ; \bar{\gamma})$.

Proof. By De Giorgi's theorem ([2, 12]), there exists a constant $C=C$ (data), such that for any $u \in[D G]_{p}^{ \pm}(E ; \gamma)$, there holds

$$
\begin{equation*}
\left\|(u-k)_{ \pm}\right\|_{\infty, K_{\rho}(y)} \leq \frac{C}{(R-\rho)^{N}} \int_{K_{R}(y)}(u-k)_{ \pm} d x \tag{4}
\end{equation*}
$$

for every pair of cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$ and all $k \in \mathbb{R}$. It suffices to prove the first statement for $\varphi \in C^{2}(\mathbb{R})$, and verify that $\varphi(u)$ satisfies $(3)_{+}$for cubes $K_{\rho} \subset K_{R}$ centered at the origin of $\mathbb{R}^{N}$. For any such $\varphi$ and all $h \leq k$

$$
\begin{equation*}
(\varphi(u)-\varphi(h))_{+}-\varphi^{\prime}(h)(u-h)_{+}=\int_{\mathbb{R}^{+}}(u-k)_{+} \chi_{[k>h]} \varphi^{\prime \prime}(k) d k \tag{5}
\end{equation*}
$$

From this, a.e. in $E$

$$
\left|D\left[(\varphi(u)-\varphi(h))_{+}-\varphi^{\prime}(h)(u-h)_{+}\right]\right|^{p} \leq\left(\int_{\mathbb{R}}\left|D(u-k)_{+}\right| \chi_{[k>h]} \varphi^{\prime \prime}(k) d k\right)^{p}
$$

Integrate over $K_{\rho}$, take the $p$ root of both sides, and majorize the resulting term on the right-hand first by the continuous version of Minkowski inequality, then by applying the definition $(1)_{+}$of the $[D G]_{p}^{+}(E ; \gamma)$-classes, and finally by using (4). This gives

$$
\begin{aligned}
\| D[(\varphi(u) & \left.-\varphi(h))_{+}-\varphi^{\prime}(h)(u-h)_{+}\right] \|_{p, K_{\rho}} \\
& \leq \int_{\mathbb{R}}\left\|D(u-k)_{+}\right\|_{p, K_{\rho}} \chi_{[k>h]} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C}{R-\rho} \int_{\mathbb{R}}\left\|(u-k)_{+}\right\|_{p, K_{\frac{R+\rho}{2}}} \chi_{[k>h]} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C R^{\frac{N}{p}}}{R-\rho} \int_{\mathbb{R}}\left\|(u-k)_{+}\right\|_{\infty, K_{\frac{R+\rho}{2}}} \chi_{[k>h]} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C R^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{\mathbb{R}}\left(\int_{K_{R}}(u-k)_{+} d x\right) \chi_{[k>h]} \varphi^{\prime \prime}(k) d k \\
& =\frac{C R^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{K_{R}}\left(\int_{\mathbb{R}}(u-k)_{+} \chi_{[k>h]} \varphi^{\prime \prime}(k) d k\right) d x \\
& =\frac{C R^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{K_{R}}\left[(\varphi(u)-\varphi(h))_{+}-\varphi^{\prime}(h)(u-h)_{+}\right] d x \\
& \leq \frac{C}{R-\rho}\left(\frac{R}{R-\rho}\right)^{N}\left\|(\varphi(u)-\varphi(h))_{+}-\varphi^{\prime}(h)(u-h)_{+}\right\|_{p, K_{R}} .
\end{aligned}
$$

In these calculations, we have denoted by $C=C(p, N, \gamma)$ a generic constant depending only upon the data, and that might be different from line to line. In the last two steps we have interchanged the order of integration with the help of Fubini's Theorem and have applied Hölder's inequality. By the convexity and monotonicity of $\varphi$,

$$
\begin{equation*}
(\varphi(u)-\varphi(h))_{+} \geq \varphi^{\prime}(h)(u-h)_{+} \geq 0 . \tag{6}
\end{equation*}
$$

Therefore,

$$
\begin{gathered}
\left\|D(\varphi(u)-\varphi(h))_{+}\right\|_{p, K_{\rho}} \leq \frac{C}{R-\rho}\left(\frac{R}{R-\rho}\right)^{N}\left\|(\varphi(u)-\varphi(h))_{+}\right\|_{p, K_{R}} \\
+\left\|\varphi^{\prime}(h) D(u-h)_{+}\right\|_{p, K_{\rho}}
\end{gathered}
$$

Upon applying the definition of $(1)_{+}$of $[D G]_{p}^{+}(E ; \gamma)$, and then (6), the last term on the right-hand side is majorized by

$$
\frac{C}{R-\rho}\left\|(\varphi(u)-\varphi(h))_{+}\right\|_{p, K_{R}} .
$$

Combining these estimates yields

$$
\begin{equation*}
\int_{K_{\rho}(y)}\left|D(\varphi(u)-k)_{+}\right|^{p} d x \leq \frac{\bar{\gamma}}{(R-\rho)^{p}}\left(\frac{R}{R-\rho}\right)^{N} \int_{K_{R}(y)}(\varphi(u)-k)_{+}^{p} d x \tag{7}
\end{equation*}
$$

for all $k \in \mathbb{R}$ and all $K_{\rho}(y) \subset K_{R}(y) \subset E$, for a constant $\bar{\gamma}=\bar{\gamma}($ data $)$.

If $u \in[D G]_{p}^{-}(E ; \gamma)$ and $\varphi$ is convex, there is no guarantee, in general, that $\varphi(u) \in[G D G]_{p}^{+}(E ; \bar{\gamma})$ for some $\bar{\gamma}=\bar{\gamma}(p, N, \gamma)$. The next lemma provides some sufficient conditions on $\varphi$ for this to occur.

Lemma 2.2. Let $\varphi:(a,+\infty) \rightarrow \mathbb{R}$, for some $a<\infty$ be convex, non-increasing, and such that

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \varphi(t)=\lim _{t \rightarrow+\infty} t \varphi^{\prime}(t)=0 \tag{8}
\end{equation*}
$$

and let $u \in[D G]_{p}^{-}(E ; \gamma)$, with range in $(a,+\infty)$. There exists a positive constant $\bar{\gamma}$ depending only on the data, such that $\varphi(u) \in[G D G]_{p}^{+}(E ; \bar{\gamma})$.

Likewise let $\psi:(-\infty, a) \rightarrow \mathbb{R}$, for some $a>-\infty$, be concave, nonincreasing, and satisfying

$$
\begin{equation*}
\lim _{t \rightarrow-\infty} \psi(t)=\lim _{t \rightarrow-\infty} t \psi^{\prime}(t)=0 \tag{9}
\end{equation*}
$$

and let $u \in[D G]_{p}^{+}(E ; \gamma)$, with range in $(-\infty, a)$. There exists a positive constant $\bar{\gamma}$ depending only on the data, such that $\psi(u) \in[G D G]_{p}^{-}(E ; \bar{\gamma})$.

Proof. It suffices to prove the first statement for $\varphi \in C^{2}(\mathbb{R})$ over congruent cubes $K_{\rho} \subset K_{R}$ centered at the origin. The starting point is the analog of (5), i.e.,

$$
\begin{equation*}
\varphi(u)=\int_{\mathbb{R}}(u-k)_{-} \varphi^{\prime \prime}(k) d k \tag{10}
\end{equation*}
$$

Since $u \in[D G]_{p}^{-}(E ; \gamma)$, by (4) the function $u$ is locally bounded below in $E$, and without loss of generality we may assume $u \geq 0$. Hence, the representation (10)
is well defined by virtue of the assumption (8) on $\varphi$. From this, by taking the gradient of both sides, then taking the $p$-power, and finally integrating over $K_{\rho}$ gives

$$
\int_{K_{\rho}}|D \varphi(u)|^{p} d x=\int_{K_{\rho}}\left|\int_{\mathbb{R}^{+}} D(u-k)_{-} \varphi^{\prime \prime}(k) d k\right|^{p} d x .
$$

The proof now parallels that of Lemma 2.1. Specifically, apply sequentially the continuous version of Minkowski's inequality, the definition (1)_ of the classes $[D G]_{p}^{-}(E ; \gamma)$, the sup-bound (4), interchange the order of integration, and use Hölder's inequality. This gives

$$
\begin{aligned}
\|D \varphi(u)\|_{p, K_{\rho}} & \leq \int_{\mathbb{R}^{+}}\left\|D(u-k)_{-}\right\|_{p, K_{\rho}} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C}{R-\rho} \int_{\mathbb{R}^{+}}\left\|(u-k)_{-}\right\|_{p, K_{\frac{R+\rho}{2}}} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C R^{\frac{N}{p}}}{R-\rho} \int_{\mathbb{R}^{+}}\left\|(u-k)_{-}\right\|_{\infty, K_{\frac{R+\rho}{2}}} \varphi^{\prime \prime}(k) d k \\
& \leq \frac{C R^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{\mathbb{R}^{+}} \int_{K_{R}}(u-k)_{-} \varphi^{\prime \prime}(k) d k \\
& =\frac{C R^{\frac{N}{p}}}{(R-\rho)^{N+1}} \int_{K_{R}} \varphi(u) d x \\
& =\frac{C}{(R-\rho)}\left(\frac{R}{R-\rho}\right)^{N}\|\varphi(u)\|_{p, K_{R}} .
\end{aligned}
$$

Now if $\varphi$ is convex, non-increasing and satisfying (8), the function $(\varphi-\ell)_{+}$, for all $\ell$ in the range of $\varphi$, shares the same properties. Hence,

$$
\int_{K_{\rho}(y)}\left|D(\varphi(u)-\ell)_{+}\right|^{p} d x \leq \frac{C}{(R-\rho)^{p}}\left(\frac{R}{R-\rho}\right)^{N p} \int_{K_{R}(y)}(\varphi(u)-\ell)_{+}^{p} d x
$$

for all cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$ and all $\ell \in \mathbb{R}$.

### 2.1. Some Consequences

The sup-bound in (4) can be given the following sharper form ([7]).
Lemma 2.3. Let $u \in[D G]_{p}^{ \pm}(E ; \gamma)$. Then for all $\sigma>0$ there exists a constant $C_{\sigma}$ depending only upon the data and $\sigma$, such that

$$
\begin{equation*}
\sup _{K_{\rho}(y)}(u-k)_{ \pm} \leq C_{\sigma}\left(\frac{R}{R-\rho}\right)^{\frac{N}{\sigma}}\left(f_{K_{R}(y)}(u-k)_{ \pm}^{\sigma} d x\right)^{\frac{1}{\sigma}} \tag{11}
\end{equation*}
$$

If $u \in[D G]_{p}^{-}(E ; \gamma)$ is non-negative, then Lemma 2.2 with $\varphi(u)=u^{-1}$ and $a=0$, implies that $u^{-1} \in[G D G]_{p}^{+}(E ; \gamma)$. Therefore Lemma 2.3, with $k=0$, implies that for all $\tau>0$,

$$
\begin{equation*}
\frac{1}{\inf _{K_{\rho}(y)} u} \leq C_{\tau}\left(\frac{R}{R-\rho}\right)^{\frac{N}{\tau}}\left(f_{K_{R}(y)} \frac{1}{u^{\tau}} d x\right)^{\frac{1}{\tau}} \tag{12}
\end{equation*}
$$

Proposition 2.4. Let $u$ be a non-negative function in the De Giorgi classes $[D G]_{p}(E ; \gamma)$. Then for any pair of positive numbers $\sigma$ and $\tau$

$$
\begin{equation*}
\frac{\sup _{K_{\rho}(y)} u}{\inf _{K_{\rho}(y)} u} \leq C_{\sigma} C_{\tau}\left(\frac{R}{R-\rho}\right)^{N\left(\frac{1}{\sigma}+\frac{1}{\tau}\right)}\left(f_{K_{R}(y)} u^{\sigma} d x\right)^{\frac{1}{\sigma}}\left(f_{K_{R}(y)} \frac{1}{u^{\tau}} d x\right)^{\frac{1}{\tau}} \tag{13}
\end{equation*}
$$

Inequalities of the form (11) are at the basis of Moser's approach to the Harnack inequality for non-negative weak solutions to quasilinear elliptic equations with bounded and measurable coefficients ([14]). The Harnack inequality will follow from (13) if $\ln u \in B M O(E)$. This fact is established by Moser for non-negative weak solutions of elliptic equations. We will establish that for non-negative functions $u \in[D G]_{p}^{-}(E ; \gamma)$, one has $\ln u \in B M O(E)$ by using the Harnack inequality established in ([7]).

## 3. De Giorgi Classes, $B M O(E)$ and Logarithmic Estimates

The proof of the following lemma is in [7].
Lemma 3.1. Let $u \in[D G]_{p}^{-}(E ; \gamma)$ be non-negative. There exist positive constants $C$ and $\sigma$, depending only upon the data, such that

$$
\begin{equation*}
f_{K_{\rho}(y)} u^{\sigma} d x \leq C \inf _{K_{\rho}(y)} u^{\sigma}, \tag{14}
\end{equation*}
$$

for any cube $K_{\rho}(y)$ such that $K_{2 \rho}(y) \subset E$.
Such an inequality, referred to as the weak Harnack inequality, was established by Moser for non-negative super-solutions of elliptic equations with bounded and measurable coefficients ([14]). It is noteworthy that it continues to hold for non-negative functions in $[D G]_{p}^{-}(E ; \gamma)$, with no further reference to equations.

Lemma 3.2. Let $u \in[D G]_{p}^{-}(E ; \gamma)$ be non-negative. Then $\ln u \in B M O$.

Proof. By Lemma 3.1

$$
\begin{align*}
f_{K_{\rho}(y)} u^{\sigma} d x f_{K_{\rho}(y)} \frac{1}{u^{\sigma}} d x & \leq f_{K_{\rho}(y)} u^{\sigma} d x \sup _{K_{\rho}(y)} \frac{1}{u^{\sigma}} \\
& =f_{K_{\rho}(y)} u^{\sigma} d x \frac{1}{\inf _{K_{\rho}(y)} u^{\sigma}} \leq C \tag{15}
\end{align*}
$$

for any cube $K_{\rho}(y)$ such that $K_{2 \rho}(y) \subset E$. Set

$$
\left(\ln u^{\sigma}\right)_{\rho}=f_{K_{\rho}(y)} \ln u^{\sigma} d x
$$

and estimate

$$
\begin{aligned}
\int_{K_{\rho}(y)} e^{\left|\ln u^{\sigma}-\left(\ln u^{\sigma}\right)_{\rho}\right|} d x \leq & e^{-\left(\ln u^{\sigma}\right)_{\rho}} \int_{K_{\rho}(y)} e^{\ln u^{\sigma}} d x \\
& +e^{\left(\ln u^{\sigma}\right)_{\rho}} f_{K_{\rho}(y)} e^{-\ln u^{\sigma}} d x .
\end{aligned}
$$

The second term on the right-hand side is estimated by Jensen's inequality and (15) and yields

$$
\begin{aligned}
e^{\left(\ln u^{\sigma}\right)_{\rho}} f_{K_{\rho}(y)} e^{-\ln u^{\sigma}} d x & \leq f_{K_{\rho}(y)} e^{\ln u^{\sigma}} d x f_{K_{\rho}(y)} \frac{1}{u^{\sigma}} d x \\
& \leq f_{K_{\rho}(y)} u^{\sigma} d x f_{K_{\rho}(y)} \frac{1}{u^{\sigma}} d x \leq C .
\end{aligned}
$$

The first term is estimated analogously. Hence, there exists a constant $\bar{C}$, depending only upon the data, such that

$$
\int_{K_{\rho}(y)} e^{\left|\ln u^{\sigma}-\left(\ln u^{\sigma}\right)_{\rho}\right|} d x \leq \bar{C}
$$

for any cube $K_{\rho}(y)$ such that $K_{2 \rho}(y) \subset E$. Thus $\ln u \in B M O(E)$.

### 3.1. Logarithmic Estimates Revisited

Let $u \in W_{\mathrm{loc}}^{1, p}(E)$ be a non-negative weak super-solution of an elliptic equation in divergence form, and with only bounded and measurable coefficients. Then there exists a constant $C$, depending only on $p, N$, and the modulus of ellipticity of the equation, such that

$$
\begin{equation*}
f_{K_{\rho}(y)}|D \ln u|^{p} d x \leq \frac{C}{(R-\rho)^{p}} \tag{16}
\end{equation*}
$$

for any pair of cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$. Such an estimate, established by Moser, permits one to prove that $\ln u \in B M O(E)$, which in turn yields the Harnack inequality. Our approach for functions in the $[D G]_{p}^{-}(E ; \gamma)$ classes is somewhat different. For non-negative functions in such classes we first establish the weak Harnack estimate (14), and then the latter is used to prove Lemma 3.2. It is not known, whether non-negative functions in $[D G]_{p}^{-}(E ; \gamma)$ satisfy (16). The next proposition is a partial result in this direction.

Proposition 3.3. Let $u \in[D G]_{p}^{-}(E ; \gamma)$ be non-negative and bounded above by some positive constant $M$. Then

$$
\begin{equation*}
\int_{K_{\rho}(y)}|D \ln u|^{p} d x \leq \frac{\gamma p}{(R-\rho)^{p}} \int_{K_{R}(y)} \ln \frac{M}{u} d x \tag{17}
\end{equation*}
$$

for any pair of cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$.

Proof. The arguments being local may assume that $y=\{0\}$. By the definition (1)_ of classes, for all $0<t<M$,

$$
\int_{K_{\rho}}\left|D(u-t)_{-}\right|^{p} d x \leq \frac{\gamma}{(R-\rho)^{p}} \int_{K_{R}}(u-t)_{-}^{p} d x
$$

Multiply both sides by $t^{-p-1}$ and integrate over $(0, M)$. The left-hand side is transformed as

$$
\begin{aligned}
\int_{0}^{M} \frac{d t}{t^{p+1}} \int_{K_{\rho}}\left|D(u-t)_{-}\right|^{p} d x & =\int_{K_{\rho}}\left(\int_{0}^{M}\left|D(u-t)_{-}\right|^{p} \frac{1}{t^{p+1}} d t\right) d x \\
& =\int_{K_{\rho}}|D u|^{p}\left(\int_{0}^{M} \frac{1}{t^{p+1}} \chi_{[u<t]} d t\right) d x \\
& =\int_{K_{\rho}}|D u|^{p}\left(\int_{u}^{M} \frac{1}{t^{p+1}} d t\right) d x \\
& =\int_{K_{\rho}}\left(-\frac{1}{p} \frac{|D u|^{p}}{M^{p}}+\frac{1}{p} \frac{|D u|^{p}}{u^{p}}\right) d x \\
& =\frac{1}{p} \int_{K_{\rho}}|D \ln u|^{p} d x-\frac{1}{p M^{p}} \int_{K_{\rho}}|D u|^{p} d x
\end{aligned}
$$

The integral on the right-hand side is transformed as

$$
\begin{aligned}
\int_{0}^{M} \frac{1}{t^{p+1}} & \left(\int_{K_{R}}(u-t)_{-}^{p} d x\right) d t \\
& =\int_{K_{R}}\left(\int_{u}^{M} \frac{(t-u)^{p}}{t^{p+1}} d t\right) d x \\
& =\int_{K_{R}}\left[-\left.\frac{1}{p} \frac{(t-u)^{p}}{t^{p}}\right|_{u} ^{M}+\int_{u}^{M} \frac{(t-u)^{p-1}}{t^{p-1}} \frac{d t}{t}\right] d x \\
& =-\frac{1}{p M^{p}} \int_{K_{R}}(M-u)^{p} d x+\int_{K_{R}}\left(\int_{u}^{M}\left(\frac{t-u}{t}\right)^{p-1} \frac{d t}{t}\right) d x \\
& \leq-\frac{1}{p M^{p}} \int_{K_{R}}(M-u)^{p} d x+\int_{K_{R}} \ln \frac{M}{u} d x
\end{aligned}
$$

Combining the previous estimates gives

$$
\begin{aligned}
\int_{K_{\rho}}|D \ln u|^{p} d x \leq & \frac{1}{M^{p}}\left(\int_{K_{\rho}}|D u|^{p} d x-\frac{\gamma}{(R-\rho)^{p}} \int_{K_{R}}(M-u)^{p} d x\right) \\
& +\frac{\gamma p}{(R-\rho)^{p}} \int_{K_{R}} \ln \frac{M}{u} d x
\end{aligned}
$$

Since $u \in[D G]_{p}^{-}(E ; \gamma)$, the term in round brackets on the right-hand side is non-positive and can be discarded.

Remark 3.4. Applying Lemma 2.2 to $\varphi(u)=\ln _{+}(M / u)$, gives the weaker estimate

$$
\begin{equation*}
\int_{K_{\rho}(y)}|D \ln u|^{p} d x \leq \frac{\bar{\gamma}}{(R-\rho)^{p}} \int_{K_{R}(y)}\left(\ln \frac{M}{u}\right)^{p} d x . \tag{18}
\end{equation*}
$$

## 4. Higher Integrability of the Gradient of Functions in the De Giorgi Classes

Proposition 4.1. Let $u \in[D G]_{p}^{ \pm}(E)$. Then there exist constants $C>1$ and $\sigma>0$, dependent only upon the data, such that, for any pair of cubes $K_{\rho}(y) \subset$ $K_{R}(y) \subset E$, there holds

$$
\begin{equation*}
\left(f_{K_{\rho}(y)}|D u|^{p(1+\sigma)} d x\right)^{\frac{1}{p(1+\sigma)}} \leq C\left(\frac{R}{\rho}\right)^{\frac{N}{p}}\left(\frac{R}{R-\rho}\right)\left(f_{K_{R}(y)}|D u|^{p} d x\right)^{\frac{1}{p}} \tag{19}
\end{equation*}
$$

Proof. Let $u$ be in the classes $[D G]_{p}(E ; \gamma)$ defined in (2). For any pair of cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$, write down $(1)_{+}$and $(1)_{-}$for the choice

$$
k=u_{R} \stackrel{\text { def }}{=} f_{K_{R}(y)} u d x .
$$

Adding the resulting inequalities gives

$$
\int_{K_{\rho}(y)}|D u|^{p} d x \leq \frac{\gamma}{(R-\rho)^{p}} \int_{K_{R}(y)}\left|u-u_{R}\right|^{p} d x .
$$

By the Sobolev-Poincaré inequality

$$
f_{K_{R}(y)}\left|u-u_{R}\right|^{p} d x \leq C_{q} R^{p}\left(f_{K_{R}(y)}|D u|^{q} d x\right)^{\frac{p}{q}}, \quad \text { for all } q \in\left[\frac{N p}{N+p}, p\right]
$$

for a constant $C_{q}=C_{q}(N, q)$. Hence, for all such $q$

$$
f_{K_{\rho}(y)}|D u|^{p} d x \leq C_{q} \gamma\left(\frac{R}{R-\rho}\right)^{p}\left(\frac{R}{\rho}\right)^{N}\left(f_{K_{R}(y)}|D u|^{q} d x\right)^{\frac{p}{q}}
$$

for all pair of congruent cubes $K_{\rho}(y) \subset K_{R}(y) \subset E$. The conclusion follows from this and the local version of Gehring's lemma ([9]), as appearing in [11].

Remark 4.2. Hence, the higher integrability of the gradient of solutions of elliptic equations with measurable coefficients ([15]), and more generally of $Q$-minima ( $[10]$ ), continues to hold for function in the De Giorgi classes. If $u \in[D G]_{p}^{ \pm}(E ; \gamma)$, the conclusion is in general false, as one can verify starting from sub(super)-harmonic functions. However, essentially the same arguments give the inequality

$$
f_{K_{\rho}(y)}\left|D(u-k)_{ \pm}\right|^{p} d x \leq C_{q} \gamma\left(\frac{R}{R-\rho}\right)^{p}\left(\frac{R}{\rho}\right)^{N}\left(f_{K_{R}}|D u|^{q} d x\right)^{\frac{p}{q}}
$$

for all $q \in\left[\frac{N p}{N+p}, p\right]$, and

$$
\begin{aligned}
& \text { all } k \geq f_{K_{R}(y)} u d x \quad \text { if } u \in[D G]_{p}^{+}(E ; \gamma), \\
& \text { all } k \leq f_{K_{R}(y)} u d x \quad \text { if } u \in[D G]_{p}^{-}(E ; \gamma) .
\end{aligned}
$$

## 5. Measure Theoretical Decay Estimates of Functions in De Giorgi Classes

For a non-negative function $f \in L_{\text {loc }}^{1}(E)$ one estimates the measure of the set $[f>t]$ relative to a cube $K_{\rho}(y) \subset E$, as $\mu\left([f>t] \cap K_{\rho}(y)\right) \leq t^{-1}\|f\|_{1, K_{\rho}(y)}$. Estimates of the measure of the set $[f<t]$ relative to $K_{\rho}(y)$ are not, in general, a consequence of the mere integrability of $f$. One of De Giorgi's estimates of [2], is that if $u$ is a non-negative function in $[D G]_{p}^{-}(E ; \gamma)$, then

$$
\begin{equation*}
\frac{\left|[u<t] \cap K_{\rho}(y)\right|}{\left|K_{\rho}\right|} \leq \frac{C(N, p, \gamma)}{|\ln t|^{1 / p}} \quad \text { asymptotically as } t \rightarrow 0 \tag{20}
\end{equation*}
$$

provided $\left|[u>t] \cap K_{\rho}(y)\right| \geq \frac{1}{2}\left|K_{\rho}\right|$. Here $|\sigma|$ denotes the Lebesgue measure of a measurable set $\sigma \subset \mathbb{R}^{N}$. The next proposition improves on this estimate.
Proposition 5.1. Let $u \in[D G]_{p}^{-}(E ; \gamma)$ be non-negative, and assume that for some $t_{o}>0$ and $\alpha \in(0,1)$, there holds

$$
\begin{equation*}
\frac{\left.\mid\left[u>t_{o}\right] \cap K_{\rho}(y)\right] \mid}{\left|K_{\rho}\right|} \geq \alpha . \tag{21}
\end{equation*}
$$

There exist positive constants $C, t_{*}, \sigma=C, t_{*}, \sigma\left(N, p, \gamma, t_{o}, \alpha\right)$, depending only on the indicated parameters and independent of $u$, such that

$$
\begin{equation*}
\frac{\left|[u<t] \cap K_{\rho}(y)\right|}{\left|K_{\rho}\right|} \leq \frac{C}{|\ln t|^{\sigma|\ln t|^{\frac{1}{2}}}}, \quad \text { for } t<t_{*} \text {. } \tag{22}
\end{equation*}
$$

Proof. In what follows we denote by $C$ a generic positive constant that can be determined a-priori only in terms of $\left\{N, p, \gamma, t_{o}, \alpha\right\}$ and that it may be different in the same context. The arguments being local to concentric cubes $K_{\rho}(y) \subset K_{2 \rho}(y) \subset E$, may assume $y=\{0\}$ and write $K_{\rho}(0)=K_{\rho}$. Let $n_{o}$ be the smallest positive integer such that $2^{-n_{o}} \leq t_{o}$, and for $n \geq n_{o}$ set

$$
A_{n, \rho} \stackrel{\text { def }}{=}\left[u<\frac{1}{2^{n}}\right] \cap K_{\rho}, \quad \text { for } n \geq n_{o}
$$

The discrete isoperimetric inequality ([3, Chapter I, Lemma 2.2]), reads

$$
(\ell-h)\left|[u<h] \cap K_{\rho}\right| \leq C(N) \frac{\rho^{N+1}}{\left|[u>\ell] \cap K_{\rho}\right|} \int_{[h<u<\ell] \cap K_{\rho}}|D u| d x
$$

for any two levels $0<h<\ell$. Applying it with

$$
\ell=\frac{1}{2^{n}}, \quad h=\frac{1}{2^{n+1}}, \quad \text { so that } \quad[h<u<\ell] \cap K_{\rho}=A_{n, \rho}-A_{n+1, \rho}
$$

and taking into account (21), yields

$$
\frac{1}{2^{n+1}}\left|A_{n+1, \rho}\right| \leq \frac{C(N)}{\alpha} \rho^{N} \int_{A_{n, \rho}-A_{n+1, \rho}}|D u| d x .
$$

Majorize the right-hand side by the Hölder inequality, then raise both terms to the power $\frac{p}{p-1}$, and majorize the right-hand side by (1)_ in the definition of the classes $[D G]_{p}^{-}(E ; \gamma)$. These sequential estimates yield

$$
\begin{aligned}
\frac{1}{2^{n \frac{p}{p-1}}}\left|A_{n+1, \rho}\right|^{\frac{p}{p-1}} & \leq C \rho^{\frac{p}{p-1}}\left(\int_{K_{\rho}}\left|D\left(u-\frac{1}{2^{n}}\right)_{-}\right|^{p} d x\right)^{\frac{1}{p-1}}\left|A_{n, \rho}-A_{n+1, \rho}\right| \\
& \leq C\left(\int_{K_{\rho}}\left(u-\frac{1}{2^{n}}\right)_{-}^{p} d x\right)^{\frac{1}{p-1}}\left|A_{n, \rho}-A_{n+1, \rho}\right| \\
& \leq \frac{C}{2^{n \frac{p}{p-1}}}\left|A_{n_{o}, 2 \rho}\right|^{\frac{1}{p-1}}\left|A_{n, \rho}-A_{n+1, \rho}\right| .
\end{aligned}
$$

This in turn yields the recursive inequalities

$$
\left|A_{n+1, \rho}\right|^{\frac{p}{p-1}} \leq C(N, p, \gamma, \alpha)\left|A_{n_{o}, 2 \rho}\right|^{\frac{1}{p-1}}\left|A_{n, \rho}-A_{n+1, \rho}\right| .
$$

Let $n_{*}$ be a positive integer to be chosen. Adding them from $n_{o}$ to $n_{*}-1$ gives

$$
\begin{equation*}
\left|A_{n_{*}, \rho}\right| \leq \frac{C(N, p, \gamma, \alpha)}{\left(n_{*}-n_{o}\right)^{\frac{p-1}{p}}}\left|A_{n_{o}, 2 \rho}\right|^{\frac{1}{p}}\left|A_{n_{o}, \rho}\right|^{\frac{p-1}{p}} . \tag{23}
\end{equation*}
$$

Return now to the assumption (21) and estimate

$$
\frac{\left.\mid\left[u>t_{o}\right] \cap K_{2 \rho}(y)\right] \mid}{\left|K_{2 \rho}\right|} \geq \frac{\left.\mid\left[u>t_{o}\right] \cap K_{\rho}(y)\right] \mid}{2^{N}\left|K_{\rho}\right|} \geq \frac{\alpha}{2^{N}} .
$$

Therefore, the same arguments leading to (23) can be repeated over the cube $K_{2 \rho}$ and give

$$
\begin{equation*}
\left|A_{n_{*}, 2 \rho}\right| \leq \frac{C(N, p, \gamma, \alpha)}{\left(n_{*}-n_{o}\right)^{\frac{p-1}{p}}}\left|A_{n_{o}, 4 \rho}\right|^{\frac{1}{p}}\left|A_{n_{o}, 2 \rho}\right|^{\frac{p-1}{p}} . \tag{24}
\end{equation*}
$$

While the constant $C$ in (24) differs from the one in (23), we may take them to be equal by taking the largest. The assumption (21) continue to hold with $t_{o}$ replaced by $2^{-n_{*}}$. Hence, the previous arguments can be repeated and yield the analogues of (23)-(24), i.e.,

$$
\begin{aligned}
\mid A_{2 n_{*}, \rho} & \leq \frac{C(N, p, \gamma, \alpha)}{\left(n_{*}-n_{o}\right)^{\frac{p-1}{p}}}\left|A_{n_{*}, 2 \rho}\right|^{\frac{1}{p}}\left|A_{n_{*}, \rho}\right|^{\frac{p-1}{p}} \\
\left|A_{2 n_{*}, 2 \rho}\right| & \leq \frac{C(N, p, \gamma, \alpha)}{\left(n_{*}-n_{o}\right)^{\frac{p-1}{p}}}\left|A_{n_{*}, 4 \rho}\right|^{\frac{1}{p}}\left|A_{n_{*}, 2 \rho}\right|^{\frac{p-1}{p}}
\end{aligned}
$$

for the same constant $C$. Combining them gives

$$
\left|A_{2 n_{*}, \rho}\right| \leq \frac{C^{2} 4^{2 N}}{\left(n_{*}-n_{o}\right)^{2 \frac{p-1}{p}}}\left|K_{\rho}\right|
$$

Iteration of this procedure yields

$$
\left|A_{j n_{*}, \rho}\right| \leq \frac{C^{j} 4^{j N}}{\left(n_{*}-n_{o}\right)^{j \frac{p-1}{p}}}\left|K_{\rho}\right| \quad \text { for all } j \in \mathbb{N}
$$

Choose $n_{*}$ so large that $n_{*}-n_{o}>\frac{1}{2} n_{*}$, and then take $j=n_{*}$. By possibly modifying the various constants, the previous inequality yields

$$
\left|A_{j^{2}, \rho}\right| \leq \frac{C^{j} 4^{j N}}{j^{j \frac{p-1}{p}}}\left|K_{\rho}\right| \quad \text { for all } j \in \mathbb{N}
$$

The constant $C$ being fixed, for each $0<\varepsilon<\frac{p-1}{p}$ there exists $j^{*}$ so large that

$$
\left|A_{j^{2}, \rho}\right| \leq \frac{1}{j^{j \varepsilon}}\left|K_{\rho}\right| \quad \text { for all } j \geq j^{*}
$$

Fix now $t \leq 2^{-j^{* 2}}$ and let $j$ be the largest integer such that $2^{-(j+1)^{2}} \leq t \leq 2^{-j^{2}}$. For such choices

$$
\frac{\left|[u<t] \cap K_{\rho}\right|}{\left|K_{\rho}\right|} \leq \frac{\left|A_{j^{2}, \rho}\right|}{\left|K_{\rho}\right|} \leq \frac{C}{|\ln t|^{\frac{\varepsilon}{2}|\ln t|^{\frac{1}{2}}}}
$$

The parabolic version of this result has been used in [6].

## 6. Boundary Behavior of Functions in the De Giorgi Classes

Let $h \in W_{\text {loc }}^{1, p}\left(\mathbb{R}^{N}\right) \cap C\left(\mathbb{R}^{N}\right)$. The De Giorgi classes $[D G]_{p}^{+(-)}(\bar{E} ; \gamma, h)$, in the closure of $E$ are the collection of functions $u \in W_{\text {loc }}^{1, p}(\bar{E})$, such that $(u-h) \in$ $W_{o}^{1, p}\left(E \cap K_{R}(y)\right)$, for all cubes $K_{R}(y)$ centered at some $y \in \partial E$, and satisfying

$$
\begin{equation*}
\int_{K_{\rho}(y) \cap E}\left|D(u-k)_{+(-)}\right|^{p} d x \leq \frac{\gamma}{(R-\rho)^{p}} \int_{K_{R}(y) \cap E}(u-k)_{+(-)}^{p} d x \tag{25}
\end{equation*}
$$

for all pairs of congruent cubes $K_{\rho}(y) \subset K_{R}(y)$, centered at some $y \in \partial E$ and all levels

$$
\begin{equation*}
k \geq \sup _{K_{R}(y) \cap \partial E} h, \quad\left(k \leq \inf _{K_{R}(y) \cap \partial E} h\right) \tag{26}
\end{equation*}
$$

We let further

$$
[D G]_{p}(\bar{E} ; \gamma, h)=[D G]_{p}^{+}(\bar{E} ; \gamma, h) \cap[D G]_{p}^{-}(\bar{E} ; \gamma, h)
$$

Functions in $[D G]_{p}(\bar{E} ; \gamma, h)$ are continuous up to points $y \in \partial E$, provided $E$ satisfies a positive geometric density at $y$, i.e., there exist $\rho_{o}$ and $\eta \in(0,1)$, such that (see [12])

$$
\left|E^{c} \cap K_{\rho}(y)\right| \geq \eta\left|K_{\rho}(y)\right|, \quad \text { for all } \rho \leq \rho_{o}
$$

For $1<p<N$, the $p$-capacity of the compact set $E^{c} \cap \bar{K}_{\rho}(y)$ is defined by

$$
\begin{equation*}
c_{p}\left[E^{c} \cap \bar{K}_{\rho}(y)\right]=\inf _{\substack{\psi \in W^{1, p}\left(\mathbb{R}^{N}\right) \cap C\left(\mathbb{R}^{N}\right) \\ E^{c} \cap \bar{K}_{\rho}(y) \subset[\psi \geq 1]}} \int_{\mathbb{R}^{N}}|D \psi|^{p} d x . \tag{27}
\end{equation*}
$$

For $1<p<N$, the relative $p$-capacity of $E^{c} \cap \bar{K}_{\rho}(y)$ with respect to $K_{\rho}(y)$ is

$$
\begin{equation*}
\delta_{y}(\rho)=\frac{c_{p}\left[E^{c} \cap \bar{K}_{\rho}(y)\right]}{\rho^{N-p}}, \quad(1<p<N) \tag{28}
\end{equation*}
$$

If $p=N$, and for $0<\rho<1$, the $N$-capacity of the compact set $E^{c} \cap \bar{K}_{\rho}(y)$, with respect to the cube $K_{2 \rho}(y)$, is defined by

$$
\begin{equation*}
c_{N}\left[E^{c} \cap \bar{K}_{\rho}(y)\right]=\inf _{\substack{\psi \in W_{o}^{1, N}\left(K_{2 \rho}(y)\right) \cap C_{o}\left(K_{2 \rho}(y)\right) \\ E^{c} \cap K_{\rho}(y) \subset[\psi \geq 1]}} \int_{K_{2 \rho}(y)}|D \psi|^{N} d x . \tag{29}
\end{equation*}
$$

The relative capacity $\delta_{y}(\rho)$ can be formally defined by (28), for all $1<p \leq N$. For $p=N$, we let $\delta_{y}(\rho) \equiv c_{N}\left[E^{c} \cap \bar{K}_{\rho}(y)\right]$, as defined by (29). For a positive parameter $\epsilon$ denote by $I_{p, \epsilon}(y, \rho)$ the Wiener integral of $\partial E$ at $y \in \partial E$, i.e.,

$$
\begin{equation*}
I_{p, \epsilon}(y, \rho)=\int_{\rho}^{1}\left[\delta_{y}(t)\right]^{\frac{1}{\epsilon}} \frac{d t}{t} . \tag{30}
\end{equation*}
$$

The celebrated Wiener criterion states that a harmonic function in $E$ is continuous up to $y \in \partial E$ if and only if the Wiener integral $I_{2,1}(y, \rho)$ diverges as $\rho \rightarrow 0$ ([16]).

It is known that weak solutions of quasilinear equations in divergence form, and with principal part exhibiting a $p$-growth with respect to $|D u|$, when given continuous boundary data $h$ on $\partial E$, are continuous up to $y \in \partial E$ if $I_{p,(p-1)}(y, \rho)$ diverges as $\rho \rightarrow 0([8])$. Since such solutions belong to the boundary $[D G]_{p}(\bar{E} ; \gamma, h)$ classes $([10])$, it is natural to ask whether the divergence of the Wiener integral $I_{p,(p-1)}(y, \rho)$, is sufficient to insure the boundary continuity for functions $u \in[D G]_{p}(\bar{E} ; \gamma, h)$.

The only result we are aware of in this direction is due to Ziemer ([17]). It states that a function $u \in[D G]_{p}(\bar{E} ; \gamma, h)$ is continuous up to $y \in \partial E$ if

$$
\begin{equation*}
\int_{\rho}^{1} \exp \left(-\frac{1}{\delta_{y}(t)^{\frac{1}{p-1}}}\right) \frac{d t}{t} \rightarrow \infty \quad \text { as } \rho \rightarrow 0 \tag{31}
\end{equation*}
$$

Ziemer's proof follows from a standard De Giorgi iteration technique. It has been recently established that local minima of variational integrals when given continuous boundary data $h$ are continuous up to $y \in \partial E$ provided ([5]) $I_{p, \varepsilon}(y, \rho)$ diverges as $\rho \rightarrow 0$. Here $\varepsilon$ is a number that can be determined apriori only in terms of the growth properties of the functional. While such minima are in the classes $[D G]_{p}(\bar{E} ; \gamma, h)$, the result is not known to hold for functions merely in such classes. Also the optimal parameter $e=(p-1)$ remains elusive. A similar result has been recently obtained with a different approach in [1].

The significance of a Wiener condition for Q -minima, is that the structure of $\partial E$ near a boundary point $y \in \partial E$, for $u$ to be continuous up to $y$, hinges on minimizing a functional, rather than solving an elliptic p.d.e.

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# Carleman estimates with two large parameters for an anisotropic system of elasticity 

Victor Isakov<br>Dedicated to Professor Giovanni Alessandrini


#### Abstract

We consider the system of partial differential equations of transversely isotropic elasticity with residual stress. Completing previous results we derive Carleman estimates for this system containing time derivatives. This permits to obtain exact observability inequalities for this system with the Cauchy data on the whole lateral boundary.


Keywords: Carleman estimates, inverse problems, elasticity theory. MS Classification 2010: 35R30, 35B60, 74B10.

## 1. Introduction

We consider a transversely isotropic elasticity system with residual stress $[2,15]$. We let $x \in \mathbf{R}^{3}$ and $(x, t) \in \Omega$ which is a bounded domain in $\mathbf{R}^{4}$. Let $\mathbf{u}(x, t)=$ $\left(u_{1}, u_{2}, u_{3}\right)^{\top}: \Omega \rightarrow \mathbf{R}^{3}$ be the displacement vector in $\Omega$. We introduce the operator of the transversely isotropic elasticity

$$
\begin{equation*}
\left(\mathbf{A}_{T} \mathbf{u}\right)_{i}=\sum_{j, k, l=1}^{3} \partial_{j}\left(C_{i j k l} \frac{1}{2}\left(\partial_{k} u_{l}+\partial_{l} u_{k}\right)\right) \tag{1}
\end{equation*}
$$

where $C_{i j l k}$ are elastic parameters. In general, they enjoy the following symmetry properties

$$
\begin{equation*}
C_{i j k l}=C_{j i k l}=C_{i j l k}=C_{k l i j} . \tag{2}
\end{equation*}
$$

In the transversely $\left(\left(x_{1}, x_{2}\right)\right.$-) isotropic case, in addition,

$$
\begin{gather*}
C_{1111}=C_{2222}=c_{11}, C_{1122}=c_{12}, C_{1133}=C_{2233}=c_{13}, C_{3333}=c_{33} \\
C_{2323}=C_{3131}=c_{23}, C_{1212}=\frac{1}{2}\left(c_{11}-c_{12}\right), C_{i j k l}=0 \text { otherwise } \tag{3}
\end{gather*}
$$

We assume that $c_{j k}$ are functions on $\bar{\Omega}$ and impose a sufficient condition of strict positivity of the elastic tensor:

$$
\begin{align*}
& \varepsilon_{0}<c_{11}, \varepsilon_{0}<c_{11}-c_{12}, \varepsilon_{0}<c_{12}+c_{11}, \\
& \varepsilon_{0}<c_{23}, \varepsilon_{0}<c_{33}, \varepsilon_{0}<c_{13}+c_{23},  \tag{4}\\
& \varepsilon_{0}+2 c_{13}^{2}<\left(c_{11}+c_{12}\right) c_{33}, \varepsilon_{0}+c_{13}^{2}<c_{11} c_{33} \quad \text { on } \Omega
\end{align*}
$$

for some $\varepsilon_{0}>0$. We also introduce the scalar partial differential operator $R=\sum_{j, k=1}^{3} r_{j k} \partial_{j} \partial_{k}$ used to model the residual stress.

To state the main results we introduce pseudo convexity condition for a general scalar partial differential operator of second order $P=\sum_{j, k=1}^{n} a_{j k} \partial_{j} \partial_{k}$ in $\Omega$ with the real-valued coefficients $a^{j k} \in C^{1}(\bar{\Omega})$. The principal symbol of this operator is $P(X ; \zeta)=\sum_{j, k=1}^{n} a_{j k}(X) \zeta_{j} \zeta_{k}, X=(x, t)$. We will assume that the coefficients of $P$ admit the following bound $\left|a_{j k}\right|_{2}(\Omega) \leq M$.

Let $K$ be a positive constant. A function $\psi$ is called $K$-pseudo-convex on $\Omega$ with respect to $P$ if $\psi \in C^{2}(\bar{\Omega}), P(X, \nabla \psi(X)) \neq 0, X \in \bar{\Omega}$, and

$$
\begin{aligned}
\sum_{j, k=1}^{4}\left(\partial_{j} \partial_{k} \psi \frac{\partial P}{\partial \zeta_{j}}\right. & \left.\frac{\partial P}{\partial \zeta_{k}}\right)(X ; \xi) \\
& +\sum_{j, k=1}^{4}\left(\left(\frac{\partial P}{\partial \zeta_{k}} \partial_{k} \frac{\partial P}{\partial \zeta_{j}}-\partial_{k} P \frac{\partial^{2} P}{\partial \zeta_{j} \partial \zeta_{k}}\right) \partial_{j} \psi\right)(X, \xi) \geq K|\xi|^{2}
\end{aligned}
$$

for any $\xi \in \mathbf{R}^{n}$ and any point $X$ of $\bar{\Omega}$ provided

$$
P(X ; \xi)=0, \quad \sum_{j=1}^{4} \frac{\partial P}{\partial \zeta_{j}}(X, \xi) \partial_{j} \psi(X)=0
$$

We use the following convention and notations. Let $\partial=\left(\partial_{1}, \ldots, \partial_{4}\right), D=$ $-i \partial, \alpha=\left(\alpha_{1}, \ldots, \alpha_{4}\right)$ is a multi-index with integer components, $\zeta^{\alpha}=\zeta_{1}^{\alpha_{1}} \cdots \zeta_{4}^{\alpha_{4}}$, $D^{\alpha}$ and $\partial^{\alpha}$ are defined similarly. $x_{4}=t . \nabla$ denotes the gradient with respect to spatial variables $x_{1}, x_{2}, x_{3}$. $\nu$ is the outward normal to the boundary of a domain. $\Omega_{\varepsilon}=\Omega \cap\{\psi(x)>\varepsilon\}$. We recall that

$$
\|u\|_{(k)}(\Omega)=\left(\sum_{|\alpha| \leq k} \int_{\Omega}\left|\partial^{\alpha} u\right|^{2}\right)^{\frac{1}{2}}
$$

is the norm in the Sobolev space $H_{(k)}(\Omega)$ and $\left\|\left\|_{2}=\right\|\right\|_{(0)}$ is the $L^{2}$-norm. Let $C$ be generic constants (different at different places) depending only on $M$, on $K$, on the function $\psi$, on $C^{2}(\Omega)$-norms of the coefficients $\rho, c_{j k}, r_{j k}$ of the elasticity system, on $\varepsilon_{0}$, and on the domain $\Omega$. Any additional dependence will be indicated.

We let

$$
\begin{align*}
& a_{1}=\frac{c_{11}-c_{12}}{c_{11}+c_{12}}, \quad a_{2}=2 \frac{c_{23}}{c_{11}+c_{12}}, \quad a_{3}=2 \frac{c_{13}+c_{23}}{c_{11}+c_{12}} \\
& a_{4}=\frac{\left(c_{11}-c_{12}\right)\left(c_{13}+c_{23}\right)}{\left(c_{11}+c_{12}\right) c_{23}}, \tag{5}
\end{align*}
$$

and

$$
\begin{align*}
& A=a_{1}\left(\partial_{1}^{2}+\partial_{2}^{2}\right)+a_{2} \partial_{3}^{3}, \operatorname{div}_{T} \mathbf{u}=\partial_{1} u_{1}+\partial_{2} u_{2}+a_{3} \partial_{3} u_{3} \\
& \operatorname{curl}_{T} \mathbf{u}=\left(\partial_{2} u_{3}-a_{4} \partial_{3} u_{2}, a_{4} \partial_{3} u_{1}-\partial_{1} u_{3}, \partial_{1} u_{2}-\partial_{2} u_{1}\right) \tag{6}
\end{align*}
$$

We introduce the following conditions

$$
\begin{gather*}
\left(c_{11}^{2}-c_{12}^{2}\right) c_{33}-2\left(c_{13}+c_{23}\right)^{2}\left(c_{11}-c_{12}\right)-2 c_{23}^{2}\left(c_{11}+c_{12}\right)=0 \\
c_{11}-c_{12}=2 c_{23} \text { on } \Omega \tag{7}
\end{gather*}
$$

and the weight and scaling functions

$$
\begin{equation*}
\varphi=e^{\gamma \psi}, \sigma=\gamma \tau \varphi \tag{8}
\end{equation*}
$$

Theorem 1.1. Let $\psi \in C^{3}(\bar{\Omega})$ be $K$ - pseudo convex with respect to $\rho \partial_{t}^{2}-A-$ $R, \rho \partial_{t}^{2}-A-\partial_{1}^{2}-\partial_{2}^{2}-a_{3} a_{4} \partial_{3}^{2}-R$ in $\bar{\Omega}$ and let $|\rho|_{2}(\Omega)+\left|c_{j k}\right|_{2}(\Omega)+\left|r_{j k}\right|_{2}(\Omega) \leq M$. Let the conditions (7) be satisfied.

Then there are constants $C, C_{0}(\gamma)$ such that

$$
\begin{align*}
\int_{\Omega}\left(\gamma \sigma^{2}|\mathbf{u}|^{2}+\sigma\left(\left|\operatorname{div}_{T} \mathbf{u}\right|^{2}+\left|\operatorname{curl}_{T} \mathbf{u}\right|^{2}\right)\right. & \left.+\gamma\left(\left|\partial_{t} \mathbf{u}\right|^{2}+|\nabla \mathbf{u}|^{2}\right)\right) e^{2 \tau \varphi} \\
& \leq C \int_{\Omega}\left|\left(\rho \partial_{t}^{2}-\mathbf{A}_{T}-R\right) \mathbf{u}\right|^{2} e^{2 \tau \varphi} \tag{9}
\end{align*}
$$

for all $\mathbf{u} \in H_{0}^{2}(\Omega), C<\gamma, C_{0}<\tau$.
This estimate for isotropic elasticity with residual stress was obtained in [11] and for more general transversely isotropic elasticity in [10] without the terms with $\gamma$ on the left side.

Let us consider the following Cauchy problem

$$
\begin{equation*}
\left(\rho \partial_{t}^{2}-\mathbf{A}_{T}-R\right) \mathbf{u}=\mathbf{f} \text { in } \Omega, \mathbf{u}=\mathbf{g}_{0}, \quad \partial_{\nu} \mathbf{u}=\mathbf{g}_{1} \quad \text { on } \quad \Gamma \subset \partial \Omega \tag{10}
\end{equation*}
$$

where $\Gamma \in C^{3}$. Let $\Omega_{\delta}=\Omega \cap\{\psi>\delta\}$. The Carleman estimate of Theorem 1.1 by standard argument ([9], section 3.2) implies the following conditional Hölder stability estimate for (10) in $\Omega(\delta)$ (and hence uniqueness in $\Omega(0)$ ).
TheOrem 1.2. Let $\psi \in C^{3}(\bar{\Omega})$ be $K$ - pseudo convex with respect to $\rho \partial_{t}^{2}-A-$ $R, \rho \partial_{t}^{2}-A-\partial_{1}^{2}-\partial_{2}^{2}-a_{3} a_{4} \partial_{3}^{2}-R$ in $\bar{\Omega}$ and let $|\rho|_{2}(\Omega)+\left|c_{j k}\right|_{2}(\Omega)+\left|r_{j k}\right|_{2}(\Omega) \leq M$. Let the condition (7) be satisfied. Assume that $\bar{\Omega}_{0} \subset \Omega \cap \Gamma$.

Then there exist $C=C(\delta), \kappa=\kappa(\delta) \in(0,1)$ such that for a solution $\mathbf{u} \in H^{2}(\Omega)$ to (10) one has

$$
\begin{equation*}
\|\mathbf{u}\|_{(0)}\left(\Omega_{\delta}\right)+\left\|\nabla_{x} \mathbf{u}\right\|_{(0)}\left(\Omega_{\delta}\right)+\left\|\partial_{t} \mathbf{u}\right\|_{(0)}\left(\Omega_{\delta}\right) \leq C\left(F+M_{1}^{1-\kappa} F^{\kappa}\right), \tag{11}
\end{equation*}
$$

where $F=\|\mathbf{f}\|_{(0)}\left(\Omega_{0}\right)+\left\|\mathbf{g}_{0}\right\|_{\left(\frac{3}{2}\right)}(\Gamma)+\left\|\mathbf{g}_{1}\right\|_{\left(\frac{1}{2}\right)}(\Gamma), M_{1}=\|\mathbf{u}\|_{(1)}(\Omega)$.
In Theorems 1.3, 1.4 we assume that $\Omega=G \times(-T, T), \partial G \in C^{3}$ and that $R=0$.

Due to (4) the system (10) is $t$-hyperbolic and from known results (e.g. [3], III.4, p.123) it follows that the first initial boundary value problem for this system is uniquely solvable in standard energy spaces, moreover the conventional energy integral

$$
E(t ; \mathbf{u})=\int_{G}\left(\left|\partial_{t} \mathbf{u}\right|^{2}+|\nabla \mathbf{u}|^{2}+|\mathbf{u}|^{2}\right)(, t)
$$

is bounded by the initial energy and the right side (more detail in the proof of Theorem 1.3). Repeating the argument in [3] one can obtain the same result when the smallest eigenvalue of the matrix $r_{j k}$ is greater than than $-\frac{\varepsilon_{0}}{2}$.
Theorem 1.3. Let $\psi \in C^{3}(\bar{\Omega})$ be $K$ - pseudo convex with respect to $\rho \partial_{t}^{2}-A-$ $R, \rho \partial_{t}^{2}-A-\partial_{1}^{2}-\partial_{2}^{2}-a_{3} a_{4} \partial_{3}^{2}-R$ in $\bar{\Omega}$ and let $|\rho|_{2}(\Omega)+\left|c_{j k}\right|_{2}(\Omega)+\left|r_{j k}\right|_{2}(\Omega) \leq M$. Assume that

$$
\begin{equation*}
\psi<0 \text { on } \bar{G} \times\{-T, T\}, 0<\psi \text { on } G \times\{0\} . \tag{12}
\end{equation*}
$$

Then there exist $C$ such that for a solution $\mathbf{u} \in H^{2}(\Omega)$ to (10) one has

$$
\begin{equation*}
E(t ; \mathbf{u}) \leq C\left(\|\mathbf{f}\|_{(0)}(\Omega)+\left\|\mathbf{g}_{0}\right\|_{\left(\frac{3}{2}\right)}(\Gamma)+\left\|\mathbf{g}_{1}\right\|_{\left(\frac{1}{2}\right)}(\Gamma)\right) \tag{13}
\end{equation*}
$$

Now we state results about identification of a source from additional boundary data.

Let $\mathbf{u}$ be a solution to

$$
\begin{gather*}
\left(\rho \partial_{t}^{2}-\mathbf{A}_{T}-R\right) \mathbf{u}=\mathcal{A} \mathbf{f} \text { in } \Omega \\
\mathbf{u}=\mathbf{0}, \partial_{t} \mathbf{u}=\mathbf{0} \text { on } G \times\{0\}, \mathbf{u}=\mathbf{0} \text { on } \partial G \times(-T, T) . \tag{14}
\end{gather*}
$$

We will assume that $\mathcal{A} \in C(\bar{\Omega})$.
We will consider the boundary stress data as measurements (observations). We introduce the norm of the of the lateral Cauchy data

$$
\begin{equation*}
F=\left\|\partial_{t}^{2} \partial_{\nu} \mathbf{u}\right\|_{\left(\frac{1}{2}\right)}(\Gamma) \tag{15}
\end{equation*}
$$

To guarantee the uniqueness, we impose some non-degeneracy condition on the matrix $\mathcal{A}$. We assume that

$$
\begin{equation*}
\operatorname{det} \mathcal{A}>\varepsilon_{0}>0 \text { on } G \times\{0\} . \tag{16}
\end{equation*}
$$

Theorem 1.4. Let $\psi \in C^{3}(\bar{\Omega})$ be $K$ - pseudo convex with respect to $\rho \partial_{t}^{2}-A-$ $R, \rho \partial_{t}^{2}-A-\partial_{1}^{2}-\partial_{2}^{2}-a_{3} a_{4} \partial_{3}^{2}-R$ in $\bar{\Omega}$. Assume that $\rho, c_{j k}, r_{j k}$ do not depend on $t$ and $|\rho|_{2}(\Omega)+\left|c_{j k}\right|_{2}(\Omega)+\left|r_{j k}\right|_{2}(\Omega)+\left|\partial_{t}^{2} \mathcal{A}\right|_{0}(\Omega) \leq M$. Assume that the condition (12) is satisfied. Let the matrix function $\mathcal{A}$ satisfy (16).

Then there exist $C$ such that

$$
\begin{equation*}
\|\mathbf{f}\|_{(0)}(\Omega) \leq C F . \tag{17}
\end{equation*}
$$

Observe that the classical isotropic elasticity is a particular case of the system under consideration, when $c_{11}=c_{33}=\lambda+2 \mu, c_{12}=c_{13}=\lambda, c_{23}=\mu$. In particular, the conditions (7) are satisfied.

Carleman estimates were introduced by Carleman in 1939 to demonstrate uniqueness in the Cauchy problem for a system of first order in $\mathbf{R}^{2}$ with non analytic coefficients. Carleman type estimates and uniqueness of the continuation theorems have been obtained for wide classes of scalar partial differential equations $[6,9]$. But useful concept of pseudo convexity is not available for systems, and Carleman estimates were derived only in particular cases, like for classical isotropic dynamical Maxwell's and elasticity systems [5] (by using principal diagonalization). Two large parameters were introduced in [8]. They were a main tool in the first proof of uniqueness and stability of all three elastic parameters in dynamical isotropic Lame system from two sets of boundary data [7]. A system of transversely isotropic elasticity with residual stress was recently studied in $[10,11,12,14]$ where there are Carleman estimates, uniqueness and stability of the continuation and of the identification of elastic coefficients.

In this paper for the transversely isotropic system with residual stress we obtain Carleman estimates including time derivative. Most advanced previous results [10] handled only spatial derivatives. Observe that our results are new for the classical isotropic elasticity system. Including temporal derivative enables to obtain exact controllability (Lipschitz) bounds in the lateral Cauchy and inverse problems under minimal regularity assumptions. So far our results need a special condition (7). The main idea is to use principal upper triangular reduction, scalar Carleman estimates with two large parameters, and spatial smoothing (pseudo-differential) operator with parameter. The crucial part is $L^{2}$ bounds on commutators of this operator and of differential operators with parameters.

We stated our basic results in section 1. In section 2 we obtain auxiliary results where crucial are bounds on commutators of multiplication and of smoothing operator and especially Lemma 2.4 on certain localization of this pseudo-differential operator. In section 3 we prove estimates of Theorem 1.1 and in section 4 apply them to stability estimates in the continuation and inverse problems. We tried to minimize technicalities and refer as much as possible to known results.

It is not easy to find functions $\psi$ which are pseudo convex with respect to a general operator. In an isotropic case explicit and verifiable conditions for $\psi(x, t)=|x-\beta|^{2}-\theta^{2} t^{2}$ were found by Isakov in 1980 and their simplifications are given in [9], section 3.4. In general anisotropic case Khaidarov [13] showed that under certain conditions the same $\psi$ is pseudo convex if the speed of the propagation determined by $A$ is monotone in a certain direction.

In the following Lemma for a general hyperbolic operator we give the condition of $K$-pseudo convexity of $\psi(x, t)=|x-\beta|^{2}-\theta^{2} t^{2}$.

Lemma 1.5. Let

$$
P=\partial_{t}^{2}-\sum_{j, k=1}^{3} a_{j k} \partial_{j} \partial_{k}, \quad a_{j k}=a_{k j}
$$

where $a_{j k} \in C^{1}$ satisfy the uniform ellipticity condition

$$
\sum_{j, k=1}^{3} a_{j k}(X) \xi_{j} \xi_{k} \geq \varepsilon_{0}|\xi|^{2}, X \in \Omega \quad \xi \in \mathbf{R}^{3}, \quad \varepsilon_{0}>0
$$

Let

$$
\psi(x, t)=|x-\beta|^{2}-\theta^{2} t^{2}, \quad \beta=\left(0,0, \beta_{3}\right) .
$$

Assume that

$$
\sum_{j, l=1}^{3}\left(\sum_{k=1}^{3} a_{3 k} \partial_{k} a_{j l}-2 \sum_{k=1}^{2} a_{l k} \partial_{k} a_{j 3}\right) \xi_{j} \xi_{l} \geq \varepsilon_{1}|\xi|^{2}, \quad \xi \in \mathbf{R}^{3} .
$$

for some $\varepsilon_{1}>0$.
Then there is large $\beta_{3}$ such that the function $\psi$ is $K$-pseudo convex with respect to $P$ in $\bar{\Omega}$.

A proof is given in [11].

## 2. Auxiliary results.

For a linear partial differential operator $\mathbf{A}$ (with matrix coefficients) we introduce $\mathbf{A}_{\varphi}$ by the equality $\left(\mathbf{A}_{\varphi} v\right) e^{-\tau \varphi}=\mathbf{A}\left(v e^{-\tau \varphi}\right)$. From the Leibniz formula it follows that $\mathbf{A}_{\varphi}$ is the linear partial differential operator with the same principal part as A. We observe that

$$
\begin{equation*}
\left(\partial_{j}\right)_{\varphi}=\partial_{j}-\sigma \partial_{j} \psi \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\mathbf{A}_{1} \mathbf{A}_{2}\right)_{\varphi}=\left(\mathbf{A}_{1}\right)_{\varphi}\left(\mathbf{A}_{2}\right)_{\varphi} \tag{19}
\end{equation*}
$$

Indeed, according to the definition,

$$
\begin{aligned}
\left(\left(\mathbf{A}_{1} \mathbf{A}_{2}\right)_{\varphi} v\right) e^{-\tau \varphi}=\mathbf{A}_{1}\left(\mathbf{A}_{2}( \right. & \left.\left.v e^{-\tau \varphi}\right)\right) \\
& =\mathbf{A}_{1}\left(\left(\left(\mathbf{A}_{2}\right)_{\varphi} v\right) e^{-\tau \varphi}\right)=\left(\left(\mathbf{A}_{1}\right)_{\varphi}\left(\mathbf{A}_{2}\right)_{\varphi} v\right) e^{-\tau \varphi}
\end{aligned}
$$

In particular,

$$
\begin{equation*}
P_{\varphi}(D)=P(D+i \tau \nabla \varphi)=P(D)+\tau P_{1}(D)+\tau^{2} P(\nabla \varphi) \tag{20}
\end{equation*}
$$

where $P_{1}$ is a first order differential operator with coefficients depending on $\gamma$. We will use the notation $\left\langle\xi>=\left(|\xi|^{2}+1\right)^{\frac{1}{2}}\right.$ and the pseudo-differential operator $\Lambda_{\tau}^{s} f=\mathcal{F}^{-1}(<\xi>+\tau)^{s} \mathcal{F} f$, where $\mathcal{F}$ is the Fourier transform in $\mathbf{R}^{3}$ and $\xi \in \mathbf{R}^{3}$. Let $\Omega^{*}$ be a bounded domain in $\mathbf{R}^{4}$ with a smooth boundary such that $\bar{\Omega} \subset \Omega^{*}$. We can extend all coefficients of the operators $\mathbf{A}_{T}, R$ and functions $\rho, \psi$ onto $\mathbf{R}^{4}$ preserving the regularity in such a way that they have support in $\Omega^{*}$ and their $C^{2}$-norms are bounded by $C$.

In next Lemmas we fix $x^{0}$ with $\left(x^{0}, t^{0}\right) \in \bar{\Omega}$ and introduce $\sigma(t)=\sigma\left(x^{0}, t\right)$
Lemma 2.1. There exists a constant $C(\gamma)$ such that

$$
\begin{gather*}
\left\|\Lambda_{\sigma(t)}^{-1} \partial_{t} u-\partial_{t} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\left\|\Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right)  \tag{21}\\
\left\|\sigma^{\frac{1}{2}}\left(\Lambda_{\sigma(t)}^{-1} \operatorname{div}_{T, \varphi} \mathbf{u}-\operatorname{div}_{T, \varphi} \Lambda_{\sigma(t)}^{-1}(\mathbf{u})\right)\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma) \tau^{-\frac{1}{2}}\|\mathbf{u}\|_{(0)}(\Omega) \\
\left\|\sigma^{\frac{1}{2}}\left(\Lambda_{\sigma(t)}^{-1} \operatorname{curl}_{T, \varphi} \mathbf{u}-\operatorname{curl}_{T, \varphi} \Lambda_{\sigma(t)}^{-1}(\mathbf{u})\right)\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma) \tau^{-\frac{1}{2}}\|\mathbf{u}\|_{(0)}(\Omega) \tag{22}
\end{gather*}
$$

and

$$
\begin{equation*}
\left\|\Lambda_{\sigma(t)}^{-1}\left(P_{\varphi} u\right)-\left(P_{\varphi} \Lambda_{\sigma(t)}^{-1} u\right)\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\left(\|u\|_{(0)}(\Omega)+\left\|\Lambda_{\sigma(t)}^{-1} \partial_{t} u\right\|_{(0)}\left(\mathbf{R}^{4}\right)\right) \tag{23}
\end{equation*}
$$

for all $u, \mathbf{u} \in H_{0}^{2}(\Omega)$.
Proof. We first prove (21). Observe that $\sigma=\tau \gamma \varphi, \partial_{t} \sigma=\gamma \sigma \partial_{t} \psi, \partial_{t}^{2} \sigma=$ $\gamma \sigma\left(\partial_{t}^{2} \psi+\gamma\left(\partial_{t} \psi\right)^{2}\right)$, that

$$
\begin{equation*}
\partial_{t} \Lambda_{\sigma(t)}^{-1} u=\mathcal{F}^{-1}\left(\frac{-\partial_{t} \sigma(t)}{(<\xi>+\sigma(t))^{2}} \mathcal{F} u+\frac{1}{<\xi>+\sigma(t)} \mathcal{F} \partial_{t} u\right) \tag{24}
\end{equation*}
$$

and

$$
\begin{align*}
& \partial_{t}^{2} \Lambda_{\sigma(t)}^{-1} u=\mathcal{F}^{-1}\left(\left(\frac{-\partial_{t}^{2} \sigma(t)}{(<\xi>+\sigma(t))^{2}}+\frac{\left(\partial_{t} \sigma(t)\right)^{2}}{(<\xi>+\sigma(t))^{3}}\right) \mathcal{F} u\right. \\
&\left.\quad+2 \frac{-\partial_{t} \sigma}{(<\xi>+\sigma(t))^{2}} \mathcal{F} \partial_{t} u+\frac{1}{<\xi>+\sigma(t)} \mathcal{F} \partial_{t}^{2} u\right) \tag{25}
\end{align*}
$$

Formula (24) implies that

$$
\Lambda_{\sigma(t)}^{-1} \partial_{t} u-\partial_{t} \Lambda_{\sigma(t)}^{-1} u=\mathcal{F}^{-1}\left(\frac{\partial_{t} \sigma(t)}{(<\xi>+\sigma(t))^{2}} \mathcal{F} u\right)
$$

so

$$
\begin{aligned}
& \left\|\Lambda_{\sigma(t)}^{-1} \partial_{t} u-\partial_{t} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}^{2}\left(\mathbf{R}^{4}\right) \\
& \quad \leq \int_{\mathbf{R}} \int_{\mathbf{R}^{3}} \left\lvert\, \mathcal{F}^{-1}\left(\left.\frac{\partial_{t} \sigma(t)}{(<\xi>+\sigma(t))^{2}} \mathcal{F} u(x, t)\right|^{2} d x d t\right.\right. \\
& \quad \leq \int_{\mathbf{R}} \int_{\mathbf{R}^{3}}\left|\frac{\partial_{t} \sigma(t)}{(<\xi>+\sigma(t))^{2}}\right|^{2}|\mathcal{F} u(\xi, t)|^{2} d \xi d t \\
& \quad \leq C(\gamma) \int_{\mathbf{R}} \int_{\mathbf{R}^{3}}\left|\frac{1}{(<\xi>+\sigma(t))} \mathcal{F} u(\xi, t)\right|^{2} d \xi d t
\end{aligned}
$$

and again using the Parseval equality we yield (21).
Due to (18), $d i v_{T, \varphi} \mathbf{u}$ is the sum of terms $a\left(\partial_{j}-\sigma \partial_{j} \psi\right) u_{k}$, where $j, k=1,2,3$, $|a|_{2}\left(\mathbf{R}^{4}\right) \leq C$, and $a=0$ outside $\Omega^{*}$. Hence it suffices to show that

$$
\begin{equation*}
\left\|\Lambda_{\sigma(t)}^{-1}\left(b \partial^{\beta} u\right)-b \partial^{\beta} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma) \tau^{-1}\|u\|_{(0)}(\Omega), \tag{26}
\end{equation*}
$$

for all $\beta$ with $|\beta|=1, \beta_{4}=0$, and that

$$
\begin{equation*}
\left\|\Lambda_{\sigma(t)}^{-1}(b u)-b \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma) \tau^{-2}\|u\|_{(0)}(\Omega), \tag{27}
\end{equation*}
$$

when $b \in C^{1}\left(\mathbf{R}^{4}\right),|b|_{1}\left(\mathbf{R}^{4}\right)<C(\gamma)$, and $b=0$ outside $\Omega^{*}$.
To prove (26) we introduce $u_{1}=\Lambda_{\sigma(t)}^{-1} \partial^{\beta} u$. Using also that $\Lambda_{\sigma}=\Lambda_{0}+\sigma$, we have

$$
\Lambda_{\sigma(t)}^{-1} b \partial^{\beta} u-b \partial^{\beta} \Lambda_{\sigma(t)}^{-1} u=\Lambda_{\sigma(t)}^{-1}\left(b \Lambda_{\sigma(t)}-\Lambda_{\sigma(t)} b\right) u_{1}=\Lambda_{\sigma(t)}^{-1}\left(b \Lambda_{0}-\Lambda_{0} b\right) u_{1} .
$$

As above, from the Parseval identity, $\left\|u_{1}\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C\|u\|_{(0)}(\Omega)$. By known (e.g. Coifman and Meyer ([1])) estimates of commutators of pseudo-differential operators and of multiplication operators

$$
\left\|\left(b \Lambda_{0}-\Lambda_{0} b\right) u_{1}(, t)\right\|_{(0)}^{2}\left(\mathbf{R}^{3}\right) \leq C(\gamma)\left\|u_{1}(, t)\right\|_{(0)}^{2}\left(\mathbf{R}^{3}\right)
$$

Using that $\left\|\Lambda_{\sigma(t)}^{-1} v\right\|_{(0)}^{2}\left(\mathbf{R}^{3}\right) \leq C(\gamma) \tau^{-2}\|v\|_{(0)}^{2}(\Omega)$ and integrating with respect to $t$ we complete the proof of (26).

Proofs of (27) and for curl are similar.
Due to (18), (19), $P_{\varphi} u$ is the sum of terms

$$
a\left(\partial_{j}-\sigma \partial_{j} \psi\right)\left(\partial_{k}-\sigma \partial_{k} \psi\right) u
$$

where $j, k=1,2,3,4,|a|_{2}\left(\mathbf{R}^{4}\right) \leq C, a=1$ when $j=k=4, a=0$ when $j=1,2,3, k=4$, and $a=0$ outside $\Omega^{*}$ otherwise. Elementary calculations show that this expression equals to

$$
a \partial_{j} \partial_{k} u-a \sigma\left(\partial_{k} \psi \partial_{j} u+\partial_{j} \psi \partial_{k} u\right)+a \sigma\left((\sigma-1) \partial_{j} \psi \partial_{k} \psi-\partial_{j} \partial_{k} \psi\right) u
$$

Hence it suffices to show that

$$
\begin{equation*}
\left\|\Lambda_{\sigma(t)}^{-1} a \partial^{\alpha} u-a \partial^{\alpha} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\left(\|u\|_{(0)}(\Omega)+\left\|\Lambda_{\sigma(t)}^{-1} \partial_{t} u\right\|_{(1)}\left(\mathbf{R}^{4}\right)\right), \tag{28}
\end{equation*}
$$

for all $\alpha$ with $|\alpha| \leq 2$, that

$$
\begin{equation*}
\tau\left\|\Lambda_{\sigma(t)}^{-1} b \partial^{\beta} u-b \partial^{\beta} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\|u\|_{(0)}(\Omega), \quad \text { for all }|\beta| \leq 1 \tag{29}
\end{equation*}
$$

for all $\beta$ with $|\beta| \leq 1$, and that

$$
\begin{equation*}
\tau^{2}\left\|\Lambda_{\sigma(t)}^{-1}(b u)-b \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\|u\|_{(0)}(\Omega) \tag{30}
\end{equation*}
$$

when $b \in C^{1}\left(\overline{\Omega^{*}}\right),|b|_{1}(\bar{\Omega})<C$, and $b=0$ outside $\Omega^{*}$.
To show (28) let first $\alpha_{4}=2$.
As above, (25) implies that

$$
\left\|\Lambda_{\sigma(t)}^{-1} \partial_{t}^{2} u-\partial_{t}^{2} \Lambda_{\sigma(t)}^{-1} u\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C(\gamma)\left(\|u\|_{(0)}(\Omega)+\left\|\Lambda_{\sigma(t)}^{-1} \partial_{t} u\right\|_{(0)}\left(\mathbf{R}^{4}\right)\right)
$$

To complete a proof of (28) we now consider $\alpha_{4}=0$. Let $\alpha_{j}>0$ and $\beta_{j}=1$ while other components of $\beta$ be zero. We introduce $u_{1}=\Lambda_{\sigma(t)}^{-1} \partial^{\alpha-\beta} u$. Using also that $\Lambda_{\sigma}=\Lambda_{0}+\sigma$, we have

$$
\begin{aligned}
& \Lambda_{\sigma(t)}^{-1} a \partial^{\alpha} u-a \partial^{\alpha} \Lambda_{\sigma(t)}^{-1} u=\Lambda_{\sigma(t)}^{-1}\left(a \Lambda_{\sigma(t)}-\Lambda_{\sigma(t)} a\right) \partial_{j} u_{1} \\
&=\Lambda_{\sigma(t)}^{-1}\left(a \Lambda_{0}-\Lambda_{0} a\right) \partial_{j} u_{1}=\Lambda_{\sigma(t)}^{-1}\left(a \partial_{j} \Lambda_{0}-\partial_{j}\left(\Lambda_{0} a\right)+\Lambda_{0} \partial_{j} a\right) u_{1} \\
&=\Lambda_{\sigma(t)}^{-1}\left(\partial_{j}\left(a \Lambda_{0}-\Lambda_{0} a\right)+\left(\Lambda_{0} \partial_{j} a-\partial_{j} a \Lambda_{0}\right)\right) u_{1}
\end{aligned}
$$

As above, from the Parseval identity, $\left\|u_{1}\right\|_{(0)}\left(\mathbf{R}^{4}\right) \leq C\|u\|_{(0)}(\Omega)$. By known (e.g. Coifman and Meyer [1]) estimates of commutators of pseudo-differential operators and of multiplication operators

$$
\left\|\left(a \Lambda_{0}-\Lambda_{0} a\right) u_{1}(, t)\right\|_{(0)}\left(\mathbf{R}^{3}\right) \leq C(\gamma)\left\|u_{1}(, t)\right\|_{(0)}\left(\mathbf{R}^{3}\right)
$$

A similar estimate is valid when we replace $a$ by $\partial_{j} a$. Using, as above, that the norm of the operator $\Lambda_{\sigma(t)}^{-1} \partial_{j}$ from $L^{2}\left(\mathbf{R}^{3}\right)$ into itself is bounded by $C(\gamma)$ and integrating with respect to $t$ we complete the proof of (28).

Next we demonstrate (29). Let first $\beta=(0,0,0,1)$. Using (24) we have

$$
\left\|\partial_{t} \Lambda_{\sigma(t)}^{-1} u-\Lambda_{\sigma(t)}^{-1} \partial_{t} u\right\|_{(0)}\left(\mathbf{R}^{3}\right) \leq C(\gamma)\|u\|_{(0)}\left(\mathbf{R}^{3}\right)
$$

so it suffices to bound $\Lambda_{\sigma(t)}^{-1} b \partial_{t} u-b \Lambda_{\sigma(t)}^{-1} \partial_{t} u$. To do this, let $u_{2}=\Lambda_{\sigma(t)}^{-1} \partial_{t} u$, then we need to bound

$$
\Lambda_{\sigma(t)}^{-1} b \Lambda_{\sigma(t)} u_{2}-b u_{2}=\Lambda_{\sigma(t)}^{-1}\left(b \Lambda_{\sigma(t)} u_{2}-\Lambda_{\sigma(t)}\left(b u_{2}\right)\right)=\Lambda_{\sigma(t)}^{-1}\left(b \Lambda_{0} u_{2}-\Lambda_{0}\left(b u_{2}\right)\right)
$$

because $\Lambda_{\sigma}=\Lambda_{0}+\sigma$. As above, from known bounds of commutators and the definition of $u_{2}$ it follows that

$$
\tau\left\|\Lambda_{\sigma(t)}^{-1}\left(b \Lambda_{0} u_{2}-\Lambda_{0}\left(b u_{2}\right)\right)\right\|_{(0)}\left\|\left(\mathbf{R}^{4}\right) \leq C(\gamma)\right\| \Lambda_{\sigma(t)}^{-1} \partial_{t} u \|_{(0)}\left(\mathbf{R}^{4}\right)
$$

Proofs of (29) for general $\beta$ and of (30) are similar.
Lemma 2.2. Let $K(x, y ; t)$ be the Schwartz kernel of the pseudo-differential operator $\Lambda_{\sigma(t)}^{-1}$ with $\tau>1$.

Then

$$
\left|\partial_{x}^{\alpha} K(x, y ; t)\right| \leq C(\gamma) \tau^{-2}|x-y|^{-8}
$$

provided $|\alpha| \leq 2$.
A proof is similar to [7], Lemma 3.4.
Proof. The Schwartz kernel $K(x, y ; t)$ is the oscillatory integral

$$
\begin{aligned}
\int_{\mathbf{R}^{3}} e^{i(x-y) \cdot \xi} & (<\xi>+\sigma(t))^{-1} d \xi \\
= & -|x-y|^{-2} \int_{\mathbf{R}^{3}}\left(\Delta_{\xi} e^{i(x-y) \cdot \xi}\right)(<\xi>+\sigma(t))^{-1} d \xi \\
= & -|x-y|^{-2} \int_{\mathbf{R}^{3}} e^{i(x-y) \cdot \xi} \Delta_{\xi}(<\xi>+\sigma(t))^{-1} d \xi \\
& =\cdots=(-1)^{l}|x-y|^{-2 l} \int_{\mathbf{R}^{3}} e^{i(x-y) \cdot \xi} \Delta_{\xi}^{l}(<\xi>+\sigma(t))^{-1} d \xi
\end{aligned}
$$

where we did integrate by parts. Observing that

$$
\left|\Delta_{\xi}^{l}(<\xi>+\sigma(t))^{-1}\right| \leq C(l)(<\xi>+\sigma(t))^{-2}<\xi>^{-2 l+1}, l=1,2, \ldots
$$

and letting $l=4$ we complete the proof.
We denote by $S^{\prime}$ the orthogonal projection of a set $S$ in $\mathbf{R}^{4}$ onto $\mathbf{R}^{3}$ and let $C y l\left(x^{0} ; \delta\right)=\left(B^{\prime}\left(x^{0} ; \delta\right) \times \mathbf{R}\right) \cap \Omega^{*}$.
Lemma 2.3. We have

$$
\begin{align*}
& \int_{\mathbf{R}^{4} \backslash \operatorname{Cyl}\left(x^{0} ; 3 \delta\right)}\left(\tau^{3}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\tau \sum_{|\alpha|=1}\left|\partial^{\alpha} \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) \\
& \leq C(\gamma, \delta) \tau^{-1} \int_{\mathbf{R}^{4}} \sigma\left(|v|^{2}+\left|\partial_{t} \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) \tag{31}
\end{align*}
$$

for all $v \in H_{0}^{1}\left(C y l\left(x^{0} ; \delta\right)\right), x^{0} \in \bar{\Omega}^{\prime}$.

Proof. We can assume that $x^{0}=0$ and drop $x^{0}$.
We first consider the case when $\alpha_{4}=0$. Since suppv $\subset C y l(\delta)$,

$$
\begin{aligned}
&\left|\partial^{\alpha} \Lambda_{\sigma(t)}^{-1} v(x, t)\right| \leq \int_{B(\delta)}|v(y, t)|\left|\partial^{\alpha} K(x, y ; t)\right| d y \\
& \leq C(\gamma, \delta) \tau^{-2} \int_{B(\delta)}|x-y|^{-8}|v(y, t)| d y
\end{aligned}
$$

by Lemma 2.2, provided $x \in \mathbf{R}^{3} \backslash B(3 \delta)$. When $y \in B(2 \delta)$,

$$
\begin{equation*}
|x-y| \geq \frac{1}{2}|x-y|+\frac{1}{8}|x-y| \geq \frac{\delta}{2}+\frac{1}{8}|x|-\frac{1}{8}|y| \geq \frac{\delta}{4}+\frac{1}{8}|x| \geq \frac{1+|x|}{C(\delta)} \tag{32}
\end{equation*}
$$

Hence by using the Schwarz inequality

$$
\left|\partial^{\alpha} \Lambda_{\sigma(t)}^{-1} v(x, t)\right| \leq C(\gamma, \delta) \tau^{-2}(1+|x|)^{-8}\left(\int_{B(\delta)}|v(, t)|^{2}\right)^{\frac{1}{2}} \quad \text { for all }|\alpha| \leq 1
$$

provided $x \in \mathbf{R}^{3} \backslash B(3 \delta)$. Using this estimate we conclude that the last integral on the left side of (31) is less than $C(\gamma, \delta) \int_{C y l(\delta)}|v|^{2}$. Similarly we bound the first integral.

Now we will handle the most delicate case of $\alpha=(0,0,0,1)$, i.e. $\partial^{\alpha}=\partial_{t}$. Let $w=\partial_{t} v$. Due to (21), it suffices to show that

$$
\begin{equation*}
\tau \int_{\mathbf{R}^{3} \backslash B(3 \delta)}\left|\Lambda_{\sigma(t)}^{-1} w(, t)\right|^{2} \leq C(\gamma) \int_{\mathbf{R}^{3}}\left|\Lambda_{\sigma(t)}^{-1} w\right|^{2}(, t) . \tag{33}
\end{equation*}
$$

To do so we will make use of the integral operator $\Lambda_{\sigma(t)}^{*} w=\mathcal{F}^{-1}\left(|\xi|^{2}+\right.$ $\sigma(t))^{-1} \mathcal{F} w$ which is obviously a fundamental solution of the differential operator $-\Delta+\sigma(t)$ in $\mathbf{R}^{3}$. So for $W=\Lambda_{\sigma(t)}^{*} w$,

$$
(-\Delta+\sigma(t)) W=w \text { in } \mathbf{R}^{3}
$$

We have

$$
\left(|\xi|^{2}+\sigma(t)\right)^{-1} \leq C(\gamma)(<\xi>+\sigma(t))^{-1}
$$

and hence

$$
\begin{equation*}
\int_{\mathbf{R}^{3}}\left|\Lambda_{\sigma(t)}^{*} w\right|^{2} \leq C(\gamma) \int_{\mathbf{R}^{3}}\left|\Lambda_{\sigma(t)}^{-1} w\right|^{2} \tag{34}
\end{equation*}
$$

Let a cut-off function $\chi_{\delta}=1$ on $B(\delta), \operatorname{supp} \chi_{\delta} \subset B(2 \delta)$ and $\left|\partial^{\alpha} \chi_{\delta}\right| \leq C(\delta)$
when $|\alpha| \leq 2$. Due to the definition of $K$,

$$
\begin{aligned}
\Lambda_{\sigma(t)}^{-1} w(x, t)= & \int_{B(\delta)} K(x-y ; \sigma(t)) w(y, t) d y \\
= & \int_{B(2 \delta)} \chi_{\delta}(y) K(x-y ; \sigma(t))(-\Delta+\sigma(t)) W(y, t) d y \\
& =\int_{B(2 \delta)}(-\Delta+\sigma(t))\left(\chi_{\delta}(y) K(x-y ; \sigma(t))\right) W(y, t) d y
\end{aligned}
$$

Therefore, by Lemma 2.2 and (32)

$$
\begin{aligned}
\left|\Lambda_{\sigma(t)}^{-1} w(x, t)\right| \leq C(\gamma, \delta) \tau^{-1} & \int_{B(2 \delta)}|x-y|^{-8}|W(y, t)| d y \\
& \leq C(\gamma, \delta) \tau^{-1}(1+|x|)^{-8}\left(\int_{B(2 \delta)}|W(y, t)|^{2} d y\right)^{\frac{1}{2}}
\end{aligned}
$$

This combined with (34) completes the proof of (33) and hence of Lemma 2.3.

Since $\psi \in C^{2}$, using (8) we will choose $\delta(\gamma)$ so that

$$
\begin{equation*}
\frac{\sigma(t)}{2} \leq \sigma \leq 2 \sigma(t) \tag{35}
\end{equation*}
$$

on $\operatorname{Cyl}\left(x^{0} ; 4 \delta(\gamma)\right)$.
Lemma 2.4. There is $C$ such that

$$
\begin{equation*}
\int_{\mathbf{R}^{4}}\left|\partial_{j} \Lambda_{\sigma(t)}^{-1} v\right|^{2}+\int_{\mathbf{R}^{4}}\left|\Lambda_{\sigma(t)}^{-1}(a \sigma v)\right|^{2} \leq C \int_{\mathbf{R}^{4}}|v|^{2}, j=1,2,3, \tag{36}
\end{equation*}
$$

for all $v \in H_{0}^{2}\left(C y l\left(x^{0} ; 4 \delta(\gamma)\right)\right), x^{0} \in \bar{\Omega}^{\prime}$, provided $|a|_{1}\left(\mathbf{R}^{4}\right)<C$ and $a$ is constant outside $\Omega^{*}$.

Proof. As above we let $x^{0}=0$ and drop it. Due to the Parseval identity

$$
\begin{aligned}
& \int_{\mathbf{R}^{4}}\left|\partial^{\alpha} \Lambda_{\sigma(t)}^{-1} v\right|^{2} \leq \int_{\mathbf{R}}\left(\int_{\mathbf{R}^{3}} \frac{|\xi|^{2}}{(<\xi>+\sigma(t))^{2}}|\mathcal{F} v(\xi, t)|^{2} d \xi\right) d t \\
& \leq \int_{\mathbf{R}}\left(\int_{\mathbf{R}^{3}}|\mathcal{F} v(\xi, t)|^{2} d \xi\right) d t=\int_{\mathbf{R}}\left(\int_{\mathbf{R}^{3}}|v(x, t)|^{2} d x\right) d t=\int_{C y l(\delta(\gamma))}|v|^{2}
\end{aligned}
$$

when $|\alpha|=1, \alpha_{4}=0$.

Similarly,

$$
\begin{aligned}
\int_{\mathbf{R}^{4}}\left|\Lambda_{\sigma(t)}^{-1}(a \sigma v)\right|^{2}= & \int_{\mathbf{R}}\left(\int_{\mathbf{R}^{3}} \frac{1}{(<\xi>+\sigma(t))^{2}}|\mathcal{F}(a \sigma v)(\xi, t)|^{2} d \xi\right) d t \\
\leq & \int_{\mathbf{R}} \frac{1}{\sigma(t)^{2}}\left(\int_{\mathbf{R}^{3}}|\mathcal{F}(a \sigma v)(\xi, t)|^{2} d \xi\right) d t \\
& =\int_{\mathbf{R}} \frac{1}{\sigma(t)^{2}}\left(\int_{\mathbf{R}^{3}}|(a \sigma v)(x, t)|^{2} d x\right) d t \\
& =\int_{C y l(\delta(\gamma))}\left(\frac{\sigma}{\sigma(t)}\right)^{2}|(a v)|^{2} \leq C \int_{C y l(\delta(\gamma))}|v|^{2}
\end{aligned}
$$

since, due to the definition of $\delta(\gamma)$, we have (35).
Lemma 2.5. Let $\psi$ be $K$ pseudo-convex with respect to $P$ on $\bar{\Omega}$.
Then there is $C$ such that

$$
\int_{\mathbf{R}^{4}} \sigma\left(|v|^{2}+\left|\Lambda_{\sigma(t)}^{-1} \partial_{t} v\right|^{2}\right) \leq C \int_{\mathbf{R}^{4}}\left|\Lambda_{\sigma(t)}^{-1} P_{\varphi} v\right|^{2}
$$

for all $v \in H_{0}^{2}\left(C y l\left(x^{0} ; \delta(\gamma)\right)\right)$ provided $\tau>C, x^{0} \in \bar{\Omega}^{\prime}$.
Proof. We can assume that $x^{0}=0$ and we let $C y l(\delta)=C y l\left(x^{0} ; \delta(\gamma)\right)$. By Theorem 1.1 in [11] there exists $C$ such that the following Carleman estimate holds

$$
\sum_{|\alpha|=0}^{1} \int_{C y l(4 \delta)} \sigma^{3-2|\alpha|}\left|\partial^{\alpha} v_{0}\right|^{2} \leq C \int_{C y l(4 \delta)}\left|P_{\varphi} v_{0}\right|^{2} \quad \text { for all } v_{0} \in H_{0}^{2}(C y l(4 \delta))
$$

provided $C<\gamma, C(\gamma)<\tau$.
Let $\chi \in C_{0}^{\infty}(C y l(4 \delta))$, is determined only by $\gamma, 0 \leq \chi \leq 1$, annd $\chi=1$ on $C y l(3 \delta)$. Using this Carleman type estimate for $v_{0}=\chi \Lambda_{\sigma(t)}^{-1} v$, we obtain

$$
\begin{align*}
& \int_{C y l(4 \delta)}\left(\sigma^{3} \chi^{2}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\sigma \sum_{|\alpha|=1}\left|\chi \partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)+\partial^{\alpha} \chi \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) \\
& \leq C \int_{C y l(4 \delta)}\left|P_{\varphi}\left(\chi \Lambda_{\sigma(t)}^{-1} v\right)\right|^{2} \\
& \leq C \int_{C y l(4 \delta)}\left(\left|P_{\varphi}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2}+C(\gamma)\left(\tau^{2}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\sum_{|\alpha|=1}\left|\partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2}\right)\right) . \tag{37}
\end{align*}
$$

where we used (20), the Leibniz' formulas

$$
P(\chi w)=\chi P w+P_{1}(; \chi) w+P(\chi) w, P_{1}(\chi w ; \varphi)=\chi P_{1}(w ; \varphi)+P_{1}(\chi ; \varphi) w
$$

and the triangle inequality.
Using these inequalities, Lemma 2.4, and recalling that $\chi=1$ on $\operatorname{Cyl}(3 \delta)$ we derive from the bound (37) that

$$
\begin{align*}
& \int_{C y l(3 \delta)}\left(\sigma^{3}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\sigma \sum_{|\alpha|=1}\left|\partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2}\right)-C(\gamma) \int_{C y l(\delta)}|v|^{2} \\
& \leq C \int_{C y l(4 \delta)}\left(\left|P_{\varphi}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2}+C(\gamma)\left(|v|^{2}\right)+\left|\partial_{t} \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) \tag{38}
\end{align*}
$$

The Parseval identity, (35), and the definition of $\Lambda_{\sigma}$ yield

$$
\begin{aligned}
\int_{C y l(\delta)} \sigma v^{2} \leq & 2 \int_{C y l(\delta)} \sigma(t) v^{2} \\
= & \int_{\mathbf{R}} \sigma(0) \int_{\mathbf{R}^{3}} \frac{\sigma(t)^{2}+1}{<\xi>^{2}+\sigma(t)^{2}}|\hat{v}(\xi, t)|^{2} d \xi d t \\
& +\int_{\mathbf{R}} \sigma(t) \int_{\mathbf{R}^{3}} \frac{|\xi|^{2}}{<\xi>^{2}+\sigma(t)^{2}}|\hat{v}(\xi, t)|^{2} d \xi d t \\
=C & \int_{\mathbf{R}}(\sigma(t))^{3} \int_{\mathbf{R}^{3}}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\sum_{|\alpha|=1, \alpha_{4}=0} \int_{\mathbf{R}} \sigma(t) \int_{\mathbf{R}^{3}}\left|\partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2} \\
\leq C & \int_{C y l(3 \delta)} \sigma^{3}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+C \sum_{|\alpha|=1, \alpha_{4}=0} \int_{C y l(3 \delta)} \sigma\left|\partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2} \\
& +C \int_{\mathbf{R}^{4} \backslash C y l(3 \delta)}\left((\sigma(t))^{3}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\sigma(t) \sum_{|\alpha|=1, \alpha_{4}=0}\left|\partial^{\alpha}\left(\Lambda_{\sigma(t)}^{-1} v\right)\right|^{2}\right) .
\end{aligned}
$$

Choosing $\tau>C(\gamma)$ and using Lemma 2.3 we will have from (38)

$$
\begin{align*}
& \int_{\mathbf{R}^{4}} \sigma\left(|v|^{2}+\left|\partial_{t} \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) \leq C \int_{C y l(4 \delta)}\left|\Lambda_{\sigma(t)}^{-1} P_{\varphi} v\right|^{2} \\
&+C(\gamma) \int_{\mathbf{R}^{4} \backslash C y l(3 \delta)}\left(\tau^{3}\left|\Lambda_{\sigma(t)}^{-1} v\right|^{2}+\tau \sum_{|\alpha|=1}\left|\partial^{\alpha} \Lambda_{\sigma(t)}^{-1} v\right|^{2}\right) . \tag{39}
\end{align*}
$$

Now, by using Lemma 2.3 and choosing again $\tau>C(\gamma)$ we will eliminate the last integral in this bound and complete the proof.

## 3. Proof of Theorem 1.1

Lemma 3.1. Let $|\nabla \psi|>0$ on $\bar{\Omega}$.
Then for any $x \in \bar{\Omega}$ there are $\delta(\gamma)$ and $C$ such that
$\gamma \int_{B(\delta)}\left(\sigma^{2}\left|\mathbf{v}^{*}\right|^{2}+\left|\partial_{j} \mathbf{v}^{*}\right|^{2}\right) \leq C \int_{B(\delta)} \sigma\left(\left|d i v_{T, \varphi} \mathbf{v}^{*}\right|^{2}+\left|\operatorname{curl}_{T, \varphi} \mathbf{v}^{*}\right|^{2}\right), j=1,2,3$,
for all $\mathbf{v}^{*} \in H_{0}^{1}\left(C y l\left(x^{0} ; \delta(\gamma)\right)\right.$ provided $\tau>C$.
Proof is available in [10], Lemma 5, where the spatial bound need to be integrated with respect to $t$ like in the proof of Lemma 2.1.

Proof of Theorem 1.1. In [10], (25) it was shown that the system (10) implies

$$
\begin{gathered}
P(1) \mathbf{u}=\frac{\mathbf{f}}{\rho}+\mathbf{A}(1) \mathbf{u} \\
P(2) v=\operatorname{div}_{T} \frac{\mathbf{f}}{\rho}+A(2) \mathbf{u} \\
P(1) \mathbf{w}=\operatorname{curl}_{T} \frac{\mathbf{f}}{\rho}+\mathbf{A}(3) \mathbf{u}
\end{gathered}
$$

where

$$
P(1)=\partial_{t}^{2}-\rho^{-1}(A+R), P(2)=\partial_{t}^{2}-\rho^{-1}\left(A+R+\partial_{1}^{2}+\partial_{2}^{2}+a_{3} a_{4} \partial_{3}^{2}\right)
$$

$\mathbf{A}(j)$ are sums of $\partial_{k}\left(\mathbf{A}_{1} \partial_{t} \mathbf{u}\right), \partial_{m}\left(\mathbf{A}_{1} \partial_{k} \mathbf{u}\right), \mathbf{A} \partial_{k} \mathbf{u}, \mathbf{A} \partial_{t} \mathbf{u} \mathbf{A u}$ with the (matrix) coefficients $\mathbf{A}, \mathbf{A}_{1},|\mathbf{A}|_{1}(\Omega)+\left|\mathbf{A}_{1}\right|_{0}(\Omega) \leq C, j, k, m=1,2,3$.

Using the the substitution $\mathbf{u}^{*}=e^{\tau \varphi} \mathbf{u}, v^{*}=e^{\tau \varphi} v, \mathbf{w}^{*}=e^{\tau \varphi} \mathbf{w}, \mathbf{f}^{*}=e^{\tau \varphi} \mathbf{f}$ this system is transformed into

$$
\begin{gather*}
P_{\varphi}(1) \mathbf{u}^{*}=\frac{\mathbf{f}^{*}}{\rho}+\mathbf{A}_{\varphi}(1) \mathbf{u}^{*} \\
P_{\varphi}(2) v^{*}=\operatorname{div}_{T, \varphi} \frac{\mathbf{f}^{*}}{\rho}+A_{\varphi}(2) \mathbf{u}^{*}  \tag{40}\\
P_{\varphi}(1) \mathbf{w}^{*}=\operatorname{curl}_{T, \varphi} \frac{\mathbf{f}^{*}}{\rho}+\mathbf{A}_{\varphi}(3) \mathbf{u}^{*} .
\end{gather*}
$$

Let $x^{0} \in \bar{\Omega}^{\prime}$ and $C y l(\delta)=C y l\left(x^{0} ; \delta(\gamma)\right)$ with $\delta(\gamma)$ defined in (35). Let a cut off function $\chi=1$ on $\operatorname{Cyl}\left(\frac{\delta}{2}\right)$, supp $\chi \subset \operatorname{Cyl}(\delta)^{\prime}, 0 \leq \chi \leq 1,|\chi|_{2} \mid\left(\mathbf{R}^{4}\right) \leq$ $C(\gamma), \partial_{t} \chi=0$, then the system (40) implies

$$
\begin{gather*}
P_{\varphi}(1)\left(\chi \mathbf{u}^{*}\right)=\chi\left(\frac{\mathbf{f}^{*}}{\rho}+\mathbf{A}_{\varphi}(1) \mathbf{u}^{*}\right)+\mathbf{A}(1,1)\left(\mathbf{u}^{*}\right) \\
P_{\varphi}(2)\left(\chi v^{*}\right)=\chi\left(\operatorname{div}_{T, \varphi} \frac{\mathbf{f}^{*}}{\rho}+A_{\varphi}(2) \mathbf{u}^{*}\right)+A(2,1)\left(v^{*}\right),  \tag{41}\\
P_{\varphi}(1)\left(\chi \mathbf{w}^{*}\right)=\chi\left(\operatorname{curl}_{T, \varphi} \frac{\mathbf{f}^{*}}{\rho}+\mathbf{A}_{\varphi}(3) \mathbf{u}^{*}\right)+\mathbf{A}(3,1)\left(\mathbf{w}^{*}\right),
\end{gather*}
$$

where $\mathbf{A}(j, 1)$ are sums of the terms $a(\gamma) \partial_{j} \mathbf{u}, a(\gamma) \partial_{j} v, a(\gamma) \partial_{j} \mathbf{w}, \sigma a(\gamma) \mathbf{u}, \sigma a(\gamma) v$, $\sigma a(\gamma) \mathbf{w}$ with $|a(\gamma)|_{2}(\Omega)<C(\gamma)$.

Using that $v^{*}=\operatorname{div}_{T, \varphi} \mathbf{u}^{*}, \mathbf{w}^{*}=\operatorname{curl}_{T, \varphi} \mathbf{u}^{*}$, applying Lemma 2.5 to each of 7 scalar equations in this system, and adding the resulting inequalities we yield

$$
\begin{align*}
& \int_{\mathbf{R}^{4}} \sigma\left(\left|\chi \mathbf{u}^{*}\right|^{2}+\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\left|\chi v^{*}\right|^{2}+\left|\chi \operatorname{div}_{T, \varphi}\left(\mathbf{u}^{*}\right)\right|^{2}\right. \\
& +\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \operatorname{div}_{T, \varphi}\left(\mathbf{u}^{*}\right)\right)\right|^{2}+\left|\chi \mathbf{w}^{*}\right|^{2}+\left|\chi \operatorname{curl}_{T, \varphi}\left(\mathbf{u}^{*}\right)\right|^{2} \\
& \left.\quad+\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \operatorname{curl}_{T, \varphi}\left(\mathbf{u}^{*}\right)\right)\right|^{2}\right) \\
& \leq C \int_{\mathbf{R}^{4}} \chi^{2}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& +C(\gamma) \tau^{-2} \int_{\mathbf{R}^{4}}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& \quad+C(\gamma) \int_{\mathbf{R}^{4}}\left(\left|\mathbf{u}^{*}\right|^{2}+\left|v^{*}\right|^{2}+\left|\mathbf{w}^{*}\right|^{2}\right) \tag{42}
\end{align*}
$$

Observe that for a first order operator $P_{1} v=\sum_{j=1}^{3} b_{j} \partial_{j} v$ we have $\chi P_{1, \varphi} v=$ $P_{1, \varphi}(\chi v)-P_{1}(\chi) v$. Since $\operatorname{div}_{T, \varphi}\left(\rho^{-1} \mathbf{f}\right)$ is the sum of terms $a \partial_{j} f_{j}, \sigma a f_{j}$ with $|a|_{1}\left(\Omega^{*}\right)<C$, by using Lemma 2.4 we will have the terms with $\mathbf{f}^{*}$ on the right side of (42). Moreover, $\chi \mathbf{A}_{\varphi}(m) u$ is the sum of terms $\left(\partial_{k}-\sigma \partial_{k} \psi\right)\left(\chi\left(\mathbf{A}_{1}\left(\partial_{1}-\right.\right.\right.$ $\left.\left.\sigma \partial_{j} \psi\right) \mathbf{u}^{*}\right)$ and of $\partial_{k}\left(\mathbf{A}_{1}\left(\partial_{j}-\sigma \partial_{j} \psi\right) \mathbf{u}^{*}\right)$, so again using Lemma 2.4 we will have remaining terms of the first two integrals on the right side of (42).

By standard calculations $\partial_{t} \operatorname{div}_{T, \varphi} \mathbf{u}^{*}=\operatorname{div}_{T, \varphi} \partial_{t} \mathbf{u}^{*}+r(1)$ where $r(1)$ is the sum of terms $a(\gamma) \sigma u_{j}^{*}$ and $a \partial_{k} u^{*}$ with $|a(\gamma)|_{1}(\Omega)<C(\gamma),|a|_{1}(\Omega)<C$ and $\chi d i v_{T, \varphi} \partial_{t} \mathbf{u}^{*}=\operatorname{div}_{T, \varphi}\left(\chi \partial_{t} \mathbf{u}^{*}\right)+r(2)$ where $r(2)$ is the sum of terms $a(\gamma) \partial_{t} u_{j}^{*}$ with $|a(\gamma)|_{1}(\Omega)<C(\delta)$. Hence

$$
\begin{aligned}
& \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t}\left(\operatorname{div}_{T, \varphi} \mathbf{u}^{*}\right)\right)-\operatorname{div}_{T, \varphi}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right) \\
& \quad=\Lambda_{\sigma(t)}^{-1}\left(\chi \operatorname{div}_{T, \varphi} \partial_{t} \mathbf{u}^{*}\right)+\Lambda_{\sigma(t)}^{-1}(\chi r(1))-\operatorname{div}_{T, \varphi}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right) \\
& \left.\quad=\Lambda_{\sigma(t)}^{-1}\left(\operatorname{div}_{T, \varphi}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)-\operatorname{div}_{T, \varphi}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)+\Lambda_{\sigma(t)}^{-1}(\chi r(1)+r(2))\right) .
\end{aligned}
$$

So using Lemma 2.1 we yield

$$
\begin{aligned}
& \| \sigma^{\frac{1}{2}}\left(\Lambda_{\sigma(t)}^{-1}( \right.\left.\left(\chi \partial_{t}\left(\operatorname{div}_{T, \varphi} \mathbf{u}^{*}\right)\right)-\operatorname{div}_{T, \varphi}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)\right) \|_{(0)}\left(\mathbf{R}^{4}\right) \\
& \leq C(\gamma) \tau^{-\frac{1}{2}}\left\|\chi \partial_{t} \mathbf{u}^{*}\right\|_{(0)}\left(\mathbf{R}^{4}\right)+C(\gamma)\left(\tau^{-\frac{1}{2}}\left\|\partial_{t} \mathbf{u}^{*}\right\|_{(0)}\left(\mathbf{R}^{4}\right)\right. \\
&+\left.\tau^{-\frac{1}{2}}\left\|\chi \nabla \mathbf{u}^{*}\right\|_{(0)}\left(\mathbf{R}^{4}\right)+\tau^{\frac{1}{2}}\left\|\chi \mathbf{u}^{*}\right\|_{(0)}\left(\mathbf{R}^{4}\right)\right) .
\end{aligned}
$$

Therefore from (42) we obtain

$$
\begin{align*}
& \int_{\mathbf{R}^{4}} \sigma\left(\left|\chi \mathbf{u}^{*}\right|^{2}+\left|\chi v^{*}\right|^{2}+\left|\chi \mathbf{u}^{*}\right|^{2}+\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\left|\operatorname{div}_{T, \varphi}\left(\chi \mathbf{u}^{*}\right)\right|^{2}\right. \\
& \left.+\left|\operatorname{curl}_{T, \varphi}\left(\chi \mathbf{u}^{*}\right)\right|^{2}+\left|\operatorname{div}_{T, \varphi} \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t}\left(\mathbf{u}^{*}\right)\right)\right|^{2}+\left|\operatorname{curl}_{T, \varphi} \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t}\left(\mathbf{u}^{*}\right)\right)\right|^{2}\right) \\
& \leq C \int_{\mathbf{R}^{4}} \chi^{2}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& \quad+C(\gamma) \tau^{-1} \int_{\mathbf{R}^{4}}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
&  \tag{43}\\
& \quad+C(\gamma) \int_{\mathbf{R}^{4}}\left(\left|\mathbf{u}^{*}\right|^{2}+\left|v^{*}\right|^{2}+\left|\mathbf{w}^{*}\right|^{2}\right)
\end{align*}
$$

Introducing another cut off function $\chi_{1}, \partial_{t} \chi_{1}=0$, supported in $B^{\prime}(4 \delta) \times \mathbf{R}$ with $\chi_{1}=1$ on $B^{\prime}(3 \delta) \times \mathbf{R},\left|\chi_{1}\right|_{1}\left(\mathbf{R}^{4}<C(\gamma)\right.$, and applying Lemma 3.1 we yield

$$
\begin{aligned}
& \int_{\mathbf{R}^{4}} \sigma\left(\left|\operatorname{div}_{T, \varphi}\left(\chi_{1}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)\right)\right|^{2}+\left|\operatorname{curl}_{T, \varphi}\left(\chi_{1}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)\right)\right|^{2}\right) \\
& \geq C^{-1} \gamma \int_{\mathbf{R}^{4}}\left(\sigma^{2}\left|\chi_{1} \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\sum_{j=1}^{3}\left|\partial_{j}\left(\chi_{1}\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)\right)\right|^{2}\right) \\
& \geq C^{-1} \gamma \int_{\mathbf{R}^{4}} \sigma^{2}\left(\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\sum_{j=1}^{3}\left|\partial_{j} \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}\right) \\
& \left.-C(\gamma) \tau^{-2} \int_{\mathbf{R}^{4}} \sigma^{2} \mid \chi \partial_{t} \mathbf{u}^{*}\right)\left.\right|^{2}
\end{aligned}
$$

because

$$
\begin{aligned}
\int_{\mathbf{R}^{4}}\left(\sigma^{2}\left|\left(1-\chi_{1}\right) \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\mid \partial_{j}\left(\left(1-\chi_{1}\right)\right.\right. & \left.\left.\left(\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right)\right)\left.\right|^{2}\right) \\
& \leq C(\gamma) \tau^{-2} \int_{\mathbf{R}^{4}} \sigma^{2}\left|\chi \partial_{t} \mathbf{u}^{*}\right|^{2}
\end{aligned}
$$

due to Lemma 2.1.
As above, by using the basic Fourier analysis we yield

$$
C \int_{\mathbf{R}^{4}} \sigma^{2}\left(\left|\Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}+\sum_{j=1}^{3}\left|\partial_{j} \Lambda_{\sigma(t)}^{-1}\left(\chi \partial_{t} \mathbf{u}^{*}\right)\right|^{2}\right) \geq \int_{\mathbf{R}^{4}} \sigma^{2}\left|\chi \partial_{t} \mathbf{u}^{*}\right|^{2}
$$

Using the two previous inequalities, from (43) we obtain

$$
\begin{align*}
& \int_{\mathbf{R}^{4}}\left(\gamma\left(\left|\partial_{j}\left(\chi \mathbf{u}^{*}\right)\right|^{2}+\sigma^{2}\left|\chi \mathbf{u}^{*}\right|^{2}\right)+\sigma \chi^{2}\left(\left|v^{*}\right|^{2}+\left|\mathbf{w}^{*}\right|^{2}\right)\right) \\
& \leq C \int_{\mathbf{R}^{4}} \chi^{2}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& +C(\gamma) \tau^{-2} \int_{\mathbf{R}^{4}}\left(\left|\mathbf{f}^{*}\right|^{2}+\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& +C(\gamma) \int_{\mathbf{R}^{4}}\left(\left|\mathbf{u}^{*}\right|^{2}+\left|v^{*}\right|^{2}+\left|\mathbf{w}^{*}\right|^{2}\right) . \tag{44}
\end{align*}
$$

Now the claim follows by partition of the unity argument. Since our choice of $\delta$ depends on $\gamma$ we give this argument in some detail.

The balls $B^{\prime}\left(x^{0} ; \delta(\gamma)\right)$ form an open covering of the compact set $\bar{\Omega}^{\prime}$, so we can find a finite covering of $\bar{\Omega}^{\prime}$ by balls $B^{\prime}(x(k), \delta(\gamma)), k=1, \ldots, K(\gamma)$. Let $\chi(; k)$ be a $C^{\infty}$ - partition of the unity subordinated to this covering, i.e. $\operatorname{supp} \chi(; k) \subset B^{\prime}\left(x(k) ; \delta(\gamma)\right.$ with $\sum_{k=1}^{K} \chi^{2}(; k)=1$ on $\Omega$.

Summing (44) with $x=x(k), \delta=\delta(\gamma, k)$ over $k=1, \ldots, K$ and choosing $\tau>C(\gamma)$ we get

$$
\begin{aligned}
& \int_{\Omega}\left(\gamma\left(\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right)+\sigma\left(\left|v^{*}\right|^{2}+\left|\mathbf{w}^{*}\right|^{2}\right)\right) \\
& \leq\left(C+C(\gamma) \tau^{-2}\right) \int_{\Omega}\left|\mathbf{f}^{*}\right|^{2}+C \int_{\Omega}\left(\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right) \\
& \quad+C(\gamma) \tau^{-1} \int_{\Omega}\left(\sum_{j=1}^{4}\left|\partial_{j} \mathbf{u}^{*}\right|^{2}+\sigma^{2}\left|\mathbf{u}^{*}\right|^{2}\right)+C(\gamma) \int_{\Omega}\left(\left|\mathbf{u}^{*}\right|^{2}+v^{* 2}+\left|\mathbf{w}^{*}\right|^{2}\right)
\end{aligned}
$$

By choosing $\gamma>2 C$ we can absorb the second integral in the right side by the left side. Then we fix $\gamma$ and choosing $\tau>C(\gamma)$ absorb the third and the fourth integral by the left side and complete the proof of (9).

## 4. Proofs of stability estimates

In this section we will prove Theorems 1.3, 1.4.
Proof of Theorem 1.3. By extension theorems for Sobolev spaces we can find $\mathbf{u}^{*} \in H^{2}(\Omega)$ so that

$$
\mathbf{u}^{*}=\mathbf{g}_{0}, \partial_{\nu} \mathbf{u}^{*}=\mathbf{g}_{1} \text { on } \Gamma
$$

and

$$
\begin{equation*}
\left\|\mathbf{u}^{*}\right\|_{(2)}(\Omega) \leq C F \tag{45}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathbf{v}=\mathbf{u}-\mathbf{u}^{*} \tag{46}
\end{equation*}
$$

The function $\mathbf{v}$ solves the Cauchy problem

$$
\begin{equation*}
\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right) \mathbf{v}=\mathbf{f}^{*} \text { in } \Omega, \mathbf{v}=0, \quad \partial_{\nu} \mathbf{v}=0 \quad \text { on } \quad \partial G \times(-T, T), \tag{47}
\end{equation*}
$$

where $\mathbf{f}^{*}=\mathbf{f}-\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right) \mathbf{u}^{*}$.
Due to the strict positivity condition (4) by standard energy estimates for hyperbolic systems (i.e. [3], p. 128 ) we have

$$
\begin{equation*}
C^{-1}\left(E(0 ; \mathbf{v})-\left\|\mathbf{f}^{*}\right\|_{(0)}(\Omega)\right) \leq E(t ; \mathbf{v}) \leq C\left(E(0 ; \mathbf{v})+\left\|\mathbf{f}^{*}\right\|_{(0)}(\Omega)\right) \tag{48}
\end{equation*}
$$

when $t \in(-T, T)$.
Let us fix $\gamma$ in Theorem 1.1. By using (12) we choose $\delta_{0}$ depending on the same parameters as $C$ so that $\varphi<1-2 \delta_{0}$ on $\left\{t: T-\delta_{0}<|t|<T\right\}$ and $1-\delta_{0}<\varphi$ on $\left(-\delta_{0}, \delta_{0}\right)$. We choose a smooth cut-off function $0 \leq \chi_{0}(t) \leq 1$ such that $\chi_{0}(t)=1$ when $|t|<T-2 \delta_{0}$ and $\chi_{0}(t)=0$ when $|t|>T-\delta_{0}$. It is clear that

$$
\begin{equation*}
\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right)\left(\chi_{0} \mathbf{v}\right)=\chi_{0} \mathbf{f}^{*}+2 \rho \partial_{t} \chi_{0} \partial_{t} \mathbf{v}+\rho \partial_{t}^{2} \chi_{0} \mathbf{v} \tag{49}
\end{equation*}
$$

Obviously, $\chi_{0} \mathbf{v} \in H_{0}^{2}(\Omega)$, hence by Theorem 1.1

$$
\begin{aligned}
& \int_{\Omega}\left(\left|\partial_{t}^{2}\left(\chi_{0} \mathbf{v}\right)\right|^{2}+\left|\nabla\left(\chi_{0} \mathbf{v}\right)\right|^{2}+\left|\chi_{0} \mathbf{v}\right|^{2}\right) e^{2 \tau \varphi} \\
& \leq C \\
& \leq \int_{\Omega}\left(\left|\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right)\left(\chi_{0} \mathbf{v}\right)\right|^{2}\right) e^{2 \tau \varphi} \\
& \leq
\end{aligned}
$$

by (47).
Shrinking the integration domain $\Omega$ on the left side to $G \times\left(0, \delta_{0}\right)$ and using our choice of $\delta_{0}$ we yield

$$
\begin{aligned}
& e^{2 \tau\left(1-\delta_{0}\right)} \int_{0}^{\delta_{0}} E(t ; \mathbf{v}) d t \leq C \int_{G \times\left(-\delta_{0}, \delta_{0}\right)}\left(\left|\partial_{t} \mathbf{v}\right|^{2}+|\nabla \mathbf{v}|^{2}+|\mathbf{v}|^{2}\right) e^{2 \tau \varphi} \\
& \leq C \int_{\Omega}\left|\mathbf{f}^{*}\right|^{2} e^{2 \tau \varphi}+C e^{2 \tau\left(1-2 \delta_{0}\right)} \int_{\left\{T-2 \delta_{0}<|t|<T\right\}} \int_{G}\left(\left|\partial_{t} \mathbf{v}\right|^{2}+|\nabla \mathbf{v}|^{2}+|\mathbf{v}|^{2}\right) \\
& \quad \leq C \int_{\Omega}\left|\mathbf{f}^{*}\right|^{2} e^{2 \tau \varphi}+C e^{2 \tau(1-2 \delta)} \int_{T-2 \delta_{0}}^{T} E(t ; \mathbf{v}) d t
\end{aligned}
$$

Choosing $\Phi=\sup _{\Omega} \varphi$ and using (48)

$$
\begin{aligned}
e^{2 \tau\left(1-\delta_{0}\right)} \frac{\delta}{C} E(0 ; \mathbf{v})-C e^{2 \tau \Phi}\left\|\mathbf{f}^{*}\right\|_{(1)}^{2} & (\Omega) \\
& \leq C e^{2 \tau\left(1-2 \delta_{0}\right)} E(0 ; \mathbf{v})+C e^{2 \tau \Phi}\left\|\mathbf{f}^{*}\right\|_{(0)}^{2}(\Omega)
\end{aligned}
$$

To eliminate the first term on the right side we choose $\tau$ (depending on $C$ ) so large that $e^{-2 \tau \delta_{0}}<\frac{1}{C^{2}}$ and by using energy estimates (48) we finally get

$$
E(t ; \mathbf{v}) \leq C\left\|\mathbf{f}^{*}\right\|_{(0)}(\Omega)
$$

and

$$
\begin{aligned}
& E(t ; \mathbf{u}) \leq C\left(\left\|\mathbf{f}^{*}\right\|_{(0)}(\Omega)+E\left(t ; \mathbf{u}^{*}\right)\right) \\
& \leq C\left(\left\|\mathbf{f}^{*}\right\|_{(0)}(\Omega)\right. \\
& \left.\quad+\left\|\mathbf{u}^{*}\right\|_{\left(\frac{3}{2}\right)}(\Gamma)+\left\|\partial_{\nu} \mathbf{u}^{*}\right\|_{\left(\frac{1}{2}\right)}(\Gamma)\right) \\
& \quad \leq C\left(\|\mathbf{f}\|_{(0)}(\Omega)+\left\|\mathbf{g}_{0}\right\|_{\left(\frac{3}{2}\right)}(\Gamma)+\left\|\mathbf{g}_{1}\right\|_{\left(\frac{1}{2}\right)}(\Gamma)\right)
\end{aligned}
$$

The proof is complete.
Proof of Theorem 1.4. By extension theorems for Sobolev spaces we can find $\mathbf{U}^{*} \in H^{2}(\Omega)$ so that

$$
\mathbf{U}^{*}=\mathbf{0}, \partial_{\nu} \mathbf{U}^{*}=\partial_{t}^{2} \partial_{\nu} \mathbf{u} \text { on } \Gamma
$$

and

$$
\begin{equation*}
\left\|\mathbf{U}^{*}\right\|_{(2)}(\Omega) \leq C F \tag{50}
\end{equation*}
$$

Let

$$
\begin{equation*}
\mathbf{V}=\partial_{t}^{2} \mathbf{u}-\mathbf{U}^{*} \tag{51}
\end{equation*}
$$

Differentiating (14) in $t$ and using time-independence of the coefficients of the system, we get

$$
\begin{array}{ll}
\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right) \mathbf{V}=\partial_{t}^{2} \mathcal{A} \mathbf{f}-\mathbf{F}^{*} & \text { in } \Omega  \tag{52}\\
\mathbf{v}=0, \quad \partial_{\nu} \mathbf{v}=0 & \text { on } \partial G \times(-T, T)
\end{array}
$$

where $\mathbf{F}^{*}=\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right) \mathbf{U}^{*}$.
By standard energy estimates for hyperbolic systems (i.e. [3])

$$
\begin{align*}
& C^{-1} E(0 ; \mathbf{V})-C\left(\int_{G}|\mathbf{f}|^{2}+\int_{\Omega}\left|\mathbf{F}^{*}\right|^{2}\right) \\
& \quad \leq E(t ; \mathbf{V}) \leq C E(0 ; \mathbf{V})+C\left(\int_{G}|\mathbf{f}|^{2}+\int_{\Omega}\left|\mathbf{F}^{*}\right|^{2}\right), \tag{53}
\end{align*}
$$

when $t \in(-T, T)$.

Using (12) we choose $\delta_{0}$ depending on the same parameters as $C$ so that $\psi<-\delta_{0}$ on $G \times\left\{t: T-\delta_{0}<|t|<T\right\}$ and $0<\psi$ on $G \times\left(-\delta_{0}, \delta_{0}\right)$. Then we fix a smooth cut-off function $\chi_{0}, 0 \leq \chi_{0}(t) \leq 1$ such that $\chi_{0}(t)=1$ when $|t|<T-2 \delta_{0}$ and $\chi_{0}(t)=0$ when $|t|>T-\delta_{0}, 0 \leq \chi_{0} \leq 1,\left|\partial_{t}^{j} \chi_{0}\right| \leq C, j=0,1,2$. By the Leibniz formula

$$
\left(\rho \partial_{t}^{2}-\left(\mathbf{A}_{T}+R\right)\right)\left(\chi_{0} \mathbf{V}\right)=\chi_{0}\left(\partial_{t}^{2} \mathcal{A} \mathbf{f}-\partial_{t}^{j} \mathbf{f}^{*}\right)+2 \rho \partial_{t} \chi_{0} \partial_{t} \mathbf{V}+\rho \partial_{t}^{2} \chi_{0} \mathbf{V}
$$

Obviously, $\chi_{0} \mathbf{V} \in H_{0}^{2}(\Omega)$, hence by Theorem 1.1

$$
\begin{align*}
& \int_{\Omega} \gamma\left(\left|\partial_{t}\left(\chi_{0} \mathbf{V}\right)\right|^{2}+\left|\nabla\left(\chi_{0} \mathbf{V}\right)\right|^{2}+\sigma^{2}\left|\left(\chi_{0} \mathbf{V}\right)\right|^{2}\right) e^{2 \tau \varphi} \\
& \quad \leq C\left(\int_{\Omega}\left(|\mathbf{f}|^{2}+\left|\mathbf{F}^{*}\right|^{2}\right) e^{2 \tau \varphi}+\int_{G \times\left\{T-2 \delta_{0}<|t|<T\right\}}\left(\left|\partial_{t} \mathbf{V}\right|^{2}+|\mathbf{V}|^{2}\right) e^{2 \tau \varphi}\right) \tag{54}
\end{align*}
$$

We have

$$
\begin{aligned}
\mathbf{V}(, 0) e^{\tau \varphi(, 0)}=-\int_{0}^{T} & \partial_{s}\left(\left(\chi_{0} \mathbf{V}(, s)\right) e^{\tau \varphi(, s)}\right) d s \\
& =-\int_{0}^{T}\left(\partial_{s}\left(\chi_{0} \mathbf{V}(, s)\right)+\sigma \partial_{s} \psi(, s) \chi_{0} \mathbf{V}(, s)\right) e^{\tau \varphi(, s)} d s
\end{aligned}
$$

So by splitting the left side in (54) into two equal terms and using the CauchySchwarz inequality we obtain

$$
\begin{aligned}
& \left.\gamma \int_{G} \mid \mathbf{V}(, 0)\right)\left.\right|^{2} e^{2 \tau \varphi(, 0)}+e^{2 \tau} \int_{-\delta_{0}}^{\delta_{0}} E(, t ; \mathbf{V}) d t \\
& \quad \leq C\left(\int_{\Omega}\left(|\mathbf{f}|^{2}+\left|\mathbf{F}^{*}\right|^{2}\right) e^{2 \tau \varphi}+e^{2 \tau \theta} \int_{\left\{T-2 \delta_{0}<|t|<T\right\}} E(t ; \mathbf{V}) d t\right)
\end{aligned}
$$

where $\theta=e^{-\gamma \delta_{0}}<1$. Using (53) and the inequality

$$
|f| \leq C\left|\partial_{t}^{2} \mathbf{u}(, 0)\right| \leq C\left(\left|\mathbf{U}^{*}(, 0)\right|+|\mathbf{V}(, 0)|\right)
$$

(due to (52) at $t=0$, the condition (16), and to (51)) we yield

$$
\begin{aligned}
\gamma \int_{G}|\mathbf{f}|^{2} e^{2 \tau \varphi(, 0)} & -C \gamma \int_{G}\left|\mathbf{U}^{*}(, 0)\right|^{2} e^{2 \tau \varphi(, 0)} \\
& \quad+e^{2 \tau} E(0 ; \mathbf{V})-C e^{2 \tau}\left(\int_{G}|\mathbf{f}|^{2}+\int_{\Omega}\left|\mathbf{F}^{*}\right|^{2}\right) \\
\leq & C \int_{G}|\mathbf{f}|^{2} e^{2 \tau \varphi(, 0)}+C e^{2 \tau \theta} \int_{\Omega}\left|\mathbf{F}^{*}\right|^{2}+C e^{2 \tau \theta} E(0 ; \mathbf{V})+C e^{2 \tau \theta} \int_{G}|\mathbf{f}|^{2}
\end{aligned}
$$

We choose and fix large $\gamma$ (depending on $C$ only) to absorb three other terms with $\mathbf{f}$ by the first term on the left side. Then we choose and fix $\tau$ (depending on $C$ ) so large that $C e^{2 \tau \theta}<e^{2 \tau}$ to absorb the term with $\mathbf{V}$ on the right by the term with $\mathbf{V}$ on the left and arrive at

$$
\int_{G}|\mathbf{f}|^{2} \leq C \int_{G}\left|\mathbf{U}^{*}(, 0)\right|^{2}+C \int_{\Omega}\left|\mathbf{F}^{*}\right|^{2} .
$$

Using the bound (50), Trace theorems and the definition of $\mathbf{F}^{*}$, we complete the proof of Theorem 1.4.

## 5. Conclusion

One can use Carleman estimates of Theorem 1.1 for coefficients identification as in $[7,11]$. However, for systems in divergent form (like the elasticity system) most precise results need a weak form of Carleman estimate which is expected to follow from Theorem 1.1 by using smoothing operators similar to $\Lambda_{\sigma}^{-1}$. At present, the condition (7) is essential for our proofs. Geometrical or mechanical meaning of this condition is not clear. A challenge is to obtain Carleman estimates and identification results for the system of transversely isotropic elasticity, without (or with relaxed) condition (7).

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# Generic controllability of the bilinear Schrödinger equation on 1-D domains: the case of measurable potentials 

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#### Abstract

Several sufficient conditions for the controllability of the Schrödinger equation have been proposed in the last few years. In this article we discuss the genericity of these conditions with respect to the variation of the controlled or the uncontrolled potential. In the case where the Schrödinger equation is set on a domain of dimension one, we improve the results in the literature, removing from the previously known genericity results some unnecessary technical assumptions on the regularity of the potentials.


Keywords: bilinear control, Schrödinger equation, approximate controllability, genericity.
MS Classification 2010: 35Q40, 35Q93, 81Q93.

## 1. Introduction

In this paper we consider controlled Schrödinger equations of the type

$$
\begin{equation*}
i \frac{\partial \psi}{\partial t}(t, x)=(-\Delta+V(x)+u(t) W(x)) \psi(t, x), \quad u(t) \in U \tag{1}
\end{equation*}
$$

where $\psi:[0,+\infty) \times \Omega \rightarrow \mathbf{C}$ for some domain $\Omega$ of $\mathbf{R}^{d}, d \geq 1, V, W$ are realvalued functions and $U=[0, \delta)$ for some $\delta>0$. We will assume either that $\Omega, V, W$ are bounded and that $\psi$ satisfies Dirichlet boundary conditions on $\partial \Omega$ or that $\Omega=\mathbf{R}^{d}$ and $-\Delta+V+u W$ has discrete spectrum for every $u \in U$. We look at (1) as at a control system evolving in the unit sphere of $L^{2}(\Omega, \mathbf{C})$, whose state $\psi(t, \cdot)$ is called the wavefunction of the Schrödinger equation. When $W$ is in $L^{\infty}(\Omega, \mathbf{R})$, the multiplication operator $L^{2}(\Omega, \mathbf{C}) \ni \psi \rightarrow W \psi \in L^{2}(\Omega, \mathbf{C})$ is bounded and then it is known that the Schrödinger equation (1) is not exactly controllable (see $[5,33]$ ). In certain cases, when $d=1$, a complete description of reachable sets has been provided (see [7, 9]). In the general case, however, such a description seems unattainable and one focuses on the analysis of the approximate controllability of system (1).

Several approaches have been developed to identify conditions on $V$ and $W$ which guarantee the approximate controllability of (1). Let us mention in particular the approaches based on: Lyapunov functions [8, 10, 23, 24, 25, 27], adiabatic evolution [1, 15, 18], Lie-bracket conditions in Banach spaces and in partially invariant finite-dimensional subspaces [11, 12, 21].

In this paper we focus on the approach developed in $[13,14,16,17]$, which is based on the idea of dropping the invariance requirement for the finitedimensional spaces and replacing it with some motion planning strategy within the finite-dimensional space which make it "almost invariant", in the sense that the norm of the projection of the solution in the finite-dimensional subspace stays as close to one as desired. (An analogous approach for the Navier-Stokes equation has been developed in $[3,31]$.) An advantage of this approach is that it also guarantees stronger notions of controllability than approximate controllability among wavefunctions. Indeed, it also implies approximate controllability between density matrices, simultaneous controllability for several initial conditions, tracking up to phases, etc (for details, see [14]). Let us mention that similar notions of controllability have also be obtained in [21].

The aim of this paper is to show that, in the case $d=1$, the approximate controllability of (1) is generic with respect to $V$ (for some suitable topology), once some non-constant potential $W$ is fixed and, similarly, that it is also generic with respect to $W$, once $V$ is fixed. Such results are proved by showing that the sufficient conditions proposed in $[13,14,17]$ are generic.

We improve here the results obtained in [22], by removing the assumption that the fixed potential (either $V$ or $W$ in the two cases presented above) is absolutely continuous. We should mention, however, that the results in [22] concern any dimension $d \in \mathbf{N}$ of the domain $\Omega$, while the technique developed here requires $d$ to be equal to 1 .

Let us mention that some other genericity results for the approximate controllability of the Schrödinger equation exist in the literature ( $[6,25,26,28]$ ). These are typically obtained by allowing variations of the pair $(V, W)$ or the triple $(\Omega, V, W)$, instead of a single element.

Our approach, shared with [22, 28], is based on analytic long-range perturbations. The idea is the following: denote by $\Gamma$ the class of systems on which the genericity of a certain property $P$ is studied. If we are able to prove the existence of at least one element of $\Gamma$ satisfying $P$, then we can propagate $P$ if some analytic dependence properties hold true. In this way we can prove that the property holds in a dense subset of $\Gamma$. A key property which allows this propagation to be performed is a result by Teytel in [32], which guarantees that between any two discrete-spectrum operators $-\Delta+V_{1}$ and $-\Delta+V_{2}$ (in a suitably defined class) there exist an analytic path $\mu \mapsto-\Delta+V_{\mu}$ such that all eigenvalues of $-\Delta+V_{\mu}$ are simple for all $\mu \in(1,2)$.

The paper is organized as follows. In Section 2 we fix the mathematical
framework, introducing the notion of solutions to the control system (1) and the notion of genericity that is investigated in the paper. In Section 3 we recall the sufficient conditions for approximate controllability obtained in [17] and in [14] and the genericity results already proved in [22]. Section 4 contains the main technical argument which allows us to improve the results in [22]. In terms of the discussion above, it contains the proof of the existence of the element whose good properties can be propagated globally by analytic perturbation. Finally, in Section 5 we develop the analytic propagation argument showing that the approximate controllability is generic separately with respect to $V$ and $W$.

## 2. Mathematical framework

### 2.1. Notation and basic definitions

Let $\mathbf{N}$ be the set of positive integers. For $d \in \mathbf{N}$, denote by $\Xi_{d}$ the set of nonempty, open, bounded and connected subsets of $\mathbf{R}^{d}$ and let $\Xi_{d}^{\infty}=\Xi_{d} \cup\left\{\mathbf{R}^{d}\right\}$. Take $U=[0, \delta) \subset \mathbf{R}$ for some $\delta>0$.

In the following we consider the Schrödinger equation (1) assuming that the potentials $V, W$ are taken in $L^{\infty}(\Omega, \mathbf{R})$ if $\Omega$ belongs to $\Xi_{d}$ and that $V, W \in$ $L_{\text {loc }}^{\infty}\left(\mathbf{R}^{d}, \mathbf{R}\right)$ and $\lim _{\|x\| \rightarrow \infty} V(x)+u W(x)=+\infty$ for every $u \in U$ if $\Omega=\mathbf{R}^{d}$. Then, for every $u \in U,-\Delta+V+u W$ (with Dirichlet boundary conditions if $\Omega$ is bounded) is a skew-adjoint operator on $L^{2}(\Omega, \mathbf{C})$ with compact resolvent and discrete spectrum (see [19, 29]). We denote by $\sigma(\Omega, V+u W)=\left(\lambda_{j}(\Omega, V+\right.$ $u W))_{j \in \mathbf{N}}$ the non-decreasing sequence of eigenvalues of $-\Delta+V+u W$, counted according to their multiplicities, and by $\left(\phi_{j}(\Omega, V+u W)\right)_{j \in \mathbf{N}}$ a corresponding sequence of eigenfunctions. Without loss of generality we can assume that $\phi_{j}(\Omega, V+u W)$ is real-valued for every $j \in \mathbf{N}$. Recall moreover that $\left(\phi_{j}(\Omega, V+\right.$ $u W))_{j \in \mathbf{N}}$ forms an orthonormal basis of $L^{2}(\Omega, \mathbf{C})$. If $j \in \mathbf{N}$ is such that $\lambda_{j}(\Omega, V+u W)$ is simple, then $\phi_{j}(\Omega, V+u W)$ is uniquely defined up to sign.

For every $\Omega \in \Xi_{d}$ let $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ be equal to $L^{\infty}(\Omega, \mathbf{R})$. For $\Omega=\mathbf{R}^{d}$ let

$$
\begin{aligned}
\mathcal{V}\left(\mathbf{R}^{d}\right) & =\left\{V \in L_{\mathrm{loc}}^{\infty}\left(\mathbf{R}^{d}, \mathbf{R}\right) \mid \lim _{\|x\| \rightarrow \infty} V(x)=+\infty\right\}, \\
\mathcal{W}\left(\mathbf{R}^{d}\right) & =\left\{W \in L_{\mathrm{loc}}^{\infty}\left(\mathbf{R}^{d}, \mathbf{R}\right) \left\lvert\, \operatorname{esssup}_{x \in \mathbf{R}^{d}}^{\operatorname{ess}} \frac{\log (|W(x)|+1)}{\|x\|+1}<\infty\right.\right\} .
\end{aligned}
$$

For every $\Omega \in \Xi_{d}^{\infty}$ let, moreover,
$\mathcal{Z}(\Omega, U)=\{(V, W) \mid V \in \mathcal{V}(\Omega), W \in \mathcal{W}(\Omega), V+u W \in \mathcal{V}(\Omega)$ for every $u \in U\}$.
If $(V, W) \in \mathcal{Z}(\Omega, U)$, each operator $-\Delta+V+u W, u \in U$, generates a group of unitary transformations $e^{i t(-\Delta+V+u W)}: L^{2}(\Omega, \mathbf{C}) \rightarrow L^{2}(\Omega, \mathbf{C})$. In
particular, $e^{i t(-\Delta+V+u W)}(\mathcal{S})=\mathcal{S}$ where $\mathcal{S}$ denotes the unit sphere of $L^{2}(\Omega, \mathbf{C})$. For every piecewise constant control function $u(\cdot)$ with values in $U$ and every initial condition $\psi_{0} \in L^{2}(\Omega, \mathbf{R})$, we can associate a solution

$$
\begin{aligned}
\psi\left(t ; \psi_{0}, u\right)= & e^{-i\left(t-\sum_{l=1}^{j-1} t_{l}\right)\left(-\Delta+V+u_{j} W\right)} \circ e^{-i t_{j-1}\left(-\Delta+V+u_{j-1} W\right)} \circ \cdots \\
& \cdots \circ e^{-i t_{1}\left(-\Delta+V+u_{1} W\right)}\left(\psi_{0}\right),
\end{aligned}
$$

where $0 \leq \sum_{l=1}^{j-1} t_{l} \leq t<\sum_{l=1}^{j} t_{l}$ and

$$
u(\tau)=u_{k} \quad \text { if } \quad \sum_{l=1}^{k-1} t_{l} \leq \tau<\sum_{l=1}^{k} t_{l}
$$

for $k=1, \ldots, j$.
Definition 2.1. Given $(V, W) \in \mathcal{Z}(\Omega, U)$ we say that the quadruple $(\Omega, V, W, U)$ is approximately controllable if for every $\psi_{0}, \psi_{1} \in \mathcal{S}$ and every $\varepsilon>0$ there exist $T>0$ and $u:[0, T] \rightarrow U$ piecewise constant such that $\left\|\psi_{1}-\psi\left(T ; \psi_{0}, u\right)\right\|<\varepsilon$.

### 2.2. Topologies and genericity

Let us endow $\mathcal{V}(\Omega), \mathcal{W}(\Omega)$ with the topology induced by the $L^{\infty}$ distance and $\mathcal{Z}(\Omega, U)$ with the corresponding product topology.

We also introduce, for every $V \in \mathcal{V}(\Omega)$ and every $W \in \mathcal{W}(\Omega)$, the topological subspaces of $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ defined, with a slight abuse of notation, by

$$
\begin{aligned}
\mathcal{V}(\Omega, W, U) & =\{\tilde{V} \in \mathcal{V}(\Omega) \mid(\tilde{V}, W) \in \mathcal{Z}(\Omega, U)\} \\
\mathcal{W}(\Omega, V, U) & =\{\tilde{W} \in \mathcal{W}(\Omega) \mid(V, \tilde{W}) \in \mathcal{Z}(\Omega, U)\}
\end{aligned}
$$

Notice that neither $\mathcal{V}(\Omega, W, U)$ nor $\mathcal{W}(\Omega, V, U)$ is empty. Moreover, both $\mathcal{V}(\Omega, W, U)$ and $\mathcal{W}(\Omega, V, U)$ are invariant by the set addition with $L^{\infty}(\Omega)$. In particular, they are open in $\mathcal{V}(\Omega)$ and $\mathcal{W}(\Omega)$ respectively and they coincide with $L^{\infty}(\Omega)$ when $\Omega \in \Xi_{d}$.

Let us recall that a topological space $X$ is called a Baire space if the intersection of countably many open and dense subsets of $X$ is dense in $X$. Every complete metric space is a Baire space. (In particular, $\mathcal{V}(\Omega), \mathcal{W}(\Omega), \mathcal{Z}(\Omega, U)$, $\mathcal{V}(\Omega, W, U)$, and $\mathcal{W}(\Omega, V, U)$ are Baire spaces.) The intersection of countably many open and dense subsets of a Baire space is called a residual subset of $X$. Given a Baire space $X$, a boolean function $P: X \rightarrow\{0,1\}$ is said to be a generic property if there exists a residual subset $Y$ of $X$ such that every $x$ in $Y$ satisfies property $P$, that is, $P(x)=1$.

## 3. Controllability of the discrete-spectrum Schrödinger equation: sufficient conditions and their genericity

The theorem below recalls the controllability result obtained in [17, Theorem 3.4]. Here and in the following a map $h: \mathbf{N} \rightarrow \mathbf{N}$ is called a reordering of $\mathbf{N}$ if it is a bijection.

Theorem 3.1 ([17]). Let $\Omega \in \Xi_{d}^{\infty}$ and $(V, W) \in \mathcal{Z}(\Omega, U)$. Assume that the elements of $\left(\lambda_{k+1}(\Omega, V)-\lambda_{k}(\Omega, V)\right)_{k \in \mathbf{N}}$ are $\mathbf{Q}$-linearly independent and that there exists a reordering $h: \mathbf{N} \rightarrow \mathbf{N}$ such that for infinitely many $n \in \mathbf{N}$ the matrix

$$
B_{n}^{h}(\Omega, V, W):=\left(\int_{\Omega} W(x) \phi_{h(j)}(\Omega, V)(x) \phi_{h(k)}(\Omega, V)(x) d x\right)_{j, k=1}^{n}
$$

is connected. Then $(\Omega, V, W, U)$ is approximately controllable.
Remark 3.2. Notice that, even in the unbounded case, each of the integrals $\int_{\Omega} W(x) \phi_{j}(\Omega, V)(x) \phi_{k}(\Omega, V)(x) d x$ is well defined. Indeed, when $\Omega=\mathbf{R}^{d}$, the growth of $|W|$ is at most exponential and $e^{a|x|} \phi_{j}\left(\mathbf{R}^{d}, V\right) \in L^{2}\left(\mathbf{R}^{d}, \mathbf{R}\right)$ for every $a>0$ and $j \in \mathbf{N}$ (see [2]).

The papers [13] and [14] present relaxed conditions on $V$ and $W$ which are enough to prove approximate controllability. In particular the $\mathbf{Q}$-linearly independence of $\left(\lambda_{k+1}(\Omega, V)-\lambda_{k}(\Omega, V)\right)_{k \in \mathbf{N}}$ can be replaced by the assumption that the elements of the sequence $\left(\left|\lambda_{k}(\Omega, V)-\lambda_{j}(\Omega, V)\right|\right)_{k, j \in \mathbf{N}}$ are pairwise distinct, or even less under some additional assumption (see in particular [13, Theorem 2.6]). However, since our goal is to prove the genericity of the sufficient conditions implying approximate controllability, we prefer to focus on the conditions stated in Theorem 3.1, which contain more informations on the potentials $V$ and $W$. We therefore introduce the following definition.

Definition 3.3. Let $V \in \mathcal{V}(\Omega)$ and $W \in \mathcal{W}(\Omega)$. We say that $(\Omega, V, W)$ is fit for control if $\left(\lambda_{k+1}(\Omega, V)-\lambda_{k}(\Omega, V)\right)_{k \in \mathbf{N}}$ is $\mathbf{Q}$-linearly independent and there exists a reordering $h$ such that $B_{n}^{h}(\Omega, V, W)$ is connected for infinitely many $n \in \mathbf{N}$. Let $(V, W)$ be an element of $\mathcal{Z}(\Omega, U)$. We say that the quadruple $(\Omega, V, W, U)$ is effective if $(\Omega, V+u W, W)$ is fit for control for some $u \in U$.

Theorem 3.1 can then be rephrased by saying that being effective is a sufficient condition for approximate controllability.

Let us recall the following result, which can be found in [22, Theorem 3.4].
Theorem 3.4 ([22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$. Then, generically with respect to $(V, W) \in \mathcal{Z}(\Omega, U)$ the triple $(\Omega, V, W)$ is fit for control.

In the present paper we give new results on the genericity of controllability when one of the two potentials $V$ and $W$ is fixed. We recall that in [22, Corollaries 4.4, 4.5, Proposition 4.6] the following was proved.

Theorem 3.5 ([22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$. Given any absolutely continuous function $V \in \mathcal{W}(\Omega)$, one has that generically with respect to $W \in \mathcal{V}(\Omega, W, U)$ the quadruple $(\Omega, V, W, U)$ is effective. Similarly, given $W \in \mathcal{V}(\Omega)$ non-constant and absolutely continuous on $\Omega$, one has that generically with respect to $V \in$ $\mathcal{V}(\Omega, V, U)$ the quadruple $(\Omega, V, W, U)$ is effective.

The goal of this paper is to show that the absolute continuity assumption on the potential that is fixed is purely technical and can be removed. We succeed in our goal at least in the case $d=1$. Our main result is the following.

Theorem 3.6. Let $\Omega$ belong to $\Xi_{1}^{\infty}$. Given any $V \in \mathcal{W}(\Omega)$, one has that generically with respect to $W \in \mathcal{V}(\Omega, W, U)$ the quadruple $(\Omega, V, W, U)$ is effective. Similarly, given $W \in \mathcal{V}(\Omega)$ non-constant, one has that generically with respect to $V \in \mathcal{V}(\Omega, V, U)$ the quadruple $(\Omega, V, W, U)$ is effective.

## 4. The basic one-dimensional technical result

The main goal of the section is to generalize the following result from [22], in the sense of dropping the assumption of absolute continuity on the function $Z$.

Lemma 4.1 ([22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$ and $Z$ be a non-constant absolutely continuous function on $\Omega$. Then there exist $\omega \in \Xi_{d}$ compactly contained in $\Omega$ with Lipschitz continuous boundary and a reordering $h: \mathbf{N} \rightarrow \mathbf{N}$ such that $\sigma(\omega, 0)$ is simple and

$$
\begin{equation*}
\int_{\omega} Z(x) \phi_{h(l)}(\omega, 0)(x) \phi_{h(l+1)}(\omega, 0)(x) d x \neq 0 \tag{2}
\end{equation*}
$$

for every $l \in \mathbf{N}$.
We are going to obtain such an extension in the case $d=1$, assuming that $Z$ is just measurable, bounded and non-constant. Let us stress that a function $Z \in L^{\infty}(\Omega, \mathbf{R})$ is said to be non-constant if no constant function on $\Omega$ coincides with $Z$ almost everywhere.

Proposition 4.2. Let $\Omega$ belong to $\Xi_{1}^{\infty}$ and $Z$ be a non-constant function in $L^{\infty}(\Omega, \mathbf{R})$. Then there exists a nonempty interval $\omega$ compactly contained in $\Omega$ such that

$$
\begin{equation*}
\int_{\omega} Z(x) \phi_{l}(\omega, 0)(x) \phi_{l+1}(\omega, 0)(x) d x \neq 0 \tag{3}
\end{equation*}
$$

for every $l \in \mathbf{N}$.

Proof. We look for $\omega$ in the form $(a, a+r)$, for some $a \in \Omega$ and $r>0$ such that $(a, a+r)$ is compactly contained in $\Omega$.

In particular, the simplicity of $\sigma(\omega, 0)$ is guaranteed and

$$
\phi_{l}(\omega, 0)(x)=\phi_{l}^{a, r}(x)=\sqrt{\frac{2}{r}} \sin \left(\frac{l \pi(x-a)}{r}\right), \quad l \in \mathbf{N} .
$$

We also define

$$
\psi_{l}^{a, r}(x)=\sqrt{\frac{2}{r}} \cos \left(\frac{l \pi(x-a)}{r}\right), \quad l \in \mathbf{N}
$$

Let us first show that it is enough to prove that there exists $(a, r)$ as above such that

$$
\begin{equation*}
\int_{a}^{a+r} Z(x) \psi_{1}^{a, r}(x) d x \neq 0 \tag{4}
\end{equation*}
$$

Indeed, assume that (4) is true and consider a neighbourhood $\mathcal{N}$ of $(a, r)$ in $\Omega \times(0,+\infty)$ such that, for every $(\alpha, \rho) \in \mathcal{N},(\alpha, \alpha+\rho)$ is compactly contained in $\Omega$ and

$$
\begin{equation*}
\int_{\alpha}^{\alpha+\rho} Z(x) \psi_{1}^{\alpha, \rho}(x) d x \neq 0 \tag{5}
\end{equation*}
$$

Such neighbourhood exists since the map $(\alpha, \rho) \mapsto \int_{\alpha}^{\alpha+\rho} Z(x) \psi_{1}^{\alpha, \rho}(x) d x$ is continuous. Assume by contradiction that there exists $l \in \mathbf{N}$ such that $(\alpha, \rho) \mapsto$ $\int_{\alpha}^{\alpha+\rho} Z(x) \phi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x) d x \equiv 0$ on a nonempty open subset $\mathcal{N}^{\prime}$ of $\mathcal{N}$.

Set

$$
F_{l}(\alpha, \rho)=\int_{\alpha}^{\alpha+\rho} Z(x) \phi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x) d x
$$

By differentiating $F_{j}$ with respect to its first variable, we get that

$$
\begin{align*}
0 & \equiv \frac{\partial}{\partial \alpha} F_{l}(\alpha, \rho) \\
& =-\frac{\pi}{\rho} \int_{\alpha}^{\alpha+\rho} Z(x)\left(l \psi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x)+(l+1) \phi_{l}^{\alpha, \rho}(x) \psi_{l+1}^{\alpha, \rho}(x)\right) d x \tag{6}
\end{align*}
$$

on $\mathcal{N}^{\prime}$. We used in this computation the fact that each function $\psi_{j}^{\alpha, \rho}$ annihilates at $\alpha$ and $\alpha+\rho$.

Differentiating once more with respect to $\alpha$, we get that, for every $(\alpha, \rho) \in$ $\mathcal{N}^{\prime}$,

$$
\begin{aligned}
0 & \equiv \frac{\partial^{2}}{\partial \alpha^{2}} F_{l}(\alpha, \rho) \\
& =\frac{\pi^{2}}{\rho^{2}} \int_{\alpha}^{\alpha+\rho} Z(x)\left(l(l+1) \psi_{l}^{\alpha, \rho}(x) \psi_{l+1}^{\alpha, \rho}(x)-\left(l^{2}+(l+1)^{2}\right) \phi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x)\right) d x \\
& =\frac{l(l+1) \pi^{2}}{\rho^{2}} \int_{\alpha}^{\alpha+\rho} Z(x) \psi_{l}^{\alpha, \rho}(x) \psi_{l+1}^{\alpha, \rho}(x) d x
\end{aligned}
$$

where the last identity follows from the relation $F_{l}(\alpha, \rho)=0$.
Differentiating once more $\frac{\partial^{2}}{\partial \alpha^{2}} F_{l}(\alpha, \rho)$ with respect to $\alpha$, we get, for almost every $(\alpha, \rho) \in \mathcal{N}^{\prime}$,

$$
\begin{aligned}
0 \equiv & -\frac{2 l(l+1) \pi^{2}}{\rho^{3}}(Z(\alpha+\rho)+Z(\alpha)) \\
& +\frac{l(l+1) \pi^{3}}{\rho^{3}} \int_{\alpha}^{\alpha+\rho} Z(x)\left(l \phi_{l}^{\alpha, \rho}(x) \psi_{l+1}^{\alpha, \rho}(x)+(l+1) \psi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x)\right) d x .
\end{aligned}
$$

Combining with (6), we deduce that

$$
\begin{align*}
Z(\alpha+\rho)+Z(\alpha) & =\frac{\pi}{2} \int_{\alpha}^{\alpha+\rho} Z(x)\left(-\phi_{l}^{\alpha, \rho}(x) \psi_{l+1}^{\alpha, \rho}(x)+\psi_{l}^{\alpha, \rho}(x) \phi_{l+1}^{\alpha, \rho}(x)\right) d x \\
& =\frac{\pi \sqrt{2}}{2 \sqrt{\rho}} \int_{\alpha}^{\alpha+\rho} Z(x) \phi_{1}^{\alpha, \rho}(x) d x \tag{7}
\end{align*}
$$

almost everywhere on $\mathcal{N}^{\prime}$, where the last equality follows by standard trigonometric identities.

Let us rewrite (7) as

$$
\begin{equation*}
Z(\beta)+Z(\alpha)=\frac{\pi \sqrt{2}}{2 \sqrt{\beta-\alpha}} \int_{\alpha}^{\beta} Z(x) \phi_{1}^{\alpha, \beta-\alpha}(x) d x \tag{8}
\end{equation*}
$$

Since the right-hand side of (8) is $\mathcal{C}^{1}$ with respect to $(\alpha, \beta)$ on $\left\{(\alpha, \beta) \in \Omega^{2} \mid\right.$ $\alpha<\beta\}$, we deduce that $Z$ is $\mathcal{C}^{1}$ on the open set $\mathcal{N}_{1}^{\prime} \cup \mathcal{N}_{2}^{\prime}$, where

$$
\mathcal{N}_{1}^{\prime}=\left\{\alpha \mid(\alpha, \rho) \in \mathcal{N}^{\prime} \text { for some } \rho>0\right\}, \quad \mathcal{N}_{2}^{\prime}=\left\{\alpha+\rho \mid(\alpha, \rho) \in \mathcal{N}^{\prime}\right\}
$$

If there exist $x \in \mathcal{N}_{1}^{\prime} \cup \mathcal{N}_{2}^{\prime}$ such that $\frac{d}{d x} Z(x) \neq 0$, then the conclusion follows from Lemma 4.1. (The fact that the reordering $h$ in the statement of Lemma 4.1 can be taken equal to the identity in the case $d=1$ follows directly from the proof given in [22].) Otherwise, $\frac{d}{d x} Z \equiv 0$ on $\mathcal{N}_{1}^{\prime} \cup \mathcal{N}_{2}^{\prime}$.

Differentiating (7) with respect to $\alpha$, we get that

$$
\int_{\alpha}^{\alpha+\rho} Z(x) \psi_{1}^{\alpha, \rho}(x) d x \equiv 0
$$

for every $(\alpha, \rho) \in \mathcal{N}^{\prime}$, contradicting (5).
We are left to prove that either $Z$ is absolutely continuous on $\Omega$ (and hence Lemma 4.1 applies) or there exists $(a, r) \in \Omega \times(0, \infty)$ such that $(a, a+r)$ is compactly contained in $\Omega$ and (4) holds true.

By contradiction, assume that for every $(a, r) \in \Omega \times(0, \infty)$ such that $(a, a+r)$ is compactly contained in $\Omega$ we have $\int_{a}^{a+r} Z(x) \psi_{1}^{a, r}(x) d x=0$. By
differentiating with respect to $a$, we get that for every $r>0$ and almost every $a$ such that $(a, a+r) \subset \Omega$,

$$
Z(a+r)+Z(a)=\frac{\pi}{r} \int_{a}^{a+r} Z(x) \phi_{1}^{a, r}(x) d x
$$

Reasoning as above, we deduce that $Z$ is $\mathcal{C}^{1}$ on $\Omega$. In particular, $Z$ is absolutely continuous and the proof of the proposition in concluded.

## 5. The genericity argument by analytic perturbation

Before proving Theorem 3.6 by considering separately the cases where $V$ or $W$ is fixed, let us recall some useful result from the literature.

The first is a technical result allowing to obtain the spectral decomposition of a Laplace-Dirichlet operator on a bounded domain $\omega$ as the limit for operators defined on larger spacial domains, whose potential converge uniformly to infinity outside $\omega$.

Lemma 5.1 ([22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$ and $\omega$ be a nonempty, open subset compactly contained in $\Omega$ and whose boundary is Lipschitz continuous. Let $v \in L^{\infty}(\omega, \mathbf{R})$ and $\left(V_{k}\right)_{k \in \mathbf{N}}$ be a sequence in $\mathcal{V}(\Omega)$ such that $\left.V_{k}\right|_{\omega} \rightarrow v$ in $L^{\infty}(\omega, \mathbf{R})$ as $k \rightarrow \infty$ and $\lim _{k \rightarrow \infty} \operatorname{ess}_{\inf }^{\Omega \backslash \omega} V_{k}=+\infty$. Then, for every $j \in$ $\mathbf{N}$, $\lim _{k \rightarrow \infty} \lambda_{j}\left(\Omega, V_{k}\right)=\lambda_{j}(\omega, v)$. Moreover, if $\lambda_{j}(\omega, v)$ is simple then (up to the sign) $\phi_{j}\left(\Omega, V_{k}\right)$ and $\sqrt{\left|V_{k}\right|} \phi_{j}\left(\Omega, V_{k}\right)$ converge respectively to $\phi_{j}(\omega, v)$ and $\sqrt{|v|} \phi_{j}(\omega, v)$ in $L^{2}(\Omega, \mathbf{C})$ as $k$ goes to infinity, where $\phi_{j}(\omega, v)$ is identified with its extension by zero outside $\omega$.

The second result states that the $\mathbf{Q}$-linear independence of the spectrum of $-\Delta+V$ is a generic property with respect to $V$. It generalises a classical result on the generic simplicity of the spectrum of $-\Delta+V$ obtained by Albert in [4]. It implies in particular that the spectral gaps $\lambda_{j+1}(\Omega, V)-\lambda_{j}(\Omega, V)$ of $-\Delta+V$ form generically a $\mathbf{Q}$-linear independent family, as required in the hypotheses of Theorem 3.1.

Proposition 5.2 ([4] and [22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$. For every $K \in \mathbf{N}$ and $q=\left(q_{1}, \ldots, q_{K}\right) \in \mathbf{Q}^{K} \backslash\{0\}$, the set

$$
\begin{equation*}
\mathcal{O}_{q}(\Omega)=\left\{V \in \mathcal{V}(\Omega) \mid \lambda_{1}(\Omega, V), \ldots, \lambda_{K}(\Omega, V) \text { simple, } \sum_{j=1}^{K} q_{j} \lambda_{j}(\Omega, V) \neq 0\right\} \tag{9}
\end{equation*}
$$

is open and dense in $\mathcal{V}(\Omega)$.
The third result, based on the contributions in [32], states the existence of analytic paths of potentials such that the spectrum is simple along them.

Proposition 5.3 ([32] and [22]). Let $\Omega$ belong to $\Xi_{d}^{\infty}$ and $V, Z \in \mathcal{V}(\Omega)$ be such that $Z-V \in L^{\infty}(\Omega, \mathbf{R})$. Then there exists an analytic function $\mu \mapsto W_{\mu}$ from $[0,1]$ into $L^{\infty}(\Omega, \mathbf{R})$ such that $W_{0}=0, W_{1}=Z-V$ and the spectrum of $-\Delta+V+W_{\mu}$ is simple for every $\mu \in(0,1)$.

### 5.1. Proof of Theorem 3.6 in the case where $W$ is fixed

Let $\Omega \in \Xi_{1}^{\infty}$ and fix $W \in \mathcal{W}(\Omega)$. Let us consider the following subspace of $\mathcal{V}(\Omega)$

$$
\hat{\mathcal{V}}(\Omega, W)=\left\{V \in \mathcal{V}(\Omega) \left\lvert\, \underset{x \in \Omega}{\operatorname{esssup}} \frac{|W(x)|}{|V(x)|+1}<+\infty\right.\right\} .
$$

Notice that $\hat{\mathcal{V}}(\Omega, W)$ is open in $\mathcal{V}(\Omega, W)$.
Proposition 5.4. Let $\Omega$ belong to $\Xi_{1}^{\infty}$ and $W \in \mathcal{W}(\Omega)$ be non-constant. Then, generically with respect to $V$ in $\hat{\mathcal{V}}(\Omega, W)$, the triple $(\Omega, V, W)$ is fit for control.

Proof. By applying Proposition 4.2 to $Z=W$, we deduce that there exists a nonempty interval $\omega$ compactly contained in $\Omega$ such that

$$
\begin{equation*}
\int_{\omega} W(x) \phi_{l}(\omega, 0)(x) \phi_{l+1}(\omega, 0)(x) d x \neq 0 \tag{10}
\end{equation*}
$$

for every $l \in \mathbf{N}$.
Denote by $\mathcal{Q}_{n}(\Omega, W)$ the set of potentials $V \in \hat{\mathcal{V}}(\Omega, W)$ such that for every $j \in\{1, \ldots, n\}$ the eigenvalue $\lambda_{j}(\Omega, V)$ is simple and

$$
\int_{\Omega} W(x) \phi_{j}(\Omega, V)(x) \phi_{j+1}(\Omega, V)(x) d x \neq 0 \quad \text { for } j=1, \ldots, n-1
$$

In the case where $\Omega$ is bounded the openness of $\mathcal{Q}_{n}(\Omega, W)$ follows from classical results on the continuity of eigenvalues and eigenfunctions (see, e.g., [20]). For the unbounded case, one should use the fact that each eigenfunction $\phi_{r}(\Omega, V)$ goes to zero at infinity faster than any exponential. Since $W$ has at most exponential growth, one then deduces that $V \mapsto \sqrt{|W|} \phi_{r}(\Omega, V)$ is continuous, as a function from the open subset of $\mathcal{V}(\Omega)$ of potentials for which the $r$-th eigenvalue of $-\Delta+V$ is simple into $L^{2}(\Omega, \mathbf{C})$ (for details, see [22, Proposition 2.9]).

Let us now prove the density of $\mathcal{Q}_{n}(\Omega, W)$. Fix $\bar{V} \in \hat{\mathcal{V}}(\Omega, W)$. We should prove that $\bar{V}$ is in the closure of $\mathcal{Q}_{n}(\Omega, W)$.

Let $\left(V_{k}\right)_{k \in \mathbf{N}}$ be the sequence in $\mathcal{V}(\Omega)$ defined by $V_{k}=0$ in $\omega$ and $V_{k}=\bar{V}+k$ in $\Omega \backslash \omega$. Then, for every $j \in \mathbf{N}$, the sequence $\left\|\sqrt{\left|V_{k}\right|} \phi_{j}\left(\Omega, V_{k}\right)\right\|_{L^{2}(\Omega \backslash \omega, \mathbf{C})}$ converges to 0 as $k$ goes to infinity (Lemma 5.1). By definition of $\hat{\mathcal{V}}(\Omega, W)$, $|W|<C(|\bar{V}|+1)$ on $\Omega$ for some $C>0$. Hence, for every $j \in \mathbf{N}$, also the sequence $\left\|\sqrt{|W|} \phi_{j}\left(\Omega, V_{k}\right)\right\|_{L^{2}(\Omega \backslash \omega, \mathbf{C})}$ converges to 0 as $k$ goes to infinity.

Moreover, Lemma 5.1 also implies that, for every $j \in \mathbf{N}$, the sequences $\lambda_{j}\left(\Omega, V_{k}\right)$ and $\phi_{j}\left(\Omega, V_{k}\right)$ converge, respectively, to $\lambda_{j}(\omega, 0)$ and $\phi_{j}(\Omega, 0)$ (up to sign) as $k$ goes to infinity. In particular, $\lambda_{1}\left(\Omega, V_{k}\right), \ldots, \lambda_{n}\left(\Omega, V_{k}\right)$ are simple for $k$ large enough and equation (10) allows to conclude that $V_{k} \in \mathcal{Q}_{n}(\Omega, W)$ for $k$ large enough.

Fix $\bar{k}$ such that $V_{\bar{k}} \in \mathcal{Q}_{n}(\Omega, W)$. It follows from Proposition 5.3 that there exists an analytic function $\mu \mapsto W_{\mu}$ from $[0,1]$ into $L^{\infty}(\Omega, \mathbf{R})$ such that $W_{0}=0$, $W_{1}=V_{\bar{k}}-\bar{V}$ and the spectrum of $-\Delta+\bar{V}+W_{\mu}$ is simple for every $\mu \in(0,1)$.

The conclusion of the proof is based on the analytic dependence of the eigenpairs of $-\Delta+\bar{V}+W_{\mu}$ with respect to $\mu$. The analytic dependence of a finite set of eigenpairs in a neighbourhood of a given $\mu$ is a consequence of the classical Kato-Rellich theorem (see [20, Chapter VII]). The global analyticity on the interval $(0,1)$ of the entire (infinite) family of eigenpairs of $-\Delta+\bar{V}+W_{\mu}$ can be deduced from the analyticity of $(0,1) \ni \mu \mapsto W_{\mu}$ in $L^{\infty}(\Omega)$, which prevents an analytic branch of the spectrum of $-\Delta+\bar{V}+W_{\mu}$ to go to infinity as $\mu$ tends to some $\mu_{0} \in[0,1]$. Indeed, since each $\lambda_{j}\left(\Omega, \bar{V}+W_{\mu}\right), j \in \mathbf{N}$, $\mu \in(0,1)$, is an isolated eigenvalue, one can compute by classical formulas the derivative of $\lambda_{j}\left(\Omega, \bar{V}+W_{\mu}\right)$ with respect to $\mu$ and get

$$
\left|\frac{d}{d \mu} \lambda_{j}\left(\Omega, \bar{V}+W_{\mu}\right)\right|=\left|\int_{\Omega} \phi_{j}^{2}\left(\Omega, \bar{V}+W_{\mu}\right)(x) \frac{d}{d \mu} W_{\mu}(x) d x\right| \leq\left\|\frac{d}{d \mu} W_{\mu}\right\|_{\infty}
$$

(see, for instance, [4]).
We are going to use a stronger analytic dependence property, namely, that each function $\mu \mapsto \phi_{j}\left(\Omega, \bar{V}+W_{\mu}\right)$ is analytic from $(0,1)$ to the domain $D(-\Delta+\bar{V})$ endowed with the graph norm (see [30, Theorem 5.6] and also [22, Proposition 2.11]).

Recalling that, by definition of $\hat{\mathcal{V}}(\Omega, W)$ and by boundedness of $W_{\mu},|W|<$ $C\left(\left|\bar{V}+W_{\mu}\right|+1\right)$ on $\Omega$ for some $C>0$, we deduce that

$$
\mu \mapsto \int_{\Omega} W(x) \phi_{j}\left(\Omega, \bar{V}+W_{\mu}\right)(x) \phi_{j+1}\left(\Omega, \bar{V}+W_{\mu}\right)(x) d x
$$

is analytic from $(0,1)$ to $\mathbf{R}$, for every $j \in \mathbf{N}$.
Since, moreover, $V_{\bar{k}}=\bar{V}+W_{1} \in \mathcal{Q}_{n}(\Omega, W)$, we get that $\bar{V}+W_{\mu} \in \mathcal{Q}_{n}(\Omega, W)$ for almost every $\mu \in(0,1)$. Hence $\bar{V}=\bar{V}+W_{0}$ is in the closure $\mathcal{Q}_{n}(\Omega, W)$. We proved that $\mathcal{Q}_{n}(\Omega, W)$ is dense in $\hat{\mathcal{V}}(\Omega, W)$.

The set $\cap_{n \in \mathbf{N}} \mathcal{Q}_{n}(\Omega, W)$ is then residual in $\hat{\mathcal{V}}(\Omega, W)$ and, for every $n \in \mathbf{N}$, the matrix $B_{n}^{\text {id }_{\mathrm{N}}}(\Omega, V, W)$ is connected.

The triple $(\Omega, V, W)$ is then fit for control if $V$ belongs to

$$
\left(\cap_{n \in \mathbf{N}} \mathcal{Q}_{n}(\Omega, W)\right) \cap\left(\cap_{q \in \cup_{K \in \mathbf{N}} \mathbf{Q}^{K} \backslash\{0\}} \mathcal{O}_{q}(\Omega)\right)
$$

where the sets $\mathcal{O}_{q}(\Omega)$ are those introduced in Proposition 5.2 , which is the intersection of countably many open and dense subsets of $\hat{\mathcal{V}}(\Omega, W)$.

The next corollary follows immediately from Proposition 5.4.
Corollary 5.5. Let $\Omega \in \Xi_{1}$ and $W \in L^{\infty}(\Omega, \mathbf{R})$ be non-constant. Then, generically with respect to $V$ in $L^{\infty}(\Omega, \mathbf{R})$, the triple $(\Omega, V, W)$ is fit for control.

In the unbounded case we deduce the following.
Corollary 5.6. Let $\Omega=\mathbf{R}$ and $W \in \mathcal{W}(\mathbf{R})$ be non-constant. Then, generically with respect to $V$ in $\mathcal{V}(\mathbf{R}, W, U)$, the quadruple $(\mathbf{R}, V, W, U)$ is effective.

Proof. Fix $u \in(0, \delta)$. Let $\eta>0$ be such that $[u-\eta, u+\eta] \subset U$. If $V \in$ $\mathcal{V}(\mathbf{R}, W, U)$ then both $V+(u-\eta) W$ and $V+(u+\eta) W$ are positive outside some bounded subset $\Omega_{0}$ of $\mathbf{R}$. In particular $|W| \leq \frac{1}{2 \eta}|V+u W|$ outside $\Omega_{0}$. Since, moreover, $W$ is bounded on $\Omega_{0}$, then $V+u W$ is in $\hat{\mathcal{V}}(\mathbf{R}, W)$.

Since $u W+\mathcal{V}(\mathbf{R}, W, U)$ is an open subset of $\hat{\mathcal{V}}(\mathbf{R}, W)$, we deduce from Proposition 5.4 that there exists a residual subset $\mathcal{R}$ of $u W+\mathcal{V}(\mathbf{R}, W, U)$ such that $(\Omega, \widetilde{V}, W)$ is fit for control for every $\widetilde{V} \in \mathcal{R}$. This means that the triple $(\mathbf{R}, V+u W, W)$ is fit for control, generically with respect to $V \in \mathcal{V}(\mathbf{R}, W, U)$. In particular, the quadruple $(\mathbf{R}, V, W, U)$ is effective, generically with respect to $V$ in $\mathcal{V}(\mathbf{R}, W, U)$.

### 5.2. Proof of Theorem 3.6 in the case where $V$ is fixed

We prove in this section that for a fixed potential $V$, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple $(\Omega, V, W, U)$ is effective. Notice that $(\Omega, V, W)$ cannot be fit for control if the spectrum of $-\Delta+V$ is resonant, independently of $W$. In this regard the result is necessarily weaker than Proposition 5.4 and Corollary 5.5 , where the genericity of the fitness for control is proved.

Proposition 5.7. Let $\Omega$ belong to $\Xi_{1}^{\infty}$ and $V \in \mathcal{V}(\Omega)$. Then, generically with respect to $W \in \mathcal{W}(\Omega, V, U)$, the quadruple $(\Omega, V, W, U)$ is effective.

Proof. Fix $u \in(0, \delta)$. Notice that $V+u \mathcal{W}(\Omega, V, U)$ is an open subset of $\mathcal{V}(\Omega)$ and that the map $W \mapsto V+u W$ is a homeomorphism between $\mathcal{W}(\Omega, V, U)$ and $V+u \mathcal{W}(\Omega, V, U)$. In particular, due to Proposition 5.2 , for every $K \in \mathbf{N}$ and $q \in \mathbf{Q}^{K} \backslash\{0\}$, the set $\left\{W \in \mathcal{W}(\Omega, V, U) \mid V+u W \in \mathcal{O}_{q}(\Omega)\right\}$ is open and dense in $\mathcal{W}(\Omega, V, U)$.

For every $W \in \mathcal{W}(\Omega, V, U)$ let, as in the previous section, $\mathcal{Q}_{n}(\Omega, W)$ be the open and dense subset of $\hat{\mathcal{V}}(\Omega, W)$ made of all potentials $\widetilde{V} \in \hat{\mathcal{V}}(\Omega, W)$ such that for every $j \in\{1, \ldots, n\}$ the eigenvalue $\lambda_{j}(\Omega, \widetilde{V})$ is simple and

$$
\int_{\Omega} W(x) \phi_{j}(\Omega, \tilde{V})(x) \phi_{j+1}(\Omega, \tilde{V})(x) d x \neq 0 \quad \text { for } j=1, \ldots, n-1
$$

As proved in Corollary 5.6, for every $W \in \mathcal{W}(\Omega, V, U)$ one has $V+u W \in$ $\hat{\mathcal{V}}(\Omega, W)$. We are going to prove the proposition by showing that for every

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$n \in \mathbf{N}$, for each $W$ in a open and dense subset of $\mathcal{W}(\Omega, V, U), V+u W$ belongs to $\mathcal{Q}_{n}(\Omega, W)$.

Define

$$
\mathcal{P}_{n}=\left\{W \in \mathcal{W}(\Omega, V, U) \mid V+u W \in \mathcal{Q}_{n}(\Omega, W)\right\}
$$

Since

$$
W \mapsto \int_{\Omega} W(x) \phi_{j}(\Omega, V+u W)(x) \phi_{k}(\Omega, V+u W)(x) d x
$$

is continuous on $\left\{W \in \mathcal{W}(\Omega, V, U) \mid \lambda_{j}(\Omega, V+u W), \lambda_{k}(\Omega, V+u W)\right.$ are simple $\}$ for every $j, k \in \mathbf{N}$ (see [22, Proposition 2.9] for details), we deduce that $\mathcal{P}_{n}$ is open.

Fix $\widetilde{W} \in \mathcal{W}(\Omega, V, U)$. We are left to prove that $\widetilde{W}$ belongs to the closure of $\mathcal{P}_{n}$.

Consider first the case in which $V$ is constant. In particular, $\Omega \in \Xi_{1}$, $\mathcal{W}(\Omega, V, U)=V+u \mathcal{W}(\Omega, V, U)=L^{\infty}(\Omega, \mathbf{R})$, and

$$
\begin{aligned}
\int_{\Omega} W(x) \phi_{j}(\Omega, V+u W)(x) & \phi_{k}(\Omega, V+u W)(x) d x \\
& =\int_{\Omega} W(x) \phi_{j}(\Omega, u W)(x) \phi_{k}(\Omega, u W)(x) d x
\end{aligned}
$$

for every $j, k \in \mathbf{N}$ and $W \in L^{\infty}(\Omega, \mathbf{R})$. Fix an interval $\omega$ compactly contained in $\Omega$. In particular, the spectrum $\sigma(\omega, 0)$ is simple.

Let $z \in L^{\infty}(\omega, \mathbf{R})$ be such that

$$
\int_{\omega} z(x) \phi_{j}(\omega, 0)(x) \phi_{k}(\omega, 0)(x) d x \neq 0
$$

for every $j, k \in \mathbf{N}$. Then, for every $j, k \in \mathbf{N}$, the derivative of

$$
\varepsilon \mapsto \int_{\omega} \varepsilon z(x) \phi_{j}(\omega, \varepsilon z)(x) \phi_{k}(\omega, \varepsilon z)(x) d x
$$

at $\varepsilon=0$ is equal to

$$
\int_{\omega} z(x) \phi_{j}(\omega, 0)(x) \phi_{k}(\omega, 0)(x) d x \neq 0
$$

By analyticity, there exists $\tilde{\varepsilon} \in \mathbf{R}$ such that the spectrum $\sigma(\omega, \tilde{\varepsilon} z)$ is simple and

$$
\int_{\omega} \tilde{\varepsilon} z(x) \phi_{j}(\omega, \tilde{\varepsilon} z)(x) \phi_{k}(\omega, \tilde{\varepsilon} z)(x) d x \neq 0
$$

for every $j, k \in \mathbf{N}$. Set $\tilde{z}=\tilde{\varepsilon} z$.

Let $\left(W_{l}\right)_{l \in \mathbf{N}}$ be a sequence in $L^{\infty}(\Omega, \mathbf{R})$ such that $\left.\lim _{l \rightarrow \infty} W_{l}\right|_{\omega}=\tilde{z} / u$ in
 there exists $\bar{l}$ large enough such that

$$
\int_{\Omega} W_{\bar{l}}(x) \phi_{j}\left(\Omega, u W_{\bar{l}}\right)(x) \phi_{k}\left(\Omega, u W_{\bar{l}}\right)(x) d x \neq 0 \quad \text { for } j, k=1, \ldots, n
$$

By Proposition 5.3 we can consider an analytic curve $\mu \mapsto \hat{W}_{\mu}$ in $L^{\infty}(\Omega, \mathbf{R})$ such that $\hat{W}_{0}=\widetilde{W}, \hat{W}_{1}=W_{\bar{l}}$ and the spectrum of $-\Delta+u \hat{W}_{\mu}$ is simple for every $\mu \in(0,1)$. Since $V$ is constant and by analytic dependence with respect to $\mu$, we have

$$
\begin{aligned}
\int_{\Omega} \hat{W}_{\mu}(x) \phi_{j}\left(\Omega, V+u \hat{W}_{\mu}\right) & (x) \phi_{k}\left(\Omega, V+u \hat{W}_{\mu}\right)(x) d x \\
& =\int_{\Omega} \hat{W}_{\mu}(x) \phi_{j}\left(\Omega, u \hat{W}_{\mu}\right)(x) \phi_{k}\left(\Omega, u \hat{W}_{\mu}\right)(x) d x \neq 0
\end{aligned}
$$

for almost every $\mu \in(0,1)$ and in particular for some $\mu$ arbitrarily small, implying that $\widetilde{W}$ belongs to the closure of $\mathcal{P}_{n}$.

Let now $V$ be non-constant. Let $\omega \subset \Omega$ be as in the statement of Proposition 4.2, with $V$ playing the role of $Z$.

Take a sequence $\left(W_{k}\right)_{k \in \mathbf{N}}$ in $\mathcal{W}(\Omega, V, U)$ such that $W_{k}-\widetilde{W}$ belongs to $L^{\infty}(\Omega, \mathbf{R})$ for every $k$ and

$$
\lim _{k \rightarrow+\infty}\left\|V+u W_{k}\right\|_{L^{\infty}(\omega, \mathbf{R})}=0, \quad \lim _{k \rightarrow+\infty} \underset{\Omega \backslash \omega}{\operatorname{ess} \inf }\left(u W_{k}\right)=+\infty
$$

According to Lemma 5.1,

$$
\lim _{k \rightarrow+\infty} \phi_{m}\left(\Omega, V+u W_{k}\right)=\phi_{m}(\omega, 0), \lim _{k \rightarrow+\infty} \sqrt{V+u W_{k}} \phi_{m}\left(\Omega, V+u W_{k}\right)=0
$$

in $L^{2}(\Omega, \mathbf{C})$ for every $m \in \mathbf{N}$, where $\phi_{m}(\omega, 0)$ is identified with its extension by zero on $\Omega \backslash \omega$. In particular, we have that $\sqrt{V} \phi_{m}\left(\Omega, V+u W_{k}\right)$ converges in $L^{2}(\Omega, \mathbf{C})$ as $k$ tends to infinity to the extension by zero of $\sqrt{V} \phi_{m}(\omega, 0)$ on $\Omega \backslash \omega$. Hence,

$$
\begin{aligned}
& \lim _{k \rightarrow+\infty} \int_{\Omega} W_{k}(x) \phi_{l}\left(\Omega, V+u W_{k}\right)(x) \phi_{l+1}\left(\Omega, V+u W_{k}\right)(x) d x \\
&=-\frac{1}{u} \int_{\omega} V(x) \phi_{l}(\omega, 0)(x) \phi_{l+1}(\omega, 0)(x) d x \neq 0
\end{aligned}
$$

for every $l \in \mathbf{N}$. For a fixed $n \in \mathbf{N}$, we can choose $\bar{k}$ large enough so that

$$
\int_{\Omega} W_{\bar{k}}(x) \phi_{l}\left(\Omega, V+u W_{\bar{k}}\right)(x) \phi_{l+1}\left(\Omega, V+u W_{\bar{k}}\right)(x) d x \neq 0
$$

for $l=1, \ldots, n-1$, in order to guarantee that $W_{\bar{k}} \in \mathcal{P}_{n}$.
Using again Proposition 5.3, we deduce that there exists an analytic path $\mu \mapsto \hat{W}_{\mu}$ from $[0,1]$ into $L^{\infty}(\Omega, \mathbf{R})$ such that $\hat{W}_{0}=0, \hat{W}_{1}=W_{\bar{k}}-\widetilde{W}$ and the spectrum of $-\Delta+V+u \widetilde{W}+u \hat{W}_{\mu}$ is simple for every $\mu \in(0,1)$. Therefore, by analyticity, we get that

$$
\int_{\Omega}\left(\widetilde{W}(x)+\hat{W}_{\mu}(x)\right) \phi_{l}\left(\Omega, V+u \widetilde{W}+u \hat{W}_{\mu}\right)(x) \phi_{l+1}\left(\Omega, V+u \widetilde{W}+u \hat{W}_{\mu}\right)(x) d x \neq 0
$$

for almost every $\mu \in(0,1)$. Hence, $\widetilde{W}$ belongs to the closure of $\mathcal{P}_{n}$.

## 6. Conclusion

In this paper we proved that once $(\Omega, V)$ or $(\Omega, W)$ is fixed (with $\Omega$ a onedimensional domain and $W$ non-constant), the bilinear Schrödinger equation on $\Omega$ having $V$ as uncontrolled and $W$ as controlled potential is generically approximately controllable with respect to the other element of the triple $(\Omega, V, W)$. This improves the results in [22], where a technical regularity assumption was imposed on the potentials. It remains to prove that the regularity assumption can be dropped also when $\Omega$ has dimension larger than one.

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# Nondestructive evaluation of inaccessible surface damages by means of active thermography 

Gabriele Inglese and Roberto Olmi

Dedicated to Giovanni Alessandrini for his 60th birthday


#### Abstract

We derive and test a formal explicit approximated rule for the reconstruction of a damaged inaccessible portion of the boundary of a thin conductor from thermal data collected on the opposite accessible face.


Keywords: Inverse problems, heat equation, nondestructive evaluation, thin plate approximation.
MS Classification 2010: 65M32, 80A20.

## 1. Introduction

Let $\Omega=\left\{(x, y, z) \in[-L, L]^{2} \times[0, a] \quad a<1 \ll L\right\}$ represent a uniform thin plate of given thermal conductivity $\kappa$. We are modeling the following experimental framework:

The half plane $z>a$ is a forbidden aggressive environment, while $z<0$ is an accessible laboratory. We are able to heat the specimen $\Omega$ from below by means of a controlled flux of density $\Phi_{0}$ generated by lamps or a laser device and we are able to get temperature maps at $z=0$ by means of an infrared camera (TMC in Figure 1).

Small corrosion damages due to chemical or mechanical aggression may appear on the upper inaccessible boundary of $\Omega$. Since they are not accessible to direct inspection, they must be identified through operations carried out on the laboratory side. If the defect consists of a loss of matter (LOM), the damaged domain is modeled by

$$
\Omega_{\epsilon \theta}=\left\{(x, y, z):(x, y) \in[-L, L]^{2}, \quad 0 \leq z \leq a-\epsilon \theta(x, y)\right\}
$$

We assume that the geometry of the damage is described by a continuous function $(x, y) \rightarrow \epsilon \theta(x, y)$. Here, $\epsilon \ll a$ is a constant dimensional scale factor while $\theta(x, y) \in[0,1]$ is dimensionless.


Figure 1: Sketch of the experimental setup

The temperature of the damaged domain solves the following Initial Boundary Value Problem in $D_{T}=\Omega_{\epsilon \theta} \times(0, T]$ :

$$
\begin{gather*}
u_{t}=\alpha \Delta u,  \tag{1}\\
\kappa u_{n}(x, y, a-\epsilon \theta(x, y), t)+a h\left(u(x, y, a-\epsilon \theta(x, y), t)-U_{0}\right)=0,  \tag{2}\\
-\kappa u_{z}(x, y, 0, t)=\Phi,  \tag{3}\\
u_{x}(-L, y, z, t)=u_{x}(L, y, z, t)=0, \quad y \in[-L, L], z \in[0, a], t \in(0, T],  \tag{4}\\
u_{y}(x,-L, z, t)=u_{y}(x, L, z, t)=0, \quad x \in[-L, L], z \in[0, a], t \in(0, T] \tag{5}
\end{gather*}
$$

and

$$
\begin{equation*}
u(x, y, z, 0)=U_{0} \tag{6}
\end{equation*}
$$

for all $(x, y, z) \in \Omega_{\epsilon \theta}$ ( $U_{0}$ is a positive constant). Here, $\alpha$ is the thermal diffusivity, $a h$ is the heat transfer coefficient between the specimen and the upper half-space (see for example [5] and [14]). The positive constant $U_{0}$ is both the initial temperature of the specimen and the temperature of the outern environment. The heat flux density $\Phi$ is taken constant in space and time for simplicity. In what follows we will refer to (4) and (5) as to "adiabatic conditions on the vertical sides".

Direct model. If $\epsilon \theta$ is given and it is sufficiently smooth, the IBVP (1)-(6) is well posed and it has a unique classical solution $u^{\epsilon}$ [12]. This notation stresses the dependence of the solution on the damage. Hence, the solution $u^{0}$ (corresponding to $\epsilon=0$ ) is called the background temperature of the undamaged specimen.

Inverse Problem. If $\epsilon \theta$ is not known, our goal is to identify it from the knowledge of the thermal contrast $G(x, y, t)=u^{\epsilon}(x, y, 0, t)-u^{0}(x, y, 0, t)$ measured
from the laboratory side $z=0$.
This method is called Active Thermography. Thermography is "Active" when an external heat source (in our case, the heat flux $\Phi$ ) stimulates the specimen for inspection.

Bibliographic remark. See [10] for a complete reference book about thermography. Amongst hundreds of research articles about thermal imaging, we mention [3] because, in our knowledge, it is one of the oldest and [4] because of the close relationship with the present paper. Since the mathematics of stationary thermography is the same used in a class of electrostatic models in nondestructive evaluation, we cite also $[1,7,8]$ and references therein.

The idea of loss of matter used in (1)-(6) is very intuitive because LOM is something real and, possibly, measurable in practice.

However, thermal effects of damages on the inaccessible surface can be modeled also by means of perturbed boundary conditions. In this case, the boundary is left unaltered so that the domain (and consequently the mesh in numerical solution with finite elements!) is not dependent on the unknown $\epsilon \theta$.

Here, we assume that $\frac{\epsilon}{a}$ is small enough to use the idea of Domain Derivative ( $[4,13]$ ). The domain derivative of $u^{\epsilon}$ can be obtained by formal differentiation as $u^{\prime}=\frac{d u^{\epsilon}}{d \epsilon}(\epsilon=0)$ or derived by means of straightforward calculations as done in [4]. The LOM model (1)-(6) in $\Omega_{\epsilon \theta}$ is turned into an Initial Boundary Value Problem in the undamaged domain $\Omega$ for the scaled domain derivative $w=\epsilon u^{\prime}$. It is remarkable that the unknown damage $\epsilon \theta$ appears now in the top boundary condition.

Furthermore, in subsection 2.1, we rescale $z$ and transform it in the new variable $\zeta=\frac{z}{a}$. Since the temperature of the specimen reaches a stationary regime for $t \rightarrow \infty$, after a time interval $T_{\alpha}$ (inversely proportional to the diffusivity $\alpha$ ) we focus our attention on the following stationary BVP on the parallelepiped $[-L, L]^{2} \times[0,1]$ (see section 3 ):

$$
\begin{gather*}
a^{2}\left(w_{x x}+w_{y y}\right)+w_{\zeta, \zeta}=0  \tag{7}\\
\kappa w_{\zeta}(x, y, 1)+a^{2} h w(x, y, 1)=-a^{2} \epsilon \theta(x, y) h \frac{\Phi}{k},  \tag{8}\\
\kappa w_{\zeta}(x, y, 0)=0 \tag{9}
\end{gather*}
$$

with adiabatic conditions on the vertical sides.

We expand $w$ and $\theta$ in powers of $a^{2}$ and plug them into the BVP above: In this way we obtain a perturbative hierarchy of relations amongst their coefficients. This procedure is called Thin Plate Approximation and improves what was done in [9] where perturbations of the heat transfer coefficient $h$ were identified.

In section 3, we derive the TPA formally in any order and implement the following approximated inversion formula for the identification of the damage:

$$
\begin{equation*}
\epsilon \theta(x, y) \approx \frac{\kappa^{2}}{h \Phi}\left(G_{x x}+G_{y y}\right)-\frac{\kappa}{\Phi} G(x, y) \tag{10}
\end{equation*}
$$

We succesfully tested this formula using synthetic data. Any difficulties arising from numerical differentiation of an approximately given function like the thermal contrast $G$ are handled by using local weighted regeression [6]. A seminal paper about regularized numerical differentiation is [2].

## 2. Domain derivative

Domain derivative, introduced in [13], is a techinque for studying PDEs on geometrically perturbed domains. In our case the domain derivative of $u^{\epsilon}$ is the Gateaux derivative of $u^{\epsilon}$ in the direction $\theta$ taken for $\epsilon=0$. This derivative is a function $u^{\prime}$ that satisfies the heat equation in $\Omega$ with boundary conditions

$$
\kappa u_{z}^{\prime}(x, y, a, t)+a h u^{\prime}(x, y, a, t)=\theta(x, y)\left(a h u_{z}^{0}(x, y, a, t)+\kappa \frac{u_{t}^{0}(x, y, a, t)}{\alpha}\right)
$$

(derived in [4] in agreement with Theorem 3.2 in [13]),

$$
\kappa u_{z}^{\prime}(x, y, 0, t)=0
$$

and "adiabatic conditions on the vertical sides".
Since we assume $\Phi$ constant (in $t$ and $(x, y)$ ), the background solution is constant in space variables and, for increasing $t$, it approaches a stationary value that, after a suitable time interval $T_{\alpha}$, is very close to the linear function $u_{\text {stat }}^{0}(z)=U_{0}+\frac{\Phi}{h}+\frac{\Phi}{\kappa}(a-z)$ (stationary background temperature).

### 2.1. Final form of the BVP: domain derivative and scaling

Since we have to recover $\epsilon \theta$ from the thermal contrast $G(x, y, t)=u^{\epsilon}(x, y, 0, t)-$ $u^{0}(x, y, 0, t) \approx \epsilon u^{\prime}(x, 0, t)$, it is convenient to introduce the scaled function $w=\epsilon u^{\prime}$. Moreover, Thin Plate Approximation (see for example [9]) requires the expansion of $w$ in powers of $a^{2}$. For this reason, we scale the variable
$z \rightarrow \zeta=\frac{z}{a}$ so that the domain becomes independent of $a$. Since $w_{\zeta}=\frac{w_{z}}{a}$ we have finally

$$
\begin{gathered}
\frac{a^{2}}{\alpha} w_{t}=a^{2}\left(w_{x x}+w_{y y}\right)+w_{\zeta \zeta} \\
\kappa w_{\zeta}(x, y, 1, t)+a^{2} h w(x, y, 1, t)=\epsilon \theta(x, y)\left(a^{2} h u_{z}^{0}+a \kappa \frac{u_{t}^{0}(x, y, a, t)}{\alpha}\right), \\
\kappa w_{\zeta}(x, y, 0, t)=0
\end{gathered}
$$

with adiabatic conditions on the vertical sides. Moreover, we have

$$
w(x, y, 0, t) \approx G(x, y, t)
$$

## 3. Stationary model, Thin Plate Approximation of the domain derivative

Here, we focus our attention to the stationary heat equation. In what follows, we remove the time variable but, as a rule, we keep the same function names. The stationary heat equation describes well the behavior of the temperature in our model for $t>T_{\alpha}$. Hence, we introduce a new function $w$ that does not depend on $t$ and solve the elliptic BVP in $[-L, L]^{2} \times[0,1]$

$$
\begin{gather*}
a^{2}\left(w_{x x}+w_{y y}\right)+w_{\zeta \zeta}=0  \tag{11}\\
\kappa w_{\zeta}(x, y, 1)+a^{2} h w(x, y, 1)=\epsilon \theta(x, y) a^{2} h u_{z}^{0} \\
\kappa w_{y}(x, y, 0)=0
\end{gather*}
$$

with adiabatic conditions on the vertical sides. Moreover, we have

$$
\begin{equation*}
w(x, y, 0) \approx G(x, y) \tag{12}
\end{equation*}
$$

Remark. The remainder $R_{2}(h, \epsilon)=\max _{x, y}\left|u^{\epsilon}(x, y, 0)-u^{0}-w(x, y, 0)\right|$ measures the precision of (12). In Figure 2 we plot $R_{2}(h, \epsilon)$ for $\epsilon \in\{.003, .005, .007\}$ and $a h \in[20,200]$ in the framework of the 2D example described in section 4. Observe that the domain derivative is very close to thermal contrast not only for small $\epsilon$ (as obviously expected), but also for large values of the heat transfer coefficient $a h$. We believe that the stabilizing role of increasing $a h$ is related to the instability expected when $h$ goes to zero (it is well known that for $h=0$ the IBVP (1)-(6) has no stationary solution).


Figure 2: $R_{2}(h, \epsilon)$ measures how much the scaled domain derivative for $\zeta=0$ is a good approximation of the thermal contrast.

### 3.1. Thin Plate Approximation

Plugging the formal expansions

$$
\begin{gather*}
w=w_{0}+a^{2} w_{1}+O\left(a^{4}\right),  \tag{13}\\
\theta=\theta_{0}+a^{2} \theta_{1}+O\left(a^{4}\right) \tag{14}
\end{gather*}
$$

in the BVP, we obtain a hierarchy of relations amongst coefficients which allows us to derive an approximate formula for the unknown $\epsilon \theta$.

Zeroth order relations give $w_{0 \zeta}(x, y, 1)=w_{0 \zeta}(x, y, 0)=w_{0 \zeta \zeta}(x, y, \zeta)=0$ so that $w_{0}$ is actually independent on $\zeta$. Hence, we set $w_{0}(x, y, \zeta) \equiv w_{0}(x, y) \approx$ $u^{\epsilon}(x, y, 0)-u^{0}(x, y, 0)$ as suggested by Figure 2.

First order relations are

$$
\begin{gathered}
w_{0 x x}+w_{0 y y}+w_{1 \zeta \zeta}=0, \\
\kappa w_{1 \zeta}(x, y, 1)+h w_{0}(x, y, 1)=-\epsilon \theta^{0}(x, y) \frac{\Phi}{\kappa}, \\
\kappa w_{1 \zeta}(x, y, 0)=0,
\end{gathered}
$$

so that (from the fundamental theorem of calculus)

$$
-h w_{0}(x, y)-\epsilon \theta^{0}(x, y) \frac{\Phi}{\kappa}=-\kappa\left(w_{0 x x}(x, y)+w_{0 y y}(x, y)\right)
$$

Hence, we have the following approximation of the boundary damage

$$
\epsilon \theta_{0}(x, y)=\frac{\kappa^{2}}{h \Phi_{0}}\left(w_{0 x x}(x, y)+w_{0 y y}(x, y)\right)-\frac{\kappa}{\Phi} w_{0}(x, y)
$$

### 3.2. The complete hierachic scheme in 2 D

We can iterate the perturbative step just described. For simplicity we limit ourselves to the 2D model in the variables $(x, \zeta)$. We have

$$
w_{0 x x}+w_{1 \zeta \zeta}=0
$$

so that it is easy to see that

$$
w_{1}(x, \zeta)=-w_{0 x x}(x) \frac{\zeta^{2}}{2}
$$

Since for all $n \geq 1$ we have

$$
w_{n x x}+w_{n+1}{ }_{\zeta \zeta}=0
$$

we obtain $w_{n}(x, \zeta)=\frac{d^{2 n} w_{0}(x)}{d x^{2 n}}(-1)^{n} \frac{\zeta^{2 n}}{(2 n)!}$.
Hence, the coefficients of the expansion of $\theta$ are derived plugging expansions (13), (14) in the BVP (11). We have

$$
\begin{align*}
\epsilon \theta_{n}(x)=(-1)^{(n+1)} \frac{\kappa^{2}}{h \Phi} \frac{d^{2 n} w_{0}}{d x^{2 n}}(x) & \frac{1}{(2 n-1)!} \\
& +(-1)^{n} \frac{\kappa}{\Phi_{0}} \frac{d^{2(n-1)} w_{0}}{d x^{2(n-1)}}(x) \frac{1}{2(n-1)!} \tag{15}
\end{align*}
$$

Since $z=a \zeta$, the formal expansion in (13) becomes

$$
w(x, z)=\sum_{n=0}^{\infty} \frac{d^{2 n} w_{0}(x)}{d x^{2 n}}(-1)^{n} \frac{z^{2 n}}{(2 n)!}
$$

For $x$ fixed in $[-L, L]$, this is a power series in $z$ that converges uniformly in $[-a, a]$ if, for a positive real number $\delta, S=\sum_{n=0}^{\infty} \frac{d^{2 n} w_{0}}{d x^{2 n}} \frac{(a+\delta)^{2 n}}{(2 n)!}<\infty$. Although the Neumann condition $w_{z}(x, 0)=0$ allows us to prove the analiticity of $w(x, 0)$, we do not know anything about the convergence of $S$. In agreement with [11] we must be content of convergence in a smaller interval. It is not a big drawback as long as we keep the formal character of our result.

## 4. Recovering surface damages using formal TPA. A numerical example.

In our numerical experiment, we fix the following geometrical and physical parameters. The values $L=-.5 m, a=.05 m, \epsilon=.005 m, \theta=e^{-90 x^{2}}$ define the domain $\Omega_{\epsilon \theta}$ in $R^{2}$. As for the conducting material we have $\kappa=100 \frac{W}{m K}$ and $\alpha=10^{-4} \frac{m^{2}}{s}$ while the heat exchange coefficient is $a h=100 \frac{W}{m^{2} K}$. The controlled heat flux is $\Phi=1000 \frac{W}{m^{2}}$.

Here we limit ourselves to the second order formal approximation and show some numerical result. The formula comes directly from (15):

$$
\begin{align*}
\epsilon \theta \approx \frac{\kappa^{2}}{h \Phi} \frac{d^{2} G}{d x^{2}}-\frac{\kappa}{\Phi} G(x)+a^{2}\left(-\frac{\kappa^{2}}{3!h \Phi} \frac{d^{4} G}{d x^{4}}\right. & \left.+\frac{\kappa}{2!\Phi} \frac{d^{2} G}{d x^{2}}\right) \\
& +a^{4}\left(\frac{\kappa^{2}}{5!h \Phi} \frac{d^{6} G}{d x^{6}}-\frac{\kappa}{4!\Phi} \frac{d^{4} G}{d x^{4}}\right) . \tag{16}
\end{align*}
$$

We produce syntetic data of thermal contrast by solving numerically the IBVP (1)-(6). If $t>T_{\alpha}$, we assume that the thermal contrast is the stationary difference $G(x) \approx u^{\epsilon}(x, 0, t)-U_{0}-\left(\frac{a}{\kappa}+\frac{1}{h}\right) \Phi$.

Formula (16) gives a good approximation of $\epsilon \theta$ : In Figure 3a we show what we obtained by means of (16) when $w(x, 0)=u^{\epsilon}(x, 0)-u^{0}(x, 0)$. Convergence at orders $>2$ seems to be slow in the neigborhood of $x=0$ (maximum of the damage size). In Figure 3b it is $w(x, 0)=u^{\epsilon}(x, 0)-u^{0}(x, 0)+R_{2}(h, \epsilon)$. Although the contrast is now affected by noise, TPA still indentify the damage.

We remark that temperature maps at $z=0$ allow us to localize the inaccessible defect. On the other hand, our goal is to evaluate the health of the specimen. For this reason, we could consider acceptable also the 3D estimate of zeroth order that gives a precise evaluation of the scale parameter $\epsilon$ (Figure 4).

(b)

Figure 3: (a) When the thermal contrast is equal to the Domain Derivative $\left(R_{2}(h, \epsilon)=0\right)$, the unknown defect (bold line) is well approximated by the zeroth order TPA (dashed). The reconstruction is improved using the first order TPA (full thin line). The correction due to the second order term (the pointed line overlaps the first order line) seems to be neglectable. (b)Here the TPA is constructed from the thermal contrast (that is $w_{0}(x)=w(x, 0)+R_{2}(h, \epsilon)$ ). It is equivalent to using noisy data. TPA gives anyhow a quite good approximation of the defect. When further noise added, some regularization is required.


Figure 4: (a) Temperature map on the accessibile side: there is a damage spread around the origin of axes. It seems a regular gaussian hole. This image gives an idea of the diameter but we have no information about its depth $\epsilon$. (b) Level sets of the damage as reconstructed in (10). The damaged area is clearly revealed. (c) Section $y=0$ of the damage (full line) compared to the reconstruction mapped in Figure 4b. The depth is fully identified by (10).

## 5. Conclusions

We derive here an explicit formal inversion rule for recovering an unknown surface damage from uncomplete thermal data. Our formula is based on the Thin Plate Approximation of the Direct Model. Numerical results are encouraging, but much work is still required: in particular, regularization of numerical differentiation and Cauchy problem for Laplace's equation are expected in the perspective of using real data.

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# Discrete approximation and regularisation for the inverse conductivity problem 

Luca Rondi

## Dedicated to Giovanni Alessandrini on the occasion of his 60th birthday


#### Abstract

We study the inverse conductivity problem with discontinuous conductivities. We consider, simultaneously, a regularisation and a discretisation for a variational approach to solve the inverse problem. We show that, under suitable choices of the regularisation and discretisation parameters, the discrete regularised solutions converge, as the noise level on the measurements goes to zero, to the looked for solution of the inverse problem.


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## 1. Introduction

In this paper we consider the inverse conductivity problem with discontinuous conductivity. For a given conducting body contained in a bounded domain $\Omega \subset \mathbb{R}^{N}, N \geq 2$, we call $X$ the space of admissible conductivities, or better conductivity tensors, in $\Omega$. For any $\sigma \in X$, we call $\Lambda(\sigma)$ either the Dirichlet-to-Neumann map, or the Neumann-to-Dirichlet map, corresponding to $\sigma$. It is a well-known fact that $\Lambda(\sigma)$ is a bounded linear operator between suitable Banach spaces defined on the boundary of $\Omega$, and we call $Y$ the space of these bounded linear operators. The forward operator $\Lambda: X \rightarrow Y$ is the one that to each $\sigma \in X$ associates $\Lambda(\sigma) \in Y$.

The aim of the inverse problem is to determine an unknown conductivity in $\Omega$ by performing suitable electrostatic measurements of current and voltage type on the boundary. If $\sigma_{0}$ is the conductivity we aim to recover by solving our inverse problem, then we measure its corresponding $\Lambda\left(\sigma_{0}\right) \in Y$. Due to the noise that is present in the measurements, actually the information that we are able to collect is $\hat{\Lambda} \in Y$, which is a perturbation of $\Lambda\left(\sigma_{0}\right)$. We call $\left\|\hat{\Lambda}-\Lambda\left(\sigma_{0}\right)\right\|_{Y}$
the noise level of the measurements and we notice that the choice of the space $Y$ corresponds to the way we measure the errors in our measurements.

The inverse problem may be stated, at least formally, in the following way. Given our measurements $\hat{\Lambda}$, we wish to find $\sigma \in X$ such that

$$
\begin{equation*}
\Lambda(\sigma)=\hat{\Lambda} \tag{1}
\end{equation*}
$$

Due to the noise, such a problem may not have any solution, therefore we better consider a least-square formulation

$$
\begin{equation*}
\min _{\sigma \in X}\|\Lambda(\sigma)-\hat{\Lambda}\|_{Y} \tag{2}
\end{equation*}
$$

Unfortunately, the inverse conductivity problem is ill-posed, therefore to solve (2) numerically, a regularisation strategy need to be implemented. Considering a regularisation à la Tikhonov, this means to choose a regularisation operator $R$, usually a norm or a seminorm, and a regularisation parameter $a$ and solve

$$
\begin{equation*}
\min _{\sigma \in X}\|\Lambda(\sigma)-\hat{\Lambda}\|_{Y}+a R(\sigma) \tag{3}
\end{equation*}
$$

A solution to (3) is called a regularised solution. A good regularisation operator need to satisfy the following two criteria. First of all, it should make the minimisation process stable from a numerical point of view. Second, the regularised solution should be a good approximation of the looked for solution of the inverse problem.

For the nonsmooth case, often this second requirement is not proved analytically but rather it is (not rigorously) justified by numerical tests only. However, a convergence analysis, using techniques inspired by variational convergences such as $\Gamma$-convergence, allows to rigorously justify the choice of the regularisation operator, [38]. For the inverse conductivity problem with discontinuous conductivity, by this technique, in the same paper [38], the use of some of the usually employed regularisation methods was rigorously justified. For instance, a convergence analysis was developed for regularisations such as the total variation penalisation or the Mumford-Shah functional. Several other works followed this approach, for instance it was extended to smoothness or sparsity penalty regularisations for the inverse conductivity problem in [28], whereas in [27] the analysis for the Mumford-Shah functional was slightly refined and applied to other inverse problems.

Once the regularisation operator is chosen, and proved to be effective, the issue of the numerical approximation for the regularised problem comes into play. One of the key points of the numerical approximation is represented by the discretisation of the regularised minimum problem. Again, two issues come forward. The first one is the choice of the kind of space of discrete unknowns we intend to use. The second important issue is how fine the discretization should
be. A compromise is necessary between a better resolution (finer discretization) and a more stable reconstruction (coarser discretization). Again, the discrete regularised solution, that is, the solution to the regularised problem (3) with $\sigma$ varying in such a discrete subset, should be a good approximation of the solution of the inverse problem. Actually, for inverse problems, this may not be necessarily so, as an example in [36] shows. Therefore, studying the effect of the discretisation when solving an inverse problem is not at all an easy task. This fundamental and nontrivial issue went rather unlooked, at least for the inverse conductivity problem and other classical inverse problems dealing with nonsmooth unknowns.

The crucial point we wish to address here is the following. We want to simultaneously fix both the regularisation parameter and the discretisation parameter, in correspondence to the given noise level, such that the discrete regularised solutions converge, as the noise level goes to zero, to the solution of the inverse problem. Previously, only the analysis of the approximation of the regularised problem with discrete ones, with a fixed regularisation parameter, was performed. For instance, a nice finite element approximation for the inverse conductivity problem, with the total variation as regularisation, may be found in [23]. In [40], instead, it was proved that the regularised inverse conductivity problem, with the Mumford-Shah as a regularisation term, could be well approximated by replacing the Mumford-Shah with its approximating Ambrosio-Tortorelli functionals developed in [6, 7]. Here the approximating parameter for the Ambrosio-Tortorelli functionals may be seen as another version of the discretisation parameter.

Actually, the first attempt to vary, in a suitable way, the regularisation and discretisation parameters simultaneously, may be found in a Master thesis supervised by the author, [14]. There the Ambrosio-Tortorelli functionals were considered, and their approximating parameter and the regularisation parameter were chosen accordingly to the noise level to guarantee the required convergence of this type of regularised solutions. For the convenience of the reader, we present a brief summary of this result in Subsection 3.2 of the present paper.

The main result of the paper, Theorems 3.5 and 3.6, is contained in Subsection 3.1. We consider the inverse conductivity problem and its regularisation by a total variation penalisation. We consider a discrete subset of admissible conductivities which is simply given by standard conforming piecewise linear finite elements over a regular triangulation. The triangulation is characterised by a discretisation parameter $h$, which is an upper bound for the diameter of any simplex forming the triangulation.

We show that, if we choose the regularisation parameter $a$ and the discretisation parameter $h$ according to the noise level, then the discrete regularised solution would converge to a solution of the inverse problem. An interesting
feature of this result is that it shows that the discretisation parameter should go to zero in a polynomial way with respect to the noise level.

We remark that in this paper we limit ourselves to a very simple scenario but we believe that this is just a first step to tackle a full discretisation of the inverse conductivity problem, in a more general setting as well. This will be the object of future work.

It would also be very interesting to address the issue of convergence estimates. In the smooth case they may be obtained by using Tikhonov regularisation for nonlinear operators, see for instance [20]. Actually, for the inverse conductivity problem in the smooth case, some convergence estimates are available for the regularised solutions, without adding the discretisation, see for instance [31] and [28]. We notice that our technique involves $\Gamma$-convergence, which is of a qualitative nature thus does not lead easily to convergence estimates.

Finally we wish to mention that, for discrete sets of unknowns, that is, unknowns depending on a finite number of parameters, the usual ill-posedness of these kinds of inverse problems considerably reduces. In fact, Lipschitz stability estimates may be obtained instead of the classical logarithmic ones. Such an important line of research was initiated in [3] and pursued in several other paper (let us mention the recent one [2] which is the closest to the setting we use in this paper). Unfortunately, the behaviour of the Lipschitz constant as the discretisation parameter approaches zero is extremely bad, as it explodes exponentially with respect to $h$, a fact firstly noted in [37]. This fact seems to prevent the use of these kinds of estimates at the discrete level to prove convergence estimate, or even just convergence, of discrete regularised solutions.

The plan of the paper is the following. In Section 2, besides fixing the notation and stating the inverse conductivity problem, we present a rather complete introduction to the regularisation issue for this inverse problem. Most of the material here is not new, a part from a few instances that we point out in a while, but our aim is to present a self-contained review to this line of research that is scattered in several papers. We begin with uniqueness results for scalar conductivities, that is, for the isotropic case, and nonuniqueness for symmetric conductivity tensors, that is, for the anisotropic case, Subsection 2.1. We recall that nonuniqueness is due to the invariance of the boundary operators by smooth changes of variables of the domain $\Omega$ that keep fixed the boundary.

In Subsection 2.2, we study the existence of a solution to (1). This part is mostly from [39]. We show that existence is true in the anisotropic case, whereas it may fail in the isotropic case, see Example 2.5. We notice that Example 2.5 appeared in a Master thesis supervised by the author, [18], and it is a slight generalisation of a similar example in [39]. The crucial ingredient for both is a nice construction due to Giovanni that may be found in [39, Example 4.4]. Even if existence of (1) is guaranteed, the ill-posedness nature of this inverse problem implies that minimiser to (1) may fail to converge to the
looked for solution to the inverse problem, as the noise level goes to zero. This is shown in three different examples, Examples 2.8, 2.10 and 2.11. Example 2.8 shows how nonuniqueness in the anisotropic case leads to instability, see also Proposition 2.9 which is taken from [22] for a corresponding partial stability result. Examples 2.10 and 2.11 deal with the isotropic case. The latter is new and slightly improves the former, which is taken from [1].

In Subsection 2.3, we recall the approach to regularisation for inverse problems with nonsmooth unknowns, and in particular for the inverse conductivity problem with discontinuous conductivities, that was developed in [38].

Section 3 is the main of the paper. We investigate simultaneous numerical approximation and regularisation for the inverse conductivity problem with discontinuous conductivities. In Subsection 3.1, we present our main result, the convergence analysis of the discretisation by the finite element method coupled with a total variation regularisation. Finally, in Subsection 3.2, we present the result of [14], that is, the convergence analysis for the regularisation by Ambrosio-Tortorelli functionals.

## 2. Statement of the inverse problem, preliminary considerations, and previous results

Throughout the paper we shall keep fixed positive constants $\lambda_{0}, \lambda_{1}$, and $\tilde{\lambda}_{1}$, with $0<\lambda_{0} \leq \lambda_{1}, \tilde{\lambda}_{1}$. The integer $N \geq 2$ will always denote the space dimension and we recall that we shall usually drop the dependence of any constant on $N$. For any Borel set $E \subset \mathbb{R}^{N}$, we denote with $|E|$ its Lebesgue measure, whereas $\mathcal{H}^{N-1}(E)$ denotes its $(N-1)$-dimensional Hausdorff measure.

Throughout the paper we also fix $\Omega$, a bounded connected open set contained in $\mathbb{R}^{N}, N \geq 2$. We assume that $\Omega$ has a Lipschitz boundary in the following usual sense. For any $x \in \partial \Omega$ there exist $r>0$ and a Lipschitz function $\varphi: \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ such that, up to a rigid change of coordinates, we have

$$
\Omega \cap B_{r}(x)=\left\{y=\left(y_{1}, \ldots, y_{N-1}, y_{N}\right) \in B_{r}(x): y_{N}<\varphi\left(y_{1}, \ldots, y_{N-1}\right)\right\}
$$

We call $\mathbb{M}^{N \times N}(\mathbb{R})$ the space of real valued $N \times N$ matrices. For any $\sigma \in$ $\mathbb{M}^{N \times N}(\mathbb{R})$, with $N \geq 2$, several equivalent ellipticity conditions may be used. For example

$$
\begin{cases}\sigma \xi \cdot \xi \geq \lambda_{0}\|\xi\|^{2} & \text { for any } \xi \in \mathbb{R}^{N}  \tag{4}\\ \sigma^{-1} \xi \cdot \xi \geq \lambda_{1}^{-1}\|\xi\|^{2} & \text { for any } \xi \in \mathbb{R}^{N}\end{cases}
$$

Otherwise we can use

$$
\left\{\begin{array}{l}
\sigma \xi \cdot \xi \geq \lambda_{0}\|\xi\|^{2} \quad \text { for any } \xi \in \mathbb{R}^{N}  \tag{5}\\
\|\sigma\| \leq \tilde{\lambda}_{1}
\end{array}\right.
$$

where $\|\sigma\|$ denotes its norm as a linear operator of $\mathbb{R}^{N}$ into itself.
The following remark shows that these two conditions are equivalent. If $\sigma$ satisfies (4) with constants $\lambda_{0}$ and $\lambda_{1}$, then it also satisfies (5) with constants $\lambda_{0}$ and $\tilde{\lambda}_{1}=\lambda_{1}$. If $\sigma$ satisfies (5) with constants $\lambda_{0}$ and $\tilde{\lambda}_{1}$, then it also satisfies (4) with constants $\lambda_{0}$ and $\lambda_{1}=\tilde{\lambda}_{1}^{2} / \lambda_{0}$. If $\sigma$ is symmetric then, picking $\tilde{\lambda}_{1}=\lambda_{1}$, (4) and (5) are exactly equivalent and coincide with the condition

$$
\lambda_{0}\|\xi\|^{2} \leq \sigma \xi \cdot \xi \leq \lambda_{1}\|\xi\|^{2} \quad \text { for any } \xi \in \mathbb{R}^{N}
$$

that we write in short as follows

$$
\lambda_{0} I_{N} \leq \sigma \leq \lambda_{1} I_{N}
$$

where $I_{N}$ is the $N \times N$ identity matrix. Finally, if $\sigma=s I_{N}$, where $s$ is a real number, the condition simply reduces to

$$
\lambda_{0} \leq s \leq \lambda_{1}
$$

We use the following classes of conductivity tensors in $\Omega$. For positive constants $\lambda_{0} \leq \lambda_{1}$ we call $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ the set of $\sigma=\sigma(x), x \in \Omega$, an $N \times N$ matrix whose entries are real valued measurable functions in $\Omega$, such that, for almost any $x \in \Omega, \sigma(x)$ satisfies (4). We call $\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, respectively $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, the set of $\sigma \in \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ such that, for almost any $x \in \Omega, \sigma(x)$ is symmetric, respectively $\sigma(x)=s(x) I_{N}$ with $s(x)$ a real number. We say that $\sigma$ is a conductivity tensor in $\Omega$ if $\sigma \in \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ for some constants $0<\lambda_{0} \leq \lambda_{1}$. We call $\mathcal{M}$ the class of conductivity tensors in $\Omega$. We say that $\sigma$ is a symmetric conductivity tensor in $\Omega$ if $\sigma \in \mathcal{M}$ and $\sigma(x)$ is symmetric for almost any $x \in \Omega$. We call $\mathcal{M}_{\text {sym }}$ the class of conductivity tensors in $\Omega$. We say that $\sigma$ is a scalar conductivity in $\Omega$ if $\sigma \in \mathcal{M}$ and $\sigma(x)=s(x) I_{N}$, with $s(x) \in \mathbb{R}$, for almost any $x \in \Omega$. We call $\mathcal{M}_{\text {scal }}$ the class of scalar conductivities in $\Omega$.

Since $\mathcal{M} \subset L^{\infty}\left(\Omega, \mathbb{M}^{N \times N}(\mathbb{R})\right)$, we may measure the distance between any two conductivity tensors $\sigma_{1}$ and $\sigma_{2}$ in $\Omega$ with an $L^{p}$ metric, for any $p, 1 \leq p \leq$ $+\infty$, as follows

$$
\left\|\sigma_{1}-\sigma_{2}\right\|_{L^{p}(\Omega)}=\left\|\left(\left\|\sigma_{1}-\sigma_{2}\right\|\right)\right\|_{L^{p}(\Omega)}
$$

With any of these $L^{p}$ metrics, any of the classes $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right), \mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, and $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ is a complete metric space.

For any $p, 1 \leq p \leq+\infty$, we denote with $p^{\prime}$ its conjugate exponent, that is $1 / p+1 / p^{\prime}=1$. For any $p, 1<p<+\infty$, we call $W^{1-1 / p, p}(\partial \Omega)$ the space of traces of $W^{1, p}(\Omega)$ functions on $\partial \Omega$. We recall that $W^{1-1 / p, p}(\partial \Omega) \subset$ $L^{p}(\partial \Omega)$, with compact immersion. For simplicity, we denote $H^{1}(\Omega)=W^{1,2}(\Omega)$, $H^{1 / 2}(\partial \Omega)=W^{1 / 2,2}(\partial \Omega)$ and $H^{-1 / 2}(\partial \Omega)$ its dual.

We call $L_{*}^{2}(\partial \Omega)$ the subspace of functions $f \in L^{2}(\partial \Omega)$ such that $\int_{\partial \Omega} f=0$. We set $H_{*}^{-1 / 2}(\partial \Omega)$ the subspace of $g \in H^{-1 / 2}(\partial \Omega)$ such that

$$
\langle g, 1\rangle_{\left(H^{-1 / 2}(\partial \Omega), H^{1 / 2}(\partial \Omega)\right)}=0
$$

We recall that $L_{*}^{2}(\partial \Omega) \subset H_{*}^{-1 / 2}(\partial \Omega)$, with compact immersion, if for any $g \in L_{*}^{2}(\partial \Omega)$ and any $\psi \in H^{1 / 2}(\partial \Omega)$ we define

$$
\begin{equation*}
\langle g, \psi\rangle_{\left(H^{-1 / 2}(\partial \Omega), H^{1 / 2}(\partial \Omega)\right)}=\int_{\partial \Omega} g \psi \tag{6}
\end{equation*}
$$

Analogously, $H_{*}^{1 / 2}(\partial \Omega)$ is the subspace of $\psi \in H^{1 / 2}(\partial \Omega)$ such that $\int_{\partial \Omega} \psi=0$. We have $H_{*}^{1 / 2}(\partial \Omega) \subset L_{*}^{2}(\partial \Omega)$, with compact immersion.

For any two Banach spaces $B_{1}, B_{2}, \mathcal{L}\left(B_{1}, B_{2}\right)$ will denote the Banach space of bounded linear operators from $B_{1}$ to $B_{2}$ with the usual operator norm.

### 2.1. Statement of the problem and uniqueness results

For any conductivity tensor $\sigma$ in $\Omega$, we define its Dirichlet-to-Neumann map $D N(\sigma): H^{1 / 2}(\partial \Omega) \rightarrow H^{-1 / 2}(\partial \Omega)$ where for each $\varphi \in H^{1 / 2}(\partial \Omega)$,

$$
D N(\sigma)(\varphi)[\psi]=\int_{\Omega} \sigma \nabla u \cdot \nabla \tilde{\psi} \quad \text { for any } \psi \in H^{1 / 2}(\partial \Omega)
$$

where $u$ solves

$$
\begin{cases}\operatorname{div}(\sigma \nabla u)=0 & \text { in } \Omega  \tag{7}\\ u=\varphi & \text { on } \partial \Omega\end{cases}
$$

and $\tilde{\psi} \in H^{1}(\Omega)$ is such that $\tilde{\psi}=\psi$ on $\partial \Omega$ in the trace sense. We have that $D N(\sigma)$ is a well-defined bounded linear operator, whose norm is bounded by a constant depending on $N, \Omega, \lambda_{0}$, and $\lambda_{1}$ only, for any $\sigma \in \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$. Let us notice that, actually, we have $D N(\sigma): H^{1 / 2}(\partial \Omega) \rightarrow H_{*}^{-1 / 2}(\partial \Omega)$. Moreover, since for any constant function $\varphi$ on $\Omega$ we have that $\varphi \in H^{1 / 2}(\Omega)$ and $D N(\sigma)(\varphi)=0$, no matter what $\sigma$ is, without loss of generality, we actually define

$$
\begin{equation*}
D N(\sigma): H_{*}^{1 / 2}(\partial \Omega) \rightarrow H_{*}^{-1 / 2}(\partial \Omega) \tag{8}
\end{equation*}
$$

For any conductivity tensor $\sigma$ in $\Omega$, we define its Neumann-to-Dirichlet map

$$
N D(\sigma): H_{*}^{-1 / 2}(\partial \Omega) \rightarrow H_{*}^{1 / 2}(\partial \Omega)
$$

where for each $g \in H_{*}^{-1 / 2}(\partial \Omega)$,

$$
N D(\sigma)(g)=\left.v\right|_{\partial \Omega}
$$

where $v$ solves

$$
\begin{cases}\operatorname{div}(\sigma \nabla v)=0 & \text { in } \Omega  \tag{9}\\ \sigma \nabla v \cdot \nu=g & \text { on } \partial \Omega \\ \int_{\partial \Omega} v=0 & \end{cases}
$$

We have that $N D(\sigma)$ is a well-defined bounded linear operator, it is the inverse of $D N(\sigma)$ as defined in (8), and its norm is bounded by a constant depending on $N, \Omega$, and $\lambda_{0}$ only, for any $\sigma \in \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$.

We consider the following forward operators

$$
D N: \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right) \rightarrow \mathcal{L}\left(H_{*}^{1 / 2}(\partial \Omega), H_{*}^{-1 / 2}(\partial \Omega)\right)
$$

and

$$
N D: \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right) \rightarrow \mathcal{L}\left(H_{*}^{-1 / 2}(\partial \Omega), H_{*}^{1 / 2}(\partial \Omega)\right)
$$

We can state the inverse conductivity problem in the following way. We wish to determine an unknown conductivity tensor $\sigma$ in $\Omega$ by performing electrostatic measurements at the boundary of voltage and current type. If all boundary measurements are performed, this is equivalent to say that we are measuring either its Dirichlet-to-Neumann map $D N(\sigma)$ or its Neumann-to-Dirichlet map $N D(\sigma)$. In other words, given either $D N(\sigma)$ or $N D(\sigma)$, we wish to recover $\sigma$.

Such an inverse problem has a long history, it was in fact proposed by Calderón [11] in 1980. About uniqueness, there are several result for scalar, that is isotropic, conductivities. In dimension 3 and higher, already in the 80 's, uniqueness was proved in $[29,30]$ for the determination of the conductivity at the boundary and for the analytic case, and then in [42] for $C^{2}$ conductivities. Slightly later it appeared the first uniqueness result for smooth conductivities in dimension 2, [35].

Recently, the two dimensional case was completely solved, [8], for $L^{\infty}$ scalar conductivities. Also for the $N$ dimensional case, with $N \geq 3$, there has been a great improvement. In [26], the regularity has been reduced to $C^{1}$ or Lipschitz but close to a constant. The case of general Lipschitz conductivities is treated in [12]. The most general result is the one in [25], where conductivities with unbounded gradient are allowed and uniqueness is shown for $W^{1, N}$ conductivities, at least for $N=3,4$.

For what concerns anisotropic conductivities, for instance when we consider symmetric conductivity tensors in $\mathcal{M}_{s y m}$, uniqueness is never achieved. In fact, let $\varphi: \Omega \rightarrow \Omega$ be a bi-Lipschitz mapping, that is a bijective map such that $\varphi$ and its inverse $\varphi^{-1}$ are Lipschitz functions. Clearly $\varphi$ can be extended to a Lipschitz function defined on $\bar{\Omega}$. For any $\sigma \in \mathcal{M}_{s y m}$ in $\Omega$ and any of these bi-Lipschitz mapping $\varphi$ from $\Omega$ onto itself, we define the push-forward of the conductivity tensor $\sigma$ by $\varphi$ as

$$
\begin{equation*}
\varphi_{*}(\sigma)(y)=\frac{J(x) \sigma(x) J(x)^{T}}{|\operatorname{det} J(x)|} \quad \text { for almost any } y \in \Omega \tag{10}
\end{equation*}
$$

where $J(x)=J \varphi(x)$ is the Jacobian matrix of $\varphi$ in $x$ and $x=\varphi^{-1}(y)$. We have that $\varphi_{*}(\sigma) \in \mathcal{M}_{\text {sym }}$ and that

$$
\begin{equation*}
D N(\sigma)=D N\left(\varphi_{*}(\sigma)\right) \text { and } N D(\sigma)=N D\left(\varphi_{*}(\sigma)\right) \quad \text { if }\left.\varphi\right|_{\partial \Omega}=I d \tag{11}
\end{equation*}
$$

In dimension $N=2$ and for $\Omega$ simply connected, (10) and (11) still hold even if we consider $\varphi: \Omega \rightarrow \Omega$ to be a quasiconformal mapping. We recall that, for $\Omega \subset \mathbb{R}^{2}$, simply connected bounded open set with Lipschitz boundary, we say that $\varphi: \Omega \rightarrow \Omega$ is a quasiconformal mapping if $\varphi$ is bijective, $\varphi \in W^{1,2}(\Omega)$, and, for some $K \geq 1$, we have

$$
\|J \varphi(x)\|^{2} \leq K \operatorname{det}(J \varphi(x)) \quad \text { for a.e. } x \in \Omega
$$

By (11), it is immediate to notice that our inverse problem can not have a unique solution if we consider symmetric conductivity tensors. On the other hand, in dimension 2 , this is the only obstruction to uniqueness for symmetric conductivity tensors, as proved in [41] in the smooth case and in [9] in the general $L^{\infty}$ case.

We summarise these results in the following theorem.
Theorem 2.1. Let $\Omega \subset \mathbb{R}^{N}, N=2,3,4$, be a bounded, connected domain with Lipschitz boundary. Let $\sigma_{1}$ and $\sigma_{2}$ belong to $\mathcal{M}_{\text {scal }}$.

If $N=3,4$ and $\sigma_{1}, \sigma_{2} \in W^{1, N}(\Omega)$, then we have, see [25],

$$
D N\left(\sigma_{1}\right)=D N\left(\sigma_{2}\right) \text { or } N D\left(\sigma_{1}\right)=N D\left(\sigma_{2}\right) \text { implies } \sigma_{1}=\sigma_{2}
$$

If $N=2$ and $\Omega$ is simply connected, then we have, see [8],

$$
D N\left(\sigma_{1}\right)=D N\left(\sigma_{2}\right) \text { or } N D\left(\sigma_{1}\right)=N D\left(\sigma_{2}\right) \text { implies } \sigma_{1}=\sigma_{2}
$$

If $N=2$ and $\Omega$ is simply connected, for any $\sigma \in \mathcal{M}_{\text {sym }}$ we define

$$
\begin{aligned}
\Sigma(\sigma)= & \left\{\sigma_{1} \in \mathcal{M}_{\text {sym }}: \sigma_{1}=\varphi_{*}(\sigma)\right. \\
& \text { where } \left.\varphi: \Omega \rightarrow \Omega \text { is a quasiconformal mapping and }\left.\varphi\right|_{\partial \Omega}=I d\right\} .
\end{aligned}
$$

Then $D N(\sigma)$, or equivalently $N D(\sigma)$, uniquely determines the class $\Sigma(\sigma)$, see [9].

### 2.2. Variational formulation and ill-posedness

In practice, the inverse problem consists in the following. Let $\sigma_{0}$ be a conductivity tensor in $\Omega$ that we wish to determine. Considering for example the Dirichlet-to-Neumann case, we measure $D N\left(\sigma_{0}\right)$. Since our measurements are obviously noisy, the information that is actually available is a perturbation of $D N\left(\sigma_{0}\right)$, that we may call $\hat{\Lambda}$. Therefore our inverse problem consists in finding a conductivity $\sigma$ such that $D N(\sigma)=\hat{\Lambda}$. Due to the noise in the measurements this problem may not have any solution. We should therefore solve the problem in a least-square-type way, namely solve

$$
\min _{\sigma}\|D N(\sigma)-\hat{\Lambda}\|
$$

The fact that such a minimum problem admits a solution depends on several aspects. In particular it depends on the class of conductivity tensors on which we consider the minimisation and, in part, also on the kind of norm we use to measure the distance between $D N(\sigma)$ and $\hat{\Lambda}$. Next we discuss in details these issues.

Occasionally, we shall use the so-called $H$-convergence. For a definition and its basic properties we refer to $[4,34,33]$. We recall that $G$ - or $H$-convergence was shown to be quite useful for the inverse conductivity problem, see for instance [1, 22, 39]. Here we just remark a few of its properties. This is a very weak kind of convergence, in fact it is weaker than $L_{l o c}^{1}$ convergence. For symmetric conductivity tensors $H$-convergence reduces to the more usual $G$-convergence. The most important fact is that $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ is compact with respect to $H$-convergence and $\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$ is also compact with respect to $H$-convergence, or equivalently $G$-convergence. Furthermore, $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ is not closed with respect to $G$-convergence, actually any symmetric conductivity tensor is the limit, in the $G$-convergence sense, of scalar conductivities assuming only two different positive values.

We use the following notation. Let $B_{1}$ and $B_{2}$ be two Banach spaces such that $B_{1} \subset H_{*}^{1 / 2}(\partial \Omega)$ and $H_{*}^{-1 / 2}(\partial \Omega) \subset B_{2}$, with continuous immersions. Moreover, let $\tilde{B}_{1}$ and $\tilde{B}_{2}$ be two Banach spaces such that $\tilde{B}_{1} \subset H_{*}^{-1 / 2}(\partial \Omega)$ and $H_{*}^{1 / 2}(\partial \Omega) \subset \tilde{B}_{2}$, with continuous immersions.

We denote with $X$ the space $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right), \mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, or $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$. The natural metric on $X$ will be the one induced by the $L^{1}$ metric.

In the Dirichlet-to-Neumann case, we call $Y=\mathcal{L}\left(B_{1}, B_{2}\right)$, with the distance induced by its norm, and denote $\Lambda=D N: X \rightarrow Y$.

We speak of the natural norm of the Dirichlet-to-Neumann map when $B_{1}=$ $H_{*}^{1 / 2}(\partial \Omega)$ and $B_{2}=H_{*}^{-1 / 2}(\partial \Omega)$ and we denote it with $\|\cdot\|_{\text {nat }}$ or $\|\cdot\|_{H^{1 / 2}, H^{-1 / 2}}$. We have a canonical continuous linear map from $\mathcal{L}\left(H_{*}^{1 / 2}(\partial \Omega), H_{*}^{-1 / 2}(\partial \Omega)\right)$ into $Y$. If we assume that $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$, then this map is injective, thus $\mathcal{L}\left(H_{*}^{1 / 2}(\partial \Omega), H_{*}^{-1 / 2}(\partial \Omega)\right) \subset Y$, with continuous immersion, and, if $y \in Y$ is such that $\|y\|_{Y}=0$, then $y \in \mathcal{L}\left(H_{*}^{1 / 2}(\partial \Omega), H_{*}^{-1 / 2}(\partial \Omega)\right)$ and also $\|y\|_{\text {nat }}=0$.

In the Neumann-to-Dirichlet case, we call $Y=\mathcal{L}\left(\tilde{B}_{1}, \tilde{B}_{2}\right)$, with the distance induced by its norm, and denote $\Lambda=N D: X \rightarrow Y$.

We speak of the natural norm of the Neumann-to-Dirichlet map when $\tilde{B}_{1}=$ $H_{*}^{-1 / 2}(\partial \Omega)$ and $\tilde{B}_{2}=H_{*}^{1 / 2}(\partial \Omega)$ and we denote it with $\|\cdot\|_{n a t}$ or $\|\cdot\|_{H^{-1 / 2}, H^{1 / 2}}$. We have a canonical continuous linear map from $\mathcal{L}\left(H_{*}^{-1 / 2}(\partial \Omega), H_{*}^{1 / 2}(\partial \Omega)\right)$ into $Y$. If we assume that $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, then this map is injective, thus $\mathcal{L}\left(H_{*}^{-1 / 2}(\partial \Omega), H_{*}^{1 / 2}(\partial \Omega)\right) \subset Y$, with continuous immersion, and, if if $y \in Y$ is such that $\|y\|_{Y}=0$, then $y \in \mathcal{L}\left(H_{*}^{-1 / 2}(\partial \Omega), H_{*}^{1 / 2}(\partial \Omega)\right)$ and also $\|y\|_{\text {nat }}=0$. Another interesting and useful choice for $\tilde{B}_{1}$ and $\tilde{B}_{2}$ is given by $\tilde{B}_{1}=\tilde{B}_{2}=$
$L_{*}^{2}(\partial \Omega)$, see the discussion in [39], and we denote its norm with $\|\cdot\|_{L^{2}, L^{2}}$. We remark that $L_{*}^{2}(\partial \Omega)$ is clearly dense in $H_{*}^{-1 / 2}(\partial \Omega)$.

Let us notice in the following remark that, when we consider the natural norms, then all results related to the Dirichlet-to-Neumann maps may be proved also for the Neumann-to-Dirichlet maps, and viceversa.

Remark 2.2. Let $\sigma_{1}, \sigma_{2} \in \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$. Then there exist positive constants $C_{1}$ and $C_{2}$, depending on $N, \Omega, \lambda_{0}$, and $\lambda_{1}$ only, such that

$$
\begin{aligned}
& C_{1}\left\|N D\left(\sigma_{1}\right)-N D\left(\sigma_{2}\right)\right\|_{H^{-1 / 2}, H^{1 / 2}} \leq\left\|D N\left(\sigma_{1}\right)-D N\left(\sigma_{2}\right)\right\|_{H^{1 / 2}, H^{-1 / 2}} \\
& \leq C_{2}\left\|N D\left(\sigma_{1}\right)-N D\left(\sigma_{2}\right)\right\|_{H^{-1 / 2}, H^{1 / 2}} .
\end{aligned}
$$

In fact, we have

$$
D N\left(\sigma_{1}\right)-D N\left(\sigma_{2}\right)=D N\left(\sigma_{1}\right)\left(N D\left(\sigma_{2}\right)-N D\left(\sigma_{1}\right)\right) D N\left(\sigma_{2}\right)
$$

and the same formula holds if we swap $D N$ with $N D$.
If we call $\hat{\Lambda} \in Y$ either the measured Dirichlet-to-Neumann map or the measured Neumann-to-Dirichlet map, then the inverse problem consists in finding $\sigma \in X$ such that $\Lambda(\sigma)=\hat{\Lambda}$. However, since $\hat{\Lambda}$ is a measured, therefore noisy, quantity, this problem may not have any solution and we thus solve the problem in a least-square-type way, namely solve

$$
\begin{equation*}
\min \left\{\|\Lambda(\sigma)-\hat{\Lambda}\|_{Y}: \sigma \in X\right\} \tag{12}
\end{equation*}
$$

Such a problem always admits a solution either if $X=\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ or if $X=\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$. In fact the following is proved in [39].

Proposition 2.3. Under the previous notation and assumptions, let us consider a sequence of conductivity tensors $\left\{\sigma_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ and a conductivity tensor $\sigma$ in the same set.

If, as $n \rightarrow \infty, \sigma_{n}$ converges to $\sigma$ strongly in $L_{l o c}^{1}$ or in the $H$-convergence sense, then

$$
\|\hat{\Lambda}-\Lambda(\sigma)\|_{Y} \leq \liminf _{n}\left\|\hat{\Lambda}-\Lambda\left(\sigma_{n}\right)\right\|_{Y}
$$

If $X$ is equal to $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ or to $\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, by compactness of $X$ with respect to $H$-convergence, we deduce that (12) admits a solution.

On the other hand, if $X$ is equal to $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ then (12) may fail to have a solution as we shall see later on in Example 2.5.

We notice that Proposition 2.3 contains a lower semicontinuity result. For certain application, instead, continuity is needed. For our purposes it will be enough the following result, proved in [1].

Proposition 2.4. Under the previous notation and assumptions, let us consider a sequence of symmetric conductivity tensors $\left\{\sigma_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$ and a conductivity tensor $\sigma$ in the same set. We assume that for some $\Omega^{\prime}$ compactly contained in $\Omega$ we have $\sigma_{n}=\sigma$ almost everywhere in $\Omega \backslash \Omega^{\prime}$ for any $n \in \mathbb{N}$.

If, as $n \rightarrow \infty, \sigma_{n}$ converges to $\sigma$ strongly in $L_{l o c}^{1}$ or in the $G$-convergence sense, then

$$
\lim _{n}\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{n}\right)\right\|_{n a t}=0
$$

as well as in $\|\cdot\|_{Y}$ for any $Y$ as above.
We notice that a certain control of the conductivity tensors near the boundary is indeed needed, see [22, Theorem 4.9]. In the same paper a more general and essentially optimal version of Proposition 2.4 is proved, see [22, Theorem 1.1].

Proposition 2.4 is enough to show that (12) may fail to have a solution if $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$. We slightly generalise [39, Example 3.4], which is based on a nice remark by Giovanni, which is presented in [39] as Example 4.4. This generalisation shows that existence may fail for both the Dirichlet-to-Neumann and Neumann-to-Dirichlet case and for the natural norms, as well as for any $\|\cdot\|_{Y}$ with $Y$ as above, if $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively. It firstly appeared in [18], and we present its proof here for the convenience of the reader.

Example 2.5. Let $\Omega=B_{1}(0) \subset \mathbb{R}^{2}$. Under the previous notation and assumptions, let us assume that $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively.

Let $a>0$ be a positive constant with $a \neq 1$. We define the conductivity tensor $\tilde{\sigma} \in \mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right) \backslash \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ in $B_{1}(0) \subset \mathbb{R}^{2}$ as follows

$$
\tilde{\sigma}= \begin{cases}I_{2} & \text { in } B_{1}(0) \backslash B_{1 / 2}(0)  \tag{13}\\
{\left[\begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array}\right]} & \text { in } B_{1 / 2}(0)\end{cases}
$$

Let us set $\hat{\Lambda}=\Lambda(\tilde{\sigma})$. There exist $0<\lambda_{0}<\lambda_{1}$ such that the minimum problem

$$
\min _{\sigma \in \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)}\|\Lambda(\tilde{\sigma})-\Lambda(\sigma)\|_{Y}
$$

does not have any solution, for any $Y$ as above, thus including the natural norms.

Proof. The crucial point is the following. By density of scalar conductivities inside symmetric conductivity tensors that follows by the results in [33], see [39, Proposition 2.2] for a convenient version, we can find $0<\lambda_{0}<\lambda_{1}$ and
$\left\{\sigma_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ such that $\sigma_{n} G$-converges to $\tilde{\sigma}$ as $n \rightarrow \infty$ and $\sigma_{n}=I_{2}$ in $B_{1}(0) \backslash B_{1 / 2}(0)$ for any $n \in \mathbb{N}$. Therefore, by Proposition 2.4 , we immediately conclude that

$$
\inf _{\sigma \in \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)}\|\Lambda(\tilde{\sigma})-\Lambda(\sigma)\|_{Y}=0
$$

In order for a minimiser to exist, then we need to find a scalar conductivity $\hat{\sigma}$ such that $\|\Lambda(\hat{\sigma})-\Lambda(\tilde{\sigma})\|_{Y}=0$, hence, by our density assumptions, such that $\Lambda(\hat{\sigma})=\Lambda(\tilde{\sigma})$. By the main result of [9], recalled in Theorem 2.1, there exists a quasiconformal mapping $\varphi: B_{1}(0) \rightarrow B_{1}(0)$ such that $\left.\varphi\right|_{\partial B_{1}(0)}=I d$ and $\varphi_{*}(\hat{\sigma})=\tilde{\sigma}$. We recall that actually $\varphi: \overline{B_{1}(0)} \rightarrow \overline{B_{1}(0)}$, it is continuous, bijective and its inverse is continuous as well. We assume that $\hat{\sigma}(x)=s(x) I_{2}$, $x \in B_{1}(0)$, with $s \in L^{\infty}\left(B_{1}(0)\right)$ and bounded away from 0 . Then $\varphi_{*}(\hat{\sigma})=\tilde{\sigma}$ means that for almost any $y \in B_{1}(0)$ we have

$$
\begin{aligned}
& \tilde{\sigma}(y)=\frac{J(x)\left(s(x) I_{2}\right) J(x)^{T}}{|\operatorname{det} J(x)|} \\
&=\frac{s(x)}{|\operatorname{det} J(x)|}\left[\begin{array}{cc}
\left|\nabla \varphi_{1}(x)\right|^{2} & \nabla \varphi_{1}(x) \cdot \nabla \varphi_{2}(x) \\
\nabla \varphi_{1}(x) \cdot \nabla \varphi_{2}(x) & \left|\nabla \varphi_{2}(x)\right|^{2}
\end{array}\right]
\end{aligned}
$$

where $\varphi=\left(\varphi_{1}, \varphi_{2}\right), J(x)$ is the Jacobian matrix of $\varphi$ in $x$, and $x=\varphi^{-1}(y)$. Since $\operatorname{det}(\tilde{\sigma}(y))=1$ for almost any $y \in B_{1}(0)$, we conclude that, for almost any $x \in B_{1}(0), s(x)=1$, that is $\hat{\sigma} \equiv I_{2}$ in $B_{1}(0)$. We also note that, since $\varphi$ is quasiconformal, then $\operatorname{det} J(x)>0$ for almost any $x \in B_{1}(0)$.

By the structure of $\tilde{\sigma}$, we infer that for almost any $x \in B_{1}(0)$ we have $\nabla \varphi_{2}(x)=\lambda(x)\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right] \nabla \varphi_{1}(x)$ with $\lambda(x)>0$, since $\operatorname{det} J(x)>0$, satisfying the following

$$
\lambda= \begin{cases}1 & \text { in } D=\varphi^{-1}\left(B_{1}(0) \backslash B_{1 / 2}(0)\right) \\ a^{-1} & \text { in } D_{1}=B_{1}(0) \backslash D=\varphi^{-1}\left(B_{1 / 2}(0)\right)\end{cases}
$$

We conclude that

$$
\Delta \varphi_{1}=\Delta \varphi_{2}=0 \quad \text { in } D \text { and in } D_{1}
$$

More precisely, we have that $\varphi_{1}+\mathrm{i} \varphi_{2}$ is holomorphic in $D$. Since $\varphi_{1}\left(x_{1}, x_{2}\right)=x_{1}$ and $\varphi_{2}\left(x_{1}, x_{2}\right)=x_{2}$ on $\partial B_{1}(0)$, by the unique continuation from Cauchy data, we infer that $\varphi=I d$ in $D$ as well. Therefore $B_{1}(0) \backslash B_{1 / 2}(0)=\varphi(D)=D$. We conclude that $D_{1}=B_{1 / 2}(0), \varphi_{1}$ and $\varphi_{2}$ are harmonic in $B_{1 / 2}(0)$, and $\varphi_{1}\left(x_{1}, x_{2}\right)=x_{1}$ and $\varphi_{2}\left(x_{1}, x_{2}\right)=x_{2}$ on $\partial B_{1 / 2}(0)$. We immediately conclude that $\varphi=I d$ on the whole $B_{1}(0)$ and we obtain a contradiction.

If we have no control on the conductivity tensors near the boundary, then continuity of our forward operators may be achieved by suitably choosing the spaces $B_{1}, B_{2}$, and $\tilde{B}_{1}, \tilde{B}_{2}$, that is by changing the distance, thus the space $Y$,
with respect to which we measure the error on our measurements. Namely we have the following results, see [39].

Proposition 2.6. Under the previous notation and assumptions, there exists $Q_{1}>2$, depending on $N, \Omega, \lambda_{0}$, and $\lambda_{1}$ only, such that the following holds for any $2<p<Q_{1}$.

In the Dirichlet-to-Neumann case, we assume that $B_{1} \subset W_{*}^{1-1 / p, p}(\partial \Omega)$, with continuous immersion.

In the Neumann-to-Dirichlet case, we assume that $\tilde{B}_{1} \subset\left(W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)\right)_{*}^{\prime}$, with continuous immersion, where $\left(W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)\right)_{*}^{\prime}$ is the subspace of $g$ belonging to the dual of $W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)$ such that $\langle g, 1\rangle=0$.

Then $\Lambda$ is Hölder continuous with respect to the $L^{1}$ distance in $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ and the distance $d$ on $Y$ given by its norm. The Hölder exponent $\beta$ is equal to $(p-2) /(2 p)$.

A particularly interesting case for Neumann-to-Dirichlet maps is to choose $\tilde{B}_{1}=\tilde{B}_{2}=L_{*}^{2}(\partial \Omega)$ since $L^{2}(\partial \Omega)$ is contained in the dual of $W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)$ for some $p, 2<p<Q_{1}$, with $p$ close enough to 2 , and $H_{*}^{1 / 2}(\partial \Omega) \subset L_{*}^{2}(\partial \Omega)$, with continuous immersions. Moreover, $L_{*}^{2}(\partial \Omega)$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$. In this case we also have continuity with respect to $H$-convergence, see again [39].

Proposition 2.7. Under the previous notation and assumptions, let us consider a sequence of conductivity tensors $\left\{\sigma_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$ and a conductivity tensor $\sigma$ in the same set.

If, as $n \rightarrow \infty, \sigma_{n}$ converges to $\sigma$ strongly in $L_{l o c}^{1}$ or in the $H$-convergence sense, then

$$
\lim _{n}\left\|N D(\sigma)-N D\left(\sigma_{n}\right)\right\|_{L^{2}, L^{2}}=0
$$

Let us consider that $\sigma_{0} \in X$ is the conductivity tensor in $\Omega$ that we wish to determine. Given the noise level $\varepsilon>0$, our measurement is given by $\hat{\Lambda}_{\varepsilon} \in Y$, satisfying

$$
\begin{equation*}
\left\|\hat{\Lambda}_{\varepsilon}-\Lambda\left(\sigma_{0}\right)\right\|_{Y} \leq \varepsilon \tag{14}
\end{equation*}
$$

For consistency, we call $\hat{\Lambda}_{0}=\Lambda\left(\sigma_{0}\right)$. Assume that our minimisation problem

$$
\begin{equation*}
\min \left\{\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}: \sigma \in X\right\} \tag{15}
\end{equation*}
$$

admits a solution and let us call $\tilde{\sigma}_{\varepsilon}$ a minimiser for (15). The main question is whether $\tilde{\sigma}_{\varepsilon}$ is a good approximation of the looked for conductivity tensor $\sigma_{0}$, namely we ask whether $\lim _{\varepsilon \rightarrow 0^{+}} \tilde{\sigma}_{\varepsilon}=\sigma_{0}$, where the limit is to be intended in a suitable sense. Unfortunately this may not be true, in fact our inverse problem is ill-posed, that is, we have no stability. There are two serious obstructions to stability. In the anisotropic case, that is, when $X=\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, for instance, the obstruction is due to invariance by changes of coordinates that
keep fixed the boundary. In the isotropic case, that is, when $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, the obstruction is due to the fact that this class is not closed with respect to $G$-convergence.

Let us illustrate these difficulties in the following three examples.
EXAMPLE 2.8. Let $\Omega=B_{1}(0) \subset \mathbb{R}^{2}$. Let $X=\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, for some $0<$ $\lambda_{0}<1<\lambda_{1}$ to be fixed later. We set $\tilde{\sigma}_{0} \equiv I_{2}$ in $B_{1}(0) \subset \mathbb{R}^{2}$. We fix a $C^{1}$ diffeomorphism $\varphi: B_{1}(0) \rightarrow B_{1}(0)$ such that $\varphi$ is identically equal to the identity in $B_{1}(0) \backslash B_{1 / 2}(0)$. We call $\sigma_{0}=\varphi_{*}\left(\tilde{\sigma}_{0}\right)$ and we assume that $\sigma_{0}$ is the conductivity tensor to be recovered. We notice that, if $\varphi$ is not trivial, we have that $\sigma_{0} \neq \tilde{\sigma}_{0}$.

Let $\tilde{\sigma}_{\varepsilon}, 0<\varepsilon \leq \varepsilon_{0}$, be a scalar conductivity satisfying $\left\|\tilde{\sigma}_{\varepsilon}-\tilde{\sigma}_{0}\right\|_{L^{\infty}(\Omega)} \leq \varepsilon$. We notice that, choosing in a suitable way $\lambda_{0}$ and $\lambda_{1}$, we have $\sigma_{0} \in X$, and, for any $0 \leq \varepsilon \leq \varepsilon_{0}$, also $\tilde{\sigma}_{\varepsilon} \in X$.

We notice that $\Lambda\left(\sigma_{0}\right)=\Lambda\left(\tilde{\sigma}_{0}\right)$ and, for some constant $C$, depending on $\lambda_{0}$, $\lambda_{1}$, and $Y$ only, we have

$$
\left\|\Lambda\left(\tilde{\sigma}_{\varepsilon}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y} \leq C \varepsilon
$$

If, for any $0<\varepsilon \leq \varepsilon_{0}$, we assume that $\hat{\Lambda}_{\varepsilon}=\Lambda\left(\tilde{\sigma}_{\varepsilon}\right)$, then, unfortunately, we have

$$
\tilde{\sigma}_{\varepsilon} \in \underset{\sigma \in X}{\arg \min }\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y},
$$

and, obviously, for any sequence $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \varepsilon_{0}\right]$ such that $\lim _{n} \varepsilon_{n}=0$, $\tilde{\sigma}_{\varepsilon_{n}}$ does not converge, not even in the $G$-convergence sense or in the weak $L^{1}$ sense, to $\sigma_{0}$.

In dimension 2, in [22], it has been proved that this is the only obstruction in the symmetric conductivity tensor case, if we consider the natural norms. Namely, from [22, Theorem 1.3], we can immediately deduce the following.

Proposition 2.9. Let $N=2$ and let $\Omega \subset \mathbb{R}^{2}$ be a bounded, simply connected open set with Lipschitz boundary. Let $\sigma_{0} \in X=\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$. We pick either $Y=\mathcal{L}\left(H_{*}^{1 / 2}(\partial \Omega), H_{*}^{-1 / 2}(\partial \Omega)\right)$, for the Dirichlet-to-Neumman case, or $Y=$ $\mathcal{L}\left(H_{*}^{-1 / 2}(\partial \Omega), H_{*}^{1 / 2}(\partial \Omega)\right)$, for the Neumann-to-Dirichlet case, respectively. For any $n \in \mathbb{N}$, let $\hat{\Lambda}_{n} \in Y$ be such that

$$
\left\|\hat{\Lambda}_{n}-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=\left\|\hat{\Lambda}_{n}-\Lambda\left(\sigma_{0}\right)\right\|_{n a t} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

Let $\sigma_{n} \in \arg \min _{\sigma \in X}\left\|\hat{\Lambda}_{n}-\Lambda\left(\sigma_{0}\right)\right\|_{Y}, n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, there exists a quasiconfomal mapping $\varphi_{n}: \Omega \rightarrow \Omega$ such that $\left.\left(\varphi_{n}\right)\right|_{\partial \Omega}=I d$ and

$$
\left(\varphi_{n}\right)_{*}\left(\sigma_{n}\right) \rightarrow \sigma_{0} \quad \text { as } n \rightarrow \infty
$$

in the $G$-convergence sense.

Proof. We have that

$$
\begin{aligned}
\left\|\Lambda\left(\sigma_{n}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y} \leq\left\|\hat{\Lambda}_{n}-\Lambda\left(\sigma_{n}\right)\right\|_{Y} & +\left\|\Lambda\left(\sigma_{0}\right)-\hat{\Lambda}_{n}\right\|_{Y} \\
& \leq 2\left\|\Lambda\left(\sigma_{0}\right)-\hat{\Lambda}_{n}\right\|_{Y} \rightarrow 0 \quad \text { as } n \rightarrow \infty
\end{aligned}
$$

Then the conclusion follows by [22, Theorem 1.3].
We notice that the kind of convergence we have in Proposition 2.9 is really weak, in several respects. First, it is only up to a change of variables, second it is in the sense of $G$-convergence, only. We recall that $G$-convergence does not imply convergence not even in the weak $L^{1}$ sense. In fact, let us consider the following example. Let $D$ be an open set such that $D \subset Q=(0,1)^{N} \subset \mathbb{R}^{N}$, $N \geq 2$, and let us consider, for two given constants $0<a<b$,

$$
\sigma= \begin{cases}a & \text { in } D  \tag{16}\\ b & \text { in } Q \backslash D .\end{cases}
$$

We also assume that $D$ and $Q \backslash D$ have positive measure. Then we have

$$
a<m_{h}=\left(\int_{Q} \sigma^{-1}\right)^{-1}<\int_{Q} \sigma=m<b
$$

where $m_{h}$ is the so-called harmonic mean of $\sigma$ on $Q$ and $m$ is the usual mean of $\sigma$ on $Q$.

We extend $\sigma$ all over $\mathbb{R}^{N}$ by periodicity and define, for any $\varepsilon>0$,

$$
\sigma_{\varepsilon}(x)=\sigma(x / \varepsilon) I_{N}, \quad x \in \mathbb{R}^{N}
$$

Given $\Omega$ a bounded connected open set with Lipschitz boundary, it is a classical fact in homogenisation theory that in $\Omega$

$$
\sigma_{\varepsilon} G \text {-converges to } \sigma_{\text {hom }} \text { as } \varepsilon \rightarrow 0^{+}
$$

where $\sigma_{h o m}$ is a constant symmetric matrix satisfying

$$
m_{h} I_{N} \leq \sigma_{\text {hom }}<m I_{N}
$$

On the other hand, $\sigma_{\varepsilon}$ converges to $a_{m} I_{N}$ in the weak $L^{\infty}(\Omega)$ sense, therefore also weakly in $L^{1}(\Omega)$.

Moreover, if $N=2$ and

$$
\begin{equation*}
D=\left\{\left(x_{1}, x_{2}\right) \in Q:\left(x_{1}-1 / 2\right)\left(x_{2}-1 / 2\right)>0\right\} \tag{17}
\end{equation*}
$$

then $\sigma_{h o m}$ can be computed explicitly and we have that $\sigma_{h o m}=\sqrt{a b} I_{2}$.

Instead, if $N=2$ and

$$
\begin{equation*}
D=\left\{\left(x_{1}, x_{2}\right) \in Q:\left(x_{1}-1 / 2\right)>0\right\} \tag{18}
\end{equation*}
$$

then also in this case $\sigma_{\text {hom }}$ can be computed explicitly and we have that

$$
\sigma_{h o m}=\left[\begin{array}{cc}
m_{h} & 0 \\
0 & m
\end{array}\right] .
$$

These explicit formulas are the bases for the next examples. The next one was introduced in [1] and we state it here. This and the next example show that in the scalar case, when, at least in dimension 2 , uniqueness is not an issue, instability phenomena may occur, no matter what we choose as $Y$.

Example 2.10. Let $\Omega=B_{1}(0) \subset \mathbb{R}^{2}$. Let $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, for some $0<$ $\lambda_{0}<1<\lambda_{1}$ to be fixed later. Let us assume that $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively.

We fix $N=2$ and two positive constants $0<a<b$. We take $Q=(0,1)^{2}$ and $D$ as in (17). We call

$$
\sigma_{0}= \begin{cases}I_{2} & \text { in } B_{1}(0) \backslash B_{1 / 2}(0) \\ \sqrt{a b} & \text { in } B_{1 / 2}(0)\end{cases}
$$

We define $\sigma$ as in (16), we extend it by periodicity all over $\mathbb{R}^{2}$, and define, for any $\varepsilon, 0<\varepsilon \leq 1 / 2$,

$$
\sigma_{\varepsilon}= \begin{cases}I_{2} & \text { in } B_{1}(0) \backslash B_{1 / 2}(0) \\ \sigma(x / \varepsilon) I_{N} & \text { if } x \in B_{1 / 2}(0)\end{cases}
$$

We have that $\sigma_{\varepsilon} G$-converges to $\sigma_{0}$ as $\varepsilon \rightarrow 0^{+}$, therefore, by Proposition 2.4, we immediately conclude that

$$
\left\|\Lambda\left(\tilde{\sigma}_{\varepsilon}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y} \rightarrow 0 \quad \text { as } \varepsilon \rightarrow 0^{+}
$$

Therefore, if $\sigma_{0}$ is the conductivity to be determined, and our measured data are $\hat{\Lambda}_{\varepsilon}=\Lambda\left(\sigma_{\varepsilon}\right)$, for any $\varepsilon \in(0,1 / 2]$, then we have that

$$
\left\{\sigma_{\varepsilon}\right\}=\underset{\sigma \in \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)}{\arg \min }\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}
$$

On the other hand, we have that $\sigma_{\varepsilon}$ converges to $m I_{N}$ in the weak* $L^{\infty}(\Omega)$ sense, therefore also weakly in $L^{1}(\Omega)$. Since $\sqrt{a b}<m$, we obtain that, as $\varepsilon \rightarrow 0^{+}, \sigma_{\varepsilon}$ does not converge to $\sigma_{0}$ even weakly in $L^{1}(\Omega)$, but only $G$-converges to $\sigma_{0}$.

The third and final example, inspired by the one in [1] we just presented, shows that even $G$-convergence may not be guaranteed.

Example 2.11. Let $\Omega=B_{1}(0) \subset \mathbb{R}^{2}$. Let $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, for some $0<$ $\lambda_{0}<1<\lambda_{1}$ to be fixed later. Let us assume that $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively.

We set the conductivity to be determined as $\sigma_{0} \equiv I_{2}$ in $B_{1}(0) \subset \mathbb{R}^{2}$. Let $\varphi: B_{1}(0) \rightarrow B_{1}(0)$ be a $C^{1}$ diffeomorphism such that $\varphi \equiv I d$ in $B_{1}(0) \backslash B_{1 / 2}(0)$ and $\varphi\left(x_{1}, x_{2}\right)=\left(x_{1} / 2, x_{2}\right)$ on $B_{1 / 4}(0)$. We call $\tilde{\sigma}_{0}=\varphi_{*}\left(\sigma_{0}\right)$. We have that $\tilde{\sigma}_{0} \neq \sigma_{0}$. In particular, $\tilde{\sigma}_{0}=I_{2}$ in $B_{1}(0) \backslash B_{1 / 2}(0)$ and $\tilde{\sigma}_{0}(y)=\left[\begin{array}{cc}1 / 2 & 0 \\ 0 & 2\end{array}\right]$ for any $y \in B_{1 / 8}(0)$.

We pick $Q=(0,1)^{2}, D$ as in (18), and $\sigma$ as in (16), with $a=2-\sqrt{3}$ and $b=2+\sqrt{3}$, so that $\sigma_{\text {hom }}=\left[\begin{array}{rrr}1 / 2 & 0 \\ 0 & 2\end{array}\right]$.

Then, again by density of scalar conductivities inside symmetric conductivity tensors, we can find $0<\lambda_{0}<\lambda_{1}$ and $\left\{\tilde{\sigma}_{n}\right\}_{n \in \mathbb{N}} \subset \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ such that $\tilde{\sigma}_{n} G$-converges to $\tilde{\sigma}_{0}$ as $n \rightarrow \infty$ and such that, for any $n \in \mathbb{N}$, $\tilde{\sigma}_{n}=I_{2}$ in $B_{1}(0) \backslash B_{1 / 2}(0)$ and

$$
\tilde{\sigma}_{n}(y)=\sigma(n y) \quad \text { for any } y \in B_{1 / 8}(0)
$$

where as usual $\sigma$ is extended by periodicity all over $\mathbb{R}^{2}$.
We notice that, as $n \rightarrow \infty$, in $B_{1 / 8}(0), \tilde{\sigma}_{n}$ converges to $2 I_{N}$ in the weak* $L^{\infty}$ sense, hence also weakly in $L^{1}\left(B_{1 / 8}(0)\right)$. Therefore, $\sigma_{n}$ can not converge, not even up to subsequences, to $\sigma_{0}$, not even weakly in $L^{1}\left(B_{1}(0)\right)$.

By Proposition 2.4, we immediately conclude that

$$
\left\|\Lambda\left(\tilde{\sigma}_{n}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=\left\|\Lambda\left(\tilde{\sigma}_{n}\right)-\Lambda\left(\tilde{\sigma}_{0}\right)\right\|_{Y} \rightarrow 0 \quad \text { as } n \rightarrow \infty
$$

If we pick as our measured data $\hat{\Lambda}_{n}=\Lambda\left(\tilde{\sigma}_{n}\right)$, for any $n \in \mathbb{N}$, then we have that

$$
\left\{\tilde{\sigma}_{n}\right\}=\underset{\sigma \in \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)}{\arg \min }\left\|\Lambda(\sigma)-\hat{\Lambda}_{n}\right\|_{Y}
$$

Then we have that, as $n \rightarrow \infty, \tilde{\sigma}_{n}$ can not converge, not even up to subsequences, to the looked for scalar conductivity $\sigma_{0}$ either in the $G$-convergence sense or locally weakly in $L^{1}$, hence, a fortiori, in the $L_{l o c}^{1}$ sense as well.

### 2.3. Regularisation

The issues for this inverse problem previous highlighted, in particular the illposedness, lead naturally to consider a suitable regularisation of the minimisation problem (12). To fix the ideas we consider a regularisation à la Tikhonov. For a general introduction to Tikhonov regularisation, we refer for instance to [20]. Here we are interested in the case of nonsmooth and possibly discontinuous unknown conductivity tensors, therefore we shall follow the approach
developed in [38]. We notice that, in the smooth case, the general theory for convergence of Tikhonov regularised solutions for nonlinear operators, as it was developed in [21], see also [20], may be used and leads also to convergence estimates. For example, for the electrical impedance tomography this approach was used in [31], see also [28].

Instead, in the nonsmooth case, our starting point is the regularisation strategy proved in [38], which we recall now. The key ingredient is $\Gamma$-convergence, see [17] for a detailed introduction. Here we just recall the definition and basic properties of $\Gamma$-convergence.

Let $(X, d)$ be a metric space. Then a sequence $F_{n}: X \rightarrow[-\infty,+\infty], n \in \mathbb{N}$, $\Gamma$-converges as $n \rightarrow \infty$ to a function $F: X \rightarrow[-\infty,+\infty]$ if for every $x \in X$ we have

$$
\begin{align*}
& \text { for every sequence }\left\{x_{n}\right\}_{n \in \mathbb{N}} \text { converging to } x \text { we have }  \tag{19}\\
& \qquad F(x) \leq \lim _{n} \inf F_{n}\left(x_{n}\right) ; \\
& \text { there exists a sequence }\left\{x_{n}\right\}_{n \in \mathbb{N}} \text { converging to } x \text { such that }  \tag{20}\\
& \qquad F(x)=\lim _{n} F_{n}\left(x_{n}\right)
\end{align*}
$$

The function $F$ will be called the $\Gamma$-limit of the sequence $\left\{F_{n}\right\}_{n \in \mathbb{N}}$ as $n \rightarrow \infty$ with respect to the metric $d$ and we denote it by $F=\Gamma-\lim _{n} F_{n}$. We recall that condition (19) above is usually called the $\Gamma$-liminf inequality, whereas condition (20) is usually referred to as the existence of a recovery sequence.

We say that the functionals $F_{n}, n \in \mathbb{N}$, are equicoercive if there exists a compact set $K \subset X$ such that $\inf _{K} F_{n}=\inf _{X} F_{n}$ for any $n \in \mathbb{N}$.

The following theorem, usually known as the Fundamental Theorem of $\Gamma$ convergence, illustrates the motivations for the definition of such a kind of convergence.

THEOREM 2.12. Let $(X, d)$ be a metric space and let $F_{n}: X \rightarrow[-\infty,+\infty]$, $n \in \mathbb{N}$, be a sequence of functions defined on $X$. If the functionals $F_{n}, n \in \mathbb{N}$, are equicoercive and $F=\Gamma-\lim _{n} F_{n}$, then $F$ admits a minimum over $X$ and we have

$$
\min _{X} F=\lim _{n} \inf _{X} F_{n} .
$$

Furthermore, if $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ is a sequence of points in $X$ which converges to a point $x \in X$ and satisfies $\lim _{n} F_{n}\left(x_{n}\right)=\lim _{n} \inf _{X} F_{n}$, then $x$ is a minimum point for $F$.

The definition of $\Gamma$-convergence may be extended in a natural way to families depending on a continuous parameter. The family of functions $F_{\varepsilon}$, defined for every $\varepsilon>0, \Gamma$-converges to a function $F$ as $\varepsilon \rightarrow 0^{+}$if for every sequence $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ of positive numbers converging to 0 as $n \rightarrow \infty$, we have $F=\Gamma-\lim _{n} F_{\varepsilon_{n}}$.

We begin with an abstract framework. We consider two metric spaces $\left(X, d_{X}\right)$ and $\left(Y, d_{Y}\right)$ and a continuous function $\Lambda: X \rightarrow Y$. We also fix $x_{0} \in X$ and $\Lambda_{0}=\Lambda\left(x_{0}\right) \in Y$.

For any $\varepsilon>0$, we consider a perturbation of $\Lambda_{0}$ given by $\Lambda_{\varepsilon} \in Y$ such that $d_{Y}\left(\Lambda_{\varepsilon}, \Lambda_{0}\right) \leq \varepsilon$. Here, and in the sequel, $\varepsilon$ plays the role of the noise level.

A function $R: X \rightarrow \mathbb{R} \cup\{+\infty\}$ is called a regularisation operator for the metric space $X$ if $R \not \equiv+\infty$ and, with respect to the metric induced by $d_{X}, R$ is a lower semicontinuous function such that for any constant $C>0$ the set $\{x \in X: R(x) \leq C\}$ is a compact subset of $X$.

We consider the following regularised minimum problem, for some $\varepsilon>0$,

$$
\begin{equation*}
\min _{x \in X}\left(d_{Y}\left(\Lambda(x), \Lambda_{\varepsilon}\right)\right)^{\alpha}+\tilde{a} R(x) \tag{21}
\end{equation*}
$$

where $\tilde{a}>0$ is the regularisation parameter and $\alpha$ is a positive parameter. In order to make the regularisation meaningful, we need to choose the regularisation parameter in terms of the noise level $\varepsilon$, namely we choose $\tilde{a}=\tilde{a}(\varepsilon)$. A solution to (21) will be called a regularised solution. To fix the ideas, given $\varepsilon_{0}>0$, we assume that for any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}, \tilde{a}(\varepsilon)=\tilde{a} \varepsilon^{\gamma}$, for some positive constants $\tilde{a}$ and $\gamma$. By a simple rescaling argument the minimisation problem (21) is equivalent to solve

$$
\begin{equation*}
\min _{x \in X} F_{\varepsilon}(x) \tag{22}
\end{equation*}
$$

where $F_{\varepsilon}: X \rightarrow \mathbb{R} \cup\{+\infty\}$ is defined as follows

$$
\begin{equation*}
F_{\varepsilon}(x)=\frac{\left(d_{Y}\left(\Lambda(x), \Lambda_{\varepsilon}\right)\right)^{\alpha}}{\varepsilon^{\gamma}}+\tilde{a} R(x) \quad \text { for any } x \in X \tag{23}
\end{equation*}
$$

We also define $F_{0}: X \rightarrow \mathbb{R} \cup\{+\infty\}$ as follows

$$
F_{0}(x)= \begin{cases}\tilde{a} R(x) & \text { if } \Lambda(x)=\Lambda\left(x_{0}\right)=\Lambda_{0} \text { in } Y  \tag{24}\\ +\infty & \text { otherwise }\end{cases}
$$

for any $x \in X$.
The following result is proved in [38], by exploiting $\Gamma$-convergence techniques.
Theorem 2.13. Let $\Lambda$ be continuous and $R$ be a regularisation operator for $X$. Let us also assume that $R\left(x_{0}\right)<+\infty$ and $\gamma<\alpha$.

Then we have that there exists $\min _{X} F_{\varepsilon}$, for any $\varepsilon, 0 \leq \varepsilon \leq \varepsilon_{0}$, and

$$
\min _{X} F_{0}=\lim _{\varepsilon \rightarrow 0^{+}} \min _{X} F_{\varepsilon}<+\infty
$$

Let $\left\{\tilde{x}_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ satisfy $\lim _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon}\left(\tilde{x}_{\varepsilon}\right)=\lim _{\varepsilon \rightarrow 0^{+}} \min _{X} F_{\varepsilon}$ (for example we may pick as $\left\{\tilde{x}_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ a family $\left\{x_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ of minimisers of $F_{\varepsilon}$ ).

Let $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0 as $n \rightarrow \infty$. Then, up to a subsequence, $\tilde{x}_{\varepsilon_{n}}$ converges to a point $\tilde{x} \in X$ such that $\tilde{x}$ is a minimiser of $F_{0}$, that is, in particular, $\Lambda(\tilde{x})=\Lambda\left(x_{0}\right)$ in $Y$ and $R(\tilde{x})=$ $\min \left\{R(x): x \in X\right.$ such that $\Lambda(x)=\Lambda\left(x_{0}\right)$ in $\left.Y\right\}$.

Furthermore, if $F_{0}$ admits a unique minimiser $\tilde{x}$, then we have that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0^{+}} \tilde{x}_{\varepsilon}=\tilde{x} \tag{25}
\end{equation*}
$$

Finally, if on the set $\{x \in X: R(x)<+\infty\}$ the map $\Lambda$ is injective, then we have

$$
\lim _{\varepsilon \rightarrow 0^{+}} \tilde{x}_{\varepsilon}=x_{0}
$$

even if we only have $\lim \sup _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon}\left(\tilde{x}_{\varepsilon}\right)<+\infty$.
Following again [38] we show the applicability of this abstract result to the inverse conductivity problem with discontinuous conductivities.

We observe that, in order to guarantee convergence of the regularised solutions to the looked for solution, we need to find a metric on the space $X$ such that the following properties are satisfied:

1) the forward operator $\Lambda$ is continuous;
2) $R$ is a regularisation operator for $X$;
3) $\Lambda$ is injective (uniqueness of the inverse problem).

We consider in this subsection $X$ equal to $\mathcal{M}\left(\lambda_{0}, \lambda_{1}\right)$, or $\mathcal{M}_{\text {sym }}\left(\lambda_{0}, \lambda_{1}\right)$, or $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$.

On $X$ we consider the metric given by the $L^{1}$ norm, in all cases. In fact we wish to have a convergence in a rather strong sense, being for instance $H$-convergence too weak for applications.

Therefore, we take as $Y$ the usual space where we assume that, for some $p>2$, in the Dirichlet-to-Neumann case, $B_{1} \subset W^{1-1 / p, p}(\partial \Omega)$, with continuous immersion, and, in the Neumann-to-Dirichlet case, we assume that $\tilde{B}_{1} \subset\left(W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)\right)_{*}^{\prime}$, with continuous immersion.

As a regularisation operator, there are several possibilities. One is to consider a kind of total variation regularisation. For instance, we define, for any $\sigma \in X, T V(\sigma)$ as the matrix such that $T V(\sigma)_{i j}=T V\left(\sigma_{i j}\right)=\left|D \sigma_{i j}\right|(\Omega)$ and set $|\sigma|_{B V(\Omega)}=\|T V(\sigma)\|$ for any $\sigma \in X$. For any $\sigma \in X$ we define

$$
\|\sigma\|_{B V(\Omega)}=\|\sigma\|_{L^{1}(\Omega)}+|\sigma|_{B V(\Omega)}
$$

Then we may pick as $R$ either $|\cdot|_{B V(\Omega)}$ or $\|\cdot\|_{B V(\Omega)}$.
The total variation regularisation has been widely used in the literature for solving numerically the inverse conductivity problem, for example in [19], with a discretisation method, and in $[13,15]$, with level set methods.

Another option is the so-called Mumford-Shah operator. In this case we limit ourselves to scalar conductivities, that is, to $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, and define, for any $\sigma \in \mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$,

$$
R(\sigma)= \begin{cases}b \int_{\Omega}\|\nabla \sigma\|^{2}+\mathcal{H}^{N-1}(J(\sigma)) & \text { if } \sigma \in S B V(\Omega)  \tag{26}\\ +\infty & \text { otherwise }\end{cases}
$$

Here $b$ is a positive constant, $\mathcal{H}^{N-1}$ denotes the ( $N-1$ )-dimensional Hausdorff measure, $J(\sigma)$ is the jump set of $\sigma$, and $S B V$ denotes the space of special functions of bounded variations. The functional $R$ here defined is referred to as the Mumford-Shah functional and was introduced in the context of image segmentation in [32]. We refer, for instance, to [5] for a detailed discussion on these topics. The compactness and semicontinuity theorem for special functions of bounded variation due to Ambrosio, see for instance [5, Theorem 4.7 and Theorem 4.8], guarantees that also in this case $R$ is a regularisation operator for $X$. In the context of inverse problems, and in particular for the inverse conductivity problem, the Mumford-Shah functional has been used as regularisation for the first time in [40], with an implementation exploiting the approximation of the Mumford-Shah functional by functionals defined on smoother functions due to Ambrosio and Tortorelli, $[6,7]$.

We now recall the results in [38], that immediately follows from the previous abstract results.

Theorem 2.14. Under the previous notation and assumptions, let $\Lambda: X \rightarrow Y$ be the forward operator. Let $R$ be either $|\cdot|_{B V(\Omega)}$ or $\|\cdot\|_{B V(\Omega)}$. If $X=$ $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right), R$ may be also chosen as in (26).

Let $\sigma_{0} \in X$ be such that $R\left(\sigma_{0}\right)<+\infty$ and $\hat{\Lambda}_{0}=\Lambda\left(\sigma_{0}\right)$. For any $\varepsilon, 0<\varepsilon \leq$ $\varepsilon_{0}$, let $\hat{\Lambda}_{\varepsilon} \in Y$ be such that $\left\|\hat{\Lambda}_{\varepsilon}-\hat{\Lambda}_{0}\right\| \leq \varepsilon$.

Let us fix positive constants $\alpha, \gamma$, and $\tilde{a}$, such that $0<\gamma<\alpha$. For any $\varepsilon$, $0<\varepsilon \leq \varepsilon_{0}$, let $F_{\varepsilon}$ be defined as in (23) and $F_{0}$ be defined as in (24).

Then we have that there exists $\min _{X} F_{\varepsilon}$, for any $\varepsilon, 0 \leq \varepsilon \leq \varepsilon_{0}$, and

$$
\min _{X} F_{0}=\lim _{\varepsilon \rightarrow 0^{+}} \min _{X} F_{\varepsilon}<+\infty
$$

Let $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0 as $n \rightarrow \infty$.
Let $\left\{\tilde{\sigma}_{n}\right\}_{n \in \mathbb{N}}$ be such that $\lim \sup _{n} F_{\varepsilon_{n}}\left(\tilde{\sigma}_{n}\right)<+\infty$. Then, up to a subsequence, $\tilde{\sigma}_{n}$ converges in the $L^{1}$ norm to $\tilde{\sigma} \in X$ such that $\tilde{\sigma}$ satisfies $\| \Lambda(\tilde{\sigma})-$ $\Lambda\left(\sigma_{0}\right) \|_{Y}=0$.

Let $\left\{\tilde{\sigma}_{n}\right\}_{n \in \mathbb{N}}$ be such that $\lim F_{\varepsilon_{n}}\left(\tilde{\sigma}_{n}\right)=\lim _{n} \min _{X} F_{\varepsilon_{n}}$. Then, up to a subsequence, $\tilde{\sigma}_{n}$ converges in the $L^{1}$ norm to $\tilde{\sigma} \in X$ such that $\tilde{\sigma}$ is a minimizer of $F_{0}$, that is, in particular, $\left\|\Lambda(\tilde{\sigma})-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$ and $R(\tilde{\sigma})=\min \{R(\sigma): \sigma \in$ $X$ such that $\left.\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0\right\}$.

In dimension 2 and for scalar conductivities we have the following.
Theorem 2.15. Under the notation and assumptions of Theorem 2.14, let us further assume that the space dimension is 2 , that is $N=2$. We pick $X=$ $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ and we assume that either $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively.

Let $\left\{\tilde{\sigma}_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ satisfy $\lim \sup _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon}\left(\tilde{\sigma}_{\varepsilon}\right)<+\infty$. Then we have that

$$
\lim _{\varepsilon \rightarrow 0^{+}} \int_{\Omega}\left|\tilde{\sigma}_{\varepsilon}-\sigma_{0}\right|=0
$$

We notice that, when $N \geq 3$, even if, recently, a great improvement has been achieved in the uniqueness issue, still we do not have a uniqueness result for scalar $B V$ or $S B V$ functions. To prove uniqueness, or nonuniqueness, in this case is an extremely interesting and challenging open problem.

We recall that the approach to regularisation developed in [38] has been followed in other works. In [27] the Mumford-Shah approach has been made slightly more precise, for instance it was proved convergence of the jump sets, and it has been applied to other inverse problems, such as image deblurring or X-ray tomography. In [28], instead, other regularisation strategies for the inverse conductivity problem have been considered, for example the sparsity or smoothness penalty was used. In this case the theory for convergence of Tikhonov regularised solutions for nonlinear operators may be used and, in fact, in [28] some convergence estimates were derived.

## 3. Numerical approximation and regularisation for the inverse conductivity problem

After the regularisation strategy has been decided, and it has been proved to be effective, the second step is to proceed in finding a suitable numerical approach to solve the regularised minimum problem. For example, in [40], the Ambrosio and Tortorelli approximation of the Mumford-Shah functional was used to tackle numerically the minimisation problem. For total variation regularisation, besides the early paper where a discretisation method, [19], or level set methods, [13, 15], were used, an interesting analysis of a finite element approximation has been developed in [23].

However, the approximations in [40] and in [23] have been performed just for the regularised minimum problem, that is, for a fixed regularisation parameter. Instead, we believe that it is very important to study how the approximation parameter (for example the size of the mesh in the finite element approximation) and the regularisation parameter interact. In other words, we wish to find, for a corresponding noise level $\varepsilon$, what are the right regularisation and
approximation parameters that allow to prove that the solutions to the approximated regularised minimum problems converge, in a suitable sense, to the looked for solution of the inverse problem. Therefore we wish to include in the convergence analysis developed in [38], and here recalled in Subsection 2.3, the approximation of the regularised minimum problem, simultaneously.

Such an approach has been developed for the Ambrosio-Tortorelli approximation of the Mumford-Shah functional in [14]. For the convenience of the reader we recall the result of [14] in Subsection 3.2.

In the next Subsection 3.1, we consider the approximation by finite element discretisation and we investigate how the discretisation parameter should be linked to the noise level and the regularisation parameter. We present here a very simple setting, in future work we will consider a much more general and complete discretisation of the inverse conductivity problem.

Let us begin by introducing the common setting for the whole section.
Throughout this section we fix $\Omega$, a bounded connected open set with Lipschitz boundary, contained in $\mathbb{R}^{N}, N \geq 2$, and two constants $\lambda_{0}$, $\lambda_{1}$, with $0<\lambda_{0} \leq \lambda_{1}$.

We consider only the case of scalar conductivities, namely we call $X=$ $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$.

We fix a real number $p>2$. In the Dirichlet-to-Neumann case, we assume that $B_{1} \subset W_{*}^{1-1 / p, p}(\partial \Omega)$ and $H_{*}^{-1 / 2}(\partial \Omega) \subset B_{2}$, with continuous immersions. In the Neumann-to-Dirichlet case, we assume that $\tilde{B}_{1} \subset\left(W^{1-1 / p^{\prime}, p^{\prime}}(\partial \Omega)\right)_{*}^{\prime}$ and $H_{*}^{1 / 2}(\partial \Omega) \subset \tilde{B}_{2}$, with continuous immersions.

In the Dirichlet-to-Neumann case, we call $Y=\mathcal{L}\left(B_{1}, B_{2}\right)$ and define $\Lambda$ : $X \rightarrow Y$ as follows

$$
\Lambda(\sigma)=\left.D N(\sigma)\right|_{B_{1}}: B_{1} \rightarrow B_{2}
$$

In the Neumann-to-Dirichlet case, we call $Y=\mathcal{L}\left(\tilde{B}_{1}, \tilde{B}_{2}\right)$ and define $\Lambda$ : $X \rightarrow Y$ as follows

$$
\Lambda(\sigma)=\left.N D(\sigma)\right|_{\tilde{B}_{1}}: \tilde{B}_{1} \rightarrow \tilde{B}_{2}
$$

The important fact is the following. We know that $\Lambda: X \rightarrow Y$ is Hölder continuous, that is, there exists constant $C_{0}>0$ and $\beta, 0<\beta<1$, such that, for any $\sigma, \tilde{\sigma} \in X$, we have

$$
\begin{equation*}
\|\Lambda(\sigma)-\Lambda(\tilde{\sigma})\|_{Y} \leq C_{0}\|\sigma-\tilde{\sigma}\|_{L^{1}(\Omega)}^{\beta} \tag{27}
\end{equation*}
$$

Here $\beta$ depends on $p$ only and it will play a crucial role in the next analysis.
We consider $\sigma_{0} \in X$ to be the scalar conductivity in $\Omega$ that we wish to determine and call $\hat{\Lambda}_{0}=\Lambda\left(\sigma_{0}\right) \in Y$.

Fixed a positive constant $\varepsilon_{0}$, for any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, let us assume that there exists $\hat{\Lambda}_{\varepsilon} \in Y$ such that

$$
\begin{equation*}
\left\|\hat{\Lambda}_{\varepsilon}-\hat{\Lambda}_{0}\right\|_{Y} \leq \varepsilon \tag{28}
\end{equation*}
$$

Here $\varepsilon$ plays the role of the noise level and $\hat{\Lambda}_{\varepsilon}$ plays the role of the measured Dirichlet-to-Neumann, or Neumann-to-Dirichlet respectively, map.

### 3.1. Discrete approximation and regularisation of the inverse conductivity problem

Since we wish to consider a discretisation of the problem, we shall make the following assumptions on $\Omega$. We further assume that $\Omega$ is polygonal, that is, $\Omega$ is a polyhedron in $\mathbb{R}^{N}$.

We use standard conforming piecewise linear finite elements, for which we refer for instance to [16]. We shall keep fixed a positive parameter $s$ and a positive constant $h_{0}$. We consider, for a fixed parameter $h, 0<h \leq h_{0}$, a triangulation $\mathcal{T}_{h}$ of $\bar{\Omega}$, that is, $\bar{\Omega}=\bigcup_{K \in \mathcal{T}_{h}} K$, where each $K \in \mathcal{T}_{h}$ is a nondegenerate $N$-simplex, and $\mathcal{T}_{h}$ satisfies assumption (FEM 1) in [16, Chapter 2]

We then define the finite element space $X_{h}$ as follows

$$
X_{h}=\left\{v_{h} \in C(\bar{\Omega}):\left.v_{h}\right|_{K} \in P_{1}(K) \text { for any } K \in \mathcal{T}_{h}\right\}
$$

where $P_{1}(K)$ is the space of polynomials of order at most 1 restricted to $K$, that is, $X_{h}$ is the finite element space associated to $N$-simplices of type (1). By [16, Theorem 2.2.3] we have that $X_{h} \subset C(\bar{\Omega}) \cap H^{1}(\Omega)$. It is also clear that $X_{0 h}=\left\{v_{h} \in X_{h}:\left.v_{h}\right|_{\partial \Omega}=0\right\}$ is contained in $H_{0}^{1}(\Omega)$. We call $\Pi_{h}$ the associated interpolation operator defined on $C(\bar{\Omega})$.

We assume that $\mathcal{T}_{h}$ is regular in the following classical sense. For any $K \in \mathcal{T}_{h}$ we call $h_{K}=\operatorname{diam}(K)$ and $\rho_{K}=\sup \{\operatorname{diam}(B): B$ is a ball contained in $K\}$. Then we assume that

$$
\begin{equation*}
h_{K} \leq h \text { and } h_{K} \leq s \rho_{K} \quad \text { for any } K \in \mathcal{T}_{h} . \tag{29}
\end{equation*}
$$

The following estimate is an immediate consequence of [16, Theorem 3.1.6]. Theorem 3.1. Let us consider $q \geq 2$ such that $q>N / 2$. Then there exists a constant $C$ such that for any $u \in \bar{W}^{2, q}(\Omega)$ we have

$$
\begin{equation*}
\left\|u-\Pi_{h} u\right\|_{W^{1, q}(\Omega)} \leq C h\left\|D^{2} u\right\|_{L^{q}(\Omega)} \tag{30}
\end{equation*}
$$

Our approach to discretisation is the following. As a regularisation operator we consider a total variation penalisation, that is $R$ is given by, for any $\sigma \in X$,

$$
\begin{equation*}
R(\sigma)=|\sigma|_{B V(\Omega)}=T V(\sigma)=|D \sigma|(\Omega) \quad \text { or } \quad R(\sigma)=\|\sigma\|_{B V(\Omega)} \tag{31}
\end{equation*}
$$

Furthermore we shall assume that $R\left(\sigma_{0}\right)<+\infty$, that is, $\sigma_{0} \in B V(\Omega)$.
For fixed $\tilde{a}, 0<\gamma<\alpha$, let us define, for any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, and $h$, $0<h \leq h_{0}$, the functional $F_{\varepsilon, h}: X \rightarrow \mathbb{R} \cup\{+\infty\}$ such that for any $\sigma \in X$

$$
F_{\varepsilon, h}(\sigma)= \begin{cases}\frac{\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}^{\alpha}}{\varepsilon^{\gamma}}+\tilde{a} R(\sigma) & \text { if } \sigma \in X_{h}  \tag{32}\\ +\infty & \text { otherwise }\end{cases}
$$

Let us immediately notice that any of these functionals $F_{\varepsilon, h}$ admits a minimum over $X$.

We also define $F_{0}: X \rightarrow \mathbb{R} \cup\{+\infty\}$ as before

$$
F_{0}(\sigma)= \begin{cases}\tilde{a} R(\sigma) & \text { if }\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0  \tag{33}\\ +\infty & \text { otherwise }\end{cases}
$$

for any $\sigma \in X$.
Our aim is to choose $h=h(\varepsilon)$ such that $F_{\varepsilon, h}$ are equicoercive and $\Gamma$ converge, as $\varepsilon \rightarrow 0^{+}$, to $F_{0}$.

Therefore, let us consider two sequences $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \varepsilon_{0}\right]$ and $\left\{h_{n}\right\}_{n \in \mathbb{N}} \subset$ $\left(0, h_{0}\right]$ and we assume that $\lim _{n} \varepsilon_{n}=0$. We define $F_{n}=F_{\varepsilon_{n}, h_{n}}$.

The $\Gamma$-liminf inequality is easy to prove. In fact we have the following.
Proposition 3.2. Let $\left\{\sigma_{n}\right\}_{n \in \mathbb{N}} \subset X$ be such that $\lim _{n} \sigma_{n}=\sigma$ in $X$, that is, $\lim _{n}\left\|\sigma_{n}-\sigma\right\|_{L^{1}(\Omega)}=0$.

Then

$$
F_{0}(\sigma) \leq \liminf _{n} F_{n}\left(\sigma_{n}\right)
$$

Proof. If $\lim \inf _{n} F_{n}\left(\sigma_{n}\right)=+\infty$, then there is nothing to prove. We therefore assume, without loss of generality, that $\liminf _{n} F_{n}\left(\sigma_{n}\right)=\lim _{n} F_{n}\left(\sigma_{n}\right)<+\infty$. In particular, for some constant $C$, we have $F_{n}\left(\sigma_{n}\right) \leq C$ for any $n \in \mathbb{N}$. Therefore, $\sigma_{n} \in X_{h_{n}}$ for any $n \in \mathbb{N}$.

By semicontinuity of the total variation, it is easy to see that

$$
\tilde{a} R(\sigma) \leq \liminf _{n}\left(\tilde{a} R\left(\sigma_{n}\right)\right) \leq \liminf _{n} F_{n}\left(\sigma_{n}\right)
$$

It remains to prove that $\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$. But, by continuity of $\Lambda$, it is easy to see that

$$
\begin{aligned}
& \left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=\lim _{n}\left\|\Lambda\left(\sigma_{n}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y} \\
& \quad \leq \liminf _{n}\left[\left\|\Lambda\left(\sigma_{n}\right)-\hat{\Lambda}_{\varepsilon_{n}}\right\|_{Y}+\left\|\hat{\Lambda}_{\varepsilon_{n}}-\Lambda\left(\sigma_{0}\right)\right\|_{Y}\right] \leq \liminf _{n}\left[\left(C \varepsilon_{n}^{\gamma}\right)^{1 / \alpha}+\varepsilon_{n}\right]
\end{aligned}
$$

which is obviously equal to 0 .
The difficult part is to find a recovery sequence. Clearly the existence of the recovery sequence is trivial, by the $\Gamma$-liminf inequality, if $F_{0}(\sigma)=+\infty$. Therefore, it is enough to prove the existence of a recovery sequence when $F_{0}(\sigma)$ is finite.

Proposition 3.3. We define $h(\varepsilon)=\varepsilon^{3 / \beta}$, for any $\varepsilon$, $0<\varepsilon \leq \varepsilon_{0}$, and recall that $\gamma<\alpha$.

Let $\sigma \in X$ be such that $F_{0}(\sigma)<+\infty$, that is, $\sigma \in B V(\Omega) \cap X$ and it satisfies $\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$.

Then there exists $\sigma_{\varepsilon} \in X$, for any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, such that

$$
F_{0}(\sigma)=\lim _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon, h(\varepsilon)}\left(\sigma_{\varepsilon}\right)
$$

Before proving this proposition, let us observe that it implies the following corollary.

Corollary 3.4. Under the notation and assumptions of Proposition 3.3, we have that $F_{\varepsilon, h(\varepsilon)} \Gamma$-converges to $F_{0}$ as $\varepsilon \rightarrow 0^{+}$.

Moreover, the family of functionals $\left\{F_{\varepsilon, h(\varepsilon)}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ is equicoercive.
Proof. The $\Gamma$-convergence result follows immediately from Propositions 3.2 and 3.3.

About equicoerciveness, we start with the following remark. By Proposition 3.3 , we can find a constant $C$ such that

$$
\begin{equation*}
\min _{X} F_{\varepsilon, h(\varepsilon)} \leq C \quad \text { for any } 0<\varepsilon \leq \varepsilon_{0} \tag{34}
\end{equation*}
$$

Then we define $K=\left\{\sigma_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$, where $\sigma_{\varepsilon}$ is a minimiser for $F_{\varepsilon, h(\varepsilon)}$, for any $0<\varepsilon \leq \varepsilon_{0}$. We prove that $K$ is relatively compact in $X$. In fact, by (34), we obtain that, for some constant $C_{1}, R\left(\sigma_{\varepsilon}\right) \leq C_{1}$ for any $0<\varepsilon \leq \varepsilon_{0}$. Then the fact that $K$ is relatively compact follows immediately by the properties of the regularisation operator $R$.

We now complete the proof of the existence of the recovery sequence.
Proof of Proposition 3.3. The difficult part is that we need to build the function $\sigma_{\varepsilon}$ in such a way that it belongs to the discrete space $X_{h(\varepsilon)}$, for any $0<\varepsilon \leq \varepsilon_{0}$.

The construction is the following. First of all we use the fact that $\Omega$ is an extension domain, since it has Lipschitz boundary. Therefore, for any $u \in$ $B V(\Omega) \cap X$, we can find a function $\tilde{u} \in L^{\infty}\left(\mathbb{R}^{N}\right)$ such that $\left.\tilde{u}\right|_{\Omega}=u, \lambda_{0} \leq \tilde{u} \leq \lambda_{1}$ almost everywhere in $\mathbb{R}^{N}$, and, for a constant $C$ depending on $\Omega$ only,

$$
|D \tilde{u}|\left(\mathbb{R}^{N}\right) \leq C|D u|(\Omega),
$$

and, moreover, $|D \tilde{u}|(\partial \Omega)=0$. This follows immediately by using [5, Definition 3.20], for instance.

We consider our function $\sigma$ and, by a slight abuse of notation, we still call $\sigma$ its extension $\tilde{\sigma}$ to the whole $\mathbb{R}^{N}$. We fix a positive symmetric mollifier $\eta$, that is, $\eta \in C_{0}^{\infty}\left(B_{1}(0)\right), \eta \geq 0, \int_{B_{1}(0)} \eta=1$, and such that $\eta(x)$ depends only
on $\|x\|$ for any $x \in B_{1}(0)$. Clearly $\eta \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ by extending it to 0 oustide $B_{1}(0)$. For any $\delta>0$, we call

$$
\eta_{\delta}(x)=\delta^{-N} \eta(x / \delta), \quad x \in \mathbb{R}^{N}
$$

and, for any $u \in L_{l o c}^{1}\left(\mathbb{R}^{N}\right)$, we call

$$
u_{\delta}=\eta_{\delta} * u
$$

where as usual $*$ denotes the convolution.
We immediately obtain that, for any $\delta>0, \sigma_{\delta} \in C^{\infty}\left(\mathbb{R}^{N}\right)$ and $\lambda_{0} \leq \sigma_{\delta} \leq \lambda_{1}$ almost everywhere in $\mathbb{R}^{N}$. We also have that, locally, $\sigma_{\delta}$ converges to $\sigma$ as $\delta \rightarrow 0^{+}$in the $L^{1}$ norm. By [24, Proposition 1.15], we conclude that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0^{+}}\left\|\sigma_{\delta}-\sigma\right\|_{L^{1}(\Omega)}=0 \quad \text { and } \quad \lim _{\delta \rightarrow 0^{+}}\left|D \sigma_{\delta}\right|(\Omega)=|D \sigma|(\Omega) \tag{35}
\end{equation*}
$$

Actually, by [5, Lemma 3.24], the $L^{1}$ convergence may be made much more precise. In fact, for a constant $C_{1}$ depending on $\Omega$ only, we have, for any $\delta$, $0<\delta \leq 1$,

$$
\begin{equation*}
\left\|\sigma-\sigma_{\delta}\right\|_{L^{1}(\Omega)} \leq C_{1}|D \sigma|(\Omega) \delta \tag{36}
\end{equation*}
$$

We choose $q$ as in Theorem 3.1. Since $\sigma_{\delta} \in C^{\infty}\left(\mathbb{R}^{N}\right)$, we obviously have that $\sigma_{\delta} \in W^{2, q}(\Omega)$, for any $\delta>0$. We need to control its norm in dependence of $\delta$. We notice that, for any multiindex $\alpha$, we have $D^{\alpha} \sigma_{\delta}=\left(D^{\alpha} \eta_{\delta}\right) * \sigma$. Therefore, for any $\delta, 0<\delta \leq 1$, and any $p, 1 \leq p \leq+\infty$,

$$
\left\|D^{\alpha} \sigma_{\delta}\right\|_{L^{p}(\Omega)} \leq C_{2} \delta^{-|\alpha|}
$$

where $C_{2}$ depends on $\Omega, p,|\alpha|, \eta$, and $\lambda_{1}$ only. We conclude that, for a constant $C_{3}$ depending on $\Omega, q, \eta$, and $\lambda_{1}$ only, we have, for any $0<\delta \leq 1$,

$$
\begin{equation*}
\left\|\sigma_{\delta}\right\|_{W^{2, q}(\Omega)} \leq C_{3} \delta^{-2} \tag{37}
\end{equation*}
$$

By Theorem 3.1, we obtain that

$$
\begin{equation*}
\left\|\sigma_{\delta}-\Pi_{h}\left(\sigma_{\delta}\right)\right\|_{W^{1, q}(\Omega)} \leq C_{4} h \delta^{-2} \tag{38}
\end{equation*}
$$

where $C_{4}=C_{3} C$, with $C$ as in (30). We have that $\Pi_{h}\left(\sigma_{\delta}\right) \in X_{h}$. Furthermore,

$$
\begin{aligned}
\left\|\sigma-\Pi_{h}\left(\sigma_{\delta}\right)\right\|_{L^{1}(\Omega)} \leq\left\|\sigma-\sigma_{\delta}\right\|_{L^{1}(\Omega)}+\| \sigma_{\delta}- & \Pi_{h}\left(\sigma_{\delta}\right) \|_{L^{1}(\Omega)} \\
& \leq C_{1}|D \sigma|(\Omega) \delta+C_{5} C_{4} h \delta^{-2}
\end{aligned}
$$

with $C_{5}$ depending on $\Omega$ and $q$ only. By picking $\delta=h^{1 / 3}$, we conclude that, for the constant $C_{6}=C_{1}|D \sigma|(\Omega)+C_{5} C_{4}$,

$$
\begin{equation*}
\left\|\sigma-\Pi_{h}\left(\sigma_{\delta}\right)\right\|_{L^{1}(\Omega)} \leq C_{6} h^{1 / 3} \tag{39}
\end{equation*}
$$

Furthermore,

$$
\begin{aligned}
&\left|D\left(\Pi_{h}\left(\sigma_{\delta}\right)\right)\right|(\Omega)=\int_{\Omega}\left\|\nabla\left(\Pi_{h}\left(\sigma_{\delta}\right)\right)\right\| \\
&=\left(\int_{\Omega}\left\|\nabla\left(\Pi_{h}\left(\sigma_{\delta}\right)\right)\right\|-\int_{\Omega}\left\|\nabla \sigma_{\delta}\right\|\right)+\int_{\Omega}\left\|\nabla \sigma_{\delta}\right\|
\end{aligned}
$$

The first term of the right hand side goes to 0 , as $h$, and thus $\delta$, goes to 0 , by (38). The second term of the right hand side is exactly $\left|D \sigma_{\delta}\right|(\Omega)$, therefore it goes to $|D \sigma|(\Omega)$, as $h$ goes to 0 , by (35).

We have therefore constructed, for any $0<h \leq h_{0}, \sigma_{h} \in X_{h}$ such that

$$
\begin{equation*}
\left\|\sigma-\sigma_{h}\right\|_{L^{1}(\Omega)} \leq C_{6} h^{1 / 3} \quad \text { and } \quad \lim _{h \rightarrow 0^{+}}\left|D \sigma_{h}\right|(\Omega)=|D \sigma|(\Omega) \tag{40}
\end{equation*}
$$

By (27), we conclude that

$$
\begin{equation*}
\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{h}\right)\right\|_{Y} \leq C_{0} C_{6}^{\beta} h^{\beta / 3} \quad \text { and } \quad \lim _{h \rightarrow 0^{+}} R\left(\sigma_{h}\right)=R(\sigma) \tag{41}
\end{equation*}
$$

Then we easily compute, since $\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$,

$$
\begin{aligned}
& \left\|\Lambda\left(\sigma_{h}\right)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}^{\alpha} \leq\left(\left\|\Lambda\left(\sigma_{h}\right)-\Lambda(\sigma)\right\|_{Y}+\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}\right)^{\alpha} \\
& \qquad\left(C_{0} C_{6}^{\beta} h^{\beta / 3}+\varepsilon\right)^{\alpha}
\end{aligned}
$$

If we choose $\gamma<\alpha$ and $h(\varepsilon)$ such that $h(\varepsilon)=\varepsilon^{3 / \beta}$, then we obtain that

$$
\lim _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon, h(\varepsilon)}\left(\sigma_{h(\varepsilon)}\right)=F_{0}(\sigma)
$$

The proof is concluded.
By Corollary 3.4 and the Fundamental Theorem of $\Gamma$-convergence, Theorem 2.12, the next theorems, which are the main results of the paper, immediately follow.

TheOrem 3.5. Under the previous notation and assumptions, we consider $X=$ $\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$ and let $\Lambda: X \rightarrow Y$ be the forward operator. Let $R$ be either $|\cdot|_{B V(\Omega)}$ or $\|\cdot\|_{B V(\Omega)}$.

Let $\sigma_{0} \in X$ be such that $R\left(\sigma_{0}\right)<+\infty$ and $\hat{\Lambda}_{0}=\Lambda\left(\sigma_{0}\right)$. For any $\varepsilon, 0<\varepsilon \leq$ $\varepsilon_{0}$, let $\hat{\Lambda}_{\varepsilon} \in Y$ be such that $\left\|\hat{\Lambda}_{\varepsilon}-\hat{\Lambda}_{0}\right\| \leq \varepsilon$.

Let us fix positive constants $\alpha, \gamma$, and $\tilde{a}$, such that $0<\gamma<\alpha$. For any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, let $h=h(\varepsilon)$ be given by $h(\varepsilon)=\varepsilon^{3 / \beta}$, and $F_{\varepsilon, h(\varepsilon)}$ be defined as in (32) and $F_{0}$ be defined as in (33).

Then we have that there exists $\min _{X} F_{\varepsilon, h(\varepsilon)}$, for any $\varepsilon, 0 \leq \varepsilon \leq \varepsilon_{0}$, and

$$
\min _{X} F_{0}=\lim _{\varepsilon \rightarrow 0^{+}} \min _{X} F_{\varepsilon, h(\varepsilon)}<+\infty
$$

Let $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0 as $n \rightarrow \infty$.
Let $\left\{\tilde{\sigma}_{n}\right\}_{n \in \mathbb{N}}$ be such that $\limsup { }_{n} F_{\varepsilon_{n}, h\left(\varepsilon_{n}\right)}\left(\tilde{\sigma}_{n}\right)<+\infty$. Then, up to a subsequence, $\tilde{\sigma}_{n}$ converges in the $L^{1}$ norm to $\tilde{\sigma} \in X$ such that $\tilde{\sigma}$ satisfies $\left\|\Lambda(\tilde{\sigma})-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$.

Let $\left\{\tilde{\sigma}_{n}\right\}_{n \in \mathbb{N}}$ be such that $\lim _{n} F_{\varepsilon_{n}, h\left(\varepsilon_{n}\right)}\left(\tilde{\sigma}_{n}\right)=\lim _{n} \min _{X} F_{\varepsilon_{n}, h(\varepsilon)}$. Then, up to a subsequence, $\tilde{\sigma}_{n}$ converges in the $L^{1}$ norm to $\tilde{\sigma} \in X$ such that $\tilde{\sigma}$ is a minimiser of $F_{0}$, that is, in particular, $\left\|\Lambda(\tilde{\sigma})-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$ and $R(\tilde{\sigma})=$ $\min \left\{R(\sigma): \sigma \in X\right.$ such that $\left.\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0\right\}$.

In the two dimensional case, as before, the result may be made more precise.
ThEOREM 3.6. Under the notation and assumptions of Theorem 3.5, let us further assume that the space dimension is 2 , that is $N=2$. We assume that either $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively.

Let $\left\{\tilde{\sigma}_{\varepsilon}\right\}_{0<\varepsilon \leq \varepsilon_{0}}$ satisfy $\lim \sup _{\varepsilon \rightarrow 0^{+}} F_{\varepsilon, h(\varepsilon)}\left(\tilde{\sigma}_{\varepsilon}\right)<+\infty$. Then we have that

$$
\lim _{\varepsilon \rightarrow 0^{+}} \int_{\Omega}\left|\tilde{\sigma}_{\varepsilon}-\sigma_{0}\right|=0
$$

### 3.2. Regularisation by the Ambrosio-Tortorelli functionals

In this subsection we present the approach to regularisation by using the socalled Ambrosio-Tortorelli functionals that was developed in [14]. These functionals were introduced in $[6,7]$ in order to solve numerically the difficult task of minimising the Mumford-Shah functional. In fact the Ambrosio-Tortorelli functionals are a good approximation, in the $\Gamma$-convergence sense, of the MumfordShah functional and they are much easier to compute with.

We recall that $\Omega$ is a fixed bounded connected open set with Lipschitz boundary, contained in $\mathbb{R}^{N}, N \geq 2$. We consider only the case of scalar conductivities, namely we call $X=\mathcal{M}_{\text {scal }}\left(\lambda_{0}, \lambda_{1}\right)$, for two constants $\lambda_{0}, \lambda_{1}$, with $0<\lambda_{0} \leq \lambda_{1}$.

Let us begin with the following definition. We fix a continuous function $V: \mathbb{R} \rightarrow \mathbb{R}$ such that $V \geq 0$ everywhere in $\mathbb{R}$ and $V(t)=0$ if and only if $t=1$. We call $c_{V}=\int_{0}^{1} \sqrt{V(t)} \mathrm{d} t$. Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a lower semicontinuous, nondecreasing function such that $\psi(0)=0, \psi(1)=1$, and $\psi(t)>0$ for any $t>0$. For any $\eta>0$, we fix $o_{\eta} \geq 0$ such that $\lim _{\eta \rightarrow 0^{+}} o_{\eta} / \eta=0$, and we call $\psi_{\eta}=\psi+o_{\eta}$. Given a positive parameter $b$, and for any $\eta>0$, we define the functional $A T_{\eta}: L^{1}(\Omega) \times L^{1}(\Omega) \rightarrow[0,+\infty]$ as follows, for any

$$
\begin{align*}
& (u, v) \in L^{1}(\Omega) \times L^{1}(\Omega), \\
& = \begin{cases}\int_{\Omega}\left(b \psi_{\eta}(v)\|\nabla u\|^{2}+\frac{1}{\eta} V(v)+\eta\|\nabla v\|^{2}\right) & \text { if } u \in H^{1}(\Omega) \cap X \\
\text { and } v \in H^{1}(\Omega,[0,1]) \\
+\infty & \text { otherwise. }\end{cases}
\end{align*}
$$

Here $H^{1}(\Omega,[0,1])=\left\{v \in H^{1}(\Omega): 0 \leq v \leq 1\right.$ a.e. in $\left.\Omega\right\}$.
We define a new version of the Mumford-Shah functional as follow. We call $M S: L^{1}(\Omega) \times L^{1}(\Omega) \rightarrow[0,+\infty]$ the functional such that, for any $(u, v) \in$ $L^{1}(\Omega) \times L^{1}(\Omega)$,

$$
M S(u, v)=\left\{\begin{array}{lc}
b \int_{\Omega}\|\nabla u\|^{2}+4 c_{V} \mathcal{H}^{N-1}(J(u)) & \text { if } u \in S B V(\Omega) \cap X  \tag{43}\\
+\infty & \text { and } v=1 \text { a.e. in } \Omega \\
\text { otherwise. }
\end{array}\right.
$$

Notice that here $v$ just plays the role of a formal variable.
We have the following result.
Theorem 3.7. We have that, as $\eta \rightarrow 0^{+}, A T_{\eta} \Gamma$-converges to $M S$ in the $L^{1}(\Omega) \times L^{1}(\Omega)$ distance.

Moreover, we assume that, for a positive constant $C_{0}$, we have $\psi(t) \geq C_{0} t^{2}$ for any $t \in[0,1]$. We consider two sequences $\left\{\eta_{n}\right\}_{n \in \mathbb{N}} \subset(0,1]$, such that $\lim _{n} \eta_{n}=0$, and $\left\{\left(u_{n}, v_{n}\right)\right\}_{n \in \mathbb{N}} \subset L^{1}(\Omega) \times L^{1}(\Omega)$. If there exists a constant $C$ such that $A T_{\eta_{n}}\left(u_{n}, v_{n}\right) \leq C$ for any $n \in \mathbb{N}$, then, as $n \rightarrow \infty$, $v_{n}$ converges to $v \equiv 1$ in $L^{1}(\Omega)$ and, up to a subsequence, $u_{n}$ converges to $u \in X$ in $L^{1}(\Omega)$.

Proof. The $\Gamma$-convergence follows from [6, 7], see also [10].
For the compactness result of the second part, the argument is the following. The fact that $\lim _{n} v_{n}=v \equiv 1$ in $L^{1}(\Omega)$ is trivial. For the compactness of the sequence $\left\{u_{n}\right\}_{n \in \mathbb{N}}$, first of all we notice that $\lambda_{0} \leq u_{n} \leq \lambda_{1}$ for any $n \in \mathbb{N}$. We call $\tilde{V}(t)=\int_{0}^{t} \sqrt{V(s)} \mathrm{d} s$, for any $t \in[0,1]$. We notice that, for any $t \in[0,1]$, we have $c_{1} t \leq \tilde{V}(t) \leq C_{1} t$, for some constants $0<c_{1}<C_{1}$. Therefore, for any $t \in(0,1]$, we have

$$
\frac{\tilde{V}(t)}{\sqrt{\psi(t)}} \leq C_{2}
$$

For any $n \in \mathbb{N}$, we define the auxiliary function $w_{n}=\tilde{V}\left(v_{n}\right) u_{n}$ and notice that $\left\|w_{n}\right\|_{L^{\infty}(\Omega)}$ is uniformly bounded. Then $\nabla w_{n}=\sqrt{V\left(v_{n}\right)} u_{n} \nabla v_{n}+$
$\tilde{V}\left(v_{n}\right) \nabla u_{n}$. We obtain that

$$
\begin{aligned}
\int_{\Omega}\left\|\nabla w_{n}\right\| & \leq\left\|u_{n}\right\|_{L^{\infty}(\Omega)}\left(\int_{\Omega} \frac{1}{\eta_{n}} V\left(v_{n}\right)\right)^{1 / 2}\left(\int_{\Omega} \eta_{n}\left\|\nabla v_{n}\right\|^{2}\right)^{1 / 2} \\
& \left.+\left(\int_{\left\{x \in \Omega: v_{n}(x)>0\right\}}\left(\frac{\tilde{V}\left(v_{n}\right)}{\sqrt{\psi\left(v_{n}\right)}}\right)\right)^{2}\right)^{1 / 2}\left(\int_{\Omega} \psi\left(v_{n}\right)\left\|\nabla u_{n}\right\|^{2}\right)^{1 / 2} .
\end{aligned}
$$

We easily conclude that $\left\{w_{n}\right\}_{n \in \mathbb{N}}$ is bounded in $W^{1,1}(\Omega)$, therefore, up to a subsequence that we do not relabel, we have that $w_{n} \rightarrow w \in L^{1}(\Omega)$ and $v_{n} \rightarrow v \equiv 1$, in both cases in $L^{1}(\Omega)$ and almost everywhere in $\Omega$. For almost $\underset{\tilde{V}}{ }$ ny $x \in \Omega$, we have that, as $n \rightarrow \infty, w_{n}(x) \rightarrow w(x)$ and $v_{n}(x) \rightarrow 1$, thus $\tilde{V}\left(v_{n}(x)\right) \rightarrow \tilde{V}(1)>0$. Therefore, for any of these $x \in \Omega$, we have $\lim _{n} u_{n}(x)=w(x) / \tilde{V}(1)=u(x)$. We conclude that $u \in X$ and that, up to the same subsequence, as $n \rightarrow \infty, u_{n}$ converges to $u$ almost everywhere in $\Omega$, thus, by the uniform $L^{\infty}$ bound and the Lebesgue theorem, in $L^{1}(\Omega)$ as well.

We now consider the following definition. For fixed $\tilde{a}, 0<\gamma<\alpha$, let us define, for any $\varepsilon, 0<\varepsilon \leq \varepsilon_{0}$, and $\eta, 0<\eta \leq \eta_{0}$, the functional $F_{\varepsilon, \eta}$ : $X \times L^{1}(\Omega) \rightarrow \mathbb{R} \cup\{+\infty\}$ such that, for any $(\sigma, v) \in X \times L^{1}(\Omega)$, we have

$$
\begin{equation*}
F_{\varepsilon, \eta}(\sigma, v)=\frac{\left\|\Lambda(\sigma)-\hat{\Lambda}_{\varepsilon}\right\|_{Y}^{\alpha}}{\varepsilon^{\gamma}}+\tilde{a} A T_{\eta}(\sigma, v) \tag{44}
\end{equation*}
$$

We also define $F_{0}: X \times L^{1}(\Omega) \rightarrow \mathbb{R} \cup\{+\infty\}$ as follows, for any $(\sigma, v) \in$ $X \times L^{1}(\Omega)$,

$$
F_{0}(\sigma, v)= \begin{cases}\tilde{a} M S(\sigma, v) & \text { if }\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0  \tag{45}\\ +\infty & \text { otherwise }\end{cases}
$$

where $M S$ is defined in (43). We notice that, equivalently, we can consider $\tilde{F}_{0}: X \rightarrow \mathbb{R} \cup\{+\infty\}$ such that, for any $\sigma \in X$,

$$
\begin{align*}
& \tilde{F}_{0}(\sigma)= \\
& =\left\{\begin{array}{lc}
\tilde{a}\left(b \int_{\Omega}\|\nabla \sigma\|^{2}+4 c_{V} \mathcal{H}^{N-1}(J(\sigma))\right) & \text { if } \sigma \in S B V(\Omega) \cap X \\
+\infty & \text { and }\left\|\Lambda(\sigma)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0
\end{array}\right.  \tag{46}\\
& =\text { otherwise. }
\end{align*}
$$

Remark 3.8. We notice that $F_{0}$, or equivalently $\tilde{F}_{0}$, admits a minimum over $X \times L^{1}(\Omega)$, or $X$ respectively. Notice that $(\tilde{\sigma}, \tilde{v})$ is a minimiser for $F_{0}$ if and only if $\tilde{\sigma}$ is a minimiser for $\tilde{F}_{0}$ and $\tilde{v} \equiv 1$. Moreover, any of these functionals $F_{\varepsilon, \eta}$ admits a minimum over $X \times L^{1}(\Omega)$ provided $o_{\eta}>0$.

We shall need the following definition.
Definition 3.9. For any Borel set $E \subset \mathbb{R}^{N}$, we define its $(N-1)$-dimensional Minkowski content as

$$
\mathcal{M}^{N-1}(E)=\lim _{\delta \rightarrow 0^{+}} \frac{1}{2 \delta}\left|\left\{x \in \mathbb{R}^{N}: \operatorname{dist}(x, E)<\delta\right\}\right|
$$

provided the limit exists.
We say that a conductivity $\sigma \in X$ is admissible if $\sigma \in S B V(\Omega)$, and it satisfies

$$
\int_{\Omega}\|\nabla \sigma\|^{2}+\mathcal{H}^{N-1}(J(\sigma))<+\infty \quad \text { and } \quad \mathcal{M}^{N-1}(J(\sigma))=\mathcal{H}^{N-1}(J(\sigma))
$$

With this definition at hand, we consider the following lemma.
Lemma 3.10. Let $\sigma \in X$ be admissible in the sense of Definition 3.9. Then we can find $\left(\sigma_{\eta}, v_{\eta}\right) \in L^{1}(\Omega) \times L^{1}(\Omega), 0<\eta \leq \eta_{0}$, such that, for some constant $C$,

$$
\begin{equation*}
\left\|\sigma_{\eta}-\sigma\right\|_{L^{1}(\Omega)} \leq C \eta \quad \text { and } \quad \lim _{\eta \rightarrow 0^{+}}\left\|v_{\eta}-1\right\|_{L^{1}(\Omega)}=0 \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\eta \rightarrow 0^{+}} A T_{\eta}\left(\sigma_{\eta}, v_{\eta}\right)=M S(\sigma, 1) \tag{48}
\end{equation*}
$$

Proof. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a $C^{\infty}$ function that is nondecreasing and such that $\phi(t)=0$ for any $t \leq 1 / 8$ and $\phi(t)=1$ for any $t \geq 7 / 8$.

For the time being, we consider the case in which $o_{\eta}>0$ and we define $\xi_{\eta}=\sqrt{\eta O_{\eta}}$.

We define, for any $\eta, 0<\eta \leq \eta_{0}$, and any $x \in \Omega$,

$$
\phi_{\eta}(x)=\phi\left(\frac{\operatorname{dist}(x, J(\sigma))}{\xi_{\eta}}\right)
$$

Then we define

$$
\sigma_{\eta}=\phi_{\eta} \sigma+\left(1-\phi_{\eta}\right) \lambda_{0}
$$

and, for any $x \in \Omega$, and any $\delta>0$,

$$
v_{\eta}^{\delta}(x)= \begin{cases}0 & \text { if } \operatorname{dist}(x, J(\sigma))<\xi_{\eta} \\ v^{\delta}\left(\frac{\operatorname{dist}(x, J(\sigma))-\xi_{\eta}}{\eta}\right) & \text { if } \xi_{\eta} \leq \operatorname{dist}(x, J(\sigma))<\xi_{\eta}+T \eta \\ 1 & \text { if } \operatorname{dist}(x, J(\sigma)) \geq \xi_{\eta}+T \eta\end{cases}
$$

Here the function $v=v^{\delta}$, and the constant $T>0$, are chosen in such a way $v \in C^{1}([0, T]), v(0)=0, v(T)=1$, and

$$
\int_{0}^{T}\left(V(v)+\left|v^{\prime}\right|^{2}\right) \leq 2 c_{V}+\delta
$$

We call, for any positive $r>0, S_{r}=\left\{x \in \mathbb{R}^{N}: \operatorname{dist}(x, J(\sigma))<r\right\}$. First of all we notice that, for some constant $C$,

$$
\left\|\sigma_{\eta}-\sigma\right\|_{L^{1}(\Omega)} \leq\left(\lambda_{1}-\lambda_{0}\right)\left|S_{\xi_{\eta}} \cap \Omega\right| \leq C \xi_{\eta} \leq C \eta \quad \text { for any } 0<\eta \leq \eta_{0}
$$

Since $\left|S_{\xi_{\eta}+T \eta}\right| \rightarrow 0$, as $\eta \rightarrow 0^{+}$, we also deduce that $v_{\eta}^{\delta} \rightarrow 1$ almost everywhere in $\Omega$ and in $L^{1}(\Omega)$ as well, in a completely independent way from the constant $\delta$ that may be chosen as depending from $\eta$.

Then we can compute, since obviously we have that $\sigma_{\eta} \in H^{1}(\Omega) \cap X$ and $v_{\eta} \in H^{1}(\Omega,[0,1])$,

$$
\begin{aligned}
& A T_{\eta}\left(\sigma_{\eta}, v_{\eta}^{\delta}\right)=\int_{\Omega}\left(b \psi_{\eta}\left(v_{\eta}^{\delta}\right)\left\|\nabla \sigma_{\eta}\right\|^{2}+\frac{1}{\eta} V\left(v_{\eta}^{\delta}\right)+\eta\left\|\nabla v_{\eta}^{\delta}\right\|^{2}\right) \\
&=b \int_{\Omega \backslash S_{\xi_{\eta}}} \psi_{\eta}\left(v_{\eta}^{\delta}\right)\|\nabla \sigma\|^{2}+b o_{\eta} \int_{S_{\xi_{\eta}}}\left\|\nabla \sigma_{\eta}\right\|^{2}+\frac{1}{\eta} V(0)\left|S_{\xi_{\eta}} \cap \Omega\right| \\
&+\int_{\left(S_{\xi_{\eta}+T \eta} \backslash S_{\xi_{\eta}}\right) \cap \Omega}\left(\frac{1}{\eta} V\left(v_{\eta}^{\delta}\right)+\eta\left\|\nabla v_{\eta}^{\delta}\right\|^{2}\right) .
\end{aligned}
$$

Since $v_{\eta}^{\delta}$ converges to 1 almost everywhere in $\Omega$, it is straightforward to see that the first three terms converge, as $\eta \rightarrow 0^{+}$, to $\int_{\Omega} b\|\nabla \sigma\|^{2}$, in a completely independent way from the constant $\delta$ that may be chosen as depending from $\eta$.

By the coarea formula, the definition of the Minkowski content, and the properties of $\sigma$, we can prove that

$$
\begin{aligned}
\lim _{\eta \rightarrow 0^{+}} \int_{S_{\xi_{\eta}+T \eta} \backslash S_{\xi_{\eta}}}\left(\frac{1}{\eta} V\left(v_{\eta}^{\delta}\right)+\eta \|\right. & \left.\nabla v_{\eta}^{\delta} \|^{2}\right) \\
& =2\left(\int_{0}^{T} V\left(v^{\delta}\right)+\left|\left(v^{\delta}\right)^{\prime}\right|^{2}\right) \mathcal{M}^{N-1}(J(\sigma))
\end{aligned}
$$

Since $\mathcal{M}^{N-1}(J(\sigma))=\mathcal{H}^{N-1}(J(\sigma))$, we easily deduce that, even if $o_{\eta}=0$,

$$
\limsup _{\eta \rightarrow 0^{+}} A T_{\eta}\left(\sigma_{\eta}, v_{\eta}^{\delta}\right) \leq \int_{\Omega} b\|\nabla \sigma\|^{2}+\left(4 c_{V}+2 \delta\right) \mathcal{H}^{N-1}(J(\sigma))
$$

It is then easy to choose $\delta=\delta(\eta)$ and define $v_{\eta}=v_{\eta}^{\delta(\eta)}, 0<\eta \leq \eta_{0}$, in such a way that

$$
\limsup _{\eta \rightarrow 0^{+}} A T_{\eta}\left(\sigma_{\eta}, v_{\eta}\right) \leq \int_{\Omega} b\|\nabla \sigma\|^{2}+4 c_{V} \mathcal{H}^{N-1}(J(\sigma))
$$

Clearly $\lim _{\eta \rightarrow 0^{+}}\left\|v_{\eta}-1\right\|_{L^{1}(\Omega)}=0$. Hence, by the corresponding $\Gamma$-liminf inequality proved in [10, Proposition 4.5], we conclude that

$$
\lim _{\eta \rightarrow 0^{+}} A T_{\eta}\left(\sigma_{\eta}, v_{\eta}\right)=\int_{\Omega} b\|\nabla \sigma\|^{2}+4 c_{V} \mathcal{H}^{N-1}(J(\sigma))=M S(\sigma, 1) .
$$

Thus the proof is complete.
We are ready to state the final convergence result.
ThEOREM 3.11. Under the previous assumptions, let us assume that $\sigma_{0}$ is admissible in the sense of Definition 3.9. Let us also assume that, for a positive constant $C_{0}$, we have $\psi(t) \geq C_{0} t^{2}$ for any $t \in[0,1]$.

If we pick $\eta=\eta(\varepsilon)=\varepsilon^{1 / \beta}$, and we call $F_{\varepsilon}=F_{\varepsilon, \eta(\varepsilon)}$ as in (44), then we obtain that

$$
\min _{X} \tilde{F}_{0} \leq \liminf _{\varepsilon \rightarrow 0^{+}}\left(\inf _{X \times L^{1}(\Omega)} F_{\varepsilon}\right) \leq \limsup _{\varepsilon \rightarrow 0^{+}}\left(\inf _{X \times L^{1}(\Omega)} F_{\varepsilon}\right)<+\infty
$$

Furthermore, let us consider two sequences $\left\{\varepsilon_{n}\right\}_{n \in \mathbb{N}} \subset\left(0, \varepsilon_{0}\right]$, such that $\lim _{n} \varepsilon_{n}=0$, and $\left\{\left(\sigma_{n}, v_{n}\right)\right\}_{n \in \mathbb{N}} \subset X \times L^{1}(\Omega)$.

If there exists a constant $C$ such that $F_{\varepsilon_{n}}\left(\sigma_{n}, v_{n}\right) \leq C$ for any $n \in \mathbb{N}$, then, as $n \rightarrow \infty$, $v_{n}$ converges to $v \equiv 1$ in $L^{1}(\Omega)$ and, up to a subsequence, $\sigma_{n}$ converges to $\tilde{\sigma} \in X$ in $L^{1}(\Omega)$. Moreover, $\tilde{\sigma} \in S B V(\Omega), M S(\tilde{\sigma}, 1)$ is finite, and $\left\|\Lambda(\tilde{\sigma})-\Lambda\left(\sigma_{0}\right)\right\|_{Y}=0$. Finally, if $N=2$ and we assume that either $B_{1}$ is dense in $H_{*}^{1 / 2}(\partial \Omega)$ or $\tilde{B}_{1}$ is dense in $H_{*}^{-1 / 2}(\partial \Omega)$, respectively, the whole sequence $\sigma_{n}$ converges, as $n \rightarrow \infty$, to $\sigma_{0}$ in $L^{1}(\Omega)$.

Proof. First of all, by applying Lemma 3.10 to $\sigma=\sigma_{0}$, we conclude that

$$
\limsup _{\varepsilon \rightarrow 0^{+}}\left(\inf _{X \times L^{1}(\Omega)} F_{\varepsilon}\right)<+\infty
$$

In fact, for any $0<\varepsilon \leq \varepsilon_{0}$, we have
$\left\|\Lambda\left(\sigma_{\eta(\varepsilon)}\right)-\hat{\Lambda}_{\varepsilon}\right\|_{Y} \leq\left\|\Lambda\left(\sigma_{\eta(\varepsilon)}\right)-\Lambda\left(\sigma_{0}\right)\right\|_{Y}+\left\|\Lambda\left(\sigma_{0}\right)-\hat{\Lambda}_{\varepsilon}\right\|_{Y} \leq C(\eta(\varepsilon))^{\beta}+\varepsilon \leq C_{1} \varepsilon$, for some constants $C$ and $C_{1}$.

By the $\Gamma$-limif inequality, [10, Proposition 4.5], and the compactness stated in the second part of Theorem 3.7, we can immediately prove that

$$
\min _{X} \tilde{F}_{0} \leq \liminf _{\varepsilon \rightarrow 0^{+}}\left(\inf _{X \times L^{1}(\Omega)} F_{\varepsilon}\right) .
$$

The second part of the theorem follows immediately, again by exploiting the compactness result in Theorem 3.7.

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# On coated inclusions neutral to bulk strain fields in two dimensions 

Hyeonbae Kang<br>"Dedicated to Giovanni Alessandrini on his 60th birthday"


#### Abstract

The neutral inclusion problem in two dimensional isotropic elasticity is considered. The neutral inclusion, when inserted in a matrix having a uniform applied field, does not disturb the field outside the inclusion. The inclusion consists of the core and shell of arbitrary shapes, and their elasticity tensors are isotropic. We show that if the coated inclusion is neutral to a uniform bulk field, then the core and shell must be concentric disks, provided that the shear and bulk moduli satisfy certain conditions.


Keywords: Elastic neutral inclusion, bulk strain field, concentric disk.
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## 1. Introduction

Some inclusions, when inserted in a matrix having a uniform field, do not disturb the field outside the inclusion. Such inclusions are called neutral inclusions (to the given field). A typical neutral inclusion consists of a core coated by a shell having the material property different from that of the core.

It is easy to construct neutral inclusions of circular shapes in the context of conductivity (or anti-plane elasticity). Let $D=\left\{|\mathbf{x}|<r_{1}\right\}$ and $\Omega=\left\{|\mathbf{x}|<r_{2}\right\}$ ( $r_{1}<r_{2}$ ) so that $D$ is the core and $\Omega \backslash D$ is the shell. The conductivity is $\sigma_{c}$ in the core, $\sigma_{s}$ in the shell, and $\sigma_{m}$ in the matrix $\left(\mathbb{R}^{2} \backslash \Omega\right)$. So the conductivity distribution is given by

$$
\sigma=\sigma_{c} \chi(D)+\sigma_{s} \chi(\Omega \backslash D)+\sigma_{m} \chi\left(\mathbb{R}^{2} \backslash \Omega\right)
$$

where $\chi$ is the characteristic function. If $\sigma_{c}, \sigma_{s}$ and $\sigma_{m}$ satisfy the relation

$$
\begin{equation*}
r_{2}^{2}\left(\sigma_{s}+\sigma_{c}\right)\left(\sigma_{m}-\sigma_{s}\right)-r_{1}^{2}\left(\sigma_{s}-\sigma_{c}\right)\left(\sigma_{m}+\sigma_{s}\right)=0 \tag{1}
\end{equation*}
$$

then $\Omega$ is neutral to uniform fields. In other words, for any constant vector a,
the solution $u$ to the problem

$$
\begin{cases}\nabla \cdot \sigma \nabla u=0 & \text { in } \mathbb{R}^{2} \\ u(\mathbf{x})-\mathbf{a} \cdot \mathbf{x}=O\left(|\mathbf{x}|^{-1}\right) & \text { as }|\mathbf{x}| \rightarrow \infty\end{cases}
$$

satisfies $u(\mathbf{x})-\mathbf{a} \cdot \mathbf{x}=0$ in $\mathbb{R}^{2} \backslash \Omega$.
Much interest in neutral inclusions was aroused by the work of Hashin [6, 7], where it is shown that since insertion of neutral inclusions does not perturb the outside uniform field, the effective conductivity of the assemblage filled with neutral inclusions of many different scales is $\sigma_{m}$ satisfying (1). It is also proved that this effective conductivity is a bound of the Hashin-Shtrikman bounds on the effective conductivity of arbitrary two phase composites. We refer to a book of Milton [13] for development on neutral inclusions in relation to theory of composites.

Another interest in neutral inclusions has aroused in relation to the invisibility cloaking by transformation optics. In this regard, we first observe that in general the solution $u$ to

$$
\begin{cases}\nabla \cdot \sigma \nabla u=0 & \text { in } \mathbb{R}^{2}  \tag{2}\\ u(\mathbf{x})-h(\mathbf{x})=O\left(|\mathbf{x}|^{-1}\right) & \text { as }|\mathbf{x}| \rightarrow \infty\end{cases}
$$

for a given harmonic function $h$ satisfies $u(\mathbf{x})-h(\mathbf{x})=O\left(|\mathbf{x}|^{-1}\right)$ as $|\mathbf{x}| \rightarrow \infty$. But, if the inclusion is neutral to all uniform fields, then the linear part of $h$ is unperturbed and one can show using multi-polar expansions that $u(\mathbf{x})-$ $h(\mathbf{x})=O\left(|\mathbf{x}|^{-2}\right)$ as $|\mathbf{x}| \rightarrow \infty$ for any $h$ (not necessarily linear). Recently, Ammari et al [2] extended the idea of neutral inclusions to construct multicoated circular structures which are neutral not only to uniform fields but also to fields of higher order, so that the solution $u$ to (2) satisfies $u(\mathbf{x})-h(\mathbf{x})=$ $O\left(|\mathbf{x}|^{-N}\right)$ as $|\mathbf{x}| \rightarrow \infty$ for any given $N$ and any $h$ (such structures are called GPT vanishing structures). Such structures have a strong connection to the cloaking by transformation optics. The transformation optics proposed by Pendry et al [16] transforms a punctured disk (or a sphere) to an annulus to achieve perfect cloaking. The same transform was used to show non-uniqueness of the Calderón's problem by Greenleaf et al [5]. Kohn et al [12] showed that if one transforms a disk with small hole, then one can avoid singularity of the conductivity which occurs on the inner boundary of the annulus and achieve near-cloaking instead of perfect cloaking. In [2] it is shown that if we coat the core by multiple layers so that the structure becomes neutral to fields of higher order (and transform the structure), then the near-cloaking effect is dramatically improved.

All above mentioned neutral inclusions have circular shapes and it is of interest to consider neutral inclusions of arbitrary shapes. For a given core of arbitrary shape, the shape of the outer boundary of the shell has been constructed by Milton \& Serkov [14] so that the coated inclusion is neutral to a
single uniform field. This is done when the conductivity $\sigma_{c}$ of the core is either 0 or $\infty$. See [9] for an extension to the case when $\sigma_{c}$ is finite. It is also proved in [14] that if an inclusion is neutral to all uniform field (or equivalently, to two linearly independent uniform fields), then the inclusion is concentric disks (confocal ellipses if the conductivity of the matrix is anisotropic), when $\sigma_{c}$ is 0 or $\infty$. In recent paper [10], Kang and Lee proved that this is the case even when $\sigma_{c}$ is finite. See also [11] for an extension to three dimensions.

In this paper the problem of neutral inclusions in two dimensional linear isotropic elasticity is considered. Let the shear and bulk moduli of the core, the shell, and the matrix be $\left(\mu_{c}, \kappa_{c}\right),\left(\mu_{s}, \kappa_{s}\right)$, and $\left(\mu_{m}, \kappa_{m}\right)$, respectively, and let $\mu$ and $\kappa$ denote their distributions in $\mathbb{R}^{2}$. Define the elasticity tensor $\mathbb{C}=\left(C_{i j k l}\right)$ by

$$
\begin{equation*}
C_{i j k l}=(\kappa-\mu) \delta_{i j} \delta_{k \ell}+\mu\left(\delta_{i k} \delta_{j \ell}+\delta_{i \ell} \delta_{j k}\right), \quad i, j, k, l=1,2 \tag{3}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker's delta. Let $\mathbf{h}(\mathbf{x})=\mathbf{x}$, whose gradient represents the bulk strain field, and consider the following interface problem:

$$
\begin{cases}\operatorname{div} \mathbb{C} \hat{\nabla} \mathbf{u}=0 & \text { in } \mathbb{R}^{2}  \tag{4}\\ \mathbf{u}(\mathbf{x})-\mathbf{h}(\mathbf{x})=O\left(|\mathbf{x}|^{-1}\right) & \text { as }|\mathbf{x}| \rightarrow \infty\end{cases}
$$

where $\widehat{\nabla} \mathbf{u}$ is the symmetric gradient (or the strain tensor), i.e.,

$$
\widehat{\nabla} \mathbf{u}:=\frac{1}{2}\left(\nabla \mathbf{u}+(\nabla \mathbf{u})^{T}\right) \quad(T \text { for transpose })
$$

The inclusion is neutral to the (strain) field $\nabla \mathbf{h}$ if the solution $\mathbf{u}$ to (4) satisfies $\mathbf{u}(\mathbf{x})-\mathbf{h}(\mathbf{x})=0$ in $\mathbb{R}^{2} \backslash \Omega$. Inclusions neutral to the bulk field was found using the exact effective bulk modulus of the assemblage of coated disks which was derived by Hashin and Rosen [8]. The purpose of this paper is to prove that concentric disks are the only coated inclusions neutral to bulk fields under some conditions on the shear and bulk moduli.

The following is the main theorem of this paper.
Theorem 1.1. Let $\Omega$ and $D$ be bounded simply connected domains in $\mathbb{R}^{2}$ with Lipschitz boundaries such that $\bar{D} \subset \Omega$. Suppose that

$$
\begin{equation*}
\mu_{c} \neq \mu_{s}, \quad \kappa_{m} \neq \kappa_{s}, \quad \text { and } \quad \kappa_{c}<2 \kappa_{s}+\mu_{s} \tag{5}
\end{equation*}
$$

If $(\Omega, D)$ is neutral to the bulk field, or equivalently, if the solution $\mathbf{u}$ to (4) with $\mathbf{h}(\mathbf{x})=\mathbf{x}$ satisfies $\mathbf{u}(\mathbf{x})-\mathbf{x}=0$ in $\mathbb{R}^{2} \backslash \Omega$, then $D$ and $\Omega$ are concentric disks.

The conditions in (5) are required to show that the solution is linear in the core. The first two conditions seem natural because the elasticity properties of
the core, the shell, and the matrix must be different. However, we don't know if the third condition is necessary.

It is worth mentioning that inclusions consisting of the concentric disks are not neutral to shear fields: for example, if $\mathbf{h}(\mathbf{x})=(y, x)^{T}$, then $u(\mathbf{x})-\mathbf{h}(\mathbf{x})$ has a term of order $|\mathbf{x}|^{-1}$ and a term of order $|\mathbf{x}|^{-3}$ as $|\mathbf{x}| \rightarrow \infty$, and it is not possible to make both terms vanish. Christensen and Lo [3] constructed circular inclusions such that the term of order $|\mathbf{x}|^{-1}$ vanishes and derived an effective transverse shear modulus of the assemblage of coated disks. It is interesting to construct coated inclusions neutral to shear fields or to prove non-existence of such inclusions.

The rest of the paper is organized as follows: In the next section we show that if $(\Omega, D)$ is neutral to the bulk field, then $\nabla \mathbf{u}$ is symmetric and divu is constant in the shell. The main theorem is proved in section 3 by showing that $\mathbf{u}$ is linear in the core. To do so we use a complex representation of the displacement vector.

## 2. Properties of the solution in the shell

In this section we prove the following proposition. We emphasize that (5) is not required for this proposition.

Proposition 2.1. Let $\Omega$ and $D$ be bounded simply connected domains in $\mathbb{R}^{2}$ with Lipschitz boundaries such that $\bar{D} \subset \Omega$. If $(\Omega, D)$ is neutral to the bulk field, then the solution $\mathbf{u}$ to (4) satisfies the following:
(i) $\Delta \mathbf{u}=0$, or equivalently div $\mathbf{u}=$ constant in $\Omega \backslash \bar{D}$.
(ii) $\nabla \mathbf{u}$ is symmetric in $\Omega \backslash \bar{D}$, namely, $\partial_{1} u_{2}=\partial_{2} u_{1}$.

To prove Proposition 2.1, we need some preparartion. The Kelvin matrix $\boldsymbol{\Gamma}(\mathbf{x})=\left(\Gamma_{i j}(\mathbf{x})\right)_{i, j=1}^{2}$ of the fundamental solution to the Lamé operator $\operatorname{div} \mathbb{C} \widehat{\nabla}$ in two dimensions is given by

$$
\begin{equation*}
\Gamma_{i j}(\mathbf{x}):=\frac{\alpha_{1}}{2 \pi} \delta_{i j} \log |\mathbf{x}|-\frac{\alpha_{2}}{2 \pi} \frac{x_{i} x_{j}}{|\mathbf{x}|^{2}}, \quad \mathbf{x} \neq 0 \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{1}=\frac{1}{2}\left(\frac{1}{\mu}+\frac{1}{\mu+\kappa}\right) \quad \text { and } \quad \alpha_{2}=\frac{1}{2}\left(\frac{1}{\mu}-\frac{1}{\mu+\kappa}\right) . \tag{7}
\end{equation*}
$$

A straight-forward computation shows that

$$
\begin{equation*}
\operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y})=\frac{\alpha_{2}-\alpha_{1}}{2 \pi} \nabla_{\mathbf{x}} \log |\mathbf{x}-\mathbf{y}|=-\frac{1}{2 \pi(\mu+\kappa)} \nabla_{\mathbf{x}} \log |\mathbf{x}-\mathbf{y}| \tag{8}
\end{equation*}
$$

In particular, we have

$$
\begin{equation*}
\int_{\Omega} \operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) d \mathbf{y}=-\frac{1}{2 \pi(\mu+\kappa)} \nabla \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y} \tag{9}
\end{equation*}
$$

Since

$$
\frac{1}{2 \pi} \Delta \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}= \begin{cases}1 & \text { if } \mathbf{x} \in \Omega \\ 0 & \text { if } \mathbf{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}\end{cases}
$$

we have

$$
\operatorname{div} \int_{\Omega} \operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) d \mathbf{y}= \begin{cases}-\frac{1}{\mu+\kappa} & \text { if } \mathbf{x} \in \Omega  \tag{10}\\ 0 & \text { if } \mathbf{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}\end{cases}
$$

We also have

$$
\begin{equation*}
\operatorname{rot} \int_{\Omega} \operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}(\mathbf{x}-\mathbf{y}) d \mathbf{y}=0 \tag{11}
\end{equation*}
$$

Proof of Proposition 2.1. Suppose that $(\Omega, D)$ is neutral to the bulk field. Then the following over-determined problem admits a solution:

$$
\begin{cases}\nabla \cdot(\mathbb{C} \widehat{\nabla} \mathbf{u})=0 & \text { in } \Omega  \tag{12}\\ \mathbf{u}(\mathbf{x})=\mathbf{x}, \quad\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{u}\right) \mathbf{n}=\left(\mathbb{C}_{m} \mathbf{I}\right) \mathbf{n} & \text { on } \partial \Omega\end{cases}
$$

Here and throughout this paper, $\mathbf{n}$ denotes the outward normal to $\partial \Omega$ (and $\partial D$ ). Let $\mathbf{u}_{c}$ and $\mathbf{u}_{s}$ denote the solution on $D$ and $\Omega \backslash \bar{D}$, respectively. Then the transmission conditions along $\partial D$ are given by

$$
\begin{equation*}
\mathbf{u}_{c}=\mathbf{u}_{s} \quad \text { and } \quad\left(\mathbb{C}_{c} \widehat{\nabla} \mathbf{u}_{c}\right) \mathbf{n}=\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{u}_{s}\right) \mathbf{n} \quad \text { on } \partial D . \tag{13}
\end{equation*}
$$

Let $\mathbf{v}$ be a smooth vector field in $\Omega$. Then we have

$$
\int_{\partial \Omega}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{u}\right) \mathbf{n} \cdot \mathbf{v} d \sigma=\int_{\Omega} \mathbb{C} \widehat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}
$$

Here and afterwards, $\mathbf{A}: \mathbf{B}$ denotes the contraction of two matrices $\mathbf{A}$ and $\mathbf{B}$, i.e., $\mathbf{A}: \mathbf{B}=\sum a_{i j} b_{i j}=\operatorname{tr}\left(\mathbf{A}^{T} \mathbf{B}\right)$. On the other hand, we have from the Neumann boundary condition in (12)

$$
\int_{\partial \Omega}(\mathbb{C} \widehat{\nabla} \mathbf{u}) \mathbf{n} \cdot \mathbf{v} d \sigma=\int_{\Omega} \mathbb{C}_{m} \mathbf{I}: \hat{\nabla} \mathbf{v} d \mathbf{y}
$$

So, we have

$$
\begin{equation*}
\int_{\Omega \backslash D} \mathbb{C}_{s} \widehat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}+\int_{D} \mathbb{C}_{c} \widehat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}=\int_{\Omega} \mathbb{C}_{m} \mathbf{I}: \widehat{\nabla} \mathbf{v} d \mathbf{y} . \tag{14}
\end{equation*}
$$

Using the Dirichlet boundary condition in (12) we have for any elasticity tensor $\mathbb{C}_{0}$

$$
\int_{\partial \Omega} \mathbf{u} \cdot\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) \mathbf{n} d \sigma=\int_{\Omega} \mathbb{C}_{0} \hat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}+\int_{\Omega} \mathbf{u} \cdot \operatorname{div}\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) d \mathbf{y}
$$

and

$$
\begin{aligned}
\int_{\partial \Omega} \mathbf{u} \cdot\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) \mathbf{n} d \sigma & =\int_{\partial \Omega} \mathbf{y} \cdot\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) \mathbf{n} d \sigma \\
& =\int_{\Omega} \mathbb{C}_{0} \mathbf{I}: \hat{\nabla} \mathbf{v} d \mathbf{y}+\int_{\Omega} \mathbf{y} \cdot \operatorname{div}\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) d \mathbf{y}
\end{aligned}
$$

Thus we have

$$
\begin{align*}
& \int_{\Omega} \mathbb{C}_{0} \widehat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}+\int_{\Omega} \mathbf{u} \cdot \operatorname{div}\left(\mathbb{C}_{0} \hat{\nabla} \mathbf{v}\right) d \mathbf{y} \\
&=\int_{\Omega} \mathbb{C}_{0} \mathbf{I}: \widehat{\nabla} \mathbf{v} d \mathbf{y}+\int_{\Omega} \mathbf{y} \cdot \operatorname{div}\left(\mathbb{C}_{0} \widehat{\nabla} \mathbf{v}\right) d \mathbf{y} \tag{15}
\end{align*}
$$

Subtracting (15) with $\mathbb{C}_{0}=\mathbb{C}_{s}$ from (14) we obtain

$$
\begin{align*}
& \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \hat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}-\int_{\Omega} \mathbf{u} \cdot \operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{v}\right) d \mathbf{y} \\
&=\int_{\Omega}\left(\mathbb{C}_{m}-\mathbb{C}_{s}\right) \mathbf{I}: \widehat{\nabla} \mathbf{v} d \mathbf{y}-\int_{\Omega} \mathbf{y} \cdot \operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{v}\right) d \mathbf{y} \tag{16}
\end{align*}
$$

Let $\boldsymbol{\Gamma}^{s}$ and $\boldsymbol{\Gamma}^{c}$ be the Kelvin matrices for $\operatorname{div} \mathbb{C}_{s} \widehat{\nabla}$ and $\operatorname{div} \mathbb{C}_{c} \widehat{\nabla}$, respectively. For $\mathbf{x} \in \Omega$, let $\mathbf{v}(\mathbf{y})$ be a column of $\boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y})$. Then we may apply the same argument of integration by parts (over $\Omega$ with an $\epsilon$ ball around $\mathbf{x}$ deleted) as above and obtain from (16) the following representation of the solution:

$$
\begin{align*}
\mathbf{u}(\mathbf{x})=\mathbf{x} & +\int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \widehat{\nabla} \mathbf{u}(\mathbf{y}): \widehat{\nabla} \boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& +\int_{\Omega}\left(\mathbb{C}_{s}-\mathbb{C}_{m}\right) \mathbf{I}: \widehat{\nabla} \boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y}) d \mathbf{y}, \quad \mathbf{x} \in \Omega \tag{17}
\end{align*}
$$

Since $\left(\mathbb{C}_{m}-\mathbb{C}_{s}\right) \mathbf{I}: \widehat{\nabla} \mathbf{v}=2\left(\kappa_{m}-\kappa_{s}\right) \operatorname{div} \mathbf{v}$, the identity (16) takes the form

$$
\begin{align*}
& \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \hat{\nabla} \mathbf{u}: \widehat{\nabla} \mathbf{v} d \mathbf{y}-\int_{\Omega} \mathbf{u} \cdot \operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{v}\right) d \mathbf{y} \\
&=2\left(\kappa_{m}-\kappa_{s}\right) \int_{\Omega} \operatorname{div} \mathbf{v} d \mathbf{y}-\int_{\Omega} \mathbf{y} \cdot \operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{v}\right) d \mathbf{y} \tag{18}
\end{align*}
$$

One can also see from (9) that the representation formula (17) takes the form

$$
\begin{align*}
\mathbf{u}(\mathbf{x})=\mathbf{x} & +\int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \widehat{\nabla} \mathbf{u}(\mathbf{y}): \widehat{\nabla}_{\mathbf{y}} \boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& +\frac{\kappa_{m}-\kappa_{s}}{\pi\left(\mu_{s}+\kappa_{s}\right)} \nabla \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}, \quad \mathbf{x} \in \Omega \tag{19}
\end{align*}
$$

Let $\boldsymbol{\Gamma}^{s, j}$ be the $j$-th column of $\boldsymbol{\Gamma}^{s}$. Let $\mathbf{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}$. Substitute $\mathbf{v}_{j}(\mathbf{y}):=$ $\frac{\partial}{\partial x_{j}} \boldsymbol{\Gamma}^{s, j}(\mathbf{x}-\mathbf{y})$ for $\mathbf{v}$ in (18) and add the identities for $j=1,2$. Note that $\operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{v}_{j}\right)=0$ in $\Omega$ since $\mathbf{x} \notin \Omega$. We infer from (10) that

$$
\sum_{j=1}^{2} \int_{\Omega} \operatorname{div} \mathbf{v}_{j}=\sum_{j=1}^{2} \frac{\partial}{\partial x_{j}} \int_{\Omega} \operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}^{s, j}(\mathbf{x}-\mathbf{y}) d \mathbf{y}=0
$$

It then follows from (18) that

$$
\begin{equation*}
\sum_{j=1}^{2} \frac{\partial}{\partial x_{j}} \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \widehat{\nabla} \mathbf{u}(\mathbf{y}): \widehat{\nabla} \boldsymbol{\Gamma}^{s, j}(\mathbf{x}-\mathbf{y}) d \mathbf{y}=0, \quad \mathbf{x} \in \mathbb{R}^{2} \backslash \bar{\Omega} \tag{20}
\end{equation*}
$$

Observe that the left-hand side in the above is a real analytic function in $\mathbb{R}^{2} \backslash \bar{D}$. So, by unique continuation (20) holds for all $\mathbf{x} \in \mathbb{R}^{2} \backslash \bar{D}$. We then infer from (10) and (19) that

$$
\begin{equation*}
\operatorname{div} \mathbf{u}=\alpha \quad \text { in } \Omega \backslash \bar{D} \tag{21}
\end{equation*}
$$

where $\alpha$ is the constant given by

$$
\begin{equation*}
\alpha=2+\frac{2\left(\kappa_{m}-\kappa_{s}\right)}{\mu_{s}+\kappa_{s}} \tag{22}
\end{equation*}
$$

Since $\operatorname{div}\left(\mathbb{C}_{s} \widehat{\nabla} \mathbf{u}\right)=\mu_{s} \Delta \mathbf{u}+\kappa_{s} \nabla \operatorname{div} \mathbf{u}=0$, we also have

$$
\begin{equation*}
\Delta \mathbf{u}=0 \quad \text { in } \Omega \backslash D \tag{23}
\end{equation*}
$$

We now prove (ii). Let $\mathbf{x} \in \mathbb{R}^{2} \backslash \bar{\Omega}$ and substitute $\frac{\partial}{\partial x_{2}} \boldsymbol{\Gamma}^{s, 1}(\mathbf{x}-\mathbf{y})$ for $\mathbf{v}$ in (18) to obtain from (9) that

$$
\begin{aligned}
& \frac{\partial}{\partial x_{2}} \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \hat{\nabla} \mathbf{u}(\mathbf{y}): \widehat{\nabla} \boldsymbol{\Gamma}^{s, 1}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& \quad=2\left(\kappa_{m}-\kappa_{s}\right) \frac{\partial}{\partial x_{2}} \int_{\Omega} \operatorname{div}_{\mathbf{y}} \boldsymbol{\Gamma}^{s, 1}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& \quad=-\frac{\left(\kappa_{m}-\kappa_{s}\right)}{\pi\left(\mu_{s}+\kappa_{s}\right)} \frac{\partial^{2}}{\partial x_{1} \partial x_{2}} \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}
\end{aligned}
$$

By substituting $\frac{\partial}{\partial x_{1}} \boldsymbol{\Gamma}^{s, 2}(\mathbf{x}-\mathbf{y})$ for $\mathbf{v}$ in (18), we also obtain

$$
\begin{aligned}
& \frac{\partial}{\partial x_{1}} \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \widehat{\nabla} \mathbf{u}(\mathbf{y}): \widehat{\nabla} \boldsymbol{\Gamma}^{s, 2}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& \quad=-\frac{\left(\kappa_{m}-\kappa_{s}\right)}{\pi\left(\mu_{s}+\kappa_{s}\right)} \frac{\partial^{2}}{\partial x_{1} \partial x_{2}} \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}
\end{aligned}
$$

So, we have using unique continuation again that

$$
\begin{aligned}
& \frac{\partial}{\partial x_{2}} \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \hat{\nabla} \mathbf{u}(\mathbf{y}): \hat{\nabla} \boldsymbol{\Gamma}^{s, 1}(\mathbf{x}-\mathbf{y}) d \mathbf{y} \\
& \quad=\frac{\partial}{\partial x_{1}} \int_{D}\left(\mathbb{C}_{c}-\mathbb{C}_{s}\right) \hat{\nabla} \mathbf{u}(\mathbf{y}): \hat{\nabla} \boldsymbol{\Gamma}^{s, 2}(\mathbf{x}-\mathbf{y}) d \mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^{2} \backslash \bar{D}
\end{aligned}
$$

and so (ii) is proved.

## 3. Neutral Inclusions to the bulk field

### 3.1. Complex representation of the solution and a lemma

Let $\mathbf{u}=\left(u_{1}, u_{2}\right)^{T}$ be the solution to (4). There are functions $\varphi$ and $\psi$ which are analytic in $D, \Omega \backslash \bar{D}$, and $\mathbb{C} \backslash \bar{D}$, separately, such that

$$
\begin{equation*}
u_{1}+i u_{2}=\frac{1}{2 \mu}\left(k \varphi(z)-z \overline{\varphi^{\prime}(z)}-\overline{\psi(z)}\right) \tag{24}
\end{equation*}
$$

where

$$
\begin{equation*}
k=1+\frac{2 \mu}{\kappa} \tag{25}
\end{equation*}
$$

See for example $[1,15]$ for derivation of (24). Conversely, one can see that $\mathbf{u}=\left(u_{1}, u_{2}\right)^{T}$ of the form (24) with $k>1$ for a pair of analytic functions $\varphi$ and $\psi$ in $D$ is a solution in $D$ of the Lamé system determined by the shear modulus $\mu$ and the bulk modulus $\kappa=2 \mu /(k-1)$.

We denote $\varphi$ and $\psi$ by $\varphi_{c}$ and $\psi_{c}$ in the core, $\varphi_{s}$ and $\psi_{s}$ in the shell, and $\varphi_{m}$ and $\psi_{m}$ in the matrix. Then the transmission conditions (12) and (13) along the interfaces $\partial D$ and $\partial \Omega$ take the following forms: along $\partial D$,

$$
\begin{aligned}
\frac{1}{2 \mu_{s}}\left(k_{s} \varphi_{s}(z)-z \overline{\varphi_{s}^{\prime}(z)}-\overline{\psi_{s}(z)}\right) & =\frac{1}{2 \mu_{c}}\left(k_{c} \varphi_{c}(z)-z \overline{\varphi_{c}^{\prime}(z)}-\overline{\psi_{c}(z)}\right) \\
d\left(\varphi_{s}(z)+z \overline{\varphi_{s}^{\prime}(z)}+\overline{\psi_{s}(z)}\right) & =d\left(\varphi_{c}(z)+z \overline{\varphi_{c}^{\prime}(z)}+\overline{\psi_{c}(z)}\right)
\end{aligned}
$$

and similar conditions on $\partial \Omega$, where $d$ is the exterior differential. The first condition is the continuity of the displacement and the second one is that of
the traction. Using complex notation $d z=d x+i d y$ and $d \bar{z}=d x-i d y$, the exterior differential is given by

$$
d f=\frac{\partial f}{\partial z} d z+\frac{\partial f}{\partial \bar{z}} d \bar{z}
$$

It is convenient to use notation

$$
\begin{equation*}
U(z):=u_{1}+i u_{2}=\frac{1}{2 \mu}\left(k \varphi(z)-z \overline{\varphi^{\prime}(z)}-\overline{\psi(z)}\right) \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
D U(z):=d\left(\varphi+z \overline{\varphi^{\prime}}+\bar{\psi}\right)=\left(\varphi^{\prime}+\overline{\varphi^{\prime}}\right) d z+\left(z \overline{\varphi^{\prime \prime}}+\overline{\psi^{\prime}}\right) d \bar{z} \tag{27}
\end{equation*}
$$

Then the transmission conditions read

$$
\begin{equation*}
U_{c}=U_{s}, \quad D U_{c}=D U_{s} \quad \text { on } \partial D \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
U_{m}=U_{s}, \quad D U_{m}=D U_{s} \quad \text { on } \partial \Omega \tag{29}
\end{equation*}
$$

The proofs in the subsequent subsection use the following lemma, which may be well-known. We include a short proof for readers' sake.

Lemma 3.1. Let $D$ be a simply connected bounded domain with the Lipschitz boundary, and let $g$ be a square integrable function on $\partial D$. If

$$
\begin{equation*}
\int_{\partial D} g(z) f^{\prime}(z) d z=0 \tag{30}
\end{equation*}
$$

for any function $f$ analytic in a neighborhood of $\bar{D}$, then there is an analytic function $G$ in $D$ such that $G=g$ on $\partial D$.

Proof. Define the Cauchy transform by

$$
C[g](w):=\frac{1}{2 \pi i} \int_{\partial D} \frac{g(z)}{z-w} d z, \quad w \in \mathbb{C} \backslash \partial D
$$

Then by Plemelj's jump formula (see [15]), we have

$$
g(w)=\left.C[g]\right|_{-}(w)-\left.C[g]\right|_{+}(w), \quad w \in \partial D
$$

where $\left.C[g]\right|_{-}$and $\left.C[g]\right|_{+}$denote the limits from inside and outside of $D$, respectively. Since $D$ is simply connected, $f(z)=\log (z-w)$ is well-defined and analytic in a neighborhood of $\bar{D}$ if $w \notin \bar{D}$. So, $C[g](w)=0$ if $w \notin \bar{D}$ by (30). Thus, we have

$$
g(w)=\left.C[g]\right|_{-}(w), \quad w \in \partial D
$$

So, $G:=C[g]$ in $D$ is the desired analytic function.

### 3.2. Proof of Theorem 1.1

Let us prove the following proposition first.
Proposition 3.2. Let $\Omega$ and $D$ be bounded simply connected domains in $\mathbb{R}^{2}$ with Lipschitz boundaries such that $\bar{D} \subset \Omega$, and assume that (5) holds. If $(\Omega, D)$ is neutral to the bulk field, then the solution $\mathbf{u}$ to (4) is linear in $D$ and of the form

$$
\begin{equation*}
\mathbf{u}(\mathbf{x})=a \mathbf{x}+\mathbf{b} \tag{31}
\end{equation*}
$$

for $a$ constant $a$ and $a$ constant vector $\mathbf{b}$.
Proof. Let $\mathbf{u}$ be the solution to (4) when $\mathbf{h}(\mathbf{x})=\mathbf{x}$, and $U$ be defined by (26). Since $(\Omega, D)$ is neutral to the bulk field, $\mathbf{u}(\mathbf{x})=\mathbf{x}$ in $\mathbb{R}^{2} \backslash \bar{\Omega}$, and hence we have

$$
U_{m}(z)=z, \quad \varphi_{m}(z)=\kappa_{m} z, \quad \psi_{m}(z)=0, \quad D U_{m}(z)=2 \kappa_{m} d z
$$

Moreover, Proposition 2.1 implies that

$$
\begin{equation*}
\varphi_{s}(z)=\beta z+\text { constant }, \quad z \in \Omega \backslash D \tag{32}
\end{equation*}
$$

where $\beta$ is a real constant. In fact, we see from Proposition 2.1 that

$$
\frac{\partial}{\partial z} U_{s}=\frac{1}{2}\left(\frac{\partial}{\partial x_{1}}-i \frac{\partial}{\partial x_{2}}\right)\left(u_{1}+i u_{2}\right)=\frac{1}{2}\left(\operatorname{div} \mathbf{u}+i\left(\frac{\partial u_{2}}{\partial x_{1}}-\frac{\partial u_{1}}{\partial x_{2}}\right)\right)=\frac{\alpha}{2}
$$

where $\alpha$ is the constant in (22). Thus we have

$$
\frac{\alpha}{2}=\frac{\partial}{\partial z} U_{s}(z)=\frac{1}{2 \mu_{s}}\left(k_{s} \varphi_{s}^{\prime}(z)-\overline{\varphi_{s}^{\prime}(z)}\right),
$$

which implies that $\varphi_{s}^{\prime}(z)=\beta=\kappa_{s} \alpha / 2$ by (25). One can see from (22) that

$$
\begin{equation*}
\kappa_{m}-\beta=\frac{\left(\kappa_{m}-\kappa_{s}\right)\left(2 \kappa_{s}+\mu_{s}\right)}{\kappa_{s}+\mu_{s}} \tag{33}
\end{equation*}
$$

Let $f$ and $g$ be functions analytic on $\bar{\Omega}$, and let $F(z)=f(z)+\overline{g(z)}$. We have from the first identity in (29), Cauchy's theorem and Stokes' theorem that

$$
\int_{\partial \Omega} U_{s} d F=\int_{\partial \Omega} U_{m} d F=\int_{\partial \Omega} z \overline{g^{\prime}} d \bar{z}=-\int_{\Omega} \overline{g^{\prime}} d m
$$

where $d m:=d \bar{z} \wedge d z$. We also have from Stokes' theorem that

$$
\begin{align*}
\int_{\partial \Omega} U_{s} d F & =\int_{\Omega} d(U d F)=\int_{\Omega}\left[\frac{\partial}{\partial \bar{z}}\left(U f^{\prime}\right)-\frac{\partial}{\partial z}\left(U \overline{g^{\prime}}\right)\right] d m \\
& =-\int_{\Omega} \frac{1}{2 \mu}\left[\left(z \overline{\varphi^{\prime \prime}}+\overline{\psi^{\prime}}\right) f^{\prime}+\left(k \varphi^{\prime}-\overline{\varphi^{\prime}}\right) \overline{g^{\prime}}\right] d m \tag{34}
\end{align*}
$$

Equating above two identities, we have

$$
\begin{aligned}
\int_{\Omega} \overline{g^{\prime}} d m= & \frac{1}{2 \mu_{s}} \int_{\Omega \backslash \bar{D}}\left[\left(z \overline{\varphi_{s}^{\prime \prime}}+\overline{\psi_{s}^{\prime}}\right) f^{\prime}+\left(k_{s} \varphi_{s}^{\prime}-\overline{\varphi_{s}^{\prime}}\right) \overline{g^{\prime}}\right] d m \\
& +\frac{1}{2 \mu_{c}} \int_{D}\left[\left(z \overline{\varphi_{c}^{\prime \prime}}+\overline{\psi_{c}^{\prime}}\right) f^{\prime}+\left(k_{c} \varphi_{c}^{\prime}-\overline{\varphi_{c}^{\prime}}\right) \overline{g^{\prime}}\right] d m
\end{aligned}
$$

It then follows from (25) and (32) that

$$
\begin{aligned}
& \int_{\Omega} \overline{g^{\prime}} d m-\frac{\beta}{\kappa_{s}} \int_{\Omega \backslash \bar{D}} \overline{g^{\prime}} d m \\
& =\frac{1}{2 \mu_{s}} \int_{\Omega \backslash \bar{D}} \overline{\psi_{s}^{\prime}} f^{\prime} d m+\frac{1}{2 \mu_{c}} \int_{D}\left(z \overline{\varphi_{c}^{\prime \prime}}+\overline{\psi_{c}^{\prime}}\right) f^{\prime} d m+\frac{1}{2 \mu_{c}} \int_{D}\left(k_{c} \varphi_{c}^{\prime}-\overline{\varphi_{c}^{\prime}}\right) \overline{g^{\prime}} d m .
\end{aligned}
$$

Since $f$ and $g$ are arbitrary, we have

$$
\begin{equation*}
\frac{1}{\mu_{s}} \int_{\Omega \backslash \bar{D}} \overline{\psi_{s}^{\prime}} f^{\prime} d m+\frac{1}{\mu_{c}} \int_{D}\left(\overline{z \varphi_{c}^{\prime \prime}}+\overline{\psi_{c}^{\prime}}\right) f^{\prime} d m=0 \tag{35}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{2 \mu_{c}} \int_{D}\left(k_{c} \varphi_{c}^{\prime}-\overline{\varphi_{c}^{\prime}}\right) \overline{g^{\prime}} d m=\int_{\Omega} \overline{g^{\prime}} d m-\frac{\beta}{\kappa_{s}} \int_{\Omega \backslash \bar{D}} \overline{g^{\prime}} d m \tag{36}
\end{equation*}
$$

Similarly, we have from the second identity in (29)

$$
\int_{\partial \Omega} F D U_{s}=\int_{\partial \Omega} F D U_{m}=2 \kappa_{m} \int_{\partial \Omega} \bar{g} d z=2 \kappa_{m} \int_{\Omega} \overline{g^{\prime}} d m
$$

and hence

$$
\begin{aligned}
2 \kappa_{m} \int_{\Omega} \overline{g^{\prime}} d m & =\int_{\Omega} d(F D U) \\
& =\int_{\Omega}\left[\frac{\partial}{\partial \bar{z}}\left((f+\bar{g})\left(\varphi^{\prime}+\overline{\varphi^{\prime}}\right)\right)-\frac{\partial}{\partial z}\left((f+\bar{g})\left(z \overline{\varphi^{\prime \prime}}+\overline{\psi^{\prime}}\right)\right)\right] d m \\
& =\int_{\Omega}\left[\overline{g^{\prime}}\left(\varphi^{\prime}+\overline{\varphi^{\prime}}\right)-f^{\prime}\left(z \overline{\varphi^{\prime \prime}}+\overline{\psi^{\prime}}\right)\right] d m
\end{aligned}
$$

Since $f$ and $g$ are arbitrary and (32) holds, we obtain

$$
\begin{equation*}
\int_{\Omega \backslash \bar{D}} \overline{\psi_{s}^{\prime}} f^{\prime} d m+\int_{D}\left(z \overline{\varphi_{c}^{\prime \prime}}+\overline{\psi_{c}^{\prime}}\right) f^{\prime} d m=0 \tag{37}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{D}\left(\varphi_{c}^{\prime}+\overline{\varphi_{c}^{\prime}}\right) \overline{g^{\prime}} d m=2 \kappa_{m} \int_{\Omega} \overline{g^{\prime}} d m-2 \beta \int_{\Omega \backslash \bar{D}} \overline{g^{\prime}} d m \tag{38}
\end{equation*}
$$

Since $\mu_{s} \neq \mu_{c}$ by the assumption (5), we infer from (35) and (37) that

$$
\int_{D}\left(z \overline{\varphi_{c}^{\prime \prime}}+\overline{\psi_{c}^{\prime}}\right) f^{\prime} d m=0
$$

or equivalently,

$$
\begin{equation*}
\int_{\partial D}\left(z \overline{\varphi_{c}^{\prime}}+\overline{\psi_{c}}\right) f^{\prime} d z=0 \tag{39}
\end{equation*}
$$

Note that (39) holds for all functions $f$ analytic in $\bar{\Omega}$. However, one can infer using Runge's approximation theorem that it holds for all $f$ analytic in a neighborhood of $\bar{D}$. So, by Lemma 3.1, there is an analytic function $\eta_{1}$ in $D$ such that

$$
\begin{equation*}
z \overline{\varphi_{c}^{\prime}}+\overline{\psi_{c}}=\eta_{1} \quad \text { on } \partial D \tag{40}
\end{equation*}
$$

On the other hand, (36) can be rewritten as

$$
\begin{equation*}
\frac{1}{2 \mu_{c}} \int_{D}\left(k \varphi_{c}^{\prime}-\overline{\varphi_{c}^{\prime}}-\frac{2 \beta \mu_{c}}{\kappa_{s}}\right) \overline{g^{\prime}} d m=\left(1-\frac{\beta}{\kappa_{s}}\right) \int_{\Omega} \overline{g^{\prime}} d m \tag{41}
\end{equation*}
$$

while (38) as

$$
\begin{equation*}
\int_{D}\left(\varphi_{c}^{\prime}+\overline{\varphi_{c}^{\prime}}-2 \beta\right) \overline{g^{\prime}} d m=2\left(\kappa_{m}-\beta\right) \int_{\Omega} \overline{g^{\prime}} d m \tag{42}
\end{equation*}
$$

Since $\kappa_{m}-\beta \neq 0$ by the assumption (5) and (33), we see from (41) and (42) that

$$
\int_{D}\left(k_{c} \varphi_{c}^{\prime}-\overline{\varphi_{c}^{\prime}}-\frac{2 \beta \mu_{c}}{\kappa_{s}}\right) \overline{g^{\prime}} d m-\gamma \int_{D}\left(\varphi_{c}^{\prime}+\overline{\varphi_{c}^{\prime}}-2 \beta\right) \overline{g^{\prime}} d m=0
$$

where

$$
\begin{equation*}
\gamma:=\frac{2 \mu_{c}\left(1-\frac{\beta}{\kappa_{s}}\right)}{2\left(\kappa_{m}-\beta\right)}=\frac{2 \mu_{c}\left(1-\frac{\alpha}{2}\right)}{2\left(\kappa_{m}-\beta\right)} . \tag{43}
\end{equation*}
$$

So by the same argument as above, we infer that the function

$$
\overline{\left(k_{c} \varphi_{c}-z \overline{\varphi_{c}^{\prime}}-\frac{2 \beta \mu_{c}}{\kappa_{s}} z\right)}-\gamma \overline{\left(\varphi_{c}+z \overline{\varphi_{c}^{\prime}}-2 \beta z\right)}
$$

can be continued analytically to $D$, namely, there is an analytic function $\eta_{2}$ in $D$ such that

$$
\left(k_{c} \varphi_{c}-z \overline{\varphi_{c}^{\prime}}-\frac{2 \beta \mu_{c}}{\kappa_{s}} z\right)-\gamma\left(\varphi_{c}+z \overline{\varphi_{c}^{\prime}}-2 \beta z\right)=\overline{\eta_{2}} \quad \text { on } \partial D,
$$

which can be rephrased as

$$
\begin{equation*}
\frac{k_{c}-\gamma}{1+\gamma} \varphi_{c}-z \overline{\varphi_{c}^{\prime}}=(1+\gamma)^{-1} \overline{\eta_{2}}+\delta z \quad \text { on } \partial D \tag{44}
\end{equation*}
$$

for some real constant $\delta$. Observe that if $f$ is analytic in $D$, then $\bar{f}$ is a solution (in the complex representation) to the Lamé system for any shear modulus $\mu>0$ and bulk modulus $\kappa>0$. So, $(1+\gamma)^{-1} \overline{\eta_{2}}+\delta z$ is a solution to any Lamé system. We claim (leaving the proof to the end of this proof) that

$$
\begin{equation*}
k_{*}:=\frac{k_{c}-\gamma}{1+\gamma}>1 \tag{45}
\end{equation*}
$$

It implies that $k_{*} \varphi_{c}-z \overline{\varphi_{c}^{\prime}}$ is a solution to the Lamé system with the shear modulus $\mu=1$ and the bulk modulus $2\left(k_{*}-1\right)^{-1}$. So, it follows from (44) and uniqueness of the Dirichlet boundary value problem for the Lamé system that

$$
k_{*} \varphi_{c}-z \overline{\varphi_{c}^{\prime}}=(1+\gamma)^{-1} \overline{\eta_{2}}+\delta z \quad \text { in } D
$$

By differentiating both sides with respect to $z$, we see that $\varphi_{c}^{\prime}$ is (real) constant in $D$. We also see from (40) that $\psi_{c}$ is constant in $D$. In fact, we have from (40) that

$$
\overline{\psi_{c}}=\eta_{1}+c z \quad \text { on } \partial D
$$

for some constant $c$. Since $\psi_{c}$ and $\eta_{1}+c z$ are analytic in $D$, it implies that they are constant in $D$.

Let us now prove (45). We see easily from (22), (33) and (43) that

$$
\gamma=\frac{\mu_{c}}{2 \kappa_{s}+\mu_{s}}
$$

So we have

$$
\left(k_{c}-\gamma\right)-(1+\gamma)=2 \mu_{c}\left(\frac{1}{\kappa_{c}}-\frac{1}{2 \kappa_{s}+\mu_{s}}\right)
$$

Then (45) follows by the third condition in (5). This completes the proof.

Proof of Theorem 1.1. According to Proposition 3.2, the solution $\mathbf{u}$ takes the form (31). Substituting this into the representation formula (19) yields

$$
\begin{aligned}
\mathbf{u}(\mathbf{x}) & =\mathbf{x}+2 a\left(\kappa_{c}-\kappa_{s}\right) \int_{D} \mathbf{I}: \widehat{\nabla}_{\mathbf{y}} \boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y}) d \mathbf{y}+\frac{\kappa_{m}-\kappa_{s}}{\pi\left(\mu_{s}+\kappa_{s}\right)} \nabla \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y} \\
& =\mathbf{x}+2 a\left(\kappa_{c}-\kappa_{s}\right) \int_{D} \operatorname{div} \widehat{\nabla}_{\mathbf{y}} \boldsymbol{\Gamma}^{s}(\mathbf{x}-\mathbf{y}) d \mathbf{y}+\frac{\kappa_{m}-\kappa_{s}}{\pi\left(\mu_{s}+\kappa_{s}\right)} \nabla \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}
\end{aligned}
$$

for $\mathbf{x} \in \Omega$. It then follows from (9) that

$$
\mathbf{u}=\nabla \chi \quad \text { in } \Omega \backslash D
$$

where

$$
\chi(\mathbf{x})=\frac{1}{2}|\mathbf{x}|^{2}+\frac{a\left(\kappa_{s}-\kappa_{c}\right)}{\pi\left(\mu_{s}+\kappa_{s}\right)} \int_{D} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}+\frac{\kappa_{m}-\kappa_{s}}{\pi\left(\mu_{s}+\kappa_{s}\right)} \int_{\Omega} \log |\mathbf{x}-\mathbf{y}| d \mathbf{y}
$$

Since $\mathbf{u}=\mathbf{x}$ on $\partial \Omega, \mathbf{u}=a \mathbf{x}+\mathbf{b}$ on $\partial D$ by (31), and divu is constant in $\Omega \backslash \bar{D}, \chi$ is a solution of the following over-determined problem:

$$
\begin{cases}\Delta \chi=\text { constant } &  \tag{46}\\ \text { in } \Omega \backslash \bar{D} \\ \nabla \chi=\mathbf{x} & \\ \nabla \chi=a \mathbf{x}+\mathbf{b} & \\ \nabla \text { on } \partial D\end{cases}
$$

It is proved in [10] (see also [11]) that if the problem (46) admits a solution if and only if $\Omega$ and $D$ are concentric disks. This completes the proof.

## Conclusion

In this paper we prove that if a coated inclusion in two dimensions is neutral to a bulk field, the core and the shell are concentric disks, provided that the assumption (5) on elastic moduli holds. It is not clear whether or not there is a coated structure neutral to shear fields, and it is of interest to clarify this. The shear field is the gradient of $\mathbf{h}(\mathbf{x})=\mathbf{A x}$ where $\mathbf{A}$ is a symmetric matrix whose trace is zero. An extension to three dimensions is also interesting. One can show by the same proof that Proposition 2.1 holds to be true in three dimensions. But, we do not know how to prove Proposition 3.2 in three dimensions.

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# Optimal stability estimate of the inverse boundary value problem by partial measurements 

Horst Heck and Jenn-Nan Wang<br>We dedicate this work to Giovanni Alessandrini for his 60th birthday and for his pioneering contribution in the stability estimates of inverse problems


#### Abstract

This manuscript was originally uploaded to arXiv in 2007 (arXiv:0708.3289v1). In the current version, we expand the Introduction and the list of references which are related to the results of this paper after 2007. In this work we establish log type stability estimates for the inverse potential and conductivity problems with partial Dirichlet-to-Neumann map, where the Dirichlet data is homogeneous on the inaccessible part. The proof is based on the uniqueness result of the inverse boundary value problem in Isakov's work [17].


Keywords: Schrödinger equation, stability, inverse problems.
MS Classification 2010: 35R30, 65N21.

## 1. Introduction

In this paper we study the stability question of the inverse boundary value problem for the Schrödinger equation with a potential and the conductivity equation by partial Cauchy data. This type of inverse problem with full data, i.e., Dirichlet-to-Neumann map, were first proposed by Calderón [6]. For three or higher dimensions, the uniqueness issue was settled by Sylvester and Uhlmann [29] and a reconstruction procedure was given by Nachman [27]. For two dimensions, Calderón's problem was solved by Nachman [28] for $W^{2, p}$ conductivities and by Astala and Päivärinta [3] for $L^{\infty}$ conductivities. This inverse problem is known to be ill-posed. A log-type stability estimate was derived by Alessandrini [1]. On the other hand, it was shown by Mandache [26] that the log-type estimate is optimal.

All results mentioned above are concerned with the full data. Over the last decade, the inverse problems with partial data have received a lot of attention. We list several earlier results $[5,12,13,15,17,18,19,21,22,24]$ and refer the reader to the survey article [20] for its detailed development and for related ref-
erences. After the uniqueness proof comes stability estimates. We summarize related results in the following.

- log log type: $[8,9,10,11,16,23,31]$.
- log type: $[2,4,7,14,25]$.

The method in [16] was based on [5] and a stability estimate for the analytic continuation proved in [32]. We believe that the log type estimate should be the right estimate for the inverse boundary problem, even with partial data. In this paper, motivated by the uniqueness proof in Isakov's work [17], we prove a log type estimate for the inverse boundary value problem under the same $a$ priori assumption on the boundary as given in [17]. Precisely, the inaccessible part of the boundary is either a part of a sphere or a plane. Also, one is able to use zero data on the inaccessible part of the boundary. The strategy of the proof in [17] follows the framework in [29] where complex geometrical optics solutions are key elements. A key observation in [17] is that when $\Gamma_{0}$ is a part of a sphere or a plane, we are able to use a reflection argument to guarantee that complex geometrical optics solutions have homogeneous data on $\Gamma_{0}$. Caro in [7] also used Isakov's idea to derive a log type estimate for the Maxwell equations. The articles $[2,4,14]$ have a common feature that the undetermined coefficients are known near the boundary.

Now we would like to describe the results in this work. Let $n \geq 3$ and $\Omega \subset \mathbb{R}^{n}$ be an open domain with smooth boundary $\partial \Omega$. Given $q \in L^{\infty}(\Omega)$, we consider the boundary value problem:

$$
\begin{align*}
(\Delta-q) u=0 & \text { in } \Omega \\
u=f & \text { on } \partial \Omega, \tag{1}
\end{align*}
$$

where $f \in H^{1 / 2}(\partial \Omega)$. Assume that 0 is not a Dirichlet eigenvalue of $\Delta-q$ on $\Omega$. Then (1) has a unique solution $u \in H^{1}(\Omega)$. The usual definition of the Dirichlet-to-Neumann map is given by

$$
\Lambda_{q} f=\left.\partial_{\nu} u\right|_{\partial \Omega}
$$

where $\partial_{\nu} u=\nabla u \cdot \nu$ and $\nu$ is the unit outer normal of $\partial \Omega$.
Let $\Gamma_{0} \subset \partial \Omega$ be an open part of the boundary of $\Omega$. We set $\Gamma=\partial \Omega \backslash \Gamma_{0}$. We further set $H_{0}^{1 / 2}(\Gamma):=\left\{f \in H^{1 / 2}(\partial \Omega): \operatorname{supp} f \subset \Gamma\right\}$ and $H^{-1 / 2}(\Gamma)$ the dual space of $H_{0}^{1 / 2}(\Gamma)$. Then the partial Dirichlet-to-Neumann map $\Lambda_{q, \Gamma}$ is defined as

$$
\Lambda_{q, \Gamma} f:=\left.\partial_{\nu} u\right|_{\Gamma} \in H^{-1 / 2}(\Gamma)
$$

where $u$ is the unique weak solution of (1) with Dirichlet Data $f \in H_{0}^{1 / 2}(\Gamma)$. In what follows, we denote the operator norm by

$$
\left\|\Lambda_{q, \Gamma}\right\|_{*}:=\left\|\Lambda_{q, \Gamma}\right\|_{H_{0}^{1 / 2}(\Gamma) \rightarrow H^{-1 / 2}(\Gamma)}
$$

We consider two types of domains in this paper:
(a) $\Omega$ is a bounded domain in $\left\{x_{n}<0\right\}$ and $\Gamma_{0}=\partial \Omega \cap\left\{x_{n}=0\right\}$;
(b) $\Omega$ is a subdomain of $B(a, R)$ and $\Gamma_{0}=\partial B(a, R) \cap \partial \Omega$ with $\Gamma_{0} \neq \partial B(a, R)$, where $B(a, R)$ is a ball centered at $a$ with radius $R$.
The main result of the paper reads as follows:
Theorem 1.1. Assume that $\Omega$ is given as in either (a) or (b). Let $N>0$, $s>\frac{n}{2}$ and $q_{j} \in H^{s}(\Omega)$ such that

$$
\begin{equation*}
\left\|q_{j}\right\|_{H^{s}(\Omega)} \leq N \tag{2}
\end{equation*}
$$

for $j=1,2$, and 0 is not a Dirichlet eigenvalue of $\Delta-q_{j}$ for $j=1,2$. Then there exist constants $C>0$ and $\sigma>0$ such that

$$
\begin{equation*}
\left\|q_{1}-q_{2}\right\|_{L^{\infty}(\Omega)} \leq C\left|\log \left\|\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right\|_{*}\right|^{-\sigma} \tag{3}
\end{equation*}
$$

where $C$ depends on $\Omega, N, n, s$ and $\sigma$ depends on $n$ and $s$.
Theorem 1.1 can be generalized to the conductivity equation. Let $\gamma \in$ $H^{s}(\Omega)$ with $s>3+\frac{n}{2}$ be a strictly positive function on $\bar{\Omega}$. The equation for the electrical potential in the interior without sinks or sources is

$$
\begin{aligned}
\operatorname{div}(\gamma \nabla u)=0 & \text { in } \quad \Omega \\
u=f & \text { on } \quad \partial \Omega .
\end{aligned}
$$

As above, we take $f \in H_{0}^{1 / 2}(\Gamma)$. The partial Dirichlet-to-Neumann map defined in this case is

$$
\Lambda_{\gamma, \Gamma}:\left.f \mapsto \gamma \partial_{\nu} u\right|_{\Gamma}
$$

Corollary 1.2. Let the domain $\Omega$ satisfy (a) or (b). Assume that $\gamma_{j} \geq N^{-1}>$ $0, s>\frac{n}{2}$, and

$$
\begin{equation*}
\left\|\gamma_{j}\right\|_{H^{s+3}(\Omega)} \leq N \tag{4}
\end{equation*}
$$

for $j=1,2$, and

$$
\begin{equation*}
\left.\partial_{\nu}^{\beta} \gamma_{1}\right|_{\Gamma}=\left.\partial_{\nu}^{\beta} \gamma_{2}\right|_{\Gamma} \quad \text { on } \quad \partial \Omega, \quad \forall \quad 0 \leq \beta \leq 1 \tag{5}
\end{equation*}
$$

Then there exist constants $C>0$ and $\sigma>0$ such that

$$
\begin{equation*}
\left\|\gamma_{1}-\gamma_{2}\right\|_{L^{\infty}(\Omega)} \leq C\left|\log \left\|\Lambda_{\gamma_{1}, \Gamma}-\Lambda_{\gamma_{2}, \Gamma}\right\|_{*}\right|^{-\sigma} \tag{6}
\end{equation*}
$$

where $C$ depend on $\Omega, N, n, s$ and $\sigma$ depend on $n, s$.
Remark 1.3. For the sake of simplicity, we impose the boundary identification condition (5) on conductivities. However, using the arguments in [1] (also see [16]), this condition can be removed. The resulting estimate is still in the form of (6) with possible different constant $C$ and $\sigma$.

## 2. Preliminaries

We first prove an estimate of the Riemann-Lebesgue lemma for a certain class of functions. Let us define

$$
g(y)=\|f(\cdot-y)-f(\cdot)\|_{L^{1}\left(\mathbb{R}^{n}\right)}
$$

for any $f \in L^{1}\left(\mathbb{R}^{n}\right)$. It is known that $\lim _{|y| \rightarrow 0} g(y)=0$.
Lemma 2.1. Assume that $f \in L^{1}\left(\mathbb{R}^{n}\right)$ and there exist $\delta>0, C_{0}>0$, and $\alpha \in(0,1)$ such that

$$
\begin{equation*}
g(y) \leq C_{0}|y|^{\alpha} \tag{7}
\end{equation*}
$$

whenever $|y|<\delta$. Then there exists a constant $C>0$ and $\varepsilon_{0}>0$ such that for any $0<\varepsilon<\varepsilon_{0}$ the inequality

$$
\begin{equation*}
|\mathcal{F} f(\xi)| \leq C\left(\exp \left(-\pi \varepsilon^{2}|\xi|^{2}\right)+\varepsilon^{\alpha}\right) \tag{8}
\end{equation*}
$$

holds with $C=C\left(C_{0},\|f\|_{L^{1}}, n, \delta, \alpha\right)$.
Proof. Let $G(x):=\exp \left(-\pi|x|^{2}\right)$ and set $G_{\varepsilon}(x):=\varepsilon^{-n} G\left(\frac{x}{\varepsilon}\right)$. Then we define $f_{\varepsilon}:=f * G_{\varepsilon}$. Next we write

$$
|\mathcal{F} f(\xi)| \leq\left|\mathcal{F} f_{\varepsilon}(\xi)\right|+\left|\mathcal{F}\left(f_{\varepsilon}-f\right)(\xi)\right|
$$

For the first term on the right hand side we get

$$
\begin{align*}
\left|\mathcal{F} f_{\varepsilon}(\xi)\right| & \leq|\mathcal{F} f(\xi)| \cdot\left|\mathcal{F} G_{\varepsilon}(\xi)\right| \\
& \leq\|f\|_{1}\left|\varepsilon^{-n} \varepsilon^{n} \mathcal{F} G(\varepsilon \xi)\right|  \tag{9}\\
& \leq\|f\|_{1} \exp \left(-\pi \varepsilon^{2}|\xi|^{2}\right)
\end{align*}
$$

To estimate the second term, we use the assumption (7) and derive

$$
\begin{aligned}
\left|\mathcal{F}\left(f_{\varepsilon}-f\right)(\xi)\right| \leq & \left\|f_{\varepsilon}-f\right\|_{1} \\
\leq & \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}}|f(x-y)-f(x)| G_{\varepsilon}(y) \mathrm{d} y \mathrm{~d} x \\
= & \int_{|y|<\delta} \int_{\mathbb{R}^{n}}|f(x-y)-f(x)| G_{\varepsilon}(y) \mathrm{d} x \mathrm{~d} y \\
& \quad+\int_{|y| \geq \delta} \int_{\mathbb{R}^{n}}|f(x-y)-f(x)| G_{\varepsilon}(y) \mathrm{d} x \mathrm{~d} y \\
= & I+I I .
\end{aligned}
$$

In view of (7) we can estimate

$$
\begin{aligned}
I & =\int_{|y|<\delta} g(y) G_{\varepsilon}(y) \mathrm{d} y \\
& \leq C_{0} \int_{|y|<\delta}|y|^{\alpha} G_{\varepsilon}(y) \mathrm{d} y \\
& =C_{0} \int_{S^{n-1}} \int_{0}^{\delta} r^{\alpha} \varepsilon^{-n} \exp \left(-\pi \varepsilon^{-2} r^{2}\right) r^{n-1} \mathrm{~d} r \mathrm{~d} \psi \\
& =C_{1} \int_{0}^{\delta} \varepsilon^{\alpha} u^{\alpha} \varepsilon^{-n} \exp \left(-u^{2}\right) \varepsilon^{n-1} u^{u-1} \varepsilon \mathrm{~d} u \\
& =C_{2} \varepsilon^{\alpha} \int_{0}^{\delta} u^{n+\alpha-1} \exp \left(-u^{2}\right) \mathrm{d} u=C_{3} \varepsilon^{\alpha}
\end{aligned}
$$

where $C_{3}=C_{3}\left(C_{0}, n, \delta, \alpha\right)$.
As for II, we obtain that for $\varepsilon$ sufficiently small

$$
\begin{aligned}
I I & =\int_{|y| \geq \delta} g(y) G_{\varepsilon}(y) \mathrm{d} y \\
& \leq 2\|f\|_{L^{1}} \int_{|y| \geq \delta} G_{\varepsilon}(y) \mathrm{d} y \\
& \leq C_{4}\|f\|_{1} \int_{\delta}^{\infty} \varepsilon^{-n} \exp \left(-\pi \varepsilon^{-2} r^{2}\right) r^{n-1} \mathrm{~d} r \\
& =C_{4}\|f\|_{1} \int_{\delta \varepsilon^{-1}}^{\infty} u^{n-1} \exp \left(-\pi u^{2}\right) \mathrm{d} u \\
& \leq C_{4}\|f\|_{1} \int_{\delta \varepsilon^{-1}}^{\infty} \exp (-\pi u) \mathrm{d} u \\
& \leq C_{4}\|f\|_{1} \frac{1}{\pi} \exp \left(-\pi \delta \varepsilon^{-1}\right) \leq C_{5} \varepsilon^{\alpha},
\end{aligned}
$$

where $C_{5}=C_{5}\left(\|f\|_{L^{1}}, n, \delta, \alpha\right)$. Combining the estimates for $I$, $I I$, and (9), we immediately get (8).

We now provide a sufficient condition on $f$, defined on $\Omega$, such that (7) in the previous lemma holds.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with $C^{1}$ boundary. Let $f \in$ $C^{\alpha}(\bar{\Omega})$ for some $\alpha \in(0,1)$ and denote by $\hat{f}$ the zero extension of $f$ to $\mathbb{R}^{n}$. Then there exists $\delta>0$ and $C>0$ such that

$$
\|\hat{f}(\cdot-y)-\hat{f}(\cdot)\|_{L^{1}\left(\mathbb{R}^{n}\right)} \leq C|y|^{\alpha}
$$

for any $y \in \mathbb{R}^{n}$ with $|y| \leq \delta$.

Proof. Since $\Omega$ is bounded and of class $C^{1}$, there exist a finite number of balls, say $m \in \mathbb{N}, B_{i}\left(x_{i}\right)$ with center $x_{i} \in \partial \Omega, i=1, \ldots, m$ and associated $C^{1}$ diffeomorphisms $\varphi_{i}: B_{i}\left(x_{i}\right) \rightarrow Q$ where $Q=\left\{x^{\prime} \in \mathbb{R}^{n-1}:\left\|x^{\prime}\right\| \leq 1\right\} \times(-1,1)$. Set $d=\operatorname{dist}\left(\partial \Omega, \partial\left(\bigcup_{i=1}^{m} B_{i}\left(x_{i}\right)\right)\right)>0$ and $\tilde{\Omega}_{\varepsilon}=\bigcup_{x \in \partial \Omega} B(x, \varepsilon)$, where $B(x, \varepsilon)$ denotes the ball with center $x$ and radius $\varepsilon>0$. Obviously, for $\varepsilon<d$, it holds that $\tilde{\Omega}_{\varepsilon} \subset \bigcup_{i=1}^{m} B_{i}\left(x_{i}\right)$. Let $x \in \partial \Omega$ and $0<|y|<\delta \leq d$, then for any $z_{1}, z_{2} \in B(x,|y|) \cap B_{i}\left(x_{i}\right)$ we get that

$$
\left|\varphi_{i}\left(z_{1}\right)-\varphi_{i}\left(z_{2}\right)\right| \leq\left\|\nabla \varphi_{i}\right\|_{L^{\infty}}\left|z_{1}-z_{2}\right| \leq C|y|
$$

for some constant $C>0$. Therefore, $\varphi_{i}\left(\tilde{\Omega}_{|y|} \cap B_{i}\left(x_{i}\right)\right) \subset\left\{x^{\prime} \in \mathbb{R}^{n-1}:\left\|x^{\prime}\right\| \leq\right.$ $1\} \times(-C|y|, C|y|)$. By the transformation formula this yields $\operatorname{vol}\left(\tilde{\Omega}_{|y|}\right) \leq C|y|$. Since $|y|<\delta$ we have $\hat{f}(x-y)-\hat{f}(x)=0$ for $x \notin \Omega \cup \tilde{\Omega}_{|y|}$. Now we write

$$
\begin{aligned}
\|\hat{f}(\cdot-y)-\hat{f}\|_{L^{1}\left(\mathbb{R}^{n}\right)}= & \int_{\Omega \backslash \tilde{\Omega}_{|y|}}|\hat{f}(x-y)-\hat{f}(x)| \mathrm{d} x \\
& +\int_{\tilde{\Omega}_{|y|}}|\hat{f}(x-y)-\hat{f}(x)| \mathrm{d} x \\
\leq & C \operatorname{vol}(\Omega)|y|^{\alpha}+2\|f\|_{L^{\infty}} \operatorname{vol}\left(\tilde{\Omega}_{|y|}\right) \\
\leq & C\left(|y|^{\alpha}+|y|\right) \leq C|y|^{\alpha}
\end{aligned}
$$

for $\delta \leq 1$.
Now let $q_{1}$ and $q_{2}$ be two potentials and their corresponding partial Dirichlet-to-Neumann maps are denoted by $\Lambda_{1, \Gamma}$ and $\Lambda_{2, \Gamma}$, respectively. The following identity plays a key role in the derivation of the stability estimate.

Lemma 2.3. Let $v_{j}$ solve (1) with $q=q_{j}$ for $j=1,2$. Further assume that $v_{1}=v_{2}=0$ on $\Gamma_{0}$. Then

$$
\int_{\Omega}\left(q_{1}-q_{2}\right) v_{1} \overline{v_{2}} \mathrm{~d} x=\left\langle\left(\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right) v_{1}, v_{2}\right\rangle .
$$

Proof. Let $u_{2}$ denote the solution of (1) with $q=q_{2}$ and $u_{2}=v_{1}$ on $\partial \Omega$. Therefore

$$
\begin{aligned}
& \int_{\Omega} \nabla v_{1} \cdot \overline{\nabla v_{2}}+q_{1} v_{1} \overline{v_{2}} \mathrm{~d} x=\left\langle\partial_{\nu} v_{1}, v_{2}\right\rangle \\
& \int_{\Omega} \nabla u_{2} \cdot \overline{\nabla v_{2}}+q_{2} u_{2} \overline{v_{2}} \mathrm{~d} x=\left\langle\partial_{\nu} u_{2}, v_{2}\right\rangle
\end{aligned}
$$

Setting $v:=v_{1}-u_{2}$ and $q_{0}=q_{1}-q_{2}$ we get after subtracting these identities

$$
\int_{\Omega} \nabla v \cdot \overline{\nabla v_{2}}+q_{2} v \overline{v_{2}}+q_{0} v_{1} \overline{v_{2}}=\left\langle\left(\Lambda_{1}-\Lambda_{2}\right) v_{1}, v_{2}\right\rangle .
$$

Since $v_{2}$ solves $\left(\Delta-q_{2}\right) v_{2}=0, v=0$ on $\partial \Omega$ and $v_{2}=0$ on $\Gamma_{0}$, we have

$$
\begin{gathered}
\int_{\Omega} \nabla v \cdot \overline{\nabla v_{2}}+q_{2} v \overline{v_{2}}=0, \\
\left\langle\left(\Lambda_{1}-\Lambda_{2}\right) v_{1}, v_{2}\right\rangle=\left\langle\left(\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right) v_{1}, v_{2}\right\rangle
\end{gathered}
$$

and the assertion follows.
In treating inverse boundary value problems, complex geometrical optics solutions play a very important role. We now describe the complex geometrical optics solutions that we are going to use in our proofs. We will follow the idea in [17]. Assume that $q_{1}, q_{2} \in L^{\infty}\left(\mathbb{R}^{n}\right)$ are compactly supported and are even in $x_{n}$, i.e.

$$
q_{1}^{*}\left(x_{1}, \cdots, x_{n-1}, x_{n}\right)=q_{1}\left(x_{1}, \cdots, x_{n-1}, x_{n}\right)
$$

and

$$
q_{2}^{*}\left(x_{1}, \cdots, x_{n-1}, x_{n}\right)=q_{2}\left(x_{1}, \cdots, x_{n-1}, x_{n}\right) .
$$

Hereafter, we denote

$$
h^{*}\left(x_{1}, \cdots, x_{n-1}, x_{n}\right)=h\left(x_{1}, \cdots, x_{n-1},-x_{n}\right) .
$$

Given $\xi=\left(\xi_{1}, \cdots, \xi_{n}\right) \in \mathbb{R}^{n}$. Let us first introduce new coordinates obtained by rotating the standard Euclidean coordinates around the $x_{n}$ axis such that the representation of $\xi$ in the new coordinates, denoted by $\tilde{\xi}$, satisfies $\tilde{\xi}=\left(\tilde{\xi}_{1}, 0, \cdots, 0, \tilde{\xi}_{n}\right)$ with $\tilde{\xi}_{1}=\sqrt{\xi_{1}^{2}+\cdots+\xi_{n-1}^{2}}$ and $\tilde{\xi}_{n}=\xi_{n}$. In the following we also denote by $\tilde{x}$ the representation of $x$ in the new coordinates. Then we define for $\tau>0$

$$
\begin{align*}
& \tilde{\rho}_{1}:=\left(\frac{\tilde{\xi}_{1}}{2}-\tau \tilde{\xi}_{n}, i|\tilde{\xi}|\left(\frac{1}{4}+\tau^{2}\right)^{1 / 2}, 0, \cdots, 0, \frac{\tilde{\xi}_{n}}{2}+\tau \tilde{\xi}_{1}\right) \\
& \tilde{\rho}_{2}:=\left(\frac{\tilde{\xi}_{1}}{2}+\tau \tilde{\xi}_{n},-i|\tilde{\xi}|\left(\frac{1}{4}+\tau^{2}\right)^{1 / 2}, 0, \cdots, 0, \frac{\tilde{\xi}_{n}}{2}-\tau \tilde{\xi}_{1}\right) \tag{10}
\end{align*}
$$

and let $\rho_{1}$ and $\rho_{2}$ be representations of $\tilde{\rho}_{1}$ and $\tilde{\rho}_{2}$ in the original coordinates. Note that $x_{n}=\tilde{x}_{n}$ and $\sum_{i=1}^{n} x_{i} y_{i}=\sum_{i=1}^{n} \tilde{x}_{i} \tilde{y}_{i}$. It is clear that, for $j=1,2$, $\rho_{j} \cdot \rho_{j}=0$ as well as $\rho_{j}^{*} \cdot \rho_{j}^{*}=0$ hold.

The construction given in [29] ensures that there are complex geometrical optics solutions $u_{j}=e^{i \rho_{j} \cdot x}\left(1+w_{j}\right)$ of $\left(\Delta-q_{j}\right) u_{j}=0$ in $\mathbb{R}^{n}, j=1,2$, and the functions $w_{j}$ satisfy $\left\|w_{j}\right\|_{L^{2}(K)} \leq C_{K} \tau^{-1}$ for any compact set $K \subset \mathbb{R}^{n}$. We then set

$$
\begin{align*}
& v_{1}(x)=e^{i \rho_{1} \cdot x}\left(1+w_{1}\right)-e^{i \rho_{1}^{*} \cdot x}\left(1+w_{1}^{*}\right) \\
& v_{2}(x)=e^{-i \rho_{2} \cdot x}\left(1+w_{2}\right)-e^{-i \rho_{2}^{*} \cdot x}\left(1+w_{2}^{*}\right) \tag{11}
\end{align*}
$$

From this definition it is clear that these functions are solutions of $\left(\Delta-q_{j}\right) v_{j}=0$ in $\mathbb{R}_{+}^{n}$ with $v_{j}=0$ on $x_{n}=0$.

## 3. Stability estimate for the potential

Now we are in the position to prove Theorem 1.1. We first consider the case (a) where $\Gamma_{0}$ is a part of a hyperplane. To construct the special solutions described in the previous section, we first perform zero extension of $q_{1}$ and $q_{2}$ to $\mathbb{R}_{n}^{+}$and then even extension to the whole $\mathbb{R}^{n}$. As in the last section, we can construct special geometrical optics solutions $v_{j}$ of the form (11) to $\left(\Delta-q_{j}\right) v_{j}=0$ in $\Omega$ for $j=1,2$. Note that $v_{1}=v_{2}=0$ on $\Gamma_{0}$. We now plug in these solutions into the identity (2.3) and write $q_{0}=q_{1}-q_{2}$. This gives

$$
\begin{align*}
& \left\langle\left(\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right) v_{1}, v_{2}\right\rangle \\
& =\int_{\Omega} q_{0} v_{1} \overline{v_{2}} \mathrm{~d} x \\
& =\int_{\Omega} q_{0}(x)\left(e^{i\left(\rho_{1}+\rho_{2}\right) \cdot x}\left(1+w_{1}\right)\left(1+\overline{w_{2}}\right)+e^{i\left(\rho_{1}^{*}+\rho_{2}^{*}\right) \cdot x}\left(1+w_{1}^{*}\right)\left(1+\overline{w_{2}^{*}}\right)\right. \\
& \left.\quad-e^{i\left(\rho_{1}+\rho_{2}^{*}\right) \cdot x}\left(1+w_{1}\right)\left(1+\overline{w_{2}^{*}}\right)-e^{i\left(\rho_{1}^{*}+\rho_{2}\right) \cdot x}\left(1+w_{1}^{*}\right)\left(1+\overline{w_{2}}\right)\right) \mathrm{d} x  \tag{12}\\
& =\int_{\Omega} q_{0}(x)\left(e^{i \xi \cdot x}+e^{i \xi^{*} \cdot x}\right) \mathrm{d} x+\int_{\Omega} q_{0}(x) f\left(x, w_{1}, w_{2}, w_{1}^{*}, w_{2}^{*}\right) \mathrm{d} x \\
& \quad \quad-\int_{\Omega} q_{0}(x)\left(e^{i\left(\rho_{1}+\rho_{2}^{*}\right) \cdot x}+e^{i\left(\rho_{1}^{*}+\rho_{2}\right) \cdot x}\right) \mathrm{d} x
\end{align*}
$$

where

$$
\begin{aligned}
& f= e^{i \xi \cdot x}\left(w_{1}+\overline{w_{2}}+w_{1} \overline{w_{2}}\right)+e^{i \xi^{*} \cdot x}\left(w_{1}^{*}+\overline{w_{2}^{*}}+w_{1}^{*} \overline{w_{2}^{*}}\right) \\
& \quad-e^{i\left(\rho_{1}^{*}+\rho_{2}\right) \cdot x}\left(w_{1}^{*}+\overline{w_{2}}+w_{1}^{*} \overline{w_{2}}\right)-e^{i\left(\rho_{1}+\rho_{2}^{*}\right) \cdot x}\left(w_{1}+\overline{w_{2}^{*}}+w_{1} \overline{w_{2}^{*}}\right) .
\end{aligned}
$$

The first term on the right hand side of (12) is equal to

$$
\int_{\mathbb{R}^{n}} q_{0}(x) e^{i \xi \cdot x} \mathrm{~d} x=\mathcal{F} q_{0}(\xi)
$$

because $q_{0}$ is even in $x_{n}$. For the second term, we use the estimate

$$
\left\|w_{1}\right\|_{2}+\left\|w_{1}^{*}\right\|_{2}+\left\|\overline{w_{2}}\right\|_{2}+\left\|\overline{w_{2}^{*}}\right\|_{2} \leq C \tau^{-1}
$$

to obtain

$$
\begin{equation*}
\left|\int_{\Omega} q_{0} f\left(x, w_{1}, w_{2}, w_{1}^{*}, w_{2}^{*}\right) \mathrm{d} x\right| \leq C\left\|q_{0}\right\|_{2} \tau^{-1} \tag{13}
\end{equation*}
$$

As for the last term on the right hand side of (12), we first observe that

$$
\left(\rho_{1}+\rho_{2}^{*}\right) \cdot x=\left(\tilde{\rho}_{1}+\tilde{\rho}_{2}^{*}\right) \cdot \tilde{x}=\tilde{\xi}_{1} \tilde{x}_{1}+2 \tau \tilde{\xi}_{1} \tilde{x}_{n}=\xi^{\prime} \cdot x^{\prime}+2 \tau\left|\xi^{\prime}\right| x_{n}
$$

and

$$
\left(\rho_{1}^{*}+\rho_{2}\right) \cdot x=\left(\tilde{\rho}_{1}^{*}+\tilde{\rho}_{2}\right) \cdot \tilde{x}=\tilde{\xi}_{1} \tilde{x}_{1}-2 \tau \tilde{\xi}_{1} \tilde{x}_{n}=\xi^{\prime} \cdot x^{\prime}-2 \tau\left|\xi^{\prime}\right| x_{n}
$$

where $\xi^{\prime}=\left(\xi_{1}, \cdots, \xi_{n-1}\right)$ and $x^{\prime}=\left(x_{1}, \cdots, x_{n-1}\right)$. Therefore, we can write

$$
\int_{\Omega} q_{0}(x) e^{i\left(\rho_{1}+\rho_{2}^{*}\right) \cdot x} \mathrm{~d} x=\mathcal{F} q_{0}\left(\xi^{\prime}, 2 \tau\left|\xi^{\prime}\right|\right)
$$

as well as

$$
\int_{\Omega} q_{0}(x) e^{i\left(\rho_{1}^{*}+\rho_{2}\right) \cdot x} \mathrm{~d} x=\mathcal{F} q_{0}\left(\xi^{\prime},-2 \tau\left|\xi^{\prime}\right|\right) .
$$

The Sobolev embedding and the assumptions on $q_{j}$ ensure that $q_{0} \in C^{\alpha}(\bar{\Omega})$ for $\alpha=s-\frac{n}{2}$ and therefore $q_{0}$ satisfies the assumption of Lemma 2.2. Applying Lemma 2.1 to $q_{0}$ yields that for $\varepsilon<\varepsilon_{0}$

$$
\begin{equation*}
\left|\mathcal{F} q_{0}\left(\xi^{\prime}, 2 \tau\left|\xi^{\prime}\right|\right)\right|+\left|\mathcal{F} q_{0}\left(\xi^{\prime},-2 \tau\left|\xi^{\prime}\right|\right)\right| \leq C\left(\exp \left(-\pi \varepsilon^{2}\left(1+4 \tau^{2}\right)\left|\xi^{\prime}\right|^{2}\right)+\varepsilon^{\alpha}\right) \tag{14}
\end{equation*}
$$

Finally, we estimate the boundary integral

$$
\begin{align*}
\left|\int_{\Gamma}\left(\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right) v_{1} \cdot v_{2} \mathrm{~d} \sigma\right| & \leq\left\|\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right\|_{*}\left\|v_{1}\right\|_{H^{\frac{1}{2}(\Gamma)}}\left\|v_{2}\right\|_{H^{\frac{1}{2}}(\Gamma)} \\
& \leq\left\|\Lambda_{1, \Gamma}-\Lambda_{2, \Gamma}\right\|_{*}\left\|v_{1}\right\|_{H^{1}(\Omega)}\left\|v_{2}\right\|_{H^{1}(\Omega)}  \tag{15}\\
& \leq C \exp (|\xi| \tau)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*} .
\end{align*}
$$

Combining (12), (13), (14), and (15) leads to the inequality

$$
\begin{equation*}
\left|\mathcal{F} q_{0}(\xi)\right| \leq C\left\{\exp (|\xi| \tau)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}+\exp \left(-\pi \varepsilon^{2}\left(1+4 \tau^{2}\right)\left|\xi^{\prime}\right|^{2}\right)+\varepsilon^{\alpha}+\frac{1}{\tau}\right\} \tag{16}
\end{equation*}
$$

for all $\xi \in \mathbb{R}^{n}$ and $\varepsilon<\varepsilon_{0}$, where $C$ only depends on a priori data on the potentials.

Next we would like to estimate the norm of $q_{0}$ in $H^{-1}$. As usual, other estimates of $q_{0}$ in more regular norms can be obtained by interpolation. To begin, we set $Z_{R}=\left\{\xi \in \mathbb{R}^{n}:\left|\xi_{n}\right|<R\right.$ and $\left.\left|\xi^{\prime}\right|<R\right\}$. Note that $B(0, R) \subset$ $Z_{R} \subset B(0, c R)$ for some $c>0$. Now we use the a priori assumption on potentials and (16) and calculate

$$
\begin{align*}
\left\|q_{0}\right\|_{H^{-1}}^{2} \leq & \int_{Z_{R}}\left|\mathcal{F} q_{0}(\xi)\right|^{2}\left(1+|\xi|^{2}\right)^{-1} \mathrm{~d} \xi+\int_{Z_{R^{c}}}\left|\mathcal{F} q_{0}(\xi)\right|^{2}\left(1+|\xi|^{2}\right)^{-1} \mathrm{~d} \xi \\
\leq & \int_{Z_{R}}\left|\mathcal{F} q_{0}(\xi)\right|^{2}\left(1+|\xi|^{2}\right)^{-1} \mathrm{~d} \xi+C R^{-2}  \tag{17}\\
\leq & C\left\{R^{n} \exp (c R \tau)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}^{2}+R^{n} \varepsilon^{2 \alpha}+R^{n} \tau^{-2}+R^{-2}\right. \\
& \left.\quad+\int_{-R}^{R} \int_{B^{\prime}(0, R)} \exp \left(-2 \pi \varepsilon^{2}\left(1+4 \tau^{2}\right)\left|\xi^{\prime}\right|^{2}\right) \mathrm{d} \xi^{\prime} \mathrm{d} \xi_{n}\right\}
\end{align*}
$$

here $B^{\prime}\left(x^{\prime}, R\right)$ denotes the ball in $\mathbb{R}^{n-1}$ with center $x^{\prime}$ and radius $R>0$. For the second term on the right hand side of (17), we choose $\varepsilon=\left(1+4 \tau^{2}\right)^{-1 / 4}$ with $\tau \geq \tau_{0} \gg 1$ and integrate

$$
\begin{align*}
& \int_{-R}^{R} \int_{B^{\prime}(0, R)} \exp \left(-2 \pi \varepsilon^{2}\left(1+4 \tau^{2}\right)\left|\xi^{\prime}\right|^{2}\right) \mathrm{d} \xi^{\prime} \mathrm{d} \xi_{n} \\
& \quad=2 R \int_{B^{\prime}(0, R)} \exp \left(-2 \pi\left(1+4 \tau^{2}\right)^{1 / 2}\left|\xi^{\prime}\right|^{2}\right) \mathrm{d} \xi^{\prime} \\
& \quad=2 R \int_{S^{n-2}} \int_{0}^{R} r^{n-2} \exp \left(-2 \pi\left(\left(1+4 \tau^{2}\right)^{1 / 4} r\right)^{2}\right) \mathrm{d} r \mathrm{~d} \omega  \tag{18}\\
& \quad \leq C R\left(1+4 \tau^{2}\right)^{-(n-1) / 4} \int_{0}^{\infty} u^{n-2} \exp \left(-2 \pi u^{2}\right) \mathrm{d} u \\
& \quad \leq C R \tau^{-(n-1) / 2}
\end{align*}
$$

Plugging (18) into (17) with the choice of $\varepsilon=\left(1+4 \tau^{2}\right)^{-1 / 4}$ we get for $R>1$

$$
\begin{align*}
\left\|q_{0}\right\|_{H^{-1}}^{2} & \leq C\left\{R^{n} \exp (c R \tau)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}^{2}+R^{n} \tau^{-\alpha}+R \tau^{-(n-1) / 2}+R^{-2}\right\} \\
& \leq C\left\{R^{n} \exp (c R \tau)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}^{2}+R^{n} \tau^{-\tilde{\alpha}}+R^{-2}\right\} \tag{19}
\end{align*}
$$

where $\tilde{\alpha}=\min \{\alpha,(n-1) / 2\}$.
Observing from (19), we now choose $\tau$ such that $R^{n} \tau^{-\tilde{\alpha}}=R^{-2}$, namely, $\tau=R^{(n+2) / \widetilde{\alpha}}$. Substituting such $\tau$ back to (19) yields

$$
\begin{equation*}
\left\|q_{0}\right\|_{H^{-1}}^{2} \leq C\left\{R^{n} \exp \left(c R^{\frac{n+2}{\alpha}+1}\right)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}^{2}+R^{-2}\right\} \tag{20}
\end{equation*}
$$

Finally, we choose a suitable $R$ so that

$$
R^{n} \exp \left(c R^{\frac{n+2}{\alpha}+1}\right)\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}^{2}=R^{-2},
$$

i.e., $R=\left|\log \left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}\right|^{\gamma}$ for some $0<\gamma=\gamma(n, \tilde{\alpha})$. Thus, we obtain from (20) that

$$
\begin{equation*}
\left\|q_{1}-q_{2}\right\|_{H^{-1}(\Omega)} \leq C\left|\log \left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}\right|^{-\gamma} . \tag{21}
\end{equation*}
$$

The derivation of (21) is legitimate under the assumption that $\tau$ is large. To make sure that it is true, we need to take $R$ sufficiently large, i.e. $R>R_{0}$ for some large $R_{0}$. Consequently, there exists $\tilde{\delta}>0$ such that if $\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*}<\tilde{\delta}$ then (21) holds. For $\left\|\Lambda_{1}-\Lambda_{2}\right\|_{*} \geq \tilde{\delta}$, (21) is automatically true with a suitable constant $C$ when we take into account the a priori bound (2).

The estimate (3) is now an easy consequence of the interpolation theorem. Precisely, let $\epsilon>0$ such that $s=\frac{n}{2}+2 \epsilon$. Using that $\left[H^{t_{0}}(\Omega), H^{t_{1}}(\Omega)\right]_{\beta}=$ $H^{t}(\Omega)$ with $t=(1-\beta) t_{0}+\beta t_{1}$ (see e.g. [30, Theorem 1 in 4.3.1]) and the

Sobolev embedding theorem, we get $\left\|q_{1}-q_{2}\right\|_{L^{\infty}} \leq C\left\|q_{1}-q_{2}\right\|_{H^{\frac{n}{2}+\epsilon}} \leq C \| q_{1}-$ $q_{2}\left\|_{H^{t_{0}}}^{(1-\beta)}\right\| q_{1}-q_{2} \|_{H^{t_{1}}}^{\beta}$. Setting $t_{0}=-1$ and $t_{1}=s$ we end up with

$$
\left\|q_{1}-q_{2}\right\|_{L^{\infty}(\Omega)} \leq C\left\|q_{1}-q_{2}\right\|_{H^{-1}(\Omega)}^{\frac{\epsilon}{s+1}}
$$

which yields the desired estimate (3) with $\sigma=\gamma \frac{\epsilon}{s+1}$.
We now turn to case (b). With a suitable translation and rotation, it suffices to assume $a=(0, \cdots, 0, R)$ and $0 \notin \bar{\Omega}$. As in [17], we shall use Kelvin's transform:

$$
\begin{equation*}
y=\left(\frac{2 R}{|x|}\right)^{2} x \quad \text { and } \quad x=\left(\frac{2 R}{|y|}\right)^{2} y \tag{22}
\end{equation*}
$$

Let

$$
\tilde{u}(y)=\left(\frac{2 R}{|y|}\right)^{n-2} u(x(y))
$$

then

$$
\left(\frac{|y|}{2 R}\right)^{n+2} \Delta_{y} \tilde{u}(y)=\Delta_{x} u(x)
$$

Denote by $\tilde{\Omega}$ the transformed domain of $\Omega$. In view of this transform, $\Gamma_{0}$ now becomes $\tilde{\Gamma}_{0} \subset\left\{y_{n}=2 R\right\}$ and $\Gamma$ is transformed to $\tilde{\Gamma}$ and $\tilde{\Gamma}=\partial \tilde{\Omega} \cap\left\{y_{n}>2 R\right\}$. On the other hand, if $u(x)$ satisfies $\Delta u-q(x) u=0$ in $\Omega$, then $\tilde{u}$ satisfies

$$
\begin{equation*}
\Delta \tilde{u}-\tilde{q} \tilde{u}=0 \quad \text { in } \quad \tilde{\Omega}, \tag{23}
\end{equation*}
$$

where

$$
\tilde{q}(y)=\left(\frac{2 R}{|y|}\right)^{4} q(x(y))
$$

Therefore, for (23) we can define the partial Dirichlet-to-Neumann map $\tilde{\Lambda}_{\tilde{q}, \tilde{\Gamma}}$ acting boundary functions with homogeneous data on $\tilde{\Gamma}_{0}$.

We now want to find the relation between $\Lambda_{q, \Gamma}$ and $\tilde{\Lambda}_{\tilde{q}, \tilde{\Gamma}}$. It is easy to see that for $f, g \in H_{0}^{1 / 2}(\Gamma)$

$$
\left\langle\Lambda_{q, \Gamma} f, g\right\rangle=\int_{\Omega}(\nabla u \cdot \nabla \bar{v}+q u \bar{v}) \mathrm{d} x
$$

where $u$ solves

$$
\begin{aligned}
\Delta u-q u & =0
\end{aligned} \quad \text { in } \quad \Omega,
$$

and $v \in H^{1}(\Omega)$ with $\left.v\right|_{\partial \Omega}=g$. Defining

$$
\tilde{f}=\left.\left(\frac{2 R}{|y|}\right)^{n-2}\right|_{\partial \tilde{\Omega}} f, \quad \tilde{g}=\left.\left(\frac{2 R}{|y|}\right)^{n-2}\right|_{\partial \tilde{\Omega}} g
$$

and

$$
\tilde{v}(y)=\left(\frac{2 R}{|y|}\right)^{n-2} v(x(y))
$$

Then we have $\tilde{f}, \tilde{g} \in H_{0}^{1 / 2}(\tilde{\Gamma})$ and

$$
\left\langle\Lambda_{q, \Gamma} f, g\right\rangle=\left\langle\tilde{\Lambda}_{\tilde{q}, \tilde{\Gamma}} \tilde{f}, \tilde{g}\right\rangle,
$$

in particular,

$$
\begin{equation*}
\left\langle\left(\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right) f, g\right\rangle=\left\langle\left(\tilde{\Lambda}_{\tilde{q}_{1}, \tilde{\Gamma}}-\tilde{\Lambda}_{\tilde{q}_{2}, \tilde{\Gamma}}\right) \tilde{f}, \tilde{g}\right\rangle . \tag{24}
\end{equation*}
$$

With the assumption $0 \notin \bar{\Omega}$, the change of coordinates $x \rightarrow y$ by (22) is a diffeomorphism from $\bar{\Omega}$ onto $\bar{\Omega}$. Note that $(2 R /|y|)^{n-2}$ is a positive smooth function on $\partial \tilde{\Omega}$. Recall a fundamental fact from Functional Analysis:

$$
\begin{equation*}
\left\|\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right\|_{*}=\sup \left\{\frac{\left|\left\langle\left(\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right) f, g\right\rangle\right|}{\|f\|_{H_{0}^{1 / 2}(\Gamma)}\|g\|_{H_{0}^{1 / 2}(\Gamma)}}: f, g \in H_{0}^{1 / 2}(\Gamma)\right\} . \tag{25}
\end{equation*}
$$

The same formula holds for $\left\|\tilde{\Lambda}_{\tilde{q}_{1}, \tilde{\Gamma}}-\tilde{\Lambda}_{\tilde{q}_{2}, \tilde{\Gamma}}\right\|_{*}$. On the other hand, it is not difficult to check that $\|f\|_{H_{0}^{1 / 2}(\Gamma)}$ and $\|\tilde{f}\|_{H_{0}^{1 / 2}(\tilde{\Gamma})},\|g\|_{H_{0}^{1 / 2}(\Gamma)}$ and $\|\tilde{g}\|_{H_{0}^{1 / 2}(\tilde{\Gamma})}$ are equivalent, namely, there exists $C$ depending on $\partial \Omega$ such that

$$
\begin{align*}
& \frac{1}{C}\|f\|_{H_{0}^{1 / 2}(\Gamma)} \leq\|\tilde{f}\|_{H_{0}^{1 / 2}(\tilde{\Gamma})} \leq C\|f\|_{H_{0}^{1 / 2}(\Gamma)} \\
& \frac{1}{C}\|g\|_{H_{0}^{1 / 2}(\Gamma)} \leq\|\tilde{g}\|_{H_{0}^{1 / 2}(\tilde{\Gamma})} \leq C\|g\|_{H_{0}^{1 / 2}(\Gamma)} \tag{26}
\end{align*}
$$

Putting together (24), (25), and (26) leads to

$$
\begin{equation*}
\left\|\tilde{\Lambda}_{\tilde{q}_{1}, \tilde{\Gamma}}-\tilde{\Lambda}_{\tilde{q}_{2}, \tilde{\Gamma}}\right\|_{*} \leq C\left\|\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right\|_{*} \tag{27}
\end{equation*}
$$

with $C$ only depending on $\partial \Omega$.
With all the preparations described above, we use case (a) for the domain $\tilde{\Omega}$ with the partial Dirichlet-to-Neumann map $\tilde{\Lambda}_{\tilde{q}, \tilde{\Gamma}}$. Therefore, we immediately obtain the estimate:

$$
\left\|\tilde{q}_{1}-\tilde{q}_{2}\right\|_{L^{\infty}(\tilde{\Omega})} \leq C\left|\log \left\|\tilde{\Lambda}_{\tilde{q}_{1}, \tilde{\Gamma}}-\tilde{\Lambda}_{\tilde{q}_{2}, \tilde{\Gamma}}\right\|_{*}\right|^{-\sigma}
$$

Finally, rewinding $\tilde{q}$ and using (27) yields the estimate (3).

## 4. Stability estimate for the conductivity

We aim to prove Corollary 1.2 in this section. We recall the following wellknown relation: let $q=\frac{\Delta \sqrt{\gamma}}{\sqrt{\gamma}}$ then

$$
\Lambda_{q, \Gamma}(f)=\left.\gamma^{-1 / 2}\right|_{\Gamma} \Lambda_{\gamma, \Gamma}\left(\left.\gamma^{-1 / 2}\right|_{\Gamma} f\right)+\left.\frac{1}{2}\left(\gamma^{-1} \partial_{\nu} \gamma\right)\right|_{\Gamma} f
$$

In view of the a priori assumption (5), we have that

$$
\left(\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right)(f)=\left.\gamma^{-1 / 2}\right|_{\Gamma}\left(\Lambda_{\gamma_{1}, \Gamma}-\Lambda_{\gamma_{2}, \Gamma}\right)\left(\left.\gamma^{-1 / 2}\right|_{\Gamma} f\right)
$$

where $\left.\gamma^{-1 / 2}\right|_{\Gamma}:=\left.\gamma_{1}^{-1 / 2}\right|_{\Gamma}=\left.\gamma_{2}^{-1 / 2}\right|_{\Gamma}$, which implies

$$
\begin{equation*}
\left\|\Lambda_{q_{1}, \Gamma}-\Lambda_{q_{2}, \Gamma}\right\|_{*} \leq C\left\|\Lambda_{\gamma_{1}, \Gamma}-\Lambda_{\gamma_{2}, \Gamma}\right\|_{*} \tag{28}
\end{equation*}
$$

for some $C=C(N)>0$. Hereafter, we set $q_{j}=\frac{\Delta \sqrt{\gamma_{j}}}{\sqrt{\gamma_{j}}}, j=1,2$. The regularity assumption (4) and Sobolev's embedding theorem imply that $q_{1}, q_{2} \in C^{1}(\bar{\Omega})$. Using this and (5), we conclude that $\hat{q}_{1}-\hat{q}_{2}$ satisfies the assumptions of Lemma 2.2 with $\alpha=1$. Therefore, Theorem 1.1 and (28) imply that

$$
\begin{equation*}
\left\|q_{1}-q_{2}\right\|_{L^{\infty}(\Omega)} \leq C\left|\log \left\|\Lambda_{\gamma_{1}, \Gamma}-\Lambda_{\gamma_{2}, \Gamma}\right\|_{*}\right|^{-\sigma_{1}} \tag{29}
\end{equation*}
$$

where $C$ depend on $\Omega, N, n, s$ and $\sigma_{1}$ depend on $n, s$. Next, we recall from [1, (26) on page 168] that

$$
\begin{equation*}
\left\|\gamma_{1}-\gamma_{2}\right\|_{L^{\infty}(\Omega)} \leq C\left\|q_{1}-q_{2}\right\|_{L^{\infty}(\Omega)}^{\sigma_{2}} \tag{30}
\end{equation*}
$$

for some $0<\sigma_{2}<1$, where $C=C(N, \Omega)$ and $\sigma_{2}=\sigma_{2}(n, s)$. Finally, putting together (29) and (30) yields (6) with $\sigma=\sigma_{1} \sigma_{2}$ and the proof of Corollary 1.2 is complete.

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# Uniqueness result for an inverse conductivity recovery problem with application to EEG 

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#### Abstract

Considering a geometry made of three concentric spherical nested layers (brain, skull, scalp), each with constant homogeneous conductivity, we establish a uniqueness result in inverse conductivity estimation, from partial boundary data in presence of a known source term. We make use of spherical harmonics and linear algebra computations, that also provide us with stability results and a robust reconstruction algorithm. As an application to electroencephalography (EEG), in a spherical 3-layer head model (brain, skull, scalp), we numerically estimate the skull conductivity from available data (electrical potential at electrodes locations on the scalp, vanishing current flux) and given pointwise dipolar sources in the brain.


Keywords: elliptic and Laplace-Poisson PDE, inverse conductivity recovery problem, spherical harmonics, EEG.
MS Classification 2010: 31B20, 33C55, 35J05, 35J25, 35Q61, 65R32, 92C55.

## 1. Introduction

We study an inverse conductivity recovery problem in the particular case of a spherical 3D domain $\Omega$ (a ball in $\mathbb{R}^{3}$ ) and for piecewise constant conductivity functions, of which one value is unknown. More precisely, we assume $\Omega$ to be made of 3 nested spherical layers, whose conductivity values are known in the innermost and outermost layers. We assume that the elliptic partial differential conductivity equation (conductivity PDE) holds with a given source term in divergence form supported in the innermost layer.

Provided a single measurement as a pair of Cauchy data on the boundary (open subset of the sphere $\partial \Omega$ ), we will establish uniqueness and stability properties together with a reconstruction algorithm for the intermediate conductivity. We will also perform some analysis in order to investigate robustness
of the reconstruction with respect to available measurements and sources information.

We face a very specific version of the many inverse conductivity issues for second order elliptic PDE under study nowadays. This one is related to piecewise constant conductivities in a spherical geometry in $\mathbb{R}^{3}$, and set from a single (Cauchy pair of partial) boundary measurement. Similar inverse conductivity recovery problems may be formulated in more general (Lipschitz smooth) domains of arbitrary dimension, with more general conductivities. They are often considered from (several or) infinitely many boundary measurements (pairs of Cauchy data, related Dirichlet-to-Neumann operator), and are called after Calderón, or after medical imaging processes (Electrical Impedance Tomography). Uniqueness and stability conductivity recovery issues are deeply discussed in [1, 2, 3, 6, 19, 29, 24].

More general inverse problems for elliptic PDEs, in particular transmission issues, are discussed in [20,25]. Stability properties of Cauchy boundary value problems are described in [5] (see also references therein).

A fundamental problem in experimental neuroscience is the inverse problem of source localization, which aims at locating the sources of the electric activity of the functioning human brain using non-invasive measurements, such as electroencephalography (EEG), see $[10,14,16,17,18,21]$.

EEG measures the effect of the electric activity of active brain regions through values of the electric potential obtained by a set of electrodes placed at the surface of the scalp [14] and serves for clinical (location of epilepsy foci) and cognitive studies of the living human brain.

The inverse source localization problem in EEG is influenced by the electric conductivities of the several head tissues and especially by the conductivity of the skull [30]. The human skull is a bony tissue consisting of compact and spongy bone compartments, whose distribution and density varies across individuals, and according to age, since humidity of tissues, and therefore their conductivity tends to decrease [28]. Therefore conductivity estimation techniques are required to minimize the uncertainty in source reconstruction due to the skull conductivity.

Typically, an inverse conductivity estimation problem aims at determining an unknown conductivity value inside a domain $\Omega$ from measurements acquired on the boundary $\partial \Omega$. In the EEG case, the measurements can be modeled as pointwise values obtained on a portion of the boundary $\partial \Omega$ (the upper part of the scalp) but they are also affected by noise and measurement errors. The questions arising are: the uniqueness of the skull conductivity for known sources inside the brain; the stability of this estimation; and a constructive estimation method.

Quite frequently, for piecewise constant conductivities, the sub-domain (supporting the unknown conductivity value) is also to be determined, in some cases
more importantly than the constant conductivity value itself (for example for tumor detection, see [7, Ch. 3] and references therein, [22, 23]). But in the case of EEG, the sub-domains containing the various tissues can be considered known, because they can be extracted from magnetic resonance images. And for simplicity, we only consider the inverse skull conductivity estimation problem in a three-layer spherical head geometry, using partial boundary EEG data. The dipolar sources positions and moments will be considered to be known. This may appear to be an unrealistic assumption because sources reconstruction is itself a difficult inverse problem. But in fact, in some situations there are prior assumptions as to the positions of the sources (in primary evoked electrical potentials), and the position of a source also constrains its orientation, because to the laminar organization of pyramidal neurons in the grey matter.

The overview of this work is as follows. In Section 2, we precise the model and the considered inverse conductivity recovery issue. Our main uniqueness and stability results are stated and proved in Section 3, while an application to EEG and a numerical study are given in Section 4. We then provide a short conclusion in Section 5.

## 2. Model, problems

### 2.1. Domain geometry, conductivity

We consider the inverse conductivity estimation problem in a spherical domain $\Omega \subset \mathbb{R}^{3}$ made of 3 concentric spherical layers (centered at 0 ), a ball $\Omega_{0}$, and 2 consecutive surrounding spherical shells $\Omega_{1}, \Omega_{2}$. Their respective boundaries are the spheres denoted as $S_{0}, S_{1}$, and $S_{2}$, with $S_{i}$ of radius $r_{i}$ such that $0<r_{0}<r_{1}<r_{2}$. We also put $\Omega_{3}=\mathbb{R}^{3} \backslash \bar{\Omega}=\mathbb{R}^{3} \backslash\left(\Omega \cup S_{2}\right)$.

For $i=0,1,2$, we assume that $\sigma$ is a real valued piecewise constant conductivity coefficient with values $\sigma_{i}>0$ in $\Omega_{i}$. Let also $\sigma_{3}=0$.

Note that in the present work, the values $\sigma_{i}$ of the conductivity in $\Omega_{i}$ for $i \neq 1$ outermost layers $\Omega_{0}, \Omega_{2}$ are assumed to be known.

In the EEG framework and for spherical three-layer head models, the domains $\Omega_{i}$ respectively represent the brain, the skull and the scalp tissues for $i=0,1,2$, as shown in Figure 1, see [17, 18]. There, under isotropic assumption, it holds that $0<\sigma_{1}<\sigma_{0} \simeq \sigma_{2}$.

Throughout the present work, the geometry $\Omega$ and the conductivity $\sigma$ will be assumed to satisfy the above assumptions.

More general situations are briefly discussed in Remark 3.2 and Section 5.


Figure 1: Spherical head model, with one source $\mathbf{C}_{q}, \mathbf{p}_{q}$.

### 2.2. PDE, source terms, statement of the problem

We consider conductivity Poisson equations

$$
\begin{equation*}
\nabla \cdot(\sigma \nabla u)=\mathcal{S} \text { or } \operatorname{div}(\sigma \operatorname{grad} u)=\mathcal{S} \text { in } \mathbb{R}^{3} \tag{1}
\end{equation*}
$$

(in the distributional sense), with a source term $\mathcal{S}$ taken to be a distribution on $\mathbb{R}^{3}$ compactly supported in $\Omega_{0}$.

We investigate situations where source terms $\mathcal{S}$ are of divergence form:

$$
\mathcal{S}=\nabla \cdot \mathbf{J}^{P}=\operatorname{div} \mathbf{J}^{P}
$$

for distributions $\mathbf{J}^{P}$ made of $Q$ pointwise dipolar sources located at $\mathbf{C}_{q} \in \Omega_{0}$ with (non zero) moments $\mathbf{p}_{q} \in \mathbb{R}^{3}$ :

$$
\begin{equation*}
\mathbf{J}^{P}=\sum_{q=1}^{Q} \mathbf{p}_{q} \delta_{\mathbf{C}_{q}}, \quad \text { whence } \mathcal{S}=\sum_{q=1}^{Q} \mathbf{p}_{q} \cdot \nabla \delta_{\mathbf{C}_{q}} \tag{2}
\end{equation*}
$$

where $\delta_{\mathbf{C}_{q}}$ is the Dirac distribution supported at $\mathbf{C}_{q} \in \Omega_{0}$. Therefore, in $\mathbb{R}^{3}$,

$$
\begin{equation*}
\nabla \cdot(\sigma \nabla u)=\sum_{q=1}^{Q} \mathbf{p}_{q} \cdot \nabla \delta_{\mathbf{C}_{q}} \tag{3}
\end{equation*}
$$

For the EEG case, under the quasi-static approximation and modeling the primary cerebral current $\mathbf{J}^{P}$ as in (2), Maxwell's equations imply that the conductivity PDE (3) drives the behaviour of the electric potential $u[17]$.

In this work, we consider the following inverse conductivity estimation problem in the 3-layered spherical framework of Section 2.1.

From (a single pair of) Cauchy boundary data $u=g$ in a (non-empty) open subset $\Gamma$ of $\partial \Omega=S_{2}$ and $\partial_{n} u=0$ on $S_{2}$ of a solution to (3), and from a (known) source term $\mathcal{S}$ given by (2), we want to recover the constant value $\sigma_{1}$ of the conductivity $\sigma$ in the intermediate layer $\Omega_{1}$.

In Section 3, we establish uniqueness properties of $\sigma_{1}$ from Cauchy data $u$ on $\Gamma \subset S_{2}, \partial_{n} u$ on $S_{2}$ and from the source term $\mathcal{S}$. A stability result is also given for $\Gamma=S_{2}$ and equation (1) for more general source terms $\mathcal{S}$ will be discussed as well.

Before, we still need to describe the PDE and associated boundary value problems in each of the consecutive layers $\Omega_{i}$.

### 2.3. Laplace-Poisson PDE and transmission issues

For $i=0,1,2,3$, write $u_{\mid \Omega_{i}}=u_{i}$ for the restriction to $\Omega_{i}$ of the solution $u$ to (3). We put $\partial_{n} u_{i}$ for the normal derivative of $u_{i}$ on spheres in $\bar{\Omega}_{i}$, the unit normal vector being taken towards the exterior direction (pointing to $\Omega_{i+1}$ ). In the present spherical setting, we actually have $\partial_{n}=\partial_{r}$.

For $i=1,2,3$, the following transmission conditions hold on $S_{i-1}$, in particular in $L^{2}\left(S_{i-1}\right)$, see $[10,14,16]$ (and Section 2.4):

$$
u_{i-1}=u_{i}, \quad \sigma_{i-1} \partial_{n} u_{i-1}=\sigma_{i} \partial_{n} u_{i}
$$

Linked by those boundary conditions, the solutions $u_{i}$ to (3) in $\Omega_{i}$ satisfy the following Laplace and Laplace-Poisson equations:

$$
\begin{cases}\Delta u_{i}=0 & \text { in } \Omega_{i}, i>0  \tag{4}\\ \Delta u_{0}=\frac{1}{\sigma_{0}} \sum_{q=1}^{Q} \mathbf{p}_{q} \cdot \nabla \delta_{\mathbf{C}_{q}} & \text { in } \Omega_{0}\end{cases}
$$

We will see (in Section 3.2.2) that the transmission from $\left[\begin{array}{c}u_{i} \\ \partial_{n} u_{i}\end{array}\right]$ on $S_{i}$ to $\left[\begin{array}{c}u_{i-1} \\ \partial_{n} u_{i-1}\end{array}\right]$ on $S_{i-1}$, for $i=1,2$, may be written

$$
\left[\begin{array}{c}
u_{i-1} \\
\partial_{n} u_{i-1}
\end{array}\right]_{\mid S_{i-1}}=\left[\begin{array}{cc}
1 & 0 \\
0 & \frac{\sigma_{i}}{\sigma_{i-1}}
\end{array}\right] \mathcal{T}\left(S_{i-1}, S_{i}\right)\left[\begin{array}{c}
u_{i} \\
\partial_{n} u_{i}
\end{array}\right]_{\mid S_{i}} .
$$

for some operator $\mathcal{T}\left(S_{i-1}, S_{i}\right)$ that accounts for the harmonicity of $u_{i}$ in $\Omega_{i}$ and that we will express using spherical harmonics.

Provided (a single pair of non identically vanishing smooth enough) Cauchy boundary data $u, \partial_{n} u$ on $\Gamma \subset S_{2}$, whenever $\Gamma \neq \emptyset$ is open, and for given conductivity values $\sigma_{0}, \sigma_{1}, \sigma_{2}$, uniqueness of $u$ holds on $S_{2}$, then on $S_{1}$ and $S_{0}$, as ensured by the above formulation and Holmgren's theorem. It is enough to assume that $u \in W^{1,2}(\Gamma)$ (the Sobolev-Hilbert space of $L^{2}(\Gamma)$ functions with first derivative in $\left.L^{2}(\Gamma)\right)$ and $\partial_{n} u \in L^{2}(\Gamma)$, see $[11,13,15]$.

We then face a preliminary data transmission issue from $\Gamma$ to $S_{0}$, a Cauchy boundary value problem for Laplace equation, which needs to be regularized in order to be well-posed [20]. This is usually done by Thykonov regularization or the addition of an appropriate constraint and may be solved using boundary elements methods, see $[8,14]$ and references therein.

In EEG, data are provided as pointwise values of $g$ at points in $\Gamma$ (electrodes measurements), and yet another extension step is needed in order to compute an estimate of $g$ on $S_{2}$, using best constrained approximation, see Section 4.

Concerning the source term $\mathcal{S}$, note that it's knowledge only determines $u_{0}$ on $S_{0}$ up to the addition of a harmonic function in $\Omega_{0}$. Indeed, by convolution with a fundamental solution of Laplace equation in $\mathbb{R}^{3}$, we see that

$$
\begin{equation*}
u_{s}(\mathbf{x})=\frac{1}{4 \pi} \sum_{q=1}^{Q} \frac{<\mathbf{p}_{q}, \mathbf{x}-\mathbf{C}_{q}>}{\left|\mathbf{x}-\mathbf{C}_{q}\right|^{3}}, \quad \mathbf{x} \notin\left\{\mathbf{C}_{q}\right\} \tag{5}
\end{equation*}
$$

satisfies $u_{s}(\mathbf{x}) \rightarrow 0$ at $|\mathbf{x}| \rightarrow \infty$,

$$
\Delta u_{s}=\sum_{q=1}^{Q} \mathbf{p}_{q} \cdot \nabla \delta_{\mathbf{C}_{q}}
$$

in $\mathbb{R}^{3}$, whence in $\Omega$ and $\Omega_{0}$, and $\Delta u_{s}=0$ outside $\Omega_{0}$. Solutions $u_{0}$ to (4) in $\Omega_{0}$ are then provided by $u_{s} / \sigma_{0}$ up to the addition of a harmonic function in $\Omega_{0}$. The later is in fact (uniquely) determined by the (transmitted) boundary conditions, see [10, Sec. 1.2], [14, 16, 21] where inverse source problems in the EEG setting are discussed, together with reconstruction algorithms.

### 2.4. Associated forward Neumann problem

Let $\phi \in L^{2}\left(S_{2}\right)$ (actually it is enough to take $\phi \in W^{-1 / 2,2}\left(S_{2}\right)$ ) of vanishing mean value on $S_{2}$. Then, there exists a solution $u$ to (3) in $\Omega$, Hölder continuous in $\bar{\Omega} \backslash\left\{\mathbf{C}_{q}\right\}$, which satisfies $\partial_{n} u=\phi$ on $S_{2}$; it is unique up to an additive constant. In particular, the associated Dirichlet boundary trace $u_{\mid S_{2}}$ is Hölder continuous on $S_{2}$. Indeed, looking to $u-u_{s}$ as a (weak) solution to a strictly elliptic PDE in a bounded smooth domain $\Omega$ or to a sequence of Laplace equations in the domains $\Omega_{i}$, variational formulation and Lax-Milgram theorem imply that $u-u_{s} \in W^{1,2}(\Omega)$ and the uniqueness property, see $[11,13,15]$.

Hence $u-u_{s}$ belongs to $W^{1 / 2,2}\left(S_{2}\right)$ and actually to $W^{1,2}\left(S_{2}\right)$. That $u$ possesses yet more regularity properties is established in [10, Prop. 1], see also [5] for stability results of Cauchy boundary transmission problems.

## 3. Conductivity recovery

### 3.1. Uniqueness result

Recall that the geometry $\Omega$ and the conductivity coefficients satisfy the hypotheses of Section 2.1. Let $\Gamma \subset S_{2}$ a (non empty) open set.

Assume the source term $\mathcal{S}$ given by (2) to be known, and not to be reduced to a single dipolar pointwise source located at the origin $\left(\mathcal{S} \neq \mathbf{p} \cdot \nabla \delta_{\mathbf{0}}\right)$.

Theorem 3.1. Let $\sigma, \sigma^{\prime}$ be piecewise constant conductivities in $\Omega$ associated to two values $\sigma_{1}, \sigma_{1}^{\prime}$ in $\Omega_{1}$ and equal values $\sigma_{0}, \sigma_{2}$ in $\Omega_{0}, \Omega_{2}$. If two solutions $u, u^{\prime}$ to (3) associated with $\sigma, \sigma^{\prime}$ and such that $\partial_{n} u=\partial_{n} u^{\prime}=0$ on $S_{2}$ coincide on $\Gamma: u_{\mid \Gamma}=u_{\mid \Gamma}^{\prime}$, then $\sigma_{1}=\sigma_{1}^{\prime}$.

This implies that a single pair of partial boundary Dirichlet data $u_{\mid \Gamma}$ on $\Gamma$ and Neumann data $\partial_{n} u=0$ (vanishing) on $S_{2}$ of a solution $u$ to (3) uniquely determines $\sigma_{1}>0$.

As the proof in Section 3.3 will show, source terms $\mathcal{S}$ that guarantee uniqueness are such that associated Dirichlet data $u_{\mid \Gamma}$ on $\Gamma$ do not identically vanish. Notice also that if no source is present, uniqueness fails (boundary data identically vanish on $S_{2}$ ). However, Theorem 3.1 would also hold true for non identically vanishing Neumann on $S_{2}$. We will discuss more general statements the Theorem in Remark 3.2 after the proof, see also Section 5.

In order to establish the result, we use spherical harmonics expansions that we now precise.

### 3.2. Spherical harmonics expansions

In order to express harmonic functions in the spherical shells and balls $\Omega_{i}$ and their boundary values on $S_{i}$, we use the spherical harmonics basis $r^{k} Y_{k m}(\theta, \varphi)$, $r^{-(k+1)} Y_{k m}(\theta, \varphi), k \geq 0,|m| \leq k$, in the spherical coordinates $(r, \theta, \varphi)$. These are homogeneous harmonic and anti-harmonic polynomials for which we refer to $[9, \mathrm{Ch} .9,10],[15, \mathrm{Ch} . \mathrm{II}, \mathrm{Sec} .7 .3]$ as for their properties. (the basis functions $Y_{k m}(\theta, \varphi)$ are products beteween associated Legendre functions of indices $k \geq 0,|m| \leq k$, applied to $\cos \theta$ and elements of the Fourier basis of index $m$ on circles in $\varphi$ (real or complex valued, $\cos m \varphi, \sin m \varphi$ or $e^{ \pm i m \varphi}$ ).

### 3.2.1. Source term, boundary data

The decomposition theorem [9, Thm 9.6], [15, Ch. II, Sec. 7.3, Prop. 6], is to the effect that the restriction $u_{i}$ of $u$ to $\Omega_{i}$ for $i=1,2$ may be expanded on the spherical harmonics basis as follows, at $(r, \theta, \varphi) \in \Omega_{i}$ :

$$
\begin{equation*}
u_{i}(r, \theta, \varphi)=\sum_{k=0}^{\infty} \sum_{m=-k}^{k}\left[\alpha_{i k m} r^{k}+\beta_{i k m} r^{-(k+1)}\right] Y_{k m}(\theta, \varphi) \in \Omega_{i} \tag{6}
\end{equation*}
$$

where $\alpha_{i k m}$ and $\beta_{i k m}$ are the spherical harmonic coefficients of the harmonic and anti-harmonic parts of $u_{i}$, respectively (harmonic inside or outside $\cup_{j \leq i} \Omega_{i}$ ). Similarly, because it is harmonic in a spherical layer surrounding $S_{0}$, the restriction $u_{0}$ of $u$ to $\Omega_{0}$ is given at points $(r, \theta, \varphi)$ with $r>\max _{q}\left|\mathbf{C}_{q}\right|>0$ by

$$
u_{0}(r, \theta, \varphi)=\sum_{k=0}^{\infty} \sum_{m=-k}^{k} \alpha_{0 k m} r^{k} Y_{k m}(\theta, \varphi)+u_{s}(r, \theta, \varphi)
$$

where $u_{s}$ given by (5) is expanded there as: $r^{-(k+1)} Y_{k m}(\theta, \varphi)$ :

$$
\begin{equation*}
u_{s}(r, \theta, \varphi)=\sum_{k, m} \beta_{0 k m} r^{-(k+1)} Y_{k m}(\theta, \varphi) \tag{7}
\end{equation*}
$$

Here, $\beta_{0 k m}$ are the spherical harmonic coefficients of the anti-harmonic (harmonic outside $\Omega_{0}$ ) function $u_{S}$.

The normal derivative of $u_{i}, i=0,1,2$, is then given in $\Omega_{i}$ (with $r>$ $\max _{q}\left|\mathbf{C}_{q}\right|$ for $i=0$ ) by:

$$
\begin{equation*}
\partial_{n} u_{i}(r, \theta, \varphi)=\sum_{k, m}\left[\alpha_{i k m} k r^{k-1}-\beta_{i k m}(k+1) r^{-(k+2)}\right] Y_{k m}(\theta, \varphi) \tag{8}
\end{equation*}
$$

On $S_{i}$, we put (because $u_{i} \in L^{2}\left(S_{i}\right)$ where the spherical harmonics form an orthogonal basis [9, Thm 5.12]):

$$
\begin{aligned}
u_{i}\left(r_{i}, \theta, \varphi\right) & =\sum_{k=0}^{\infty} \sum_{m=-k}^{k} \gamma_{i k m} Y_{k m}(\theta, \varphi), \\
\partial_{n} u_{i}\left(r_{i}, \theta, \varphi\right) & =\sum_{k=0}^{\infty} \sum_{m=-k}^{k} \delta_{i k m} Y_{k m}(\theta, \varphi),
\end{aligned}
$$

with $l^{2}$ summable coefficients $\gamma_{i k m}, \delta_{i k m}$ (that may be real or complex valued depending on the choice for $Y_{k m}$ ).

In particular, once the boundary data $u_{2}=g$ is extended from $\Gamma$ to $S_{2}$ (see $[8,14]$ and the discussion in Section 2.3), we have:

$$
u_{2}\left(r_{2}, \theta, \varphi\right)=\sum_{k, m} \gamma_{2 k m} Y_{k m}(\theta, \varphi)=\sum_{k, m} g_{k m} Y_{k m}(\theta, \varphi)
$$

with $g_{k m}=\gamma_{2 k m}$, whereas the corresponding $\delta_{2 k m}=0$ since $\partial_{n} u_{2}=0$ on $S_{2}$ (because $\sigma_{3}=0$ ).

### 3.2.2. Preliminary computations

Below, we write for sake of simplicity, for $i=0,1,2: \alpha_{i k}=\alpha_{i k m}, \beta_{i k}=\beta_{i k m}$, $\gamma_{i k}=\gamma_{i k m}, \delta_{i k}=\delta_{i k m}, g_{k}=g_{k m}$, for all $k \geq 0$, and every $|m| \leq k$ (we could also take the sums over $|m| \leq k)$.

Recall from Section 2.3 that the following transmission conditions hold on $S_{i-1}$ for $i=1,2,3$ :

$$
\Sigma_{i-1}\left[\begin{array}{c}
u_{i-1}  \tag{9}\\
\partial_{n} u_{i-1}
\end{array}\right]_{\mid S_{i-1}}=\Sigma_{i}\left[\begin{array}{c}
u_{i} \\
\partial_{n} u_{i}
\end{array}\right]_{\mid S_{i-1}}
$$

with

$$
\Sigma_{i}=\left[\begin{array}{cc}
1 & 0 \\
0 & \sigma_{i}
\end{array}\right] \text { hence } \Sigma_{i}^{-1}=\left[\begin{array}{cc}
1 & 0 \\
0 & \frac{1}{\sigma_{i}}
\end{array}\right] \text { and } \sigma_{i} \Sigma_{i}^{-1}=\left[\begin{array}{cc}
\sigma_{i} & 0 \\
0 & 1
\end{array}\right]
$$

By projection of (6), (8), onto (the orthogonal $L^{2}\left(S_{i}\right)$ basis of) spherical harmonics, and with

$$
T_{k}\left(r_{i}\right)=\left[\begin{array}{cc}
r_{i}^{k} & r_{i}^{-(k+1)} \\
k r_{i}^{k-1} & -(k+1) r_{i}^{-(k+2)}
\end{array}\right],
$$

we obtain for all $k \geq 0$ the following relations on $S_{i}$ :

$$
\left[\begin{array}{c}
\gamma_{i k} \\
\delta_{i k}
\end{array}\right]=T_{k}\left(r_{i}\right)\left[\begin{array}{c}
\alpha_{i k} \\
\beta_{i k}
\end{array}\right]
$$

In particular:

$$
\begin{equation*}
\beta_{i k}=\frac{r_{i}^{k+1}}{2 k+1}\left(k \gamma_{i k}-\delta_{i k}\right) . \tag{10}
\end{equation*}
$$

The transmission conditions (9) through $S_{i-1}$ express as:

$$
\Sigma_{i-1}\left[\begin{array}{c}
\gamma_{i-1 k} \\
\delta_{i-1 k}
\end{array}\right]=\Sigma_{i} T_{k}\left(r_{i-1}\right)\left[\begin{array}{c}
\alpha_{i k} \\
\beta_{i k}
\end{array}\right]
$$

Because $T_{k}\left(r_{i}\right)$ is invertible $\left(r_{i}>0\right)$, this implies that:

$$
\left[\begin{array}{c}
\gamma_{i-1 k} \\
\delta_{i-1 k}
\end{array}\right]=\Sigma_{i-1}^{-1} \Sigma_{i} T_{k}\left(r_{i-1}\right) T_{k}\left(r_{i}\right)^{-1}\left[\begin{array}{l}
\gamma_{i k} \\
\delta_{i k}
\end{array}\right]
$$

Therefore, in the spherical geometry, $\mathcal{T}\left(S_{i-1}, S_{i}\right)=T_{k}\left(r_{i-1}\right) T_{k}\left(r_{i}\right)^{-1}$ for the operator $\mathcal{T}\left(S_{i-1}, S_{i}\right)$ introduced at the end of Section 2.3.

Hence, because $\gamma_{2 k}=g_{k}$ and $\delta_{2 k}=0$ :

$$
\left[\begin{array}{c}
\delta_{0 k}  \tag{11}\\
\gamma_{0 k}
\end{array}\right]=\Sigma_{0}^{-1} \Sigma_{1} T_{k}\left(r_{0}\right) T_{k}\left(r_{1}\right)^{-1} \Sigma_{1}^{-1} \Sigma_{2} T_{k}\left(r_{1}\right) T_{k}\left(r_{2}\right)^{-1}\left[\begin{array}{c}
g_{k} \\
0
\end{array}\right]
$$

while

$$
\beta_{0 k}=\left[\begin{array}{ll}
0 & 1
\end{array}\right] T_{k}\left(r_{0}\right)^{-1}\left[\begin{array}{c}
\delta_{0 k}  \tag{12}\\
\gamma_{0 k}
\end{array}\right] .
$$

These formula express a linear relation between the source term coefficients $\beta_{0 k}$ and the boundary Dirichlet data with coefficients $g_{k}$, which is studied in Appendix and gives rise to (13) below. We already see the particular role of $\sigma_{1}$ that appears through $\Sigma_{1}^{-1}$ and $\Sigma_{1}$. This explains why, after multiplication by $\sigma_{1}$ and algebraic manipulations, we obtain in (13) a polynomial of degree 2 in $\sigma_{1}$.

### 3.2.3. Algebraic equations

As computed in Appendix, equations (11), (12) can be rewritten, for all $k \geq 0$, as:

$$
\begin{equation*}
B_{1}(k) \sigma_{1} \beta_{0 k}=\left(A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)\right) g_{k} \tag{13}
\end{equation*}
$$

with non negative quantities $A_{i}(k), i=0,1,2, B_{1}(k)$ that depend only on the geometry, on the given conductivity values $\sigma_{0}, \sigma_{2}$, and on $k$. Actually, $A_{1}(k), B_{1}(k)>0$ for all $k \geq 0$ while $A_{0}(k), A_{2}(k)>0$ for $k>0$ but $A_{0}(0)=A_{2}(0)=0$. In particular, for all $k \geq 0$ and for $\sigma_{1}>0$, we have $A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)>0$.

This implies that $\beta_{0 k}=0 \Leftrightarrow g_{k}=0$ and that for all $k$ such that $g_{k} \neq 0$, $\beta_{0 k} / g_{k}$ is real valued positive: the spherical harmonics basis diagonalizes the transmission relations.

### 3.3. Uniqueness proof

Proof. (Theorem 3.1) Assume that there exists another value $\sigma_{1}^{\prime}>0$ of the conductivity in $\Omega_{1}$ that gives rise to the same potential (and vanishing current flux) on $\Gamma \subset S_{2}$, from the same source term $u_{s}$ (same boundary measurements and coefficients $g_{k}=g_{k m}$, same sources term coefficients $\beta_{0 k}=\beta_{0 k m}$, given). Equation (13) then holds for both $\sigma_{1}, \sigma_{1}^{\prime}>0$. We thus get that either $\beta_{0 k}=$ $g_{k}=0$ or

$$
\frac{\beta_{0 k}}{g_{k}}=\frac{A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)}{B_{1}(k) \sigma_{1}}=\frac{A_{2}(k) \sigma_{1}^{\prime, 2}+A_{1}(k) \sigma_{1}^{\prime}+A_{0}(k)}{B_{1}(k) \sigma_{1}^{\prime}}
$$

whence

$$
\frac{A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)}{B_{1}(k) \sigma_{1}}-\frac{A_{2}(k) \sigma_{1}^{\prime, 2}+A_{1}(k) \sigma_{1}^{\prime}+A_{0}(k)}{B_{1}(k) \sigma_{1}^{\prime}}=0
$$

hence multiplying by $B_{1}(k) \sigma_{1} \sigma_{1}^{\prime}>0$ :

$$
\left(\sigma_{1}-\sigma_{1}^{\prime}\right)\left(A_{2}(k) \sigma_{1} \sigma_{1}^{\prime}-A_{0}(k)\right)=0
$$

Thus either $\sigma_{1}^{\prime}=\sigma_{1}$ and uniqueness holds or, for all values of $k \geq 0$ such that $\beta_{0 k} \neq 0$,

$$
A_{0}(k)=\sigma_{1} \sigma_{1}^{\prime} A_{2}(k)
$$

This holds for $k=0$ but for $k>0$ it implies that

$$
\frac{A_{0}(k)}{A_{2}(k)}=\sigma_{1} \sigma_{1}^{\prime}
$$

which could not be true for more than a single value of $k>0$. Indeed, the product $\sigma_{1} \sigma_{1}^{\prime}$ is constant while $A_{0}(k) / A_{2}(k)$ stricly increases with $k$, as we now show. We have:

$$
\begin{equation*}
\frac{A_{0}(k)}{A_{2}(k)}=\sigma_{0} \sigma_{2} k \frac{1-\left(\frac{r_{1}}{r_{2}}\right)^{2 k+1}}{(k+1)\left(\frac{r_{1}}{r_{2}}\right)^{2 k+1}+k}=\sigma_{0} \sigma_{2} k \frac{1-\varrho^{2 k+1}}{(k+1) \varrho^{2 k+1}+k} \tag{14}
\end{equation*}
$$

with $\varrho=r_{1} / r_{2}<1$, and we put:

$$
E(k)=\frac{1}{\sigma_{0} \sigma_{2}} \frac{A_{0}(k)}{A_{2}(k)}=\frac{1-\varrho^{2 k+1}}{1+\frac{k+1}{k} \varrho^{2 k+1}}, k>0, E(0)=0
$$

Because for $k>0, k+2 /(k+1)<(k+1) / k$ and $\varrho^{2 k+3}<\varrho^{2 k+1}$, the numerator of $E(k)$ strictly increases with $k$ while its denominator strictly decreases. Thus, $E$ is a strictly increasing function of $k$, which converges to 1 as $k \rightarrow \infty$.

Hence, among the $k>0$, the equation $E(k)=\frac{\sigma_{1} \sigma_{1}^{\prime}}{\sigma_{0} \sigma_{2}}$ admits at most one solution, and pairs $\sigma_{1}, \sigma_{1}^{\prime}>0$ cannot solve $A_{2}(k) \sigma_{1} \sigma_{1}^{\prime}-A_{0}(k)=0$ for more than 1 value of $k>0$ (actually, a necessary condition for $\sigma_{1}, \sigma_{1}^{\prime}$ to solve $A_{2}(k) \sigma_{1} \sigma_{1}^{\prime}-A_{0}(k)=0$ for 1 value of $k>0$ is that $\left.\sigma_{1} \sigma_{1}^{\prime} \in\left(0, \sigma_{0} \sigma_{2}\right)\right)$. So we must have $\sigma_{1}^{\prime}=\sigma_{1}$, as soon as $\beta_{0 k}$ (or $g_{k}$ ) does not vanish for at least 2 distinct values of $k$.

Finally, we show that potentials $u_{s}$ associated to pointwise dipolar source terms $\mathcal{S} \neq \mathbf{p} \cdot \nabla \delta_{\mathbf{0}}$ have at least 2 non-null coefficients $\beta_{0 k}$ in their spherical harmonic expansion. Indeed, assume that all the coefficients $\beta_{0 \mathrm{~km}}$ are 0 , except
for a single value of $k>0$, say $k_{0}$. From (7), the function $u_{s}$ is then a antiharmonic homogeneous polynomial of degree $k_{0}$ and for $r>\max _{q}\left|\mathbf{C}_{q}\right|>0$,

$$
\begin{aligned}
u_{s}(r, \theta, \varphi)=\sum_{|m| \leq k_{0}} \beta_{0 k_{0} m} r^{-\left(k_{0}+1\right)} Y_{k_{0} m}(\theta, \varphi) & \\
& =\frac{1}{r^{2 k_{0}+1}} \sum_{|m| \leq k_{0}} \beta_{0 k_{0} m} r^{k_{0}} Y_{k_{0} m}(\theta, \varphi)
\end{aligned}
$$

From [9, Ch. 5], the distribution $\mathcal{S}=\Delta u_{s}$ also coincides far from $\{0\}$ with a polynomial divided by an odd power of $r$. This contradicts the assumptions on $\mathcal{S}$, which has a pointwise support in $\Omega_{0}$ not reduced to $\{0\}$.

Remark 3.2. Theorem 3.1 is in fact valid for solutions to Equation (1) with more general source terms $\mathcal{S}$. To ensure uniqueness, it is indeed enough to assume that $u_{s}$ does not coincide with some homogeneous anti-harmonic polynomial of positive degree, so that it admits on $S_{0}$ at least two coefficients $\beta_{0 k} \neq 0$.

In the present spherical geometry, note that $u_{s}$ is equal on $S_{0}$ to a homogeneous harmonic polynomial if and only if so is $u$ on $S_{2}$.

The last part of the proof actually implies that potentials $u_{s}$ issued from pointwise dipolar source terms $\mathcal{S}$ with support in $\Omega_{0}$ not reduced to $\{0\}$ have infiniteley many coefficients $\beta_{0 k} \neq 0$.

### 3.4. Stability properties

We now establish a stability result for the inverse conductivity estimation problem with respect to the source term whenever $\Gamma=S_{2}$.

Proposition 3.3. Assume the source terms $\mathcal{S}, \mathcal{S}^{\prime}$ and the conductivities $\sigma$, $\sigma^{\prime}$ to satisfy the assumptions of Theorem 3.1. Let $u_{s}, u_{s}^{\prime}$ be the associated potentials through (5). Let $u, u^{\prime}$ be the associated solutions to (3) such that $\partial_{n} u=\partial_{n} u^{\prime}=0$ on $S_{2}$. Put $g, g^{\prime}$ for their boundary values on $S_{2}$. Then, there exist $c, c_{s}>0$ such that

$$
\left|\sigma_{1}-\sigma_{1}^{\prime}\right| \leq c\left\|g-g^{\prime}\right\|_{L^{2}\left(S_{2}\right)}+c_{s}\left\|u_{s}-u_{s}^{\prime}\right\|_{L^{2}\left(S_{0}\right)}
$$

Whenever $0<s_{m} \leq \sigma_{1}, \sigma_{1}^{\prime} \leq s_{M}$ for constants $s_{m}$, $s_{M}$, then $c, c_{s}$ do not depend on $\sigma_{1}, \sigma_{1}^{\prime}$ but on $s_{m}, s_{M}$.
REmark 3.4. For ordered lists of sources $\left(\mathbf{p}_{q}, \mathbf{C}_{q}\right),\left(\mathbf{p}_{q}^{\prime}, \mathbf{C}_{q}^{\prime}\right)$ with length $Q$, we can define the geometric distance

$$
d\left(\mathcal{S}, \mathcal{S}^{\prime}\right)=\sum_{q=1}^{Q}\left(\left|\mathbf{p}_{q}-\mathbf{p}_{q}^{\prime}\right|+\left|\mathbf{C}_{q}-\mathbf{C}_{q}^{\prime}\right|\right)
$$

If the sources are located far enough from $S_{0}$ in the sense that $\max \left(\left|\mathbf{C}_{q}\right|,\left|\mathbf{C}_{q}^{\prime}\right|\right) \leq$ $\rho<r_{0}$, and because $u_{s}$ is on $S_{0}$ a continuous function of $\mathbf{p}_{q}, \mathbf{C}_{q}$, we can rewrite the inequality in Proposition 3.3 as:

$$
\left|\sigma_{1}-\sigma_{1}^{\prime}\right| \leq c\left\|g-g^{\prime}\right\|_{L^{2}\left(S_{2}\right)}+c_{s}^{\prime} d\left(\mathcal{S}, \mathcal{S}^{\prime}\right)
$$

with $c_{s}^{\prime}=K(\rho) c_{s}$ for some constant $K(\rho)$ which depends on $\rho$. Hence, the conductivity $\sigma_{1}$ depends continuously on the (complete) Dirichlet boundary data $g$ (in $\left.L^{2}\left(S_{2}\right)\right)$ and on the source term $\mathcal{S}$, with appropriate topology.

Notice also the relation:

$$
\beta_{0 k m}=\frac{1}{2 k+1} \sum_{q=1}^{Q}\left\langle\mathbf{p}_{q}, \nabla\left(r^{k} Y_{k m}(\theta, \varphi)\right)\left(\mathbf{C}_{q}\right)\right\rangle_{L^{2}\left(S_{0}\right)}
$$

Finally, observe that the constants $c, c_{s}, c_{s}^{\prime}$ in the above inequalities also depend on the data $g^{\prime}$ whence on $\mathcal{S}^{\prime}$. The dependence between Dirichlet data $g^{\prime}$ on $S_{2}$ and the source term $\mathcal{S}^{\prime}$ can be precised by using, for instance, the last equality together with relation (13) between their coefficients $\left(g_{k}^{\prime}\right)$ and $\left(\beta_{0 k}\right)$, and then recalling the assumption $s_{m} \leq \sigma_{1}^{\prime} \leq s_{M}$.

Proof. Let

$$
\begin{equation*}
\varepsilon_{k}\left(\sigma_{1}, \beta_{0 k}, g_{k}\right)=B_{1}(k) \sigma_{1} \beta_{0 k}-\left(A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)\right) g_{k} \tag{15}
\end{equation*}
$$

It follows from (13) that $\sigma_{1}^{\prime} \varepsilon_{k}\left(\sigma_{1}, \beta_{0 k}, g_{k}\right)-\sigma_{1} \varepsilon_{k}\left(\sigma_{1}^{\prime}, \beta_{0 k}^{\prime}, g_{k}^{\prime}\right)=0$, whence we get

$$
0=\sigma_{1}^{\prime} \varepsilon_{k}\left(\sigma_{1}, \beta_{0 k}-\beta_{0 k}^{\prime}, g_{k}-g_{k}^{\prime}\right)+\sigma_{1}^{\prime} \varepsilon_{k}\left(\sigma_{1}, \beta_{0 k}^{\prime}, g_{k}^{\prime}\right)-\sigma_{1} \varepsilon_{k}\left(\sigma_{1}^{\prime}, \beta_{0 k}^{\prime}, g_{k}^{\prime}\right)
$$

But

$$
\sigma_{1}^{\prime} \varepsilon_{k}\left(\sigma_{1}, \beta_{0 k}^{\prime}, g_{k}^{\prime}\right)-\sigma_{1} \varepsilon_{k}\left(\sigma_{1}^{\prime}, \beta_{0 k}^{\prime}, g_{k}^{\prime}\right)=g_{k}^{\prime}\left(\sigma_{1}-\sigma_{1}^{\prime}\right)\left[A_{2}(k) \sigma_{1} \sigma_{1}^{\prime}-A_{0}(k)\right]
$$

so

$$
\begin{aligned}
& g_{k}^{\prime}\left(\sigma_{1}-\sigma_{1}^{\prime}\right)\left[A_{2}(k) \sigma_{1} \sigma_{1}^{\prime}-A_{0}(k)\right]= \\
& \quad-\sigma_{1}^{\prime}\left[B_{1}(k) \sigma_{1}\left(\beta_{0 k}-\beta_{0 k}^{\prime}\right)-\left(A_{2}(k) \sigma_{1}^{2}+A_{1}(k) \sigma_{1}+A_{0}(k)\right)\left(g_{k}-g_{k}^{\prime}\right)\right] .
\end{aligned}
$$

Recall that $A_{2}(k)>0$ for $k>0$ and other arguments of Section 3.3 together with the computations in Appendix show that $A_{0}(k) / A_{2}(k), A_{1}(k) / A_{2}(k)$, $r_{0}^{k+1} B_{1}(k) / A_{2}(k)$ are uniformely bounded in $k$ from above and from below (by strictly positive constants). We can then divide by $A_{2}(k)$, in order to
obtain

$$
\begin{aligned}
& \left|\sigma_{1}-\sigma_{1}^{\prime}\right|\left[\sum_{k, m}\left|g_{k}^{\prime}\right|^{2}\left|\sigma_{1} \sigma_{1}^{\prime}-\frac{A_{0}(k)}{A_{2}(k)}\right|^{2}\right]^{\frac{1}{2}} \\
& \leq \sqrt{2} \sigma_{1}^{\prime}\left[\sum_{k, m}\left(\sigma_{1}^{2}+\frac{A_{1}(k)}{A_{2}(k)} \sigma_{1}+\frac{A_{0}(k)}{A_{2}(k)}\right)^{2}\left|g_{k}-g_{k}^{\prime}\right|^{2}\right]^{\frac{1}{2}} \\
& \\
& \quad+\sqrt{2} \sigma_{1} \sigma_{1}^{\prime}\left[\sum_{k, m} \frac{B_{1}^{2}(k)}{A_{2}^{2}(k)}\left|\beta_{0 k}-\beta_{0 k}^{\prime}\right|^{2}\right]^{\frac{1}{2}} .
\end{aligned}
$$

In order to establish an upper bound, note that

$$
\sum_{k, m} \frac{B_{1}^{2}(k)}{A_{2}^{2}(k)}\left|\beta_{0 k}-\beta_{0 k}^{\prime}\right|^{2} \leq \sup _{k} \frac{r_{0}^{2(k+1)} B_{1}^{2}(k)}{A_{2}^{2}(k)} \sum_{k, m} r_{0}^{-2(k+1)}\left|\beta_{0 k}-\beta_{0 k}^{\prime}\right|^{2}
$$

with

$$
\sum_{k, m} r_{0}^{-2(k+1)}\left|\beta_{0 k}-\beta_{0 k}^{\prime}\right|^{2}=\left\|u_{s}-u_{s}^{\prime}\right\|_{L^{2}\left(S_{0}\right)}^{2}
$$

Moreover, since $g_{k}$ (as $\beta_{0 k}$ ) possess non vanishing values for infinitely many (at least two) values of $k$, and $\sigma_{1} \sigma_{1}^{\prime}-A_{0}(k) / A_{2}(k)$ can vanish for at most one value of $k$, it holds that:

$$
\sum_{k, m}\left|g_{k}^{\prime}\right|^{2}\left|\sigma_{1} \sigma_{1}^{\prime}-A_{0}(k) / A_{2}(k)\right|^{2}>0
$$

can be bounded from below under the assumptions on $\sigma_{1} \sigma_{1}^{\prime}$.

## 4. Application to EEG

In order to illustrate Proposition 3.3, we now perform a short numerical analysis of the inverse conductivity estimation problem in the spherical domain and the EEG setting described in Section 2. Measurements of the Dirichlet data $g$ on the scalp $S_{2}$ (pointwise values at electrodes locations) and known sources activity are expanded on the spherical harmonics basis, using the FindSources3D software ${ }^{1}$ (FS3D), see also [14]. We therefore have at our disposal the spherical harmonics coefficients $\left(g_{k m}, \beta_{0 k m}\right)$ for $0 \leq k \leq K$ for some $K>0$ and $|m| \leq k$.

[^7]
### 4.1. Reconstruction algorithm

As the reconstruction of the conductivity $\sigma_{1}$ does not depend on the spherical harmonics indices $m$, in order to increase the robustness of our reconstruction algorithm, the following normalization is applied over the different spherical harmonics indices $k$ :

$$
\left\{\begin{array}{l}
\tilde{g}_{k}=\sum_{|m| \leq k} g_{k m} \bar{\beta}_{0 k m}, \\
\tilde{\beta}_{0 k}=\sum_{|m| \leq k} \beta_{0 k m} \bar{\beta}_{0 k m}=\sum_{|m| \leq k}\left|\beta_{0 k m}\right|^{2} .
\end{array}\right.
$$

There, $\bar{\beta}_{0 k m}$ is the complex conjugate number to $\beta_{0 k m}$ (indeed, $\beta_{0 k m}$ could be complex valued if the basis elements $Y_{k m}$ are taken in their complex valued form).

The procedure is a least square minimization of the error equation obtained from (15) as a truncated finite sum for $K>0$ :

$$
\begin{equation*}
\sigma_{1}^{e s t}=\arg \min _{s} \sum_{k=0}^{K}\left|\varepsilon_{k}\left(s, \tilde{\beta}_{0 k}, \tilde{g}_{k}\right)\right|^{2} \tag{16}
\end{equation*}
$$

### 4.2. Numerical illustrations

We consider the EEG framework in the spherical three-layer head model, as described in Section 2.1, where the layers represent the brain, the skull and the scalp tissues, respectively. The radii of the spheres used in the numerical analysis are normalized to the values $r_{0}=0.87, r_{1}=0.92$ and $r_{2}=1$. In the present analysis, the brain and scalp tissue conductivities are set to $\sigma_{0}=\sigma_{2}=$ $0.33 \mathrm{~S} / \mathrm{m}$, while the skull conductivity $\sigma_{1}$ is to be recovered. When generating simulated EEG data through the associated forward simulation, we will set $\sigma_{1}=0.0042 \mathrm{~S} / \mathrm{m}$.

Our study uses simulated data associated to a single dipole and the minimization of (16) for the conductivity estimation. The algorithm is written as a MATLAB code and the forward simulations are run with the FS3D software.

We validate our reconstruction algorithm using simulated EEG data by FS3D (for solving the direct EEG problem). Of course, the EEG data are subject to some ambient noise and measurements errors, and the a priori knowledge on the sources is not perfect. The inverse conductivity estimation problem is sensitive to such perturbations though it possesses the stability property described in Proposition 3.3.

To investigate the stability of our algorithm with respect to the source term, we select a source term $\mathcal{S}$ made of a single dipole located at $\mathbf{C}_{1}=$ $(0.019,0.667,0.1)$, mimicking an EEG source at the frontal lobe of the brain, with moment $\mathbf{p}_{1}=(0.027,0.959,0.28)$. The associated spherical harmonics
coefficients $\tilde{g}_{k}$ and $\tilde{\beta}_{0 k}$ are computed for $0 \leq k \leq K=30$. The original source location $\mathbf{C}_{1}$ is replaced by inexact locations $\mathbf{C}_{1}^{n}$ for $n=1, \cdots, 20$ located at a constant distance from $\mathbf{C}_{1}$ (a percentage of the inner sphere radius $r_{0}$ ), as illustrated in Figure 2, while the source moment $\mathbf{p}_{1}$ is retained. For each new dipole location $\mathbf{C}_{1}^{n}$, the associated spherical harmonics coefficients $\tilde{\beta}_{0 k}^{n}$ are simulated. We perform conductivity estimation from the pairs $\tilde{g}_{k}, \tilde{\beta}_{0 k}^{n}$ (recall that $\tilde{g}_{k}$ correspond to the actual $\left.\tilde{\beta}_{0 k}\right)$.


Figure 2: Locations (in $\Omega_{0}$ ) of $\mathbf{C}_{1}$ (red bullet) and of the 20 points $\mathbf{C}_{1}^{n}$ (blue cross) surrounding it, for $\left|\mathbf{C}_{1}-\mathbf{C}_{1}^{n}\right|$ equal to $10 \%$ of $r_{0}$.

The effect of the source mislocation on the conductivity estimation is summarized in Figure 3 and Table 4.2 which respectively shows and lists the values and other characteristics of the estimated conductivities with respect to the distance between actual and inexact sources.

These preliminary results illustrate the influence of source mislocation on conductivity estimation, and the robustness character of our algorithm, in accordance with the stability result of Proposition 3.3. In order to penalize high frequencies and to get more accurate estimations, we will in particular introduce in the above criterion (16) appropriate multiplicative weights (decreasing with the index $k$ ).


Figure 3: Conductivity estimation results for various mislocations of the actual dipole used to simulate the EEG data: 20 dipole locations $\mathbf{C}_{1}^{n}$ are selected by displacing $\mathbf{C}_{1}$ by a constant distance, computed as a percent of the brain radius $r_{0}$ (on the abcissa axis). Displayed are: $\sigma_{1}$, the actual conductivity value used in the EEG data simulation, $\sigma_{1}^{\text {est }}$, the estimated conductivity value for each dipole position $\mathbf{C}_{1}^{n}$, and $\tilde{\sigma}_{1}^{\text {est }}$, the mean value of $\sigma_{1}^{\text {est }}$ among $n=1, \cdots, 20$.

| Dipole mislocation <br> $\left(\%\right.$ of radius $\left.r_{0}\right)$ | $\tilde{\sigma}_{1}^{\text {est }}$ | Standard <br> deviation | Mean of <br> relative errors |
| :---: | :---: | :---: | :---: |
| 0 | $4.200 \mathrm{e}-03$ | 0 | $1.858 \mathrm{e}-15$ |
| 0.1 | $4.195 \mathrm{e}-03$ | $1.450 \mathrm{e}-05$ | $3.123 \mathrm{e}-03$ |
| 1 | $4.187 \mathrm{e}-03$ | $1.629 \mathrm{e}-04$ | $3.318 \mathrm{e}-02$ |
| 5 | $4.160 \mathrm{e}-03$ | $7.703 \mathrm{e}-04$ | $1.511 \mathrm{e}-01$ |
| 10 | $4.741 \mathrm{e}-03$ | $1.512 \mathrm{e}-03$ | $3.350 \mathrm{e}-01$ |

Table 1: Conductivity estimation results, continued; Listed in columns are: (i) the distance between $\mathbf{C}_{1}$ and $\mathbf{C}_{1}^{n}$, (ii) the mean estimated conductivity value $\tilde{\sigma}_{1}^{\text {est }},(i i i)$ the standard deviation of $\sigma_{1}^{e s t},(i v)$ the mean value of the relative errors between $\sigma_{1}$ and $\sigma_{1}^{\text {est }}$.

## 5. Conclusion

Observe that our uniqueness result, Theorem 3.1, may be expressed as an identifiability property of the conductivity value (model parameter) $\sigma_{1}$ in the
relation (transfer function) from boundary data to sources (control to observation), $[12,26]$. This could be useful in order to couple EEG with additional modalities, like EIT (where $\partial_{n} u \neq 0$ is known on $\Gamma$ ) or even MEG (magnetoencephalography, which measures the magnetic field outside the head), and to simultaneously estimate both $\sigma_{1}$ and the source term $\mathcal{S}$ in situations where the latter is (partially) unknown.

Following Remark 3.2, we may also wish to recover possibly unknown information about the (spherical) geometry of $\Omega_{1}$ (like $r_{1}$ or/and $r_{0}$ ).

Situations with more than 3 spherical layers could be described similarly, which may help to consider more general conductivities (smooth but non constant) by piecewise constant discretization.

As Theorem 3.1, Proposition 3.3 would still hold true under a weaker sufficient condition for the source terms, according to which the associated potential on $S_{0}$ through (5) should admit at least 2 non-null coefficients (this is equivalent to the same property for $g$ on $S_{2}$, see Remark 3.2). Moreover, it could be extended to a stability property with respect to boundary data with close and non vanishing Neumann data on $S_{2}$. However, stability properties for situations with partial Dirichlet boundary data only (on $\Gamma \subset S_{2}$ ) would be weaker, see e.g. $[8,5,14]$.

We have also begun to study the same uniqueness and stability issues in more general (non-spherical) nested geometries, see [7, 22, 23, 27], and also [4] for a number of open problems.

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## Appendix: More computations related to Section 3.2.2

From (11), (12), we get for all $k \geq 0$,

$$
\beta_{0 k}=g_{k} \times\left[\begin{array}{ll}
0 & 1
\end{array}\right] T_{k}\left(r_{0}\right)^{-1} \Sigma_{0}^{-1} \Sigma_{1} T_{k}\left(r_{0}\right) T_{k}\left(r_{1}\right)^{-1} \Sigma_{1}^{-1} \Sigma_{2} T_{k}\left(r_{1}\right) T_{k}\left(r_{2}\right)^{-1}\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

The matrices $T_{k}\left(r_{i}\right)$ and $T_{k}\left(r_{i}\right)^{-1}$ can be written:

$$
T_{k}\left(r_{i}\right)=\left[\begin{array}{cc}
1 & 0 \\
0 & \frac{1}{r_{i}}
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
k & -(k+1)
\end{array}\right]\left[\begin{array}{cc}
r_{i}^{k} & 0 \\
0 & r_{i}^{-(k+1)}
\end{array}\right]
$$

$$
T_{k}\left(r_{j}\right)^{-1}=\frac{1}{2 k+1}\left[\begin{array}{cc}
r_{j}^{-k} & 0 \\
0 & r_{j}^{(k+1)}
\end{array}\right]\left[\begin{array}{cc}
k+1 & 1 \\
k & -1
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & r_{j}
\end{array}\right] .
$$

Their products that give an expression of $\mathcal{T}\left(S_{i-1}, S_{i}\right)$ in the spherical geometry are then such that:

$$
\begin{aligned}
& T_{k}\left(r_{i-1}\right) T_{k}\left(r_{i}\right)^{-1}=\frac{1}{2 k+1} \times \\
& {\left[\begin{array}{cc}
1 & 0 \\
0 & r_{i-1}^{-1}
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
k & -(k+1)
\end{array}\right]\left[\begin{array}{cc}
\left(\frac{r_{i-1}}{r_{i}}\right)^{k} & 0 \\
0 & \left(\frac{r_{i}}{r_{i-1}}\right)^{k+1}
\end{array}\right]\left[\begin{array}{cc}
k+1 & 1 \\
k & -1
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & r_{i}
\end{array}\right]} \\
& \\
& =\frac{1}{2 k+1}\left(\frac{r_{i}}{r_{i-1}}\right)^{k+1} \times \\
& {\left[\begin{array}{cc}
1 & 0 \\
0 & r_{i-1}^{-1}
\end{array}\right]\left[\begin{array}{cc}
1 & 1 \\
k & -(k+1)
\end{array}\right]\left[\begin{array}{cc}
\left(\frac{r_{i-1}}{r_{i}}\right)^{2 k+1} & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
k+1 & 1 \\
k & -1
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
0 & r_{i}
\end{array}\right] .}
\end{aligned}
$$

We can write

$$
T_{k}\left(r_{i-1}\right) T_{k}\left(r_{i}\right)^{-1}=\rho_{k}^{(i)}\left[\begin{array}{cc}
a_{k}^{(i)} & b_{k}^{(i)} \\
c_{k}^{(i)} & d_{k}^{(i)}
\end{array}\right]
$$

with

$$
\rho_{k}^{(i)}=\frac{1}{2 k+1}\left(\frac{r_{i}}{r_{i-1}}\right)^{k+1}, i=1,2, \rho_{k}^{(0)}=\frac{r_{0}^{k+1}}{2 k+1}
$$

and the real valued quantities, with their equivalent asymptotic behaviours as $k \rightarrow \infty$ :

$$
\begin{cases}a_{k}^{(i)}=(k+1)\left(\frac{r_{i-1}}{r_{i}}\right)^{2 k+1}+k & \sim k \\ b_{k}^{(i)}=r_{i}\left[\left(\frac{r_{i-1}}{r_{i}}\right)^{2 k+1}-1\right] \\ c_{k}^{(i)}=\frac{k(k+1)}{r_{i-1} r_{i}} b_{k}^{(i)} & \sim-r_{i} \\ d_{k}^{(i)}=\frac{r_{i}}{r_{i-1}}\left[k\left(\frac{r_{i-1}}{r_{i}}\right)^{2 k+1}+k+1\right] & \sim \frac{k r_{i}}{r_{i-1}}\end{cases}
$$

Define also the real valued quantities $e_{k}^{(0)}, f_{k}^{(0)}$ :

$$
e_{k}^{(0)}=k, f_{k}^{(0)}=f^{(0)}=-r_{0} .
$$

We have

$$
\left[\begin{array}{ll}
0 & 1
\end{array}\right] T_{k}\left(r_{0}\right)^{-1}=\rho_{k}^{(0)}\left[\begin{array}{ll}
e_{k}^{(0)} & f_{k}^{(0)}
\end{array}\right]
$$

Then, equation (13) holds true with:

$$
\begin{aligned}
B_{1}(k)= & \frac{\sigma_{0}}{\rho_{k}^{(0)} \rho_{k}^{(1)} \rho_{k}^{(2)}} \\
& \quad \text { whence } r_{0}^{k+1} B_{1}(k)=\sigma_{0}(2 k+1)^{3}\left(\frac{r_{0}}{r_{2}}\right)^{k+1} \sim 8 k^{3}\left(\frac{r_{0}}{r_{2}}\right)^{k+1}
\end{aligned}
$$

and

$$
\begin{cases}A_{1}(k)=\sigma_{0} e_{k}^{(0)} a_{k}^{(1)} a_{k}^{(2)}+\sigma_{2} f_{k}^{(0)} d_{k}^{(1)} c_{k}^{(2)} & \sim k^{3}\left(\sigma_{0}+\sigma_{2}\right) \\ A_{2}(k)=f_{k}^{(0)} c_{k}^{(1)} a_{k}^{(2)} & \sim k^{3} \\ A_{0}(k)=\sigma_{0} \sigma_{2} e_{k}^{(0)} b_{k}^{(1)} c_{k}^{(2)} & \sim k^{3} \sigma_{0} \sigma_{2}\end{cases}
$$

Observe that $r_{0}^{k+1} B_{1}(k)$ acts on $r_{0}^{-(k+1)} \beta_{0 k}$ that are members of an $l^{2}$ sequence (see Section 3.2.1 and equation (10) with $i=0$ ).

As in (14), one can show with the above expressions that the behaviours of the ratios $A_{i}(k) / A_{2}(k), B_{1}(k) / A_{2}(k)$ ensure that they all are uniformly bounded from below on from above by positive constants, for $k>0$.

Note also that

$$
\left\{\begin{array}{l}
B_{1}(k)=\sigma_{0} \tilde{B}_{1}(k), \\
A_{1}(k)=\sigma_{0} \tilde{A}_{10}(k)+\sigma_{2} \tilde{A}_{12}(k), \\
A_{2}(k)=\tilde{A}_{2}(k) \\
A_{0}(k)=\sigma_{0} \sigma_{2} \tilde{A}_{0}(k)
\end{array}\right.
$$

where $\tilde{A}_{i}, \tilde{A}_{i j}, \tilde{B}_{1}$ only depend on the spherical geometry.

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# Stable determination at the boundary of the optical properties of a medium: the static case 

Romina Gaburro<br>This manuscript is dedicated to Giovanni Alessandrini on the occasion of his $60^{\text {th }}$ birthday, to honour his outstanding contribution to the field of inverse problems and mathematical analysis and to thank him for having been such an inspirational guide to the author


#### Abstract

The problem of the stable determination of the coefficients of second order elliptic partial differential equations arising in inverse problems is considered. Results of uniqueness and stability at the boundary were obtained in [3] and extended in [8, 9] for the conductivity equation. The common features of these papers are the employment of the singular solutions and the monotonicity assumption introduced in [3]. We revisit the techniques adopted in these papers to stably determine the absorption coefficient in anisotropic media by means of Optical Tomography (OT) in the so-called static case. This also shows that the monotonicity assumption is realistic at least in the context of OT.


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## 1. Introduction

We start by considering the well known inverse conductivity problem. In absence of internal sources, the electrostatic potential $u$ in a conducting body, described by a domain $\Omega \subset \mathbb{R}^{n}$, is governed by the elliptic equation

$$
\begin{equation*}
\operatorname{div}(\sigma \nabla u)=0 \quad \text { in } \quad \Omega \tag{1}
\end{equation*}
$$

where the symmetric, positive definite matrix $\sigma=\sigma(x), x \in \Omega$, represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding $\sigma$ when the so called Dirichlet-to-Neumann (D-N) map

$$
\Lambda_{\sigma}:\left.\left.u\right|_{\partial \Omega} \in H^{\frac{1}{2}}(\partial \Omega) \longrightarrow \sigma \nabla u \cdot \nu\right|_{\partial \Omega} \in H^{-\frac{1}{2}}(\partial \Omega)
$$

is given for any $u \in H^{1}(\Omega)$ solution to (1.1). Here, $\nu$ denotes the unit outer normal to $\partial \Omega$. If measurements can be taken only on one portion $\Gamma$ of $\partial \Omega$, then the relevant map is called the local D-N map.

This problem arises in electrical resistivity tomography (ERT) (or more generally electrical impedance tomography EIT), a method used for subsurface geophysical imaging, industrial process monitoring and as an experimental medical imaging technique. Different materials display different electrical properties, so that a map of the conductivity $\sigma(x), x \in \Omega$ can be used to investigate internal properties of $\Omega$. The first mathematical formulation of the inverse conductivity problem is due to A. P. Calderón [19], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity by the D-N map.
The case when measurements can be taken all over the boundary has been studied extensively in the past and fundamental papers like [3, 37, 38, 54] show that the isotropic case can be considered solved. On the other hand the anisotropic case is still open and different lines of research have been pursued. One direction has been to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see [39, 40, 41, 46, 53]). The original work of [41] assumed that the metric was real-analytic with topological assumptions subsequently relaxed in [39, 40] in the context of local data. We also refer to the work [22] which introduced methods for studying the anisotropic Calderón problem on manifolds which are not real-analytic, but where the metric has a certain form. This result is based on the concept of limiting Carleman weights, earlier introduced in [36] for the Euclidean case and partial data. We refer to [20] and [35] for related works on the stability and reconstruction respectively of anisotropic conductivities. We also mention that the results obtained in [22] have been improved in [23]. Another direction has been the one to assume that the anisotropic conductivity is a priori known to depend on a restricted number of spatially-dependent parameters (see [3, 8, 9, 24, 25, 42]).
Alessandrini [3] considered the case when $\sigma(x)$ is anisotropic and it is a priori known to have the structure $\sigma(x)=\sigma(a(x))$, where $t \rightarrow \sigma(t)$ is a given matrixvalued function and $a=a(x)$ is an unknown scalar function. In [3] results of uniqueness and stability at the boundary are proven by using the method of singular solutions under the additional assumption of monotonicity

$$
D_{t} \sigma(t) \geq \text { Const. } I>0
$$

These results have been extended in [8] and[9] to the case when $\sigma$ has the more general structure

$$
\begin{equation*}
\sigma(x)=\sigma(x, a(x)) \tag{2}
\end{equation*}
$$

where $a(x)$ is an unknown scalar function and $\sigma(x, t)$ is given and satisfies the monotonicity assumption

$$
\begin{equation*}
D_{t} \sigma(x, t) \geq \text { Const. } I>0, \tag{3}
\end{equation*}
$$

in the case of full and local data respectively. The singular solutions introduced in [3] have been extended in [51] for the more general operator of type

$$
\begin{equation*}
L u=-\operatorname{div}(\sigma \nabla u+P u)+Q \cdot \nabla u+q u \tag{4}
\end{equation*}
$$

where the leading order coefficients matrix $\sigma=\sigma(x)$ is merely Hölder continuous and some positivity condition is imposed on the lower order terms. We recall that singular solutions have also been used by Isakov [31] to determine discontinuities in the conductivity for the isotropic case. However, only Green's function type singularities were needed for this purpose.
In the present paper the author considers the inverse problem of determining the optical properties of a medium and shows that the structure (2) introduced in $[8,9]$ is appropriate in optical tomography (OT). This is the problem of determining the spacially dependent optical properties (the absorption and the scattering coefficients $\mu_{a}, \mu_{s}$ respectively) when light in a narrow-wavelength band in the near infrared is employed to transilluminate tissue (see [11, 13, 14]). We also refer to $[28,29,30]$ for related topis in OT. The resulting measurements of intensity on the tissue boundary are then used to reconstruct a map of the optical properties within the tissue. In the so-called OT static case the integral equation (Radiative Transfer Equation) typically used to model this problem can be reduced (under certain conditions) to an elliptic partial differential equation of type

$$
\begin{equation*}
\operatorname{div}(\sigma \nabla u)-q u=0 \quad \text { in } \quad \Omega \tag{5}
\end{equation*}
$$

with

$$
\begin{equation*}
\sigma=\sigma\left(x, \mu_{a}(x)\right), \quad q=\mu_{a}(x) \tag{6}
\end{equation*}
$$

where $\mu_{a}(x)$ is a function (the absorption coefficient) to be determined and $\sigma(x, t)$ is given and satisfies the monotonicity assumption

$$
\begin{equation*}
D_{t} \sigma(x, t) \leq \text { Const. } I<0 . \tag{7}
\end{equation*}
$$

Notice that although $D_{t} \sigma(x, t)$ is a negative definite matrix in (7), whereas the case of a positive definite $D_{t} \sigma(x, t)$ was considered in [8, 9], the arguments used in the current paper and in $[8,9]$ continue to work if $D_{t} \sigma(x, t)$ satisfies either (3) or (7). In other words a monotonicity assumption of either type (3) or (7) seems to be a realistic hypothesis that is satisfied for example in the OT problem considered in this manuscript. The result presented here also shows that the machinery of the stability proofs at the boundary via singular solutions introduced in [3] works also in the more general case (5), where the equation has an extra zero order term. The OT formulation given by (5), (6) is achieved in the static case if it is assumed that the scattering coefficient $\mu_{s}$ has been determined by employing a different imaging modality (like MRI) prior to the application of OT and the structural information we are interested in
is the determination of $\mu_{a}$. The main focus of the present paper is indeed on the stable determination at the boundary of $\mu_{a}$ and its derivatives by pursuing the same line of investigation of $[8,9]$. This is done by considering anisotropic diffusion tensors $\sigma(x, t)$ arising in OT that are real matrix-valued functions. The time-harmonic case where $\sigma(x, t)$ is a complex matrix-valued function will be investigated in future work. The case in which $\mu_{a}$ is known and the scattering coefficient $\mu_{s}$ is to be determined can be treated in a similar manner to the one considered in this work. In medical applications, while the scattering coefficient $\mu_{s}$ varies from tissue to tissue, it is the absorption coefficient $\mu_{a}$ that carries the more interesting physiological information as it is related to the global concentrations of certain metabolites in their oxygenated and deoxygenated states. Moreover, many tissues including parts of the brain, muscle and breast tissue have a fibrous structure on a microscopic scale which results in anisotropic physical properties on a larger scale.
We shall also emphasize that the stable determination of $\mu_{a}$ (or equivalently of $\mu_{s}$ ) and its derivatives at the boundary are useful tools to infer uniqueness and stability of $\mu_{a}$ (or $\mu_{s}$ ) in the interior, which represents the preliminary goal to achieve an image of the interior of $\Omega$ (the body under investigation). On the other hands, it is well known that the inverse boundary value problem of determining $\sigma$ in (1) from the knowledge of the D-N map is severely ill-posed. Indeed, regarding the stability of the inverse conductivity problem, Alessandrini [2] proved that, assuming $n \geq 3$ and a-priori bounds on $\sigma$ of the form

$$
\begin{equation*}
\|\sigma\|_{H^{s}(\Omega)} \leq E, \quad \text { for some } s>\frac{n}{2}+2 \tag{8}
\end{equation*}
$$

$\sigma$ depends continuously on $\Lambda_{\sigma}$ with a modulus of continuity of logarithmic type. For subsequent results of this type we also refer to [3, 4] and to [15, 16, 43] for the two-dimensional case. The common logarithmic type of stability cannot be avoided ([5, 44]). However, the ill-posed nature of this problem can be modified to be conditionally well-posed by restricting the conductivity to certain function subspaces. Well-posedness is here expressed by Lipschitz stability. A first result of this kind was established by Alessandrini and Vessella [10], where the authors proved global stability of $\sigma$ in terms of the local D-N map, for the case when $\sigma$ is isotropic and piecewise constant on a given finite partition of $\Omega$. This fundamental result was extended later on to different types of inverse problems. In the context of the inverse conductivity problem to which we refer in this work, we wish to recall the results of $[7,17]$ for the cases of real piecewise linear and complex piecewise constant isotropic conductivity respectively and to [25] for the case of a conformal class of piecewise anisotropic conductivities. All of these results are obtained in terms of local data. We also refer to [50] where it was shown that the Lipschitz stability constant appearing in the above mentioned results grows exponentially with the number of domains partitioning $\Omega$ and to [6] for a recent result of global uniqueness for anisotropic
conductivities that are piecewise constant in the context of local data too. To conclude, we shall point out that the problem of recovering the conductivity $\sigma$ by local measurements has been treated more recently. In this context we wish to recall also $[18,21,27,33,34,47,48,49]$. The results obtained in the current paper could be adapted to the case of local data too.
The paper is organized as follows. Section 2 contains the formulation of the problem in OT for the static case (subsection 2.1) and the main results (subsection 2.2, Theorems 2.5, 2.6). Section 3 is devoted to a review of the construction of singular solutions for equations of type (5) having a singularity of arbitrarily high order at a given point. This is done by following the same line of [3] (see also [51] for the more general case (4)). The proofs of Theorems 2.5, 2.6 are given in section 4.

## 2. The main result

### 2.1. Formulation of the problem

Although Maxwell's equations provide a complete model for the light propagation in a scattering medium on a micro scale, on the scale suitable for medical OT an appropriate model is given by the Radiative Transfer Equation (or Boltzmann equation) [14]. If $\Omega$ is a domain in $\mathbb{R}^{n}$, with $n \geq 2$ with smooth boundary $\partial \Omega$ and radiation is considered in the body $\Omega$, then it is well known that if the input field is modulated with a fixed harmonic frequency $\omega$, the so-called Diffusion Approximation leads to the elliptic equation (see [11]) for the energy current density $u$

$$
\begin{equation*}
\operatorname{div}(K \nabla u)-\left(\mu_{a}-\mathrm{i} k\right) u=0, \quad \text { in } \Omega, \tag{9}
\end{equation*}
$$

where $k=\frac{\omega}{c}$ is the wave number and $K$ is the complex matrix valued function

$$
K=\frac{1}{n}\left(\left(\mu_{a}-\mathrm{i} k\right) I+(I-B) \mu_{s}\right)^{-1}
$$

where $B_{i j}(x)=B_{j i}(x)$ is a real matrix valued function and $I-B$ is positive definite $([11,29,30])$. The spacially varying coefficients $\mu_{a}$ and $\mu_{s}$ are called the absorption and the scattering coefficients of the medium $\Omega$ and represent the optical properties of $\Omega$. Here we consider the simpler static case $k=0$ for which $K$ reduces to the real matrix valued function

$$
\begin{equation*}
K=\frac{1}{n}\left(\mu_{a} I+(I-B) \mu_{s}\right)^{-1} \tag{10}
\end{equation*}
$$

Although it is common practise in OT to use the Robin-to-Robin map to describe the boundary measurements (see [11]), the D-N map will be employed in this manuscript instead. The rigorous definition of this map for an equation
of type (9) will be given in subsection 2.1.1. For now, we just recall that prescribing its inverse, called the Neumann-to-Dirichlet (N-D) map, is equivalent to prescribe in OT the more commonly used Robin-to-Robin map. It can also be shown that prescribing the N-D map is insufficient to recover both coefficients $\mu_{a}$ and $\mu_{s}$ uniquely [13] unless a priori smoothness assumptions are employed [26]. In this paper we consider the problem of determining $\mu_{a}$ and its derivatives when $\mu_{s}$ and $B$ are assumed known. More precisely, we show that $\mu_{a}$ and its derivatives at the boundary depend upon $\Lambda_{K, \mu_{a}}$ with a modulus of continuity of Lipschitz and Hölder type respectively. These are the main results of this paper and are contained in Theorems 2.5, 2.6.
We rigorously formulate the problem by introducing the following notation, definitions and assumptions.
For $n \geq 3$, a point $x \in \mathbb{R}^{n}$ will be denoted by $x=\left(x^{\prime}, x_{n}\right)$, where $x^{\prime} \in \mathbb{R}^{n-1}$ and $x_{n} \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^{n}$, we will denote with $B_{r}(x)$, $B_{r}^{\prime}\left(x^{\prime}\right)$ the open balls in $\mathbb{R}^{n}, \mathbb{R}^{n-1}$ respectively centred at $x$ and $x^{\prime}$ with radius $r$ and by $Q_{r}(x)$ the cylinder

$$
Q_{r}(x)=B_{r}^{\prime}\left(x^{\prime}\right) \times\left(x_{n}-r, x_{n}+r\right) .
$$

We will also denote $B_{r}=B_{r}(0), B_{r}^{\prime}=B_{r}^{\prime}(0)$ and $Q_{r}=Q_{r}(0)$.
Definition 2.1. Let $\Omega$ be a domain in $\mathbb{R}^{n}$. We say that $\partial \Omega$ is of Lipschitz class with constants $L, r>0$ if for any $P \in \partial \Omega$ there exists a rigid transformation of $\mathbb{R}^{n}$ under which we have $P=0$ and

$$
\Omega \cap Q_{r}=\left\{x \in Q_{r}: x_{n}>\varphi\left(x^{\prime}\right)\right\}
$$

where $\varphi$ is a Lipschitz function on $B_{r_{0}}^{\prime}$ satisfying

$$
\varphi(0)=0 ; \quad\|\varphi\|_{C^{0,1}\left(B_{r}^{\prime}\right)} \leq L r
$$

Assumption (on the known parameters $\mu_{s}$ and $B$ ): we assume that $\mu_{s}, B \in$ $W^{1, \infty}(\Omega)$ and that for some positive constants $\lambda, E$

$$
\begin{equation*}
\lambda^{-1} \leq \mu_{s}(x) \leq \lambda, \quad \text { for every } \quad x \in \Omega \tag{11}
\end{equation*}
$$

and

$$
\begin{align*}
\left\|\mu_{s}\right\|_{W^{1, \infty}(\Omega)} & \leq E,  \tag{12}\\
\|B\|_{W^{1, \infty}(\Omega)} & \leq E . \tag{13}
\end{align*}
$$

We introduce the following class of matrix valued functions $\sigma(x, t)$ on $\Omega \times$ $\left[\lambda^{-1}, \lambda\right]$.

Definition 2.2. Given $p>n$, we say that $\sigma(\cdot, \cdot) \in \mathcal{H}^{\prime}{ }_{p}$ if there are positive constants $\lambda, \mathcal{E}, \mathcal{F}>0$, such that, denoting by $\operatorname{Sym}_{n}$ the class of $n \times n$ real symmetric matrices, the following conditions are satisfied

$$
\begin{align*}
& \sigma \in W^{1, p}\left(\Omega \times\left[\lambda^{-1}, \lambda\right], \operatorname{Sym}_{n}\right),  \tag{14}\\
& D_{t} \sigma \in W^{1, p}\left(\Omega \times\left[\lambda^{-1}, \lambda\right], \operatorname{Sym}_{n}\right),  \tag{15}\\
& \operatorname{ess} \sup _{t \in\left[\lambda^{-1}, \lambda\right]}\left(\|\sigma(\cdot, t)\|_{L^{p}(\Omega)}+\left\|D_{x} \sigma(\cdot, t)\right\|_{L^{p}(\Omega)}\right. \\
& \left.+\left\|D_{t} \sigma(\cdot, t)\right\|_{L^{p}(\Omega)}+\left\|D_{t} D_{x} \sigma(\cdot, t)\right\|_{L^{p}(\Omega)}\right) \leq \mathcal{E},  \tag{16}\\
& \lambda^{-1}|\xi|^{2} \leq \sigma(x, t) \xi \cdot \xi \leq \lambda|\xi|^{2}, \quad \text { for almost every } x \in \Omega, \\
& \text { for every } t \in\left[\lambda^{-1}, \lambda\right], \xi \in \mathbb{R}^{n} \text {, }  \tag{17}\\
& D_{t} \sigma(x, t) \xi \cdot \xi \leq-\mathcal{F}|\xi|^{2}, \quad \text { for almost every } x \in \Omega, \\
& \text { for every } t \in\left[\lambda^{-1}, \lambda\right], \xi \in \mathbb{R}^{n} . \tag{18}
\end{align*}
$$

Remark 2.3. We observe that properties (14) - (17) were satisfied by the oneparameter family of conductivities $\sigma(x, t)$ belonging to the class $\mathcal{H}$ introduced in $[8,9]$. Property (18), which is a property of monotonicity of $D_{t} \sigma(x, t)$ with respect to the variable t , replaces the monotonicity assumption (3) in $\mathcal{H}$. (18) states that $D_{t} \sigma(x, t)$ is a negative definite matrix for almost every $x \in \Omega$, where the monotonicity assumption (3) of $\mathcal{H}$ in $[8,9]$ required $D_{t} \sigma(x, t)$ to be positive definite instead. In this work we will show that the results obtained in $[8,9]$ can be similarly obtained when (3) is replaced by (18) and equation (1) is replaced by the more general one in (5).

Let us rigorously define the D-N map for (5).

### 2.1.1. The Dirichlet-to-Neumann map.

If $n \geq 3$ and $\Omega$ is a domain in $\mathbb{R}^{n}$ with Lipschitz boundary $\partial \Omega$ (with constants $L, r)$ as in Definition 2.1, we assume that $\sigma \in L^{\infty}\left(\Omega\right.$, Sym $\left._{n}\right), q \in L^{\infty}(\Omega)$ satisfy the ellipticity condition

$$
\begin{array}{ll}
\lambda^{-1}|\xi|^{2} \leq \sigma(x) \xi \cdot \xi \leq \lambda|\xi|^{2}, & \text { for almost every } x \in \Omega, \\
& \text { for every } \xi \in \mathbb{R}^{n} . \tag{19}
\end{array}
$$

and

$$
\begin{equation*}
\lambda^{-1} \leq q(x) \leq \lambda, \quad \text { for almost every } x \in \Omega \tag{20}
\end{equation*}
$$

respectively. We denote by $\langle\cdot, \cdot\rangle$ the $L^{2}(\partial \Omega)$-pairing between $H^{\frac{1}{2}}(\partial \Omega)$ and its dual $H^{-\frac{1}{2}}(\partial \Omega)$.

Definition 2.4. The Dirichlet-to-Neumann ( $D-N$ ) map associated with $\sigma, q$ is the operator

$$
\begin{equation*}
\Lambda_{\sigma, q}: H^{\frac{1}{2}}(\partial \Omega) \longrightarrow H^{-\frac{1}{2}}(\partial \Omega) \tag{21}
\end{equation*}
$$

defined by

$$
\begin{equation*}
\left\langle\Lambda_{\sigma, q} f, g\right\rangle=\int_{\Omega}(\sigma(x) \nabla u(x) \cdot \nabla \varphi(x)+q(x) u(x) \varphi(x)) d x \tag{22}
\end{equation*}
$$

for any $f, g \in H^{\frac{1}{2}}(\partial \Omega)$, where $u \in H^{1}(\Omega)$ is the weak solution to

$$
\begin{cases}\operatorname{div}(\sigma(x) \nabla u(x))-q(x) u(x)=0, & \text { in } \Omega, \\ u=f, & \text { on } \partial \Omega\end{cases}
$$

and $\varphi \in H^{1}(\Omega)$ is any function such that $\left.\varphi\right|_{\partial \Omega}=g$ in the trace sense.
Note that, by (22), it is easily verified that $\Lambda_{\sigma, q}$ is selfadjoint and that given $\sigma_{i} \in L^{\infty}\left(\Omega, S y m_{n}\right)$, for $i=1,2, q_{i} \in L^{\infty}(\Omega)$, satisfying (19) and (20) respectively, the well known Alessandrini's identity (see [32, (5.0.4), p.129])

$$
\begin{align*}
\left\langle\Lambda_{\sigma_{1}, q_{1}}-\Lambda_{\sigma_{2}, q_{2}} f_{1}, f_{2}\right\rangle= & \int_{\Omega}\left(\sigma_{1}(x)-\sigma_{2}(x)\right) \nabla u_{1}(x) \cdot \nabla u_{2}(x) d x \\
& +\int_{\Omega}\left(q_{1}(x)-q_{2}(x)\right) u_{1}(x) u_{2}(x) d x \tag{23}
\end{align*}
$$

holds true for any $f_{i} \in H^{\frac{1}{2}}(\partial \Omega)$, where $u_{i} \in H^{1}(\Omega)$ is the unique weak solution to the Dirichlet problem

$$
\begin{cases}\operatorname{div}\left(\sigma_{i}(x) \nabla u_{i}(x)\right)-q_{i}(x) u_{i}(x)=0, & \text { in } \Omega \\ u_{i}=f_{i}, & \text { on } \partial \Omega\end{cases}
$$

for $i=1,2$.
In the sequel we will denote the D-N map $\Lambda_{K, \mu_{a}}$ corresponding to (9) (for $k=0$ ) by

$$
\Lambda_{\mu_{a}}
$$

to simplify our notation. We will also denote by $\|\cdot\|_{*}$ the norm on the Banach space of bounded linear operators between $H^{\frac{1}{2}}(\partial \Omega)$ and $H^{-\frac{1}{2}}(\partial \Omega)$.

### 2.2. The main result

The following theorems are the main results of this paper.

THEOREM 2.5 (Lipschitz stability of boundary values). Let $n \geq 3, p>n$ and $\Omega$ be a bounded domain with Lipschitz boundary with constants $L, r$ as in Definition 2.1. Let $\mu_{s_{i}}$ satisfy (11), (12), $i=1,2$ and $B$ satisfy (13). If $\mu_{a_{i}}$ satisfies

$$
\begin{gather*}
\lambda^{-1} \leq \mu_{a_{i}}(x) \leq \lambda, \quad \text { for every } \quad x \in \Omega,  \tag{24}\\
\left\|\mu_{a_{i}}\right\|_{W^{1, p}(\Omega)} \leq E \tag{25}
\end{gather*}
$$

for $i=1,2$, then we have

$$
\begin{equation*}
\left\|\mu_{a_{1}}(x)-\mu_{a_{2}}(x)\right\|_{L^{\infty}(\partial \Omega)} \leq C\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*} . \tag{26}
\end{equation*}
$$

Here $C>0$ is a constant depending on $n, p, L, r, \operatorname{diam}(\Omega), \lambda, \mathcal{E}, \mathcal{F}$ and $E$.
THEOREM 2.6 (Hölder stability of derivatives at the boundary). Let $n \geq 3$, $p, \Omega, \mu_{a_{i}}, \mu_{s_{i}}, i=1,2$ and $B$ be as in Theorem 2.5. Given $y \in \partial \Omega$ and a neighborhood $U$ of $y$ in $\bar{\Omega}$, assume that for some positive integer $k$ and some $\alpha, 0<\alpha<1$ we have

$$
\begin{equation*}
\left\|\mu_{s_{i}}\right\|_{C^{k, \alpha}(\bar{U})},\|B\|_{C^{k, \alpha}(\bar{U})} \leq E_{k} \tag{27}
\end{equation*}
$$

for $i=1,2$ and

$$
\begin{equation*}
\left\|\mu_{a_{1}}-\mu_{a_{2}}\right\|_{C^{k, \alpha}(\bar{U})} \leq E_{k} . \tag{28}
\end{equation*}
$$

Then, for every neighborhood $W$ of $y$ in $\bar{\Omega}$ such that $\bar{W} \subset U$,

$$
\begin{equation*}
\left\|D^{k}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \leq C\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}^{\delta_{k} \alpha}, \tag{29}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta_{k}=\prod_{j=0}^{k} \frac{\alpha}{\alpha+j} \tag{30}
\end{equation*}
$$

Here $C>0$ is a constant which depends only on n, $p, L, r, \operatorname{diam}(\Omega), \operatorname{dist}(W \cap$ $\partial \Omega, \Omega \backslash U), \lambda, E, \mathcal{F}, \mathcal{E} \alpha, k$, and $E_{k}$.

## 3. Singular solutions

This section is devoted to a review of the construction of singular solutions of an elliptic equation in divergence form with a lower extra term of order zero. This type of solutions were introduced by Alessandrini in [3] for an equation of type (1) and have been extended to solutions of a more general equation of type (4). The decision to expose in this manuscript the key-points necessary for the constructions of such solutions in the OT context is driven by the willingness of keeping the manuscript as self-contained as possible. It is also hoped that the details highlighted here will be of use for the more physically
relevant time-harmonic case in OT, where the matrix valued function $K$ is complex and the zero order term in (9) is complex too. Here we consider an operator of type

$$
\begin{equation*}
L=\frac{\partial}{\partial x_{i}}\left(\sigma_{i j} \frac{\partial}{\partial x_{j}}\right)-q, \quad \text { in } \quad B_{R}, \tag{31}
\end{equation*}
$$

where the leading order coefficients $\sigma_{i j}(x), i, j=1, \ldots, n$ and the zero order coefficient $q(x)$ satisfy

$$
\begin{gather*}
\lambda^{-1}|\xi|^{2} \leq \sigma_{i j}(x) \xi_{i} \xi_{j} \leq \lambda|\xi|^{2}, \quad \text { for every } x, \xi, \quad x \in B_{R}, \xi \in \mathbb{R}^{n}  \tag{32}\\
\left\|\sigma_{i j}\right\|_{W^{1, p}\left(B_{R}\right)} \leq E, \quad i, j=1, \ldots, n \tag{33}
\end{gather*}
$$

for some $p>n$ and

$$
\begin{equation*}
\lambda^{-1} \leq q(x) \leq \lambda, \quad \text { for any } x, \quad x \in B_{R} \tag{34}
\end{equation*}
$$

Theorem 3.1 (Singular solutions for $L=\operatorname{div}(\sigma \nabla \cdot)-q)$. Let $L$ satisfy (31)(34). For any spherical harmonic $S_{m}$ of degree $m=0,1,2, \ldots$, there exists $u \in W_{\text {loc }}^{2, p}\left(B_{R} \backslash\{0\}\right)$ such that

$$
\begin{equation*}
L u=0, \text { in } B_{R} \backslash\{0\} \tag{35}
\end{equation*}
$$

and furthermore

$$
\begin{align*}
u(x) & =\log |J x| S_{0}\left(\frac{J x}{|J x|}\right)+w(x), \quad \text { when } n=2 \text { and } m=0  \tag{36}\\
u(x) & =|J x|^{2-n-m} S_{m}\left(\frac{J x}{|J x|}\right)+w(x) \quad \text { otherwise } \tag{37}
\end{align*}
$$

where $J$ is a positive definite symmetric matrix such that $J=\sqrt{\left(\sigma_{i j}(0)\right)^{-1}}$ and $w$ satisfies

$$
\begin{gather*}
|w(x)|+|x||D w(x)| \leq C|x|^{2-n-m+\alpha}, \quad \text { in } \quad B_{r} \backslash\{0\}  \tag{38}\\
\left(\int_{r<|x|<2 r}\left|D^{2} w\right|^{p}\right)^{\frac{1}{p}} \leq C r^{-n-m+\alpha+\frac{n}{p}}, \quad \text { forevery } \quad r, 0<r<R / 2 \tag{39}
\end{gather*}
$$

Here $\alpha$ is any number such that $0<\alpha<1-\frac{n}{p}$, and $C$ is a constant depending only on $\alpha, n, p, r, \lambda$, and $\mathcal{E}$.

Next we consider three technical lemmas. The proofs of these results for the case where $L=\operatorname{div}(\sigma \nabla \cdot)$ are treated in details in [2] and their extension to the more general case $L=\operatorname{div}(\sigma \nabla \cdot)-q$ is quite straightforward, therefore only the key points of their proofs will be highlighted here. In what follows $A$ denotes a positive constant.

Lemma 3.2. Let $p>n$ and $u \in W_{l o c}^{2, p}\left(B_{R} \backslash\{0\}\right)$ be such that, for some positive $s$,

$$
\begin{align*}
& |u(x)| \leq|x|^{2-s}, \quad \text { for any } \quad x \in B_{R} \backslash\{0\}  \tag{40}\\
& \left(\int_{r<|x|<2 r}|L u|^{p}\right)^{\frac{1}{p}} \leq A r^{\frac{n}{p}-s}, \quad \text { for any } r, \quad 0<r<\frac{R}{2} \tag{41}
\end{align*}
$$

Then we have

$$
\begin{align*}
& |D u(x)| \leq C|x|^{1-s}, \quad \text { for any } \quad x \in B_{R} \backslash\{0\}  \tag{42}\\
& \left(\int_{r<|x|<2 r}\left|D^{2} u\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-s} \quad \text { for any } r, \quad 0<r<\frac{R}{4} \tag{43}
\end{align*}
$$

where $C$ is a positive constant depending only on $A, n, p, \lambda$ and $E$.
Proof of Lemma 3.2. The proof is a consequence of the $L^{p}$ interior Schauder estimate

$$
\begin{equation*}
\left\|D^{2} u\right\|_{L^{p}\left(B_{\rho_{1} \rho_{2}}\right)} \leq \frac{C}{\left(1-\rho_{1}^{2}\right) \rho_{2}^{2}}\left(\rho_{2}^{2}\|L u\|_{L^{p}\left(B_{\rho_{2}}\right)}+\|u\|_{L^{p}\left(B_{\rho_{2}}\right)}\right) \tag{44}
\end{equation*}
$$

where $C=C(n, p, \lambda, E)$ is a positive constant, $0<\rho_{1}<1$ and $B_{\rho_{2}}, B_{\rho_{1} \rho_{2}}$ are two concentric balls such that $u \in W^{2, p}\left(B_{\rho_{2}}\right)$ (see [45, Lemma 5.6.1]). We refer to [2, Proof of Lemma 2.1] for a detailed proof of this lemma.
Lemma 3.3. Let $f \in L_{l o c}^{p}\left(B_{R} \backslash\{0\}\right)$ satisfy

$$
\begin{equation*}
\left(\int_{r<|x|<2 r}|f|^{p}\right)^{\frac{1}{p}} \leq A r^{\frac{n}{p}-s}, \quad \text { for any } r, \quad 0<r<\frac{R}{2} \tag{45}
\end{equation*}
$$

with $2<s<n<p$. Then there exists $u \in W_{\text {loc }}^{2, p}\left(B_{R} \backslash\{0\}\right)$ satisfying

$$
\begin{equation*}
L u=f, \quad \text { in } \quad B_{R} \backslash\{0\} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
|u(x)| \leq C|x|^{2-s}, \quad \text { for any } \quad x \in B_{R} \backslash\{0\} \tag{47}
\end{equation*}
$$

where $C$ is a positive constant depending only on $A, s, n, p, R, \lambda$ and $E$.
Proof of Lemma 3.3. The proof is based on the construction of a fundamental solution $\Gamma$ of the equation $L u=0$ so that

$$
\begin{equation*}
|\Gamma(x, y)| \leq C(n, \lambda)|x-y|^{2-n}, \quad \text { for any } x \neq y \tag{48}
\end{equation*}
$$

(see [52]). See also [1, section 4] for a brief description of this construction and [2, Proof of Lemma 2.2] for a complete proof of this lemma.

Definition 3.4. We shall denote solution $u$ of (46) by

$$
u=T_{L} u
$$

The last technical result that we recall involves pointwise estimates of some solution of the Laplace equation and we refer to [2, Proof of Lemma 2.3] for its proof.

Lemma 3.5. Let $s>n$ be a non-integer real number. Let $f$ be as in lemma 3.3 and satisfying (45) with $p>n$. Then there exists $u \in W_{l o c}^{2, p}\left(B_{R} \backslash\{0\}\right)$ satisfying

$$
\begin{equation*}
\Delta u=f, \text { in } B_{R} \backslash\{0\} \tag{49}
\end{equation*}
$$

and such that (47) holds true with $C>0$ a constant depending only on $A, s, n, p$ and $R$.

Definition 3.6. We shall denote solution $u$ of (49) by

$$
u=T_{S} u
$$

We proceed next with the proof of 3.1.
Proof of Theorem 3.1. The proof follows the same line of [2, Proof of Theorem 1.1]. We will therefore only rephrase the key points of this proof showing how it can be adapted to the more general case treated here. For simplicity we first assume that $\sigma(0)=I$, where $I$ denotes the $n \times n$ identity matrix and prove that, under this additional assumption, for any spherical harmonic $S_{m}$ of degree $m=0,1,2, \ldots$, there exists $u \in W_{\text {loc }}^{2, p}\left(B_{R} \backslash\{0\}\right)$ such that

$$
\begin{equation*}
L u=0, \text { in } B_{R} \backslash\{0\} \tag{50}
\end{equation*}
$$

and

$$
\begin{align*}
& u(x)=\log |x| S_{0}\left(\frac{x}{|x|}\right)+w(x), \quad \text { when } n=2 \text { and } m=0,  \tag{51}\\
& u(x)=|x|^{2-n-m} S_{m}\left(\frac{x}{|x|}\right)+w(x) \quad \text { otherwise, } \tag{52}
\end{align*}
$$

where $w$ satisfies (38), (39). For this, we consider in $B_{R} \backslash\{0\}$ the harmonic

$$
H(x)=|x|^{2-n-m} S_{m}\left(\frac{x}{|x|}\right)
$$

As in [2, Proof of Theorem 1.1] the idea is to find $w$ satisfying (38), (39) and such that

$$
L w=-L H, \text { in } B_{R} \backslash\{0\} .
$$

We have

$$
\begin{equation*}
-L H=(\Delta-L) H=\left(\delta_{i j}-a_{i j}\right) \frac{\partial^{2} H}{\partial x_{i} \partial x_{j}}-\frac{\partial a_{i j}}{\partial x_{i}} \frac{\partial H}{\partial x_{j}}-q H . \tag{53}
\end{equation*}
$$

From [2, Proof of Theorem 1.1] we have

$$
\begin{align*}
& \left(\int_{r<|x|<2 r}\left|\delta_{i j}-a_{i j}\right|^{p}\left|\frac{\partial^{2} H}{\partial x_{i} \partial x_{j}}\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta}  \tag{54}\\
& \left(\int_{r<|x|<2 r}\left|\frac{\partial a_{i j}}{\partial x_{i}}\right|^{p}\left|\frac{\partial H}{\partial x_{j}}\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta} \tag{55}
\end{align*}
$$

where $\beta=1-\frac{n}{p}$. Here the extra lower order term $-q H$ can be estimated as follows

$$
\begin{align*}
\left(\int_{r<|x|<2 r}|q H|^{p}\right)^{\frac{1}{p}} & \leq C(\lambda, R)\left(\int_{r<|x|<2 r}|x|^{(2-n-m) p}\right)^{\frac{1}{p}} \\
& \leq C(\lambda, R)\left(\int_{r}^{2 r} \rho^{(2-n-m) p+n-1}\right)^{\frac{1}{p}} \\
& \leq C r^{\frac{n}{p}-n-m+\beta} \tag{56}
\end{align*}
$$

and by combining (54)-(56) together we obtain

$$
\begin{equation*}
\left(\int_{r<|x|<2 r}|L H|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+\beta} \tag{57}
\end{equation*}
$$

Let $\alpha$ be an irrational number such that $0<\alpha<\beta$ and define

$$
K=\left[\frac{m}{\alpha}\right] .
$$

If $w_{0}=T_{S}(-L H)$, then we have

$$
\left|w_{0}(x)\right| \leq C|x|^{2-n-m+\beta}, \quad \text { for any } x, x \in B_{R} \backslash\{0\}
$$

We define

$$
w_{j}= \begin{cases}w_{0}, & j=0  \tag{58}\\ T_{S} f, \quad f=(\Delta-L) w_{j-1}, & j=1, \ldots, K-1\end{cases}
$$

Lemma 3.7. For any $j=0, \ldots, K-1$ we have

$$
\begin{align*}
& \left|w_{j}(x)\right| \leq C|x|^{2-n-m+(j+1) \alpha}  \tag{59}\\
& \left(\int_{r<|x|<2 r}\left|(\Delta-L) w_{j}\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+(j+2) \alpha} \tag{60}
\end{align*}
$$

Proof of Lemma 3.7. . We prove (59), (60) by induction on $j$. For $j=0$ we have

$$
\left|w_{0}(x)\right| \leq C|x|^{2-n-m+\beta} \leq C|x|^{2-n-m+\alpha}
$$

and

$$
\begin{aligned}
\left(\int_{r<|x|<2 r}\left|(\Delta-L) w_{j}\right|^{p}\right)^{\frac{1}{p}} & \leq C r^{\frac{n}{p}-n-m+2 \alpha}+C\left(\int_{r<|x|<2 r} \left\lvert\,\left(\left.c w_{0}\right|^{p}\right)^{\frac{1}{p}}\right.\right. \\
& \leq C r^{\frac{n}{p}-n-m+2 \alpha}+C\left(\int_{r<|x|<2 r}|x|^{(2-n-m+\alpha) p}\right)^{\frac{1}{p}} \\
& \leq C r^{\frac{n}{p}-n-m+2 \alpha}+C r^{\frac{n}{p}-n-m+\alpha} \\
& \leq C r^{\frac{n}{p}-n-m+\alpha} .
\end{aligned}
$$

Suppose now that $(59),(60)$ are true for $j$, i.e.

$$
\begin{aligned}
& \left|w_{j}(x)\right| \leq C|x|^{2-n-m+(j+1) \alpha} \\
& \left(\int_{r<|x|<2 r}\left|(\Delta-L) w_{j}\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+(j+2) \alpha},
\end{aligned}
$$

then if we define $s=n+m-(j+2) \alpha$, we have that $s>n$ and if we take

$$
w_{j+1}=T_{S} f, \quad \text { with } \quad f=(\Delta-L) w_{j}
$$

then

$$
\begin{equation*}
\left|w_{j+1}(x)\right| \leq C|x|^{2-n-m+(j+2) \alpha} \tag{61}
\end{equation*}
$$

and

$$
\begin{align*}
\left(\int_{r<|x|<2 r}\left|(\Delta-L) w_{j+1}\right|^{p}\right)^{\frac{1}{p}} \leq & C r^{\frac{n}{p}-n-m+(j+3) \alpha}+C\left(\int_{r<|x|<2 r}\left|c w_{j+1}\right|^{p}\right)^{\frac{1}{p}} \\
\leq & C r^{\frac{n}{p}-n-m+(j+3) \alpha} \\
& \quad+C\left(\int_{r<|x|<2 r}|x|^{(2-n-m+(j+2) \alpha)} p\right)^{\frac{1}{p}} \\
& =C r^{\frac{n}{p}-n-m+(j+3) \alpha}+C r^{\frac{n}{p}-n-m+(j+2) \alpha} \\
\leq & C r^{\frac{n}{p}-n-m+(j+3) \alpha} \tag{62}
\end{align*}
$$

which conclude the proof.
(60) with $j=K-1$ gives

$$
\left(\int_{r<|x|<2 r}\left|(\Delta-L) w_{K-1}\right|^{p}\right)^{\frac{1}{p}} \leq C r^{\frac{n}{p}-n-m+(K+1) \alpha}
$$

and if we define $s=n+m-(K+1) \alpha$, we have $s<n$. If we define

$$
W_{K}=T_{L} f, \quad \text { with } \quad f=(\Delta-L) w_{K-1}
$$

we have

$$
\begin{equation*}
\left|W_{K}(x)\right| \leq C|x|^{2-n-m+(K+1) \alpha}, \quad \text { for any } \quad x \in B_{R} \backslash\{0\} . \tag{63}
\end{equation*}
$$

We define as in [2, Proof of Theorem 1.1] the function $w$

$$
\begin{equation*}
w=\sum_{j=0}^{K-1} w_{j}+W_{K} \tag{64}
\end{equation*}
$$

$w \in W_{l o c}^{2, p}\left(B_{R} \backslash\{0\}\right)$ and satisfies

$$
|w(x)| \leq C|x|^{2-n-m+\alpha} \quad \text { for any } \quad x \in B_{R} \backslash\{0\}
$$

moreover

$$
\begin{align*}
\left(\int_{r<|x|<2 r}|L w|^{p}\right)^{\frac{1}{p}} & \leq C r^{\frac{n}{p}-n-m+\alpha}+\left(\int_{r<|x|<2 r}|q w|^{p}\right)^{\frac{1}{p}} \\
& \leq C r^{\frac{n}{p}-n-m+\alpha}+C\left(\int_{r<|x|<2 r}|x|^{(2-n-m+\alpha) p}\right)^{\frac{1}{p}} \\
& \leq C r^{\frac{n}{p}-n-m+\alpha}+C r^{\frac{n}{p}+2-n-m+\alpha} \\
& \leq C r^{\frac{n}{p}-n-m+\alpha} . \tag{65}
\end{align*}
$$

Estimate (65), together with Lemma 3.2, leads to

$$
\begin{align*}
|D w(x)| & \leq C|x|^{1-n-m+\alpha}  \tag{66}\\
\left(\int_{r<|x|<2 r}\left|D^{2} w\right|^{p}\right)^{\frac{1}{p}} & \leq C r^{\frac{n}{p}-n-m+\alpha} \tag{67}
\end{align*}
$$

In the general case in which the extra assumption $\sigma(0)=I$ is not satisfied, we consider the linear change of variable $\xi=J x$, with $J=\sqrt{\left(\sigma_{i j}(0)\right)^{-1}}$, so that in the new coordinate system the above mentioned extra assumption is satisfied. In this case (51), (52) must be replaced by (36), (37) respectively, which concludes the proof.

We shall also need the following lemma.
Lemma 3.8. Let the hypotheses of Theorem 3.1 be satisfied. For every $m=$ $1,2, \ldots$ there exists a spherical harmonic $S_{m}$ of degree $m$ such that the solution u given by Theorem 3.1 also satisfies

$$
\begin{equation*}
|D u(x)|>|x|^{1-(n+m)}, \quad \text { for every } x, 0<|x|<r_{0} \tag{68}
\end{equation*}
$$

where $r_{0}$ depends only on $\lambda, E, p, m$ and $R$.
Proof. The proof of this lemma can be obtained along the same lines as of [2, Lemma 3.1] and [8, Section 3].

## 4. Proof of the main result.

Since the boundary $\partial \Omega$ is Lipschitz, the normal unit vector field might not be defined on $\partial \Omega$. We shall therefore introduce a unitary vector field $\widetilde{\nu}$ locally defined near $\partial \Omega$ such that: (i) $\widetilde{\nu}$ is $C^{\infty}$ smooth, (ii) $\widetilde{\nu}$ is non-tangential to $\partial \Omega$. At this point we would need to quantify $\partial \Omega$ in terms of its compactness and the constants $L, r$ introduced in definition 2.1. We think that this goes beyond the scope of this paper, therefore we choose to refer to [8, Lemmas 3.1-3.3] for a precise introduction of $\widetilde{\nu}$. Here we will simply recall that the point $z_{\tau}=x^{0}+\tau \widetilde{\nu}$, where $x^{0} \in \partial \Omega$, satisfies

$$
\begin{equation*}
C \tau \leq d\left(z_{\tau}, \partial \Omega\right) \leq \tau, \quad \text { for any } \quad \tau, \quad 0 \leq \tau \leq \tau_{0} \tag{69}
\end{equation*}
$$

where $\tau_{0}$ and $C$ depend on $L$ and $r$ only.
Lemma 4.1. If $\mu_{s}$, B satisfy conditions (11), (12) and (13) respectively, then $K(x, t)$ given by (10) belongs to the class $\mathcal{H}_{\infty}^{\prime}$ with $\mathcal{E}$ being a positive constant depending only on $n, \lambda$ and $E$.

Proof of Lemma 4.1. Notice that if $\mu_{s}$ and $B$ satisfy (11), (12) and (13) respectively, then

$$
\begin{equation*}
K(x, t) \in L^{\infty}(\Omega) \tag{70}
\end{equation*}
$$

We also have

$$
\begin{align*}
D_{t} K(x, t) & =-n K^{2}(x, t)  \tag{71}\\
D_{x} K(x, t) & =n K(x, t)\left[\left(D_{x} B\right) \mu_{s}-(I-B) D_{x} \mu_{s}\right] K(x, t)  \tag{72}\\
D_{t} D_{x} K(x, t) & =-2 n^{2} K^{2}(x, t)\left[\left(D_{x} B\right) \mu_{s}-(I-B) D_{x} \mu_{s}\right] K(x, t) . \tag{73}
\end{align*}
$$

By combining (70) together with (71)-(73) and recalling that $I-B$ is positive definite, we obtain that $K \in \mathcal{H}_{\infty}^{\prime}$.

Note that if $K$ is given by (10), $\mu_{s}, B$ satisfy conditions (11), (12) and (13) respectively and $\mu_{a}$ satisfies (24), (25), then

$$
\begin{equation*}
K\left(\cdot, \mu_{a}(\cdot)\right) \in W^{1, p}\left(\Omega, \operatorname{Sym}_{n}\right) \tag{74}
\end{equation*}
$$

where $p$ is the number introduced in (25). Furthermore

$$
\begin{equation*}
\left\|K\left(\cdot, \mu_{a}(\cdot)\right)\right\|_{W^{1, p}(\Omega)} \leq C \mathcal{E}\left(1+\left\|\mu_{a}\right\|_{W^{1, p}(\Omega)}\right) \tag{75}
\end{equation*}
$$

where $C$ is a positive constant depending only on $\lambda, \Omega, n$ and $p$ (see for instance [8, Lemma 3.6]).
In the following two proofs of the main result the appearance of positive constants that depend on the various quantities $n, p, \alpha, \beta, k L, r, E, \mathcal{E}, \mathcal{F}$ and $\Omega$ will be common. These quantities represent our a-priori information, therefore, we will denote by $C$ any of these positive constants arising in the proofs in order to keep the notation simple.

Proof of Theorem 2.5. Let $x^{0} \in \partial \Omega$ be such that

$$
\left(\mu_{a_{2}}-\mu_{a_{1}}\right)\left(x^{0}\right)=\left\|\mu_{a_{1}}-\mu_{a_{2}}\right\|_{L^{\infty}(\partial \Omega)}
$$

and $z_{\tau}=x^{0}+\tau \widetilde{\nu}$, with $0<\tau \leq \min \left\{\tau_{0}, \frac{r_{0}}{4}\right\}$, where $\tau_{0}$ is the number fixed in (69) and $r_{0}$ is the number appearing in (68). We set $\sigma_{i}=K\left(\cdot, \mu_{a_{i}}\right), q_{i}=\mu_{a_{i}}$, for $i=1,2$ and $m=0$ in Theorem 3.1. The corresponding singular solution $u_{i} \in W^{2, p}(\Omega)$ of

$$
\operatorname{div}\left(K\left(\cdot, \mu_{a_{i}}\right) \nabla u_{i}\right)-\mu_{a_{i}} u_{i}=0 \quad \text { in } \Omega
$$

have a Green's function type of singularity at $z_{\tau}$ outside $\Omega$

$$
\begin{equation*}
u_{i}(x)=\left|J_{\mu_{a_{i}}}\left(x-z_{\tau}\right)\right|^{2-n}+O\left(\left|x-z_{\tau}\right|^{2-n+\alpha}\right) \tag{76}
\end{equation*}
$$

for $i=1,2$. By setting $\rho=r_{0}$ we have that $B_{\rho}\left(z_{\tau}\right) \cap \Omega \neq \emptyset$ and, recalling (23), we have

$$
\begin{align*}
& \left|\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left(K\left(x, \mu_{a_{1}}\right)-K\left(x, \mu_{a_{2}}\right)\right) \nabla u_{1} \cdot \nabla u_{2}\right| \\
& \quad \leq \int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|\mu_{a_{1}}-\mu_{a_{2}}\right|\left|u_{1}\right|\left|u_{2}\right| \\
& \quad+\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|K\left(x, \mu_{a_{1}}\right)-K\left(x, \mu_{a_{2}}\right)\right|\left|\nabla u_{1}\right|\left|\nabla u_{2}\right| \\
& \quad+\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|\mu_{a_{1}}-\mu_{a_{2}}\right|\left|u_{1} \| u_{2}\right| \\
& \quad+\left\|\Lambda \Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}\left\|u_{1}\right\|_{H^{\frac{1}{2}}(\partial \Omega)}\left\|u_{2}\right\|_{H^{\frac{1}{2}}(\partial \Omega)} . \tag{77}
\end{align*}
$$

By combining (76) with (77) and the fact that $K\left(x, \mu_{a_{i}}\right)$ is Hölder continuous with exponent $\beta=1-\frac{n}{p}$, we obtain

$$
\begin{aligned}
& \int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)} \frac{J_{\mu_{a_{2}}}^{2}\left(K\left(x^{0}, \mu_{a_{1}}\right)-K\left(x^{0}, \mu_{a_{2}}\right) J_{\mu_{a_{1}}}^{2}\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right)\right.}{\left|J_{\mu_{a_{1}}}\left(x-z_{\tau}\right)\right|^{n}\left|J_{\mu_{a_{2}}}\left(x-z_{\tau}\right)\right|^{n}} \\
& \leq C\left\{\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|x-z_{\tau}\right|^{2-2 n+\alpha}\right. \\
& +\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|x-z_{\tau}\right|^{2-2 n}\left|x-x^{0}\right|^{\beta} \\
& +\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|x-z_{\tau}\right|^{4-2 n} \\
& +\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|K\left(x, \mu_{a_{2}}\right)-K\left(x, \mu_{a_{1}}\right)\right|\left|x-z_{\tau}\right|^{2-2 n} \\
& \left.+\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|\mu_{a_{1}}-\mu_{a_{2}}\right|\left|x-z_{\tau}\right|^{4-2 n}\right\} \\
& +\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}\left\|u_{1}\right\|_{H^{\frac{1}{2}(\partial \Omega)}}\left\|u_{2}\right\|_{H^{\frac{1}{2}(\partial \Omega)}} .
\end{aligned}
$$

Since $\left|J_{\mu_{a_{i}}}-K\left(x^{0}, \mu_{a_{i}}\right)^{-1}\right| \leq C \tau^{\beta}$, for $i=1,2$, we have

$$
\begin{align*}
& J_{\mu_{a_{2}}}^{2}\left(K\left(x^{0}, \mu_{a_{1}}\right)-K\left(x^{0}, \mu_{a_{2}}\right)\right) J_{\mu_{1}}^{2}\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) \\
& \quad \geq\left(K\left(x^{0}, \mu_{a_{2}}\right)^{-1}-K\left(x^{0}, \mu_{a_{1}}\right)^{-1}\right)\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) \\
& \quad-C \tau^{\beta}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\left(x^{0}\right)\left|x-z_{\tau}\right|^{2} \tag{78}
\end{align*}
$$

and

$$
\begin{align*}
& \left(K\left(x^{0}, \mu_{a_{2}}\right)^{-1}-K\left(x^{0}, \mu_{a_{1}}\right)^{-1}\right)\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) \\
& \quad=\int_{\mu_{a_{1}}\left(x^{0}\right)}^{\mu_{a_{2}}\left(x^{0}\right)} D_{t}\left(K\left(x^{0}, t\right)\right)^{-1}\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) d t \\
& \quad=\int_{\mu_{a_{1}}\left(x^{0}\right)}^{\mu_{a_{2}}\left(x^{0}\right)}-K^{-1}\left(x^{0}, t\right) D_{t} K\left(x^{0}, t\right) K^{-1}\left(x^{0}, t\right)\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) d t \\
& \quad=\int_{\mu_{a_{2}}\left(x^{0}\right)}^{\mu_{a_{1}}\left(x^{0}\right)}-D_{t} K\left(x^{0}, t\right) K^{-1}\left(x^{0}, t\right)\left(x-z_{\tau}\right) \cdot K^{-1}\left(x^{0}, t\right)\left(x-z_{\tau}\right) d t \\
& \quad \geq \mathcal{F} \int_{\mu_{a_{1}}\left(x^{0}\right)}^{\mu_{a_{2}}\left(x^{0}\right)}\left|K^{-1}\left(x^{0}, t\right)\left(x-z_{\tau}\right)\right|^{2} d t \\
& \quad \geq \mathcal{F} \lambda^{-2}\left(\mu_{a_{2}}\left(x^{0}\right)-\mu_{a_{1}}\left(x^{0}\right)\right)\left|x-z_{\tau}\right|^{2} . \tag{79}
\end{align*}
$$

By combining (78) together with (79) we obtain

$$
\begin{align*}
& J_{\mu_{a_{2}}}^{2}\left(K\left(x^{0}, \mu_{a_{1}}\right)-K\left(x^{0}, \mu_{a_{2}}\right)\right) J_{\mu_{a_{1}}}^{2}\left(x-z_{\tau}\right) \cdot\left(x-z_{\tau}\right) \\
& \quad \geq\left(\mathcal{F} \lambda^{-2}+C \tau^{\beta}\right)\left(\mu_{a_{2}}\left(x^{0}\right)-\mu_{a_{1}}\left(x^{0}\right)\right)\left|x-z_{\tau}\right|^{2} \\
& \quad \geq C\left(\mu_{a_{2}}\left(x^{0}\right)-\mu_{a_{1}}\left(x^{0}\right)\right)\left|x-z_{\tau}\right|^{2} . \tag{80}
\end{align*}
$$

Hence, we have

$$
\begin{aligned}
& \left\|\mu_{a_{1}}-\mu_{a_{2}}\right\|_{L^{\infty}(\partial \Omega)} \int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|x-z_{\tau}\right|^{2-2 n} \\
& \leq C\{ \\
& \quad \int_{\Omega_{\cap B_{\rho}\left(z_{\tau}\right)}}\left|x-z_{\tau}\right|^{2-2 n+\alpha} \\
& \\
& \quad+\int_{\Omega_{\cap B_{\rho}\left(z_{\tau}\right)}}\left|x-z_{\tau}\right|^{2-2 n}\left|x-x^{0}\right|^{\beta} \\
& \\
& \quad+\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|\mu_{a_{2}}-\mu_{a_{1}}\right|\left|x-z_{\tau}\right|^{4-2 n} \\
& \\
& \quad+\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|K\left(x, \mu_{a_{2}}\right)-K\left(x, \mu_{a_{1}}\right)\right|\left|x-z_{\tau}\right|^{2-2 n} \\
& \\
& \left.\quad+\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|\mu_{a_{2}}-\mu_{a_{1}}\right|\left|x-z_{\tau}\right|^{4-2 n}\right\} \\
& +\|
\end{aligned}
$$

and by estimating the above integrals and the $H^{\frac{1}{2}}(\partial \Omega)$ norm of $u_{i}$ for $i=1,2$ (see $[2,8]$ ) we obtain

$$
\begin{align*}
&\left\|\mu_{a_{1}}-\mu_{a_{2}}\right\|_{L^{\infty}(\partial \Omega)} \tau^{2-n} \leq C\{ \tau^{2-n+\alpha}+\tau^{2-n+\beta}+\tau^{4-n}+C \\
&\left.+\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\left.\mu_{a_{2}}\right)}\right\|_{*} \tau^{2-n}\right\} \tag{81}
\end{align*}
$$

which leads to

$$
\begin{equation*}
\left\|\mu_{a_{1}}-\mu_{a_{2}}\right\|_{L^{\infty}(\partial \Omega)} \leq C\left\{\omega(\tau)+\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}\right\}, \tag{82}
\end{equation*}
$$

where $\omega(\tau) \rightarrow 0$ as $\tau \rightarrow 0$, which concludes the proof.
Proof of Theorem 2.6. Let $\widetilde{\nu}$ be the unit vector field introduced in this section. We shall prove that

$$
\begin{equation*}
\left\|\frac{\partial^{j}}{\partial \widetilde{\nu}^{j}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \leq C\left\|\Lambda_{1}-\Lambda_{2)}\right\|_{*}^{\delta_{j}}, \quad \text { for every } j \leq k \tag{83}
\end{equation*}
$$

where $\delta_{j}$ is given by (30). We proceed by induction on $k$ by following the same line of [8, Proof of Theorem 2.2] and therefore only the points where the two proofs differ will be highlighted. From theorem 2.5 we have that (83) holds true for $k=0$. Let us assume that (83) holds true for $j=k-1$ and prove that it is true for $j=k$ too.
Let $m$ be a positive integer and $x^{0} \in \partial \Omega \cap \bar{W}$ be such that

$$
\begin{equation*}
(-1)^{k} \frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{2}}-\mu_{a_{1}}\right)\left(x^{0}\right)=\left\|\frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \tag{84}
\end{equation*}
$$

Let $z_{\tau}=x^{0}+\tau \widetilde{\nu}$, with $\tau \leq \min \left\{\tau_{0}, \frac{\rho}{2}\right\}$, where $\tau_{0}$ is the number fixed in (69) and $\rho=\min \left\{r_{0}, \frac{h}{4 L}\right\}$, where $r_{0}$ is the number depending on the choice of $m$ which was introduced in (68). With these choices $B_{\rho}\left(z_{\tau}\right) \cap \bar{\Omega}$ is nonempty and

$$
\begin{equation*}
B_{\rho}\left(z_{\tau}\right) \cap \bar{\Omega} \subset U \tag{85}
\end{equation*}
$$

For the choice of $\rho$ and (85) we recall [8, Lemmas 3.1-33] as explained at the beginning of this section. Let $u_{i}$ be the singular solution of Theorem 3.1 corresponding to $\mu_{a_{i}}$, for $i=1,2$ and $m$. By Lagrange theorem, for every $x \in \bar{U}$ there exists $t(x), 0<t(x)<1$, such that

$$
\begin{equation*}
K\left(x, \mu_{a_{1}}\right)-K\left(x, \mu_{a_{2}}\right)=\left.\left(\mu_{a_{1}}(x)-\mu_{a_{2}}(x)\right) D_{t} K(x, t)\right|_{t=c(x)} \tag{86}
\end{equation*}
$$

where $c(x)=a(x)+t(x)\left(\mu_{a_{2}}(x)-\mu_{a_{1}}(x)\right)$ and

$$
\begin{equation*}
\left|D u_{1}-D u_{2}\right| \leq C\left(\left|x-z_{\tau}\right|^{1-n-m}\left|\mu_{a_{1}}\left(x^{0}\right)-\mu_{a_{2}}\left(x^{0}\right)\right|+\left|x-z_{\tau}\right|^{1-n-m+\alpha}\right), \tag{87}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\left.D_{t} K(x, t)\right|_{t=c(x)} D u_{1} \cdot D u_{2} \leq-C\left|x-z_{\tau}\right|^{2-2(n+m)} \tag{88}
\end{equation*}
$$

for almost every $x \in B_{\rho}\left(z_{\tau}\right) \cap \Omega$. Noting that every $x \in U$ can be uniquely represented as $x=y-s \widetilde{\nu}$, where $y \in \partial \Omega, 0 \leq s \leq \tau_{0}$, with $0<\tau_{0}<h-L r$, Taylor's formula for $\mu_{a_{2}}-\mu_{a_{1}}$ leads to

$$
\begin{align*}
k!\left(\mu_{a_{2}}-\mu_{a_{1}}\right)(x) \geq & \left\|\frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \\
& -C\left\{\sum_{j=0}^{k-1}\left\|\frac{\partial^{j}}{\partial \tilde{\nu}^{j}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\| s^{j}\right. \\
& \left.-s^{k}\left|x-x^{0}\right|^{\alpha}\right\} \tag{89}
\end{align*}
$$

and by combining Alessandrini's identity (23) together with (88) and (89) we obtain

$$
\begin{align*}
& \left.\left|\left|\Lambda_{\mu_{1}}-\Lambda_{\mu_{2}}\right|\right|_{*}\left\|u_{1}\right\|_{H^{\frac{1}{2}}(\partial \Omega)} \right\rvert\,\left\|u_{2}\right\|_{H^{\frac{1}{2}}(\partial \Omega)} \\
& \geq\left\|\frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}(d(x, \partial \Omega))^{k}\left|x-z_{\tau}\right|^{2-2(n+m)} \\
& -\sum_{j=0}^{k-1}\left\|\frac{\partial^{j}}{\partial \widetilde{\nu}^{j}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}(d(x, \partial \Omega))^{j}\left|x-z_{\tau}\right|^{2-2(n+m)} \\
& -\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}(d(x, \partial \Omega))^{k}\left|x-x^{0}\right|^{\alpha}\left|x-z_{\tau}\right|^{2-2(n+m)} \\
& -\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|K\left(x, \mu_{a_{1}}\right)-K\left(x, \mu_{a_{2}}\right)\right|\left|x-z_{\tau}\right|^{2-2(n+m)} \\
& -\int_{\Omega \cap B_{\rho}\left(z_{\tau}\right)}\left|\left(\mu_{a_{1}}-\mu_{a_{2}}\right)(x)\right|\left|x-z_{\tau}\right|^{4-2(n+m)} \\
& -\int_{\Omega \backslash B_{\rho}\left(z_{\tau}\right)}\left|\left(\mu_{a_{1}}-\mu_{a_{2}}\right)(x)\right|\left|x-z_{\tau}\right|^{4-2(n+m)} . \tag{90}
\end{align*}
$$

Estimating the above integrals and the norms $\left\|u_{i}\right\|_{H^{\frac{1}{2}}(\partial \Omega)}$, for $i=1,2$ as
in [8, Proof of Theorem 2.2] leads to

$$
\begin{align*}
\left\|\frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \tau^{2-n-2 m+k} \leq C\{ & \sum_{j=0}^{k-1}\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}^{\delta_{j}} \tau^{2-n-2 m+j} \\
& +\tau^{2-n-2 m+\alpha+k}+C+\tau^{4-n-2 m} \\
& \left.+\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}^{\delta_{j}} \tau^{2-n-2 m}\right\},(91 \tag{91}
\end{align*}
$$

therefore to

$$
\begin{equation*}
\left\|\frac{\partial^{k}}{\partial \widetilde{\nu}^{k}}\left(\mu_{a_{1}}-\mu_{a_{2}}\right)\right\|_{L^{\infty}(\partial \Omega \cap \bar{W})} \leq C\left\{\left\|\Lambda_{\mu_{a_{1}}}-\Lambda_{\mu_{a_{2}}}\right\|_{*}^{\delta_{k-1}} \tau^{-k}+\tau^{\alpha}\right\} \tag{92}
\end{equation*}
$$

(83) is then derived for $j=k$ by optimizing the choice of $\tau$ in (92). We recall for sake of completeness that (29) is obtained by combining (83) together with an iterated use of the following interpolation inequality

$$
\begin{equation*}
\|D f\|_{L^{\infty}(\partial \Omega \cap \bar{U})} \leq C\left\{\left\|\frac{\partial}{\partial \widetilde{\nu}} f\right\|_{L^{\infty}(\partial \Omega}+\|f\|_{L^{\infty}(\partial \Omega \cap \bar{U})}^{\frac{\alpha}{1-\alpha}}\|f\|_{C^{1+\alpha}(\bar{U})}^{\frac{\alpha}{1+\alpha}}\right\} \tag{93}
\end{equation*}
$$

for every $f \in C^{1, \alpha}(\bar{\Omega})$. Such an interpolation inequality can be found for example in [2, Lemma 3.2].

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# Stability analysis of the inverse inclusion problem 

Michele Di Cristo<br>Dedicated to Professor Giovanni Alessandrini on the occasion of his $60^{\text {th }}$ birthday


#### Abstract

We review some results concerning the determination of an inclusion within a body. In particular we show stability estimates, that is the continuous dependance of the inclusion from the boundary measurements. We present the cases of an electrical conductor, an elastic body and a thermal conductor.


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## 1. Introduction

In this note we consider the inverse problem of determining an inclusion $D$ contained in a domain $\Omega$. More precisely we aim to locate a region of a specimen whose physical properties are different from the properties of the surrounding material. For instance, if we consider an electrical conductor $\Omega$ of constant conductivity 1 , the inclusion $D$ has a conductivity equals to some unknown constant $k$, different from 1 .

Prescribing a voltage $f \in H^{1 / 2}(\partial \Omega)$ on the boundary of $\Omega$, the induced potential $u \in H^{1}(\Omega)$ is the solution of the problem

$$
\begin{cases}\operatorname{div}\left(\left(1+(k-1) \chi_{D}\right) \nabla u\right)=0 & \text { in } \Omega  \tag{1}\\ u=f & \text { on } \partial \Omega\end{cases}
$$

where $\chi_{D}$ denotes the characteristic function of the set $D$.
The normal derivative of the solution $u$ on the boundary $\left.\frac{\partial u}{\partial \nu} \right\rvert\, \partial \Omega$ corresponds to the current density measured. The pair of Cauchy data $\left\{f, \left.\frac{\partial u}{\partial \nu} \right\rvert\, \partial \Omega\right\}$ represents the electrostatic measurements performed on the boundary. We define the so called Dirichlet-to-Neumann map $\Lambda_{D}$ as

$$
\begin{array}{cccc}
\Lambda_{D}: H^{1 / 2}(\partial \Omega) & \rightarrow H^{-1 / 2}(\partial \Omega) \\
f & \rightarrow & \left.\frac{\partial u}{\partial \nu} \right\rvert\, \partial \Omega
\end{array}
$$

Its knowledge corresponds to performing infinitely many boundary measurements.

The inverse problem we are addressing to is to recover information on the inclusion $D$ from a knowledge of the map $\Lambda_{D}$.

This problem is a special instance of the well-known Calderon's inverse conductivity problem [11]. Uniqueness was established in 1988 by Isakov [21], whose approach makes use of the Runge approximation Theorem and solutions of the equation with Green's function type singularities.

In 2005 Alessandrini and Di Cristo [4] have studied the stability issue, that is the continuous dependance of the solution $D$ from the given data $\Lambda_{D}$. Converting Isakov's argument in a quantitative form, the authors prove that under mild a priori assumptions on the regularity and the topology of the inclusion, the modulus of continuity is of logarithmic type. Though such a modulus of continuity is weak, in [14] it is shown that, keeping as minimal as possible, the a priori information on the solution, it turns out to be optimal. To improve this rate of continuity, more a priori information on the inclusion are needed (see for instance [8]).

The argument proposed in [4] is very flexible and it can be extended to other problems like locating a scattered object by the knowledge of the near field data [13] or an inclusion in an elastic body by measuring the displacement and the traction on the boundary [5] or in a thermal conductor from the knowledge of the temperature and the heat flux on the boundary [15].

Let us mention here that in all these papers a crucial role is played by the explicit representation of the fundamental solution of the operator $\operatorname{div}(1+$ $\left.(k-1) \chi^{+} \nabla \cdot\right)$ ), where $\chi^{+}$is the characteristic function of the half space. It would be interesting generalize such argument when different information on the fundamental solution are available. Some ideas in this direction can be found in the parabolic case (see Section 4) but still it is not clear what kind of analysis is needed.

In this review note we illustrate the main step to get stability in the impedance tomography case (Section 2). Then in the subsequent Section 3 we analyze the elastic body context, emphasizing the main differences and the new tools needed. We conclude in the last Section 4 with the parabolic case.

## 2. Electrical Conductors

Let us first premise some notations and definitions we will use later on. In places we denote a point $x \in \mathbf{R}^{n}$ by $x=\left(x^{\prime}, x_{n}\right)$, where $x^{\prime} \in \mathbf{R}^{n-1}, x_{n} \in \mathbf{R}$.

Definition 2.1. Let $\Omega$ be a bounded domain in $\mathbf{R}^{n}$. Given $\alpha, 0<\alpha \leq 1$, we shall say that a portion $S$ of $\partial \Omega$ is of class $C^{1, \alpha}$ with constants $r_{0}, M_{0}>0$ if, for any $P \in S$, there exists a rigid transformation of coordinates under which
we have $P=0$ and

$$
\Omega \cap B_{r_{0}}(0)=\left\{x \in B_{r_{0}}: x_{n}>\varphi\left(x^{\prime}\right)\right\}
$$

where $\varphi$ is a $C^{1, \alpha}$ function on $B_{r_{0}}(0) \subset \mathbf{R}^{n-1}$ satisfying $\varphi(0)=|\nabla \varphi(0)|=0$ and $\|\varphi\|_{C^{1, \alpha}\left(B_{r_{0}}(0)\right)} \leq M_{0} r_{0}$.
Definition 2.2. We shall say that a portion $S$ of $\partial \Omega$ is of Lipschitz class with constants $r_{0}, M_{0}>0$ if for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P=0$ and

$$
\Omega \cap B_{r_{0}}(0)=\left\{x \in B_{\bar{r}}: x_{n}>\varphi\left(x^{\prime}\right)\right\},
$$

where $\varphi$ is a Lipschitz continuous function on $B_{r_{0}}(0) \subset \mathbf{R}^{n-1}$ satisfying $\varphi(0)=$ 0 and $\|\varphi\|_{C^{0,1}\left(B_{r_{0}}(0)\right)} \leq M_{0} r_{0}$.
Assumptions on the domain
Given $r_{0}, M_{0}, M_{1}>0$ and $0<\alpha<1$ as constants, we assume that $\Omega \subset \mathbf{R}^{n}$ is of class $C^{1, \alpha}$ class with constants $r_{0}, M_{0}$ such that

$$
|\Omega| \leq M_{1} r_{0}^{n}
$$

where $|\cdot|$ denotes the Lebesgue measure of $\Omega$.

## Assumptions on the inclusion

Let $D$ be a domain contained in $\Omega$ such that $\mathbf{R}^{n} \backslash D$ is connected, $\partial D$ is of $C^{1, \alpha}$ class with constants $r_{0}, M_{0}$ and, for a given $\delta_{0}>0, \operatorname{dist}(D, \partial \Omega) \geq \delta_{0}$.

In what follows we will refer to constants $k, n, r_{0}, M_{0}, M_{1}, \alpha, \delta_{0}$ as to the a priori data. We recall that $n \geq 2$ is the dimension and $k$ is the conductivity inside the inclusion.

We denote by $D_{1}$ and $D_{2}$ two possible inclusions in $\Omega$ both satisfying the aforementioned properties and by $\Lambda_{D_{1}}$ and $\Lambda_{D_{2}}$ the corresponding Dirichlet-to-Neumann maps.
Remark 2.3. As it is well known, the Dirichlet-to-Neumann map $\Lambda_{D}$ associated to problem (1) is defined by

$$
<\Lambda_{D} u, v>=\int_{\Omega}\left(1+(k-1) \chi_{D}\right) \nabla u \cdot \nabla v
$$

for every $u \in H^{1}(\Omega)$ solution to (1) and $v \in H^{1}(\Omega)$. Here $<\cdot, \cdot>$ denotes the duality pairing between $H^{-1 / 2}(\partial \Omega)$ and $H^{1 / 2}(\partial \Omega)$. With a slight abuse of notation, we will write

$$
<g, f>=\int_{\partial \Omega} g f d \sigma
$$

for any $f \in H^{1 / 2}(\partial \Omega)$ and $g \in H^{-1 / 2}(\partial \Omega)$.

TheOrem 2.4. Let $\Omega \subset \mathbf{R}^{n}$, $n \geq 2$, be as above, $k>0, k \neq 1$ be given and $D_{1}$ and $D_{2}$ be two inclusions in $\Omega$ as above. If, given $\varepsilon>0$, we have

$$
\begin{equation*}
\left\|\Lambda_{D_{1}}-\Lambda_{D_{2}}\right\|_{\mathcal{L}\left(H^{1 / 2}, H^{-1 / 2}\right)} \leq \varepsilon \tag{2}
\end{equation*}
$$

then

$$
\begin{equation*}
d_{\mathcal{H}}\left(\partial D_{1}, \partial D_{2}\right) \leq \omega(\varepsilon) \tag{3}
\end{equation*}
$$

where $\omega$ is an increasing function on $[0,+\infty)$, which satisfies

$$
\omega(t) \leq C|\log t|^{-\eta}, \quad \text { for every } \quad 0<t<1
$$

and $C, \eta, C>0,0<\eta \leq 1$, are constants only depending on the a priori data.
Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between bounded closed sets of $\mathbf{R}^{n}$ and $\|\cdot\|_{\mathcal{L}\left(H^{1 / 2} H^{-1 / 2}\right)}$ denotes the operator norm on the space of bounded linear operators between $H^{1 / 2}(\partial \Omega)$ and $H^{-1 / 2}(\partial \Omega)$. Let us also stress here that this theorem holds in any dimension $n \geq 2$ as the proof is based on singular solutions arguments that are not related to the dimension.

Remark 2.5. For the sake of simplicity we have chosen to present the theorem in the case of piecewise constant conductivity with the knowledge of the full Dirichlet-to-Neumann map- It is possible to consider a slightly more general case with conductivities of the form

$$
\gamma(x)=a(x)+b(x) \chi_{D},
$$

where $a \in C^{0,1}(\bar{\Omega})$ and $b \in C^{\alpha}(\Omega)$, and when only a portion of the boundary $\partial \Omega$ is available to perform measurements. We refer to [12] for a detailed study of this problem.

Let us sketch the argument to prove this theorem. For the reader convenience we divide it into several steps.

## Step 1: modified distance.

Let $\mathcal{G}$ be the connected component of $\mathbf{R}^{n} \backslash\left(\overline{D_{1} \cup D_{2}}\right)$ which contains $\mathbf{R}^{n} \backslash \bar{\Omega}$ and let us denote $\Omega_{D}=\mathbf{R}^{n} \backslash \overline{\mathcal{G}}$. As we shall see later, one of the key ingredients of the stability proof consists in propagating the smallness appearing in the measurements (2) from the boundary $\partial \Omega$ inside $\Omega$. Since the value $d_{H}\left(\partial D_{1}, \partial D_{2}\right)$ may be attained at some point not belonging to $\overline{\mathcal{G}}$ and, therefore, not reachable from the exterior, it is necessary to introduce a modified distance following the ideas developed in [4]. Precisely, let us introduce the modified distance between $D_{1}$ and $D_{2}$

$$
\begin{equation*}
d_{\mu}\left(D_{1}, D_{2}\right)=\max \left\{\max _{x \in \partial D_{1} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, D_{2}\right), \max _{x \in \partial D_{2} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, D_{1}\right)\right\} \tag{4}
\end{equation*}
$$

We remark here that $d_{\mu}$ is not a metric and, in general, it does not dominate the Hausdorff distance. However, under our a priori assumptions on the inclusion, the following lemma holds true.

Lemma 2.6. Under the assumptions of Theorem 2.4, there exists a constant $c_{0} \geq 1$ only depending on $M_{0}$ and $\alpha$ such that

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq c_{0} d_{\mu}\left(D_{1}, D_{2}\right) \tag{5}
\end{equation*}
$$

Proof. See [4, Proposition 3.3].
It is easy to verify that

$$
\begin{aligned}
& \max _{x \in \partial D_{1} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, D_{2}\right)=\max _{x \in \partial D_{1} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, \partial D_{2}\right) \\
& \max _{x \in \partial D_{2} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, D_{1}\right)=\max _{x \in \partial D_{2} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, \partial D_{1}\right),
\end{aligned}
$$

so that $d_{\mu}\left(D_{1}, D_{2}\right) \leq d_{H}\left(\partial D_{1}, \partial D_{2}\right)$, and therefore, in view of Lemma 2.6, these two quantities are comparable.

Another obstacle comes out from the fact that the propagation of smallness arguments are based on an iterated application of the three-spheres inequality for solutions of the equation over chains of balls contained in $\mathcal{G}$ and, in this step, it is crucial to control from below the radii of these balls. In the following Lemma 2.7 we treat the case of points of $\partial \Omega_{D}$ that are not reachable by such chains of balls. This problem was originally considered by [7] in the context of cracks detection in electrical conductors and was underestimated in the papers $[4,12,13,15,16]$. The procedure developed here enables to fill the possible gaps in the proofs.

Let us premise some notation. Given $O=(0, \ldots, 0)$ the origin, $v$ a unit vector, $h>0$ and $\vartheta \in\left(0, \frac{\pi}{2}\right)$, we denote

$$
\begin{equation*}
C(O, v, h, \vartheta)=\left\{x \in \mathbf{R}^{n}| | x-(x \cdot v) v|\leq \sin \vartheta| x \mid, 0 \leq x \cdot v \leq h\right\} \tag{6}
\end{equation*}
$$

the closed truncated cone with vertex at $O$, axis along the direction $v$, height $h$ and aperture $2 \vartheta$. Given $R, d, 0<R<d$ and $Q=-d e_{n}$, where $e_{n}=$ $(0, \ldots, 0,1)$, let us consider the cone $C\left(O,-e_{n}, \frac{d^{2}-R^{2}}{d}, \arcsin \frac{R}{d}\right)$.

From now on, for simplicity, we assume that

$$
\begin{equation*}
d_{\mu}\left(D_{1}, D_{2}\right)=\max _{x \in \partial D_{1} \cap \partial \Omega_{D}} \operatorname{dist}\left(x, \partial D_{2}\right) \tag{7}
\end{equation*}
$$

and we write $d_{\mu}=d_{\mu}\left(D_{1}, D_{2}\right)$.
Let us define

$$
\begin{equation*}
S_{2 \rho_{0}}=\left\{x \in \mathbf{R}^{n} \mid \rho_{0}<\operatorname{dist}(x, \bar{\Omega})<2 \rho_{0}\right\} . \tag{8}
\end{equation*}
$$

We shall make use of paths connecting points in order that appropriate tubular neighborhoods of such paths still remain within $\mathbf{R}^{n} \backslash \Omega_{D}$.

Let us pick a point $P \in \partial D_{1} \cap \partial \Omega_{D}$, let $\nu$ be the outer unit normal to $\partial D_{1}$ at $P$ and let $d>0$ be such that the segment $[(P+d \nu), P]$ is contained in $\mathbf{R}^{n} \backslash \Omega_{D}$. Given $P_{0} \in \mathbf{R}^{n} \backslash \Omega_{D}$, let $\gamma$ be a path in $\mathbf{R}^{n} \backslash \Omega_{D}$ joining $P_{0}$ to $P+d \nu$. We consider the following neighborhood of $\gamma \cup[(P+d \nu), P] \backslash\{P\}$ formed by a tubular neighborhood of $\gamma$ attached to a cone with vertex at $P$ and axis along $\nu$

$$
\begin{equation*}
V(\gamma)=\bigcup_{S \in \gamma} B_{R}(S) \cup C\left(P, \nu, \frac{d^{2}-R^{2}}{d}, \arcsin \frac{R}{d}\right) \tag{9}
\end{equation*}
$$

Note that two significant parameters are associated to such a set, the radius $R$ of the tubular neighborhood of $\gamma, \cup_{S \in \gamma} B_{R}(S)$, and the half-aperture $\arcsin \frac{R}{d}$ of the cone $C\left(P, \nu, \frac{d^{2}-R^{2}}{d}\right.$, $\left.\arcsin \frac{R}{d}\right)$. In other terms, $V(\gamma)$ depends on $\gamma$ and also on the parameters $R$ and $d$. At each of the following steps, such two parameters shall be appropriately chosen and shall be accurately specified. For the sake of simplicity we convene to maintain the notation $V(\gamma)$ also when different values of $R, d$ are introduced.

Also we warn the reader that it will be convenient at various stages to use a reference frame such that $P=O=(0, \ldots, 0)$ and $\nu=-e_{n}$.
Lemma 2.7. Under the above notation, there exist positive constants $\bar{d}, c_{1}$, where $\frac{\bar{d}}{\rho_{0}}$ only depends on $M_{0}$ and $\alpha$, and $c_{1}$ only depends on $M_{0}, \alpha, M_{1}$, and there exists a point $P \in \partial D_{1}$ satisfying

$$
\begin{equation*}
c_{1} d_{\mu} \leq \operatorname{dist}\left(P, D_{2}\right) \tag{10}
\end{equation*}
$$

and such that, giving any point $P_{0} \in S_{2 \rho_{0}}$, there exists a path $\gamma \subset\left(\overline{\Omega^{\rho_{0}}} \cup S_{2 \rho_{0}}\right) \backslash$ $\overline{\Omega_{D}}$ joining $P_{0}$ to $P+\bar{d} \nu$, where $\nu$ is the unit outer normal to $D_{1}$ at $P$, such that, choosing a coordinate system with origin $O$ at $P$ and axis $e_{n}=-\nu$, the set $V(\gamma)$ introduced in (9) satisfies

$$
\begin{equation*}
V(\gamma) \subset \mathbf{R}^{n} \backslash \Omega_{D} \tag{11}
\end{equation*}
$$

provided $R=\frac{\bar{d}}{\sqrt{1+L_{0}^{2}}}$, where $L_{0}, 0<L_{0} \leq M_{0}$, is a constant only depending on $M_{0}$ and $\alpha$.

In order to prove Lemma 2.7, we shall use the following results.
Lemma 2.8 (Lemma 5.5 in [6]). Let $U$ be a Lipschitz domain in $\mathbf{R}^{n}$ with constants $\rho_{0}, M_{0}$. There exists $h_{0}, 0<h_{0}<1$, only depending on $M_{0}$, such that

$$
\begin{equation*}
U_{h \rho_{0}} \text { is connected for every } h, 0<h \leq h_{0} \tag{12}
\end{equation*}
$$

Theorem 2.9 (Theorem 3.6 in [3]). There exist positive constants $d_{0}, r_{0}, L_{0}$, $L_{0} \leq M_{0}$, with $\frac{d_{0}}{\rho_{0}}, \frac{r_{0}}{\rho_{0}}$ only depending on $M_{0}$ and $L_{0}$ only depending on $\alpha$ and $M_{0}$, such that if

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq d_{0} \tag{13}
\end{equation*}
$$

then $\partial \Omega_{D}$ is Lipschitz with constants $r_{0}$ and $L_{0}$. Moreover, for every $P \in$ $\partial \Omega_{D} \cap \partial D_{1}$, up to a rigid transformation of coordinates which maps $P$ into the origin and $e_{n}=-\nu$, where $\nu$ is the outer unit normal to $D_{1}$ at $P$, we have

$$
\begin{gather*}
D_{i} \cap B_{r_{0}}(P)=\left\{x \in B_{r_{0}}(0) \mid x_{n}>\varphi_{i}\left(x^{\prime}\right)\right\}, \quad i=1,2,  \tag{14}\\
\varphi_{1}(0)=0, \quad \nabla \varphi_{1}(0)=0  \tag{15}\\
\left\|\varphi_{i}\right\|_{C^{0,1}\left(B_{r_{0}}^{\prime}(0)\right)} \leq L_{0} r_{0}, \quad i=1,2 \tag{16}
\end{gather*}
$$

An analogous representation holds for every $P \in \partial \Omega_{D} \cap \partial D_{2}$.
Proof of Lemma 2.7. Let

$$
\begin{equation*}
d_{1}=\frac{d_{0}}{c_{0}} \tag{17}
\end{equation*}
$$

where $c_{0}$ is the constant introduced in Lemma 2.6, and let

$$
\begin{equation*}
d_{2}=\min \left\{d_{1}, h_{0} \rho_{0}\right\} \tag{18}
\end{equation*}
$$

where $h_{0}, 0<h_{0}<1$, only depending on $M_{0}$, has been introduced in Lemma 2.8. We shall distinguish two cases.

Case i) Let $d_{\mu} \leq d_{1}$.
Then, by Lemma 2.6 we have $d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq d_{0}$. Therefore, by Theorem 2.9, $\partial \Omega_{D}$ is Lipschitz with constants $r_{0}, L_{0}$, where $\frac{r_{0}}{\rho_{0}}$ only depends on $M_{0}$, and $L_{0}$ only depends on $M_{0}$ and $\alpha$. We may apply Lemma 2.8 to $\mathbf{R}^{n} \backslash \Omega_{D}$ obtaining that there exists $\widetilde{h}_{0}, 0<\widetilde{h}_{0}<1$, only depending on $\alpha$ and $M_{0}$, such that $\left(\mathbf{R}^{n} \backslash \Omega_{D}\right)_{h r_{0}}$ is connected for every $h \leq \widetilde{h}_{0}$.

Let $P \in \partial D_{1} \cap \partial \Omega_{D}$ be such that

$$
\begin{equation*}
d_{\mu}\left(D_{1}, D_{2}\right)=\operatorname{dist}\left(P, D_{2}\right) \tag{19}
\end{equation*}
$$

Under the coordinate system introduced in Theorem 2.9, let us consider the point $Q=P-\frac{\widetilde{h}_{0} r_{0}}{2} e_{n}$. We have that

$$
\begin{equation*}
\operatorname{dist}\left(Q, \Omega_{D}\right) \geq \frac{\widetilde{h}_{0} r_{0}}{2 \sqrt{1+L_{0}^{2}}} \tag{20}
\end{equation*}
$$

Let us denote $h_{1}=\frac{\widetilde{h}_{0}}{2 \sqrt{1+L_{0}^{2}}}$. Since $h_{1}<\widetilde{h}_{0}$, the set $\overline{\left(\mathbf{R}^{n} \backslash \Omega_{D}\right)_{h_{1} r_{0}}}$ is connected and contains $Q$. Therefore, there exists a path $\gamma \subset \overline{\left(\mathbf{R}^{n} \backslash \Omega_{D}\right)_{h_{1} r_{0}}}$ joining any
point $P_{0} \in S_{2 \rho_{0}}$ with $Q$. Therefore, in the above coordinate system, the set $V(\gamma)$ satisfies

$$
\begin{equation*}
V(\gamma) \subset \mathbf{R}^{n} \backslash \Omega_{D} \tag{21}
\end{equation*}
$$

provided

$$
\begin{equation*}
d=\frac{\widetilde{h}_{0} r_{0}}{2}, \quad R=\frac{d}{\sqrt{1+L_{0}^{2}}} \tag{22}
\end{equation*}
$$

Case ii) Let $d_{\mu} \geq d_{1}$.
Then, trivially, $d_{\mu} \geq d_{2}$. Let $\widetilde{P} \in \partial D_{1} \cap \partial \Omega_{D}$ be such that

$$
\begin{equation*}
d_{\mu}\left(D_{1}, D_{2}\right)=\operatorname{dist}\left(\widetilde{P}, D_{2}\right) \tag{23}
\end{equation*}
$$

Since $d_{2} \leq h_{0} \rho_{0}$, by Lemma 2.8, $\left(\mathbf{R}^{n} \backslash D_{2}\right)_{d_{2}}$ is connected. Therefore, given any point $P_{0} \in S_{2 \rho_{0}}$, there exists a path $\gamma, \gamma:[0,1] \rightarrow\left(\mathbf{R}^{n} \backslash D_{2}\right)_{d_{2}}$ such that $\gamma(0) \in S_{2 \rho_{0}}$ and $\gamma(1)=\widetilde{P}$. Let $\bar{t}=\inf _{t \in[0,1]}\left\{t \left\lvert\, \operatorname{dist}\left(\gamma(t), \partial D_{1}\right)>\frac{d_{2}}{2}\right.\right\}$. By definition, $\operatorname{dist}\left(\gamma(\bar{t}), \partial D_{1}\right)=\frac{d_{2}}{2}$, so that there exists $P \in \partial D_{1}$ satisfying $|P-\gamma(\bar{t})|=\frac{d_{2}}{2}$. We have that

$$
\begin{equation*}
\operatorname{dist}\left(P, D_{2}\right) \geq \operatorname{dist}\left(\gamma(\bar{t}), D_{2}\right)-|\gamma(\bar{t})-P| \geq d_{2}-\frac{d_{2}}{2}=\frac{d_{2}}{2} \tag{24}
\end{equation*}
$$

Let $\bar{\gamma}=\left.\gamma\right|_{[0, \bar{t}]}$ and let us choose a cartesian coordinate system with origin $O$ at $P$, and $e_{n}=-\nu$, where $\nu$ is the outer unit normal to $D_{1}$ at $P$. We have that

$$
\begin{equation*}
V(\bar{\gamma}) \subset \mathbf{R}^{n} \backslash \Omega_{D} \tag{25}
\end{equation*}
$$

assuming

$$
\begin{equation*}
d=\frac{d_{2}}{2}, \quad R=\frac{d}{\sqrt{1+M_{0}^{2}}} \tag{26}
\end{equation*}
$$

Let

$$
\begin{equation*}
\bar{d}=\min \left\{\frac{\widetilde{h}_{0} r_{0}}{2}, \frac{d_{0}}{2 c_{0}}, \frac{h_{0} \rho_{0}}{2}\right\} \tag{27}
\end{equation*}
$$

and let us notice that $\frac{\bar{d}}{\rho_{0}}$ only depends on $M_{0}, \alpha$. Observing that $L_{0} \leq M_{0}$, formula (11) follows with $\bar{d}$ given in (27). Since there exists a positive constant $C$ only depending on $M_{0}, M_{1}$ such that $\operatorname{diam}(\Omega) \leq C \rho_{0}$, we have that

$$
\begin{equation*}
d_{\mu} \leq\left(\frac{\operatorname{diam}(\Omega)}{\frac{d_{2}}{2}}\right) \frac{d_{2}}{2} \leq \widetilde{c}_{1} \frac{d_{2}}{2} \tag{28}
\end{equation*}
$$

with $\widetilde{c}_{1}$ only depending on $M_{0}, \alpha$ and $M_{1}$. Letting $c_{1}=\min \left\{1, \frac{1}{\widetilde{c}_{1}}\right\}$, inequality (10) follows.

From now on we will denote by $P=O \in \partial D_{1} \cap \partial \Omega$ the point such that

$$
\begin{equation*}
d_{\mu}\left(D_{1}, D_{2}\right)=\operatorname{dist}\left(P, D_{2}\right) \tag{29}
\end{equation*}
$$

## Step 2: Alessandrini's identity.

Let $u_{i} \in H^{1}(\Omega), i=1,2$, be solutions to (1) when $D=D_{1}, D_{2}$ respectively, the following identity holds.

$$
\begin{align*}
& \int_{\Omega}\left(1+(k-1) \chi_{D_{1}}\right) \nabla u_{1} \cdot \nabla u_{2}-\int_{\Omega}\left(1+(k-1) \chi_{D_{2}}\right) \nabla u_{1} \cdot \nabla u_{2} \\
& =\int_{\partial \Omega} u_{1}\left[\Lambda_{D_{1}}-\Lambda_{D_{2}}\right] u_{2} \tag{30}
\end{align*}
$$

This identity can be obtained by using repeatedly Green's formula. In the context of inverse problems, the prototype of this identity can be traced back to Alessandrini, who first used in [1].

Let $\Gamma_{D}(x, y)$ be the fundamental solution for the operator $\operatorname{div}((1+(k-$ 1) $\left.\left.\chi_{D}\right) \nabla \cdot\right)$, thus

$$
\begin{equation*}
\operatorname{div}\left(\left(1+(k-1) \chi_{D}\right) \nabla \Gamma_{D}(\cdot, y)\right)=-\delta(\cdot-y) \tag{31}
\end{equation*}
$$

where $y \in \mathbf{R}^{n}, \delta$ denotes the Dirac distribution. We shall denote by $\Gamma_{D_{1}}, \Gamma_{D_{2}}$ such fundamental solutions when $D=D_{1}, D_{2}$ respectively. Replacing $u_{1}, u_{2}$ with $\Gamma_{D_{1}}, \Gamma_{D_{2}}$ in (30), we get

$$
\begin{align*}
& \int_{\Omega}\left(1+(k-1) \chi_{D_{1}}\right) \nabla \Gamma_{D_{1}}(\cdot, y) \cdot \nabla \Gamma_{D_{2}}(\cdot, w) \\
& -\int_{\Omega}\left(1+(k-1) \chi_{D_{2}}\right) \nabla \Gamma_{D_{1}}(\cdot, y) \cdot \nabla \Gamma_{D_{2}}(\cdot, w) \\
& \quad=\int_{\partial \Omega} \Gamma_{D_{1}}(\cdot, y)\left[\Lambda_{D_{1}}-\Lambda_{D_{2}}\right]\left(\Gamma_{D_{2}}(\cdot, w)\right) d \sigma \tag{32}
\end{align*}
$$

for any singularities $y$ and $w$ taken in the complement $\mathcal{C} \bar{\Omega}$ of $\bar{\Omega}$. Let us define, for $y, w \in \mathcal{G} \cup \mathcal{C} \Omega$

$$
\begin{align*}
S_{D_{1}}(y, w) & =(k-1) \int_{D_{1}} \nabla \Gamma_{D_{1}}(\cdot, y) \cdot \nabla \Gamma_{D_{2}}(\cdot, w)  \tag{33}\\
S_{D_{2}}(y, w) & =(k-1) \int_{D_{2}} \nabla \Gamma_{D_{1}}(\cdot, y) \cdot \nabla \Gamma_{D_{2}}(\cdot, w)  \tag{34}\\
f(y, w) & =S_{D_{1}}(y, w)-S_{D_{2}}(y, w) \tag{35}
\end{align*}
$$

Thus (32) can be rewritten as

$$
\begin{equation*}
f(y, w)=\int_{\partial \Omega} \Gamma_{D_{1}}(\cdot, y)\left[\Lambda_{D_{1}}-\Lambda_{D_{2}}\right]\left(\Gamma_{D_{2}}(\cdot, w)\right) d \sigma \quad \forall y, w \in \mathcal{C} \bar{\Omega} \tag{36}
\end{equation*}
$$

For $y, w \in \mathcal{C} \bar{\Omega}$, since $(2), f(y, w)$ is small. The idea to get stability is to evaluate how this smallness propagates as $y$ and $w$ move toward the inclusion To perform such analysis, a crucial step is the study of the behavior of the fundamental solution.

## Step 3: fundamental solutions.

For $x=\left(x^{\prime}, x_{n}\right)$, where $x^{\prime} \in \mathbf{R}^{n-1}$ and $x_{n} \in \mathbf{R}$, we set $x^{\star}=\left(x^{\prime},-x_{n}\right)$. We shall denote with $\chi^{+}$the characteristic function of the half-space $\left\{x_{n}>0\right\}$ and with $\Gamma_{+}$the fundamental solution of the operator $\operatorname{div}\left(\left(1+(k-1) \chi^{+}\right) \nabla \cdot\right)$. If $\Gamma$ is the standard fundamental solution of the Laplace operator, we have that

$$
\Gamma_{+}(x, y)= \begin{cases}\frac{1}{k} \Gamma(x, y)+\frac{k-1}{k(k+1)} \Gamma\left(x, y^{\star}\right) & \text { for } x_{n}>0, y_{n}>0  \tag{37}\\ \frac{2}{k+1} \Gamma(x, y) & \text { for } x_{n} y_{n}<0, \\ \Gamma(x, y)-\frac{k-1}{k+1} \Gamma\left(x, y^{\star}\right) & \text { for } x_{n}<0, y_{n}<0\end{cases}
$$

The following Proposition holds.
Proposition 2.10. Let $D \subset \mathbf{R}^{n}$ be an open set whose boundary is of class $C^{1, \alpha}$, with constants $r_{0}, M_{0}$.
(i) There exists a constant $c_{1}>0$ depending on $k, n, \alpha$ and $M_{0}$ only, such that

$$
\begin{equation*}
\left|\nabla_{x} \Gamma_{D}(x, y)\right| \leq c_{1}|x-y|^{1-n} \tag{38}
\end{equation*}
$$

for every $x, y \in \mathbf{R}^{n}$,
(ii) There exist constants $c_{2}, c_{3}>0$ depending on $k, n$, $\alpha$ and $M_{0}$ only, such that

$$
\begin{align*}
& \left|\Gamma_{D}(x, y)-\Gamma_{+}(x, y)\right| \leq \frac{c_{2}}{\bar{r}^{\alpha}}|x-y|^{2-n+\alpha}  \tag{39}\\
& \left|\nabla_{x} \Gamma_{D}(x, y)-\nabla_{x} \Gamma_{+}(x, y)\right| \leq \frac{c_{3}}{\bar{r}^{\alpha^{2}}}|x-y|^{1-n+\alpha^{2}} \tag{40}
\end{align*}
$$

for every $x \in D \cap B_{r}(P)$, and for every $y=h \nu(P)$, with $0<r<\bar{r}_{0}$, $0<h<\bar{r}_{0}$, where $\bar{r}_{0}=\left(\min \left\{\frac{1}{2}\left(8 M_{0}\right)^{-1 / \alpha}, \frac{1}{2}\right\}\right) \frac{r_{0}}{2}$.
Proof. The proof of $i$ ) is based on the $C^{1, \alpha}$ regularity of $\Gamma_{D}$ proved in [17], see also [24], and the pointwise bounds of $\Gamma_{D}$ with $\Gamma$ contained in [25].

To prove $i i$ ) we first flatten the boundary $\partial D$ around the point $P$ through a $C^{1, \alpha}$ diffeomorphism $\Phi$ from $\mathbf{R}^{n}$ into itself. Defining $\tilde{\Gamma}_{D}(\xi, \eta)=\Gamma_{D}(x, y)$ where $\xi=\Phi(x), \eta=\Phi(y)$, it is not difficult to check that $\tilde{\Gamma}_{D}$ solves

$$
\operatorname{div}_{\xi}\left(\left(1+(k-1) \chi^{+}\right) B(\xi) \nabla_{\xi} \widetilde{\Gamma}_{D}(\xi, \eta)\right)=-\delta(\xi-\eta)
$$

where $B$ is a $C^{\alpha}$ matrix such that $B(0)=I$. Considering

$$
\tilde{R}(x, y)=\tilde{\Gamma}_{D}(x, y)-\Gamma_{+}(x, y)
$$

by the properties of $\Gamma_{+}, \tilde{R}$ satisfies

$$
\operatorname{div}_{x}\left(\left(1+(k-1) \chi^{+}\right) \nabla_{x} \widetilde{R}(x, y)\right)=\operatorname{div}_{x}\left(\left(1+(k-1) \chi^{+}\right)(I-B) \nabla_{x} \widetilde{\Gamma}_{D}(x, y)\right)
$$

Using the fundamental solution $\Gamma_{+}$of the above operator and estimating the integral that represents the solution $\tilde{R}$, it is possible to show that

$$
|\tilde{R}(x, y)| \leq c|x-y|^{\alpha+2-n}
$$

Estimate (39) follows going back to the original coordinates and estimate (40) follows by using the interpolation inequality

$$
\|\nabla \widetilde{R}(\cdot, y)\|_{L^{\infty}(Q)} \leq c\|\widetilde{R}(\cdot, y)\|_{L^{\infty}(Q)}^{1-\delta}|\nabla \widetilde{R}(\cdot, y)|_{\alpha, Q}^{\delta}
$$

where $\delta=\frac{1}{1+\alpha}$ and

$$
|\nabla \widetilde{R}|_{\alpha, Q}=\sup _{x, x^{\prime} \in Q, x \neq x^{\prime}} \frac{\left|\nabla \widetilde{R}(x, y)-\nabla \widetilde{R}\left(x^{\prime}, y\right)\right|}{\left|x-x^{\prime}\right|^{\alpha}}
$$

We refer to [4, Proposition 3.4] for details.

## Step 4: quantitative estimates.

The next two Propositions provide quantitative estimates on $f$ and $S_{D_{1}}$ when we move $y$ towards $O$, along $\nu(O)$.
Proposition 2.11. Let $\Omega$ be an open set in $\mathbf{R}^{n}$ satisfying the above properties. Let $D_{1}, D_{2}$ be two inclusions in $\Omega$ verifying the above properties and let $y=$ $h \nu(O)$, with $O$ defined in (29). If, given $\varepsilon>0$, we have

$$
\left\|\Lambda_{D_{1}}-\Lambda_{D_{2}}\right\|_{\mathcal{L}\left(H^{1 / 2}, H^{-1 / 2}\right)} \leq \varepsilon
$$

then for every $h, 0<h<\bar{c} r_{0}$, where $0<\bar{c}<1$, depends on $M_{0}$,

$$
\begin{equation*}
|f(y, y)| \leq C \frac{\varepsilon^{B h^{F}}}{h^{A}} \tag{41}
\end{equation*}
$$

where $0<A<1$ and $C, B, F>0$ are constants that depend only on the a priori data.

Proof. To get this upper bound, the procedure is to fix one of the two singularities, say $w$, in $\mathcal{C} \bar{\Omega}$. It is not difficult to check that $f(y, w)$ is harmonic with respect to $y$ in $\mathcal{C} \bar{\Omega}_{D}$ and, therefore, we can apply iteratively the three spheres inequality to evaluate the propagation of the $\varepsilon$-smallness as we drag $y$ toward $\Omega_{D}$. Finally employing this procedure for $w$, we get the bound. We refer the reader to [4, Proposition 3.5] for details.

Proposition 2.12. Let $\Omega$ be an open set in $\mathbf{R}^{n}$ satisfying the above properties. Let $D_{1}, D_{2}$ be two inclusions in $\Omega$ verifying the above properties and $y=h \nu(O)$. Then for every $h, 0<h<\bar{r}_{0} / 2$,

$$
\begin{equation*}
\left|S_{D_{1}}(y, y)\right| \geq c_{1} h^{2-n}-c_{2} d_{\mu}^{2-2 n}+c_{3} \tag{42}
\end{equation*}
$$

where $c_{1}, c_{2}$ and $c_{3}$ are positive constants only depending on the a priori data. Here $\bar{r}_{0}$ is the number introduced in Proposition 2.10.

Proof. Choosing $y=h \nu(O)$, where $\nu(O)$ is the exterior outer normal to $\Omega_{D}$ in $O$, with $O$ defined as in (29), with $h$ sufficiently small, to get the lower bound (42), the crucial ingredient is the following inequality

$$
\nabla_{x} \Gamma_{D_{1}}(x, y) \cdot \nabla_{x} \Gamma_{D_{1}}(x, y) \geq c|x-y|^{2-2 n}
$$

with $x \in D_{1}$ sufficiently close to $y$. This estimate can be derived from [2, Lemma 3.1] once one has at disposal the asymptotic behavior (40) (see [4, Proposition 3.6] for details).

## Step 5: proof of Theorem 2.4.

Let $O \in \partial D_{1}$ satisfying (29), that is

$$
d_{\mu}\left(D_{1}, D_{2}\right)=\operatorname{dist}\left(O, D_{2}\right)=d_{\mu}
$$

Then, for $y=h \nu(O)$, with $0<h<h_{1}$, where $h_{1}=\min \left\{d_{\mu}, \bar{c} r_{0}, \bar{r}_{0} / 2\right\}$, using (38), we have

$$
\begin{equation*}
\left|S_{D_{2}}(y, y)\right| \leq c \int_{D_{2}} \frac{1}{\left(d_{\mu}-h\right)^{n-1}} \frac{1}{\left(d_{\mu}-h\right)^{n-1}} d x=c \frac{1}{\left(d_{\mu}-h\right)^{2 n-2}}\left|D_{2}\right| \tag{43}
\end{equation*}
$$

Using Proposition 2.11, we have

$$
\begin{aligned}
\left|S_{D_{1}}(y, y)\right|-\left|S_{D_{2}}(y, y)\right| & \leq\left|S_{D_{1}}(y, y)-S_{D_{2}}(y, y)\right| \\
& =|f(y, y)| \leq c \frac{\varepsilon^{B h^{F}}}{h^{A}}
\end{aligned}
$$

On the other hand, by Proposition 2.12 and (43)

$$
\left|S_{D_{1}}(y, y)\right|-\left|S_{D_{2}}(y, y)\right| \geq c_{1} h^{2-n}-c_{2}\left(d_{\mu}-h\right)^{2-2 n}
$$

Thus we have

$$
c_{3} h^{2-n}-c_{4}\left(d_{\mu}-h\right)^{2-2 n} \leq \frac{\varepsilon^{B h^{F}}}{h^{A}} .
$$

That is

$$
\begin{align*}
c_{4}\left(d_{\mu}-h\right)^{2-2 n} & \geq c_{3} h^{2-n}-\frac{\varepsilon^{B h^{F}}}{h^{A}}=h^{2-n}\left(c_{3}-\varepsilon^{B h^{F}} h^{\widetilde{A}}\right) \\
& \geq c_{5} h^{2-n}\left(1-\varepsilon^{B h^{F}} h^{\widetilde{A}}\right), \tag{44}
\end{align*}
$$

where $\widetilde{A}=n-2-A, \widetilde{A}>0$. Let $h=h(\varepsilon)$ where $h(\varepsilon)=\min \left\{|\ln \varepsilon|^{-\frac{1}{2 F}}, d_{\mu}\right\}$, for $0<\varepsilon \leq \varepsilon_{1}$, with $\varepsilon_{1} \in(0,1)$ such that $\exp \left(-B\left|\ln \varepsilon_{1}\right|^{1 / 2}\right)=1 / 2$. If $d_{\mu} \leq$ $|\ln \varepsilon|^{-\frac{1}{2 F}}$, since, by Lemma 2.6, the Hausdorff distance is dominated by $d_{\mu}$, estimate (3) follows trivially. In the other case we have

$$
\varepsilon^{B h(\varepsilon)^{F}} h(\varepsilon)^{\widetilde{A}} \leq \varepsilon^{B|\ln \varepsilon|^{-1 / 2}} \leq \exp \left(-B|\ln \varepsilon|^{1 / 2}\right)
$$

Then, for any $\varepsilon, 0<\varepsilon<\varepsilon_{1}$,

$$
\left(d_{\mu}-h(\varepsilon)\right)^{2-2 n} \geq c_{6} h(\varepsilon)^{2-n}
$$

that is, solving for $d_{\mu}$, and recalling that, in this case, $h(\varepsilon)=|\ln \varepsilon|^{-\frac{1}{2 F}}$

$$
\begin{equation*}
d_{\mu} \leq c_{7}|\ln \varepsilon|^{-\delta \frac{n-2}{2 n-2}} \tag{45}
\end{equation*}
$$

where $\delta=1 /(2 F)$. When $\varepsilon \geq \varepsilon_{1}$, then

$$
d_{\mu} \leq \operatorname{diam} \Omega
$$

and, in particular when $\varepsilon_{1} \leq \varepsilon<1$

$$
d_{\mu} \leq \operatorname{diam} \Omega \frac{|\ln \varepsilon|^{-\frac{1}{2 F}}}{\left|\ln \varepsilon_{1}\right|^{-\frac{1}{2 F}}}
$$

Finally, using Lemma 2.6, the theorem follows.

## 3. Elastic Bodies

Let us consider now the determination of an inclusion $D$ in an elastic body $\Omega$ by measuring the displacements and traction on the boundary $\partial \Omega$. More precisely, let $\Omega$ be a bounded domain in $\mathbf{R}^{3}$ and let $D$ be an open set contained in $\Omega$. We deal with the dimension $n=3$ as it is more relevant for applications. Everything works in any dimension. Assume that both the body $\Omega$ and the inclusion $D$ are made by different homogeneous, isotropic, elastic materials, with Lamé moduli $\mu, \lambda$ and $\mu^{D}, \lambda^{D}$, respectively, satisfying the strong convexity conditions $\mu>0,2 \mu+3 \lambda>0, \mu^{D}>0,2 \mu^{D}+3 \lambda^{D}>0$. For a given $f \in$ $H^{\frac{1}{2}}(\partial \Omega)$, consider the weak solution $u \in H^{1}(\Omega)$ to the Dirichlet problem

$$
\left\{\begin{array}{lr}
\operatorname{div}\left(\left(\mathbf{C}+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D}\right) \nabla u\right)=0, & \text { in } \Omega  \tag{46}\\
u=f, & \text { on } \partial \Omega
\end{array}\right.
$$

where $\mathbf{C}, \mathbf{C}^{D}$ are the elastic tensors of the body and of the inclusion, respectively, and $\chi_{D}$ is the characteristic function of $D$. We denote by $\Lambda_{D}: H^{\frac{1}{2}} \rightarrow$ $H^{-\frac{1}{2}}$ the Dirichlet-to-Neumann map associated to the problem (46)-(47), that is the operator which maps the Dirichlet data $\left.u\right|_{\partial \Omega}$ onto the corresponding Neumann data $\left.(\mathbf{C} \nabla u) \nu\right|_{\partial \Omega}$, where $\nu$ is the outer unit normal to $\Omega$. The inverse problem is to determine $D$ when $\Lambda_{D}$ is given. In the recent paper [5] it is shown the modulus of continuity of the continuous dependance of the inclusion $D$ from the map $\Lambda_{D}$ under mild a priori assumptions on the regularity and the topology. In this section we review the main steps of the proof that is inspired by the argument shown in Section 2. Let us mention here that one of the main difference between the scalar conductivity equation and the vector Lamé is the study of the asymptotic of the fundamental solution. In fact in the scalar case it was possible to prove that $\left(\Gamma^{D_{1}}-\Gamma^{D_{2}}\right)(y, y)$ blows up as $y=w$ tends nontangentially to $P \in \partial D_{1} \backslash \overline{D_{2}}$, and to evaluate quantitatively the blowup rate. In the present case the situation is more complicated for a number of reasons. First of all the fundamental solutions of the elastic operator are matrix valued (not scalar) functions and, therefore, it is crucial to understand which of the entries of $\Gamma^{D_{1}}-\Gamma^{D_{2}}$ has the desired blowup behavior. Second, we are assuming that either $\mu^{D} \neq \mu$ or $\lambda^{D} \neq \lambda$ with no order condition between such parameters. Hence, we cannot expect, in general, that the difference matrix $\Gamma^{D_{1}}-\Gamma^{D_{2}}$ may satisfy any positivity condition. For these reasons we have chosen to examine each diagonal entry of $\Gamma^{D_{1}}-\Gamma^{D_{2}}$ separately. Similarly to the scalar case, we can show that, as $y, w$ tend to $P \in \partial D_{1} \backslash \overline{D_{2}},\left(\Gamma^{D_{1}}-\Gamma^{D_{2}}\right)(y, w)$ has, in a suitable reference frame, the same asymptotic behavior of $\left(\Gamma^{+}-\Gamma\right)(y, w)$. Here $\Gamma$ is the standard Kelvin fundamental solution with Lamé moduli $\mu, \lambda$ and $\Gamma^{+}$is the fundamental solution $\Gamma^{D}$ when $D$ is replaced by the upper half plane $\left\{x_{3}>0\right\}$.

We can take advantage of the fact that $\Gamma^{+}$is explicitly known, in fact its expression, although complicated, was calculated by Rongved [26] in 1955. With the aid of Rongved's formulas it is possible to estimate the blowup rate of $\left(\Gamma^{+}-\Gamma\right)_{i i}(y, w), i=1,2,3$, as $y, w \rightarrow 0$ vertically along the line $\left\{x_{1}=x_{2}=0\right\}$ for suitable choices of $y, w$. The peculiar fact is that we are obliged to pick up very specific choices of $y, w$, with $w \neq y$. In fact we have found explicit examples of moduli $(\lambda, \mu) \neq\left(\lambda^{D}, \mu^{D}\right)$ for which $\left(\Gamma^{+}-\Gamma\right)_{i i}(y, y)=0$.

Let us consider a elastic body $\Omega \subset \mathbf{R}^{3}$ and an inclusion $D$ satisfying the assumptions of the previous sections. Moreover we assume the following conditions.

## Assumptions on the domain

The body $\Omega$ is assumed to be made of linearly elastic, isotropic and homoge-
neous material, with elastic tensor $\mathbf{C}$ of components

$$
\begin{equation*}
C_{i j k l}=\lambda \delta_{i j} \delta_{k l}+\mu\left(\delta_{k i} \delta_{l j}+\delta_{l i} \delta_{k j}\right), \tag{48}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker's delta. The constant Lamé moduli $\lambda, \mu$ satisfy the strong convexity conditions

$$
\begin{equation*}
\mu \geq \alpha_{0}, \quad 2 \mu+3 \lambda \geq \gamma_{0} \tag{49}
\end{equation*}
$$

where $\alpha_{0}>0, \gamma_{0}>0$ are given constants. We shall also assume upper bounds on the Lamé moduli

$$
\begin{equation*}
\mu \leq \bar{\mu}, \quad \lambda \leq \bar{\lambda} \tag{50}
\end{equation*}
$$

where also $\bar{\mu}>0, \bar{\lambda} \in \mathbf{R}$ are known quantities. In some points of our analysis, we will express the constitutive equation (48) in terms of $\mu$ and of Poisson's ratio $\nu$, instead of the Lamé moduli $\mu, \lambda$. Recalling that

$$
\begin{equation*}
\nu=\frac{\lambda}{2(\lambda+\mu)}, \tag{51}
\end{equation*}
$$

by (49), (50) we have

$$
\begin{equation*}
-1<\nu_{0} \leq \nu \leq \nu_{1}<\frac{1}{2} \tag{52}
\end{equation*}
$$

where $\nu_{0}, \nu_{1}$ only depend on $\alpha_{0}, \gamma_{0}, \bar{\mu}, \bar{\lambda}$. Let us notice that (48) trivially implies that

$$
\begin{equation*}
C_{i j k l}=C_{k l i j}=C_{l k i j}, \quad i, j, k, l=1,2,3 . \tag{53}
\end{equation*}
$$

We recall that the first equality in (53) is usually named as the major symmetry of the tensor $\mathbf{C}$, whereas the second equality is called the minor symmetry.

Also we note that (49) is equivalent to

$$
\begin{equation*}
\mathbf{C} A \cdot A \geq \xi_{0}|A|^{2} \tag{54}
\end{equation*}
$$

for every $3 \times 3$ symmetric matrix $A$, where $\xi_{0}=\min \left\{2 \alpha_{0}, \gamma_{0}\right\}$.

## Assumptions on the inclusion

The inclusion $D$ is made of isotropic homogeneous material having elasticity tensor $\mathbf{C}^{D}$, with constant Lamé moduli $\lambda^{D}, \mu^{D}$ satisfying the conditions (49), (50) and such that

$$
\begin{equation*}
\left(\lambda-\lambda^{D}\right)^{2}+\left(\mu-\mu^{D}\right)^{2} \geq \eta_{0}^{2}>0 \tag{55}
\end{equation*}
$$

for a given constant $\eta_{0}>0$.
In what follows we shall refer to the constants $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\mu}, \bar{\lambda}, \eta_{0}$ as to the $a$-priori data.

Observe that, in view of (51) and of the a-priori bounds on the Lamé moduli, from (55) it also follows

$$
\begin{equation*}
\left(\nu-\nu^{D}\right)^{2}+\left(\mu-\mu^{D}\right)^{2} \geq C \eta_{0}^{2}>0 \tag{56}
\end{equation*}
$$

where $C$ only depends on $\alpha_{0}, \gamma_{0}, \bar{\mu}, \bar{\lambda}$.
Finally, note that the jump condition (55) does not imply any kind of monotonicity relation between $\mathbf{C}$ and $\mathbf{C}^{D}$.

Before state the stability theorem, we remind that the Dirichlet-to-Neumann map associated to problem (46)-(47) is defined similarly as in Remark 2.3. The stability theorem reads as follows.

Theorem 3.1. Let $\Omega \subset \mathbf{R}^{3}$ and let $D_{1}, D_{2}$ be as above Let $\mathbf{C}$ and $\mathbf{C}^{D}$ be the constant elastic tensors of the material of $\Omega$ and of the inclusions $D_{i}$, $i=1,2$, respectively, where $\mathbf{C}$ and $\mathbf{C}^{D}$ satisfy (48)-(50) and (55). If, for some $\varepsilon, 0<\varepsilon<1$,

$$
\begin{equation*}
\left\|\Lambda_{D_{1}}-\Lambda_{D_{2}}\right\|_{\mathcal{L}\left(H^{1 / 2}(\partial \Omega), H^{-1 / 2}(\partial \Omega)\right)} \leq \frac{\varepsilon}{r_{0}} \tag{57}
\end{equation*}
$$

then

$$
\begin{equation*}
d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq r_{0} \omega(\varepsilon) \tag{58}
\end{equation*}
$$

where $\omega$ is an increasing function on $[0,+\infty)$ satisfying

$$
\begin{equation*}
\omega(t) \leq C|\log t|^{-\eta}, \text { for every } 0<t<1 \tag{59}
\end{equation*}
$$

where $C>0$ and $\eta, 0<\eta \leq 1$, are constants only depending on the a-priori data.

We will go through the proof of the theorem dividing it in to the same steps of the conductivity problems and underlying the main differences.

## Step 1: modified distance.

This part does not change with respect to the impedance tomography case.

## Step 2: Alessandrini's identity.

Also in this framework, using Green's formula and the symmetry properties of $\mathbf{C}, \mathbf{C}^{D}$, it is not difficult to get

$$
\begin{align*}
\int_{\Omega}\left(\mathbf{C}+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D_{1}}\right) \nabla u_{1} \cdot \nabla u_{2}-\int_{\Omega}(\mathbf{C} & \left.+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D_{2}}\right) \nabla u_{1} \cdot \nabla u_{2}= \\
& =\int_{\partial \Omega} u_{1} \cdot\left(\Lambda_{D_{1}}-\Lambda_{D_{2}}\right) u_{2} \tag{60}
\end{align*}
$$

Arguing similarly as in the previous case, we want to use (60) replacing solutions $u_{1}, u_{2}$ with fundamental solutions with singularities outside $\Omega$. For this purpose
let us define them precisely. Given $y \in \mathbf{R}^{3}$ and a concentrated force $l \in \mathbf{R}^{3}$ applied at $y,|l|=1$, let us consider the normalized fundamental solution $u^{D} \in$ $L_{l o c}^{1}\left(\mathbf{R}^{3}, \mathbf{R}^{3}\right)$ defined by

$$
\left\{\begin{array}{l}
\operatorname{div}_{x}\left(\left(\mathbf{C}+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D}\right) \nabla_{x} u^{D}(x, y ; l)\right)=-l \delta(x-y), \quad \text { in } \mathbf{R}^{3} \backslash\{y\}  \tag{61}\\
\lim _{|x| \rightarrow \infty} u^{D}(x, y ; l)=0
\end{array}\right.
$$

where $\delta(\cdot-y)$ is the Dirac distribution supported at $y$, that is

$$
\begin{align*}
& \int_{\mathbf{R}^{3}}\left(\mathbf{C}+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D}\right) \nabla_{x} u^{D}(x, y ; l) \cdot \nabla_{x} \varphi(x)=l \cdot \varphi(y) \\
& \text { for every } \varphi \in C_{c}^{\infty}\left(\mathbf{R}^{3}, \mathbf{R}^{3}\right) . \tag{62}
\end{align*}
$$

It is well-known that

$$
\begin{equation*}
u^{D}(x, y ; l)=\Gamma^{D}(x, y) l \tag{63}
\end{equation*}
$$

where $\Gamma^{D}=\Gamma^{D}(\cdot, y) \in L_{l o c}^{1}\left(\mathbf{R}^{3}, \mathcal{L}\left(\mathbf{R}^{3}, \mathbf{R}^{3}\right)\right)$ is the normalized fundamental matrix for the operator $\operatorname{div}_{x}\left(\left(\mathbf{C}+\left(\mathbf{C}^{D}-\mathbf{C}\right) \chi_{D}\right) \nabla_{x}(\cdot)\right)$. The existence of $\Gamma^{D}$ is ensured by the following Proposition.

Proposition 3.2. Under the above assumptions, there exists a unique fundamental matrix $\Gamma^{D}(\cdot, y) \in C^{0}\left(\mathbf{R}^{3} \backslash\{y\}\right)$. Moreover, we have

$$
\begin{gather*}
\Gamma^{D}(x, y)=\left(\Gamma^{D}(y, x)\right)^{T}, \quad \text { for every } x \in \mathbf{R}^{3}, x \neq y  \tag{64}\\
\left|\Gamma^{D}(x, y)\right| \leq C|x-y|^{-1}, \quad \text { for every } x \in \mathbf{R}^{3}, x \neq y  \tag{65}\\
\left|\nabla_{x} \Gamma^{D}(x, y)\right| \leq C|x-y|^{-2}, \quad \text { for every } x \in \mathbf{R}^{3}, x \neq y \tag{66}
\end{gather*}
$$

where the constant $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$.
Proof. Using a result contained in [23] combined with the results presented in [20] it is possible to get the thesis. See [5, Proposition 5.1] for details.

Let us choose $y, w \in \mathbf{R}^{3}, y \neq w$, and $l, m \in \mathbf{R}^{3}$ such that $|l|=|m|=1$. We define the functions

$$
\begin{gather*}
S_{D_{1}}(y, w ; l, m)=\int_{D_{1}}\left(\mathbf{C}^{D}-\mathbf{C}\right) \nabla_{x}\left(\Gamma^{D_{1}}(x, y) l\right) \cdot \nabla_{x}\left(\Gamma^{D_{2}}(x, w) m\right)  \tag{67}\\
S_{D_{2}}(y, w ; l, m)=\int_{D_{2}}\left(\mathbf{C}^{D}-\mathbf{C}\right) \nabla_{x}\left(\Gamma^{D_{1}}(x, y) l\right) \cdot \nabla_{x}\left(\Gamma^{D_{2}}(x, w) m\right)  \tag{68}\\
f(y, w ; l, m)=S_{D_{1}}(y, w ; l, m)-S_{D_{2}}(y, w ; l, m) \tag{69}
\end{gather*}
$$

Again the leading argument to get stability is to evaluate the function $f$ as we move the singularities $y, w$ quantifying the propagation of the boundary
information we have from the measurements. A key ingredient in this analysis is the behavior of fundamental solutions.

## Step 3: fundamental solutions.

Let $O \in \partial D$ and $\nu=\nu(O)$ the outer unit normal to $D$ at $O$. Let us choose a coordinate system with origin $O$ and axis $e_{3}=-\nu$, and let $\Gamma^{+}(x, y)=\Gamma^{\mathbf{R}_{+}^{3}}(x, y)$ the normalized fundamental matrix associated to $D=\mathbf{R}_{+}^{3}$. We recall that its explicit expression was found by Rongved [26].

Recalling the notation $u^{D}(x, y)=\Gamma^{D}(x, y) l$ (see (63)) and defining similarly $u^{+}(x, y)=\Gamma^{+}(x, y) l$, for any $l \in \mathbf{R}^{3},|l|=1$, the asymptotic approximation of $u^{D}$ in terms of $u^{+}$reads as follows.

Theorem 3.3. Let $y=(0,0,-h), 0<h<\frac{r_{0} M_{0}}{8 \sqrt{1+M_{0}^{2}}}$. Under the above assumptions and notation, we have

$$
\begin{align*}
& \left|u^{D}(x, y)-u^{+}(x, y)\right| \leq \frac{C}{r_{0}}\left(\frac{|x-y|}{r_{0}}\right)^{-1+\alpha}, \\
& \text { for every } x \in Q \frac{r_{0}}{8 \sqrt{1+M_{0}^{2}}}, \frac{r_{0} M_{0}}{8 \sqrt{1+M_{0}^{2}}} \cap D,  \tag{70}\\
& \left|\nabla_{x} u^{D}(x, y)-\nabla_{x} u^{+}(x, y)\right| \leq \frac{C}{r_{0}^{2}}\left(\frac{|x-y|}{r_{0}}\right)^{-2+\frac{\alpha^{2}}{3 \alpha+2}}, \\
& \text { for every } x \in Q_{\frac{r_{0}}{+}}^{12 \sqrt{1+M_{0}^{2}}}, \frac{r_{0} M_{0}}{12 \sqrt{1+M_{0}^{2}}} \cap D, \tag{71}
\end{align*}
$$

where $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$.
Proof. The thesis can be obtained defining the function

$$
R(x, y)=u^{d}(x, y)-u^{+}(x, y)
$$

and flattening the boundary $\partial D$. See [5, Theorem 8.1] for details.

## Step 4: quantitative estimates.

As in the impedance tomography case, in this step we show how the boundary information and the asymptotic behavior of the fundamental solution can be used to estimate the auxiliary function $f$.
Theorem 3.4 (Upper bound on the function $f$ ). Under the notation of Lemma 2.7, let

$$
\begin{align*}
& y_{h}=P-h e_{3}  \tag{72}\\
& w_{h}=P-\lambda_{w} h e_{3}, \quad 0<\lambda_{w}<1 \tag{73}
\end{align*}
$$

with

$$
\begin{equation*}
0<h \leq \bar{d}\left(1-\frac{\sin \widetilde{\vartheta}_{0}}{4}\right) \tag{74}
\end{equation*}
$$

where $\widetilde{\vartheta}_{0}=\arctan \frac{1}{L_{0}}$ and $\nu=-e_{3}$ is the outer unit normal to $D_{1}$ at $P$. Then, for every $l, m \in \mathbf{R}^{3},|l|=|m|=1$, we have

$$
\begin{equation*}
\left|f\left(y_{h}, w_{h} ; l, m\right)\right| \leq \frac{C}{\lambda_{w} h} \epsilon^{C_{1}\left(\frac{h}{\rho_{0}}\right)^{C_{2}}} \tag{75}
\end{equation*}
$$

where the constant $C>0$ only depends on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$;

$$
\begin{equation*}
C_{1}=\gamma \delta^{2+2 \frac{|\log A|}{|\log \chi|}}, C_{2}=2 \frac{|\log \delta|}{|\log \chi|}, A=\frac{\lambda_{w}}{\frac{\bar{d}}{\rho_{0}}\left(1-\vartheta^{*} \frac{\sin \tilde{\vartheta}_{0}}{8}\right)}, \chi=\frac{1-\frac{\sin \tilde{\vartheta}_{0}}{8}}{1+\frac{\sin \widetilde{\vartheta}_{0}}{8}} \tag{76}
\end{equation*}
$$

where $\delta, 0<\delta<1, \vartheta^{*}, 0<\vartheta^{*} \leqq 1$, only depend on $\alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu} ; \gamma>0$ only depends on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$.
Proof. Similarly to the impedance tomography case, the proof is based on the use of the three spheres inequality for solution to the Lamé system. We refer to [5, Theorem 6.4] for details.

Theorem 3.5 (Lower bound on the function $f$ ). Under the notation of Lemma 2.7, let

$$
\begin{equation*}
y_{h}=P-h e_{3} . \tag{77}
\end{equation*}
$$

For every $i=1,2,3$, there exists $\lambda_{w} \in\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ and there exists $\bar{h} \in\left(0, \frac{1}{2}\right)$ only depending on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \eta_{0}$, such that

$$
\begin{equation*}
\left|f\left(y_{h}, w_{h} ; e_{i}, e_{i}\right)\right| \geq \frac{C}{h}, \quad \text { for every } h, 0<h<\bar{h} \rho \tag{78}
\end{equation*}
$$

where

$$
\begin{gather*}
w_{h}=P-\lambda_{w} h e_{3}  \tag{79}\\
\rho=\min \left\{\operatorname{dist}\left(P, D_{2}\right), \frac{r_{0}}{12 \sqrt{1+M_{0}^{2}}} \cdot \min \left\{1, M_{0}\right\}\right\}, \tag{80}
\end{gather*}
$$

and $C>0$ only depends on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \eta_{0}$.
Proof. To obtain such a bound we refer to Theorem 6.5 of [5]. Let us only mention that besides the use of the asymptotic os $\Gamma_{D}$ (Theorem 3.3) other ingredients are needed. In particular we point out the identity

$$
\begin{array}{r}
\int_{\mathbf{R}_{+}^{3}}\left(\mathbf{C}^{D}-\mathbf{C}\right) \nabla_{x}\left(\Gamma^{+}\left(x, y_{0}\right) l\right) \cdot \nabla_{x}\left(\Gamma\left(x, w_{0}\right) m\right)=\left(\Gamma\left(y_{0}, w_{0}\right)-\Gamma^{+}\left(y_{0}, w_{0}\right)\right) m \cdot l \\
\text { for every } y_{0}, w_{0} \in \mathbf{R}^{3}, y_{0} \neq w_{0}
\end{array}
$$

(See [5, Lemma 9.2]) that is a special case of [10, Prposition 3.2] and the bound

$$
\left|\left(\Gamma^{+}\left(y_{0}, w_{0}\right)-\Gamma\left(y_{0}, w_{0}\right)\right) e_{i} \cdot e_{i}\right| \geq \mathcal{C}
$$

where $y_{0}=(0,0,-1)$, $w_{0}=\left(0,0,-\lambda_{w}\right)$, with $\lambda_{w} \in\left\{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}\right\}$ for $i=1,2,3$ (see [5, Proposition 9.3]).

## Step 5: proof of Theorem 3.1.

From the combination of the upper bound (75), with $l=m=e_{i}$ for $i \in\{1,2,3\}$, and from the lower bound (78), we have

$$
\begin{equation*}
C \leq \varepsilon^{C_{1}\left(\frac{h}{r_{0}}\right)^{C_{2}}}, \quad \text { for every } h, 0<h \leq \bar{h} \rho \tag{81}
\end{equation*}
$$

where $\rho$ is given in (80), the constants $C_{1}>0, C_{2}>0$ are defined in (76) and depend only on $M_{0}, \alpha, M_{1}, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}$, and the constants $C \in(0,1)$, $\bar{h} \in\left(0, \frac{1}{2}\right)$ only depend on $M_{0}, \alpha, \alpha_{0}, \gamma_{0}, \bar{\lambda}, \bar{\mu}, \eta_{0}$.

Passing to the logarithm and recalling that $\varepsilon \in(0,1)$, we have

$$
\begin{equation*}
h \leq C r_{0}\left(\frac{1}{|\log \varepsilon|}\right)^{\frac{1}{C_{2}}}, \quad \text { for every } h, 0<h \leq \bar{h} \rho \tag{82}
\end{equation*}
$$

In particular, choosing $h=\bar{h} \rho$, we have

$$
\begin{equation*}
\rho \leq C r_{0}\left(\frac{1}{|\log \varepsilon|}\right)^{\frac{1}{C_{2}}} \tag{83}
\end{equation*}
$$

If $\rho=\operatorname{dist}\left(P, D_{2}\right)$, by Lemmas 2.6 and 2.7 , the thesis follows. If, otherwise, $\rho=\frac{r_{0}}{12 \sqrt{1+M_{0}^{2}}} \min \left\{1, M_{0}\right\}$, the thesis follows by noticing that $d_{H}\left(\partial D_{1}, \partial D_{2}\right) \leq$ $\operatorname{diam}(\Omega) \leq C r_{0}$, with $C>0$ only depending on $M_{0}, M_{1}$.

## 4. Thermal Conductors

In this section we go through the problem of determining an inclusion, whose shape can vary with the time, within a thermal conductor. Let $T$ be a given positive number. Let $\Omega$ be a bounded domain of $\mathbf{R}^{n}$ with a sufficiently smooth boundary and let $Q$ be a domain contained in $\Omega \times(0, T)$. Assume that for every $\tau \in(0, T)$ the intersection of $Q$ with the hyperplane $t=\tau$ is a nonempty set and denote by $k, k \neq 1$ a positive constant. Let $u$ be the weak solution to the following parabolic initial-boundary value problem

$$
\begin{cases}\partial_{t} u-\operatorname{div}\left(\left(1+(k-1) \chi_{Q}\right) \nabla u\right)=0 & \text { in } \Omega \times(0, T),  \tag{84}\\ u(\cdot, 0)=0 & \text { in } \bar{\Omega} \\ u=g & \text { on } \partial \Omega \times(0, T)\end{cases}
$$

where $g$ is a prescribed function on $\partial \Omega \times(0, T)$. The inverse problem we are addressing to is to determine the region $Q$ when infinitely many boundary measurements $\left\{g, \frac{\partial u}{\partial \nu}{ }_{\partial \Omega \times(0, T)}\right\}$ are available. A uniqueness result was proved in 1997 by Elayyan and Isakov [18]. We want to discuss the stability issue proved in [15]. We will show that also in this case the stability estimates are of logarithmic type. The argument to get such a rate of continuity follows the line of the impedance tomography case, using singular solutions of Green's type. Let us emphasize here that one of the main difference with respect to the previous cases is the lack of an explicit representation of the fundamental solution when the interface is flat. To overcome this difficulty we will use some formulas proved by [22] involving the Fourier transform of the fundamental solution that will lead to an estimate from below (see Proposition 4.5).

Another difficulty that characterizes the parabolic case consists in employing a precise evaluation of the smallness propagation based on the two-sphere and one-cylinder inequality for solution of parabolic equation [19], [27] (see Theorem 4.7 below).

Let us first premise a definition.
Definition 4.1. Let $Q$ be a domain in $\mathbf{R}^{n+1}$. We shall say that $Q$ (or equivalently $\partial Q$ ) is of class $\mathcal{K}$ with constants $r_{0}, M_{0}$ if for all $X_{0} \in \partial Q$ there exists a rigid transformation of space coordinates under which we have $X_{0}=(0,0)$ such that

$$
Q \cap\left(B_{r_{0}}(0) \times\left(-r_{0}^{2}, r_{0}^{2}\right)\right)=\left\{X \in B_{r_{0}}(0) \times\left(-r_{0}^{2}, r_{0}^{2}\right): x_{n}>\varphi\left(x^{\prime}, t\right)\right\}
$$

where $\varphi$ is endowed with second derivatives with respect to $x_{i}, i=1, \cdots, n$, with the $t$-derivative and with second derivatives with respect to $x_{i}$ and $t$ and it satisfies the following conditions $\varphi(0,0)=\left|\nabla_{x^{\prime}} \varphi(0,0)\right|=0$ and

$$
\begin{aligned}
& r_{0}^{2}\left\|D_{x^{\prime}}^{2} \varphi\right\|_{L^{\infty}\left(B_{r_{0}}^{\prime} \times\left(-r_{0}^{2}, r_{0}^{2}\right)\right)}+r_{0}^{2}\left\|\partial_{t} \varphi\right\|_{L^{\infty}\left(B_{r_{0}}^{\prime} \times\left(-r_{0}^{2}, r_{0}^{2}\right)\right)} \\
&+r_{0}^{3}\left\|\nabla_{x^{\prime}} \partial_{t} \varphi\right\|_{L^{\infty}\left(B_{r_{0}}^{\prime} \times\left(-r_{0}^{2}, r_{0}^{2}\right)\right)} \leq M_{0} r_{0} .
\end{aligned}
$$

## Assumptions on the domain

Let $r_{0}, M_{0}, M_{1}$ be given positive numbers. We assume that $\Omega$ is a bounded domain in $\mathbf{R}^{n}$ satisfying

$$
\begin{equation*}
|\Omega| \leq M_{1} r_{0}^{n} \tag{85a}
\end{equation*}
$$

where $|\Omega|$ denotes the Lebesgue measure of $\Omega$. We also assume that

$$
\begin{equation*}
\partial \Omega \text { is of class } \mathcal{C}^{1,1} \text { with constants } r_{0}, M_{0} \tag{85b}
\end{equation*}
$$

## Assumptions on the inclusion

Denoting by $Q=\bigcup_{t \in \mathbf{R}} D(t) \times\{t\}$, we assume the following conditions

$$
\begin{equation*}
\partial Q \text { is of class } \mathcal{K} \text { with constans } r_{0}, M_{0} \tag{86a}
\end{equation*}
$$

$$
\begin{align*}
& \operatorname{dist}(D(t), \partial \Omega) \geq r_{0} \quad \forall t \in[0, T]  \tag{86b}\\
& \Omega \backslash \overline{D(t)} \text { is connected } \forall t \in[0, T] \tag{86c}
\end{align*}
$$

Before stating the stability result, let us define the Dirichlet-to-Neumann map in this framework. We denote by $H=H_{, 0}^{3 / 2,3 / 4}(\partial \Omega \times(0, T))$, its dual $H^{\prime}=$ $H^{-3 / 2,-3 / 4}(\partial \Omega \times(0, T))$, and

$$
W(\Omega \times(0, T))=\left\{v \in L^{2}\left((0, T), H^{1}(\Omega)\right): \partial_{t} v \in L^{2}\left((0, T), H^{-1}(\Omega)\right)\right\}
$$

For any $g \in H$, let $u \in W(\Omega \times(0, T))$ be the weak solution of the initialboundary value problem

$$
\begin{cases}\partial_{t} u-\operatorname{div}\left(\left(1+(k-1) \chi_{Q}\right) \nabla u\right)=0, & \text { in } \Omega \times(0, T)  \tag{87}\\ u(x, 0)=0, & x \in \bar{\Omega}, \\ u(x, t)=g(x, t), & \text { on } \partial \Omega \times(0, T)\end{cases}
$$

where $\chi_{Q}$ is the characteristic function of the set $Q$. Then for any $g \in H$, we set

$$
\Lambda_{Q} g=\left.\frac{\partial u}{\partial \nu}\right|_{\partial \Omega \times(0, T)}, \quad u \text { solution to (87) }
$$

We can also consider $\Lambda_{Q}$ as a linear and bounded operator between $H$ and $H^{\prime}$, by setting

$$
\left\langle\Lambda_{Q} g, \phi\right\rangle_{H^{\prime}, H}=\left\langle\left.\frac{\partial u}{\partial \nu}\right|_{\partial \Omega \times(0, T)}, \phi\right\rangle_{H^{\prime}, H}=\int_{\partial \Omega \times(0, T)} \frac{\partial u}{\partial \nu} \phi, \quad \text { for any } g, \phi \in H
$$

where $u$ solves (87) and $\langle\cdot, \cdot\rangle_{H^{\prime}, H}$ is the duality pairing between $H^{\prime}$ and $H$.
Theorem 4.2. Let $\Omega \subset \mathbf{R}^{n}$ satisfying (85). Let $k>0, k \neq 1$ be given. Let $\left\{D_{1}(t)\right\}_{t \in \mathbf{R}},\left\{D_{2}(t)\right\}_{t \in \mathbf{R}}$ be two families of domains satisfying (86). Assume that $D_{1}(0)=D_{2}(0)$ and, for $\varepsilon>0$,

$$
\begin{equation*}
\left\|\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right\|_{\mathcal{L}\left(H, H^{\prime}\right)} \leq \varepsilon \tag{88}
\end{equation*}
$$

where $Q_{i}=D_{i}((-\infty,+\infty)), i=1,2$. Then

$$
\begin{equation*}
d_{\mathcal{H}}\left(D_{1}(t), D_{2}(t)\right) \leq \omega_{t}(\varepsilon), \quad t \in(0, T] \tag{89}
\end{equation*}
$$

where $\omega_{t}(s)$ is such that

$$
\begin{equation*}
\omega_{t}(s) \leq C|\log s|^{-\eta}, \quad 0<s<1, \tag{90}
\end{equation*}
$$

with $C=C(t)>0$ and $0<\eta \leq 1$ depend on the a priori data only.

Remark 4.3. Let us observe that for the case of more general thermal conductivities with local Dirichlet-to-Neumann map has been studied in [16].

## Step 1: modified distance.

This part can be obtained through minor modifications form the impedance tomography case (see [15, Proposition 3.2, 3.3] for further details).

## Step 2: Alessandrini's identity.

For the sake of brevity we name $a_{j}=1+(k-1) \chi_{Q_{j}}, j=1,2$. We fix $g \in H$. We shall denote by $u_{j}, j=1,2$ the solution of (84) when $Q=Q_{j}$. For $\psi \in H^{1,1}(\Omega \times(0, T))$ such that

$$
\begin{equation*}
\psi(\cdot, T)=0 \quad \text { in } \Omega \tag{91}
\end{equation*}
$$

using the weak formulation of (84) we have

$$
\begin{aligned}
\int_{\partial \Omega \times(0, T)} a_{j} \frac{\partial u_{j}}{\partial \nu} & \psi d S+\int_{\Omega} u_{j}(x, 0) \psi(x, 0) d x \\
& -\int_{\Omega \times(0, T)}\left(a_{j} \nabla u_{j} \cdot \nabla \psi-u_{j} \partial_{t} \psi\right) d x d t=0 \quad \text { for } j=1,2 .
\end{aligned}
$$

Subtracting the two equations we obtain

$$
\begin{align*}
& \int_{\Omega \times(0, T)}\left(a_{1} \nabla\left(u_{1}-u_{2}\right) \cdot \nabla \psi-\left(u_{1}-u_{2}\right) \partial_{t} \psi\right) d x d t \\
&+\int_{\Omega \times(0, T)}\left(a_{1}-a_{2}\right) \nabla u_{2} \cdot \nabla \psi=<\left(\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right) g, \psi>_{H^{\prime}, H} \tag{92}
\end{align*}
$$

(we notice here that in these identities it is possible to have $u_{i}(\cdot, 0) \neq 0$ for $i=1,2$ ). Taking $\psi$ such that it satisfies (91) and

$$
\begin{equation*}
\partial_{t} \psi+\operatorname{div}\left(a_{1} \nabla \psi\right)=0 \quad \text { in } \Omega \times(0, T), \tag{93}
\end{equation*}
$$

by (92) we have (recalling that on $\partial \Omega \times(0, T) u_{1}=u_{2}=g$ )

$$
\int_{\Omega \times(0, T)}\left(a_{1}-a_{2}\right) \nabla u_{2} \cdot \nabla \psi=<\left(\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right) g, \psi>_{H^{\prime}, H}, \quad \forall g \in H
$$

or, equivalently,

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left(\chi_{Q_{1}}-\chi_{Q_{2}}\right) \nabla u_{2} \cdot \nabla \psi d x d t=\frac{1}{k-1}<\left(\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right) u_{2}, \psi>_{H^{\prime}, H} \tag{94}
\end{equation*}
$$

Let us denote by $\Gamma_{2}(x, t ; y, s)$ and $\Gamma_{1}^{*}(x, t ; y, s)$ the fundamental solutions of the operator $\partial_{t}-\operatorname{div}\left(\left(1+(k-1) \chi_{Q_{2}}\right) \nabla\right)$ and $\partial_{t}+\operatorname{div}\left(\left(1+(k-1) \chi_{Q_{1}}\right) \nabla\right)$ respectively, that is

$$
\begin{aligned}
& \partial_{t} \Gamma_{2}(x, t ; y, s)-\operatorname{div}\left(\left(1+(k-1) \chi_{Q_{2}}\right) \nabla_{x} \Gamma_{2}(x, t ; y, s)\right)=-\delta(x-y, t-s) \\
& \partial_{t} \Gamma_{1}^{*}(x, t ; y, s)+\operatorname{div}\left(\left(1+(k-1) \chi_{Q_{1}}\right) \nabla_{x} \Gamma_{1}^{*}(x, t ; y, s)\right)=-\delta(x-y, t-s)
\end{aligned}
$$

where $\delta$ denotes the Dirac distribution. Choosing in (94) $u_{2}(x, t)=\Gamma_{2}(x, t ; y, s)$ and $\psi(x, t)=\Gamma_{1}^{*}(x, t ; \xi, \tau)$, with $(y, s)$ and $(\xi, \tau) \notin \Omega \times(0, T), 0 \leq s<\tau \leq T$, we obtain

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega}\left(\chi_{Q_{1}}-\right. & \left.\chi_{Q_{2}}\right) \nabla_{x} \Gamma_{2}(x, t ; y, s) \cdot \nabla_{x} \Gamma_{1}^{*}(x, t ; \xi, \tau) d x d t \\
& =\frac{1}{k-1}<\left(\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right) \Gamma_{2}(\cdot, \cdot ; y, s), \Gamma_{1}^{*}(\cdot, \cdot ; \xi, \tau)>_{H^{\prime}, H} \tag{95}
\end{align*}
$$

$\underline{\text { For } t} \in[0, T]$ we shall define $\mathcal{G}(t)$ as the connected component of $\Omega \backslash\left(\overline{D_{\tilde{1}}(t)} \cup\right.$ $\left.\overline{D_{2}(t)}\right)$ that contains $\partial \Omega, \tilde{\mathcal{G}}(t)=\left(\mathbf{R}^{n} \backslash \Omega\right) \cup \mathcal{G}(t)$ and $\tilde{\mathcal{G}}((0, T)):=\bigcup_{t \in(0, T)} \tilde{\mathcal{G}}(t) \times$ $\{t\}$. For $(y, s),(\xi, \tau) \in \tilde{\mathcal{G}}((0, T))$ with $0 \leq s<\tau \leq T$, we set

$$
\begin{aligned}
& S_{1}(y, s ; \xi, \tau)=\int_{Q_{1}} \nabla_{x} \Gamma_{2}(x, t ; y, s) \cdot \nabla_{x} \Gamma_{1}^{*}(x, t ; \xi, \tau) d x d t \\
& S_{2}(y, s ; \xi, \tau)=\int_{Q_{2}} \nabla_{x} \Gamma_{2}(x, t ; y, s) \cdot \nabla_{x} \Gamma_{1}^{*}(x, t ; \xi, \tau) d x d t \\
& \mathcal{U}(y, s ; \xi, \tau):=S_{1}(y, s ; \xi, \tau)-S_{2}(y, s ; \xi, \tau)
\end{aligned}
$$

By (95) we have

$$
\begin{equation*}
\mathcal{U}(y, s ; \xi, \tau)=\frac{1}{k-1}<\left(\Lambda_{Q_{1}}-\Lambda_{Q_{2}}\right) \Gamma_{2}(\cdot, \cdot ; y, s), \Gamma_{1}^{*}(\cdot, \cdot ; \xi, \tau)>_{H^{\prime}, H} \tag{96}
\end{equation*}
$$

for all $y, \xi \notin \Omega, 0 \leq s<\tau \leq T$.

## Step 3: fundamental solutions.

We denote by $\Gamma_{0}(x-y, t-s)$ the standard fundamental solution of $\partial_{t}-\Delta$ which is

$$
\Gamma_{0}(x-y, t-s)=\frac{1}{[4 \pi(t-s)]^{n / 2}} \mathrm{e}^{-\frac{|x-y|^{2}}{4(t-s)}}, \quad t>s
$$

and by denote by $\Gamma(x, t ; y, s)$ the fundamental solution of the operator $\partial_{t}-$ $\operatorname{div}\left(\left(1+(k-1) \chi_{Q}\right) \nabla_{x}\right)($ see $[9])$. We recall that $\Gamma$ satisfies the following properties

$$
\begin{equation*}
\Gamma(x, t ; y, s)=\Gamma(y, s ; x, t) \quad \forall(x, t),(y, s) \in Q,(x, t) \neq(y, s) \tag{97}
\end{equation*}
$$

and

$$
\begin{equation*}
0<\Gamma(x, t ; y, s) \leq \frac{C}{[4 \pi(t-s)]^{n / 2}} \mathrm{e}^{-\frac{|x-y|^{2}}{C(t-s)}} \chi_{[s,+\infty)}(t) \tag{98}
\end{equation*}
$$

where $C \geq 1$ depends on $k$ and $M_{0}$ only. Furthermore we have also the following estimate for the gradient of $\Gamma$.
Proposition 4.4. Let $\Gamma(x, t ; y, s)$ be the fundamental solution of the operator $\partial_{t}-\operatorname{div}\left(\left(1+(k-1) \chi_{Q}\right) \nabla_{x}\right)$. There exists $C \geq 1$, depending on $k$ and $E$ only such that

$$
\begin{equation*}
\left|\nabla_{x} \Gamma(x, t ; y, s)\right| \leq \frac{C}{(t-s)^{\frac{n+1}{2}}} \mathrm{e}^{-\frac{|x-y|^{2}}{C(t-s)}} \tag{99}
\end{equation*}
$$

for almost every $x, y \in \mathbf{R}^{n}$ and $t, s \in \mathbf{R}, t>s$.
Proof. See [15, Proposition 3.6].
In the sequel we need the fundamental solution of the operator $\mathcal{L}_{+}=\partial_{t}-$ $\operatorname{div}\left(\left(1+(k-1) \chi_{+}\right) \nabla\right)$ where $\chi_{+}=\chi_{\left\{(x, t) \in \mathbf{R}^{n+1}: x_{n}>0\right\} \text {. We shall denote by }}$ $\Gamma_{+}$such a fundamental solution and by $\Gamma_{+}^{*}$ the fundamental solution of the adjoint operator of $\mathcal{L}_{+}$. Observe that $\Gamma_{+}(x, t ; y, s)=\Gamma_{+}(x, t-s ; y, 0)$ and $\Gamma_{+}^{*}(x, t ; y, s)=-\Gamma_{+}(x, s-t ; y, 0)$. For a given function $f\left(x^{\prime}, x_{n}\right), \mathcal{F}_{\zeta^{\prime}}\left(f\left(\cdot, x_{n}\right)\right)$ will be the Fourier transform of $f$ with respect to the variable $x^{\prime}$. Thus

$$
\mathcal{F}_{\zeta^{\prime}}\left(f\left(\cdot, x_{n}\right)\right)=\int_{\mathbf{R}^{n-1}} f\left(x^{\prime}, x_{n}\right) \mathrm{e}^{-i x^{\prime} \cdot \zeta^{\prime}} d x^{\prime}
$$

for every $\zeta^{\prime} \in \mathbf{R}^{n-1}$.
In [22] it has been proved some formulas for $\mathcal{F}_{\zeta^{\prime}}\left(\Gamma_{+}\left(., x_{n}, t ; y\right)\right)$. The technique to prove such formulas is rather classical and lengthy. For this reason we display only the ones that we need corresponding to the case in which $x_{n}>0$, $y_{n}<0$.

Case $k>1$. Denote by

$$
\begin{align*}
& E\left(\zeta^{\prime}, x_{n}, t ; \rho\right)=\exp \left[-t(k-(k-1) \rho)\left|\zeta^{\prime}\right|^{2}-\sqrt{\frac{k-1}{k}} x_{n}\left|\zeta^{\prime}\right| \sqrt{\rho}\right]  \tag{100}\\
& F\left(\zeta^{\prime}, y_{n} ; \rho\right)=\operatorname{Im}\left(A_{1}(\rho) \mathrm{e}^{i y_{n} \sqrt{k-1} \sqrt{1-\rho}\left|\xi^{\prime}\right|}\right) \tag{101}
\end{align*}
$$

where, for complex number $z=a+i b, \operatorname{Im}(z)$ denotes the imaginary part $b$ of $z$, and

$$
\begin{equation*}
A_{1}(\rho)=\frac{\sqrt{k-1}}{\pi} \frac{1}{i \sqrt{k-1} \sqrt{1-\rho}+\sqrt{k} \sqrt{\rho}} \tag{102}
\end{equation*}
$$

Then

$$
\begin{equation*}
\mathcal{F}_{\zeta^{\prime}}\left(\Gamma_{+}\left(\cdot, x_{n}, t ; y, 0\right)\right)=\int_{0}^{1}\left|\zeta^{\prime}\right| \mathrm{e}^{-i y^{\prime} \cdot \zeta^{\prime}} E\left(\zeta^{\prime}, x_{n}, t ; \rho\right) F\left(\zeta^{\prime}, y_{n} ; \rho\right) d \rho \tag{103}
\end{equation*}
$$

for every $x_{n}>0, y_{n}<0$.
Case $0<k<1$. Denote by

$$
\begin{aligned}
& G\left(\zeta^{\prime}, y_{n}, t ; \rho\right)=\exp \left[-t(1-(1-k) \rho)\left|\zeta^{\prime}\right|^{2}+\sqrt{1-k} y_{n}\left|\zeta^{\prime}\right| \sqrt{\rho}\right] \\
& H\left(\zeta^{\prime}, x_{n} ; \rho\right)=\operatorname{Im}\left(A_{2}(\rho) \mathrm{e}^{-i x_{n} \sqrt{\frac{1-k}{k}} \sqrt{1-\rho}\left|\zeta^{\prime}\right|}\right)
\end{aligned}
$$

where

$$
A_{2}(\rho)=\frac{\sqrt{1-k}}{\pi} \frac{1}{\sqrt{k} \sqrt{\rho}-i \sqrt{1-k} \sqrt{1-\rho}}
$$

Then

$$
\mathcal{F}_{\zeta^{\prime}}\left(\Gamma_{+}\left(\cdot, x_{n}, t ; y, 0\right)\right)=\int_{0}^{1}\left|\zeta^{\prime}\right| \mathrm{e}^{-i y^{\prime} \cdot \zeta^{\prime}} G\left(\zeta^{\prime}, y_{n}, t ; \rho\right) H\left(\zeta^{\prime}, x_{n} ; \rho\right) d \rho
$$

for every $x_{n}>0, y_{n}<0$.
Proposition 4.5. For every $\lambda_{0} \in(0,1]$ there exist $\lambda_{1}, \lambda_{2}, \lambda_{3} \in\left(0, \lambda_{0}\right]$ such that for every $h>0$ the following inequality holds true

$$
\begin{align*}
& I^{(h)}:=\mid \int_{0}^{\lambda_{2} h^{2}} d t \int_{\mathbf{R}_{+}^{n}} \nabla_{x} \Gamma_{+}^{*}\left(x, t ;-\lambda_{1} h e_{n}, \lambda_{2} h^{2}\right) \\
& \cdot \nabla_{x} \Gamma_{0}\left(x, t ;-\lambda_{3} h e_{n}, 0\right) d x \left\lvert\, \geq \frac{1}{C h^{n}}\right. \tag{104}
\end{align*}
$$

where $C, C \geq 1$, depends on $\lambda_{1}, \lambda_{2}, \lambda_{3}$ and $k$ only.
Proof. See [15, Proposition 3.7].

## Step 4: quantitative estimates.

For $\bar{t} \in(0, T]$ fixed, we can assume, without loosing generality, that there exists $O \in \partial D_{1}(\bar{t}) \cap \Omega_{D}(\bar{t})$ such that

$$
\begin{equation*}
d_{\mu}(\bar{t})=\operatorname{dist}\left(O, D_{2}(\bar{t})\right) \tag{105}
\end{equation*}
$$

Denote by

$$
\rho=\min \left\{d_{\mu}(\bar{t}), \rho_{0}\right\}
$$

Furthermore, denote by $\nu(O, \bar{t})$ the exterior unit normal to $\partial D_{1}(\bar{t})$ in $O$. Choosing parameters $\lambda_{1}, \lambda_{2}, \lambda_{3} \in(0,1]$ satisfying inequality (104) and $\delta \in(0,1]$, we set

$$
\begin{equation*}
t_{1}=\bar{t}-\lambda_{2} h^{2}, \quad \bar{y}=\lambda_{1} h \nu(0, \bar{t}), \quad y_{1}=\lambda_{3} h \nu(0, \bar{t}) \tag{106}
\end{equation*}
$$

where

$$
\begin{equation*}
0<h \leq \delta \min \{\rho, \sqrt{\bar{t}}\} . \tag{107}
\end{equation*}
$$

By using (86a) it is simple to check that there exists $C_{1}, C_{1} \geq 1$, depending on $M_{0}$ only such that if

$$
\begin{equation*}
0<\delta \leq \frac{\lambda_{3}}{C_{1}} \tag{108}
\end{equation*}
$$

then, for every $t \in\left[t_{1}, \vec{t}\right]$, we have

$$
\begin{align*}
& \operatorname{dist}\left(\bar{y}, D_{1}(t)\right) \geq \frac{1}{2} \min \left\{\lambda_{1}, \lambda_{2}, \lambda_{3}\right\} h  \tag{109}\\
& \operatorname{dist}\left(y_{1}, D_{1}(t)\right) \geq \frac{1}{2} \min \left\{\lambda_{1}, \lambda_{2}, \lambda_{3}\right\} h \tag{110}
\end{align*}
$$

On the other side, using the inequality [27, Proposition 4.1.6]

$$
\begin{equation*}
\left|\operatorname{dist}\left(O, D_{2}(t)\right)-\operatorname{dist}\left(O, D_{2}(\bar{t})\right)\right| \leq \frac{C_{0}}{\rho_{0}}|t-\bar{t}| \tag{111}
\end{equation*}
$$

where $C_{0}$ depends on $M_{0}$ and $M_{1}$ only, for $t \in\left[t_{1}, \bar{t}\right]$ and by using the triangle inequality we have that there exists $C_{2}, C_{2} \geq 1$, depending on $M_{0}$ and $M_{1}$ only such that if

$$
\begin{equation*}
0<\delta \leq \frac{1}{C_{2}} \tag{112}
\end{equation*}
$$

then

$$
\begin{equation*}
\operatorname{dist}\left(z, D_{2}(t)\right) \geq \frac{1}{2} \rho, \quad \text { with } z=\bar{y}, y_{1} \tag{113}
\end{equation*}
$$

Proposition 4.6. Let $\left\{D_{1}(t)\right\}_{t \in \mathbf{R}},\left\{D_{2}(t)\right\}_{t \in \mathbf{R}}$ be two families of domains satisfying (86) and let $\lambda_{1}, \lambda_{2}, \lambda_{3} \in(0,1)$ be such that the inequality (104) is satisfied. Then there exist $C, C \geq 1$, and $\tilde{C}, \tilde{C} \geq 1, C$ depending on $k$ only and $\tilde{C}$ depending on $k, M_{0}, M_{1}, \lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ only such that

$$
\begin{equation*}
\left|\mathcal{U}\left(y_{1}, t_{1} ; \bar{y}, \bar{t}\right)\right| \geq \frac{1}{C h^{n}} \tag{114}
\end{equation*}
$$

for $0<h \leq \frac{1}{\bar{C}} \min \{\rho, \sqrt{\bar{t}}\}$, where $y_{1}, t_{1}, \bar{y}, \bar{t}$, and $\rho$ are defined in (106).
Proof. See [15].
Theorem 4.7 (Two-spheres and one-cylinder inequality). Let $\lambda, \Lambda$ and $M$ positive numbers with $\lambda \in(0,1]$. Let $P$ be the parabolic operator

$$
\begin{equation*}
P=\partial_{t}-\partial_{i}\left(a^{i j} \partial_{j}\right), \tag{115}
\end{equation*}
$$

where $\left\{a^{i j}(x, t)\right\}_{i, j=1}^{n}$ is a symmetric $n \times n$ matrix. For $\xi \in \mathbf{R}^{n}$ and $(x, t),(y, s) \in$ $\mathbf{R}^{n+1}$ assume that

$$
\begin{equation*}
\lambda|\xi|^{2} \leq \sum_{i, j=1}^{n} a^{i j}(x, t) \xi_{i} \xi_{j} \leq \lambda^{-1}|\xi|^{2} \tag{116a}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\sum_{i, j=1}^{n}\left(a^{i j}(x, t)-a^{i j}(y, s)\right)^{2}\right)^{1 / 2} \leq \frac{\Lambda}{R}\left(|x-y|^{2}+|t-s|\right)^{1 / 2} \tag{116b}
\end{equation*}
$$

Let $u$ be a function in $H^{2,1}\left(B_{R} \times\left(0, R^{2}\right)\right)$ satisfying the inequality

$$
\begin{equation*}
|P u| \leq \Lambda\left(\frac{|\nabla u|}{R}+\frac{|u|}{R^{2}}\right)^{1 / 2} \quad \text { in } B_{R} \times\left[0, R^{2}\right) \tag{117}
\end{equation*}
$$

Then there exist constants $\eta_{1} \in(0,1)$ and $C \in[1,+\infty)$, depending on $\lambda, \Lambda$ and $n$ only such that for every $r_{1}, r_{2}, 0<r_{1} \leq r_{2} \leq \eta_{1} R$ we have

$$
\begin{equation*}
\|u(\cdot, 0)\|_{L^{2}\left(B_{r_{2}}\right)} \leq \frac{C R}{r_{2}}\|u\|_{L^{2}\left(B_{R} \times\left(0, R^{2}\right)\right)}^{1-\theta_{1}}\|u(\cdot, 0)\|_{L^{2}\left(B_{r_{1}}\right)}^{\theta_{1}} \tag{118}
\end{equation*}
$$

where $\theta_{1}=\frac{1}{C \log \frac{R}{r_{1}}}$.
Proof. See [27].

## Step 5: proof of Theorem 4.2.

For the proof of the theorem we refer to [15, Theorem 2.7] as it is rather technical and lengthy.

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# On the boundary behaviour of solutions to parabolic equations of $p$-Laplacian type 

Ugo Gianazza and Sandro Salsa<br>Dedicated to Giovanni Alessandrini for his 60th Birthday


#### Abstract

We describe some recent results on the boundary behavior of non-negative solutions to a class of degenerate/singular parabolic equations, whose prototype is the parabolic p-Laplacian. More precisely we focus on Carleson-type estimates and boundary Harnack principles.


Keywords: Degenerate and singular parabolic equations, parabolic p-Laplacian, Carleson estimates, boundary Harnack principle.
MS Classification 2010: 35K65, 35K67, 35B65, 35K92.

## 1. Introduction

Carleson and boundary Harnack estimates are among the most important tools in the study of the boundary behaviour of solutions to elliptic and parabolic equation. Carleson estimates apply to nonnegative solutions $u$ continuously vanishing on some distinguished part $S$ of the boundary with the goal of showing that nearby, $u$ is controlled above in a non-tangential fashion. More precisely, this means that an inequality of the following type

$$
\begin{equation*}
u \leq \gamma u\left(P_{\rho}\right) \tag{1}
\end{equation*}
$$

holds in a box $\psi_{\rho}$ of size $\rho$, based on $S$, where $P_{\rho}$ approaches $S$ in a nontangential fashion as $\rho \rightarrow 0$ and $\gamma$ depends only on the dimension and the structure of the equation. The first results of this kind are due to Carleson [13], for harmonic functions in sawtooth regions and to Kemper [38] for solutions of the heat equation in parabolic $C^{1,1 / 2}$ domains. Since then, an inequality like (1) is known as a Carleson estimate.

A Boundary Comparison Principle or Boundary Harnack Inequality is a relation of the type

$$
\begin{equation*}
u / v \approx u\left(P_{r}\right) / v\left(P_{r}\right) \text { in } \psi_{\rho} . \tag{2a}
\end{equation*}
$$

where both $u$ and $v$ are nonnegative solutions vanishing on $S$. It implies that $u$ and $v$ vanish at the same speed approaching $S$. For linear equations, it also implies the Hölder continuity up to the boundary of the quotient $u / v$.

Both (1) and (2a) have been generalized to more general contexts and operators. In the elliptic context we mention [37] for the Laplace operator in non-tangentially accessible domains, [11], and [3, 7, 26] for elliptic operators in divergence and non-divergence form, respectively, [44, 46], for the $p$-Laplace operator, $[16,15]$ for the Kolmogorov operator.

Actually, for uniformly elliptic linear equations, the Carleson estimate has been proved to be equivalent to the boundary Harnack principle as shown in [1]. It would be interesting to explore this connection between the two inequalities also in the nonlinear setting.

For parabolic operators, we quote $[28,29,35,50]$ for cylindrical domains, and [27] for parabolic Lipschitz domains.

A classical application of the two inequalities is to Fatou-type theorems, but even more remarkable is their use in the regularity theory of two-phase free boundary problems, as shown in the two seminal papers [9, 10], where a general strategy to attack the regularity of the free boundary governed by the Laplace operator has been set up.

This technique has been subsequently extended to stationary problems governed by variable coefficients linear and semilinear operators [14, 32], to fully nonlinear operators [30, 31], and to the $p$-Laplace operator [45, 47].

The free boundary regularity theory for two-phase parabolic problems is less developed. For Stefan type problems we mention [12, 17, 33, 34] and the references therein.

In this brief review we describe and comment recent results concerning a class of singular/degenerate equations whose prototype is the parabolic $p$ Laplace equation

$$
\begin{equation*}
u_{t}-\operatorname{div}\left(|D u|^{p-2} D u\right)=0 \tag{1}
\end{equation*}
$$

where $D w$ denotes the gradient of $w$ with respect to the space variables. Precisely, let $\Omega$ be an open set in $\mathbb{R}^{N}$ and for $T>0$ let $\Omega_{T}$ denote the cylindrical domain $\Omega \times(0, T]$. Moreover let

$$
S_{T}=\partial \Omega \times(0, T), \quad \partial_{P} \Omega_{T}=S_{T} \cup(\Omega \times\{0\})
$$

denote the lateral, and the parabolic boundary respectively.
We shall consider quasi-linear, parabolic partial differential equations of the form

$$
\begin{equation*}
u_{t}-\operatorname{div} \mathbf{A}(x, t, u, D u)=0 \quad \text { weakly in } \Omega_{T} \tag{3}
\end{equation*}
$$

where the function $\mathbf{A}: \Omega_{T} \times \mathbb{R} \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ is only assumed to be measurable and subject to the structure conditions

$$
\left\{\begin{array}{l}
\mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_{o}|\xi|^{p}  \tag{4}\\
|\mathbf{A}(x, t, u, \xi)| \leq C_{1}|\xi|^{p-1}
\end{array} \quad \text { a.e. }(x, t) \in \Omega_{T}, \forall u \in \mathbb{R}, \forall \xi \in \mathbb{R}^{N}\right.
$$

where $C_{o}$ and $C_{1}$ are given positive constants, and $p>1$. We refer to the parameters $N, p, C_{o}, C_{1}$ as our structural data. We say that a constant is universal if it depends only on the structural data and on the Lipschitz (or $C^{k}$, if it is the case) character of the domain $\Omega$.

A function

$$
\begin{equation*}
u \in C\left([0, T] ; L^{2}(\Omega)\right) \cap L^{p}\left(0, T ; W^{1, p}(\Omega)\right) \tag{5}
\end{equation*}
$$

is a weak sub(super)-solution to $(3)-(4)$ if for every sub-interval $\left[t_{1}, t_{2}\right] \subset(0, T]$

$$
\begin{equation*}
\left.\int_{\Omega} u v d x\right|_{t_{1}} ^{t_{2}}+\int_{t_{1}}^{t_{2}} \int_{\Omega}\left[-u v_{t}+\mathbf{A}(x, t, u, D u) \cdot D v\right] d x d t \leq(\geq) 0 \tag{6}
\end{equation*}
$$

for all non-negative test functions

$$
v \in W^{1,2}\left(0, T ; L^{2}(\Omega)\right) \cap L^{p}\left(0, T ; W_{o}^{1, p}(\Omega)\right)
$$

Under the conditions (4), equation (3) is degenerate when $p>2$ and singular when $1<p<2$, since the modulus of ellipticity $|D u|^{p-2}$ respectively tends to 0 or to $+\infty$ as $|D u| \rightarrow 0$. In the latter case, we further distinguish between singular super-critical range (when $\frac{2 N}{N+1}<p<2$ ), and singular critical and sub-critical range (when $1<p \leq \frac{2 N}{N+1}$ ).

Let us first focus on Carleson's estimate and, in particular, on the approach developed for linear elliptic equations in [11] and for linear parabolic equations in [50]. Two are the main tools: the Harnack inequality and the geometric decay of the oscillation of $u$ up to the boundary. Let us sketch the main strategy. Consider a non-negative solution $u$ in a cylinder, and assume further that the solution vanishes on a part of the lateral boundary, which we assume to be a part of the hyperplane $\left\{x_{N}=0\right\}$, containing the origin. One wants to show that

$$
\begin{equation*}
u(P) \leq \gamma \tag{7}
\end{equation*}
$$

$\gamma$ universal, for all $P \in \Psi_{1}$, where

$$
\Psi_{r}=\left\{\left(x^{\prime}, x_{N}\right) \in \mathbb{R}^{N-1} \times \mathbb{R},\left|x^{\prime}\right| \leq r, 0<x_{N}<r\right\} \times\left(-2 r^{2},-r^{2}\right)
$$

Observe that, if $\operatorname{dist}\left(P, \partial \Omega_{T}\right) \approx 2^{-k}$, by the Harnack inequality we infer $u\left(P_{1}\right) \leq$ $H^{k}$. Suppose now that (7) is not true. Then, given an integer $h$, there must exist $P_{1} \in \Psi_{1}$, such that $u\left(P_{1}\right)>H^{h}$, which forces dist $\left(P_{1}, \partial \Omega_{T}\right)<2^{-h}$. By the geometric decay of the oscillation of $u$ up to the boundary, one deduces the existence of $P_{2}$ such that $u\left(P_{2}\right)>H^{h+1}$, and $\operatorname{dist}\left(P_{2}, \partial \Omega_{T}\right)<2^{-(h+1)}$.

If $h$ is chosen large enough, an iteration of this procedure yields a sequence of points $\left\{P_{j}\right\}_{j=1}^{\infty}$ all belonging to $\Psi_{2 / 3}$ (say) and approaching the boundary, whereas the sequence $\left\{u\left(P_{j}\right)\right\}_{j=1}^{\infty}$ blows up. This contradicts the assumption that $u$ vanishes continuously on the boundary, and we conclude

$$
\sup _{\Psi_{1}} u \leq H^{h} \equiv \gamma .
$$

Due to recent development in the field of Harnack inequalities for the above class of equations [20, 21, 23, 39], it is possible to prove suitable versions of Carleson estimate for non-negative solutions to (3)-(4) both in cylindrical Lipschitz domains and in time-independent NTA-cylinders (non-tangentially accessible domain). For more particulars on these sets, we refer the reader to [12, § 12.3].

According to the theory developed in the above papers, a Carleson type estimate makes sense only for $p>2 N /(N+1)$.

Indeed, in the critical and sub-critical range, explicit counterexamples rule out the possibility of a Harnack inequality. Only so-called Harnack-type estimates are possible, where, however, the ratio of infimum over supremum in proper space-time cylinders depends on the solution itself (for more details, see [24, Chapter 6, § 11-15]).

Although the overall strategy in the nonlinear setting follows the same kind of arguments of the linear case, its implementation presents a difficulty due to the lack of homogeneity of the equations. Also there is a striking difference between the singular and the degenerate case; this is already reflected in the intrinsic character of the interior Harnack inequality, and it is amplified when approaching the boundary through dyadically shrinking intrinsic cylinders. Concerning the Carleson estimate, its statement in the degenerate case can be considered as the intrinsic version of the analogous statement in the linear uniformly parabolic case. Things are different in the singular super-critical case, where, in general, one can only prove a somewhat weaker estimate, due to the possibility for a solution to extinguish in finite time. Indeed, we exhibit some counterexamples which show that one cannot do any better, unless some control of the interior oscillation of the solution is available.

The difference between the two cases, degenerate and singular super-critical, becomes more evident when one considers the validity of a boundary Harnack principle, even in smooth cylinders. In the singular case, for $C^{2}$ cylinders, the existence of suitable barriers provides a linear behavior. Together with Carleson's estimate, this fact implies almost immediately a Hopf principle and the boundary Harnack inequality. The extension of the boundary Harnack principle to Lipschitz cylinders remains an open question.

On the other hand, solutions to the parabolic $p$-Laplace equations can vanish arbitrarily fast in the degenerate case $p>2$, so that no possibility exists to prove a boundary Harnack principle in its generality. Indeed, when $p>2$, two explicit solutions to the parabolic $p$-Laplacian in the half space $\left\{x_{N} \geq 0\right\}$, that vanish at $x_{N}=0$, are given by

$$
\begin{equation*}
u_{1}(x, t)=x_{N}, u_{2}(x, t)=\left(\frac{p-2}{p^{\frac{p-1}{p-2}}}\right)(T-t)^{-\frac{1}{p-2}} x_{N}^{\frac{p}{p-2}} \tag{8}
\end{equation*}
$$

The power-like behavior, as exhibited in the second one of (8), is not the
"worst" possible case. Indeed, let $\Omega=\left\{-1 \leq x_{i} \leq 1,0 \leq x_{N} \leq \frac{1}{4}\right\}$, and consider the following Cauchy-Dirichlet Problem in $\Omega \times[0, T[$ :

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}\left(|D u|^{p-2} D u\right)=0  \tag{9}\\
u(x, 0)=C T^{-\frac{1}{p-2}} \exp \left(-\frac{1}{x_{N}}\right) \\
u\left(x^{\prime}, 0, t\right)=0 \\
u\left(x^{\prime}, \frac{1}{4}, t\right)=C(T-t)^{-\frac{1}{p-2}} e^{-4} \\
u(x, t)=C(T-t)^{-\frac{1}{p-2}} \exp \left(-\frac{1}{x_{N}}\right), \quad x \in \partial \Omega \cap\left\{0<x_{N}<\frac{1}{4}\right\}
\end{array}\right.
$$

where

$$
C=\frac{1}{2(p-1)(p-2)}\left(\frac{\Omega(p-2)}{2 p}\right)^{\frac{2 p}{p-2}}
$$

It is easy to check that the function

$$
\begin{equation*}
u_{3}=C(T-t)^{-\frac{1}{p-2}} \exp \left(-\frac{1}{x_{N}}\right), \quad x_{N}>0 \tag{10}
\end{equation*}
$$

is a super-solution to such a problem. Therefore, the solution to the same problem (which is obviously positive) lies below $u_{3}$ and approaches the zero boundary value at $x_{N}=0$ at least with exponential speed.

There is more. Let $\gamma \in(0,1), \Omega=\left\{x_{N}>0\right\}, T=\frac{2}{\gamma}-1$ : then

$$
\begin{equation*}
u(x, t)=\left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}}(t+1)\left(\gamma+\frac{x_{N}-2}{t+1}\right)_{+}\right]^{\frac{p-1}{p-2}} \tag{11}
\end{equation*}
$$

is a solution to $(1)_{o}$ in $\Omega_{T}$, and vanishes not only on the boundary $\left\{x_{N}=0\right\}$, but also in the set $\left\{0<x_{N}<2-\gamma(t+1), \quad 0<t<T\right\}$, which has positive measure.

Therefore, if one wants to prove an estimate like (2a), one needs to be able to rule out examples like the ones we have just discussed.

## 2. The Degenerate Case $p>2$

### 2.1. Harnack inequality and Harnack chains

As we mentioned in the Introduction, our results are strongly based on the interior Harnack inequalities proved in [20, 21, 22, 39], that we recall below.

First we need to introduce further notations. $D^{\prime} w$ stands for the gradient of $w$ with respect to $x^{\prime}$.

For $y \in \mathbb{R}^{N}$ and $\rho>0, K_{\rho}(y)$ denotes the cube of edge $2 \rho$, centered at $y$ with faces parallel to the coordinate planes. When $y$ is the origin of $\mathbb{R}^{N}$ we simply write $K_{\rho} ; K_{\rho}^{\prime}\left(y^{\prime}\right)$ denotes the $(N-1)$-dimensional cube $\left\{\left(x^{\prime}:\left|x_{i}-y_{i}\right|<\right.\right.$ $\rho, i=1,2, \ldots, N-1\}$; we write for short $\left\{\left|x_{i}-y_{i}\right|<\rho\right\}$.

For $\theta>0$ we also define

$$
Q_{\rho}^{-}(\theta)=K_{\rho} \times\left(-\theta \rho^{p}, 0\right], \quad Q_{\rho}^{+}(\theta)=K_{\rho} \times\left(0, \theta \rho^{p}\right]
$$

and for $(y, s) \in \mathbb{R}^{N} \times \mathbb{R}$,

$$
(y, s)+Q_{\rho}^{-}(\theta)=K_{\rho}(y) \times\left(s-\theta \rho^{p}, s\right], \quad(y, s)+Q_{\rho}^{+}(\theta)=K_{\rho}(y) \times\left(s, s+\theta \rho^{p}\right] .
$$

Now fix $\left(x_{o}, t_{o}\right) \in \Omega_{T}$ such that $u\left(x_{o}, t_{o}\right)>0$ and construct the cylinders

$$
\begin{equation*}
\left(x_{o}, t_{o}\right)+Q_{\rho}^{ \pm}(\theta) \quad \text { where } \quad \theta=\left(\frac{c}{u\left(x_{o}, t_{o}\right)}\right)^{p-2} \tag{12}
\end{equation*}
$$

and $c$ is a given positive constant. These cylinders are "intrinsic" to the solution, since their height is determined by the value of $u$ at $\left(x_{o}, t_{o}\right)$. Cylindrical domains of the form $K_{\rho} \times\left(0, \rho^{p}\right]$ reflect the natural, parabolic space-time dilations that leave the homogeneous, prototype equation (1) oinvariant. The latter however is not homogeneous with respect to the solution $u$. The time dilation by a factor $u\left(x_{o}, t_{o}\right)^{2-p}$ is intended to restore the homogeneity. Most of the results we describe in this paper hold in such geometry.

Here is the Harnack inequality.
THEOREM 2.1. Let $u$ be a non-negative, weak solution to (3)-(4) in $\Omega_{T}$ for $p>2,\left(x_{o}, t_{o}\right) \in \Omega_{T}$ such that $u\left(x_{o}, t_{o}\right)>0$. There exist positive universal constants $c$ and $\gamma$, such that for all intrinsic cylinders $\left(x_{o}, t_{o}\right)+Q_{2 \rho}^{ \pm}(\theta)$ as in (12), contained in $\Omega_{T}$,

$$
\begin{equation*}
\gamma^{-1} \sup _{K_{\rho}\left(x_{o}\right)} u\left(\cdot, t_{o}-\theta \rho^{p}\right) \leq u\left(x_{o}, t_{o}\right) \leq \gamma \inf _{K_{\rho}\left(x_{o}\right)} u\left(\cdot, t_{o}+\theta \rho^{p}\right) . \tag{13}
\end{equation*}
$$

The constants $\gamma$ and $c$ deteriorate as $p \rightarrow \infty$ in the sense that $\gamma(p), c(p) \rightarrow \infty$ as $p \rightarrow \infty$; however, they are stable as $p \rightarrow 2$.

Some comments are in order. It could be interesting to examine the existence of a so-called Harnack chain allowing the control of the value of $u(x, t)$ by the value of $u\left(x_{o}, t_{o}\right)$ with $t<t_{o}$, thanks to the repeated application of the Harnack inequality. A Harnack chain argument is indeed one of the usual tools for proving a Carleson estimate.

In [21], the authors show that such a result actually holds for solutions defined in $\mathbb{R}^{N} \times(0, T)$, and not in a smaller domain $\Omega_{T}$. Although the correct form of the Harnack chain for solutions defined in $\Omega_{T}$, when $\Omega \subset \mathbb{R}^{N}$, can be given, nevertheless, such a result is of no use in the proof of Carleson's estimates, as there are two different, but equally important obstructions.

First of all $u$ can vanish and hence prevent any further application of the Harnack inequality. Indeed, let us consider the following two examples.

Let $\gamma \in(0,1)$; the function

$$
\begin{aligned}
u(x, t)= & {\left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}}(t+1)\left(\gamma+\frac{x_{N}-2}{t+1}\right)_{+}\right]^{\frac{p-1}{p-2}} } \\
& +\left[\frac{p-2}{p-1} \gamma^{\frac{1}{p-1}}(t+1)\left(\gamma-\frac{x_{N}+2}{t+1}\right)_{+}\right]^{\frac{p-1}{p-2}}
\end{aligned}
$$

is a solution to the parabolic $p$-Laplacian in the set $\mathbb{R}^{N} \times\left(0, \frac{2}{\gamma}-1\right)$ and vanishes in the cone

$$
\left\{\begin{array}{c}
0<t<\frac{2}{\gamma}-1 \\
-(2-\gamma(t+1))<x_{N}<2-(\gamma(t+1))
\end{array}\right.
$$

If we take $(x, t)$ and $\left(x_{o}, t_{o}\right)$ with $t<t_{o}$ on opposite sides of the cone, there is no way to build a Harnack chain that connects the two points.

Let $\gamma_{p}=\left(\frac{1}{\lambda}\right)^{\frac{1}{p-1}} \frac{p-2}{p}$, with $\lambda=N(p-2)+p$, consider the cylinder $\left\{x_{N}>\right.$ $0\} \times\left(0,\left(2 \gamma_{p}\right)^{\lambda}\right)$ and let $x_{1}=(0,0, \ldots, 2), x_{2}=(0,0, \ldots, 6)$. The function
$u(x, t)=t^{-\frac{N}{\lambda}}\left[1-\gamma_{p}\left(\frac{\left|x-x_{1}\right|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right]_{+}^{\frac{p-1}{p-2}}+t^{-\frac{N}{\lambda}}\left[1-\gamma_{p}\left(\frac{\left|x-x_{2}\right|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right]_{+}^{\frac{p-1}{p-2}}$
is a solution to the parabolic $p$-Laplacian in the indicated cylinder and vanishes on its parabolic boundary. Notice that such a solution is the sum of two Barenblatt functions with poles respectively at $x_{1}$ and $x_{2}$ and masses $M_{1}=$ $M_{2}=1$ : in the interval $0<t<\left(2 \gamma_{p}\right)^{\lambda}$ the support of $u$ is given by two disjoint regions $R_{1}$ and $R_{2}$, and only at time $T=\left(2 \gamma_{p}\right)^{\lambda}$ the support of $u$ finally becomes a simply connected set. Once more, taking $(x, t)$ and $\left(x_{o}, t_{o}\right)$ respectively in $R_{1}$ and $R_{2}$, there is no way to connect them with a Harnack chain. As a matter of fact, before the two supports touch, each Barenblatt function does not feel in any way the presence of the other one. In particular, we can change the mass of the two Barenblatt functions: this will modify the time $T$ the two supports touch, but up to $T$, there is no way one Barenblatt component can detect the change performed on the other one.

On the other hand, one could think that if we have a solution vanishing on a flat piece of the boundary and strictly positive everywhere in the interior, then one could build a Harnack chain extending arbitrarily close to the boundary. However, this is not the case, as clearly shown by the following example.

Let us consider a domain $\Omega \subset \mathbb{R}^{N}$, which has a part of its boundary that coincides with the hyperplane $\left\{x_{N}=0\right\}$, and let $\Gamma=\partial \Omega \cap\left\{x_{N}=0\right\}$. Let $\bar{T}>0$, be given and consider a non-negative solution $u$ to

$$
\left\{\begin{array}{l}
u_{t}-\operatorname{div}\left(|D u|^{p-2} D u\right)=0, \quad \text { in } \Omega_{\bar{T}} \\
u>0, \quad \text { in } \Omega_{\bar{T}} \\
u=0, \quad \text { on } \Gamma \times(0, \bar{T}]
\end{array}\right.
$$

Let $u$ be such that its value is bounded above by the distance to the flat boundary piece raised to some given power $a>0$, i.e.

$$
\begin{equation*}
u(x, t) \leq \gamma \operatorname{dist}(x, \Gamma)^{a}, \quad a>0, \quad(x, t) \in \Omega_{\bar{T}} \tag{14}
\end{equation*}
$$

where $\gamma>0$ is a proper parameter.
Let $\left(x_{o}, t_{o}\right)=\left(x_{o}^{\prime}, x_{o, N}, t_{o}\right) \in \Omega_{T}$ be such that $\gamma\left(x_{o}, \Gamma\right)=1$. The goal is to form a Harnack chain of dyadic non-tangential cylinders approaching the boundary, while the chain stays inside $\Omega_{\bar{T}}$ : we want to control the size of the time interval, which we need to span in order to complete the chain. Let

$$
\begin{aligned}
u_{o} & =u\left(x_{o}, t_{o}\right) \\
r_{k} & =2^{-k} \\
x_{k} & =\left(\hat{x}_{o}^{\prime}, 2^{-k}\right) \\
t_{k} & =t_{o}-c^{p-2} \sum_{i=0}^{k-1} u_{i}^{2-p} r_{i}^{p} \\
u_{k} & =u\left(x_{k}, t_{k}\right) \approx\left(2^{-k}\right)^{a}
\end{aligned}
$$

for $k=1, \ldots$ Assuming that at each step one can use Harnack's inequality, we get an estimate on the size of $t_{k}$ from above

$$
t_{k} \leq t_{o}-c^{p-2} \sum_{i=0}^{k-1}\left(2^{-a i}\right)^{2-p} 2^{-i p} \leq t_{o}-c^{p-2} \sum_{i=0}^{k-1} 2^{a i(p-2)-i p}
$$

which diverges to $-\infty$ as $k \rightarrow \infty$ and $x_{k} \rightarrow \Gamma$, if $a \geq p /(p-2)$. Considering the solution $u_{2}$ from (8), we see that the above dyadic Harnack chain would diverge for such a solution as $a=\frac{p}{p-2}$.

The infinite length of the time interval needed to reach the boundary, is just one face (i. e. consequence) of the finite speed of propagation when $p>2$. Points $(x, t)$ that lie inside a proper $p$-paraboloid centered at $\left(x_{o}, t_{o}\right)$ can be reached, starting from $\left(x_{o}, t_{o}\right)$ : if $u_{o}$ is very small, and therefore the $p$-paraboloid is very narrow, with small values of $r$ one ends up with very large values of $t$. On the other hand, points $(x, t)$ that lie outside the same $p$-paraboloid centered at $\left(x_{o}, t_{o}\right)$ cannot be reached.

These difficulties have been recently overcome in [6], where a sequence of Harnack chain estimates has been proved. The authors develop Harnack chains based on the weak Harnack inequality of [39], valid for supersolutions to the $p$ parabolic equation. As truncations of solutions are supersolutions, the authors achieve a finer control of the waiting times (for further details, see $\S 3$ of [6]).

### 2.2. The Carleson estimates

We need to introduce some further notation. Let $\Omega_{T}$ be a Lipschitz cylinder and fix $\left(x_{o}, t_{o}\right) \in S_{T}$; in a neighbourhood of such a point, the cross section
is represented by the graph $\left\{\left(x^{\prime}, x_{N}\right): x_{N}=\Phi\left(x^{\prime}\right)\right\}$, where $\Phi$ is a Lipschitz function with Lipschitz constant $L$. Without loss of generality, from here on we assume $\Phi\left(x_{o}^{\prime}\right)=0$ and $L \geq 1$.

For $\rho \in\left(0, r_{o}\right)$, let $x_{\rho}=\left(x_{o}^{\prime}, 2 L \rho\right), P_{\rho}=P_{\rho}\left(x_{o}, t_{o}\right)=\left(x_{o}^{\prime}, 2 L \rho, t_{o}\right) \in \Omega_{T}$ such that $u\left(P_{\rho}\right)>0$. Note that $\operatorname{dist}\left(x_{\rho}, \partial \Omega\right)$ is of order $\rho$. Set
$\Psi_{\rho}^{-}\left(x_{o}, t_{o}\right)=\Omega_{T} \cap\left\{\left|x_{i}-x_{o, i}\right|<\frac{\rho}{4},\left|x_{N}\right|<2 L \rho, t \in\left(t_{o}-\frac{\alpha+\beta}{2} \theta \rho^{p}, t_{o}-\beta \theta \rho^{p}\right\}\right.$
where $\theta=\left[\frac{c}{u\left(P_{\rho}\right)}\right]^{p-2}$, with $c$ given in Theorem 2.1, and $\alpha>\beta$ are two positive parameters. We are now ready to state our main result in the degenerate case $p>2$ (see [5]).
Theorem 2.2. (Carleson's Estimate, $p>2$ ) Let $u$ be a non-negative, weak solution to (3)-(4) in $\Omega_{T}$. Assume that

$$
\left(t_{o}-\theta(4 \rho)^{p}, t_{o}+\theta(4 \rho)^{p}\right] \subset(0, T]
$$

and that $u$ vanishes continuously on

$$
\partial \Omega \cap\left\{\left|x_{i}-x_{o, i}\right|<2 \rho,\left|x_{N}\right|<8 L \rho\right\} \times\left(t_{o}-\theta(4 \rho)^{p}, t_{o}+\theta(4 \rho)^{p}\right) .
$$

Then there exist two universal positive parameters $\alpha>\beta$, and a constant $\tilde{\gamma}>0$, such that

$$
\begin{equation*}
u(x, t) \leq \tilde{\gamma} u\left(P_{\rho}\right) \quad \text { for every }(x, t) \in \Psi_{\rho}^{-}\left(x_{o}, t_{o}\right) \tag{15}
\end{equation*}
$$

Without going too much into details here, let us point out that for the prototype equation (1) , estimate (15) have been extended in [4] from Lipschitz cylinders to a wider class of cylinders $\Omega_{T}$, whose cross section $\Omega$ is a NTA domain.

Weak solutions to (3) with zero Dirichlet boundary conditions on a Lipschitz domain are Hölder continuous up to the boundary (see, for example, [19, Chapter III, Theorem 1.2]). Combining this result with the previous Carleson estimate, yields a quantitative estimate on the decay of $u$ at the boundary, invariant by the intrinsic rescaling

$$
x=x_{o}+\rho y, \quad t=t_{o}+\frac{\rho^{p}}{u\left(P_{\rho}\right)^{p-2}} \tau .
$$

Corollary 2.3. Under the same assumption of Theorem 2.2, we have

$$
0 \leq u(x, t) \leq \gamma\left(\frac{\operatorname{dist}(x, \partial \Omega)}{\rho}\right)^{\mu} u\left(P_{\rho}\right)
$$

for every $(x, t) \in \Psi_{\frac{\rho}{2}}^{-}\left(x_{o}, t_{o}\right)$, where $\mu \in(0,1)$ is universal.

If we restrict our attention to solutions to the model equation $(1)_{o}$, the result of Corollary 2.3 was strengthened for $C^{2}$ cylinders in [5].

Theorem 2.4. (Lipschitz Decay) Let $\Omega_{T}$ be a $C^{2}$ cylinder and u a non-negative, weak solution to (1) oin $\Omega_{T}$. Let the other assumptions of Theorem 2.2 hold. Then there exist two positive parameters $\alpha>\beta$, and a constant $\gamma>0$, depending only on $p, N$, and the $C^{2}$-constant $M_{2}$ of $\Omega$, such that

$$
\begin{equation*}
0 \leq u(x, t) \leq \gamma\left(\frac{\operatorname{dist}(x, \partial \Omega)}{\rho}\right) u\left(P_{\rho}\right) \tag{16}
\end{equation*}
$$

for every $(x, t)$ in the set

$$
\Omega_{T} \cap\left\{\left|x_{i}-x_{o, i}\right|<\frac{\rho}{4}, 0<x_{N}<2 M_{2} \rho\right\} \times\left(t_{o}-\frac{\alpha+3 \beta}{4} \theta \rho^{p}, t_{o}-\beta \theta \rho^{p}\right]
$$

Following Definition 2.2 of [6], let us recall that for a bounded domain $\Omega \subset \mathbb{R}^{N}$, we say that it satisfies the ball condition with radius $r_{o}>0$, if for each point $y \in \partial \Omega$ there exist points $x^{+} \in \Omega$ and $x^{-} \in \Omega^{c}$ such that $B_{r_{o}}\left(x^{+}\right) \subset \Omega$, $B_{r_{o}}\left(x^{-}\right) \subset \Omega^{c}, \partial B_{r_{o}}\left(x^{+}\right) \cap \partial \Omega=\{y\}=\partial B_{r_{o}}\left(x^{-}\right) \cap \partial \Omega$, and $x^{+}(y), x^{-}(y)$, and $y$ are collinear for each $y \in \partial \Omega$; the previous result has been further extended to $C^{1,1}$ domains satisfying the ball condition with radius $r_{o}$ : in such a case it is shown that $u$ has a linear decay at the boundary (see Theorem 9.3 of [6]), giving proper decay estimates both from above and from below.

Relying on the recent papers [8, 40, 41, 42], these results can be extended both to a wider class of degenerate equations with differentiable principal part which have the same structure of the $p$-Laplacian.

### 2.3. The Boundary Harnack Inequality

For $x_{o} \in \partial \Omega$, let $a_{r}\left(x_{o}\right):=x_{o}+\frac{r}{2} \frac{x^{+}-x_{o}}{\left|x^{+}-x_{o}\right|}$. In [6], the following result is proven.
THEOREM 2.5. Let $u$ and $v$ be two non-negative, weak solutions to (1) oin $\Omega_{T}$, where $\Omega$ is a $C^{1,1}$ domain satisfying the ball condition with radius $r_{o}$. Let $x_{o} \in \partial \Omega, t_{o} \in(0, T)$, and $r \in\left(0, r_{o}\right)$ be fixed. Let $A_{-}=\left(a_{r}\left(x_{o}\right), t_{o}\right)$, and assume that $u\left(A_{-}\right)=v\left(A_{-}\right)$. There exist constants $c_{4}, c_{5}, c_{6}$, which depend only on the data, which satisfy the following. Let $\theta_{-}=u\left(A_{-}\right)^{2-p}$, and assume

$$
\theta_{-} r^{p}<t_{o}, \quad \text { and } \quad t_{o}+2 c_{4} \theta_{-} r^{p}<T
$$

Set

$$
A_{+}=\left(a_{r}\left(x_{o}\right), t_{o}+2 c_{4} \theta_{-} r^{p}\right), \quad \theta_{+, u}=c_{6}^{-1} u\left(A_{+}\right)^{2-p}
$$

Assume that $v\left(A_{+}\right) \geq u\left(A_{+}\right)$. Then there exists a time $t_{+}^{*}$, depending on $v$, satisfying

$$
\begin{aligned}
& t_{+}^{*} \in\left(t_{o}+\left(2 c_{4} \theta_{-}-\theta_{+, u}\right) r^{p}, t_{o}+2 c_{4} \theta_{-} r^{p}\right) \\
& A_{+}^{*}=\left(a_{r}\left(x_{o}\right), t_{+}^{*}\right), \quad \theta_{+, v}^{*}=c_{6}^{-1} v\left(A_{+}^{*}\right)^{2-p}
\end{aligned}
$$

such that the following holds. If both $u$ and $v$ vanish continuously on

$$
S_{T} \cap\left(B_{r}\left(x_{o}\right) \times\left(t_{o}+\left[2 c_{4} \theta_{-}-5 \theta_{+, u}\right] r^{p}, t_{o}+\left[2 c_{4} \theta_{-}-\theta_{+, u}\right] r^{p}\right)\right),
$$

then

$$
\frac{1}{c_{5}} \frac{u\left(A_{-}\right)}{v\left(A_{+}^{*}\right)} \leq \frac{u(x, t)}{v(x, t)} \leq c_{5} \frac{u\left(A_{+}\right)}{v\left(A_{-}\right)}
$$

whenever $(x, t)$ belongs to the set

$$
\left(B_{r}\left(x_{o}\right) \cap \Omega\right) \times\left(t_{o}+\left[2 c_{4} \theta_{-}-\left(\theta_{+, v}^{*}+\theta_{+, u}\right)\right] r^{p}, t_{o}+\left[2 c_{4} \theta_{-}-\theta_{+, u}\right] r^{p}\right)
$$

It is important to notice that $t_{+}^{*}$ cannot be precisely controlled, and the only information at disposal is the interval it lies in. Moreover, the previous theorem reduces to the classical Boundary Harnack inequality for linear parabolic equations, whenever $p=2$.

Finally, in [6] a global Harnack inequality is established as well; we refer the interested reader to $\S 8$ of this work.

## 3. The Singular Super-critical Case $\frac{2 N}{N+1}<p<2$

### 3.1. The Harnack inequality

As already mentioned in the introduction, in the singular case, Harnack inequality exhibits different features with respect the degenerate case. The following theorem is proved in [23] (see also [24] for a thorough presentation).

For fixed $\left(x_{o}, t_{o}\right) \in \Omega_{T}$ and $\rho>0$, set $\mathcal{M}=\sup _{K_{\rho}\left(x_{o}\right)} u\left(x, t_{o}\right)$, and require that

$$
\begin{equation*}
K_{8 \rho}\left(x_{o}\right) \times I\left(t_{o}, 8 \rho, \mathcal{M}^{2-p}\right) \subset \Omega_{T} \tag{17}
\end{equation*}
$$

Theorem 3.1. (Harnack Inequality) Let u be a non-negative, weak solution to (3)-(4), in $\Omega_{T}$ for $p \in\left(\frac{2 N}{N+1}, 2\right)$. There exist universal constants $\bar{\epsilon} \in(0,1)$ and $\bar{\gamma}>1$ such that for all intrinsic cylinders $\left(x_{o}, t_{o}\right)+Q_{8 \rho}^{ \pm}(\theta)$ for which (17) holds,

$$
\begin{equation*}
\bar{\gamma}^{-1} \sup _{K_{\rho}\left(x_{o}\right)} u(\cdot, \sigma) \leq u\left(x_{o}, t_{o}\right) \leq \bar{\gamma} \inf _{K_{\rho}\left(x_{o}\right)} u(\cdot, \tau) \tag{18}
\end{equation*}
$$

for any pair of time levels $\sigma, \tau$ in the range

$$
\begin{equation*}
t_{o}-\bar{\epsilon} u\left(x_{o}, t_{o}\right)^{2-p} \rho^{p} \leq \sigma, \tau \leq t_{o}+\bar{\epsilon} u\left(x_{o}, t_{o}\right)^{2-p} \rho^{p} \tag{19}
\end{equation*}
$$

The constants $\bar{\epsilon}$ and $\bar{\gamma}^{-1}$ tend to zero as either $p \rightarrow 2$ or as $p \rightarrow \frac{2 N}{N+1}$.

With respect to the degenerate case, we now have $c=1$ for the size of the intrinsic cylinders. The upper bound $\mathcal{M}$ has only the qualitative role to insure that $\left(x_{o}, t_{o}\right)+Q_{8 \rho}^{ \pm}(\mathcal{M})$ are contained within the domain of definition of $u$.

### 3.2. A Weak Carleson Estimate

Relying on the above Harnack inequality, one can first prove a weak form of Carleson estimate. Let $\Omega_{T}, u,\left(x_{o}, t_{o}\right), \rho, x_{\rho}, P_{\rho}$ be as in Theorem 2.2 and set

$$
I\left(t_{o}, \rho, h\right)=\left(t_{o}-h \rho^{p}, t_{o}+h \rho^{p}\right) .
$$

Moreover, let $u$ be a weak solution to (3)-(4) such that

$$
\begin{equation*}
0<u \leq M \quad \text { in } \quad \Omega_{T} \tag{20}
\end{equation*}
$$

and assume that

$$
\begin{equation*}
I\left(t_{o}, 9 \rho, M^{2-p}\right) \subset(0, T] \tag{21}
\end{equation*}
$$

Then we define

$$
\begin{gathered}
\tilde{\Psi}_{\rho}=\Omega_{T} \cap\left\{(x, t):\left|x_{i}-x_{o, i}\right|<2 \rho,\left|x_{N}<4 L \rho\right|, t \in I\left(t_{o}, 9 \rho, \eta_{\rho}^{2-p}\right)\right\} \\
\bar{\Psi}_{\rho}=\Omega_{T} \cap\left\{(x, t):\left|x_{i}-x_{o, i}\right|<\frac{\rho}{4},\left|x_{N}<2 L \rho\right|, t \in I\left(t_{o}, \rho, \eta_{\rho}^{2-p}\right)\right\}
\end{gathered}
$$

where $\eta_{\rho}$ is the first root of the equation

$$
\begin{equation*}
\max _{\widetilde{\Psi}_{\rho}\left(x_{o}, t_{o}\right)} u=\eta_{\rho} . \tag{22}
\end{equation*}
$$

Notice that both the functions $y_{1}\left(\eta_{\rho}\right)=\max _{\widetilde{\Psi}_{\rho}\left(x_{o}, t_{o}\right)} u, y_{2}\left(\eta_{\rho}\right)=\eta_{\rho}$ are monotone increasing. Moreover

$$
\left\{\begin{array} { l } 
{ y _ { 1 } ( 0 ) \geq u ( P _ { \rho } ) > 0 , } \\
{ y _ { 2 } ( 0 ) = 0 , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
y_{1}(M) \leq M \\
y_{2}(M)=M
\end{array}\right.\right.
$$

Therefore, it is immediate to conclude that at least one root of (22) actually exists. Moreover, by $(21) \widetilde{\Psi}_{\rho}\left(x_{o}, t_{o}\right) \subset \Omega_{T}$.

A weak form of the Carleson estimate, is expressed by the following theorem (see [5]).

Theorem 3.2. (Carleson-type Estimate, weak form, $\frac{2 N}{N+1}<p<2$ ). Let u be a weak solution to (3)-(4), that satisfies (20). Assume that (21) holds true and $u$ vanishes continuously on

$$
\partial \Omega \cap\left\{\left|x_{i}-x_{o, i}\right|<2 \rho,\left|x_{N}\right|<8 L \rho\right\} \times I\left(t_{o}, 9 \rho, M^{2-p}\right) .
$$

Then there exist universal constants $\gamma>0$ and $\alpha \in(0,1)$, such that

$$
u(x, t) \leq \gamma\left(\frac{\operatorname{dist}(x, \partial \Omega)}{\rho}\right)^{\alpha} \times \sup _{\tau \in I\left(t_{o}, \rho, 2 \eta_{\rho}^{2-p}\right)} u\left(x_{\rho}, \tau\right)
$$

for every $(x, t) \in \bar{\Psi}_{\rho}\left(x_{o}, t_{o}\right)$.
If we let

$$
\Psi_{\rho, M}\left(x_{o}, t_{o}\right)=\Omega_{T} \cap\left\{(x, t):\left|x_{i}-x_{o, i}\right|<\frac{\rho}{4},\left|x_{N}\right|<2 L \rho, t \in I\left(t_{o}, \rho, M^{2-p}\right)\right\}
$$

we have a second statement.
Corollary 3.3. Under the same assumptions of Theorem 3.2, we have

$$
u(x, t) \leq \gamma\left(\frac{\operatorname{dist}(x, \partial \Omega)}{\rho}\right)^{\alpha} \times \sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right),
$$

for every $(x, t) \in \Psi_{\rho, M}\left(x_{o}, t_{o}\right)$.
The quantity $\eta_{\rho}$ is known only qualitatively through (22), whereas $M$ is a datum. Therefore, Corollary 3.3 can be viewed as a quantitative version of a purely qualitative statement. On the other hand, since $\eta_{\rho}$ could be attained in $P_{\rho}$, Theorem 3.2 gives the sharpest possible statement, and is genuinely intrinsic.

Moreover, with respect to Theorem 2.2 and Corollary 2.3, Theorem 3.2 combines two distinct statements in a single one (mainly for simplicity), and presents two fundamental differences: when $p>2$, the value of $u$ at a point above controls the values of $u$ below, whereas when $\frac{2 N}{N+1}<p<2$, the maximum of $u$ over a proper time interval centered at $t_{o}$ controls the values of $u$ both above and below the time level $t_{o}$. These are consequences of the different statements of the Harnack inequality in the two cases.

Can we improve the result of Theorem 3.2, namely can we substitute the supremum of $u$ on $I\left(t_{o}, \rho, 2 \eta_{\rho}^{2-p}\right)$ with the pointwise value $u\left(P_{\rho}\right)$ ? This would certainly be possible, if there existed a universal constant $\gamma$ such that

$$
\forall t \in I\left(t_{o}, \rho, 2 \eta_{\rho}^{2-p}\right) \quad u\left(x_{\rho}, t\right) \leq \gamma u\left(P_{\rho}\right)
$$

Under a geometrical point of view, this amounts to building a Harnack chain connecting $\left(x_{\rho}, t\right)$ and $P_{\rho}$, for all $t \in I\left(t_{o}, \rho, 2 \eta_{\rho}^{2-p}\right)$. In general, without further assumptions on $u$, this is not possible, as the following counterexample shows.

Let $u$ be the unique non-negative solution to

$$
\left\{\begin{array}{l}
u \in C\left(\mathbb{R}_{+} ; L^{2}(\Omega)\right) \cap L^{p}\left(\mathbb{R}_{+} ; W_{o}^{1, p}(\Omega)\right) \\
u_{t}-\operatorname{div}\left(|D u|^{p-2} D u\right)=0 \quad \text { in } \Omega_{T} \\
u(\cdot, 0)=u_{o} \in C^{o}(\bar{\Omega}),
\end{array}\right.
$$

with $u_{o}>0$ in $\Omega$, and $u_{o}=0$ on $\partial \Omega$.
By Proposition 2.1, Chapter VII of [19], there exists a finite time $T_{*}$, depending only on $N, p, u_{o}$, such that $u(\cdot, t) \equiv 0$ for all $t \geq T_{*}$. By the results of [19, Chapter IV], $u \in C^{o}\left(\overline{\Omega \times\left(0, T_{*}\right)}\right)$. Suppose now that at time $t=T_{*}+1$, we modify the boundary value and for any $t>T_{*}+1$ we let $u(\cdot, t)=g(\cdot, t)$ on $\partial \Omega$, where $g$ is continuous and strictly positive. It is immediate to verify that $u$ becomes strictly positive for any $t>T_{*}+1$. Therefore, the positivity set for $u$ is not a connected set, $u(x, t) \equiv 0$ for all $\forall(x, t) \in \overline{\Omega \times\left(T_{*}, T_{*}+1\right)}$, and if $\left(x_{\rho}, t\right)$ and $P_{\rho}$ lie on opposite sides of the vanishing layer for $u$, by the intrinsic nature of Theorem 3.1, there is no way to connect them with a Harnack chain.

The previous counterexample allows $u$ to vanish identically for $t$ in a proper interval, but by suitably modifying the boundary values, it is clear that we can have $u$ strictly positive, and as close to zero as we want. Therefore, the impossibility of connecting two arbitrary points by a Harnack chain, does not depend on the vanishing of $u$, but it is a general property of solutions to (3)(4), whenever $\Omega \neq \mathbb{R}^{N}$. Moreover, by properly adjusting the boundary value, one can even create an arbitrary number of oscillations for $u$ between positivity and null regions.

We considered solutions to the $p$-Laplacian just for the sake of simplicity, but everything continues to hold, if we consider the same boundary value problem for (3)-(4).

Notice that if we deal with weak solutions to (3)-(4) in $\mathbb{R}^{N} \times(0, T]$, then we do not have boundary values any more, the situation previously discussed cannot occur, and therefore any two points $(x, t)$ and ( $x_{o}, t_{o}$ ) can always be connected by a Harnack chain, provided both $u(x, t)$ and $u\left(x_{o}, t_{o}\right)$ are strictly positive, and $0<t-t_{o}<\frac{\epsilon}{8^{p}} t_{o}$, as discussed in [24, Chapter 7, Proposition 4.1]. The sub potential lower bound discussed there is then a property of weak solutions given in the whole $\mathbb{R}^{N} \times(0, T)$.

The Harnack inequality given in Theorem 3.1 is time-insensitive, and its constants are not stable as $p \rightarrow 2$. A different statement, analogous to the one given in Theorem 2.1, could be given, and in such a case the constants would be stable (see [24, Chapter 6] for a thorough discussion of the two possible forms). However, the eventual result is the same, and independently of the kind of Harnack inequality one considers, two points $(x, t)$ and $\left(x_{o}, t_{o}\right)$ of positivity for $u$, cannot be connected by a Harnack chain.

Notice that we have a sort of dual situation: when $1<p<2$ the support of $u$ can be disconnected in time, when $p>2$, the support can be disconnected in space.

Strictly speaking, the previous counterexample only shows that we cannot replace the line with a point, but per se it does not rule out the possibility for a strong form of Carleson's estimate to hold true all the same. However, if one tries to adapt to the singular super-critical context the standard proof based
on the Harnack inequality and the boundary Hölder continuity (as we did, for example, in the degenerate context), then one quickly realizes that, one needs to know in advance the oscillation of $u$ : this suggests that only a control in terms of the supremum taken in a proper set can be feasible.

### 3.3. A Strong Carleson Estimate

With respect to the statement of Theorem 3.2, a stronger form is indeed possible, provided we allow the parameter $\gamma$ to depend not only on the data, but also on the oscillation of $u$. Let $\Omega_{T}, u,\left(x_{o}, t_{o}\right), \rho, P_{\rho}$ be as in Theorem 2.2, and for $k=0,1,2, \ldots$ set

$$
\begin{aligned}
& \rho_{k}=\left(\frac{7}{8}\right)^{k} \rho, \quad \sigma_{k}=\frac{\rho_{k}}{\gamma^{k \frac{2-p}{p}}} \\
& x_{\rho_{k}}=\left(x_{o}^{\prime}, 2 L \rho_{k}\right), \quad P_{\rho_{k}}=\left(x_{o}^{\prime}, 2 L \rho_{k}, t_{o}\right), \\
& \Psi_{\rho_{k}, M}\left(x_{o}, t_{o}\right) \\
& \quad=E_{T} \cap\left\{(x, t):\left|x_{i}-x_{o, i}\right|<\frac{\rho_{k}}{4},\left|x_{N}\right|<2 L \rho_{k}, t \in I\left(t_{o}, \sigma_{k}, M^{2-p}\right)\right\}, \\
& m_{o}=\inf _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right), \quad M_{o}=\sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right)
\end{aligned}
$$

Corollary 3.4. (Carleson-type Estimate, strong form, $\frac{2 N}{N+1}<p<2$ ). Let $u$ be a weak solution to (3)-(4) such that $0<u \leq M$ in $\Omega_{T}$. Assume that $I\left(t_{o}, 9 \rho, M^{2-p}\right) \subset(0, T]$ and that $u$ vanishes continuously on

$$
\partial \Omega \cap\left\{\left|x_{i}-x_{o, i}\right|<2 \rho,\left|x_{N}\right|<8 L \rho\right\} \times I\left(t_{o}, 9 \rho, M^{2-p}\right) .
$$

Then there exists a constant $\gamma$, depending only on $\rho, N, C_{o}, C_{1}$, L, and $\frac{M}{m_{o}}$, such that

$$
\begin{equation*}
u(x, t) \leq \gamma u\left(P_{\rho_{k}}\right) \tag{23}
\end{equation*}
$$

for every $(x, t) \in \Psi_{\rho_{k}, M}\left(x_{o}, t_{o}\right)$, for all $k=0,1,2, \ldots$.
The strong form of the Carleson-type estimate is derived from Corollary 3.3. An analogous statement can be derived from Theorem 3.2.

Estimate (23) has the same structure as the backward Harnack inequality for caloric functions that vanish just on a disk at the boundary (see [12, Theorem 13.7, page 234]). This is not surprising, because (23) is indeed a backward Harnack inequality, due to the specific nature of the Harnack inequality for the singular case. However, it is worth mentioning that things are not completely equivalent; indeed, the constants we have in the time-insensitive Harnack inequality (18)-(19) are not stable (and cannot be stabilized), and therefore, the result for caloric functions cannot be recovered from the singular case, by simply letting $p \rightarrow 2$ (as it is instead the case for many other results).

### 3.3.1. Hopf Principle and Boundary Harnack inequality

Another striking difference with respect to the degenerate case appears when we consider $C^{1,1}$ cylinders and (mainly for simplicity) the prototype equation $(1)_{o}$. In this case, indeed, weak solutions vanishing on the lateral part enjoy a linear behavior at the boundary with implications expressed in the following result. Note that the role of $L$ in the definition of $\Psi_{\rho, M}$ is now played by $C^{1,1}$ constant $M_{1,1}$ of $\Omega$.

Theorem 3.5. Let $\frac{2 N}{N+1}<p<2$. Assume $\Omega_{T}$ is a $C^{1,1}$ cylinder, and $\left(x_{o}, t_{o}\right)$, $\rho, P_{\rho}$ are as in Theorem 2.2. Let $u, v$ be two weak solutions to (1) oin $\Omega_{T}$, satisfying the hypotheses of Theorem 3.2, $0<u, v \leq M$ in $\Omega_{T}$. Then there exist positive constants $\bar{s}, \gamma, \beta, 0<\beta \leq 1$, depending only on $N$, $p$, and $M_{1,1}$, and $\rho_{o}, c_{o}>0$, depending also on the oscillation of $u$, such that the following properties hold.
(a) Hopf Principle:

$$
\begin{equation*}
|D u| \geq c_{o} \quad \text { in } \quad \Psi_{\rho_{o}, M}\left(x_{o}, t_{o}\right) . \tag{24}
\end{equation*}
$$

(b) Boundary Harnack Inequality:

$$
\begin{equation*}
\gamma^{-1} \frac{\inf _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right)}{\sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} v\left(x_{\rho}, \tau\right)} \leq \frac{u(x, t)}{v(x, t)} \leq \gamma \frac{\sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right)}{\inf _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} v\left(x_{\rho}, \tau\right)} \tag{25}
\end{equation*}
$$

for all $(x, t) \in\left\{x \in K_{\bar{s} \frac{\rho}{4}}\left(x_{o}\right) \cap \Omega: \operatorname{dist}(x, \partial \Omega)<\bar{s} \frac{\rho}{8}\right\} \times I\left(t_{o}, \rho, \frac{1}{2} M^{2-p}\right)$, with $\rho<\rho_{o}$.
(c) The quotient $u / v$ is Hölder continuous with exponent $\beta$ in $\Psi_{\frac{\rho_{o}, M}{2}\left(x_{o}, t_{o}\right)}$

Since

$$
\begin{aligned}
& \frac{\sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right)}{\inf _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} v\left(x_{\rho}, \tau\right)} \leq \frac{M_{o, u} u\left(P_{\rho}\right)}{m_{o, u}} \frac{M_{o, v}}{m_{o, v} v\left(P_{\rho}\right)} \\
& \frac{\inf _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} u\left(x_{\rho}, \tau\right)}{\sup _{\tau \in I\left(t_{o}, \rho, 2 M^{2-p}\right)} v\left(x_{\rho}, \tau\right)} \leq \frac{m_{o, u} u\left(P_{\rho}\right)}{M_{o, u}} \frac{m_{o, v}}{M_{o, v} v\left(P_{\rho}\right)}
\end{aligned}
$$

the Boundary Harnack Inequality (25) can be rewritten as

$$
\tilde{\gamma}^{-1} \frac{u\left(P_{\rho}\right)}{v\left(P_{\rho}\right)} \leq \frac{u(x, t)}{v(x, t)} \leq \tilde{\gamma} \frac{u\left(P_{\rho}\right)}{v\left(P_{\rho}\right)}
$$

where now $\tilde{\gamma}$ depends not only on $N, p, M_{1,1}$, but also on $M_{o, u} / m_{o, u}$ and $M_{o, v} / m_{o, v}$.

Note that (a) implies that near a part of the lateral boundary, where a non-negative solution vanishes, the parabolic $p$-Laplace operator is uniformly elliptic. Since we do not have an estimate at the boundary of the type

$$
|D u(x, t)| \geq c \frac{u(x, t)}{\operatorname{dist}(x, \partial \Omega)}
$$

(a) and (c) hold only in a small neighbourhood of $S_{T}$, whose size depends on the solution, as both $c_{o}$ and the oscillation of the gradient $D u$ depend on the oscillation of $u$ : this is precisely the meaning of $\rho_{o}$.

The proof relies on proper estimates from above and below, which were originally proved in $[25, \S 4]$ for solutions to the singular porous medium equations in $C^{2}$ domains by building explicit barriers.

We recast these estimates in the lemma below, in a form tailored to our purposes. Indeed, the Hopf Principle and a weak version of the Boundary Harnack Inequality follow easily from these estimates. Our improvement lies in the use of the Carleson estimates, that allow a more precise bound for $\frac{u(x, t)}{v(x, t)}$ in terms of $\frac{u\left(P_{\rho}\right)}{v\left(P_{\rho}\right)}$. The restriction to $\frac{2 N}{N+1}<p<2$ comes into play only in this last step.

Thus, let $\partial \Omega$ be of class $C^{1,1}$ and $u$ be a non-negative, weak solution to $(1)_{o}$ in $\Omega_{T}$, for $1<p<2$. Assume that $u \leq M$ in $\Omega_{T}$. For $x \in \mathbb{R}^{N}$, set $d(x)=\operatorname{dist}(x, \partial \Omega)$, and for $s>0$, let

$$
\Omega^{s}=\left\{x \in \Omega: \quad \frac{s}{2} \leq d(x) \leq 2 s\right\}
$$

Lemma 3.6. Let $\tau \in(0, T)$ and fix $x_{o} \in \partial \Omega$. Assume that $u$ vanishes on

$$
\partial \Omega \cap K_{2 \rho}\left(x_{o}\right) \times(\tau, T)
$$

For every $\nu>0$, there exist positive constants $\gamma_{1}, \gamma_{2}$, and $0<\bar{s}<\frac{1}{2}$, depending only on $N$, $p, \nu$, and $M_{1,1}$, such that for all $\tau+\nu M^{2-p} \rho^{p}<t<T$, and for all $x \in \Omega \cap K_{2 \bar{s} \rho}\left(x_{o}\right)$ with $d(x)<\bar{s} \rho$,

$$
\gamma_{2}\left(\frac{d(x)}{\rho}\right) \inf _{K_{2 \rho}\left(x_{o}\right) \cap \Omega^{\bar{s} \rho} \times(\tau, T)} u \leq u(x, t) \leq \gamma_{1}\left(\frac{d(x)}{\rho}\right) \sup _{\Omega \cap K_{2 \rho}\left(x_{o}\right) \times(\tau, T)} u
$$

Relying on the above lemma, the proof of Theorem 3.5 follows rather easily.

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Section 2

# On the improved Massera's theorem for the unique existence of the limit cycle for a Liénard equation 

Makoto Hayashi


#### Abstract

We further generalize a recent improvement obtained by G. Villari of the classical Massera's theorem about the unique existence of the limit cycle of a Liénard equation.


Keywords: Liénard equation, limit cycle.
MS Classification 2010: 34C25.

## 1. Background for the improved Massera's theorem

In this paper, we consider the well-known Liénard equation

$$
\ddot{x}+f(x) \dot{x}+x=0 \text {. }
$$

Throughout, we assume for the above equation that the function $f(x)$ satisfies smoothness conditions in order to guarantee the uniqueness of solutions of initial value problems. This equation has been widely investigated in the literature (for instance see [9]). We are interested in the unique existence of the limit cycle of the equation under the following Property (A) (see [8]):

$$
\begin{aligned}
& f(x) \text { is continuous and there exist } a<0<b \text { such that } f(x)<0 \text { for } a<x<b, \\
& f(x)>0 \text { for } x \leq a \text { or } x \geq b \text {; moreover, } x F(x)>0 \text { for }|x| \text { large, where } F(x) \\
& =\int_{0}^{x} f(t) d t .
\end{aligned}
$$

Note that $F(x)$ has three zeros at $\alpha<0,0, \beta>0$ and is monotone increasing for $x<\alpha$ and for $x>\beta$.

It is well-known that the Liénard equation is equivalent to the Liénard system

$$
\begin{equation*}
\dot{x}=y-F(x), \quad \dot{y}=-x . \tag{L}
\end{equation*}
$$

First, we recall some previous results for system (L). Levinson-Smith [3] in 1942 and Sansone [5] in 1949 (see also the paper of Villari [7] in 1985) have proved the following

Proposition 1.1. Under the property (A) a limit cycle intersecting both the lines $x=\alpha$ and $x=\beta$ is at most one.

Afterwards Massera [4] in 1954 improved a result of Sansone [6] in 1951 by using the phase-plane analysis as follows.

Proposition 1.2. (Massera's Theorem) System (L) has at most one limit cycle which is stable if $f(x)$ is monotone decreasing for $x<0$ and $f(x)$ is monotone increasing for $x>0$.

We remark that the existence of a limit cycle is not guaranteed in the above theorem.

Recently, Villari [8] in 2012, on these bases, has presented the following
Proposition 1.3. Under the property (A) system (L) has exactly one limit cycle, which is stable, provided that

- if $|\alpha|>\beta$, then $f(x)$ is monotone decreasing for $\alpha<x<0$, $f(x)$ is monotone increasing for $0<x<\delta$,
- if $|\alpha|<\beta$, then $f(x)$ is monotone decreasing for $\delta_{1}<x<0$, $f(x)$ is monotone increasing for $0<x<\beta$,
where $\delta=\sqrt{\left(1+F(a)+\frac{\alpha^{2}}{2}\right)^{2}+\beta^{2}}$ and $\delta_{1}=-\sqrt{\left(1-F(b)+\frac{\beta^{2}}{2}\right)^{2}+\alpha^{2}}$.
Our aim is to give a new criterion for the unique existence of the limit cycle of system (L) by combining Proposition 1.3 with our result [2] in 2000 below.

Proposition 1.4. Assume that $f(x)$ is continuous, $f(a)=f(b)=0$ for $a<$ $0<b, f(0)<0$ and $x F(x)>0$ for $|x|$ large. System (L) has exactly one limit cycle, which is stable, provided that
(i) $|\alpha|=\beta$ and $f(x)>0$ for $|x| \geq \beta$,
(ii) $|a| \leq \beta<|\alpha|$ and $f(x)>0$ for $|x| \geq \beta$,
(iii) $\quad b \leq|\alpha|<\beta$ and $f(x)>0$ for $|x| \geq|\alpha|$.

We produce the proof of the above proposition in the Appendix.

## 2. Main results

We show in this section that our method yields an improvement of the result of Villari [8]. Instead of the Property(A), assume the following Property (B):
$f(x)$ is continuously differentiable and $F(0)=F(\alpha)=F(\beta)=0, \frac{F(x)}{x}<0$ for $\alpha<0<\beta, f(x)>0$ for $x \leq p$ and $x \geq \beta$, or $x \leq \alpha$ and $x \geq q$, where

$$
p=\min \left\{x \in(\alpha, 0) \mid F^{\prime}(x)=0, F^{\prime \prime}(x) \neq 0\right\}
$$

and

$$
q=\max \left\{x \in(0, \beta) \mid F^{\prime}(x)=0, F^{\prime \prime}(x) \neq 0\right\}
$$

Remark that Property (B) includes Property (A). We now state our result concerning the unique existence of limit cycles of system (L).

Theorem 2.1. Under the property (B), if system (L) satisfies one of the conditions :
(1) $|\alpha|=\beta$ and $f(x)>0$ for $|x| \geq \beta$,
(2) $|p| \leq \beta<|\alpha|$ and $f(x)>0$ for $|x| \geq \beta$,
(3) $\quad q \leq|\alpha|<\beta$ and $f(x)>0$ for $|x| \geq|\alpha|$,
(4) $|\alpha|>\beta$ and $\beta<|p|, f(x)>0$ for $x \leq p$ and $x \geq \beta, f(x)$ is monotone decreasing for $p \leq x<0, f(x)$ is monotone increasing for $0<x<\delta^{*}$, where $\delta^{*}=\sqrt{\left(1+F\left(a^{*}\right)+\frac{p^{2}}{2}\right)^{2}+\beta^{2}}$ for $a^{*}=\min \left\{x \mid \max _{x \in(\alpha, 0)} F(x)\right\}$,
(5) $|\alpha|<\beta$ and $|\alpha|<q, f(x)>0$ for $x \leq \alpha$ and $x \geq q, f(x)$ is monotone decreasing for $\delta_{1}^{*}<x<0, f(x)$ is monotone increasing for $0<x \leq q$, where $\delta_{1}^{*}=-\sqrt{\left(1-F\left(b^{*}\right)+\frac{q^{2}}{2}\right)^{2}+\alpha^{2}}$ for $b^{*}=\max \left\{x \mid \min _{x \in(0, \beta)} F(x)\right\}$,
then it has a unique stable limit cycle.
Remark 2.2. In [8] the case of $p=a=a^{*}$ or $q=b=b^{*}$ is treated.
Remark 2.3. In Theorem 2.1 the unique limit cycle intersects the lines $x= \pm \beta$ in the case (1) or (2). In the case (3) it intersects the lines $x= \pm \alpha$, in the case (4) $x=p$ and $x=\beta$, in the case (5) $x=\alpha$ and $x=q$.

We now apply Theorem 2.1 to the Liénard equation with a positive parameter $\lambda$ :

$$
\ddot{x}+\lambda f(x) \dot{x}+x=0 .
$$

It is equivalent to the Liénard system

$$
\dot{x}=y-\lambda F(x), \quad \dot{y}=-x .
$$

TheOrem 2.4. Under each condition in Theorem 2.1 system $\left(L_{\lambda}\right)$ satisfies the following:
$(1)^{\prime} \quad$ if $|\alpha|=\beta$, then it has a unique stable limit cycle intersecting the lines $x=\alpha$ and $x=\beta$, for all $\lambda>0$,
(2) ${ }^{\prime} \quad$ if $|p| \leq \beta<|\alpha|$, then it has a unique stable limit cycle intersecting the lines $x= \pm \beta$, for all $\lambda>0$.
(3) ${ }^{\prime}$ if $q \leq|\alpha|<\beta$, then it has a unique stable limit cycle intersecting the lines $x= \pm \alpha$, for all $\lambda>0$.
(4) ${ }^{\prime} \quad$ if $|\alpha|>\beta$ and $\beta<|p|$, then it has a unique stable limit cycle intersecting the lines $x=p$ and $x=\beta$, for all $\lambda>\tilde{\lambda}_{1}=\sqrt{\frac{p^{2}-\beta^{2}}{F^{2}\left(b^{*}\right)}}$.
(5) ${ }^{\prime} \quad$ if $|\alpha|<\beta$ and $|\alpha|<q$, then it has a unique stable limit cycle intersecting the lines $x=\alpha$ and $x=q$, for all $\lambda>\tilde{\lambda}_{2}=\sqrt{\frac{q^{2}-\alpha^{2}}{F^{2}\left(a^{*}\right)}}$.

## 3. Proofs of theorems

Proof of Theorem 2.1. First, the cases of (1), (2) and (3) follow from [1] and [2]. So we omit the details. Next, we prove the case (4). By the Property (B), the existence of the limit cycle for system (L) is guaranteed. From [2] system (L) has at most one limit cycle intersecting the lines $x=p$ and $x=\beta$. Further it is stable. On the other hand, the limit cycle of system (L) contained in the region $D=\left\{(x, y) \mid p \leq x \leq \delta^{*}, y \in \mathbb{R}\right\}$ is at most one, by the monotonicity condition on the function $f(x)$, and is stable (see [8]). Thus we conclude from the stability of the limit cycle that system (L) has exactly one limit cycle, either intersecting the lines $x=p$ and $x=\beta$, or in $D$. Similarly, we can prove the case (5).

Proof of Theorem 2.2. The case (1) ${ }^{\prime}$ is well-known from [1] or [8]. In the case $(2)^{\prime}$ or $(3)^{\prime}$ the result in [2] applies. So we consider the case (4) ${ }^{\prime}$. Any positive semitrajectory which starts from the point $\left(\beta, \lambda F\left(b^{*}\right)\right)$ must intersect the line $x=p$ for the positive number $\lambda$ such that

$$
\sqrt{\lambda^{2} F^{2}\left(b^{*}\right)+\beta^{2}} \geq|p|,
$$

namely, for all $\lambda>\tilde{\lambda}_{1}$. Then, as was mentioned in Theorem 2.1, the unique limit cycle intersecting $x=p$ and $x=\beta$ exists. Further $\delta^{*}$ is given by

$$
\delta^{*}=\sqrt{\left(1+\lambda F\left(a^{*}\right)+\frac{p^{2}}{2}\right)^{2}+\beta^{2}}
$$

for each $\lambda$ satisfying $\lambda>\tilde{\lambda}_{1}$. Similarly, the case (5) ${ }^{\prime}$ is discussed, where

$$
\delta_{1}^{*}=-\sqrt{\left(1-\lambda F\left(b^{*}\right)+\frac{q^{2}}{2}\right)^{2}+\alpha^{2}}
$$

for all $\lambda>\tilde{\lambda}_{2}$.

## 4. An example

We shall apply our results to some polynomial system.
Example 4.1. Consider the function

$$
F(x)= \begin{cases}\frac{1}{3} x^{3}+\frac{3}{2} x^{2}-4 x & \text { for } x \leq-4, x \geq 0 \\ -\frac{1}{2} x^{2}-4 x & \text { for }-4<x<0\end{cases}
$$

for system (L). This system has a unique stable limit cycle. Indeed, we have $\alpha=(-9-\sqrt{273}) / 4<p\left(=a^{*}\right)=-4<b=1<\beta=(-9+\sqrt{273}) / 4$ and all conditions of the case (4) in Theorem 2.1 hold. For instance we have that $F^{\prime}(x)$ is monotone decreasing for $-4<x<0$ and $F^{\prime}(x)$ is monotone increasing for $x>0$.

## 5. Appendix

We give the outline of the proof of Theorem 2 in our result in [2]. This is a special case of Theorem 1 in [2]. It is well-known from the Poincaré-Bendixson's theorem that if System (L) satisfies the conditions that $f(0)<0$ and $x F(x)>0$ for $|x|$ large, then it has at least one limit cycles.

We consider the case of $|a| \leq \beta \leq|\alpha|$ and $f(x)=F^{\prime}(x)>0$ for $|x| \geq|\beta|$. The other case can be discussed similarly. Letting $G(x)=(1 / 2) x^{2}$, there exists a negative number $-\beta \in[\alpha, 0)$ such that $G(-\beta)=G(\beta)$. Then System (L) has no limit cycles in the strip domain $\Omega=\{(x, y)| | x \mid \leq \beta, y \in \mathbb{R}\}$ because of $x F(x)<0$ for $|x|<\beta$ (for instance see [1]). Thus, we know that there is a closed orbit which $C$ surrounds the origin and meets $\Omega^{c}$.

We show its uniqueness. Without loss of generality we can assume that $\tilde{C}$ is outside $C$. We define Lyapunov-type functions by

$$
V(x, y, t)=\left\{\begin{array}{l}
V_{1}(x, y)=(1 / 2) y^{2}+G(x) \text { if } x \geq \beta \\
V_{2}(x, y, t)=(1 / 2) y^{2}+G(x)+\gamma_{1} t \quad \text { if }|x|<\beta \text { and } y<F(x) \\
V_{3}(x, y)=(1 / 2)(y-F(a))^{2}+G(x) \quad \text { if } x \leq-\beta \\
V_{4}(x, y, t)=(1 / 2) y^{2}+G(x)+\gamma_{2} t \quad \text { if }|x|<\beta \text { and } y>F(x)
\end{array}\right.
$$

We use the same notations as in [2]. Let $(x(t), y(t))$ be a periodic solution which starts from a point on the positive half of the vertical line $x=\beta, T>0$ be its smallest period and

$$
A=y\left(T_{2}\right)-y\left(T_{3}\right)-\delta_{1} \quad \text { and } \quad \tilde{A}=\tilde{y}\left(\tilde{T}_{2}\right)-\tilde{y}\left(\tilde{T}_{3}\right)-\delta_{2}
$$

for some constants $\delta_{1}$ and $\delta_{2}$.
We assume $M=\left(T-T_{3}\right)\left(\tilde{T}_{2}-\tilde{T}_{1}\right)-\left(\tilde{T}-\tilde{T}_{3}\right)\left(T_{2}-T_{1}\right)>0$. Then the constants $\gamma_{1}$ and $\gamma_{2}$ are defined by

$$
\gamma_{1}=\frac{F(a)\left\{\left(\tilde{T}-\tilde{T}_{3}\right) A-\left(T-T_{3}\right) \tilde{A}\right\}}{M}
$$

and

$$
\gamma_{2}=\frac{F(a)\left\{\left(\tilde{T}_{2}-\tilde{T}_{1}\right) A-\left(T_{2}-T_{1}\right) \tilde{A}\right\}}{M}
$$

Since $\tilde{y}\left(\tilde{T}_{2}\right)-\tilde{y}\left(\tilde{T}_{3}\right)<y\left(T_{2}\right)-y\left(T_{3}\right)<0$ and $F(a)>0$, we can take the numbers $\delta_{1}$ and $\delta_{2}$ such that $\gamma_{1}>0, \gamma_{2}>0$ and $\delta_{1} \leq \delta_{2}$.

Then it follows from the same calculations as in [2] that $I_{i}=\int_{C_{i}} d V_{i}>\tilde{I}_{i}=$ $\int_{\tilde{C}_{i}} d V_{i}$ for $i=1, \ldots, 4$. Hence we have $I=\sum_{i=1}^{4} I_{i}>\tilde{I}=\sum_{i=1}^{4} \tilde{I}$.

On the other hand, we have from the choice of $\delta_{1}$ and $\delta_{2}$ that

$$
\begin{aligned}
I & =\oint_{C} d V=F(a)\left\{y\left(T_{2}\right)-y\left(T_{3}\right)\right\}+\gamma_{1}\left(T_{2}-T_{1}\right)-\gamma_{2}\left(T-T_{3}\right) \\
& =F(a)\left(A+\delta_{1}\right)+\gamma_{1}\left(T_{2}-T_{1}\right)-\gamma_{2}\left(T-T_{3}\right)=F(a) \delta_{1} .
\end{aligned}
$$

Similarly we have

$$
\tilde{I}=F(a)\left(\tilde{A}+\delta_{2}\right)+\gamma_{1}\left(\tilde{T}_{2}-\tilde{T}_{1}\right)-\gamma_{2}\left(\tilde{T}-\tilde{T}_{3}\right)=F(a) \delta_{2} .
$$

Thus we have $I \leq \tilde{I}$. This contradicts $I>\tilde{I}$.
In the case $M<0$, by replacing with $V_{2}(x, y, t)=(1 / 2) y^{2}+G(x)-\gamma_{1} t$ and $V_{4}(x, y, t)=(1 / 2) y^{2}+G(x)-\gamma_{2} t$, we can take the numbers $\delta_{1}$ and $\delta_{2}$ satisfying $\gamma_{1}<0, \gamma_{2}<0$ and $\delta_{1} \leq \delta_{2}$. In the case $M=0$, we have by taking $\delta_{1}=\delta_{2}$ that $I=\tilde{I}$ for some numbers $\gamma_{1}>0$ and $\gamma_{2}>0$. These contradict $I>\tilde{I}$ too.

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# On elliptic curves of bounded degree in a polarized Abelian surface 

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#### Abstract

For a polarized complex Abelian surface $A$ we study the function $N_{A}(t)$ counting the number of elliptic curves in $A$ with degree bounded by $t$. We describe elliptic curves as solutions of an explicit Diophantine equation, and we show that computing the number of solutions is reduced to the classical problem in Number Theory of counting lattice points lying on an explicit bounded subset of Euclidean space. We obtain in this way some asymptotic estimate for the counting function.


Keywords: Elliptic curve, Abelian surface, polarization, lattice points.
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## 1. Introduction

Let $A$ be a complex Abelian surface. With the expression 'elliptic curve in an Abelian surface' we mean a one-dimensional subtorus. The collection of all elliptic curves in $A$ is (at most) countable (and possibly empty). Assume that $A$ is endowed with a polarization. Every algebraic curve in $A$ has a degree with respect to the polarization, and the following finiteness theorem holds: for every integer $t \geq 1$ the collection of elliptic curves $E \subset A$ such that $\operatorname{deg}(E) \leq t$ is finite. This was known to Bolza and Poincaré, and a modern account is in the paper of Kani [4].

Denote by $N_{A}(t)$ the number of elliptic curves in $A$ with degree bounded by $t$. The aim in the present paper is to present an approach to the counting function $N_{A}(t)$. The problem of bounding this function is invariant under isogenies, and the most relevant case is when $A$ is the product $E \times E^{\prime}$ of two elliptic curves, with a split polarization (the sum of two pullback polarizations from the factors). When we consider $E \times E^{\prime}$ as a polarized Abelian surface we always assume that it is endowed with such a split polarization.

We show (see $\S 4$ ) that computing elliptic curves in $E \times E^{\prime}$ is reduced to solving some explicit Diophantine equation, in terms of coordinates in the Néron Severi group $N S\left(E \times E^{\prime}\right)$. It turns out that computing $N_{E \times E^{\prime}}(t)$ is reduced to counting points of the lattice $\mathbb{Z}^{r}$ lying on an explicit bounded subset of $\mathbb{R}^{r}$, where $r$ is the rank of the Néron Severi group. This is a classical topic in Number Theory, originating from Gauss' circle problem and still a field of active
research. So we are lead to apply some result from that field, and in this way we obtain an asymptotic estimate for the counting function.

Clearly when $r=2$ then $N_{E \times E^{\prime}}(t)=2$. So assume that $r \geq 3$. Denote by $m$ the minimum of $\operatorname{deg}(E)$ and $\operatorname{deg}\left(E^{\prime}\right)$, the degrees with respect to the polarization, and assume that $m=\operatorname{deg}\left(E^{\prime}\right)$. When $r=3$ then $E$ and $E^{\prime}$ are isogenous, so let $d$ be the degree of a primitive isogeny $E \rightarrow E^{\prime}$. When $r=4$, $E$ and $E^{\prime}$ are isogenous elliptic curves with complex multiplication; we denote by $\delta$ the discriminant of the relevant imaginary quadratic field (see $\S 3.2$ ). In terms of these properties, we prove (in $\S 5.2$ ) the following main result.

Theorem 1.1. Assume that $r \geq 3$. There is an asymptotic estimate

$$
N_{E \times E^{\prime}}(t)=C t^{r-1}+O\left(t^{e}\right),
$$

the constant $C$ being given by

$$
\frac{\pi}{4 \sqrt{d} m^{2}} \quad \text { for } r=3, \quad \frac{\pi}{3 \sqrt{-\delta} m^{3}} \quad \text { for } r=4
$$

the exponent e being

$$
0 \text { for } r=3, \quad \frac{85}{52}=1.634 \ldots \text { for } r=4
$$

Finally we show that the result for a product Abelian surface implies some result holding for an arbitrary polarized Abelian surface (Proposition 6.1), and we observe that the estimates for $N_{A}(t)$ obtained in this way are, at least asymptotically, sharper than an existing upper bound (Remark 6.2).

## 2. Some preliminary material

### 2.1. Elliptic curves as divisor classes

Let $A$ be an Abelian surface. Every curve $C \subset A$ determines a divisor class $[C]$ in the Néron Severi group $N S(A)$. For elliptic curves (subgroups), the induced correspondence

$$
\{\text { elliptic curves in } A\} \longrightarrow N S(A)
$$

is injective and the divisor classes in $N S(A)$ corresponding to elliptic curves in $A$ are characterized by the following properties (cf. [4], Theorem 1.1):

- $D$ is primitive (indivisible),
$-D \cdot D=0$,
- $D \cdot H>0$ for some (every) ample divisor $H$.


### 2.2. Degree with respect to a polarization

Let $L$ in $N S(A)$ be an ample divisor class, representing a polarization of $A$. For every curve $C \subset A$ the degree with respect to the polarization is

$$
\operatorname{deg}(C):=C \cdot L
$$

Let $A$ be a polarized Abelian surface (we usually omit an explicit reference to the polarization). The following is a classical result: for every integer $t \geq 1$ the collection of elliptic curves $E \subset A$ such that $\operatorname{deg}(E) \leq t$ is finite (cf. [4], Corollary 1.3). We define the function

$$
N_{A}(t)
$$

counting the number of elliptic curves in $A$ with degree bounded by $t$.
An important special case is when $A=J(C)$ is the Jacobian variety of a curve of genus 2, with the canonical polarization. Elliptic curves $E \subset J(C)$ correspond bijectively to isomorphism classes of non-constant morphisms $f$ : $C \rightarrow E$ to an elliptic curve $E$, which do not factor as $C \rightarrow E^{\prime} \rightarrow E$ where $E^{\prime} \rightarrow$ $E$ is a non-isomorphic isogeny, and the degree $\operatorname{deg}(E)$ in $J(C)$ coincides with the degree $\operatorname{deg}(f)$ of the corresponding morphism. As a corollary of the theorem above, it follows that: for every integer $t \geq 1$ the collection of isomorphism classes of morphisms $f: C \rightarrow E$ which do not factor through a non-trivial isogeny of $E$ and have $\operatorname{deg}(f) \leq t$ is finite.

### 2.3. Product Abelian surfaces

Consider an Abelian surface of the form $E \times E^{\prime}$ where $E, E^{\prime}$ are elliptic curves. There is a natural isomorphism

$$
\mathbb{Z}^{2} \oplus \operatorname{Hom}\left(E, E^{\prime}\right) \xrightarrow{\sim} N S\left(E \times E^{\prime}\right)
$$

induced by the homomorphism

$$
D: \mathbb{Z}^{2} \oplus \operatorname{Hom}\left(E, E^{\prime}\right) \longrightarrow \operatorname{Div}\left(E \times E^{\prime}\right)
$$

that is defined by

$$
D(a, b, f):=(b-1) E_{h}+(a-\operatorname{deg} f) E_{v}^{\prime}+\Gamma_{-f}
$$

where $E_{h}:=E \times\{0\}$ and $E_{v}^{\prime}:=\{0\} \times E^{\prime}$ are the 'horizontal' and the 'vertical' factor, and $\Gamma_{-f}$ is the graph of the homomorphism $-f$. The intersection form on $N S\left(E \times E^{\prime}\right)$ is expressed as

$$
D(a, b, f) \cdot D\left(a^{\prime}, b^{\prime}, f^{\prime}\right)=a b^{\prime}+b a^{\prime}-\left(\operatorname{deg}\left(f+f^{\prime}\right)-\operatorname{deg}(f)-\operatorname{deg}\left(f^{\prime}\right)\right)
$$

This is a special case of the description of correspondences between two curves in terms of homomorphisms between the associated Jacobian varieties (cf. e.g. [1], Theorem 11.5.1) and also is a special case of a result of Kani ([5], Proposition 61) for the Néron Severi group of a product Abelian variety.

### 2.4. Reducibility

We will make use of the Poincaré reducibility theorem with respect to a polarization, in the following form.

If $A$ is a polarized Abelian variety and $B$ is an Abelian subvariety of $A$, there is a unique Abelian subvariety $B^{\prime}$ of $A$ such that the sum homomorphism $B \times B^{\prime} \rightarrow A$ is an isogeny and the pullback polarization on $B \times B^{\prime}$ is the sum of the pullback polarizations from $B$ and $B^{\prime}$ (cf. [1], Theorem 5.3.5 and Corollary 5.3.6).

## 3. The homomorphism group

Let $E$ and $E^{\prime}$ be elliptic curves, that we identify with $E_{\tau}$ and $E_{\tau^{\prime}}$ for suitable moduli $\tau$ and $\tau^{\prime}$, and denote by $\Lambda:=\langle 1, \tau\rangle$ and $\Lambda^{\prime}:=\left\langle 1, \tau^{\prime}\right\rangle$ the corresponding lattices in $\mathbb{C}$. There is the natural identification

$$
\operatorname{Hom}\left(E, E^{\prime}\right) \longleftrightarrow\left\{\alpha \in \mathbb{C} \text { s.t. } \alpha \Lambda \subseteq \Lambda^{\prime}\right\}=: \mathcal{H}
$$

### 3.1. In presence of an isogeny

Assume that there is an isogeny $E \rightarrow E^{\prime}$.
Lemma 3.1. In this case, we can choose $\tau$ such that $\Lambda=\langle 1, \tau\rangle$ and such that for some $\ell \in \mathbb{Q}_{>0}$ the complex number $\ell \tau$ is the modulus of an elliptic curve $E^{\prime \prime}$ isomorphic to $E^{\prime}$.

Proof. Assume that $\alpha \in \mathbb{C}$ represents an isogeny $\mathbb{C} / \Lambda \rightarrow \mathbb{C} / \Lambda^{\prime}$. In the present setting the lattice $\Lambda$ is of the form $\langle 1, \tau\rangle$. Hence $\alpha \in \Lambda^{\prime}$. Write $\alpha=p \beta$ with $\beta \in \Lambda^{\prime}$ primitive and $p \in \mathbb{Z}_{>0}$.

In $\Lambda^{\prime} /\langle\alpha\rangle$ the torsion submodule is $\langle\beta\rangle /\langle\alpha\rangle \cong \mathbb{Z}_{p}$. Since $\Lambda^{\prime} /\langle\beta\rangle$ is torsion free of rank 1 , one can find $\omega^{\prime} \in \Lambda^{\prime}$ such that

$$
\Lambda^{\prime}=\left\langle\beta, \omega^{\prime}\right\rangle
$$

The module $\alpha \Lambda /\langle\alpha\rangle$ is a free module of rank 1 (isomorphic to $\Lambda / \mathbb{Z}$ ). Therefore the induced homomorphism

$$
\alpha \Lambda /\langle\alpha\rangle \longrightarrow \Lambda^{\prime} /\langle\beta\rangle
$$

is injective. It follows that there is some multiple $q \omega^{\prime}$ with $q \in \mathbb{Z}_{>0}$ such that $q \omega^{\prime} \in \alpha \Lambda$, thus $q \omega^{\prime}=\alpha \omega$ with $\omega \in \Lambda$, and moreover

$$
\alpha \Lambda=\left\langle\alpha, q \omega^{\prime}\right\rangle
$$

whence it follows that

$$
\Lambda=\langle 1, \omega\rangle
$$

Note that $\Lambda^{\prime} / \alpha \Lambda \cong \mathbb{Z}_{p} \times \mathbb{Z}_{q}$, so the degree of the given isogeny is $p q$.
We can choose $\omega^{\prime}$ with $\operatorname{im}\left(\omega^{\prime}\right)>0$ and, replacing $\alpha$ with $-\alpha$ if necessary, we obtain that $\operatorname{im}(\omega)>0$. We can replace the initial $\tau$ with this $\omega$. Then define $\omega^{\prime \prime}:=\omega^{\prime} / \beta=(p / q) \omega$ and define $\Lambda^{\prime \prime}:=\left\langle 1, \omega^{\prime \prime}\right\rangle$. Clearly $\beta$ represents an isomorphism $\mathbb{C} / \Lambda^{\prime \prime} \rightarrow \mathbb{C} / \Lambda^{\prime}$ and the modulus $\omega^{\prime \prime}$ for $\mathbb{C} / \Lambda^{\prime \prime}$ is as in the statement. (Note, by the way, that $p$ represents an isogeny $\mathbb{C} / \Lambda \rightarrow \mathbb{C} / \Lambda^{\prime \prime}$ that, followed by the isomorphism $\beta$, gives the initial isogeny $\alpha$.)

Remark 3.2. In the setting of the proof above, we see that $\alpha$ defines a primitive isogeny if and only if it defines a cyclic isogeny, and both conditions are equivalent to $p, q$ being coprime integers.

It is enough to observe that: if $t$ is an integer, then $(1 / t) \alpha$ sends $\Lambda=\langle 1, \omega\rangle$ into $\Lambda^{\prime}=\left\langle\beta, \omega^{\prime}\right\rangle$ if and only if $t$ is a common divisor of $p, q$; on the other hand, the quotient $\Lambda^{\prime} / \alpha \Lambda \cong \mathbb{Z}_{p} \times \mathbb{Z}_{q}$ is a cyclic group if and only if $p, q$ are coprime.

It is well known that an isogeny of minimum degree between two given elliptic curves is a cyclic isogeny (cf. [6], Lemma 6.2).

Assume now that $E$ and $E^{\prime}$ are isogenous elliptic curves, and assume that they have moduli $\tau$ and $\tau^{\prime}$ as in the Lemma, with

$$
\tau^{\prime}=\ell \tau
$$

and $\ell=p / q$ with $p, q$ coprime positive integers. Then clearly $\mathcal{H}$ contains the integer $p$ (corresponding to some primitive isogeny of degree $p q$ ) and also the subset $p \mathbb{Z}$.

Remark 3.3. If $f$ is the homomorphism corresponding to $x \in \mathbb{Z}$, then

$$
\operatorname{deg}(f)=x^{2}(p q)
$$

Because $f$ is just multiplication by $x$ in $E$ followed by the given isogeny $E \rightarrow E^{\prime}$, of degree $p q$.

### 3.2. In presence of complex multiplication

Let us continue with the same setting ( $E$ and $E^{\prime}$ isogenous, $\Lambda=\langle 1, \tau\rangle$ and $\Lambda^{\prime}=\left\langle 1, \tau^{\prime}\right\rangle$, with $\tau^{\prime}=\ell \tau$ ). We may assume that the given isogeny is primitive.

Assume now that the homomorphism group $\operatorname{Hom}\left(E, E^{\prime}\right)$ has rank $>1$. Then $E$ has complex multiplication, and the same is for $E^{\prime}$. Therefore the modulus $\tau$ is algebraic of degree 2 over $\mathbb{Q}$ (cf. e.g. [9], Chapter VI, Theorem 5.5). So, assume that $\tau$ satisfies the equation

$$
\tau^{2}+\frac{u}{w} \tau+\frac{v}{w}=0
$$

with $u, v, w$ in $\mathbb{Z}$ such that $w>0$ and $(u, v, w)=(1)$ and moreover

$$
\delta:=u^{2}-4 v w<0
$$

as $\tau$ is an imaginary complex number. Note that $\delta \equiv 0,1(\bmod 4)$.
Lemma 3.4. In the equation above we also have that $p \mid w$ and $q \mid v$.
Proof. The quadratic equation for $\tau$ is related to the quadratic equation for $\ell \tau$ over $\mathbb{Q}$, that we write as $\left(\frac{p}{q} \tau\right)^{2}+\frac{u^{\prime}}{w^{\prime}}\left(\frac{p}{q} \tau\right)+\frac{v^{\prime}}{w^{\prime}}=0$, where $u^{\prime}, v^{\prime}, w^{\prime}$ are coprime integers with $w^{\prime}$ positive. Divide both $w^{\prime}, q$ by their greatest common divisor and denote by $\tilde{w}, \tilde{q}$ the resulting coprime pair, and similarly define a coprime pair $\tilde{v}, \tilde{p}$ obtained from $v^{\prime}, p$. So we have

$$
\tau^{2}+\frac{u^{\prime} \tilde{p} \tilde{q}}{\tilde{w} \tilde{p} p} \tau+\frac{\tilde{v} \tilde{q} q}{\tilde{w} \tilde{p} p}=0
$$

and it is easily seen that $u=u^{\prime} \tilde{p} \tilde{q}, v=\tilde{v} \tilde{q} q, w=\tilde{w} \tilde{p} p$ have no common divisor: because $\tilde{v} \tilde{q} q, p$ and $\tilde{w} \tilde{p} p, q$ are coprime pairs, and because $u^{\prime}, v^{\prime}, w^{\prime}$ have no common divisor.

Thus we define

$$
\bar{w}:=w / p \quad \text { and } \quad \bar{v}:=v / q .
$$

Moreover, since $p, q$ are coprime, we can write

$$
u=p p^{\prime}+q q^{\prime}
$$

for suitable integers $p^{\prime}, q^{\prime}$.
Proposition 3.5. In the present setting, an explicit isomorphism $\mathbb{Z}^{2} \rightarrow \mathcal{H}$ is given by

$$
(x, y) \longmapsto\left(x p+y q^{\prime}\right)+(y \bar{w})(\ell \tau) .
$$

Proof. Let $\alpha \in \mathbb{C}$ represent an homomorphism $\mathbb{C} / \Lambda \rightarrow \mathbb{C} / \Lambda^{\prime}$, i.e. both $\alpha$ and $\alpha \tau$ belong to $\Lambda^{\prime}$. Write $\alpha=a+b(\ell \tau)$ with $a, b$ integers. Then

$$
\alpha \tau=-(b \ell v / w)+((a / \ell)-(b u / w))(\ell \tau)
$$

Hence $\alpha \tau \in \Lambda^{\prime}$ if and only if

$$
b \ell(v / w), a / \ell-b(u / w) \in \mathbb{Z}
$$

So the set $\mathcal{H}$ consists of the complex numbers $\alpha \in \Lambda^{\prime}$ which satisfy the two conditions above.

The map $\mathbb{Z}^{2} \rightarrow \mathbb{C}$ defined in the statement restricts to $\mathbb{Z}^{2} \rightarrow \mathcal{H}$, as is easily checked using the conditions above. It is clearly an injective homomorphism and we have to show that it is surjective.

Let $\alpha$ be an element of $\mathcal{H}$. In the representation given above we have that $b(\bar{v} / \bar{w}) \in \mathbb{Z}$ and $a q-b(u / \bar{w}) \in p \mathbb{Z}$, and in particular $b(\bar{v} / \bar{w})$ and $b(u / \bar{w})$ are integers. Since $u, v, w$ are coprime, it follows that $\bar{w} \mid b$, and the first condition above is satisfied. So write

$$
b=y \bar{w}
$$

with $y \in \mathbb{Z}$. Then the second condition above requires that $y u=a q+a^{\prime} p$ for some integer $a^{\prime}$. Since $p, q$ are coprime, the solutions are of the form $\left(a^{\prime}, a\right)=$ $y\left(p^{\prime}, q^{\prime}\right)+x(-q, p)$ with $x \in \mathbb{Z}$. Thus

$$
a=x p+y q^{\prime} .
$$

This proves that $\alpha$ belongs to the image of the map in the statement.
Proposition 3.6. The degree of the homomorphism $f: E \rightarrow E^{\prime}$ corresponding to $(x, y) \in \mathbb{Z}^{2}$ is given by

$$
\operatorname{deg}(f)=x^{2}(p q)+x y\left(q q^{\prime}-p p^{\prime}\right)+y^{2}\left(-p^{\prime} q^{\prime}+\bar{v} \bar{w}\right)
$$

The discriminant of the quadratic form $f \mapsto \operatorname{deg}(f)$ on $\operatorname{Hom}\left(E, E^{\prime}\right)$ is equal to $\delta$.

Proof. With the notation of the preceding proof, the degree is given by the absolute value of the determinant of the submodule $\alpha \Lambda$ in $\Lambda^{\prime}$, that is

$$
\left|\begin{array}{cc}
a & -b \ell(v / w) \\
b & (a / \ell)-b(u / w)
\end{array}\right|=\left|\begin{array}{cc}
x p+y q^{\prime} & -y \bar{v} \\
y \bar{w} & x q-y p^{\prime}
\end{array}\right|
$$

where we used the expressions for $a, b$ given in the preceding proof. It is then easy to calculate that the determinant is equal to the expression given in the statement. It is also easy to check that the discriminant of this quadratic form in $x, y$ is given by $u^{2}-4(p q)(\bar{v} \bar{w})=u^{2}-4 v w=\delta$.

## 4. Elliptic curves in a product Abelian surface

Consider an Abelian surface of the form $E \times E^{\prime}$ where $E, E^{\prime}$ are elliptic curves. Let $r$ be the rank of the Néron Severi group $N S\left(E \times E^{\prime}\right)$. We have (see §2.3) a natural isomorphism

$$
\mathbb{Z}^{2} \oplus \operatorname{Hom}\left(E, E^{\prime}\right) \xrightarrow{\sim} N S\left(E \times E^{\prime}\right)
$$

and we can describe (see §2.1) the collection of elements $(a, b, f)$ in the group $\mathbb{Z}^{2} \oplus \operatorname{Hom}\left(E, E^{\prime}\right)$ such that the corresponding divisor class $[D(a, b, f)]$ is the class of an elliptic curve in $E \times E^{\prime}$.

Besides the condition of primitivity of the element $(a, b, f)$, the numerical condition $D \cdot D=0$ becomes

$$
a b=\operatorname{deg}(f)
$$

and the positivity condition $D \cdot H>0$ is equivalent to

$$
a+b>0
$$

(using the ample divisor $H:=E_{h}+E_{v}^{\prime}$ ).
If on $E \times E^{\prime}$ we choose a split polarization $L=m E_{h}+n E_{v}^{\prime}=D(n, m, 0)$, where $m, n$ are positive integers, then the degree $\operatorname{deg}(D)=D \cdot L$ with respect to the polarization is given by the linear function

$$
a m+b n
$$

Furthermore, we have (see §3) a description of the group of homomorphisms between two elliptic curves, i.e. an explicit isomorphism

$$
\operatorname{Hom}\left(E, E^{\prime}\right) \longleftrightarrow \mathbb{Z}^{h}
$$

where $h$ is the rank of the homomorphism group. So we have an explicit isomorphism

$$
N S\left(E \times E^{\prime}\right) \longleftrightarrow \mathbb{Z}^{r}
$$

where $r=h+2$ is the rank of the Néron Severi group, and in terms of coordinates in $\mathbb{Z}^{r}$ the description of elliptic curves in $E \times E^{\prime}$ can be written as a Diophantine equation, with some limitation. We will study the equation according to the values of the rank $r$.

The case $r=2$, i.e. $h=0$, is when $E$ and $E^{\prime}$ are not isogenous. Clearly $E_{h}$ and $E_{v}^{\prime}$ are the only elliptic curves in $E \times E^{\prime}$. When $E$ and $E^{\prime}$ are isogenous, i.e. $r \geq 3$ and $h \geq 1$, there are infinitely many elliptic curves in $E \times E^{\prime}$, the graphs of homomorphisms $E \rightarrow E^{\prime}$. Then (see $\S 3$ ) we have $r=4$ if and only if both $E$ and $E^{\prime}$ have complex multiplication.

For small values of the degree $a m+b n$, it is sometimes possible to compute all solutions of the Diophantine equation.

Example 4.1. Elliptic curves of degree at most 2. The maximum number is attained only if on $E \times E^{\prime}$ is given the principal split polarization ( $m=n=$ 1). So assume this is the case. The only elliptic curves of degree 1 are $E_{h}$ and $E_{v}^{\prime}$. An elliptic curve of degree 2 must be the graph of an isomorphism $E \xrightarrow{\sim} E^{\prime}$ (follows from $a b=\operatorname{deg}(f)$ ). Hence, without loss of generality, we may assume that $E=E^{\prime}$ (and $\ell=1$ ). In the self product $E^{2}$ the diagonal and the anti-diagonal are elliptic curves of degree 2. If $E$ has no complex multiplication, these are the only ones. If $E$ has complex multiplication, the
maximum number of elliptic curves of degree 2 in $E^{2}$ is equal to 6 , and is attained if and only if $\delta=-3$. The degree form is written as $x^{2}-u x y+v w y^{2}$, equal to $\left((2 x-u y)^{2}-\delta t^{2}\right) / 4$, and we only have to compute the solutions of $(2 x-u y)^{2}-\delta t^{2}=4($ where $\delta \equiv 0,1(\bmod 4))$. Two solutions are $( \pm 1,0)$ for every $\delta$, that give the diagonal and the anti-diagonal; for more solutions we must have $-\delta=3,4$; if $-\delta=4$ two more are $\pm(u / 2,1)$, if $-\delta=3$ four more are $\pm((u \pm 1) / 2,1)$.

Remark 4.2. The following result is found in a recent paper by Rosen and Schnidman ([8], Lemma 2.10): in a polarized Abelian surface with polarization degree $\geq 5$ there is at most one elliptic curve of degree 2 .

## 5. On the number of elliptic curves

### 5.1. A result from Number Theory

The following is a classical problem in Number Theory, originating from Gauss' circle problem. Given a compact convex subset $K$ in $\mathbb{R}^{2}$, estimate the number $N:=\operatorname{card}\left(\mathbb{Z}^{2} \cap K\right)$ of integer vectors (or lattice points) belonging to the convex set. This number is naturally approximated by the area $A$ of the subset, and then the question is to estimate the (error or) discrepancy $N-A$. The following estimate is due to Nosarzewska [7]. If $K$ is a compact convex region in $\mathbb{R}^{2}$ of area $A$ whose boundary is a Jordan curve of length $L$ then

$$
N \leq A+\frac{1}{2} L+1
$$

We will apply this result through the following consequence. For every scale factor $t \in \mathbb{R}_{\geq 0}$ denote by $N(t)$ the number of lattice points in the deformed region $\sqrt{t} K$. Then

$$
N(t) \leq A t+\frac{L}{2} t^{1 / 2}+1
$$

The inequality above is valid for arbitrary $t$. But in an asymptotic estimate

$$
N(t)=A t+O\left(t^{e}\right)
$$

(an implicit inequality holding for $t \gg 0$ ) the exponent $e$ may be lowered, and precisely one can take $e=33 / 104=0.317 \ldots$. This follows from a result of Huxley [2].

### 5.2. Estimate for the counting function

Let $E \times E^{\prime}$ be a product Abelian surface, endowed with a split polarization. Let $r$ be the rank of the Néron Severi group $N S\left(E \times E^{\prime}\right)$ and assume that
$r \geq 3$. Here we prove the result in the introduction, asserting that there is an asymptotic estimate $N_{E \times E^{\prime}}(t)=C t^{r-1}+O\left(t^{e}\right)$, with the constant $C$ and the exponent $e$ as given in the statement.

Proof of Theorem 1.1. We work in terms of coordinates, as explained in $\S 4$. The degree with respect to the polarization is given by the linear function $a m+b n$. We assume that $m \leq n$, so that $m=\operatorname{deg}\left(E_{v}^{\prime}\right)$ is the minimum degree occurring in the statement. Define $t^{\prime}:=[t / m]$, and assume that $t^{\prime} \geq 1$ since otherwise the inequality $a m+b n \leq t$ has no nonzero solution.

We have to estimate the collection of primitive vectors $(a, b, f)$ in $\mathbb{Z}^{2} \times$ $\operatorname{Hom}\left(E, E^{\prime}\right)$ such that $a b=\operatorname{deg}(f)$ and $a+b>0$ and $a m+b n \leq t$. Note that $a+b>0$ may be replaced with $a, b \geq 0$. There are at most two such vectors with $a b=0$, since then $f=0$. The subcollection with $a b \neq 0$ is mapped, forgetting $b$, to the collection

$$
\left\{(a, f) \text { s.t. } f \neq 0,0<a<t^{\prime}, \operatorname{deg}(f) \leq a\left(t^{\prime}-a\right)\right\}
$$

and the map is injective. Therefore we have

$$
N_{E \times E^{\prime}}(t) \leq 2+\sum_{a=0}^{t^{\prime}} R(a, t)
$$

where $R(a, t)$ is the number of nonzero $f$ such that $\operatorname{deg}(f) \leq a\left(t^{\prime}-a\right)$. The function $R(a, t)$ can be estimated, according to the values of the rank $r=h+2$, using the description of the quadratic form $\operatorname{deg}(f)$ given in $\S 3$.

When $r=3$ then $R(a, t)$ is the number of nonzero $x \in \mathbb{Z}$ such that $x^{2} d \leq$ $a\left(t^{\prime}-a\right)$, where $d$ is the degree of a primitive isogeny $E \rightarrow E^{\prime}$, by Remark 3.3, and hence $R(a, t) \leq \frac{2}{\sqrt{d}}\left(a\left(t^{\prime}-a\right)\right)^{1 / 2}$. We will show in Remark 5.1 below that

$$
\sum_{a=0}^{t^{\prime}}\left(a\left(t^{\prime}-a\right)\right)^{1 / 2}=\frac{\pi}{8} t^{2}+O(1)
$$

Therefore, since $t^{\prime} \leq t / m$, in this case we have the asymptotic estimate

$$
N_{E \times E^{\prime}}(t)=\frac{\pi}{4 \sqrt{d} m^{2}} t^{2}+O(1)
$$

When $r=4$ then $R(a, t)$ is the number of nonzero vectors $(x, y) \in \mathbb{Z}^{2}$ such that $Q(x, y) \leq a\left(t^{\prime}-a\right)$, where $Q(x, y)$ is the coordinate expression for the quadratic form $\operatorname{deg}(f)$, given in Proposition 3.6, whose determinant is equal to $-\delta$. Applying the result in $\S 5.1$ we have

$$
R(a, t)=A a\left(t^{\prime}-a\right)+O\left(\left(a\left(t^{\prime}-a\right)\right)^{e}\right)
$$

with $e=33 / 104$, where $A=2 \pi / \sqrt{-\delta}$ is the area of the region $Q(x, y) \leq 1$ in $\mathbb{R}^{2}$. Remark here that for every $a$ the discrepancy above arises from a single discrepancy function $N(t)-A t$. It follows that

$$
\sum_{a=0}^{t^{\prime}} R(a, t)=A\left(\sum_{a=0}^{t^{\prime}} a\left(t^{\prime}-a\right)\right)+O\left(\sum_{a=0}^{t^{\prime}}\left(a\left(t^{\prime}-a\right)\right)^{e}\right)
$$

We have to estimate the summations occurring in this formula. For one summation we have an exact formula

$$
\sum_{a=0}^{t^{\prime}} a\left(t^{\prime}-a\right)=\frac{1}{6} t^{\prime}\left(t^{\prime}+1\right)\left(t^{\prime}-1\right)
$$

For the other summation, using a basic approximation method as explained in Remark 5.1 below, we find the asymptotic estimate

$$
\sum_{a=0}^{t^{\prime}}\left(a\left(t^{\prime}-a\right)\right)^{e}=O\left(t^{t^{2 e+1}}\right)
$$

Summing up, we obtain for the function $N_{E \times E^{\prime}}(t)$ an estimate that is a function of $t^{\prime}$ and then, using $t^{\prime} \leq t / m$, we obtain one that is a function of $t$. Explicitely, we find the asymptotic estimate

$$
N_{E \times E^{\prime}}(t)=\frac{A}{6}\left(\frac{t^{3}}{m^{3}}-\frac{t}{m}\right)+O\left(t^{2 e+1}\right)=\left(\frac{2 \pi}{6 \sqrt{-\delta} m^{3}}\right) t^{3}+O\left(t^{2 e+1}\right)
$$

with $e=33 / 104$, as in the statement.
Remark 5.1. In the interval $[0, t]$, with $t$ a positive integer, for the function $f(x):=(x(t-x))^{e}$ with $0<e<1$, applying the approximation method known as the 'trapezoidal rule', in the interval $[1, t-1]$ and for $t \geq 2$, we have that

$$
\int_{1}^{t-1} f(x) d x-\sum_{n=1}^{t-1} f(n)=-\frac{t-2}{12} f^{\prime \prime}(\xi)
$$

for some $\xi$ in $[1, t-1]$; since for $f^{\prime \prime}$ the maximum value is $f^{\prime \prime}(t / 2)=-c / t^{2-2 e}$ where $c=(e / 2) 4^{2-e}$, and since

$$
\int_{0}^{t} f(x) d x=H t^{2 e+1}
$$

where $H=\int_{0}^{1}(y(1-y))^{e} d y$, it follows that

$$
\sum_{n=0}^{t} f(n) \leq H t^{2 e+1}-\frac{c}{12} \frac{t-2}{t^{2-2 e}}
$$

For $e=1 / 2$ the special value $H=\pi / 8$ is used in the proof above.

## 6. Arbitrary polarized Abelian surfaces

### 6.1. Behavior under isogenies

Let $A, B$ be polarized Abelian surfaces and let $\varphi: B \rightarrow A$ be an isogeny, preserving the polarizations (the polarization on $B$ is the pullback of the polarization on $A$ ), whose degree we call $d$. There is a one to one correspondence

$$
\{\text { elliptic curves in } A\} \xrightarrow{\sim}\{\text { elliptic curves in } B\} .
$$

Given $E \subset A$ the corresponding $E^{*}$ in $B$ is the connected component of 0 in the pre-image $\varphi^{-1}(E)$. The restricted isogeny $E^{*} \rightarrow E$ has degree $d_{E} \leq d$ (in fact a divisor of $d$ ), and the degree of $E^{*}$ is given by

$$
\operatorname{deg}\left(E^{*}\right)=d_{E} \operatorname{deg}(E)
$$

(by the projection formula: $E^{*} \cdot \varphi^{*} L=\varphi_{*} E^{*} \cdot L=d_{E} E \cdot L$ ). Therefore:

$$
\operatorname{deg}(E) \leq \operatorname{deg}\left(E^{*}\right) \leq d \operatorname{deg}(E)
$$

It follows that the functions counting elliptic curves in $A$ and in $B$ are related by the following inequalities:

$$
N_{A}(t) \leq N_{B}(d t) \quad \text { and } \quad N_{B}(t) \leq N_{A}(t)
$$

### 6.2. On the counting function

Let $A$ be a polarized Abelian surface. Let $r$ be the rank of the Néron Severi group $N S(A)$. We may assume that $A$ is a non-simple Abelian surface, so it contains an elliptic curve $E$. It follows from the reducibility theorem (see §2.4) that $A$ also contains a complementary elliptic curve $E^{\prime}$ and there is an isogeny $E \times E^{\prime} \rightarrow A$, where the pullback polarization on $E \times E^{\prime}$ is a split polarization. Let $d$ be the minimum degree of such an isogeny. Choose an isogeny $E \times E^{\prime} \rightarrow A$ as above of degree $d$.

The rank of the Néron Severi group $N S\left(E \times E^{\prime}\right)$ is also equal to $r$, and there is a bijective correspondence

$$
\{\text { elliptic curves in } A\} \xrightarrow{\sim}\left\{\text { elliptic curves in } E \times E^{\prime}\right\}
$$

described in the previous subsection. Clearly, as $A$ is non-simple, then $r \geq 2$ and if $r=2$ then $N_{A}(t)=2$. Note that: when $r \geq 3$ there are in $A$ infinitely many elliptic curves, as is in $E \times E^{\prime}$.

Proposition 6.1. Assume that $r \geq 3$. The function $N_{A}(t)$ can be given an asymptotic estimate of the form

$$
N_{A}(t)=C t^{r-1}+O\left(t^{e}\right)
$$

for some constant $C$ and exponent $e<r-2$.
Proof. If $A$ is non-simple, and $E \times E^{\prime} \rightarrow A$ is an isogeny of degree $d$, as in the description above, then

$$
N_{A}(t) \leq N_{E \times E^{\prime}}(d t)
$$

(see $\S 6.1$ ); the estimate for the function $N_{E \times E^{\prime}}(t)$ is given in Theorem 1.1, and so the statement follows.

Remark 6.2. When $A=J(C)$ is the Jacobian of a curve of genus $g>1$, there is an effective bound for the function $N_{A}(t)$ due to Kani (cf. [3], Theorem 4), which is of order $O\left(t^{2 g^{2}-2}\right)$, in particular for $g=2$ of order $O\left(t^{6}\right)$. As the order found in the present paper is smaller, we are encouraged to believe that our approach may lead to some sharper asymptotic estimate for arbitrary $g$.

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# Schrödinger model and Stratonovich-Weyl correspondence for Heisenberg motion groups 

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#### Abstract

We introduce a Schrödinger model for the unitary irreducible representations of a Heisenberg motion group and we show that the usual Weyl quantization then provides a Stratonovich-Weyl correspondence.


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## 1. Introduction

There are different ways to extend the usual Weyl correspondence between functions on $\mathbb{R}^{2 n}$ and operators on $L^{2}\left(\mathbb{R}^{n}\right)$ to the general setting of a Lie group acting on a homogeneous space $[1,14,31,34]$. Here we are concerned with Stratonovich-Weyl correspondences. The notion of Stratonovich-Weyl correspondence was introduced in [51] and its systematic study began with the work of J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers (see [26, 29, 32, 33] and also [12]). The following definition is taken from [32], see also [33].

Definition 1.1. Let $G$ be a Lie group and $\pi$ be a unitary representation of $G$ on a Hilbert space $\mathcal{H}$. Let $M$ be a homogeneous $G$-space and let $\mu$ be a $G$-invariant measure on $M$. Then a Stratonovich-Weyl correspondence for the triple $(G, \pi, M)$ is an isomorphism $\mathcal{W}$ from a vector space of operators on $\mathcal{H}$ to $a$ vector space of functions on $M$ satisfying the following properties:

1. the function $\mathcal{W}\left(A^{*}\right)$ is the complex-conjugate of $\mathcal{W}(A)$;
2. Covariance: we have $\mathcal{W}\left(\pi(g) A \pi(g)^{-1}\right)(x)=\mathcal{W}(A)\left(g^{-1} \cdot x\right)$;
3. Traciality: we have

$$
\int_{M} \mathcal{W}(A)(x) \mathcal{W}(B)(x) d \mu(x)=\operatorname{Tr}(A B)
$$

Stratonovich-Weyl correspondences were constructed for various Lie group representations, see [26, 32]. In particular, in [20], Stratonovich-Weyl correspondences for the holomorphic representations of quasi-Hermitian Lie groups were obtained by taking the isometric part in the polar decomposition of the Berezin quantization map, see also $[3,4,16,17,24,29]$.

The basic example is the case when $G$ is the ( $2 n+1$ )-dimensional Heisenberg group acting on $\mathbb{R}^{2 n} \cong \mathbb{C}^{n}$ by translations. Each non-degenerate unitary irreducible representation of $G$ has then two classical realizations: the Schrödinger model on $L^{2}\left(\mathbb{R}^{n}\right)$ and the Bargmann-Fock model on the Fock space [30], an intertwining operator between these realizations being the Segal-Bargmann transform [27, 30]. In this context, it is well-known that the usual Weyl correspondence provides a Stratonovich-Weyl correspondence for the Schrödinger realization $[6,49,54]$. It is also known that this Stratonovich-Weyl correspondence is connected by the Segal-Bargmann transform to the Stratonovich-Weyl correspondence for the Bargmann-Fock realization which was obtained by polarization of the Berezin quantization map [43, 44]. In [22], we obtained similar results for the $(2 n+2)$-dimensional real diamond group. This group, also called oscillator group, is a semidirect product of the Heisenberg group by the real line.

The aim of the present paper is to extend the preceding results to the Heisenberg motion groups. An Heisenberg motion group is the semidirect product of the $(2 n+1)$-dimensional Heisenberg group $H_{n}$ by a compact subgroup $K$ of the unitary group $U(n)$. Note that Heisenberg motion groups play an important role in the theory of Gelfand pairs, since the study of a Gelfand pair of the form $\left(K_{0}, N\right)$ where $K_{0}$ is a compact Lie group acting by automorphisms on a nilpotent Lie group $N$ can be reduced to that of the form $\left(K_{0}, H_{n}\right)$, see [8, 9].

More precisely, we introduce a Schrödinger realization for the unitary irreducible representations of a Heisenberg motion group and we prove that we obtain a Stratonovich-Weyl correspondence by combining the usual Weyl correspondence and the unitary part of the Berezin calculus for $K$.

Let us briefly describe our construction. First notice that each Heisenberg motion group is, in particular, a quasi-Hermitian Lie group and that we can obtain its unitary irreducible representations as holomorphically induced representations on some generalized Fock space by the general method of [46], Chapter XII. Then we can get Schrödinger realizations for these representations by using, as in the case of the Heisenberg group, a (generalized) BargmannFock transform. Hence we obtain a Stratonovich-Weyl correspondence for such a Schrödinger realization by introducing a generalization of the usual Weyl correspondence.

Note that, in [45], a Schrödinger model and a generalized Segal-Bargmann transform for the scalar highest weight representations of an Hermitian Lie group of tube type were introduced and studied. Let us also mentioned that
B. Hall has obtained some generalized Segal-Bargmann transforms in various situations by means of the heat kernel, see [36] and references therein. Then one can hope for futher generalizations of our results to quasi-Hermitian Lie groups.

This paper is organized as follows. In Section 2, we review some wellknown facts about the Fock model and the Schrödinger model of the unitary irreducible representations of an Heisenberg group and about the corresponding Berezin calculus and Weyl correspondence. In Section 3, we introduce the Heisenberg motion groups and, in Section 4 and Section 5, we describe their unitary irreducible representations in the Fock model and the associated Berezin calculus. We introduce the (generalized) Segal-Bargmann transform and the Schrödinger model in Section 6. In Section 7, we show that the usual Weyl correspondence also gives a Stratonovich-Weyl correspondence for the Schrödinger model. Moreover, we compare it with the Stratonovich-Weyl correspondence for the Fock model which is directly obtained by polarization of the Berezin quantization map.

## 2. Heisenberg groups

In this section, we review some well-known results about the the Schrödinger model and the Fock model of the unitary irreducible (non-degenerated) representations of the Heisenberg group. We follow the presentation of [22] in a large extend.

Let $G_{0}$ be the Heisenberg group of dimension $2 n+1$ and $\mathfrak{g}_{0}$ be the Lie algebra of $G_{0}$. Let $\left\{X_{1}, \ldots, X_{n}, Y_{1}, \ldots, Y_{n}, \tilde{Z}\right\}$ be a basis of $\mathfrak{g}_{0}$ in which the only non trivial brackets are $\left[X_{k}, Y_{k}\right]=\tilde{Z}, 1 \leq k \leq n$ and let

$$
\left\{X_{1}^{*}, \ldots, X_{n}^{*}, Y_{1}^{*}, \ldots, Y_{n}^{*}, \tilde{Z}^{*}\right\}
$$

be the corresponding dual basis of $\mathfrak{g}_{0}^{*}$.
For $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right) \in \mathbb{R}^{n}, b=\left(b_{1}, b_{2}, \ldots, b_{n}\right) \in \mathbb{R}^{n}$ and $c \in \mathbb{R}$, we denote by $[a, b, c]$ the element $\exp _{G_{0}}\left(\sum_{k=1}^{n} a_{k} X_{k}+\sum_{k=1}^{n} b_{k} Y_{k}+c \tilde{Z}\right)$ of $G_{0}$. Similarly, for $\alpha=\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{n}\right) \in \mathbb{R}^{n}, \beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{n}\right) \in \mathbb{R}^{n}$ and $\gamma \in \mathbb{R}$, we denote by $(\alpha, \beta, \gamma)$ the element $\sum_{k=1}^{n} \alpha_{k} X_{k}^{*}+\sum_{k=1}^{n} \beta_{k} Y_{k}^{*}+\gamma \tilde{Z}^{*}$ of $\mathfrak{g}_{0}^{*}$. The coadjoint action of $G_{0}$ is then given by

$$
\operatorname{Ad}^{*}([a, b, c])(\alpha, \beta, \gamma)=(\alpha+\gamma \beta, \beta-\gamma \alpha, \gamma)
$$

Now we fix a real number $\lambda>0$ and denote by $\mathcal{O}_{\lambda}$ the orbit of the element $\lambda \tilde{Z}^{*}$ of $\mathfrak{g}_{0}^{*}$ under the coadjoint action of $G_{0}$ (the case $\lambda<0$ can be treated similarly). By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) unitary irreducible representation of $G_{0}$ whose restriction to the center of $G_{0}$ is the character $[0,0, c] \rightarrow e^{i \lambda c}[7,30]$. Note that this
representation is associated with the coadjoint orbit $\mathcal{O}_{\lambda}$ by the Kirillov-Kostant method of orbits [41, 42]. More precisely, if we choose the real polarization at $\lambda \tilde{Z}^{*}$ to be the space spanned by the elements $Y_{k}$ for $1 \leq k \leq n$ and $\tilde{Z}$ then we obtain the Schrödinger representation $\sigma_{0}$ realized on $L^{2}\left(\mathbb{R}^{n}\right)$ as

$$
\sigma_{0}([a, b, c])(f)(x)=e^{i \lambda\left(c-b x+\frac{1}{2} a b\right)} f(x-a)
$$

see [30] for instance. Here we denote $x y:=\sum_{k=1}^{n} x_{k} y_{k}$ for $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and $y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$ in $\mathbb{R}^{n}$.

The differential of $\sigma_{0}$ is then given by

$$
d \sigma_{0}\left(X_{k}\right) f(x)=-\partial_{k} f(x), d \sigma_{0}\left(Y_{k}\right) f(x)=-i \lambda x_{k} f(x), d \sigma_{0}(\tilde{Z}) f(x)=i \lambda f(x)
$$

where $k=1,2, \ldots, n$.
On the other hand, if we consider the complex polarization at $\lambda \tilde{Z}^{*}$ to be the space spanned by the elements $X_{k}+i Y_{k}$ for $1 \leq k \leq n$ and $\tilde{Z}$ then the method of orbits leads to the Bargmann-Fock representation $\pi_{0}$ defined as follows [13].

Let $\mathcal{F}_{0}$ be the Hilbert space of holomorphic functions $F$ on $\mathbb{C}^{n}$ such that

$$
\|F\|_{\mathcal{F}_{0}}^{2}:=\int_{\mathbb{C}^{n}}|F(z)|^{2} e^{-|z|^{2} / 2 \lambda} d \mu_{\lambda}(z)<+\infty
$$

where $d \mu_{\lambda}(z):=(2 \pi \lambda)^{-n} d x d y$. Here $z=x+i y$ with $x$ and $y$ in $\mathbb{R}^{n}$.
Let us consider the action of $G_{0}$ on $\mathbb{C}^{n}$ defined by $g \cdot z:=z+\lambda(b-i a)$ for $g=[a, b, c] \in G_{0}$ and $z \in \mathbb{C}^{n}$. Then $\pi_{0}$ is the representation of $G_{0}$ on $\mathcal{F}_{0}$ given by

$$
\pi_{0}(g) F(z)=\alpha\left(g^{-1}, z\right) F\left(g^{-1} \cdot z\right)
$$

where the map $\alpha$ is defined by

$$
\alpha(g, z):=\exp (-i c \lambda+(1 / 4)(b+a i)(-2 z+\lambda(-b+a i)))
$$

for $g=[a, b, c] \in G_{0}$ and $z \in \mathbb{C}^{n}$.
The differential of $\pi_{0}$ is then given by

$$
\left\{\begin{aligned}
d \pi_{0}\left(X_{k}\right) F(z) & =\frac{1}{2} i z_{k} F(z)+\lambda i \frac{\partial F}{\partial z_{k}} \\
d \pi_{0}\left(Y_{k}\right) F(z) & =\frac{1}{2} z_{k} F(z)-\lambda \frac{\partial F}{\partial z_{k}} \\
d \pi_{0}(\tilde{Z}) F(z) & =i \lambda F(z)
\end{aligned}\right.
$$

As in [35, Section 6] or [27, Section 1.3] we can verify by using the previous formulas for $d \pi_{0}$ and $d \sigma_{0}$ that the Segal-Bargmann transform $B_{0}: L^{2}\left(\mathbb{R}^{n}\right) \rightarrow$ $\mathcal{F}_{0}$ defined by

$$
B_{0}(f)(z)=(\lambda / \pi)^{n / 4} \int_{\mathbb{R}^{n}} e^{(1 / 4 \lambda) z^{2}+i x z-(\lambda / 2) x^{2}} f(x) d x
$$

is a (unitary) intertwining operator between $\sigma_{0}$ and $\pi_{0}$. The inverse SegalBargmann transform $B_{0}^{-1}=B_{0}^{*}$ is then given by

$$
B_{0}^{-1}(F)(x)=(\lambda / \pi)^{n / 4} \int_{\mathbb{C}^{n}} e^{(1 / 4 \lambda) \bar{z}^{2}-i x \bar{z}-(\lambda / 2) x^{2}} F(z) e^{-|z|^{2} / 2 \lambda} d \mu_{\lambda}(z)
$$

For $z \in \mathbb{C}^{n}$, consider the coherent state $e_{z}(w)=\exp (\bar{z} w / 2 \lambda)$. Then we have the reproducing property $F(z)=\left\langle F, e_{z}\right\rangle_{\mathcal{F}_{0}}$ for each $F \in \mathcal{F}_{0}$ where $\langle\cdot, \cdot\rangle_{\mathcal{F}_{0}}$ denotes the scalar product on $\mathcal{F}_{0}$.

Now, we introduce the Berezin quantization map and we review some of its properties. Let $\mathcal{C}_{0}$ be the space of all operators (not necessarily bounded) $A$ on $\mathcal{F}_{0}$ whose domain contains $e_{z}$ for each $z \in \mathbb{C}^{n}$. Then the Berezin symbol of $A \in \mathcal{C}_{0}$ is the function $S^{0}(A)$ defined on $\mathbb{C}^{n}$ by

$$
S^{0}(A)(z):=\frac{\left\langle A e_{z}, e_{z}\right\rangle_{\mathcal{F}_{0}}}{\left\langle e_{z}, e_{z}\right\rangle_{\mathcal{F}_{0}}} .
$$

We have the following result, see for instance [22].
Proposition 2.1. 1. Each $A \in \mathcal{C}_{0}$ is determined by $S^{0}(A)$;
2. For each $A \in \mathcal{C}_{0}$ and each $z \in \mathbb{C}^{n}$, we have $S^{0}\left(A^{*}\right)(z)=\overline{S^{0}(A)(z)}$;
3. For each $z \in \mathbb{C}^{n}$, we have $S^{0}\left(I_{\mathcal{F}_{0}}\right)(z)=1$. Here $I_{\mathcal{F}_{0}}$ denotes the identity operator of $\mathcal{F}_{0}$;
4. For each $A \in \mathcal{C}_{0}, g \in G_{0}$ and $z \in \mathbb{C}^{n}$, we have $\pi_{0}(g)^{-1} A \pi_{0}(g) \in \mathcal{C}_{0}$ and

$$
S^{0}(A)(g \cdot z)=S^{0}\left(\pi_{0}(g)^{-1} A \pi_{0}(g)\right)(z)
$$

5. The map $S^{0}$ is a bounded operator from $\mathcal{L}_{2}\left(\mathcal{F}_{0}\right)$ (endowed with the HilbertSchmidt norm) to $L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ which is one-to-one and has dense range.
Proof. For 1 and 2, see [10] and [25]. Note that 4 follows from the following property: For each $g \in G_{0}$ and each $z \in \mathbb{C}^{n}$, we have $\pi_{0}(g) e_{z}=\overline{\alpha(g, z)} e_{g \cdot z}$, see [20]. Finally, 5 is a particular case of [52, Proposition 1.19].

Recall that the Berezin transform is then the operator $\mathcal{B}^{0}$ on $L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ defined by $\mathcal{B}^{0}=S^{0}\left(S^{0}\right)^{*}$. Thus we have the integral formula

$$
\mathcal{B}^{0}(F)(z)=\int_{\mathbb{C}^{n}} F(w) e^{|z-w|^{2} / 2 \lambda} d \mu_{\lambda}(w),
$$

see $[10,11,48,52]$ for instance. Recall also that we have $\mathcal{B}^{0}=\exp (\lambda \Delta / 2)$ where $\Delta=4 \sum_{k=1}^{n} \partial^{2} / \partial z_{k} \partial \bar{z}_{k}$, see [44, 52].

Note that Berezin transforms have been studied, in the general setting, by many authors, see in particular [28, 47, 48, 52, 56].

Note also that $S^{0}$ allows us to connect $\pi_{0}$ to $\mathcal{O}_{\lambda}$ as shown by the following proposition. Here we denote by $\mathfrak{g}_{0}^{c}$ the complexification of $\mathfrak{g}_{0}$.

Proposition 2.2 ([22]). Let $\Phi_{\lambda}$ be the map defined by

$$
\Phi_{\lambda}(z):=\sum_{k=1}^{n}\left(\operatorname{Re} z_{k} X_{k}^{*}+\operatorname{Im} z_{k} Y_{k}^{*}\right)+\lambda \tilde{Z}^{*}
$$

Then

1. For each $X \in \mathfrak{g}_{0}^{c}$ and each $z \in \mathbb{C}^{n}$, we have

$$
S^{0}\left(d \pi_{0}(X)\right)(z)=i\left\langle\Phi_{\lambda}(z), X\right\rangle
$$

2. For each $g \in G_{0}$ and each $z \in \mathbb{C}^{n}$, we have $\Phi_{\lambda}(g \cdot z)=\operatorname{Ad}^{*}(g) \Phi_{\lambda}(z)$.
3. The map $\Phi_{\lambda}$ is a diffeomorphism from $\mathbb{C}^{n}$ onto $\mathcal{O}_{\lambda}$.

Now we aim to transfer $S^{0}$ to operators on $L^{2}\left(\mathbb{R}^{n}\right)$. To this goal, we define $S^{1}(A):=S^{0}\left(B_{0} A B_{0}^{-1}\right)$ for $A$ operator on $L^{2}\left(\mathbb{R}^{n}\right)$. Of course, the properties of $S^{0}$ give rise to similar properties of $S^{1}$. In particular, $S^{1}$ is a bounded operator from $\mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ to $L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ and $S^{1}$ is $G_{0}$-covariant with respect to $\sigma_{0}$.

Moreover, denoting by $I_{B_{0}}$ the (unitary) map from $\mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}\right)\right)$ onto $\mathcal{L}_{2}\left(\mathcal{F}_{0}\right)$ defined by $I_{B_{0}}(A)=B_{0} A B_{0}^{-1}$, we have $S^{1}=S^{0} I_{B_{0}}$ then

$$
S^{1}\left(S^{1}\right)^{*}=\left(S^{0} I_{B_{0}}\right)\left(S^{0} I_{B_{0}}\right)^{*}=S^{0} I_{B_{0}} I_{B_{0}}^{*}\left(S^{0}\right)^{*}=S^{0}\left(S^{0}\right)^{*}=\mathcal{B}^{0}
$$

This shows that the Berezin transform corresponding to $S^{1}$ is the same as the Berezin transform corresponding to $S^{0}$. Then we can write the polar decompositions of $S^{0}$ and $S^{1}$ as $S^{0}=\left(\mathcal{B}^{0}\right)^{1 / 2} U^{0}$ and $S^{1}=\left(\mathcal{B}^{0}\right)^{1 / 2} U^{1}$ where the maps $U^{0}: \mathcal{L}_{2}\left(\mathcal{F}_{0}\right) \rightarrow L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ and $U^{1}: \mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}\right)\right) \rightarrow L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ are unitary.

Moreover, as in the proof of [17], Proposition 3.1, we can verify that $U^{0}$ is a Stratonovich-Weyl correspondence for $\left(G_{0}, \pi_{0}, \mathbb{C}^{n}\right)$ and that $U^{1}$ is a Strato-novich-Weyl correspondence for $\left(G_{0}, \sigma_{0}, \mathbb{C}^{n}\right)$. Note that $G_{0}$-covariance of $U^{0}$ and $U^{1}$ immediately follows from $G_{0}$-covariance of $S^{0}$ and $S^{1}$. Note also that we have $U^{1}=U^{0} I_{B_{0}}$.

Now, we show how to use the usual Weyl correspondence in order to get another Stratonovich-Weyl correspondence for $\sigma_{0}$. The Weyl correspondence on $\mathbb{R}^{2 n}$ is defined as follows. For each $f$ in the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2 n}\right)$, let $W_{0}(f)$ be the operator on $L^{2}\left(\mathbb{R}^{n}\right)$ defined by

$$
W_{0}(f) \phi(p)=(2 \pi)^{-n} \int_{\mathbb{R}^{2 n}} e^{i s q} f(p+(1 / 2) s, q) \phi(p+s) d s d q
$$

The Weyl calculus can be extended to much larger classes of symbols (see for instance [38]). In particular, if $f(p, q)=u(p) q^{\alpha}$ where $u \in C^{\infty}\left(\mathbb{R}^{n}\right)$ then we have, see [53],

$$
W_{0}(f) \varphi(p)=\left.\left(i \frac{\partial}{\partial s}\right)^{\alpha}(u(p+(1 / 2) s) \phi(p+s))\right|_{s=0} .
$$

From this, we can deduce the following proposition. Consider the action of $G_{0}$ on $\mathbb{R}^{2 n}$ given by $g \cdot(p, q):=(p+a, q+\lambda b)$ where $g=[a, b, c]$.

Proposition 2.3 ([22]). Let $\Psi_{\lambda}$ be the map defined by

$$
\Psi_{\lambda}(p, q):=\sum_{k=1}^{n}\left(q_{k} X_{k}^{*}-\lambda p_{k} Y_{k}^{*}\right)+\lambda \tilde{Z}^{*} .
$$

Then

1. For each $X \in \mathfrak{g}_{0}^{c}$ and each $(p, q) \in \mathbb{R}^{2 n}$, we have

$$
W_{0}^{-1}\left(d \sigma_{0}(X)\right)(p, q)=i\left\langle\Psi_{\lambda}(p, q), X\right\rangle
$$

2. For each $g \in G_{0}$ and $(p, q) \in \mathbb{R}^{2 n}$, we have $\Psi_{\lambda}(g \cdot(p, q))=\operatorname{Ad}^{*}(g) \Psi_{\lambda}(p, q)$.
3. The map $\Psi_{\lambda}$ is a diffeomorphism from $\mathbb{R}^{2 n}$ onto $\mathcal{O}_{\lambda}$.
4. For each $(p, q) \in \mathbb{R}^{2 n}$, we have $\Phi_{\lambda}(q-\lambda p i)=\Psi_{\lambda}(p, q)$.

Now, we assume that $\mathbb{R}^{2 n}$ is equipped with the $G_{0}$-invariant measure $\tilde{\mu}:=$ $(2 \pi)^{-n} d p d q$. Then one has the following result.

Proposition 2.4 ([22, 30]). The map $W_{0}^{-1}$ is a Stratonovich-Weyl correspondence for $\left(G_{0}, \sigma_{0}, \mathbb{R}^{2 n}\right)$.

The following proposition asserts that if we identify $\mathbb{R}^{2 n}$ with $\mathbb{C}^{n}$ by the map $j:(p, q) \rightarrow q-\lambda p i$ then the unitary part in the polar decomposition of $S^{1}$ coincides with the inverse of the Weyl transform, see [44] and [48].

Proposition 2.5. Let $J$ be the map from $L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ onto $L^{2}\left(\mathbb{R}^{2 n}\right)$ defined by $J(F)=F \circ j$. Then we have $U^{1}=\left(W_{0} J\right)^{-1}$.

Finally, note that we can obtain Stratonovich-Weyl correspondences for $\left(G_{0}, \sigma_{0}, \mathcal{O}_{\lambda}\right)$ and $\left(G_{0}, \pi_{0}, \mathcal{O}_{\lambda}\right)$ by transferring $W_{0}^{-1}$ and $U^{0}$ by using $\Phi_{\lambda}$ and $\Psi_{\lambda}$. More precisely, let $\nu_{\lambda}$ be the $G_{0}$-invariant measure on $\mathcal{O}_{\lambda}$ defined by $\nu_{\lambda}:=\left(\Phi_{\lambda}^{-1}\right)^{*}\left(\mu_{\lambda}\right)=\left(\Psi_{\lambda}^{-1}\right)^{*}(\tilde{\mu})$. Then the maps $\tau_{\Phi_{\lambda}}: F \rightarrow F \circ \Phi_{\lambda}^{-1}$ from $L^{2}\left(\mathbb{C}^{n}, \mu_{\lambda}\right)$ onto $L^{2}\left(\mathcal{O}_{\lambda}, \nu_{\lambda}\right)$ and $\tau_{\Psi_{\lambda}}: F \rightarrow F \circ \Psi_{\lambda}^{-1}$ from $L^{2}\left(\mathbb{R}^{2 n}\right)$ onto $L^{2}\left(\mathcal{O}_{\lambda}, \nu_{\lambda}\right)$ are unitary and we have $\tau_{\Phi_{\lambda}}=\tau_{\Psi_{\lambda}} J$. Hence we can assert the following proposition.

Proposition 2.6. The map $\mathcal{W}_{1}:=\tau_{\Psi_{\lambda}} W_{0}^{-1}$ is a Stratonovich-Weyl correspondence for $\left(G_{0}, \sigma_{0}, \mathcal{O}_{\lambda}\right)$, the map $\mathcal{W}_{2}:=\tau_{\Phi_{\lambda}} U^{0}$ is a Stratonovich-Weyl correspondence for $\left(G_{0}, \pi_{0}, \mathcal{O}_{\lambda}\right)$ and we have $\mathcal{W}_{1}=\mathcal{W}_{2} I_{B_{0}}$.

## 3. Generalities on Heisenberg motion groups

In order to introduce the Heisenberg motion groups, it is convenient to write the elements of the Heisenberg group $G_{0}$ and its multiplication law as follows. For each $z \in \mathbb{C}^{n}, c \in \mathbb{R}$, we denote here by $(z, \bar{z}, c)$ the element $G_{0}$ which is denoted by $[\operatorname{Re} z, \operatorname{Im} z, c]$ in Section 2. Moreover, for each $z, w \in \mathbb{C}^{n}$, we denote $z w:=\sum_{k=1}^{n} z_{k} w_{k}$ and we consider the symplectic form $\omega$ on $\mathbb{C}^{2 n}$ defined by

$$
\omega\left((z, w),\left(z^{\prime}, w^{\prime}\right)\right)=\frac{i}{2}\left(z w^{\prime}-z^{\prime} w\right)
$$

for $z, w, z^{\prime}, w^{\prime} \in \mathbb{C}^{n}$. Then the multiplication of $G_{0}$ is given by

$$
\begin{equation*}
((z, \bar{z}), c) \cdot\left(\left(z^{\prime}, \bar{z}^{\prime}\right), c^{\prime}\right)=\left(\left(z+z^{\prime}, \bar{z}+\bar{z}^{\prime}\right), c+c^{\prime}+\frac{1}{2} \omega\left((z, \bar{z}),\left(z^{\prime}, \bar{z}^{\prime}\right)\right)\right) \tag{1}
\end{equation*}
$$

the complexification $G_{0}^{c}$ of $G_{0}$ is $G_{0}^{c}=\left\{((z, w), c): z, w \in \mathbb{C}^{n}, c \in \mathbb{C}\right\}$ and the multiplication of $G_{0}^{c}$ is obtained by replacing $(z, \bar{z})$ by $(z, w)$ and $\left(z^{\prime}, \bar{z}^{\prime}\right)$ by $\left(z^{\prime}, w^{\prime}\right)$ in Eq. 1.

Now, let $K$ be a closed subgroup of $U(n)$. Then $K$ acts on $G_{0}$ by $k$. $((z, \bar{z}), c)=((k z, \overline{k z}), c)$ and we can form the semidirect product $G:=G_{0} \rtimes K$ which is called a Heisenberg motion group. The elements of $G$ can be written as $((z, \bar{z}), c, k)$ where $z \in \mathbb{C}^{n}, c \in \mathbb{R}$ and $k \in K$. The multiplication of $G$ is then given by

$$
\begin{aligned}
&((z, \bar{z}), c, k) \cdot\left(\left(z^{\prime}, \bar{z}^{\prime}\right), c^{\prime}, k^{\prime}\right)= \\
& \quad\left((z, \bar{z})+\left(k z^{\prime}, k \bar{z}^{\prime}\right), c+c^{\prime}+\frac{1}{2} \omega\left((z, \bar{z}),\left(k z^{\prime}, k \bar{z}^{\prime}\right)\right), k k^{\prime}\right) .
\end{aligned}
$$

We denote by $K^{c}$ the complexification of $K$ and we consider the action of $K^{c}$ on $\mathbb{C}^{n} \times \mathbb{C}^{n}$ given by $k \cdot(z, w)=\left(k z,\left(k^{t}\right)^{-1} w\right)$ (here, the subscript $t$ denotes transposition). The group $G^{c}$ is then the semidirect product $G^{c}=G_{0}^{c} \rtimes K^{c}$. The elements of $G^{c}$ can be written as $((z, w), c, k)$ where $z, w \in \mathbb{C}^{n}, c \in \mathbb{C}$ and $k \in K^{c}$ and the multiplication law of $G^{c}$ is given by

$$
\begin{aligned}
& ((z, w), c, k) \cdot\left(\left(z^{\prime}, w^{\prime}\right), c^{\prime}, k^{\prime}\right)= \\
& \quad\left((z, w)+k \cdot\left(z^{\prime}, w^{\prime}\right), c+c^{\prime}+\frac{1}{2} \omega\left((z, w), k \cdot\left(z^{\prime}, w^{\prime}\right)\right), k k^{\prime}\right) .
\end{aligned}
$$

We denote by $\mathfrak{k}, \mathfrak{k}^{c}, \mathfrak{g}$ and $\mathfrak{g}^{c}$ the Lie algebras of $K, K^{c}, G$ and $G^{c}$. The derived action of $\mathfrak{k}^{c}$ on $\mathbb{C}^{n} \times \mathbb{C}^{n}$ is then $A \cdot(z, w):=\left(A z,-A^{t} w\right)$ and the Lie brackets of $\mathfrak{g}^{c}$ are given by

$$
\begin{aligned}
& {\left[((z, w), c, A),\left(\left(z^{\prime}, w^{\prime}\right), c^{\prime}, A^{\prime}\right)\right]=} \\
& \quad\left(A \cdot\left(z^{\prime}, w^{\prime}\right)-A^{\prime} \cdot(z, w), \omega\left((z, w),\left(z^{\prime}, w^{\prime}\right)\right),\left[A, A^{\prime}\right]\right) .
\end{aligned}
$$

Let $\tilde{K}$ be the subgroup of $G$ defined by $\tilde{K}:=\{((0,0), c, k): c \in \mathbb{R}, k \in K\}$. Also, let $\mathfrak{h}_{0}$ be a Cartan subalgebra of $\mathfrak{k}$. Then the Lie algebra $\tilde{\mathfrak{k}}$ of $\tilde{K}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consisting of all elements of the form $((0,0), c, A)$ where $c \in \mathbb{R}$ and $A \in \mathfrak{h}_{0}$ is a compactly embedded Cartan subalgebra of $\mathfrak{g}$ [46], p. 250.

Following [46, Chapter XII.1], we set $\mathfrak{p}^{+}=\left\{((z, 0), 0,0): z \in \mathbb{C}^{n}\right\}$ and $\mathfrak{p}^{-}=\left\{((0, w), 0,0): w \in \mathbb{C}^{n}\right\}$ and we denote by $P^{+}$and $P^{-}$the corresponding analytic subgroups of $G^{c}$, that is, $P^{+}=\left\{\left((z, 0), 0, I_{n}\right): z \in \mathbb{C}^{n}\right\}$ and $P^{-}=$ $\left\{\left((0, w), 0, I_{n}\right): w \in \mathbb{C}^{n}\right\}$.

Note that $G$ is a group of the Harish-Chandra type [46, p. 507] (see also [50] and [37, Chapter VIII]), that is, the following properties are satisfied:

1. $\mathfrak{g}^{c}=\mathfrak{p}^{+} \oplus \tilde{\mathfrak{k}}^{c} \oplus \mathfrak{p}^{-}$is a direct sum of vector spaces, $\left(\mathfrak{p}^{+}\right)^{*}=\mathfrak{p}^{-}$and $\left[\tilde{\mathfrak{k}}^{c}, \mathfrak{p}^{ \pm}\right] \subset \mathfrak{p}^{ \pm} ;$
2. The multiplication map $P^{+} \tilde{K}^{c} P^{-} \rightarrow G^{c},(z, k, y) \rightarrow z k y$ is a biholomorphic diffeomorphism onto its open image;
3. $G \subset P^{+} \tilde{K}^{c} P^{-}$and $G \cap \tilde{K}^{c} P^{-}=\tilde{K}$.

We denote by $p_{\mathfrak{p}^{+}}, p_{\tilde{\mathfrak{k}}^{c}}$ and $p_{\mathfrak{p}^{-}}$the projections of $\mathfrak{g}^{c}$ onto $\mathfrak{p}^{+}, \tilde{\mathfrak{k}}^{c}$ and $\mathfrak{p}^{-}$ associated with the above direct decomposition.

We can easily verify that each $g=\left(\left(z_{0}, w_{0}\right), c_{0}, k\right) \in G^{c}$ has a $P^{+} \tilde{K}^{c} P^{-}$decomposition given by

$$
g=\left(\left(z_{0}, 0\right), 0, I_{n}\right) \cdot((0,0), c, k) \cdot\left(\left(0, w_{0}\right), 0, I_{n}\right)
$$

where $c=c_{0}-\frac{i}{4} z_{0} w_{0}$. We denote by $\zeta: P^{+} \tilde{K}^{c} P^{-} \rightarrow P^{+}, \kappa: P^{+} \tilde{K}^{c} P^{-} \rightarrow K^{c}$ and $\eta: P^{+} \tilde{K}^{c} P^{-} \rightarrow P^{-}$the projections onto $P^{+}, \tilde{K}^{c}$ - and $P^{-}$-components.

We can introduce an action (defined almost everywhere) of $G$ on $\mathfrak{p}^{+}$as follows. For $Z \in \mathfrak{p}^{+}$and $g \in G^{c}$, we define $g \cdot Z \in \mathfrak{p}^{+}$by $g \cdot Z:=\log \zeta(g \exp Z)$. From the above formula for the $P^{+} \tilde{K}^{c} P^{-}$-decomposition, we deduce that if $g=\left(\left(z_{0}, w_{0}\right), c_{0}, k\right) \in G$ and $Z=((z, 0), 0,0) \in \mathfrak{p}^{+}$then we have $g \cdot Z=$ $\log \zeta(g \exp Z)=\left(\left(z_{0}+k z, 0\right), 0,0\right)$. Note that $\mathcal{D}:=G \cdot 0=\mathfrak{p}^{+} \simeq \mathbb{C}^{n}$ here.

A useful section $Z \rightarrow g_{Z}$ for the action of $G$ on $\mathcal{D}$ can be obtained by using [21, Proposition 4.5]. Here we get $g_{Z}=\left((z, \bar{z}), 0, I_{n}\right)$ for each $Z=((z, 0), 0,0)$, $z \in \mathbb{C}^{n}$.

Now we compute the adjoint and coadjoint actions of $G^{c}$. Consider $g=$ $\left(v_{0}, c_{0}, k_{0}\right) \in G^{c}$ where $v_{0} \in \mathbb{C}^{2 n}, c_{0} \in \mathbb{C}, k_{0} \in K^{c}$ and $X=(w, c, A) \in \mathfrak{g}^{c}$ where $w \in \mathbb{C}^{2 n}, c \in \mathbb{C}$ and $A \in \mathfrak{k}^{c}$. We can easily verify that

$$
\begin{aligned}
& \operatorname{Ad}(g) X=\left.\frac{d}{d t}\left(g \exp (t X) g^{-1}\right)\right|_{t=0}=\left(k_{0} w-\left(\operatorname{Ad}\left(k_{0}\right) A\right) \cdot v_{0}, c\right. \\
&\left.+\omega\left(v_{0}, k_{0} w\right)-\frac{1}{2} \omega\left(v_{0},\left(\operatorname{Ad}\left(k_{0}\right) A\right) \cdot v_{0}\right), \operatorname{Ad}\left(k_{0}\right) A\right)
\end{aligned}
$$

Now, let us denote by $\xi=(u, d, \phi)$, where $u \in \mathbb{C}^{2 n}, d \in \mathbb{C}$ and $\phi \in\left(\mathfrak{k}^{c}\right)^{*}$, the element of $\left(\mathfrak{g}^{c}\right)^{*}$ defined by

$$
\langle\xi,(w, c, A)\rangle=\omega(u, w)+d c+\langle\phi, A\rangle
$$

Also, for $u, v \in \mathbb{C}^{2 n}$, we denote by $v \times u$ the element of $\left(\mathfrak{k}^{c}\right)^{*}$ defined by $\langle v \times$ $u, A\rangle:=\omega(u, A \cdot v)$ for $A \in \mathfrak{k}^{c}$. Then, from the above formula for the adjoint action, we deduce that for each $\xi=(u, d, \phi) \in\left(\mathfrak{g}^{c}\right)^{*}$ and $g=\left(v_{0}, c_{0}, k_{0}\right) \in G^{c}$ we have

$$
\operatorname{Ad}^{*}(g) \xi=\left(k_{0} u-d v_{0}, d, \operatorname{Ad}^{*}\left(k_{0}\right) \phi+v_{0} \times\left(k_{0} u-\frac{d}{2} v_{0}\right)\right)
$$

By restriction, we also get the analogous formula for the coadjoint action of $G$. From this, we see that if a coadjoint orbit of $G$ contains a point $(u, d, \phi)$ with $d \neq 0$ then it also contains a point of the form $\left(0, d, \phi_{0}\right)$. Such an orbit is called generic.

## 4. Fock model for Heisenberg motion groups

In this section, we introduce the Fock model of the unitary irreducible representations of $G$ by using the general method of [46, Chapter XII] that we describe here briefly.

Let $\rho$ be a unitary irreducible representation of $K$ on a (finite-dimensional) Hilbert space $V$ and $\lambda \in \mathbb{R}$. Let $\tilde{\rho}$ be the representation of $\tilde{K}$ on $V$ defined by $\tilde{\rho}((0,0), c, k)=e^{i \lambda c} \rho(k)$ for each $c \in \mathbb{R}$ and $k \in K$.

For each $Z, W \in \mathcal{D}$, let $K(Z, W):=\tilde{\rho}\left(\kappa\left(\exp W^{*} \exp Z\right)\right)^{-1}$ and for each $g \in G, Z \in \mathcal{D}$, let $J(g, Z):=\tilde{\rho}(\kappa(g \exp Z))$, [46, Chapter XII.1]. Consider the Hilbert space $\tilde{\mathcal{F}}$ of all holomorphic functions on $\mathcal{D}$ with values in $V$ such that

$$
\|f\|_{\tilde{\mathcal{F}}}^{2}:=\int_{\mathcal{D}}\left\langle K(Z, Z)^{-1} f(Z), f(Z)\right\rangle_{V} d \mu(Z)<+\infty
$$

where $\mu$ denotes an invariant $G$-measure on $\mathcal{D}$. Then the equation

$$
\tilde{\pi}(g) f(Z)=J\left(g^{-1}, Z\right)^{-1} f\left(g^{-1} \cdot Z\right)
$$

defines a unitary representation of $G$ on $\tilde{\mathcal{F}}$. This representation can be also obtained by holomorphic induction from $\tilde{\rho}$, that is, it corresponds to the natural action of $G$ on the square-integrable holomorphic sections of the Hilbert $G$ bundle $G \times_{\tilde{\rho}} V$ over $G / K \cong \mathcal{D}$ [22]. Note also that $\tilde{\pi}$ is irreducible since $\tilde{\rho}$ is irreducible, [46, p. 515].

Here we can easily compute $K$ and $J$. For each $Z=((z, 0), 0,0), W=$ $((w, 0), 0,0) \in \mathcal{D}$, we have $K(Z, W)=e^{\lambda z \bar{w} / 2} I_{V}$. Moreover, for each $g=$ $\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$ and $Z=((z, 0), 0,0) \in \mathcal{D}$, we have

$$
J(g, Z)=\exp \left(i \lambda c_{0}+\frac{\lambda}{2} \bar{z}_{0}(k z)+\frac{\lambda}{4}\left|z_{0}\right|^{2}\right) \rho(k)
$$

Note that $\mu$ can be taken to be the $G$-invariant measure on $\mathcal{D} \simeq \mathbb{C}^{n}$ defined by $d \mu(Z):=\lambda^{n}(2 \pi)^{-n} d x d y$. Here $Z=((z, 0), 0,0)$ and $z=x+i y$ with $x$ and $y$ in $\mathbb{R}^{n}$. From now on, we identify $Z=((z, 0), 0,0) \in \mathcal{D}$ with $z \in \mathbb{C}^{n}$ and each function on $\mathcal{D}$ with the corresponding function on $\mathbb{C}^{n}$.

Consequently, the Hilbert product on $\tilde{\mathcal{F}}$ is given by

$$
\langle f, g\rangle_{\tilde{\mathcal{F}}}=\int_{\mathbb{C}^{n}}\langle f(z), g(z)\rangle_{V} e^{-\lambda|z|^{2} / 2}\left(\frac{\lambda}{2 \pi}\right)^{n} d x d y
$$

and we get the following formula for $\tilde{\pi}$ :

$$
(\tilde{\pi}(g) f)(z)=\exp \left(i \lambda c_{0}+\frac{\lambda}{2} \bar{z}_{0} z-\frac{\lambda}{4}\left|z_{0}\right|^{2}\right) \rho(k) f\left(k^{-1}\left(z-z_{0}\right)\right)
$$

where $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$ and $z \in \mathbb{C}^{n}$.
In fact, in order to use the results of Section 2, it is convenient to replace $\tilde{\pi}$ by an equivalent representation $\pi$ whose restriction to $G_{0}$ is precisely $\pi_{0}$. To this aim, we consider the Fock space $\mathcal{F}$ of all holomorphic functions $f: \mathbb{C}^{n} \rightarrow V$ such that

$$
\|f\|_{\mathcal{F}}^{2}:=\int_{\mathbb{C}^{n}}\|f(z)\|_{V}^{2} e^{-|z|^{2} / 2 \lambda} d \mu_{\lambda}(z)<+\infty
$$

Let $\mathcal{J}: \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ be the unitary operator defined by $\mathcal{J}(f)(z)=f\left(i \lambda^{-1} z\right)$ and set $\pi(g):=\mathcal{J} \tilde{\pi}(g) \mathcal{J}^{-1}$ for each $g \in G$. Then we have

$$
(\pi(g) f)(z)=\exp \left(i \lambda c_{0}+\frac{1}{2} i \bar{z}_{0} z-\frac{\lambda}{4}\left|z_{0}\right|^{2}\right) \rho(k) f\left(k^{-1}\left(z+i \lambda z_{0}\right)\right)
$$

where $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$ and $z \in \mathbb{C}^{n}$.
We can easily compute the differential of $\pi$ :
Proposition 4.1. Let $X=((a, \bar{a}), c, A) \in \mathfrak{g}$. Then, for each $f \in \mathcal{F}$ and each $z \in \mathbb{C}^{n}$, we have

$$
(d \pi(X) f)(z)=d \rho(A) f(z)+i\left(\lambda c+\frac{1}{2} \bar{a} z\right) f(z)+d f_{z}(-A z+i \lambda a)
$$

Clearly, one has $\mathcal{F}=\mathcal{F}_{0} \otimes V$. For $f_{0} \in \mathcal{F}_{0}$ and $v \in V$, we denote by $f_{0} \otimes v$ the function $z \rightarrow f_{0}(z) v$. Moreover, if $A_{0}$ is an operator of $\mathcal{F}_{0}$ and $A_{1}$ is an operator of $V$ then we denote by $A_{0} \otimes A_{1}$ the operator of $\mathcal{F}$ defined by $\left(A_{0} \otimes A_{1}\right)\left(f_{0} \otimes v\right)=A_{0} f_{0} \otimes A_{1} v$.

Let $\tau$ be the left-regular representation of $K$ on $\mathcal{F}_{0}$, that is, $\left(\tau(k) f_{0}\right)(z)=$ $f_{0}\left(k^{-1} z\right)$. Then we have

$$
\begin{equation*}
\pi\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right)=\pi_{0}\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}\right) \tau(k) \otimes \rho(k) \tag{2}
\end{equation*}
$$

for each $z_{0} \in \mathbb{C}^{n}, c_{0} \in \mathbb{R}$ and $k \in K$. Note that this is precisely Formula (3.18) in [8].

## 5. Stratonovich-Weyl correspondence via Berezin quantization

In this section, we introduce the Berezin quantization map associated with $\pi$ and the corresponding Stratonovich-Weyl correspondence. We consider first the Berezin quantization map associated with $\rho[5,15,55]$.

Let us fix a positive root system of $\mathfrak{k}$ relative to $\mathfrak{h}_{0}$ and denote by $\Lambda \in\left(\mathfrak{h}_{0}^{c}\right)^{*}$ the highest weight of $\rho$ and by $\mathfrak{k}^{c}=\mathfrak{n}^{+} \oplus \mathfrak{h}_{0}^{c} \oplus \mathfrak{n}^{-}$the corresponding triangular decomposition of $\mathfrak{k}^{c}$. Let $\tilde{\varphi_{0}}$ be the element of $\left(\mathfrak{k}^{c}\right)^{*}$ defined by $\tilde{\varphi_{0}}=-i \Lambda$ on $\mathfrak{h}_{0}$ and by $\tilde{\varphi_{0}}=0$ on $\mathfrak{n}^{ \pm}$. We denote by $\varphi_{0}$ the restriction of $\tilde{\varphi_{0}}$ to $\mathfrak{k}$. Then the orbit $o\left(\varphi_{0}\right)$ of $\varphi_{0}$ under the coadjoint action of $K$ is said to be associated with $\rho[14,55]$.

Here we assume that $\varphi_{0}$ is regular in the sense that the stabilizer of $\varphi_{0}$ for the coadjoint action of $K$ is precisely the connected subgroup $H_{0}$ of $K$ with Lie algebra $\mathfrak{h}_{0}$ [15].

Note that a complex structure on $o\left(\varphi_{0}\right)$ is then defined by the diffeomorphism $o\left(\varphi_{0}\right) \simeq K / H_{0} \simeq K^{c} / H_{0}^{c} N^{-}$where $H_{0}$ is the connected subgroup of $K$ with Lie algebra $\mathfrak{h}_{0}$ and $N^{-}$is the analytic subgroup of $K^{c}$ with Lie algebra $\mathfrak{n}^{-}$.

Without loss of generality, we can assume that $V$ is a space of holomorphic sections of a complex line bundle over $o\left(\varphi_{0}\right)$ as in [15]. For each $\varphi \in o\left(\varphi_{0}\right)$ there exists a unique function $e_{\varphi} \in V$ (a coherent state) such that $a(\varphi)=\left\langle a, e_{\varphi}\right\rangle_{V}$ for each $a \in V$. The Berezin calculus on $o\left(\varphi_{0}\right)$ associates with each operator $B$ on $V$ the complex-valued function $s(B)$ on $o\left(\varphi_{0}\right)$ defined by

$$
s(B)(\varphi)=\frac{\left\langle B e_{\varphi}, e_{\varphi}\right\rangle_{V}}{\left\langle e_{\varphi}, e_{\varphi}\right\rangle_{V}}
$$

which is called the symbol of $B$. We denote by $S y\left(o\left(\varphi_{0}\right)\right)$ the space of all such symbols. Then we have the following proposition, see [5, 15, 25].

Proposition 5.1. 1. The map $B \rightarrow s(B)$ is injective.
2. For each operator $B$ on $V$, we have $s\left(B^{*}\right)=\overline{s(B)}$.
3. For each $\varphi \in o\left(\varphi_{0}\right), k \in K$ and $B \in \operatorname{End}(V)$, we have

$$
s(B)\left(\operatorname{Ad}^{*}(k) \varphi\right)=s\left(\rho(k)^{-1} B \rho(k)\right)(\varphi)
$$

4. For each $A \in \mathfrak{k}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have $s(d \rho(A))(\varphi)=i\langle\varphi, A\rangle$.

In our papers [18, 19, 23], we developped a general method for constructing a Berezin quantization map associated with a unitary representation of a quasi-Hermitian Lie group which is holomorphically induced from a unitary irreducible representation of a maximal compactly embedded subgroup. This construction goes as follows.

The evaluation maps $K_{z}: \mathcal{H} \rightarrow V, f \rightarrow f(z)$ are continuous [46], p. 539. The vector coherent states of $\mathcal{F}$ are the maps $E_{z}=K_{z}^{*}: V \rightarrow \mathcal{F}$ defined by $\langle f(z), v\rangle_{V}=\left\langle f, E_{z} v\right\rangle_{\mathcal{F}}$ for $f \in \mathcal{F}$ and $v \in V$. Here we have that $E_{z} v=e_{z} \otimes v$, that is, we have $\left(E_{z} v\right)(w)=e^{\lambda \bar{z} w / 2} v$.

Let $\mathcal{F}^{s}$ be the subspace of $\mathcal{F}$ generated by the functions $e_{z} \otimes v$ for $z \in \mathbb{C}^{n}$ and $v \in V$. Then $\mathcal{F}^{s}$ is a dense subspace of $\mathcal{F}$. Let $\mathcal{C}$ be the space consisting of all operators $A$ on $\mathcal{F}$ such that the domain of $A$ contains $\mathcal{F}^{s}$ and the domain of $A^{*}$ also contains $\mathcal{F}^{s}$. Then, following an idea of [40] and [2], we first introduce the pre-symbol $S_{0}(A)$ of $A \in \mathcal{C}$ by

$$
S_{0}(A)(z)=\left(E_{z}^{*} E_{z}\right)^{-1 / 2} E_{z}^{*} A E_{z}\left(E_{z}^{*} E_{z}\right)^{-1 / 2}=e^{-\lambda z \bar{z} / 2} E_{z}^{*} A E_{z}
$$

The Berezin symbol $S(A)$ of $A$ is thus defined as the complex-valued function on $\mathbb{C}^{n} \times o\left(\varphi_{0}\right)$ given by

$$
S(A)(z, \varphi)=s\left(S_{0}(A)(z)\right)(\varphi)
$$

By applying [23, Proposition 4.4] we can see that $S$ has the following properties.

Proposition 5.2. 1. Each $A \in \mathcal{C}$ is determined by $S(A)$.
2. For each $A \in \mathcal{C}$, we have $S\left(A^{*}\right)=\overline{S(A)}$.
3. We have $S\left(I_{\mathcal{F}}\right)=1$.
4. For each $A \in \mathcal{C}, g=\left(\left(z_{0}, \bar{z}_{0}\right), c, k\right) \in G, z \in \mathbb{C}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
S(A)(g \cdot z, \varphi)=S\left(\pi(g)^{-1} A \pi(g)\right)\left(z, \operatorname{Ad}^{*}\left(k^{-1}\right) \varphi\right)
$$

Moreover, we can decompose $S$ according to the decomposition $\mathcal{F}=\mathcal{F}_{0} \otimes V$. Let $f_{0}$ be a complex-valued function on $\mathbb{C}^{n}$ and $f_{1}$ be a complex-valued function on $o\left(\varphi_{0}\right)$. Then we denote by $f_{0} \otimes f_{1}$ the function on $\mathbb{C}^{n} \times o\left(\varphi_{0}\right)$ defined by $\left(f_{0} \otimes f_{1}\right)(z, \varphi)=f_{0}(z) f_{1}(\varphi)$.
Proposition 5.3. Let $A_{0} \in \mathcal{C}_{0}$ and let $A_{1}$ be an operator on $V$. Then $A_{0} \otimes A_{1} \in$ $\mathcal{C}$ and we have $S\left(A_{0} \otimes A_{1}\right)=S^{0}\left(A_{0}\right) \otimes s\left(A_{1}\right)$.

From this, we deduce the following result. We denote by $\varphi^{0}$ the restriction to $\mathfrak{g}$ of the extension of $\tilde{\varphi_{0}} \in\left(\mathfrak{k}^{c}\right)^{*}$ to $\mathfrak{g}^{c}$ which vanishes on $\mathfrak{p}^{ \pm}$. We also denote by $\mathcal{O}\left(\varphi^{0}\right)$ the orbit of $\varphi^{0}$ for the coadjoint action of $G$.
Proposition 5.4 ([23]). 1. Let $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$. For each $z \in \mathbb{C}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
\begin{aligned}
S(\pi(g))(z, \varphi)=\exp & \left(i \lambda c_{0}+\frac{1}{2} i \bar{z}_{0} z-\frac{\lambda}{4}\left|z_{0}\right|^{2}-\frac{\lambda}{2}|z|^{2}+\frac{\lambda}{2} \bar{z} k^{-1}\left(z+i \lambda z_{0}\right)\right) \\
& \times s(\rho(k))(\varphi) .
\end{aligned}
$$

2. For each $X=((a, \bar{a}), c, A) \in \mathfrak{g}, z \in \mathbb{C}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
S(d \pi(X))(z, \varphi)=i \lambda c+\frac{i}{2}\left(\bar{a} z+\lambda^{2} a \bar{z}\right)-\frac{\lambda}{2} \bar{z}(A z)+s(d \rho(A))(\varphi) .
$$

3. For each $X=((a, \bar{a}), c, A) \in \mathfrak{g}, z \in \mathbb{C}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
S(d \pi(X))(z, \varphi)=i\langle\Phi(z, \varphi), X\rangle
$$

where the map $\Phi: \mathbb{C}^{n} \times o\left(\varphi_{0}\right) \rightarrow \mathfrak{g}^{*}$ is defined by

$$
\Phi(z, \varphi)=\left(i\left(-z, \lambda^{2} \bar{z}\right), \lambda, \varphi-\frac{\lambda}{2}(z, \bar{z}) \times(z, \bar{z})\right)
$$

Moreover $\Phi$ is a diffeomorphism from $\mathbb{C}^{n} \times o\left(\varphi_{0}\right)$ onto $\mathcal{O}\left(\varphi^{0}\right)$.
Consider now the Berezin transforms $\mathcal{B}:=S S^{*}, \mathcal{B}^{0}:=S^{0}\left(S^{0}\right)^{*}, b:=s s^{*}$ and the corresponding maps $U:=\mathcal{B}^{-1 / 2} S, U^{0}:=\left(\mathcal{B}^{0}\right)^{-1 / 2} S^{0}$ and $w:=b^{-1 / 2} s$. We fix a $K$-invariant measure $\nu$ on $o\left(\varphi_{0}\right)$ and we endow $\mathbb{C}^{n} \times o\left(\varphi_{0}\right)$ with the measure $\mu_{\lambda} \otimes \nu$. Also, we consider the action of $G$ on $\mathbb{C}^{n} \times o\left(\varphi_{0}\right)$ given by

$$
g \cdot(z, \varphi):=\left(g \cdot z, \operatorname{Ad}^{*}(k) \varphi\right)
$$

for $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$. Then we have the following results.
Proposition 5.5 ([23]). The map $U$ is a Stratonovich-Weyl correspondence for $\left(G, \pi, \mathbb{C}^{n} \times o\left(\varphi_{0}\right)\right)$.
Proposition 5.6 ([23]). For each $f \in L^{2}\left(\mathbb{C}^{n} \times o\left(\varphi_{0}\right), \mu_{\lambda} \otimes \nu\right)$, we have

$$
\mathcal{B}(f)(z, \psi)=\int_{\mathbb{C}^{n} \times o\left(\varphi_{0}\right)} k_{\mathcal{B}}(z, w, \psi, \varphi) f(w, \varphi) d \mu_{\lambda}(w) d \nu(\varphi)
$$

where

$$
k_{\mathcal{B}}(z, w, \psi, \varphi):=e^{-\lambda|z-w|^{2} / 2} \frac{\left|\left\langle e_{\psi}, e_{\varphi}\right\rangle_{V}\right|^{2}}{\left\langle e_{\varphi}, e_{\varphi}\right\rangle_{V}\left\langle e_{\psi}, e_{\psi}\right\rangle_{V}} .
$$

In particular, for each $f_{0} \in L^{2}\left(\mathbb{C}^{n}\right)$ and $f_{1} \in S y\left(o\left(\varphi_{0}\right)\right)$, we have $B\left(f_{0} \otimes\right.$ $\left.f_{1}\right)=B_{0}\left(f_{0}\right) \otimes b\left(f_{1}\right)$. Moreover for each $A_{0}$ operator on $\mathcal{F}_{0}$ and $A_{1}$ operator on $V$, we have $U\left(A_{0} \otimes A_{1}\right)=U^{0}\left(A_{0}\right) \otimes w\left(A_{1}\right)$.

Note that it is well-known that if $\Delta_{0}:=4 \sum_{k=1}^{n}\left(\partial_{z_{k}} \partial_{\bar{z}_{k}}\right)$ is the Laplace operator then we have $\mathcal{B}^{0}=\exp \left(\Delta_{0} / 2 \gamma\right)$, see [44]. Thus we get

$$
U^{0}=\exp \left(-\Delta_{0} / 4 \gamma\right) S^{0}
$$

Hence, by applying Proposition 5.4 and Proposition 5.6, we obtain the following result.
Proposition $5.7([23])$. For each $X=((a, \bar{a}), c, A) \in \mathfrak{g}, z \in \mathbb{C}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
U(d \pi(X))(z, \varphi)=i c \lambda+w(d \rho(A))(\varphi)+\frac{1}{2} \operatorname{Tr}(A)+\frac{i}{2}\left(\bar{a} z+\lambda^{2} a \bar{z}\right)-\frac{\lambda}{2} \bar{z}(A z)
$$

## 6. Schrödinger model for Heisenberg motion groups

Here we introduce the Schrödinger representations of $G$ by using a SegalBargmann transform which is obtained by a slight modification of $B_{0}$. More precisely, let us define the map $B$ from $L^{2}\left(\mathbb{R}^{n}, V\right) \cong L^{2}\left(\mathbb{R}^{n}\right) \otimes V$ to $\mathcal{F} \cong \mathcal{F}_{0} \otimes V$ by $B:=B_{0} \otimes I_{V}$ or, equivalently, by the integral formula

$$
B(f)(z)=(\lambda / \pi)^{n / 4} \int_{\mathbb{R}^{n}} e^{(1 / 4 \lambda) z^{2}+i x z-(\lambda / 2) x^{2}} f(x) d x
$$

for each $f \in L^{2}\left(\mathbb{R}^{n}, V\right)$.
Now, by analogy with the case of the Heisenberg group, we define the Schrödinger representation $\sigma$ of $G$ on $L^{2}\left(\mathbb{R}^{n}, V\right)$ by $\sigma(g):=B^{-1} \pi(g) B$. Similarly, recalling that $\tau$ is the representation of $K$ on $\mathcal{F}_{0}$ given by $(\tau(k) F)(z)=$ $F\left(k^{-1} z\right)$, we define the representation $\tilde{\tau}$ of $K$ on $L^{2}\left(\mathbb{R}^{n}\right)$ by $\tilde{\tau}(k):=B_{0}^{-1} \tau(k) B_{0}$. Then we have the following proposition.

Proposition 6.1. Let $g_{0} \in G_{0}, k \in K$ and $g=\left(g_{0}, k\right) \in G$. Then we have $\sigma(g)=\sigma_{0}\left(g_{0}\right) \tilde{\tau}(k) \otimes \rho(k)$.

Proof. Let $f_{0} \in L^{2}\left(\mathbb{R}^{n}\right)$ and $v \in V$. Then by Eq. 2 we have

$$
\begin{aligned}
\sigma(g)\left(f_{0} \otimes v\right) & =\left(B_{0}^{-1} \otimes I_{V}\right)\left(\pi_{0}\left(g_{0}\right) \tau(k) \otimes \rho(k)\right)\left(B_{0} \otimes I_{V}\right)\left(f_{0} \otimes v\right) \\
& =\left(B_{0}^{-1} \pi_{0}\left(g_{0}\right) \tau(k) B_{0}\right) f_{0} \otimes \rho(k) v \\
& =\sigma_{0}\left(g_{0}\right)\left(B_{0}^{-1} \tau(k) B_{0}\right) f_{0} \otimes \rho(k) v,
\end{aligned}
$$

hence the result.
The following proposition gives an explicit expression for $d \sigma(X)$ when $X$ is of the form $((0,0), 0, A)$ where $A \in \mathfrak{k}$.
Proposition 6.2. 1. For each $A=\left(a_{k l}\right) \in \mathfrak{k}$, we have

$$
\begin{aligned}
d \tilde{\tau}(A)= & \frac{1}{2 \lambda} \sum_{k, l} a_{k l} \frac{\partial^{2}}{\partial x_{k} \partial x_{l}}+\frac{1}{2} \sum_{k, l} a_{k l}\left(x_{k} \frac{\partial}{\partial x_{l}}-x_{l} \frac{\partial}{\partial x_{k}}\right) \\
& -\frac{\lambda}{2} x(A x)+\frac{1}{2} \operatorname{Tr}(A) .
\end{aligned}
$$

2. For each $X=((0,0), 0, A)$ with $A \in \mathfrak{k}$, we have

$$
d \sigma(X)=d \tilde{\tau}(A) \otimes I_{V}+I_{\mathcal{F}_{0}} \otimes d \rho(A)
$$

where $d \tilde{\tau}(A)$ is as in 1.

Proof. In order to prove the first statement, first note that for each $A \in \mathfrak{k}$ and $F^{0} \in \mathcal{F}_{0}$ we have

$$
\left(d \tau(A) F^{0}\right)(z)=-\left(d F^{0}\right)_{z}(A z)=-\sum_{k} \frac{\partial F^{0}}{\partial z_{k}}(z)\left(e_{k}(A z)\right)
$$

To simplify the notation we denote by $k_{B_{0}}(z, x)$ the kernel of $B_{0}$, that is,

$$
k_{B_{0}}(z, x):=\left(\frac{\lambda}{\pi}\right)^{n / 4} e^{(1 / 4 \lambda) z^{2}+i x z-(\lambda / 2) x^{2}}
$$

Then, for each $f_{0} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ we have

$$
\left(d \tau(A) B_{0} f_{0}\right)(z)=-\int_{\mathbb{R}^{n}}\left(\frac{1}{2 \lambda} z(A z)+i x(A z)\right) k_{B_{0}}(z, x) f_{0}(x) d x
$$

Thus writing $z(A z)=\sum_{k, l} a_{k l} z_{k} z_{l}$ and integrating by parts, we get

$$
\begin{aligned}
& \int_{\mathbb{R}^{n}} z(A z) k_{B_{0}}(z, x) f_{0}(x) d x \\
& \quad=-\left(\frac{\lambda}{\pi}\right)^{n / 4} \sum_{k, l} a_{k l} \int_{\mathbb{R}^{n}} e^{(1 / 4 \lambda) z^{2}+i x z} \frac{\partial^{2}}{\partial x_{k} \partial x_{l}}\left(e^{-(\lambda / 2) x^{2}} f_{0}(x)\right) d x
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\int_{\mathbb{R}^{n}} i x(A z) & k_{B_{0}}(z, x) f_{0}(x) d x \\
& =-\left(\frac{\lambda}{\pi}\right)^{n / 4} \sum_{k, l} a_{k l} \int_{\mathbb{R}^{n}} e^{(1 / 4 \lambda) z^{2}+i x z} \frac{\partial}{\partial x_{l}}\left(e^{-(\lambda / 2) x^{2}} x_{k} f_{0}(x)\right) d x
\end{aligned}
$$

The first statement hence follows. The second statement is an immediate consequence of Proposition 6.1.

Note that $\sigma$ is completely determined by the fact that $\sigma\left(g_{0}, I_{n}\right)=\sigma_{0}\left(g_{0}\right) \otimes I_{V}$ and by Proposition 6.2.

## 7. Stratonovich-Weyl correspondence via Weyl calculus

In this section we first introduce a slight modification of the usual Weyl correspondence in the spirit of our previous works, see for instance [14].

Recall that the Berezin calculus $s$ associates with each operator $B$ on $V$ a complex-valued function $s(B)$ on $o\left(\varphi_{0}\right)$ which is called the symbol of $B$ and
that the space of all such symbols is denoted by $\operatorname{Sy}\left(o\left(\varphi_{0}\right)\right)$, see Section 5 . Then the unitary part $w$ of $s$ is an isomorphism from $\operatorname{End}(V)$ onto $S y\left(o\left(\varphi_{0}\right)\right)$.

Now we say that a complex-valued smooth function $f:(p, q, \varphi) \rightarrow f(p, q, \varphi)$ is a symbol on $\mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)$ if for each $(p, q) \in \mathbb{R}^{2 n}$ the function $f(p, q, \cdot)$ : $\varphi \rightarrow f(p, q, \varphi)$ is an element of $S y\left(o\left(\varphi_{0}\right)\right)$. In that case, we denote $\hat{f}(p, q):=$ $w^{-1}(f(p, q, \cdot))$. A symbol $f$ on $\mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)$ is called an $S$-symbol if the function $\hat{f}$ belongs to the Schwartz space $\mathcal{S}\left(\mathbb{R}^{2 n}, \operatorname{End}(V)\right)$ of rapidly decreasing smooth functions on $\mathbb{R}^{2 n}$ with values in $\operatorname{End}(V)$. For each $S$-symbol on $\mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)$, we define the operator $W(f)$ on the Hilbert space $L^{2}\left(\mathbb{R}^{n}, V\right)$ by

$$
W(f) \phi(p)=(2 \pi)^{-n} \int_{\mathbb{R}^{2 n}} e^{i s q} \hat{f}(p+(1 / 2) s, q) \phi(p+s) d s d q
$$

Of course, $W$ can be extended to much larger classes of symbols as the usual Weyl calculus, see Section 2. As an immediate consequence of the definition of $W$, we have the following proposition.

Proposition 7.1. 1. The map $W$ is a unitary operator from $L^{2}\left(\mathbb{R}^{2 n}, V\right)$ onto $\mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}, V\right)\right)$;
2. For each $f_{0} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ and $f_{1} \in \operatorname{Sy}\left(o\left(\varphi_{0}\right)\right)$, we have

$$
W\left(f_{0} \otimes f_{1}\right)=W_{0}\left(f_{0}\right) \otimes w^{-1}\left(f_{1}\right)
$$

In order to compare $W$ and $U$, it is convenient to transfer $U$ to operators on $L^{2}\left(\mathbb{R}^{n}, V\right)$ in the spirit of Proposition 2.5. First, for any operator $A$ on $L^{2}\left(\mathbb{R}^{n}, V\right)$, we define $S_{1}(A):=S\left(B A B^{-1}\right)$. Clearly, one has $S_{1} S_{1}^{*}=S S^{*}=\mathcal{B}$. Then the unitary part $U_{1}$ of $S_{1}$ is given by $U_{1}(A):=U\left(B A B^{-1}\right)$ for any operator $A$ on $L^{2}\left(\mathbb{R}^{n}, V\right)$. Moreover, we have

$$
\begin{aligned}
U_{1} & =\mathcal{B}^{-1 / 2} S_{1}=\left(\left(\mathcal{B}^{0}\right)^{-1 / 2} \otimes b^{-1 / 2}\right)\left(S^{1} \otimes s\right) \\
& =\left(\mathcal{B}^{0}\right)^{-1 / 2} S^{1} \otimes b^{-1 / 2} s=U^{1} \otimes w
\end{aligned}
$$

with obvious notation. Hence we are in position to extend Proposition 2.5 to Heisenberg motion groups.

Proposition 7.2. We have $U_{1}=\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1}$.
Proof. By using Proposition 7.1 and Proposition 2.5, we get

$$
\begin{aligned}
\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1} & =\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right)\left(W_{0}^{-1} \otimes w\right) \\
& =\left(J^{-1} W_{0}^{-1}\right) \otimes w=U^{1} \otimes w=U_{1} .
\end{aligned}
$$

This is the desired result.

Now consider the action of $G$ on $\mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)$ given by

$$
g \cdot(p, q, \varphi):=\left(j^{-1}(g \cdot j(p, q)), \operatorname{Ad}^{*}(k) \varphi\right)
$$

for $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$. Then we have the following result.
Proposition 7.3. 1. The map $W^{-1}$ is a Stratonovich-Weyl correspondence for $\left(G, \sigma, \mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)\right)$.
2. For each $X=((a, \bar{a}), c, A) \in \mathfrak{g}, p, q \in \mathbb{R}^{n}$ and $\varphi \in o\left(\varphi_{0}\right)$, we have

$$
\begin{aligned}
W^{-1}(d \sigma(X))(p, q, \varphi)=i \lambda c+\frac{1}{2} & \operatorname{Tr}(A)+\frac{i}{2}\left(\bar{a} j(p, q)+\lambda^{2} a \overline{j(p, q)}\right) \\
& -\frac{\lambda}{2} \overline{j(p, q)}(A j(p, q))+w(d \rho(A))(\varphi)
\end{aligned}
$$

Proof. 1. For each $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$ let us denote by $L_{g}$ the operator of $L^{2}\left(\mathbb{C}^{n} \times o\left(\varphi_{0}\right), \mu_{\lambda} \otimes \nu\right)$ defined by

$$
\left(L_{g} F\right)(z, \varphi)=F\left(g \cdot z, \operatorname{Ad}^{*}(k) \varphi\right)
$$

Then the covariance property for $U$ can be rewritten as

$$
L_{g} U(A)=U\left(\pi(g)^{-1} A \pi(g)\right)
$$

for each $g \in G$ and $A \in \mathcal{L}_{2}(\mathcal{F})$. This gives the following covariance property for $U_{1}$ :

$$
L_{g} U_{1}(A)=U_{1}\left(\sigma(g)^{-1} A \sigma(g)\right)
$$

for each $g \in G$ and $A \in \mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}, V\right)\right)$. But by Proposition 7.2 we have $U_{1}=\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1}$. Thus we get

$$
\left(J \otimes I_{\left.S y\left(o\left(\varphi_{0}\right)\right)\right)}\right) L_{g}\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1}(A)=W^{-1}\left(\sigma(g)^{-1} A \sigma(g)\right)
$$

for each $g \in G$ and $A \in \mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}, V\right)\right)$.
Now let

$$
\left(\tilde{L}_{g} f\right)(p, q, \varphi):=f\left(j^{-1}(g \cdot j(p, q)), \operatorname{Ad}^{*}(k) \varphi\right)
$$

for each $g=\left(\left(z_{0}, \bar{z}_{0}\right), c_{0}, k\right) \in G$ and $(p, q, \varphi) \in \mathbb{R}^{2 n} \times o\left(\varphi_{0}\right)$. Since it is clear that for each $g \in G$ we have

$$
\tilde{L}_{g}=\left(J \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) L_{g}\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right)
$$

we see that

$$
\tilde{L}_{g} W^{-1}(A)=W^{-1}\left(\sigma(g)^{-1} A \sigma(g)\right)
$$

for each $g \in G$ and $A \in \mathcal{L}_{2}\left(L^{2}\left(\mathbb{R}^{n}, V\right)\right)$. Hence $W^{-1}$ is $G$-covariant. The other properties of a Stratonovich-Weyl correspondence can be easily verified.
2. For each $X \in \mathfrak{g}^{c}$, we have

$$
U(d \pi(X))=U_{1}(d \sigma(X))=\left(\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1}(d \sigma(X))\right.
$$

hence the result follows from Proposition 5.7.
Finally, we can obtain Stratonovich-Weyl correspondences for $\left(G, \pi, \mathcal{O}\left(\varphi^{0}\right)\right)$ and for $\left(G, \sigma, \mathcal{O}\left(\varphi^{0}\right)\right)$ by transferring $U$ and $W^{-1}$ by means of $\Phi$. Let

$$
\Psi:=\Phi \circ(j \otimes 1): \mathbb{R}^{2 n} \times o\left(\varphi_{0}\right) \rightarrow \mathcal{O}\left(\varphi^{0}\right)
$$

and let $\tilde{\nu}$ be the $G$-invariant measure on $\mathcal{O}\left(\varphi^{0}\right)$ defined by

$$
\tilde{\nu}:=\left(\Phi^{-1}\right)^{*}\left(\mu_{\lambda} \otimes \nu\right)=\left(\Psi^{-1}\right)^{*}(\tilde{\mu} \otimes \nu) .
$$

Consider also the unitary maps $\tau_{\Phi}: F \rightarrow F \circ \Phi^{-1}$ from $L^{2}\left(\mathbb{C}^{n} \times o\left(\varphi_{0}\right), \mu_{\lambda} \otimes \nu\right)$ onto $L^{2}\left(\mathcal{O}\left(\varphi^{0}\right), \tilde{\nu}\right)$ and $\tau_{\Psi}: F \rightarrow F \circ \Psi^{-1}$ from $L^{2}\left(\mathbb{R}^{2 n} \times o\left(\varphi_{0}\right), \tilde{\mu} \otimes \nu\right)$ onto $L^{2}\left(\mathcal{O}\left(\varphi^{0}\right), \tilde{\nu}\right)$. Then we have the following proposition.

Proposition 7.4. The map $\mathcal{W}_{1}^{\prime}:=\tau_{\Psi} W^{-1}$ is a Stratonovich-Weyl correspondence for $\left(G, \sigma, \mathcal{O}\left(\varphi^{0}\right)\right)$, the map $\mathcal{W}_{2}^{\prime}:=\tau_{\Phi} U$ is a Stratonovich-Weyl correspondence for $\left(G, \pi, \mathcal{O}\left(\varphi^{0}\right)\right)$ and we have $\mathcal{W}_{1}^{\prime}=\mathcal{W}_{2}^{\prime} I_{B}$.

Proof. The first and the second assertions immediately follow from Proposition 5.5 and Proposition 7.3. To prove the third assertion, note that we have $\tau_{\Psi}\left(J \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right)=\tau_{\Phi}$. Then, by Proposition 7.2, we can write

$$
\mathcal{W}_{2}^{\prime} I_{B}=\tau_{\Phi} U I_{B}=\tau_{\Phi} U_{1}=\tau_{\Phi}\left(J^{-1} \otimes I_{S y\left(o\left(\varphi_{0}\right)\right)}\right) W^{-1}=\tau_{\Psi} W^{-1}=\mathcal{W}_{1}^{\prime}
$$

hence the result.

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# Notes on a combinatorial identity 

Horst Alzer and Helmut Prodinger

Abstract. We present a short and simple proof by induction for

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{2}}=\frac{q^{m}}{1-q^{m}} \prod_{j=1}^{n} \frac{1-q^{j}}{1-q^{j+m}}\left(2+\sum_{j=0}^{n} \frac{q^{j+m}}{1-q^{j+m}}\right)
$$

where $n \geq 1$ is an integer and $m \neq 0,-1, \ldots,-n$ is a complex number. This is a q-analogue of a combinatorial identity obtained by Kirschenhofer (1996) and Larcombe, Fennessey, and Koepf (2004). Moreover, we show that the alternating $q$-binomial sum is completely monotonic with respect to $m$, if $m>0$ and $q \in(0,1)$. The general case where the exponent 2 is replaced by a positive integer $d$ is dealt with using the elementary technique of partial fraction decomposition.

Keywords: Combinatorial identity, $q$-binomial coefficient, completely monotonic, partial fraction decomposition.
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## 1. Introduction

The work on this note has been inspired by an interesting research paper published in 1996 by Kirschenhofer [11], who performed manipulations of generating functions to find identities for the alternating binomial sum

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} f(k) \tag{1}
\end{equation*}
$$

A well-known approach to study sums of the type (1) is attributed to Rice, who made use of Complex Analysis. The Rice method is based upon the formula

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} f(k)=-\frac{1}{2 \pi i} \int_{\mathscr{C}} B(n+1,-z) d z
$$

where $B(x, y)$ is Euler's beta function, $\mathscr{C}$ is a positively oriented closed curve surrounding $0,1,2, \ldots, n$ and $f$ is an analytic function with no poles inside the region surrounded by $\mathscr{C}$.

The main reason for the interest in alternating binomial sums is that they have remarkable applications in Computer Science and the Theory of Algorithms. For more information on this subject we refer to $[7,8,16]$.

Kirschenhofer proved that the sum

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} \frac{1}{(k+m)^{d}} \quad(d \in \mathbb{N}) \tag{2}
\end{equation*}
$$

can be expressed in terms of Bell polynomials and harmonic numbers, whereas Coffey [5] showed that this sum can be written as an infinite series involving Stirling numbers. As a special case Kirschenhofer found the elegant identity

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} \frac{1}{(k+m)^{2}}=\frac{h_{m, n}^{(1)}}{m\binom{m+n}{m}} \tag{3}
\end{equation*}
$$

where

$$
h_{m, n}^{(j)}=\sum_{k=m}^{m+n} \frac{1}{k^{j}}=H_{m+n}^{(j)}-H_{m-1}^{(j)} .
$$

Here, $H_{n}^{(j)}$ denotes the $n$-th harmonic number of order $j$.
In 2004, Larcombe et al. [13] presented a new method to find identities for (2). They used an integration technique to offer proofs for (3) and

$$
\begin{equation*}
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} \frac{1}{(k+m)^{3}}=\frac{\left(h_{m, n}^{(1)}\right)^{2}+h_{m, n}^{(2)}}{2 m\binom{m+n}{m}} \tag{4}
\end{equation*}
$$

Moreover, they demonstrated that (3) and (4) as well as corresponding identities for the sum in (2) with $d \geq 4$ can be obtained by differentiation with respect to $m$, starting with

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} \frac{1}{k+m}=\frac{1}{m\binom{m+n}{n}}
$$

Larcombe et al. [14] provided a recursive equation for a sum closely related to (2) and used their result to find new proofs for (3), (4) and similar identities. Other methods to deal with the sums in question are described in $[10,12,15]$.

In this paper we demonstrate that the identity (3) can be proved easily by using induction. More precisely, in the next section we present a short and elementary proof for a $q$-analogue of (3). Furthermore, as an application we prove a monotonicity property of the alternating $q$-binomial sum. In a final section, we show how to deal with the general case in a completely elementary fashion, using not more than partial fraction decomposition from elementary calculus.

## 2. The identity

The $q$-binomial coefficients (also known as Gaussian binomial coefficients) are defined by

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}=\prod_{j=1}^{k} \frac{1-q^{n+1-j}}{1-q^{j}} \quad \text { if } \quad 0 \leq k \leq n \quad \text { and } \quad\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}=0 \quad \text { otherwise. }
$$

If $q \rightarrow 1$, then $\left[\begin{array}{l}n \\ k\end{array}\right]_{q}$ tends to $\binom{n}{k}$. A collection of the most important properties of $q$-binomial coefficients can be found, for instance, in [4].

The following $q$-version of (3) holds.
ThEOREM 2.1. Let $n \geq 1$ be an integer and let $m$ be a complex number with $m \neq 0,-1, \ldots,-n$. Then,

$$
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n  \tag{5}\\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{2}}=\frac{q^{m}}{1-q^{m}} \prod_{j=1}^{n} \frac{1-q^{j}}{1-q^{j+m}}\left(2+\sum_{j=0}^{n} \frac{q^{j+m}}{1-q^{j+m}}\right)
$$

Throughout, we denote the sum on the left-hand side of (5) by $S(m, n, q)$.
Proof. We use induction on $n$ to prove (5). If $n=1$, then both sides of (5) are equal to

$$
\frac{(1-q) q^{m}\left(2-q^{m}-q^{m+1}\right)}{\left(1-q^{m}\right)^{2}\left(1-q^{m+1}\right)^{2}}
$$

We set

$$
T(m, n, q)=2+\sum_{j=0}^{n} \frac{q^{j+m}}{1-q^{j+m}}
$$

Applying the recurrence formula

$$
\left[\begin{array}{c}
n+1 \\
k
\end{array}\right]_{q}=\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}+q^{n+1-k}\left[\begin{array}{c}
n \\
k-1
\end{array}\right]_{q}
$$

and the induction hypothesis yields

$$
\begin{aligned}
& S(m, n+1, q)=S(m, n, q)-q^{n} S(m+1, n, q) \\
& =\frac{q^{m}}{1-q^{m}} \prod_{j=1}^{n+1} \frac{1-q^{j}}{1-q^{j+m}} \\
& \quad \times\left(\frac{1-q^{m+n+1}}{1-q^{n+1}} T(m, n, q)-\frac{q^{n+1}\left(1-q^{m}\right)}{1-q^{n+1}} T(m+1, n, q)\right) \\
& \quad=\frac{q^{m}}{1-q^{m}} \prod_{j=1}^{n+1} \frac{1-q^{j}}{1-q^{j+m}} T(m, n+1, q)
\end{aligned}
$$

This gives (2.1) with $n+1$ instead of $n$.

Remark 2.2. (i) If we multiply both sides of (5) by $(1-q)^{2}$ and let $q \rightarrow 1$, then we obtain

$$
\sum_{k=0}^{n}(-1)^{k}\binom{n}{k} \frac{1}{(k+m)^{2}}=\frac{n!\Gamma(m)}{\Gamma(m+n+1)}[\psi(m+n+1)-\psi(m)]
$$

where $\psi=\Gamma^{\prime} / \Gamma$ denotes the digamma function. This identity is given in [13]. The special case that $m$ is a natural number yields (3).
(ii) If we differentiate both sides of (5) with respect to $m$, then we obtain the following $q$-analogue of (4):

$$
\begin{align*}
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k+1}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} & \frac{1}{\left(1-q^{k+m}\right)^{3}} \\
& =\frac{1}{1-q^{m}} \prod_{j=1}^{n} \frac{1-q^{j}}{1-q^{j+m}}\left[\left(1+\sigma_{1}\right)\left(1+\frac{1}{2} \sigma_{1}\right)+\sigma_{2}\right] \tag{6}
\end{align*}
$$

where

$$
\sigma_{k}=\sigma_{k}(m, n, q)=\frac{1}{k} \sum_{j=0}^{n} \frac{q^{j+m}}{\left(1-q^{j+m}\right)^{k}}
$$

Identity (4) follows easily from (6). Indeed, if we multiply both sides of (6) by $(1-q)^{3}$ and let $q \rightarrow 1$, then we arrive at (4).

We recall that a function $f:(0, \infty) \rightarrow \mathbb{R}$ is said to be completely monotonic, if

$$
(-1)^{N} f^{(N)}(x) \geq 0 \quad(N=0,1,2, \ldots ; x>0)
$$

These functions have interesting applications, for instance, in probability theory, numerical and asymptotic analysis. In numerous papers it was proved that various functions which are defined in terms of gamma, polygamma and other special functions are completely monotonic. We refer to $[2,3]$ and the references therein. See also [17] for background information.

An application of Theorem 2.1 reveals that the alternating $q$-binomial sum $S(m, n, q)$ satisfies the following monotonicity property.

Corollary 2.3. Let $n \geq 1$ be an integer and $q \in(0,1)$ be a real number. The function $m \mapsto S(m, n, q)$ is completely monotonic on $(0, \infty)$.
Proof. Since a nonnegative constant function is completely monotonic and the sum and the product of completely monotonic functions are also completely monotonic, we conclude from (5) that in order to show that $S(m, n, q)$ is completely monotonic with respect to $m$ it suffices to show that the functions

$$
f_{1}(m)=q^{m} \quad \text { and } \quad f_{2}(m)=\frac{1}{1-q^{j+m}} \quad(j \geq 0)
$$

are completely monotonic. This follows from

$$
(-1)^{N} f_{1}^{(N)}(m)=(-\log q)^{N} q^{m}>0
$$

and

$$
(-1)^{N} f_{2}^{(N)}(m)=(-\log q)^{N} \sum_{k=0}^{\infty} q^{k(j+m)} k^{N}>0
$$

which hold for all integers $N \geq 0$.
Remark 2.4. (i) Fink [6] proved that a completely monotonic function is not only convex but even log-convex. This means that Corollary 2.3 leads to the inequality

$$
\begin{equation*}
S\left(\frac{a+b}{2}, n, q\right) \leq \sqrt{S(a, n, q) S(b, n, q)} \tag{7}
\end{equation*}
$$

(ii) A theorem of Hardy et al. [9, p. 97] states that if a function $\phi$ is twice differentiable and convex on $(0, \infty)$, then so is $x \mapsto x \phi(1 / x)$. Using this result with $\phi=\log S$ we obtain

$$
\begin{equation*}
S\left(\frac{2}{1 / a+1 / b}, n, q\right)^{a+b} \leq S(a, n, q)^{b} S(b, n, q)^{a} \tag{8}
\end{equation*}
$$

The inequalities (7) and (8) are valid for all $a, b>0, n \geq 1$ and $q \in(0,1)$.
(iii) We have shown that identity (5) can be applied to prove a monotonicity property of $S(m, n, p)$. It might be of interest to present series, product or integral representations for other binomial sums in order to find similar results. An example is given in [1].

## 3. The general case

Let, as usual, $(z ; q)_{n}=(1-z)(1-z q) \ldots\left(1-z q^{n-1}\right)$ and set

$$
F(z)=\frac{(q ; q)_{n}}{(z ; q)_{n+1}} \frac{z^{d}}{\left(z-q^{m}\right)^{d}}
$$

This rational function has poles at $q^{0}, q^{-1}, \ldots, q^{-n}$, and at $q^{m}$. We construct the partial fraction decomposition:

$$
\begin{aligned}
F(z) & =\sum_{k=0}^{n}(-1)^{k-1} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{d}} \frac{1}{z-q^{-k}} \\
& +\frac{A_{d}}{\left(z-q^{m}\right)^{d}}+\frac{A_{d-1}}{\left(z-q^{m}\right)^{d-1}}+\cdots+\frac{A_{1}}{\left(z-q^{m}\right)^{1}} .
\end{aligned}
$$

Multiplying this relation by $z$, and then letting $z \rightarrow \infty$, we get for $n \geq 1$ :

$$
0=\sum_{k=0}^{n}(-1)^{k-1} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{d}}+A_{1} .
$$

So

$$
\begin{aligned}
A_{1} & =\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{d}} \\
& =\left[\left(z-q^{m}\right)^{-1}\right] F(z)=\left[\left(z-q^{m}\right)^{d-1}\right] \frac{(q ; q)_{n} z^{d}}{(z ; q)_{n+1}}=\left[w^{d-1}\right] \frac{(q ; q)_{n}\left(w+q^{m}\right)^{d}}{\left(w+q^{m} ; q\right)_{n+1}} \\
& =\left[w^{d-1}\right] \frac{(q ; q)_{n}\left(w+q^{m}\right)^{d}}{\left(1-w-q^{m}\right)\left(1-w q-q^{m+1}\right) \ldots\left(1-w q^{n}-q^{m+n}\right)} \\
& =\left[w^{d-1}\right] \frac{(q ; q)_{n}\left(w+q^{m}\right)^{d}}{\left(1-q^{m}\right) \ldots\left(1-q^{m+n}\right)\left(1-\frac{w}{1-q^{m}}\right) \ldots\left(1-\frac{w q^{n}}{1-q^{n+m}}\right)} \\
& =\left[w^{d-1}\right] \frac{(q ; q)_{n}(q ; q)_{m-1}}{(q ; q)_{m+n}\left(1-\frac{w}{1-q^{m}}\right) \ldots\left(1-\frac{w q^{n}}{1-q^{n+m}}\right)} \sum_{j=0}^{d}\binom{d}{j} w^{d-j} q^{m j} \\
& =\frac{(q ; q)_{n}(q ; q)_{m-1}}{(q ; q)_{m+n}} \sum_{j=0}^{d-1}\binom{d}{j+1} q^{m(j+1)}\left[w^{j}\right] \frac{1}{\left(1-\frac{w}{1-q^{m}}\right) \ldots\left(1-\frac{w q^{n}}{1-q^{n+m}}\right)} .
\end{aligned}
$$

We continue with the computation of

$$
\begin{aligned}
{\left[w^{j}\right] } & \frac{1}{\left(1-\frac{w}{1-q^{m}}\right) \cdots\left(1-\frac{w q^{n}}{1-q^{n+m}}\right)} \\
& =\left[w^{j}\right] \exp \left\{\log \frac{1}{1-\frac{w}{1-q^{m}}}+\cdots+\log \frac{1}{1-\frac{w q^{n}}{1-q^{m+n}}}\right\} \\
& =\left[w^{j}\right] \exp \left\{\sum_{k \geq 1} \frac{1}{k} \frac{w^{k}}{\left(1-q^{m}\right)^{k}}+\cdots+\sum_{k \geq 1} \frac{1}{k} \frac{w^{k} q^{n k}}{\left(1-q^{m+n}\right)^{k}}\right\} \\
& =\left[w^{j}\right] \exp \left\{\sum_{k \geq 1} \tau_{k} w^{k}\right\},
\end{aligned}
$$

with

$$
\tau_{k}=\tau_{k}(m, n, q)=\frac{1}{k} \sum_{j=0}^{n}\left(\frac{q^{j}}{1-q^{j+m}}\right)^{k} .
$$

Furthermore,

$$
\begin{aligned}
{\left[w^{j}\right] \exp \left\{\sum_{k \geq 1} \tau_{k} w^{k}\right\} } & =\left[w^{j}\right] e^{\tau_{1} w} e^{\tau_{2} w^{2}} e^{\tau_{3} w^{3}} \cdots \\
& =\sum_{k_{1}+2 k_{2}+3 k_{3}+\cdots=j} \frac{\tau_{1}^{k_{1}} \tau_{2}^{k_{2}} \tau_{3}^{k_{3}} \cdots}{k_{1}!k_{2}!k_{3}!\cdots}
\end{aligned}
$$

which leads to the final formula:

$$
\begin{aligned}
\sum_{k=0}^{n} & (-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \frac{1}{\left(1-q^{k+m}\right)^{d}} \\
& =\frac{(q ; q)_{n}(q ; q)_{m-1}}{(q ; q)_{m+n}} \sum_{j=0}^{d-1}\binom{d}{j+1} q^{m(j+1)} \sum_{k_{1}+2 k_{2}+3 k_{3}+\cdots=j} \frac{\tau_{1}^{k_{1}} \tau_{2}^{k_{2}} \tau_{3}^{k_{3}} \cdots}{k_{1}!k_{2}!k_{3}!\ldots}
\end{aligned}
$$

The special case $d=2$ gives (5) and for $d=3$ we obtain the following counterpart of (6):

$$
\begin{aligned}
\sum_{k=0}^{n}(-1)^{k} q^{\binom{k}{2}}\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} & \frac{1}{\left(1-q^{k+m}\right)^{3}} \\
& =\frac{q^{m}}{1-q^{m}} \prod_{j=1}^{n} \frac{1-q^{j}}{1-q^{j+m}}\left[3+3 q^{m} \tau_{1}+q^{2 m}\left(\frac{1}{2} \tau_{1}^{2}+\tau_{2}\right)\right]
\end{aligned}
$$

Remark 3.1. If $m$ is a positive integer, then

$$
\tau_{k}=\frac{q^{-m k}}{k} \sum_{j=m}^{m+n}\left(\frac{q^{j}}{1-q^{j}}\right)^{k}=\frac{q^{-m k}}{k}\left(\mathscr{H}_{m+n}^{(k)}-\mathscr{H}_{m-1}^{(k)}\right),
$$

where $\mathscr{H}_{n}^{(k)}$ denotes the $q$-analogue of the $n$-th harmonic number of order $k$.

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# Global bifurcation for Fredholm operators 

Julián López-Gómez


#### Abstract

This paper reviews the global bifurcation theorem of $J$. López-Gómez and C. Mora-Corral [18] and derives from it a global version of the local theorem of M. G. Crandall and P. H. Rabinowitz [5] on bifurcation from simple eigenvalues, as well as a refinement of the unilateral bifurcation theorem of [14, Chapter 6].


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## 1. Introduction

The local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) and the global alternative of P. H. Rabinowitz [23] (1971) are two pioneering results that have been extensively used by applied analysts over the last fortyfive years. Undoubtedly, they have shown to be a milestone for the generation of new results in nonlinear analysis. Although the functional setting of the former is user-friendly by practitioners, as it merely involves a simple transversality condition easy to check in applications, the latter often requires to express a nonlinear equation as a fixed point equation for a nonlinear compact operator and then checking that the classical concept of algebraic multiplicity is odd, which is not always an easy task, even if possible. Among other technical troubles, the geometric multiplicity of the eigenvalue might be one while the algebraic one is even. Thus, a global bifurcation result in the functional setting of Crandall-Rabinowitz local bifurcation theorem was desirable since the early seventies, so that the local theorem could be applied directly to get global results.

Actually, the (extremely hidden) links between the several concepts of algebraic multiplicities available in the context of local and global bifurcation theory remained a mystery, almost un-explored except for some few attempts involving the cross numbers, until the papers of R. J. Magnus [20] (1976) and J. Esquinas and J. López-Gómez [8], [7] (1988) were published. Indeed, the cross number was designed to detect any change of the topological degree as the
underlying parameter, $\lambda$, crossed a singular value, $\lambda_{0}$, through the Schauder formula, i.e., by means of the total number of negative eigenvalues, counting classical algebraic multiplicities, of the linearized equation. From a practical point of view the cross number was far from useful; as merely reformulated an open problem in nonlinear analysis through another one in operator theory -of dynamical nature, but equally open-, though it certainly illuminated the underlying mathematical analysis by incorporating a (new) dynamical perspective into it.

Theorem 2.1 of J. Esquinas and J. López-Gómez [8] (1988) provided with a substantial -rather direct, but far from obvious- extension of the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) by characterizing the nonlinear eigenvalues, $\lambda_{0}$, of a Fredholm family of operators, $\mathfrak{L}(\lambda)$, $\lambda \sim \lambda_{0}$, through a new generalized concept of algebraic multiplicity, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$, which is a substantial extension of the previous one of Crandall-Rabinowitz.

Roughly, a nonlinear eigenvalue, $\lambda_{0}$, is a critical value of the parameter where a local bifurcation occurs independently of the structure of the nonlinear perturbation. According to Theorem 4.3.4 of [14] (2001), an isolated eigenvalue $\lambda_{0}$ of $\mathfrak{L}(\lambda)$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$ if and only if $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is odd. That $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ extends the concept of multiplicity of a simple eigenvalue as discussed by M. G. Crandall and P. H. Rabinowitz [5] (1971) is evident from its own definition. This becomes apparent by simply having a glance at Remark 4.2.5 of [14] (2001), or going back to the comment on the first paragraph on page 77 of [8] (1988), where it was explicitly asserted that
"In fact, $k$-genericity implies $k+1$-genericity and the genericity of Crandall and Rabinowitz is our 1-genericity."
Furthermore, by Lemma 3.2 of J. Esquinas and J. López-Gómez [8] (1988), $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ equals the generalized algebraic multiplicity of R. J. Magnus [20] (1976). Thus, thanks to the global bifurcation theorem of R. J. Magnus [20] (1976), it became also apparent that, at least for compact perturbations of the identity map, the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) is indeed global.

Later, the author considerably polished and tidied up most of the previous materials, collecting them together in the book [14] (2001). In that monograph, besides characterizing the set of singular values where the algebraic multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is well defined, through the (new) concept of algebraic eigenvalue, the author remarked on the bottom of Page 180 that
"Rabinowitz's reflection argument in the proof of Theorem 1.27, [23], was actually performed with respect to the supplement $Y$ of $N\left[\mathfrak{L}_{0}\right]$ in $U$, rather than with respect to $u=0$ within the cone $Q_{\xi, \eta}$. Therefore, the last alternative of Theorem 1.27 of [23] seems to be far from natural, though the theorem might be true."
and immediately gave a (new) unilateral bifurcation result -Theorem 6.4.3 in
[14] (2001)- widely used in the specialized literature since then. Prompted by the new findings of [14, Ch. 6] (2001), E. N. Dancer [6] (2002), using some classical devices in (topological) obstruction theory, was able to construct a counterexample to the unilateral theorems of P. H. Rabinowitz [23] (1971). According to Dancer's counterexample, the unilateral Theorems 1.27 and 1.40 of P. H. Rabinowitz [23] were wrong as stated. As a byproduct of Dancer's counterexample, Theorem 6.3.4 of [14] (2001) became the first (correct) available unilateral theorem in the literature. Many nonlinear analysts had been systematically applying -almost mutatis mutandis- the (wrong) unilateral theorems of Rabinowitz for almost four decades and most experts and reviewers were not aware of it. They are doing it just now!

Three years later, in 2004, the theory of generalized algebraic multiplicities was axiomatized and considerably sharpened by C. Mora-Corral in his PhD thesis under the supervision of the author. This thesis was judged by I. Gohberg, R. J. Magnus and J. L. Mawhin at Complutense University of Madrid on June 2004. Shortly later, C. Mora-Corral and the author completed the monograph [19] (2007) edited by I. Gohberg as the volume 177 of his prestigious series 'Operator Theory: Advances and Applications'. Reading [19] (2007) is imperative to realize the (tremendous) development of the theory of algebraic multiplicities from the seminal work of M. G. Crandall and P. H. Rabinowitz [5] up to the characterization of any local change of the topological degree through the algebraic multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$. Actually, according to the uniqueness results collected in Chapter 6 of [19] (2007), it turns out that $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is the unique normalized algebraic multiplicity satisfying the product formula; a fundamental result in Operator Theory, attributable to C. Mora-Corral, which has not received the deserved attention yet.

As far as concerns global bifurcation theory, the more general abstract bifurcation result available in the literature is Corollary 5.5 of J. López-Gómez and C. Mora-Corral [18] (2005), where the notion of orientability introduced by P. Benevieri and M. Furi [2] (2000) for Fredholm maps of index zero was combined with the algebraic multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ to establish that any compact component, $\mathfrak{C}$, of the solution set must bifurcate from the given state at exactly an even number of singular values having an odd algebraic multiplicity. So, extending the pioneering global bifurcation theorems of L. Nirenberg [21] (1974), attributed to P. H. Rabinowitz by L. Nirenberg himself in his celebrated Lecture Notes at the Courant Institute, and R. J. Magnus [20] (1976) to work, almost mutatis mutandis, in the more general setting of Fredholm maps of index zero. However, the classical multiplicities must be inter-exchanged by the concept of multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$.

Actually, since [18] (2005) did not required the linearized operators at the given state to have a discrete spectrum, but an arbitrary structure, the abstract theory of [18] (2005) can be applied not only to quasilinear problems
in bounded domains, but, more generally, to arbitrary quasilinear systems in bounded or unbounded domains. Naturally, by [18, Cor. 5.5], when the component $\mathfrak{C}$ bifurcates from a given state and cannot meet the given state at another point -or spectral interval-, $\mathfrak{C}$ must be non-compact, which in particular yields the global alternative of P. H. Rabinowitz [23] (1971). Being this alternative so user-friendly by practitioners, there is still the serious danger that many users of global bifurcation theory might be reluctant to face the few topological technicalities inherent to Corollary 5.5 of [18] (2005). This is one of the reasons why we are going to tidy up considerably some of the materials of [18] here. As a matter of fact, from the pioneering results of P. H. Rabinowitz [23] (1971) and R. J. Magnus [20] (1976) and the vibrant Lecture Notes of L. Nirenberg [21] (1974) it became apparent that Rabinowitz's global alternative was nothing more than a friendly byproduct of the global result already stated by L. Nirenberg [21] valid for nonlinear compact perturbations of the identity map in the very special case when

$$
\mathfrak{L}(\lambda)=I-\lambda K
$$

The additional information provided by Corollary 5.5 in [18] (2005) is relevant as well because a variety of nonlinear elliptic systems and semilinear weighted boundary value problems of elliptic type can possess solution components with multiple bifurcation points from a given state. It suffices to have a glance at the cover of the monograph [14] (2001), or at the numerics of Chapter 2 of [14] (2001), or at the paper of M. Molina-Meyer with the author [15] (2005), where a series of compact components possessing several bifurcation points were constructed in a systematic way in the context of semilinear elliptic equations.

The key idea behind Corollary 5.5 of [18] (2005) was exploiting a definition of orientation/parity for Fredholm maps and associated degree as developed by P. Benevieri and M. Furi [1] (1998), [2] (2000), [3] (2001). Although this idea is closely related in a number of ways to some previous notions of parity pioneered by P. M. Fitzpatrick and J. Pejsachowicz [9] (1991), [10] (1993), it certainly requires less smoothness and hence, it is more general.

By Theorem 3.3 of [18] (2005), for any isolated eigenvalue, $\lambda_{0}$, of an oriented family, $\mathfrak{L}(\lambda)$, the sign jump of $\mathfrak{L}(\lambda)$ changes as $\lambda$ crosses $\lambda_{0}$ if, and only if, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is odd. Moreover, in the context of Crandall-Rabinowitz theorem, as already commented above,

$$
\chi\left[\mathfrak{L} ; \lambda_{0}\right]=1
$$

Therefore, by Corollary 5.5 of [18] (2005), it is obvious that the local theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) must be global. Since $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is substantially more general than the pioneering concept of algebraic multiplicity for simple eigenvalues of M. G. Crandall and P. H. Rawinowitz [5] (1971), there was no any need for the authors of [18] (2005) to make any explicit reference to [5] (1971) therein. By the same reason, it was absolutely unnecessary invoking
to any other algebraic multiplicity sharper than the pioneering one of M. G. Crandall and P. H. Rabinowitz [5] (1971), because $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ had shown to be the optimal one in the context of bifurcation theory. It turns out that $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is an optimal algebraic/analytic invariant to compute any change of the degree, or parity, for Fredholm maps.

In spite of these circumstances, being already published in top mathematical journals a series of closely related papers by the author in collaboration with C. Mora-Corral, as [16] (2004) and [17] (2004), where some precursors of Corollary 5.5 of [18] (2005) had been already developed for compact perturbations of the identity map, J. Shi and X. Wang submitted [24] (2009) on May 2008, where they established, at least four years later than C. Mora-Corral and the author, that the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] (1971) is global (see Theorem 4.3 of [24] (2009)). Incidentally, J. Shi and X. Wang [24] (2009) left outside their their list of references all the previous works by the author and coworkers, except [14] (2001), which was required for paraphrasing the proof of the unilateral Theorem 6.4.3 of Chapter 6 of [14] (2001) in order to give a version of [14, Th. 6.4.3] (2001) in the context of Fredholm operators of index zero, Theorem 4.4 of [24] (2009), by imposing the additional restriction that the underlying norm in the Banach space is differentiable.

Being Chapters 3, 4 and 5 of [14] (2001) devoted to the analysis of the main properties of the algebraic multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$, J. Shi and X. Wang [24] (2009) did not say a word about $\chi\left[\mathcal{L} ; \lambda_{0}\right]$ in their discussion on page 2803 of [24] (2009). In terms of the algebraic multiplicity $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$, Theorem 4.3 of J. Shi and X. Wang [24] (2009) is a very special case of Theorem 6.3.1 of J. López-Gómez [14] (2001) for compact perturbations of the identity. Moreover, Corollary 5.5 of J. López-Gómez and C. Mora-Corral [18] (2005) had already generalized Theorem 6.3 .1 of [14] (2001) to cover the general setting of Fredholm operators with index zero four years before. Should J. Shi and X. Wang have invoked all existing results in the literature, very specially Corollary 5.5 of [18] (2005), their Section 4 in [24] (2009) might have shortened up to remarking that the author unilateral theorem [14, Th. 6.4.3] (2001) admitted an obvious extension to cover the case of Fredholm operators of index zero by imposing the differentiability of the underlying norm.

The first goal of this paper is updating the main global bifurcation theorem of J. López-Gómez and C. Mora-Corral [18] in order to derive from it, as a direct straightforward consequence, some global versions of the local theorem of M. G. Crandall and P. H. Rabinowitz [5] on bifurcation from simple eigenvalues. These versions are substantially sharper than the one given by J. Shi and S. Wang in Section 4 of [24] through the generalized parity of P. M. Fitzpatrick and J. Pejsachowicz [9].

Throughout this paper, given two real Banach spaces, $U$ and $V$, we de-
note by $\mathcal{L}(U, V)$ the space of bounded linear operators from $U$ to $V$, and by $\operatorname{Fred}_{0}(U, V)$ the subset of $\mathcal{L}(U, V)$ consisting of all Fredholm operators of index zero. Also, for any $L \in \mathcal{L}(U, V)$, we denote by $N[L]$ and $R[L]$ the null space, or kernel, and the range, or image, of $L$, respectively. We recall that $L \in \mathcal{L}(U, V)$ is said to be a Fredholm operator if

$$
\operatorname{dim} N[L]<\infty \quad \text { and } \quad \operatorname{codim} R[L]<\infty
$$

In such case, $R[L]$ is closed, and the index of $L$ is defined by

$$
\operatorname{ind}[L]:=\operatorname{dim} N[L]-\operatorname{codim} R[L]
$$

Thus, $L \in \operatorname{Fred}_{0}(U, V)$ if

$$
\operatorname{dim} N[L]=\operatorname{codim} R[L]<\infty
$$

Naturally, if $\operatorname{Fred}_{0}(U, V) \neq \emptyset$, then $U$ and $V$ are isomorphic. So, it would not be a serious restriction assuming $U=V$. In that case, we denote

$$
\operatorname{Fred}_{0}(U):=\operatorname{Fred}_{0}(U, U)
$$

The most paradigmatic class of functions in $\operatorname{Fred}_{0}(U)$ are the compact perturbations of the identity $I_{U}$. An operator $T \in \mathcal{L}(U, V)$ is said to be compact if the closure $\overline{T(B)}$ is a compact subset of $V$ for all bounded subset $B \subset U$. In this paper, we denote by $\mathcal{K}(U, V)$ the subset of $\mathcal{L}(U, V)$ of all compact operators. Another significant subset of $\mathcal{L}(U, V)$ is the set of all isomorphism from $U$ to $V$, Iso $(U, V)$. Naturally, we will denote

$$
\mathcal{L}(U):=\mathcal{L}(U, U), \quad \mathcal{K}(U):=\mathcal{K}(U, U), \quad \text { Iso }(U):=\operatorname{Iso}(U, U)
$$

The main goal of this paper is analyzing the structure of the components of the set of non-trivial solutions of

$$
\begin{equation*}
\mathfrak{F}(\lambda, u)=0, \quad(\lambda, u) \in \mathbb{R} \times U \tag{1}
\end{equation*}
$$

bifurcating from $(\lambda, 0)$, where

$$
\begin{equation*}
\mathfrak{F}: \mathbb{R} \times U \rightarrow V \tag{2}
\end{equation*}
$$

is a continuous map satisfying the following requirements:
(F1) For each $\lambda \in \mathbb{R}$, the map $\mathfrak{F}(\lambda, \cdot)$ is of class $\mathcal{C}^{1}(U, V)$ and

$$
\begin{equation*}
D_{u} \mathfrak{F}(\lambda, u) \in \operatorname{Fred}_{0}(U, V) \quad \text { for all } u \in U \tag{3}
\end{equation*}
$$

(F2) $D_{u} \mathfrak{F}: \mathbb{R} \times U \rightarrow \mathcal{L}(U, V)$ is continuous.
(A3) There exists $\theta \in \mathcal{C}(\mathbb{R}, U)$ such that $\mathfrak{F}(\lambda, \theta(\lambda))=0$ for all $\lambda \in \mathbb{R}$.
By performing the change of variable

$$
\mathfrak{G}(\lambda, u):=\mathfrak{F}(\lambda, u+\theta(\lambda)), \quad(\lambda, u) \in \mathbb{R} \times U
$$

and inter-exchanging $\mathfrak{F}$ by $\mathfrak{G}$, one can assume, instead of (A3), that
(F3) $\mathfrak{F}(\lambda, 0)=0$ for all $\lambda \in \mathbb{R}$.
By a component it is meant a closed and connected subset which is maximal for the inclusion. So, by a component it is meant a connected component. As $(\lambda, 0)$ is a given (known) zero, it is referred to as the trivial state. Given $\lambda_{0} \in \mathbb{R}$, it is said that $\left(\lambda_{0}, 0\right)$ is a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$ if there exists a sequence $\left(\lambda_{n}, u_{n}\right) \in \mathfrak{F}^{-1}(0)$, with $u_{n} \neq 0$ for all $n \geq 1$, such that

$$
\lim _{n \rightarrow \infty}\left(\lambda_{n}, u_{n}\right)=\left(\lambda_{0}, 0\right)
$$

In order to state our first result we need to introduce some notations. For every map $\mathfrak{F}$ satisfying (F1), (F2) and (F3), we denote

$$
\begin{equation*}
\mathfrak{L}(\lambda):=D_{u} \mathfrak{F}(\lambda, 0), \quad \lambda \in \mathbb{R} \tag{4}
\end{equation*}
$$

the linearization of $\mathfrak{F}$ at $(\lambda, 0)$. By ( F 2 ), $\mathfrak{L} \in \mathcal{C}(\mathbb{R}, \mathcal{L}(U, V))$. Moreover, since $\mathfrak{L}(\lambda) \in \operatorname{Fred}_{0}(U, V)$,

$$
\mathfrak{L}(\lambda) \in \operatorname{Iso}(U, V) \quad \text { if, and only if, } \quad \operatorname{dim} N[\mathfrak{L}(\lambda)]=0 .
$$

Consequently, the spectrum of $\mathfrak{L}$ can be defined as

$$
\begin{equation*}
\Sigma:=\Sigma(\mathfrak{L}) \equiv\{\lambda \in \mathbb{R}: \quad \operatorname{dim} N[\mathfrak{L}(\lambda)] \geq 1\} \tag{5}
\end{equation*}
$$

Our global version of the main theorem of [5] reads as follows.
Theorem 1.1. Suppose $\mathfrak{L} \in \mathcal{C}^{1}\left(\mathbb{R}, \operatorname{Fred}_{0}(U, V)\right)$ and $\lambda_{0} \in \mathbb{R}$ is a simple eigenvalue of $\mathfrak{L}$, as discussed by M. G. Crandall and P. H. Rabinowitz [5], i.e.,

$$
\begin{equation*}
\mathfrak{L}^{\prime}\left(\lambda_{0}\right) \varphi_{0} \notin R\left[\mathfrak{L}\left(\lambda_{0}\right)\right], \quad \text { where } N\left[\mathfrak{L}\left(\lambda_{0}\right)\right]=\operatorname{span}\left[\varphi_{0}\right] . \tag{6}
\end{equation*}
$$

Then, for every continuous function $\mathfrak{F}: \mathbb{R} \times U \rightarrow V$ satisfying (F1), (F2), (F3) and $D_{u} \mathfrak{F}(\cdot, 0)=\mathfrak{L},\left(\lambda_{0}, 0\right)$ is a bifurcation point from $(\lambda, 0)$ to a continuum of non-trivial solutions of $\mathfrak{F}=0$.

For any of these $\mathfrak{F}$ 's, let $\left\{K_{j}\right\}_{j=r}^{s}$ be an admissible family of disjoint closed subsets of $\Sigma$ with $K_{0}=\left\{\lambda_{0}\right\}$, as discussed in Definition 5.1, and let $\mathfrak{C}$ be the component of the set of nontrivial solutions with $\left(\lambda_{0}, 0\right) \in \mathfrak{C}$. Then, either
(a) $\mathfrak{C}$ is not compact; or
(b) there is another $\Sigma \ni \lambda_{1} \neq \lambda_{0}$ with $\left(\lambda_{1}, 0\right) \in \mathfrak{C}$.

Actually, if $\mathfrak{C}$ is compact, there is $N \geq 1$ such that

$$
\left(K_{j} \times\{0\}\right) \cap \mathfrak{C} \neq \emptyset \quad \text { if, and only if, } \quad j \in\left\{j_{i_{1}}, \ldots, j_{i_{N}}\right\} \subset \mathbb{Z} \cap[r, s]
$$

with $j_{i_{k}}=0$ for some $k \in\{1, \ldots, N\}$. Moreover,

$$
\sum_{k=1}^{N} \mathcal{P}\left(j_{i_{k}}\right)=0
$$

where $\mathcal{P}$ stands for the parity map introduced in Section 5. Therefore, $\mathfrak{C}$ links $\left(\lambda_{0}, 0\right)$ to an odd number of $K_{j} \times\{0\}$ 's with parity $\pm 1$.

The second goal of this paper is generalizing the unilateral bifurcation theorem of the author [14, Th. 6.4.3] to the general context of Fredholm equations, in the same vain as the version of [14, Th. 6.4.3] given by J. Shi and S. Wang in [24, Th. 4.4]. Our updated version of [14, Th. 6.4.3] has the advantage that it does not require the differentiability of the norm, as it is required in [24, Th. 4.4], but only the compact inclusion of $U$ in $V$, which is a rather natural assumption from the point of view of the applications. Precisely, the following result holds.

THEOREM 1.2. Suppose the injection $U \hookrightarrow V$ is compact, $\mathfrak{F}$ satisfies (F1)-(F3), the map

$$
\mathfrak{N}(\lambda, u):=\mathfrak{F}(\lambda, u)-D_{u} \mathfrak{F}(\lambda, 0) u, \quad(\lambda, u) \in \mathbb{R} \times U
$$

admits a continuous extension to $\mathbb{R} \times V$, the transversality condition (6) holds, and consider a closed subspace $Y \subset U$ such that

$$
U=N\left[\mathfrak{L}_{0}\right] \oplus Y .
$$

Let $\mathfrak{C}$ be the component given by Theorem 1.1 and let denote by $\mathfrak{C}^{+}$and $\mathfrak{C}^{-}$the subcomponents of $\mathfrak{C}$ in the directions of $\varphi_{0}$ and $-\varphi_{0}$, respectively. Then, for each $\nu \in\{-,+\}, \mathfrak{C}^{\nu}$ satisfies some of the following alternatives:
(a) $\mathfrak{C}^{\nu}$ is not compact in $\mathbb{R} \times U$.
(b) There exists $\lambda_{1} \neq \lambda_{0}$ such that $\left(\lambda_{1}, 0\right) \in \mathfrak{C}^{\nu}$.
(c) There exists $(\lambda, y) \in \mathfrak{C}^{\nu}$ with $y \in Y \backslash\{0\}$.

All the assumptions of Theorem 1.2 are fulfilled as soon as $U$ is a space of smooth functions and $V$ is some subspace of the space of continuous functions, or a subspace of $L^{\infty}(\Omega)$, as it occurs in most of applications. Concrete applications of these results will be given elsewhere. Naturally, as we are not imposing
the differentiability of the norm of $U$, this is a fully complementary result of [24, Th. 4.4], though the proof is also based on the proof of [14, Th. 6.4.3], the first (correct) unilateral bifurcation theorem available in the literature.

The distribution of this paper is the following. Section 2 contains some basic preliminaries. Section 3 gives the concept of orientation and degree introduced by P. Benevieri and M. Furi [1]. Section 4 collects the most relevant concepts and results of the theory of algebraic multiplicities, as they detect any change of orientation and hence, any global bifurcation phenomenon. Section 5 discusses the main global bifurcation theorem of this paper, Section 6 derives Theorem 1.1 from our main global result and, finally, in Section 7 we tidy up the unilateral bifurcation theory of $[14, \mathrm{Ch} .6]$ in order to derive Theorem 1.2.

## 2. A preliminary result

Naturally, the resolvent set of $\mathfrak{L}$ is defined by $\varrho(\mathfrak{L}):=\mathbb{R} \backslash \Sigma$. Since $\mathfrak{L} \in$ $\mathcal{C}(\mathbb{R}, \mathcal{L}(U, V))$ and Iso $(U, V)$ is an open subset of $\mathcal{L}(U, V), \varrho(\mathfrak{L})$ is open, possibly empty. Hence, $\Sigma(\mathfrak{L})$ is closed. Moreover, the next result holds. Although should be an old result in bifurcation theory, we could not find it stated in this way in the existing literature. So, we will prove it here by completeness.

Lemma 2.1. Suppose $\left(\lambda_{0}, 0\right)$ is a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$. Then, $\lambda_{0} \in \Sigma(\mathfrak{L})$.

Proof. Let $\left(\lambda_{n}, u_{n}\right) \in \mathfrak{F}^{-1}(0)$ with $u_{n} \neq 0$ for all $n \geq 1$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\lambda_{n}, u_{n}\right)=\left(\lambda_{0}, 0\right) \tag{7}
\end{equation*}
$$

Then, setting

$$
\begin{equation*}
\mathfrak{N}(\lambda, u):=\mathfrak{F}(\lambda, u)-\mathfrak{L}(\lambda) u, \quad(\lambda, u) \in \mathbb{R} \times U \tag{8}
\end{equation*}
$$

we have that

$$
\begin{equation*}
0=\mathfrak{F}\left(\lambda_{n}, u_{n}\right)=\mathfrak{L}\left(\lambda_{n}\right) u_{n}+\mathfrak{N}\left(\lambda_{n}, u_{n}\right), \quad n \geq 1 \tag{9}
\end{equation*}
$$

Note that, thanks to (F3) and (4), we also have that

$$
\begin{equation*}
\mathfrak{N}(\lambda, 0)=0, \quad D_{u} \mathfrak{N}(\lambda, 0)=0, \quad \lambda \in \mathbb{R} \tag{10}
\end{equation*}
$$

Suppose $\lambda_{0} \in \varrho(\mathfrak{L})$. Then, since (9) can be re-written as

$$
\mathfrak{L}\left(\lambda_{0}\right) u_{n}=\left[\mathfrak{L}\left(\lambda_{0}\right)-\mathfrak{L}\left(\lambda_{n}\right)\right] u_{n}-\mathfrak{N}\left(\lambda_{n}, u_{n}\right)=0, \quad n \geq 1
$$

and $\mathfrak{L}\left(\lambda_{0}\right) \in \operatorname{Iso}(U, V)$, we find that

$$
u_{n}=\mathfrak{L}^{-1}\left(\lambda_{0}\right)\left[\mathfrak{L}\left(\lambda_{0}\right)-\mathfrak{L}\left(\lambda_{n}\right)\right] u_{n}-\mathfrak{L}^{-1}\left(\lambda_{0}\right) \mathfrak{N}\left(\lambda_{n}, u_{n}\right), \quad n \geq 1
$$

Hence, dividing by $\left\|u_{n}\right\|$ and taking norms yields

$$
\begin{equation*}
1 \leq\left\|\mathfrak{L}^{-1}\left(\lambda_{0}\right)\right\|\left\|\mathfrak{L}\left(\lambda_{0}\right)-\mathfrak{L}\left(\lambda_{n}\right)\right\|+\left\|\mathfrak{L}^{-1}\left(\lambda_{0}\right)\right\| \frac{\left\|\mathfrak{N}\left(\lambda_{n}, u_{n}\right)\right\|}{\left\|u_{n}\right\|}, \quad n \geq 1 \tag{11}
\end{equation*}
$$

By the continuity of $\mathfrak{L}(\lambda)$, (7) implies that

$$
\lim _{n \rightarrow \infty}\left\|\mathfrak{L}\left(\lambda_{0}\right)-\mathfrak{L}\left(\lambda_{n}\right)\right\|=0
$$

Moreover, according to (10),

$$
\mathfrak{N}\left(\lambda_{n}, u_{n}\right)=\mathfrak{N}\left(\lambda_{n}, u_{n}\right)-\mathfrak{N}\left(\lambda_{n}, 0\right)=\int_{0}^{1} D_{u} \mathfrak{N}\left(\lambda_{n}, t u_{n}\right) u_{n} d t
$$

and hence,

$$
\left\|\mathfrak{N}\left(\lambda_{n}, u_{n}\right)\right\| \leq \int_{0}^{1}\left\|D_{u} \mathfrak{N}\left(\lambda_{n}, t u_{n}\right)\right\| d t\left\|u_{n}\right\|, \quad n \geq 1
$$

Thus, owing to (7) and (10), we find from (F2) that

$$
\limsup _{n \rightarrow \infty} \frac{\left\|\mathfrak{N}\left(\lambda_{n}, u_{n}\right)\right\|}{\left\|u_{n}\right\|} \leq \limsup _{n \rightarrow \infty} \int_{0}^{1}\left\|D_{u} \mathfrak{N}\left(\lambda_{n}, t u_{n}\right)\right\| d t=0
$$

Therefore, letting $n \rightarrow \infty$ in (11) yields $1 \leq 0$, which is impossible. This contradiction yields $\lambda_{0} \in \Sigma$ and ends the proof.

## 3. Orientation and degree for Fredholm maps

This section collects the concepts of orientation and topological degree for Fredholm maps of class $\mathcal{C}^{1}$ introduced by P. Benevieri and M. Furi [1]-[3], and a related result of J. López-Gómez and C. Mora-Corral [18]. These concepts sharpen those derived from the parity of P. M. Fitzpatrick and J. Pejsachowicz [10]. Naturally, they are far from being user-friendly by practitioners.

Given three real Banach spaces, $U, V$ and $W$, and $L \in \operatorname{Fred}_{0}(U, V)$, we will denote by $\mathcal{F}(L)$ the (non-empty) set of finite-rank operators $F \in \mathcal{L}(U, V)$ such that $L+F \in$ Iso $(U, V)$. An equivalence relation can be defined in $\mathcal{F}(L)$ by declaring that $F_{1}, F_{2} \in \mathcal{F}(L)$ are equivalent, $F_{1} \sim_{L} F_{2}$, if

$$
\operatorname{det}\left[\left(L+F_{1}\right)^{-1}\left(L+F_{2}\right)\right]>0
$$

Since

$$
\left(L+F_{1}\right)^{-1}\left(L+F_{2}\right)=I_{U}+\left(L+F_{1}\right)^{-1}\left(F_{2}-F_{1}\right)
$$

is a finite rank perturbation of the identity, its determinant can be defined as, e.g., in Section III.4.3 of T. Kato [12]. This relation has two equivalence classes.

Each of them is called an orientation of $L ; L$ is said to be oriented when an orientation has been chosen. In such case, this orientation is denoted by $\mathcal{F}_{+}(L)$ and we set

$$
\mathcal{F}_{-}(L):=\mathcal{F}(L) \backslash \mathcal{F}_{+}(L)
$$

Given two oriented operators, $L_{1} \in \operatorname{Fred}_{0}(U, V)$ and $L_{2} \in \operatorname{Fred}_{0}(V, W)$, their oriented composition is the operator $L_{2} L_{1}$ equipped with the orientation $\mathcal{F}_{+}\left(L_{2} L_{1}\right)$ generated by $L_{2} F_{1}+F_{2} F_{1}+F_{2} L_{1}$, where $F_{1} \in \mathcal{F}_{+}\left(L_{1}\right)$ and $F_{2} \in$ $\mathcal{F}_{+}\left(L_{2}\right)$. It is well defined in the sense that it does not depend on the choice of $F_{1}$ and $F_{2}$.

Let $L \in$ Iso $(U, V)$ be oriented. Its $\operatorname{sign}, \operatorname{sgn} L$, is then defined by

$$
\operatorname{sgn} L:= \begin{cases}+1, & \text { if } 0 \in \mathcal{F}_{+}(L) \\ -1, & \text { if } 0 \in \mathcal{F}_{-}(L)\end{cases}
$$

Te next result is Lemma 2.1 of J. López-Gómez and C. Mora-Corral [18].
Lemma 3.1. Let $L_{1} \in \operatorname{Iso}(U, V)$ and $L_{2} \in \operatorname{Iso}(V, W)$ be two oriented isomorphisms, and consider the oriented composition $L_{2} L_{1}$. Then,

$$
\operatorname{sgn}\left(L_{2} L_{1}\right)=\operatorname{sgn} L_{2} \cdot \operatorname{sgn} L_{1}
$$

Next, we suppose that $X$ is a topological space and $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U, V))$ satisfies $\mathfrak{L}(x) \in \operatorname{Fred}_{0}(U, V)$ for all $x \in X$. An orientation of $\mathfrak{L}$ is a map $X \ni x \mapsto \alpha(x)$ such that $\alpha(x)$ is an orientation of $\mathfrak{L}(x)$ for all $x \in X$, and the map $\alpha$ satisfies the continuity condition that for each $x_{0} \in X$ and $F \in \alpha\left(x_{0}\right)$, there is a neighborhood, $\mathcal{U}$, of $x_{0}$ in $X$ such that $F \in \alpha(x)$ for all $x \in \mathcal{U}$. Although not every $\mathfrak{L}$ admits an orientation, the next result holds (see [1]-[3]).
Proposition 3.2. Suppose $X$ is a simply connected topological space. Then, every map $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U, V))$, with $\mathfrak{L}(x) \in \operatorname{Fred}_{0}(U, V)$ for all $x \in X$, admits two orientations, $\mathcal{F}_{+}(\mathfrak{L})$ and $\mathcal{F}_{-}(\mathfrak{L})$, and each of them is uniquely determined by the orientation of $\mathfrak{L}(x)$, where $x \in X$ is arbitrary.

In this paper $X$ is simply connected because $X=\mathbb{R}$. As soon as $X$ is simply connected and $\mathfrak{L} \in \mathcal{C}(X, \mathcal{L}(U, V))$ satisfies $\mathfrak{L}(x) \in \operatorname{Fred}_{0}(U, V)$ for all $x \in X$, we will think of $\mathfrak{L}$ as oriented by $\mathcal{F}_{+}(\mathfrak{L})$. Moreover, if $g \in \mathcal{C}^{1}(U, V)$ satisfies $D g(x) \in \operatorname{Fred}_{0}(U, V)$ for all $x \in U$, then we will suppose that $g$ is oriented, which means that an orientation, $\mathcal{F}_{+}(D g)$, has been chosen for $D g$. Similarly, any operator $\mathfrak{F} \in \mathcal{C}(\mathbb{R} \times U, V)$ satisfying (F1) and (F2) is assumed to be oriented by choosing an orientation, $\mathcal{F}_{+}\left(D_{u} \mathfrak{F}\right)$, for $D_{u} \mathfrak{F}$. Finally, we denote by $\mathcal{A}$ the set of (admissible) pairs, $(g, \mathcal{U})$, formed by an oriented function $g \in \mathcal{C}^{1}(U, V)$ with $D g(x) \in \operatorname{Fred}_{0}(U, V)$ for all $x \in U$, and an open subset $\mathcal{U} \subset U$ such that $g^{-1}(0) \cap \mathcal{U}$ is compact. According to P. Benevieri and M. Furi [1], a $\mathbb{Z}$-valued degree is defined in $\mathcal{A}$, and it satisfies the same fundamental properties as the Leray-Schauder degree. Among them, the normalization, the additivity and the generalized homotopy-invariance.

## 4. The generalized algebraic multiplicity for Fredholm maps

Subsequently, given an open subinterval $J \subset \mathbb{R}$ and $r \in \mathbb{N} \cup\{\infty, \omega\}$, we will denote by $\mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ the set of maps of class $\mathcal{C}^{r}$ from $J$ to $\mathcal{L}(U, V)$ with values in $\operatorname{Fred}_{0}(U, V) ; \mathcal{C}^{\omega}$ stands for the set of real analytic maps. The next concept plays a pivotal role in the theory of algebraic multiplicities (it goes back to [14, Def. 4.3.1]).

Definition 4.1. Suppose $\mathfrak{L} \in \mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ for some integer $r \geq 1$, and $\lambda_{0} \in J$. Then, $\lambda_{0}$ is said to be an algebraic eigenvalue of $\mathfrak{L}$ if

$$
\operatorname{dim} N\left[\mathfrak{L}\left(\lambda_{0}\right)\right] \geq 1
$$

and there are $C, \delta>0$, and an integer $1 \leq k \leq r$ such that $\mathfrak{L}(\lambda) \in \operatorname{Iso}(U, V)$ and

$$
\begin{equation*}
\left\|\mathfrak{L}^{-1}(\lambda)\right\|_{\mathcal{L}(V, U)} \leq \frac{C}{\left|\lambda-\lambda_{0}\right|^{k}} \quad \text { for all } \lambda \in\left(\lambda_{0}-\delta, \lambda_{0}+\delta\right) \backslash\left\{\lambda_{0}\right\} \tag{12}
\end{equation*}
$$

$\lambda_{0}$ is said to be of order $k i f$, in addition, $k$ is minimal.
The next result is a direct consequence from Theorems 4.4.1 and 4.4.4 of [14]. Note that, in most of the applications, the dependence of $\mathfrak{L}(\lambda)$ in $\lambda$ is real analytic.

Theorem 4.2. Suppose $\mathfrak{L} \in \mathcal{C}^{\omega}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ and

$$
\Sigma:=\{\lambda \in J: \operatorname{dim} N[\mathfrak{L}(\lambda)] \geq 1\}
$$

Then, either $\Sigma=J$, or $\Sigma$ is a discrete subset of $J$. Moreover, if $\Sigma$ is discrete, any $\lambda_{0} \in \Sigma$ must be an algebraic eigenvalue of $\mathfrak{L}(\lambda)$, as discussed by Definition 4.1.

Actually, a complex counterpart of Theorem 4.2 holds (see Chapter 8 of J. López-Gómez and C. Mora-Corral [19]). In the context of the Riesz-Schauder theory, $U=V$ and $\mathfrak{L}$ is given by

$$
\mathfrak{L}(\zeta)=I_{U}-\zeta T, \quad \zeta \in \mathbb{C}
$$

for some $T \in \mathcal{K}(U)$. As $\mathfrak{L}(0)=I_{U}$ is an isomorphism, Theorem 4.2 guarantees that $\Sigma$ is a discrete subset of $\mathbb{C}$. Moreover, any characteristic value of $T$ must be a pole of the resolvent operator $\left(I_{U}-\zeta T\right)^{-1}$.

The next concept was coined by J. Esquinas and J. López-Gómez [8] to generalize the (local) theorem of M. G. Crandall and P. H. Rabinowitz [5] on
bifurcation from simple eigenvalues. Subsequently, given $\mathfrak{L} \in \mathcal{C}^{r}(J, \mathcal{L}(U, V))$ and $\lambda_{0} \in J$, we will denote

$$
\mathfrak{L}_{j}=\frac{1}{j!} \frac{d^{j} \mathfrak{L}}{d \lambda^{j}}\left(\lambda_{0}\right), \quad 0 \leq j \leq r .
$$

Definition 4.3. Suppose $\mathfrak{L} \in \mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ for some $r \geq 1$, and $\lambda_{0} \in$ $J \cap \Sigma$. Then, given an integer $1 \leq k \leq r, \lambda_{0}$ is said to be a $k$-transversal eigenvalue of $\mathfrak{L}(\lambda)$ if

$$
\begin{equation*}
\bigoplus_{j=1}^{k} \mathfrak{L}_{j}\left(N\left[\mathfrak{L}_{0}\right] \cap \cdots \cap N\left[\mathfrak{L}_{j-1}\right]\right) \oplus R\left[\mathfrak{L}_{0}\right]=V \tag{13}
\end{equation*}
$$

with

$$
\operatorname{dim} \mathfrak{L}_{k}\left(N\left[\mathfrak{L}_{0}\right] \cap \cdots \cap N\left[\mathfrak{L}_{k-1}\right]\right) \geq 1
$$

In such case, the algebraic multiplicity of $\mathfrak{L}(\lambda)$ at $\lambda_{0}$, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$, is defined by

$$
\begin{equation*}
\chi\left[\mathfrak{L} ; \lambda_{0}\right]:=\sum_{j=1}^{k} j \cdot \operatorname{dim} \mathfrak{L}_{j}\left(N\left[\mathfrak{L}_{0}\right] \cap \cdots \cap N\left[\mathfrak{L}_{j-1}\right]\right) . \tag{14}
\end{equation*}
$$

Naturally, in case $r=1$, the transversality condition of M. G. Crandall and P. H. Rabinowitz [5] holds if, and only if, $\operatorname{dim} N\left[\mathfrak{L}\left(\lambda_{0}\right)\right]=1$ and $\lambda_{0}$ is a 1-transversal eigenvalue of $\mathfrak{L}$, i.e., if

$$
\mathfrak{L}_{1} \varphi_{0} \notin R\left[\mathfrak{L}_{0}\right], \quad \text { where } N\left[\mathfrak{L}_{0}\right]=\operatorname{span}\left[\varphi_{0}\right] .
$$

Consequently, in this particular case, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]=1$.
The next fundamental result goes back to Chapters 4 and 5 of [14], where the findings of J. Esquinas and J. López-Gómez [8] and J. Esquinas [7] were substantially sharpened. It was collected as part of Theorem 5.3.1 of J. LópezGómez and C. Mora-Corral [19].

Theorem 4.4. Suppose $\mathfrak{L} \in \mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ for some integer $r \geq 1$, and $\lambda_{0} \in J$. Then, the following conditions are equivalent:
(a) $\lambda_{0}$ is an algebraic eigenvalue of order $1 \leq k \leq r$.
(b) There exists $\Phi \in \mathcal{C}^{\omega}\left(J ; \operatorname{Fred}_{0}(U)\right)$ with $\Phi\left(\lambda_{0}\right)=I_{U}$ such that $\lambda_{0}$ is a $k$-transversal eigenvalue of

$$
\mathfrak{L}^{\Phi}(\lambda):=\mathfrak{L}(\lambda) \Phi(\lambda), \quad \lambda \in J
$$

Moreover, $\chi\left[\mathfrak{L}^{\Phi} ; \lambda_{0}\right]$ is independent of the transversalizing family of isomorphisms, $\Phi(\lambda)$. Therefore, the concept of multiplicity

$$
\chi\left[\mathfrak{L} ; \lambda_{0}\right]:=\chi\left[\mathfrak{L}^{\Phi} ; \lambda_{0}\right]
$$

is consistent.
(c) There exist $k$ finite rank projections $P_{j} \in \mathcal{L}(U) \backslash\{0\}, 1 \leq j \leq k$, and a map $\mathfrak{M} \in \mathcal{C}^{r-k}\left(J, \operatorname{Fred}_{0}(U, V)\right)$, with $\mathfrak{M}\left(\lambda_{0}\right) \in \operatorname{Iso}(U, V)$, such that

$$
\begin{equation*}
\mathfrak{L}(\lambda)=\mathfrak{M}(\lambda)\left[\left(\lambda-\lambda_{0}\right) P_{1}+I_{U}-P_{1}\right] \cdots\left[\left(\lambda-\lambda_{0}\right) P_{k}+I_{U}-P_{k}\right] \tag{15}
\end{equation*}
$$

for all $\lambda \in J$. Moreover, for any choice of these projections,

$$
\begin{equation*}
\chi\left[\mathfrak{L} ; \lambda_{0}\right]=\sum_{j=1}^{k} \operatorname{rank} P_{j} . \tag{16}
\end{equation*}
$$

Based on Theorem 4.4, the next result establishes that $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ detects any sign jump of $\mathfrak{L}(\lambda)$ at any algebraic eigenvalue $\lambda_{0}$, as discussed by $P$. Benevieri and M. Furi [3]. Although it goes back to Theorem 3.3 of J. López-Gómez and C. Mora-Corral [18], the original proof will be shortened here.

Theorem 4.5. Suppose $\mathfrak{L} \in \mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ for some integer $r \geq 1$, and $\lambda_{0} \in J$ is an algebraic eigenvalue of $\mathfrak{L}$ or order $1 \leq k \leq r$. Once oriented $\mathfrak{L}$, $\operatorname{sgn} \mathfrak{L}(\lambda)$ changes as $\lambda$ crosses $\lambda_{0}$ if, and only if, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is odd.

Proof. By Theorem 4.4(c), (15) holds. The statement of the theorem is independent of the chosen orientations. For each $1 \leq i \leq k$, the orientation of

$$
\mathcal{E}_{i}(\lambda):=\left(\lambda-\lambda_{0}\right) P_{i}+I_{U}-P_{i}
$$

is defined as $P_{i} \in \mathcal{C}_{+}\left(I_{U}-P_{i}\right)$, and the orientation of $\mathfrak{M}(\lambda)$ by $0 \in \mathcal{C}_{+}\left(\mathfrak{M}\left(\lambda_{0}\right)\right)$. Naturally, the orientation of $\mathfrak{L}(\lambda)$ is defined as the product orientation from (15). Fix $1 \leq i \leq k$ and $\lambda \sim \lambda_{0}, \lambda \neq \lambda_{0}$. Then, $P_{i} \in \mathcal{C}_{+}\left(\left(\lambda-\lambda_{0}\right) P_{i}+I_{U}-P_{i}\right)$ and

$$
\operatorname{det}\left[\mathcal{E}_{i}^{-1}(\lambda)\left(\mathcal{E}_{i}(\lambda)+P_{i}\right)\right]=\operatorname{det}\left[I_{U}+\left(\lambda-\lambda_{0}\right)^{-1} P_{i}\right]=\left[1+\left(\lambda-\lambda_{0}\right)^{-1}\right]^{\operatorname{rank} P_{i}}
$$

Thus,

$$
\operatorname{sgn} \mathcal{E}_{i}(\lambda)=\operatorname{sign}\left(\lambda-\lambda_{0}\right)^{\operatorname{rank} P_{i}} \quad \text { for all } \lambda \sim \lambda_{0}, \lambda \neq \lambda_{0}
$$

Therefore, by (15) and (16), it follows from Lemma 3.1 that

$$
\operatorname{sgn} \mathfrak{L}(\lambda)=\operatorname{sign}\left(\lambda-\lambda_{0}\right)^{\sum_{i=1}^{k} \operatorname{rank} P_{i}}=\operatorname{sign}\left(\lambda-\lambda_{0}\right)^{\chi\left[\mathfrak{L}(\lambda) ; \lambda_{0}\right]}
$$

This ends the proof.
Consequently, according to Theorem 3.1, or Theorem 4.2, of P. Benevieri and M. Furi [3], the next result holds.

Theorem 4.6. Suppose $\mathfrak{L} \in \mathcal{C}^{r}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ for some integer $r \geq 1$, and $\lambda_{0} \in J$ is an algebraic eigenvalue of $\mathfrak{L}$ of order $1 \leq k \leq r$ with $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ odd. Then, for every continuous function $\mathfrak{F}: \mathbb{R} \times U \rightarrow V$ satisfying (F1), (F2), (F3) and $D_{u} \mathfrak{F}(\lambda, 0)=\mathfrak{L},\left(\lambda_{0}, 0\right)$ is a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$ to a continuum of non-trivial solutions.

The characterization theorem of J. Esquinas and J. López-Gómez [8] establishes that Theorem 4.6 is optimal, in the sense that whenever $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is even there is a smooth $\mathfrak{F}$ satisfying (F1), (F2), (F3) and $D_{u} \mathfrak{F}(\lambda, 0)=\mathfrak{L}$, for which $\left(\lambda_{0}, 0\right)$ is not a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$ (see Chapter 4 of [14]). Consequently, under the general assumptions of Theorem 4.6, the following conditions are equivalent:

- $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ is an odd integer.
- $\operatorname{sgn} \mathfrak{L}(\lambda)$ changes as $\lambda$ crosses $\lambda_{0}$.
- $\lambda_{0}$ is a nonlinear eigenvalue of $\mathfrak{L}(\lambda)$, as discussed by Definition 1.1.2 of [14].

As a direct consequence from Theorem 4.6, the next generalized version of the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [5] holds. As $\mathfrak{F}$ is not required to be of class $\mathcal{C}^{2}$, the bifurcating continuum is not necessarily a $\mathcal{C}^{1}$ curve.

Corollary 4.7. Suppose $\mathfrak{L} \in \mathcal{C}^{1}\left(J, \operatorname{Fred}_{0}(U, V)\right)$ and $\lambda_{0} \in J$ is a simple eigenvalue $\mathfrak{L}$ in the sense that

$$
\mathfrak{L}_{1} \varphi_{0} \notin R\left[\mathfrak{L}_{0}\right], \quad \text { where } N\left[\mathfrak{L}_{0}\right]=\operatorname{span}\left[\varphi_{0}\right] .
$$

Then, $\chi\left[\mathfrak{L} ; \lambda_{0}\right]=1$ and hence, for every continuous function $\mathfrak{F}: \mathbb{R} \times U \rightarrow V$ satisfying (F1), (F2), (F3) and $D_{u} \mathfrak{F}(\lambda, 0)=\mathfrak{L},\left(\lambda_{0}, 0\right)$ is a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$ to a continuum of non-trivial solutions.

When, in addition, $\mathfrak{F}$ is of class $\mathcal{C}^{2}$, then the bifurcating continuum consists of a $\mathcal{C}^{1}$ curve, as established by the theorem of M. G. Crandall and P. H. Rabinowitz [5].

## 5. A sharp global bifurcation theorem for Fredholm operators

This section polishes the main global bifurcation theorem of J. López-Gómez and C. Mora-Corral [18] and extracts some important consequences from it. It should be noted that it is a substantial extension of all available results in the literature and, in particular, of Theorem 4.2 of P. Benevieri and M. Furi [3].

Given two non-empty subsets of $\mathbb{R}, A$ and $B$, it is said that $A<B$ if $a<b$ for all $(a, b) \in A \times B$. A family, $\mathcal{A}$, whose elements are subsets of a topological space, $X$, is said to be locally finite if for every $x \in X$ there is a neighborhood, $\Omega$, of $x$ such that $\{A \in \mathcal{A}: A \cap \Omega \neq \emptyset\}$ is finite.

Subsequently, we consider

$$
\mathfrak{L}(\lambda)=D_{u} \mathfrak{F}(\lambda, 0), \quad \lambda \in \mathbb{R}
$$

and its spectrum, $\Sigma=\Sigma(\mathfrak{L})$. The following concept is very useful.
Definition 5.1. Given $r, s \in \mathbb{Z} \cup\{-\infty, \infty\}$, with $r \leq s$, a family, $\left\{K_{j}\right\}_{j=r}^{s}$, of disjoint closed subsets of $\mathbb{R}$ is said to be admissible with respect to $\Sigma$ if

$$
\begin{equation*}
\Sigma=\bigcup_{j=r}^{s} K_{j}, \quad K_{j}<K_{j+1}, \quad j \in \mathbb{Z} \cap[r, s-1] \tag{17}
\end{equation*}
$$

and each of the next conditions is satisfied:
(a) If $r \in \mathbb{Z}$, then either $K_{r}$ is compact, or $K_{r}=(-\infty, a]$ for some $a \in \mathbb{R}$.
(b) If $s \in \mathbb{Z}$, then either $K_{s}$ is compact, or $K_{s}=[b,+\infty)$ for some $b \in \mathbb{R}$.
(c) $K_{j}$ is compact for all $j \in \mathbb{Z} \cap(r, s)$.

Naturally, such a family $\left\{K_{j}\right\}_{j=r}^{s}$ is locally finite, and $\Sigma$ admits many admissible families, because $\Sigma$ is a closed subset of $\mathbb{R}$ and any bounded closed subset of $\mathbb{R}$ is compact. In most of applications, $\mathfrak{L}(\lambda)$ is real analytic in $\lambda$ and hence, thanks to Theorem 4.2, either $\Sigma=\mathbb{R}$, or $\Sigma$ is discrete. Therefore, $\Sigma$ is discrete if $\mathfrak{L}(a) \in \operatorname{Iso}(U, V)$ for some $a \in \mathbb{R}$. In such cases, each of the $K_{j}$ 's can be taken as a single point of the spectrum $\Sigma$, which is the most common situation covered in the specialized literature.

Associated to any admissible family of disjoint closed subsets with respect to $\Sigma,\left\{K_{j}\right\}_{j=r}^{s}$, there is a locally finite family of open subintervals of $\mathbb{R},\left\{J_{i}\right\}_{i=r-1}^{s}$, defined by

$$
\begin{equation*}
J_{i}:=\left(\max K_{i}, \min K_{i+1}\right), \quad i \in \mathbb{Z} \cap[r, s-1] \tag{18}
\end{equation*}
$$

if $r=-\infty$ and $s=+\infty$. When $r \in \mathbb{Z}$ and $K_{r}$ is compact, we should add the interval $J_{r-1}:=\left(-\infty, \min K_{r}\right)$ to the previous family. Similarly, when $s \in \mathbb{Z}$ and $K_{s}$ is compact, $J_{s}:=\left(\max K_{s},+\infty\right)$ should be also added to the previous ones. By construction,

$$
J_{i} \cap \Sigma=\emptyset \quad \text { for all } \quad i \in \mathbb{Z} \cap[r-1, s]
$$

and

$$
J_{i-1}<J_{i} \quad \text { for all } \quad i \in \mathbb{Z} \cap[r, s] .
$$

Moreover, the map

$$
\bigcup_{i=r-1}^{s} J_{i} \ni \lambda \mapsto \operatorname{sgn} \mathfrak{L}(\lambda) \in\{-1,1\}
$$

is continuous. Hence, for every $i \in \mathbb{Z} \cap[r-1, s]$, there exists $a_{i} \in\{-1,1\}$ such that

$$
\operatorname{sgn} \mathfrak{L}(\lambda)=a_{i} \quad \text { for all } \lambda \in J_{i} .
$$

Consequently, a parity map, $\mathcal{P}$, associated to the family $\left\{J_{i}\right\}_{i=r-1}^{s}$, or, equivalently, $\left\{K_{j}\right\}_{j=r}^{s}$, can defined through

$$
\begin{equation*}
\mathcal{P}: \mathbb{Z} \cap[r-1, s] \rightarrow\{-1,0,1\}, \quad \mathcal{P}(i):=\frac{a_{i}-a_{i-1}}{2} . \tag{19}
\end{equation*}
$$

It should be noted that, setting

$$
\Gamma_{0}:=\left\{i \in \mathbb{Z} \cap[r, s]: a_{i-1}=a_{i}\right\}, \quad \Gamma_{1}:=\left\{i \in \mathbb{Z} \cap[r, s]: a_{i-1} \neq a_{i}\right\}
$$

the parity $\mathcal{P}$ satisfies the following properties:

- $\mathcal{P}(i)=0$ if $i \in \Gamma_{0}$.
- $\mathcal{P}(i)= \pm 1$ if $i \in \Gamma_{1}$.
- $\mathcal{P}(i) \mathcal{P}(j)=-1$ if $i, j \in \Gamma_{1}$ with $i<j$ and $(i, j) \cap \Gamma_{1}=\emptyset$.

Moreover, any map defined in $\mathbb{Z} \cap[r, s]$ satisfying these properties must be either $\mathcal{P}$ or $-\mathcal{P}$. Thus, either $\Gamma_{0}$, or $\Gamma_{1}$, determines $\mathcal{P}$ up to a change of sign.

Subsequently, we consider a continuous map $\mathfrak{F}: \mathbb{R} \times U \rightarrow V$ satisfying (F1), (F2) and (F3), with $\mathfrak{L}=D_{u} \mathfrak{F}(\cdot, 0)$, and an admissible family with respect to $\Sigma,\left\{K_{j}\right\}_{j=r}^{s}$, with associated family of open intervals $\left\{J_{i}\right\}_{i=r-1}^{s}$, and we set

$$
\begin{align*}
\mathfrak{S} & :=\operatorname{closure}\left(\mathfrak{F}^{-1}(0) \cap[\mathbb{R} \times(U \backslash\{0\})]\right) \cup \bigcup_{j=r}^{s}\left[K_{j} \times\{0\}\right]  \tag{20}\\
& =\text { closure }\left(\mathfrak{F}^{-1}(0) \cap[\mathbb{R} \times(U \backslash\{0\})]\right) \cup[\Sigma \times\{0\}]
\end{align*}
$$

$\mathfrak{S}$ is usually refereed to as the set of non-trivial solutions of $\mathfrak{F}=0 . \quad$ By Lemma 2.1, it consists of the pairs $(\lambda, u) \in \mathfrak{F}^{-1}(0)$ with $u \neq 0$ plus all possible bifurcation points from $(\lambda, 0), \Sigma \times\{0\}$. Since $\Sigma$ is closed, $\mathfrak{S}$ is closed.

The next result is an easy consequence of Theorem 5.4 of J. López-Gómez and C. Mora-Corral [18], whose proof is based on the degree of P. Benevieri and M. Furi [1]-[3] sketched in Section 3. It extends some previous findings of [16] and [17].

Theorem 5.2. Suppose $\mathfrak{C}$ is a compact component of $\mathfrak{S}$. Then,

$$
\mathcal{B}:=\left\{j \in \mathbb{Z} \cap[r, s]: \mathfrak{C} \cap\left(K_{j} \times\{0\}\right) \neq \emptyset\right\}
$$

is finite, possibly empty. Moreover,

$$
\begin{equation*}
\sum_{i \in \mathcal{B}} \mathcal{P}(i)=0 \quad \text { if } \quad \mathcal{B} \neq \emptyset \tag{21}
\end{equation*}
$$

When $\mathcal{B}=\emptyset, \mathfrak{C}$ is an isola with respect to the trivial solution $(\lambda, 0)$. The existence of isolas is well documented in the context of nonlinear differential equations (see, e.g., J. López-Gómez [14, Section 2.5.2], S. Cano-Casanova et al. [4] and J. López-Gómez and M. Molina-Meyer [15]).

When $\mathcal{B} \neq \emptyset, \mathfrak{C}$ bifurcates from the trivial solution $(\lambda, 0)$. In such case, $\mathcal{B}$ provides us with the set of compact subsets, $K_{j}$ 's, of $\Sigma$ where $\mathfrak{C}$ bifurcates from $(\lambda, 0)$. Note that if $r \in \mathbb{Z}$ and $K_{r}=(-\infty, a]$ for some $a \in \mathbb{R}$, then $r \notin \mathcal{B}$. Indeed, if

$$
\mathfrak{C} \cap((-\infty, a] \times\{0\}) \neq \emptyset
$$

then $(-\infty, a] \times\{0\} \subset \mathfrak{C}$, because $\mathfrak{C}$ is a closed and connected subset of $\mathfrak{S}$ maximal for the inclusion. But this is impossible if $\mathfrak{C}$ is bounded. Therefore, $K_{j}$ is compact for all $j \in \mathcal{B}$ if $\mathfrak{C}$ is compact. In particular, $\mathcal{B}$ must be finite. J. López-Gómez [14, Section 2.5.2] and J. López-Gómez and M. Molina-Meyer [15] gave a number of examples of compact components, $\mathfrak{C}$, with $\mathcal{B} \neq \emptyset$.

REMARK 5.3. As an immediate consequence from (21), when $\mathcal{P}(i)= \pm 1$ for some $i \in \mathcal{B}$, there exists another $j \in \mathcal{B} \backslash\{i\}$ with $\mathcal{P}(j)=\mp 1$. Therefore, in such case, the component $\mathfrak{C}$ links $K_{i} \times\{0\}$ to $K_{j} \times\{0\}$. Actually, there is an even number of $i \in \mathcal{B}$ 's for which $\mathcal{P}(i)= \pm 1$.

Theorem 5.2 is a substantial generalization of Theorem 6.3.1 of J. LópezGómez [14]. Consequently, it extends to the general framework of Fredholm operators covered in this paper the most pioneering global results of P. H. Rabinowitz [23], L. Nirenberg [21], J. Ize [11] and R. J. Magnus [20]; most of them stated for the special case when $U=V$ and

$$
\begin{equation*}
\mathfrak{L}(\lambda)=I_{U}-\lambda T, \quad T \in \mathcal{K}(U), \tag{22}
\end{equation*}
$$

in the context of the local theorem of M. A. Krasnoselskij [13]. Indeed, in Theorem 3.4.1 of of L. Nirenberg [21], attributed to P. H. Rabinowitz there in, L. Nirenberg proved that if the component $\mathfrak{C}$ is compact, then
"C contains a finite number of points $\left(\lambda_{j}, 0\right)$ with $1 / \lambda_{j}$ eigenvalues of $T$. Furthermore the number of such points having odd multiplicity is even."

When (22) holds, since $\mathfrak{L}(0)=I_{U} \in$ Iso $(U)$, thanks to Theorem 4.2, $\Sigma(\mathfrak{L})$ is discrete and every $\lambda_{0} \in \Sigma$ must be an algebraic eigenvalue of $\mathfrak{L}$. Moreover,
according to Theorem 5.4.1 of J. López-Gómez [14],

$$
\chi\left[\mathfrak{L} ; \lambda_{0}\right]=\operatorname{dim} \bigcup_{k=1}^{\infty} N\left[\left(\lambda_{0}^{-1}-T\right)^{k}\right]
$$

i.e., $\chi\left[\mathfrak{L} ; \lambda_{0}\right]$ equals the classical concept of algebraic multiplicity.

More generally, by Theorems 4.2 and 4.5 , when $\mathfrak{L}(\lambda)$ is a real analytic family of Fredholm operators of index zero such that $\mathfrak{L}(a)$ is an isomorphism for some $a \in \mathbb{R}, \Sigma(\mathfrak{L})$ is discrete and if

$$
\Sigma=\left\{\lambda_{j}: j \in I\right\}
$$

for some $I \subset \mathbb{Z}$ and we take $K_{j}=\left\{\lambda_{j}\right\}$ for all $j \in I$, then $\mathcal{P}(j)= \pm 1$ if, and only if, $\chi\left[\mathfrak{L} ; \lambda_{j}\right]$ is odd. Therefore, due to Theorem 5.2, if

$$
\mathfrak{C} \cap(\mathbb{R} \times\{0\})=\left\{\left(\lambda_{i_{1}}, 0\right), \ldots .,\left(\lambda_{i_{N}}, 0\right)\right\}
$$

then

$$
\sum_{j=1}^{N} \mathcal{P}\left(\lambda_{i_{j}}\right)=0
$$

Consequently, the number of eigenvalues, $\lambda_{i_{j}}$, with an odd multiplicity must be even, likewise in the classical context of P. H. Rabinowitz [23] and L. Nirenberg [21], though in the general setting of this paper, $\Sigma$ might not be a discrete set and $\mathfrak{F}(\lambda, \cdot)$ is not assumed to be a compact perturbation of the identity map, but a general Fredholm operator of index zero.

## 6. Two obvious-for-experts consequences of Theorem 5.2

As an immediate consequence of Theorem 5.2, the next generalized version of the global alternative of P. H. Rabinowitz [23] holds. Note that it is a substantial extension of Theorem 4.2 of P. Benevieri and M. Furi [3].

Theorem 6.1. Suppose $\mathfrak{C}$ is a component of $\mathfrak{S}$ such that

$$
\mathfrak{S} \cap\left(K_{j_{0}} \times\{0\}\right) \neq \emptyset
$$

for some $j_{0} \in \mathcal{B}$ with $\mathcal{P}\left(j_{0}\right)= \pm 1$. Then, either
(A1) $\mathfrak{C}$ is not compact; or
(A2) there exists another $\mathcal{B} \ni j_{1} \neq j_{0}$ with $\mathcal{P}\left(j_{1}\right)=\mp 1$ such that

$$
\mathfrak{S} \cap\left(K_{j_{1}} \times\{0\}\right) \neq \emptyset
$$

Consequently, $\mathfrak{S}$ links $K_{j_{0}} \times\{0\}$ to $K_{j_{1}} \times\{0\}$.

As the degree of P. Benevieri and M. Furi extends the concept of parity introduced by P. M. Fitzpatrick and J. Pejsachowicz, also Theorem 6.1 of J. Pejsachowicz and P. J. Rabier [22] holds from the previous result.

As another corollary of Theorem 5.2, the following global version of the local theorem of M. G. Crandall and P H. Rabinowitz [5] holds. It should be noted that it is a substantial generalization of Theorem 4.3 of J. Shi and X. Wang [24].

Theorem 6.2. Suppose $\mathfrak{L} \in \mathcal{C}^{1}\left(\mathbb{R}, \operatorname{Fred}_{0}(U, V)\right)$ and $\lambda_{0} \in \mathbb{R}$ is a simple eigenvalue $\mathfrak{L}$, as discussed by M. G. Crandall and P. H. Rabinowitz [5], i.e.,

$$
\begin{equation*}
\mathfrak{L}^{\prime}\left(\lambda_{0}\right) \varphi_{0} \notin R\left[\mathfrak{L}\left(\lambda_{0}\right)\right], \quad \text { where } N\left[\mathfrak{L}\left(\lambda_{0}\right)\right]=\operatorname{span}\left[\varphi_{0}\right] \tag{23}
\end{equation*}
$$

Then, for every continuous function $\mathfrak{F}: \mathbb{R} \times U \rightarrow V$ satisfying (F1), (F2), (F3) and $D_{u} \mathfrak{F}(\cdot, 0)=\mathfrak{L},\left(\lambda_{0}, 0\right)$ is a bifurcation point from $(\lambda, 0)$ to a continuum of non-trivial solutions of $\mathfrak{F}=0$.

For any of these $\mathfrak{F}$ 's, let $\left\{K_{j}\right\}_{j=r}^{s}$ be an admissible family of disjoint closed subsets of $\Sigma$ with $K_{0}=\left\{\lambda_{0}\right\}$, and let $\mathfrak{C}$ be the component of $\mathfrak{S}$ such that $\left(\lambda_{0}, 0\right) \in \mathfrak{C}$. Then, either
(a) $\mathfrak{C}$ is not compact; or
(b) there is another $\Sigma \ni \lambda_{1} \neq \lambda_{0}$ with $\left(\lambda_{1}, 0\right) \in \mathfrak{C}$.

Actually, if $\mathfrak{C}$ is compact, then there exists $N \geq 1$ such that

$$
\left(K_{j} \times\{0\}\right) \cap \mathfrak{C} \neq \emptyset \quad \text { if, and only if, } \quad j \in\left\{j_{i_{1}}, \ldots, j_{i_{N}}\right\} \subset \mathbb{Z} \cap[r, s]
$$

with $j_{i_{k}}=0$ for some $k \in\{1, \ldots, N\}$. Moreover,

$$
\sum_{k=1}^{N} \mathcal{P}\left(j_{i_{k}}\right)=0
$$

Therefore, $\mathfrak{C}$ links $\left(\lambda_{0}, 0\right)$ to an odd number of $K_{j} \times\{0\}$ 's with parity $\pm 1$.
Proof. By Definition 4.3, $\lambda_{0}$ is a 1-transversal eigenvalue of $\mathfrak{L}(\lambda)$ with

$$
\chi\left[\mathfrak{L} ; \lambda_{0}\right]=1 .
$$

Thus, by Theorem 4.4, $\lambda_{0}$ is an algebraic eigenvalue of $\mathfrak{L}(\lambda)$ of order one, as discussed by Definition 4.1. In particular, $\mathfrak{L}(\lambda) \in \operatorname{Iso}(U, V)$ for $\lambda \sim \lambda_{0}, \lambda \neq \lambda_{0}$. Thus, by Theorem 4.5, sgn $\mathfrak{L}(\lambda)$ changes of sign as $\lambda$ crosses $\lambda_{0}$. Therefore, $\mathcal{P}(0)= \pm 1$. The remaining assertions of the theorem are obvious consequences of Theorem 5.2.

## 7. Unilateral bifurcation from geometrically simple eigenvalues

Throughout this section, besides (F1), (F2) and (F3), we asume that
(C) $U$ is a subspace of $V$ with compact inclusion $U \hookrightarrow V$.
(F4) The map

$$
\begin{equation*}
\mathfrak{N}(\lambda, u):=\mathfrak{F}(\lambda, u)-D_{u} \mathfrak{F}(\lambda, 0) u, \quad(\lambda, u) \in \mathbb{R} \times U \tag{24}
\end{equation*}
$$

admits a continuous extension, also denoted by $\mathfrak{N}$, to $\mathbb{R} \times V$.
As usual, we denote $\mathfrak{L}:=D_{u} \mathfrak{F}(\lambda, 0)$, and $\left\{K_{j}\right\}_{j=r}^{s}$, with $r \leq 0 \leq s$, stands for an admissible family of closed subintervals of $\mathbb{R}$ with respect to $\Sigma=\Sigma(\mathfrak{L})$ such that

$$
\begin{equation*}
K_{0}=\left\{\lambda_{0}\right\}, \quad \operatorname{dim} N\left[\mathfrak{L}_{0}\right]=1 \tag{25}
\end{equation*}
$$

In other words, $\lambda_{0}$ is assumed to be an isolated eigenvalue of $\mathfrak{L}$ with onedimensional kernel. Let $\varphi_{0} \in U$ be such that

$$
\begin{equation*}
N\left[\mathfrak{L}_{0}\right]=\operatorname{span}\left[\varphi_{0}\right], \quad\left\|\varphi_{0}\right\|=1 \tag{26}
\end{equation*}
$$

and consider a closed subspace $Y \subset U$ such that

$$
U=N\left[\mathfrak{L}_{0}\right] \oplus Y
$$

According to the Hahn-Banach theorem, there exists $\varphi_{0}^{*} \in U^{\prime}$ such that

$$
Y=\left\{u \in U:\left\langle\varphi_{0}^{*}, u\right\rangle=0\right\}=N\left[\varphi_{0}^{*}\right], \quad\left\langle\varphi_{0}^{*}, \varphi_{0}\right\rangle=1
$$

where $\langle\cdot, \cdot\rangle$ stands for the $\left\langle U^{\prime}, U\right\rangle$-duality. In particular, each $u \in U$ can be uniquely decomposed as

$$
u=s \varphi_{0}+y
$$

for some $(s, y) \in \mathbb{R} \times Y$. Necessarily, $s:=\left\langle\varphi_{0}^{*}, u\right\rangle$.
As in P. H. Rabinowitz [23] and J. López-Gómez [14, Section 6.4], for each $\varepsilon>0$ and $\eta \in(0,1)$, we consider

$$
Q_{\varepsilon, \eta}:=\left\{(\lambda, u) \in \mathbb{R} \times U:\left|\lambda-\lambda_{0}\right|<\varepsilon, \quad\left|\left\langle\varphi_{0}^{*}, u\right\rangle\right|>\eta\|u\|\right\} .
$$

Since $u \mapsto\left|\left\langle\varphi_{0}^{*}, u\right\rangle\right|-\eta\|u\|$ is continuous, $Q_{\varepsilon, \eta}$ is open. Moreover, it consists of

$$
\begin{aligned}
& Q_{\varepsilon, \eta}^{+}:=\left\{(\lambda, u) \in \mathbb{R} \times U:\left|\lambda-\lambda_{0}\right|<\varepsilon, \quad\left\langle\varphi_{0}^{*}, u\right\rangle>\eta\|u\|\right\} \\
& Q_{\varepsilon, \eta}^{-}:=\left\{(\lambda, u) \in \mathbb{R} \times U:\left|\lambda-\lambda_{0}\right|<\varepsilon, \quad\left\langle\varphi_{0}^{*}, u\right\rangle<-\eta\|u\|\right\}
\end{aligned}
$$

The next counterpart of [14, Le. 6.4.1] holds. Note that $\left(\lambda_{0}, 0\right)$ might not be a bifurcation point of $\mathfrak{F}=0$ from $(\lambda, 0)$ because we are not imposing sgn $\mathfrak{L}(\lambda)$ to change sign as $\lambda$ crosses $\lambda_{0}$.

Proposition 7.1. Suppose $\mathfrak{F}$ satisfies (F1)-(F4), (C) and (25). Then, for sufficiently small $\varepsilon>0$, there exists $\delta_{0}=\delta_{0}(\eta)>0$ such that for every $\delta \in$ $\left(0, \delta_{0}\right)$,

$$
\begin{equation*}
\mathfrak{S}_{0, \delta}:=\left[\mathfrak{S} \backslash\left\{\left(\lambda_{0}, 0\right)\right\}\right] \cap B_{\delta}\left(\lambda_{0}, 0\right) \subset Q_{\varepsilon, \eta} . \tag{27}
\end{equation*}
$$

Moreover, for each $(\lambda, u) \in \mathfrak{S}_{0, \delta}$, there are $s \in \mathbb{R}$ and $y \in Y$ (unique) such that

$$
\begin{equation*}
u=s \varphi_{0}+y \quad \text { with } \quad|s|>\eta\|u\| \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda=\lambda_{0}+o(1) \quad \text { and } \quad y=o(s) \quad \text { as } \quad s \rightarrow 0 . \tag{29}
\end{equation*}
$$

Proof. The proof of the first claim is based on the next two lemmas of technical nature.

Lemma 7.2. Suppose $\mathfrak{F}$ satisfies (F1)-(F4) and (C). Then, $\mathfrak{N}: \mathbb{R} \times U \rightarrow V$ is a compact operator, in the sense that $\overline{T(B)}$ is compact for all bounded subset $B \subset \mathbb{R} \times U$.

Lemma 7.3. Suppose $a<b$ satisfy $a, b \in \varrho(\mathfrak{L})$. Then, there exists a continuous map, $\Phi:[a, b] \rightarrow \mathcal{L}(V, U)$, such that

$$
\Phi(\lambda) \in \operatorname{Iso}(V, U) \quad \text { and } \quad \mathcal{K}(\lambda) \equiv I_{U}-\Phi(\lambda) \mathfrak{L}(\lambda) \in \mathcal{K}(U) \quad \text { for all } \lambda \in[a, b]
$$

Lemma 7.3 goes back to P. M. Fitzpatrick and J. Pejsachowicz [9], [10]. Next, we will give the proof of Lemma 7.2. Let $\left(\lambda_{n}, u_{n}\right) \in \mathbb{R} \times U, n \geq 1$, be a bounded sequence. As $\left\{\lambda_{n}\right\}_{n \geq 1}$ is bounded in $\mathbb{R}$ we can extract a subsequence, relabeled by $n$, such that $\lim _{n \rightarrow \infty} \lambda_{n}=\lambda_{\omega}$ for some $\lambda_{\omega} \in \mathbb{R}$. According to (C), we can extract a subsequence of $\left\{u_{n}\right\}_{n \geq 1}$, labeled again by $n$, such that $\lim _{n \rightarrow \infty} u_{n}=v_{\omega}$ for some $v_{\omega} \in V$. Thus, owing to (F4), we find that

$$
\lim _{n \rightarrow \infty} \mathfrak{N}\left(\lambda_{n}, u_{n}\right)=\mathfrak{N}\left(\lambda_{\omega}, v_{\omega}\right)
$$

which ends the proof of Lemma 7.2.
As $\lambda_{0}$ is an isolated point of $\Sigma$, there is $\varepsilon_{0}>0$ such that

$$
\Sigma \cap\left[\lambda_{0}-\varepsilon_{0}, \lambda_{0}+\varepsilon_{0}\right]=\left\{\lambda_{0}\right\} .
$$

Thus, by Lemma 7.3, there exists a continuous map

$$
\Phi:\left[\lambda_{0}-\varepsilon_{0}, \lambda_{0}+\varepsilon_{0}\right] \rightarrow \mathcal{L}(V, U)
$$

such that

$$
\Phi(\lambda) \in \operatorname{Iso}(V, U) \quad \text { and } \quad \mathcal{K}(\lambda) \equiv I_{U}-\Phi(\lambda) \mathfrak{L}(\lambda) \in \mathcal{K}(U) \quad \text { if } \quad\left|\lambda-\lambda_{0}\right| \leq \varepsilon_{0}
$$

As for $\left|\lambda-\lambda_{0}\right| \leq \varepsilon_{0}$ the equation $\mathfrak{F}(\lambda, u)=0$ can be equivalently written as

$$
\Phi(\lambda) \mathfrak{F}(\lambda, u)=0
$$

it becomes apparent that $\mathfrak{F}(\lambda, u)=0$ can be expressed as

$$
\begin{equation*}
u-\mathcal{K}(\lambda) u+\Phi(\lambda) \mathfrak{N}(\lambda, u)=0, \quad\left|\lambda-\lambda_{0}\right| \leq \varepsilon_{0}, \quad u \in U \tag{30}
\end{equation*}
$$

Since $\mathcal{K}(\lambda) \in \mathcal{K}(U)$ and, due to Lemma $7.2, \Phi(\lambda) \mathfrak{N}(\lambda, u): \mathbb{R} \times U \rightarrow U$ is compact, the proof of Lemma 6.4 .1 of J. López-Gómez [14] can be adapted mutatis mutandis to complete the proof.

By Proposition 7.1, the following counterpart of Proposition 6.4.2 of [14] holds.
Proposition 7.4. Suppose $\mathfrak{F}$ satisfies (F1)-(F4), (C), (25), and, once oriented $\mathfrak{L}(\lambda), \operatorname{sgn} \mathfrak{L}(\lambda)$ changes sign as $\lambda$ crosses $\lambda_{0}$. According to Theorem 3.1 of $P$. Benevieri and M. Furi [3], $\mathfrak{S}$ has a (non-trivial) component $\mathfrak{C}$ with $\left(\lambda_{0}, 0\right) \in \mathfrak{C}$. Then, for every $\varepsilon \in\left(0, \varepsilon_{0}\right), \mathfrak{C}$ possesses a subcontinuum in each of the cones $Q_{\varepsilon, \eta}^{+} \cup\left\{\left(\lambda_{0}, 0\right)\right\}$ and $Q_{\varepsilon, \eta}^{-} \cup\left\{\left(\lambda_{0}, 0\right)\right\}$ each of which links $\left(\lambda_{0}, 0\right)$ with $\partial B_{\delta}\left(\lambda_{0}, 0\right)$ for sufficiently small $\delta>0$.
Proof. Pick $\varepsilon \in\left(0, \varepsilon_{0}\right)$. By Theorem 3.1 of P. Benevieri and M. Furi [3] and Proposition 7.1, the result is true for at least one of the cones. Suppose it fails, for example, for $Q_{\varepsilon, \eta}^{-}$. Then, no continuum $\tilde{\mathfrak{C}} \subset Q_{\varepsilon, \eta}^{-} \cup\left\{\left(\lambda_{0}, 0\right)\right\}$ exists with $\left(\lambda_{0}, 0\right) \in \tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{C}} \cap \partial B_{\delta}\left(\lambda_{0}, 0\right) \neq \emptyset$ for sufficiently small $\delta>0$. Moreover, owing to Lemma 7.3, there is a continuous map $\Phi:\left[\lambda_{0}-\varepsilon, \lambda_{0}+\varepsilon\right] \rightarrow \mathcal{L}(V, U)$, such that

$$
\Phi(\lambda) \in \operatorname{Iso}(V, U) \quad \text { and } \quad \mathcal{K}(\lambda) \equiv I_{U}-\Phi(\lambda) \mathfrak{L}(\lambda) \in \mathcal{K}(U) \quad \text { if } \quad\left|\lambda-\lambda_{0}\right| \leq \varepsilon
$$

Thus, by Lemmas 7.2 and $7.3, \mathfrak{F}=0$ can be equivalently written in the form

$$
\begin{equation*}
\mathfrak{G}(\lambda, u):=u-\mathcal{K}(\lambda) u+\mathfrak{M}(\lambda, u)=0, \quad\left|\lambda-\lambda_{0}\right| \leq \varepsilon, \quad u \in U \tag{31}
\end{equation*}
$$

where $\mathcal{K}(\lambda)$ and

$$
\mathfrak{M}(\lambda, u):=\Phi(\lambda) \mathfrak{N}(\lambda, u), \quad\left|\lambda-\lambda_{0}\right| \leq \varepsilon, \quad u \in U
$$

are compact operators. Now, as in the proof of [14, Pr. 6.4.2], we define

$$
\hat{\mathfrak{G}}(\lambda, u):=u-\mathcal{K}(\lambda) u+\hat{\mathfrak{M}}(\lambda, u)
$$

as follows

$$
\hat{\mathfrak{M}}(\lambda, u):= \begin{cases}\mathfrak{M}(\lambda, u) & \text { if }(\lambda, u) \in Q_{\varepsilon, \eta}^{-}, \\ -\frac{\left\langle\varphi_{0}^{*}, u\right\rangle}{\eta\|u\|} \mathfrak{M}\left(\lambda,-\eta\|u\| \varphi_{0}+y\right) & \text { if }-\eta\|u\| \leq\left\langle\varphi_{0}^{*}, u\right\rangle \leq 0, \\ -\hat{\mathfrak{M}}(\lambda,-u) & \text { if }\left\langle\varphi_{0}^{*}, u\right\rangle \geq 0 .\end{cases}
$$

The map $\hat{\mathfrak{M}}$ satisfies the same continuity and compactness properties as $\mathfrak{M}$ and, in addition, it is odd in $u$. Thus, $\hat{\mathfrak{G}}$ also is odd in $u$.

On the other hand, as $\operatorname{sgn} \mathfrak{L}(\lambda)$ changes as $\lambda$ crosses $\lambda_{0}$, according to P . Benevieri and M. Furi [2, Sect. 5], the parity of P. M. Fitzpatrick and J. Pejsachowicz [9] of $\mathfrak{L}(\lambda)$ over $\left[\lambda_{0}-\varepsilon, \lambda_{0}+\varepsilon\right]$ equals -1 . i.e., the Leray-Schauder degree $\operatorname{Deg}\left(I_{U}-\mathcal{K}(\lambda), B_{R}(0)\right)$ changes as $\lambda$ crosses $\lambda_{0}$. Consequently, thanks to Theorem 6.2.1 of J. López-Gómez [14], there is a component, $\hat{\mathfrak{C}}$ of nontrivial solutions of $\hat{\mathfrak{G}}=0$ with $\left(\lambda_{0}, 0\right) \in \hat{\mathfrak{C}}$. According to Lemma 6.4.1 of [14], there exists $\delta_{0}>0$ such that

$$
\hat{\mathfrak{C}} \cap B_{\delta}\left(\lambda_{0}, 0\right) \subset Q_{\varepsilon, \eta} \cup\left\{\left(\lambda_{0}, 0\right)\right\} \quad \text { for all } \delta \in\left(0, \delta_{0}\right]
$$

Moreover, by the homotopy invariance of the degree, from Theorem 5.2 it becomes apparent that there exists $\delta_{1} \in\left(0, \delta_{0}\right)$ such that

$$
\begin{equation*}
\hat{\mathfrak{C}} \cap \partial B_{\delta}\left(\lambda_{0}, 0\right) \cap Q_{\varepsilon, \eta} \neq \emptyset \quad \text { for all } \delta \in\left(0, \delta_{1}\right) \tag{32}
\end{equation*}
$$

On the other hand,

$$
\hat{\mathfrak{C}} \cap Q_{\varepsilon, \eta}^{+}=\left\{(\lambda,-u): \quad(\lambda, u) \in \hat{\mathfrak{C}} \cap Q_{\varepsilon, \eta}^{-}\right\}
$$

because $\hat{\mathfrak{G}}(\lambda, u)$ is odd in $u$. Therefore,

$$
\hat{\mathfrak{C}} \cap \partial B_{\delta}\left(\lambda_{0}, 0\right) \cap Q_{\varepsilon, \eta}^{-} \neq \emptyset \quad \text { for all } \delta \in\left(0, \delta_{1}\right)
$$

which contradicts our first assumption and ends the proof.
Subsequently, likewise in [14, p. 187], we denote by $\mathfrak{C}^{+}$(resp. $\mathfrak{C}^{-}$), the component of $\mathfrak{S}$ such that $\left(\lambda_{0}, 0\right) \in \mathfrak{C}^{+}$(resp. $\left.\left(\lambda_{0}, 0\right) \in \mathfrak{C}^{-}\right)$and in a neighborhood of $\left(\lambda_{0}, 0\right)$ lies in $\mathfrak{S} \backslash Q_{\varepsilon, \eta}^{-}$(resp. $\left.\mathfrak{S} \backslash Q_{\varepsilon, \eta}^{+}\right)$. The next generalized version of Theorem 1.27 of P. H. Rabinowitz [23] holds.
Theorem 7.5. Suppose $\mathfrak{F}$ satisfies (F1)-(F4), (C), (25), and, once oriented $\mathfrak{L}(\lambda)$, sgn $\mathfrak{L}(\lambda)$ changes sign as $\lambda$ crosses $\lambda_{0}$. Let $Y \subset U$ a closed subspace such that

$$
U=N\left[\mathfrak{L}_{0}\right] \oplus Y
$$

Then, for each $\nu \in\{-,+\}$, $\mathfrak{C}^{\nu}$ satisfies some of the following alternatives:
(a) $\mathfrak{C}^{\nu}$ is not compact in $\mathbb{R} \times U$.
(b) There exists $\lambda_{1} \neq \lambda_{0}$ such that $\left(\lambda_{1}, 0\right) \in \mathfrak{C}^{\nu}$.
(c) There exists $(\lambda, y) \in \mathfrak{C}^{\nu}$ with $y \in Y \backslash\{0\}$.

The proof of this theorem follows mutatis mutandis the proof of Theorem 6.4.3 of [14]. So, the technical details are omitted here. Under the transversality condition of M. G. Crandall and P. H. Rabinowitz (see (23)), $\chi\left[\mathfrak{L} ; \lambda_{0}\right]=1$. Hence, by Theorem 4.5, sgn $\mathfrak{L}(\lambda)$ changes as $\lambda$ crosses $\lambda_{0}$. Therefore, Theorem 1.2 holds from Theorem 7.5.

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# On the singular 1-dimensional planar sheaves supported on quartics 

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#### Abstract

In the case of the fine Simpson moduli spaces of 1dimensional planar sheaves supported on quartics, the subvariety of sheaves that are not locally free on their support is connected, singular, and has codimension 2.


Keywords: Simpson moduli spaces, coherent sheaves, vector bundles on curves, singular sheaves.
MS Classification 2010: 14D20.

## 1. Introduction

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, let $V$ be a vector space over $\mathbb{k}$ of dimension 3 , and let $\mathbb{P}_{2}=\mathbb{P} V$ be the corresponding projective plane. Fix a linear polynomial $P(m)=d m+c$ with integer coprime coefficients and consider the Simpson [13] moduli space $M:=M_{P}(X)$ of stable sheaves on $X$ with Hilbert polynomial $P$. As shown in [9], $M$ is a fine moduli space, it is a smooth irreducible projective variety of dimension $d^{2}+1$. A generic sheaf in $M$ is a line bundle on its Fitting support, which is a planar projective curve of degree $d$.

## Singular sheaves

In general $M$ contains a closed subvariety $M^{\prime}$ of sheaves that are not locally free on their support. Since $M$ is irreducible, the complement $M_{B}$ of $M^{\prime}$ is an open dense subset whose points are sheaves that are locally free on their support. So, one could consider $M$ as a compactification of $M_{B}$. We call the sheaves from the boundary $M^{\prime}=M \backslash M_{B}$ singular. As one can see on the following examples for $d \leqslant 3$, the boundary $M^{\prime}$ does not have the minimal codimension in general.

## First examples

Notice that twisting with $\mathcal{O}_{\mathbb{P}_{2}}(1)$ gives the isomorphism of the moduli spaces $M_{d m+c}\left(\mathbb{P}_{2}\right) \cong M_{d m+c+d}\left(\mathbb{P}_{2}\right)$. Moreover, by the duality result from [12], there is the isomorphism $M_{d m+c}\left(\mathbb{P}_{2}\right) \cong M_{d m-c}\left(\mathbb{P}_{2}\right)$ given by $\mathcal{F} \mapsto \mathscr{E} x t^{1}\left(\mathcal{F}, \omega_{\mathbb{P}_{2}}\right)$. As
shown in [14], two moduli spaces $M_{d m+c}\left(\mathbb{P}_{2}\right)$ and $M_{d^{\prime} m+c^{\prime}}\left(\mathbb{P}_{2}\right)$ are isomorphic if and only if $\mathrm{f} d=d^{\prime}$ and $c= \pm c^{\prime} \bmod d$. Therefore, for fixed $d$, it is enough to understand $d / 2+1$ different moduli spaces.

For $d=1, M_{m+c}$ is a fine moduli space that consists of twisted structure sheaves $\mathcal{O}_{L}(c-1)$ of lines $L$ in $\mathbb{P}_{2}$. Therefore, each $M_{m+c}$ is just the dual projective plane $\mathbb{P}_{2}^{*}=\mathbb{P} V^{*}$. In this case there are no singular sheaves.

For $d=2$ and $c=2 \beta+1, M_{2 m+c}$ is a fine moduli space whose points are the isomorphism classes of twisted structure sheaves $\mathcal{O}_{C}(\beta)$ of planar conics $C \subseteq \mathbb{P}_{2}$. In this case $M_{2 m+c}$ is isomorphic to the space of conics $\mathbb{P} S^{2} V^{*}$. As in the previous case the subvariety $M_{2 m+c}^{\prime}$ of singular sheaves is empty.

The situation changes for $d=3$. For $c \in \mathbb{Z}$ with $\operatorname{gcd}(3, c)=1$ all moduli spaces $M_{3 m+c}$ are isomorphic to the universal plane cubic curve and $M_{3 m+c}^{\prime}$ is a smooth subvariety of codimension 2 isomorphic to the universal singular locus of a cubic curve. A construction that interprets in this case the blow-up of $M$ along $M^{\prime}$ as a compactification of $M_{B}$ by an irreducible divisor consisting of vector bundles of curves in certain reducible surfaces was given in [8]. Since it explicitly uses the properties of $M^{\prime}$, it seems important to understand the geometry of $M^{\prime}$ in order to perform a similar modification for other moduli spaces of planar 1-dimensional sheaves.

## The main result of the paper

The cases with $d \leqslant 3$ were the only cases where the boundary $M^{\prime}$ has been completely understood. In this note we study the subvariety of singular sheaves in the case of $M=M_{4 m+c}\left(\mathbb{P}_{2}\right), \operatorname{gcd}(4, c)=1$, i. e., for the fine Simpson moduli spaces, which consist entirely of stable points and parameterize the isomorphism classes of sheaves. As already mentioned above, it is enough to consider the case $c=-1$.

We describe all possible singular sheaves in $M$, the main result of the paper is summarized in the following:

Proposition 1.1. Let $M$ be the Simpson moduli space of stable sheaves with Hilbert polynomial $P(m)=4 m+c, \operatorname{gcd}(4, c)=1$. Let $M^{\prime} \subseteq M$ be the subvariety of singular sheaves. Then $M^{\prime}$ is a singular (path-)connected subvariety of codimension 2.

We use the merits of computer algebra computations: the most important computations in the paper are performed using Singular [1]. At the same time we comment on the restrictedness of computer algebra methods due to the complexity of the involved algorithms.

## Structure of the paper

In Section 2 we give a detailed description of the stratification from [4] of the moduli space $M$ into an open stratum $M_{0}$ and its closed complement $M_{1}$. In Section 3 we describe the open stratum of $M$ as an open subvariety of a projective bundle over the space of Kronecker modules $N=N(3 ; 2,3)$. In Section 4 we give a characterization of singular sheaves in $M_{0}$ and study the fibres of $M_{0}$ over $N$, which allows us to demonstrate in Section 5 the assertions of Proposition 1.1. In Section 6 we study, for an isomorphism class [ $\mathcal{E}$ ] in $M_{0}$, how being singular is related with the singularities of the support of $\mathcal{E}$. The computations with Singular [1] used in the paper (the code and its output) are presented in Appendix A.

## 2. Description of $M_{4 m-1}\left(\mathbb{P}_{2}\right)$

Let $M$ be the Simpson moduli space of stable sheaves on $\mathbb{P}_{2}$ with Hilbert polynomial $4 m-1$. In [4] it has been shown that $M$ can be decomposed into two strata $M_{1}$ and $M_{0}$ such that $M_{1}$ is a closed subvariety of $M$ of codimension 2 and $M_{0}$ is its open complement.

### 2.1. Closed stratum.

The closed stratum $M_{1}$ is a closed subvariety of $M$ of codimension 2 given by the condition $h^{0}(\mathcal{E}) \neq 0$ (more precisely $h^{0}(\mathcal{E})=1$ ). It consists of the isomorphism classes of sheaves with locally free resolutions

$$
0 \rightarrow 2 \mathcal{O}_{\mathbb{P}_{2}}(-3) \xrightarrow{\left(\begin{array}{c}
z_{1} q_{1}  \tag{1}\\
z_{2} \\
q_{2}
\end{array}\right)} \mathcal{O}_{\mathbb{P}_{2}}(-2) \oplus \mathcal{O}_{\mathbb{P}_{2}} \rightarrow \mathcal{E} \rightarrow 0
$$

where $z_{1}$ and $z_{2}$ are linear independent linear forms on $\mathbb{P}_{2} . M_{1}$ is a geometric quotient of the variety of injective matrices $\left(\begin{array}{ll}z_{1} \\ z_{2} & q_{1} \\ q_{2}\end{array}\right)$ as above by the non-reductive group

$$
\left(\operatorname{Aut}\left(2 \mathcal{O}_{\mathbb{P}_{2}}(-3)\right) \times \operatorname{Aut}\left(\mathcal{O}_{\mathbb{P}_{2}}(-2) \oplus \mathcal{O}_{\mathbb{P}_{2}}\right)\right) / \mathbb{k}^{*}
$$

(cf. [5]). $M_{1}$ is isomorphic to the universal quartic plane curve

$$
\left\{(p, C) \mid C \subseteq \mathbb{P}_{2} \text { is a quartic plane curve, } p \in C\right\}
$$

The latter can be explained as follows. The sheaves with resolution (1) are exactly the non-trivial extensions

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{C} \rightarrow \mathcal{E} \rightarrow \mathbb{k}_{p} \rightarrow 0 \tag{2}
\end{equation*}
$$

where $C=C_{A}=Z(\operatorname{det} A)$ is the quartic curve defined by the determinant of $A$ and $p=p_{A}=Z\left(z_{1}, z_{2}\right)$ is the point on $C$ defined by two linear independent linear forms $z_{1}$ and $z_{2}$.

### 2.2. Open stratum.

The open stratum $M_{0}$ is the complement of $M_{1}$ given by the condition $h^{0}(\mathcal{E})=$ 0 , it consists of the cokernels $\mathcal{E}_{A}$ of the injective morphisms

$$
\begin{equation*}
\mathcal{O}_{\mathbb{P}_{2}}(-3) \oplus 2 \mathcal{O}_{\mathbb{P}_{2}}(-2) \xrightarrow{A} 3 \mathcal{O}_{\mathbb{P}_{2}}(-1) \tag{3}
\end{equation*}
$$

with

$$
A=\left(\begin{array}{ccc}
q_{0} & q_{1} & q_{2} \\
z_{0} & z_{1} & z_{2} \\
w_{0} & w_{1} & w_{2}
\end{array}\right)
$$

such that the $(2 \times 2)$-minors of the linear part of $A$ are linear independent. Equivalently, the Kronecker module

$$
\alpha=\left(\begin{array}{ccc}
z_{0} & z_{1} & z_{2}  \tag{4}\\
w_{0} & w_{1} & w_{2}
\end{array}\right)
$$

is stable (cf. [6, Lemma 1], [2, Proposition 15]).

### 2.2.1. Twisted ideals of 3 non-collinear points of $C$

If the maximal minors of $\alpha$ are coprime, then $\mathcal{E}_{A} \cong \mathcal{I}_{Z}(1)$, where $\mathcal{I}_{Z}$ is the ideal sheaf of the zero dimensional subscheme $Z \subseteq C$ of length 3 defined by the maximal minors of $\alpha$. In this case the isomorphism class of $\mathcal{E}=\mathcal{E}_{A}$ is a part of the exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{C}(1) \rightarrow \mathcal{O}_{Z} \rightarrow 0 \tag{5}
\end{equation*}
$$

and is uniquely defined by $Z$ and $C$.
Let $M_{00}$ denote the open subscheme of all such sheaves in $M_{0}$.

### 2.2.2. Extensions

If the maximal minors of $\alpha$ have a linear common factor, say $l$, then $f=$ $\operatorname{det}(A)=l \cdot h$ and $\mathcal{E}_{A}$ is in this case a non-split extension

$$
\begin{equation*}
0 \rightarrow \mathcal{O}_{L}(-2) \rightarrow \mathcal{E}_{A} \rightarrow \mathcal{O}_{C^{\prime}} \rightarrow 0 \tag{6}
\end{equation*}
$$

where $L=Z(l), C^{\prime}=Z(h)$.
For fixed $L$ and $C^{\prime}$ the subscheme of the isomorphism classes of non-trivial extensions (6) can be identified with $\mathbb{k}^{2}$.

Let $M_{01}$ denote the closed subscheme of $M_{0}$ of all such sheaves. Notice that $M_{01}$ is locally closed in $M$.

### 2.2.3. $M_{0}$ as a geometric quotient.

$M_{0}$ is the geometric quotient of the variety of injective matrices as in (3) by the group

$$
G^{\prime}=\operatorname{Aut}\left(\mathcal{O}_{\mathbb{P}_{2}}(-3) \oplus 2 \mathcal{O}_{\mathbb{P}_{2}}(-2)\right) \times \operatorname{Aut}\left(3 \mathcal{O}_{\mathbb{P}_{2}}(-1)\right)
$$

As shown in [11] $M_{0}$ can be seen as an open subvariety in the projective quotient $\mathbb{B}$ of the variety of all semistable matrices (3) by the same group.

## 3. Description of $M_{0}$ as an open subvariety in $\mathbb{B}$

### 3.1. Kronecker modules

Let $\mathbb{V}$ be the affine variety of Kronecker modules

$$
\begin{equation*}
2 \mathcal{O}_{\mathbb{P}_{2}}(-1) \xrightarrow{\Phi} 3 \mathcal{O}_{\mathbb{P}_{2}} \tag{7}
\end{equation*}
$$

There is a natural group action of $G=\left(\mathrm{GL}_{2}(\mathbb{k}) \times \mathrm{GL}_{3}(\mathbb{k})\right) / \mathbb{k}^{*}$ on $\mathbb{V}$. Since $\operatorname{gcd}(2,3)=1$, all semistable points of this action are stable and $G$ acts freely on the open subset $\mathbb{V}^{s}$ of stable points. A Kronecker module (7) is stable if its maximal minors are linear independent quadratic forms. There exists a geometric quotient $N=N(3 ; 2,3)=\mathbb{V}^{s} / G$, which is a smooth projective variety of dimension 6. For more details consult [7, Section 6] and [2, Section III].

The cokernel of a stable Kronecker module $\Phi \in \mathbb{V}^{s}$ is an ideal of a zerodimensional scheme $Z$ of length 3 if the maximal minors of $\Phi$ are coprime. In this case there is a locally free resolution

$$
0 \rightarrow 2 \mathcal{O}_{\mathbb{P}_{2}}(-3) \xrightarrow{\Phi} 3 \mathcal{O}_{\mathbb{P}_{2}}(-2) \xrightarrow{\left(\begin{array}{l}
d_{0}  \tag{8}\\
d_{1} \\
d_{2}
\end{array}\right)} \mathcal{O}_{\mathbb{P}_{2}} \rightarrow \mathcal{O}_{Z} \rightarrow 0
$$

and, moreover, $Z$ does not lie on a line. Let $\mathbb{V}_{0}$ denote the open subvariety of $\Phi \in \mathbb{V}^{s}$ of Kronecker modules with coprime maximal minors. Let $N_{0} \subseteq N$ be the corresponding open subvariety in the quotient space.

This way one obtains a morphism from $N_{0} \subseteq N$ to the Hilbert scheme $H$ of zero-dimensional subschemes of $\mathbb{P}_{2}$ of length 3 , which sends a class of $\Phi \in \mathbb{V}^{s}$ to the zero scheme of its maximal minors. Since, by Hilbert-Burch theorem, every zero dimensional scheme of length 3 that does not lie on a line has a minimal resolution of type (8), this gives an isomorphism between $N_{0}$ and the open subvariety $H_{0} \subseteq H$ consisting of $Z$ that do not lie on a line.

Let $N^{\prime}=N \backslash N_{0}$, then $N^{\prime}$ is the quotient of the variety of Kronecker modules (7) whose maximal minors have a common linear factor.

Since every matrix representing a point in $N^{\prime}$ is equivalent to a matrix $\left(\begin{array}{ccc}z_{0} & 0 & z_{1} \\ 0 & z_{0} & z_{2}\end{array}\right)$ with linear independent linear forms $z_{0}, z_{1}, z_{2}$, one can see that $N^{\prime}$ is isomorphic to $\mathbb{P}_{2}^{*}=\mathbb{P} V^{*}$, the space of lines in $\mathbb{P}_{2}$, such that a line corresponds to the common linear factor of the minors of the corresponding Kronecker module.

The complement $H^{\prime}$ of $H_{0}$ is an irreducible hypersurface (cf. [3, p. 46], [7]). The isomorphism $H_{0} \rightarrow N_{0}$ can be extended to the morphism $H \xrightarrow{\pi} N$ that describes $H$ as the blowing up of $N$ along $N^{\prime}$. The fibre over $L \in \mathbb{P}_{2}^{*}$ consists of those $Z \in H$ lying on $L$, i. e., the fibre over $L$ is $L^{[3]} \cong \mathbb{P}_{3}$, the Hilbert scheme of 3 points on $L$.

## 3.2. $\mathbb{B}$ as a projective bundle over $N$

Let us provide here the argument from the proof of [11, Proposition 7.7].
Consider two vector spaces $\mathbb{U}_{1}=2 \Gamma\left(\mathbb{P}_{2}, \mathcal{O}_{\mathbb{P}_{2}}(1)\right)$ and $\mathbb{U}_{2}=3 \Gamma\left(\mathbb{P}_{2}, \mathcal{O}_{\mathbb{P}_{2}}(2)\right)$. One identifies elements of $\mathbb{V} \times \mathbb{U}_{2}$ with morphisms (3) by

$$
(\Phi, Q) \mapsto\binom{Q}{\Phi}
$$

Both $\mathbb{V} \times \mathbb{U}_{1}$ and $\mathbb{V} \times \mathbb{U}_{2}$ are trivial vector bundles over $\mathbb{V}$. Consider the morphism

$$
\mathbb{V} \times \mathbb{U}_{1} \xrightarrow{F} \mathbb{V} \times \mathbb{U}_{2}, \quad(\Phi, L) \mapsto\binom{L \cdot \Phi}{\Phi}
$$

Since the matrices from $\mathbb{V}^{s}$ have linear independent maximal minors, $F$ is injective over $\mathbb{V}^{s}$. Therefore, $\mathbb{V}^{s} \times \mathbb{U}_{1} \xrightarrow{F} \mathbb{V}^{s} \times \mathbb{U}_{2}$ is a vector subbundle and hence the cokernel $E$ of $F$ is a vector bundle of rank 12 on $\mathbb{V}^{s}$.

The group action of $\mathrm{GL}_{2}(\mathbb{k}) \times \mathrm{GL}_{3}(\mathbb{k})$ on $\mathbb{V}^{s} \times \mathbb{U}_{1}$ and $\mathbb{V}^{s} \times \mathbb{U}_{2}$ induces a group action of $\mathrm{GL}_{2}(\mathbb{k}) \times \mathrm{GL}_{3}(\mathbb{k})$ on $E$ and hence an action of $G=\left(\mathrm{GL}_{2}(\mathbb{k}) \times\right.$ $\left.\mathrm{GL}_{3}(\mathbb{k})\right) / \mathbb{k}^{*}$ on the projective bundle $\mathbb{P} E$. Finally, since the stabilizer of $\Phi \in \mathbb{V}^{s}$ under the action of $G$ is trivial, $G$ acts trivially on the fibres of $\mathbb{P} E$ and thus $\mathbb{P} E$ descends to a projective $\mathbb{P}_{11}$-bundle

$$
\mathbb{B} \xrightarrow{\nu} N=N(3 ; n-1, n)=\mathbb{V}^{s} / G
$$

which is exactly the geometric quotient of $\mathbb{V}^{s} \times \mathbb{U}_{2} \backslash \operatorname{Im} F$ with respect to $G^{\prime}$ mentioned above.

### 3.2.1. The fibres of $\mathbb{B} \xrightarrow{\nu} N$ over $N_{0}$

A fibre over a point from $N_{0}$ can be seen as the space of plane quartics through the corresponding subscheme of 3 non-collinear points. Indeed, consider a point from $N_{0}$ given by a Kronecker module $\left(\begin{array}{lll}z_{0} & z_{1} & z_{2} \\ w_{0} & w_{1} & w_{2}\end{array}\right)$ with coprime minors
$d_{0}, d_{1}, d_{2}$. The fibre over such a point consists of the orbits of injective matrices

$$
\left(\begin{array}{ccc}
q_{0} & q_{1} & q_{2} \\
z_{0} & z_{1} & z_{2} \\
w_{0} & w_{1} & w_{2}
\end{array}\right), \quad q_{0}, q_{1}, q_{2} \in S^{2} V^{*}
$$

under the group action of $G^{\prime}$. In particular such a fibre is contained in $M_{00}$. If two matrices

$$
\left(\begin{array}{ccc}
q_{0} & q_{1} & q_{2} \\
z_{0} & z_{1} & z_{2} \\
w_{0} & w_{1} & w_{2}
\end{array}\right), \quad\left(\begin{array}{lll}
Q_{0} & Q_{1} & Q_{2} \\
z_{0} & z_{1} & z_{2} \\
w_{0} & w_{1} & w_{2}
\end{array}\right)
$$

lie in the same orbit of the group action, then their determinants are equal up to a multiplication by a non-zero constant. Vice versa, if the determinants of two such matrices are equal, $q-Q=\left(q_{0}-Q_{0}, q_{1}-Q_{1}, q_{2}-Q_{2}\right)$ lies in the syzygy module of $\left(\begin{array}{l}d_{0} \\ d_{1} \\ d_{2}\end{array}\right)$, which is generated by the rows of $\left(\begin{array}{ccc}z_{0} & z_{1} & z_{2} \\ w_{0} & w_{1} & w_{2}\end{array}\right)$ by Hilbert-Burch theorem. This implies that $q-Q$ is a combination of the rows and thus the matrices lie on the same orbit.

### 3.2.2. $M_{0}$ and flags of subschemes on $\mathbb{P}_{2}$.

Let $\mathbb{B}_{0}$ denote the restriction of $\mathbb{B}$ to $N_{0}$. Then $\mathbb{B}_{0}$ coincides with $M_{00}$ as the fibres over $N_{0}$ are contained in $M_{0}$.

Let $\mathbb{P} S^{4} V^{*}=\mathbb{P}_{14}$ be the space of plane quartics. Let

$$
M \xrightarrow{\mu} \mathbb{P} S^{4} V^{*}=\mathbb{P}_{14}, \quad[\mathcal{E}] \mapsto \operatorname{Supp}(\mathcal{E})
$$

be the morphism sending an isomorphism class of sheaf $\mathcal{E}$ to its support. Then its restriction to $M_{0}$ is induced by the equivariant morphism that sends a matrix (3) defining a point in $M_{0}$ to the quartic determined by its determinant.
$\mathbb{B}_{0}$ is isomorphic to the image of the injective morphism

$$
\begin{equation*}
\mathbb{B}_{0} \xrightarrow{\mu \times \nu} \mathbb{P}\left(S^{4} V^{*}\right) \times N_{0} \cong \mathbb{P}\left(S^{4} V^{*}\right) \times H_{0}, \tag{9}
\end{equation*}
$$

which coincides with the subvariety of pairs $(C, Z)$ with $Z \subseteq C$. It is isomorphic to the open subscheme $H_{0}(3,4) \subseteq H(3,4)$ of the Hilbert flag-scheme of flags $Z \subseteq C \subseteq \mathbb{P}_{2}$ (zero-dimensional subscheme $Z$ of length 3 on a curve $C \subseteq \mathbb{P}_{2}$ of degree 4) such that $Z$ does not lie on a line.

### 3.2.3. The fibres of $\mathbb{B} \xrightarrow{\nu} N$ over $N^{\prime}$

A fibre over $L \in N^{\prime}$ can be seen as the join $J\left(L^{*}, \mathbb{P} S^{3} V^{*}\right) \cong \mathbb{P}_{11}$ of $L^{*} \cong \mathbb{P}_{1}$ and the space of plane cubic curves $\mathbb{P}\left(S^{3} V^{*}\right) \cong \mathbb{P}_{9}$. To see this assume $L=Z\left(x_{0}\right)$, i. e., $L$ is given by $\left(\begin{array}{ccc}x_{0} & 0 & x_{1} \\ 0 & x_{0} & x_{2}\end{array}\right)$. Then the fibre over $L$ is given by the orbits of matrices

$$
\left(\begin{array}{ccc}
q_{0}\left(x_{1}, x_{2}\right) & q_{1}\left(x_{1}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}, x_{2}\right)  \tag{10}\\
x_{0} & 0 & x_{1} \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

and can be identified with the projective space $\mathbb{P}\left(2 H^{0}\left(L, \mathcal{O}_{L}(2)\right) \oplus S^{2} V^{*}\right)$. Rewrite the matrix (10) as

$$
\left(\begin{array}{ccc}
l \cdot x_{2}-b\left(x_{1}, x_{2}\right) & -l \cdot x_{1}-c x_{2}^{2} & a\left(x_{0}, x_{1}, x_{2}\right) \\
x_{0} & 0 & x_{1} \\
0 & x_{0} & x_{2}
\end{array}\right), \quad l\left(x_{1}, x_{2}\right)=\xi_{1} x_{1}+\xi_{2} x_{2}, \quad \xi_{1}, \xi_{2} \in \mathbb{k} .
$$

Its determinant equals $x_{0}\left(a\left(x_{0}, x_{1}, x_{2}\right) \cdot x_{0}+b\left(x_{0}, x_{1}\right) \cdot x_{1}+c \cdot x_{2}^{3}\right)$. This allows to reinterpret the fibre as the projective space

$$
\mathbb{P}\left(H^{0}\left(L, \mathcal{O}_{L}(1)\right) \oplus S^{3} V^{*}\right) \cong J\left(L^{*}, \mathbb{P} S^{3} V^{*}\right)
$$

The intersection of the fibre with $M_{0}$ is $J\left(L^{*}, \mathbb{P}\left(S^{3} V^{*}\right)\right) \backslash L^{*}$. It is a rank 2 vector bundle over $\mathbb{P}\left(S^{3} V^{*}\right)$ whose fibre over a cubic curve $C^{\prime} \in \mathbb{P} S^{3} V^{*}$ is identified with the set of the isomorphism classes of sheaves from $M_{01}$ defined by (6) with fixed $L$ and $C^{\prime}$. This fibre corresponds to the projective plane joining $C^{\prime}$ with $L^{*}$ inside the join $J\left(L^{*}, \mathbb{P}\left(S^{3} V^{*}\right)\right)$. In the notations of the example above $\xi_{1}$ and $\xi_{2}$ are the coordinates of this affine plane.


The points of $J\left(L^{*}, \mathbb{P}\left(S^{3} V^{*}\right)\right) \backslash L^{*}$ parameterize the extensions (6) from $M_{01}$ with fixed $L$.

### 3.2.4. Description of the complement of $M_{0}$ in $\mathbb{B}$.

Let $\mathbb{B}^{\prime}=\mathbb{B} \backslash M_{0}$. Then $\mathbb{B}^{\prime}$ is a union of lines $L^{*}$ from each fibre over $N^{\prime}$ (as explained above), it is isomorphic to the tautological $\mathbb{P}_{1}$-bundle over $N^{\prime}=\mathbb{P}_{2}^{*}$

$$
\begin{equation*}
\left\{(L, x) \in \mathbb{P}_{2}^{*} \times \mathbb{P}_{2} \mid L \in \mathbb{P}_{2}^{*}, x \in L\right\} \tag{11}
\end{equation*}
$$

The fibre $\mathbb{P}_{1}$ of $\mathbb{B}^{\prime}$ over, say, line $L=Z\left(x_{0}\right) \subseteq \mathbb{P}_{2}$ can be identified with the space of classes of matrices (3) with zero determinant

$$
\left(\begin{array}{ccc}
\xi \cdot x_{2} & -\xi \cdot x_{1} & 0 \\
x_{0} & 0 & x_{1} \\
0 & x_{0} & x_{2}
\end{array}\right), \quad \xi=\alpha x_{1}+\beta x_{2}, \quad\langle\alpha, \beta\rangle \in \mathbb{P}_{1} .
$$

Let $N_{c}$ be the open subset of $N_{0}$ that corresponds to 3 different (and hence non-collinear) points. Under the isomorphism $N_{0} \cong H_{0}$ it corresponds to the open subvariety $H_{c} \subseteq H_{0}$ of the non-collinear configurations of 3 points on $\mathbb{P}_{2}$.

Let $M_{c}=\mathbb{B}_{c}$ be the restriction of $\mathbb{B}$ to $N_{c}$. Then $M_{c} \subseteq M_{00} \subseteq M_{0} \subseteq M$ are inclusions of open subvarieties of $M$.

## 4. The subvariety of singular sheaves

Let $M_{1}^{\prime}$ and $M_{0}^{\prime}$ denote the intersections of the subvariety $M^{\prime}=M_{4 m-1}^{\prime}$ of singular sheaves with $M_{1}$ and $M_{0}$ respectively.

### 4.1. Characterization of singular sheaves

### 4.1.1. Singular sheaves in $M_{1}$

As shown in [8], the subvariety $M_{1}^{\prime}$ coincides with the universal singular locus

$$
\left\{(p, C) \mid C \subseteq \mathbb{P}_{2} \text { is a quartic plane curve, } p \in \operatorname{Sing}(C)\right\}
$$

which is a smooth subvariety of $M_{1}$ of codimension 2.

### 4.1.2. Singular sheaves in $M_{0}$.

Lemma 4.1. The sheaf $\mathcal{E}_{A}$ from $M_{0}$ is singular if and only if the ideal $\mathcal{I}_{\text {min }}=$ $\mathcal{I}_{\text {min }}(A)$ generated by all $(2 \times 2)$-minors of $A$ defines a non-empty scheme.

Proof. If there are no zeros of $\mathcal{I}_{\text {min }}$, then at every point of $\mathbb{P}_{2}$ at least one of the $(2 \times 2)$-minors is invertible, hence using invertible elementary transformations one can bring $A$ to the form

$$
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \operatorname{det} A
\end{array}\right)
$$

and therefore $\mathcal{E}$ is locally isomorphic to $\mathcal{O}_{C}, C=Z(\operatorname{det} A)=\operatorname{Supp} \mathcal{E}$.
If $p$ is a zero point of $\mathcal{I}_{\text {min }}$, then the rank of $A$ is at most 1 at $p$. Therefore, the dimension of $\mathcal{E}(p)=\mathcal{E}_{p} / \mathfrak{m}_{p} \mathcal{E}_{p}$ is at least 2 . Since the rank of $\mathcal{E}$ (on support) is 1 , we conclude that $\mathcal{E}$ is a singular sheaf.

## 4.2. $M_{0}^{\prime}$ and computer algebra

Lemma 4.1 suggests the following approach to study $M_{0}^{\prime}$ using computer algebra.

Put $\mathbb{A}:=\operatorname{Hom}\left(\mathcal{O}_{\mathbb{P}_{2}}(-3) \oplus 2 \mathcal{O}_{\mathbb{P}_{2}}(-2), 3 \mathcal{O}_{\mathbb{P}_{2}}(-1)\right) \cong \mathbb{K}^{36}$ and let $\mathbb{W}_{0} \subseteq \mathbb{A}$ be the quasi-affine variety of injective matrices (3) such that $M_{0} \cong \mathbb{W}_{0} / G^{\prime}$
as mentioned in 2.2.3. Consider the ideal $I \subseteq \mathbb{k}(\mathbb{A})\left[x_{0}, x_{1}, x_{3}\right]$ of $(2 \times 2)$ minors of the universal matrix on $\mathbb{A}$. Then eliminating the variables $x_{0}, x_{1}, x_{2}$ from the saturation ideal $I:\left(x_{0}, x_{1}, x_{2}\right)^{\infty}$, we will obtain the ideal $J=(I$ : $\left.\left(x_{0}, x_{1}, x_{2}\right)^{\infty}\right) \cap \mathbb{k}[\mathbb{A}]$ defining the subvariety in $\mathbb{A}$ of the matrices whose cokernels are singular sheaves. Having this, one computes the dimension of the zero scheme of $J$, its singularities, etc.

Though all actions with the ideals mentioned above are implemented in different systems of computer algebra, the complexity of the involved algorithms have not even made it possible for us to compute $J$. Therefore, we are going to study first the fibres of $M_{0}^{\prime}$ over $N$.

### 4.3. Fibres of $M_{0}^{\prime}$ over $N$

Let us consider the restriction of $\nu$ to $M_{0}^{\prime}$ and describe its fibres. There are the following possible cases:

1. fibres over $N_{c} \cong H_{c}$, i. e., over 3 different non-collinear points;
2. fibres over $Z \in N_{0}$ consisting of a simple point and a double point;
3. fibres over curvilinear triple points $Z \in N_{0}$;
4. fibres over non-curvilinear triple points $Z \in N_{0}$;
5. fibres over $N^{\prime}$.

The corresponding fibres will be referred to as fibres of type (1), (2), (3), (4), and (5) respectively.

### 4.3.1. Fibres of type (1)

Let $Z \in H_{c} \cong N_{c}$ be a non-collinear configuration of 3 points in $\mathbb{P}_{2}$. Then, after applying an appropriate coordinate change, we can assume without loss of generality that $Z$ is the union of three points $\mathrm{pt}_{0}=\langle 1,0,0\rangle, \mathrm{pt}_{1}=\langle 0,1,0\rangle$, $\mathrm{pt}_{2}=\langle 0,0,1\rangle$, the corresponding Kronecker module is

$$
\Phi=\left(\begin{array}{ccc}
x_{0} & x_{1} & 0 \\
x_{0} & 0 & x_{2}
\end{array}\right)
$$

whose minors $d_{0}=x_{1} x_{2}, d_{1}=-x_{0} x_{2}, d_{2}=-x_{0} x_{1}$ generate the ideal $I_{Z}$ of $Z$.


The fibre of $\nu$ over the class of $\Phi$ in $N_{0}$ consists of the orbits of the matrices

$$
A=\left(\begin{array}{ccc}
q_{0}\left(x_{0}, x_{1}, x_{2}\right) & q_{1}\left(x_{0}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}\right) \\
x_{0} & x_{1} & 0 \\
x_{0} & 0 & x_{2}
\end{array}\right) .
$$

The coefficients of

$$
\begin{align*}
& q_{0}=a_{0} x_{0}^{2}+a_{1} x_{0} x_{1}+a_{2} x_{0} x_{2}+a_{3} x_{1}^{2}+a_{4} x_{1} x_{2}+a_{5} x_{2}^{2} \\
& q_{1}=b_{0} x_{0}^{2}+b_{2} x_{0} x_{2}+b_{5} x_{2}^{2}  \tag{12}\\
& q_{2}=c_{0} x_{0}^{2}+c_{1} x_{0} x_{1}+c_{3} x_{1}^{2}
\end{align*}
$$

can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.
The ideal that defines the subvariety corresponding to the singular sheaves is computed by eliminating the variables $x_{0}, x_{1}, x_{2}$ from the saturation of $\mathcal{I}_{\text {min }}$ with respect to the non-essential maximal ideal $\left(x_{0}, x_{1}, x_{2}\right)$. We perform the computations using the computer algebra system Singular (cf. [1]).

We get the ideal (see A. 1 for computations)

$$
\left(b_{0}, c_{0}\right) \cap\left(a_{3}, c_{3}\right) \cap\left(a_{5}, b_{5}\right)
$$

i. e., the fibre of $M_{0}^{\prime}$ over [ $\Phi$ ] is a union of 3 components, each being a projective subspace in $\mathbb{P}_{11}$ of codimension 2 . The components lie in a general position: each two components intersect along a projective subspace of codimension 4 and the intersection of all three of them is a projective subspace of codimension 6 .

### 4.3.2. Fibres of type (2)

Let $Z \in H_{0} \backslash H_{c}$ be a non-collinear configuration of a simple point $\mathrm{pt}_{1}$ and a double non-collinear point at $\mathrm{pt}_{2}$. The double point is defined by the underlying simple point $\mathrm{pt}_{2}$ and a tangent vector at $\mathrm{pt}_{2}$. Since $Z$ does not lie on a line, the tangent vector should be normal to the line joining $\mathrm{pt}_{1}$ and $\mathrm{pt}_{2}$. Therefore, after applying an appropriate coordinate change, we can assume without loss of generality that $\mathrm{pt}_{1}=\langle 0,0,1\rangle, \mathrm{pt}_{2}=\langle 0,1,0\rangle$, and the tangent vector at $\mathrm{pt}_{2}$ is parallel to the line given by $x_{2}$.


The ideal of $Z$ equals $\left(x_{0}, x_{1}\right) \cap\left(x_{0}^{2}, x_{2}\right)$, the corresponding Kronecker module can be taken to be

$$
\Phi=\left(\begin{array}{ccc}
x_{0} & x_{1} & 0 \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

The fibre of $\nu$ over the class of $\Phi$ in $N_{0}$ consists of the orbits of the matrices

$$
A=\left(\begin{array}{ccc}
q_{0}\left(x_{0}, x_{1}, x_{2}\right) & q_{1}\left(x_{0}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}\right) \\
x_{0} & x_{1} & 0 \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

The coefficients of $q_{0}, q_{1}, q_{2}$ as in (12) can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.

The fibre of $M_{0}^{\prime}$ over $[\Phi] \in N$ is given by the ideal

$$
\left(a_{3}^{2}, c_{3}^{2}, a_{3} c_{3}, a_{1} c_{3}-a_{3} c_{1}\right) \cap\left(a_{5}, b_{5}\right)
$$

whose radical is $\left(a_{3}, c_{3}\right) \cap\left(a_{5}, b_{5}\right)$, which means that the fibre consists of two components each of which is a projective subspace of $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$ of codimension 2. For computations see A.2.

### 4.3.3. Fibres of type (3)

Let $Z$ be a triple curvilinear point. Without loss of generality, applying an appropriate coordinate change if necessary, we can assume that $Z$ is supported at $\mathrm{pt}=\langle 1,0,0\rangle$ and the ideal of $Z$ in the affine coordinates $x=x_{1} / x_{0}, y=$ $x_{2} / x_{0}$ is in this case

$$
\left(y^{3}, x-s y-t^{-1} y^{2}\right), \quad s \in \mathbb{k}, \quad t \in \mathbb{k}^{*} .
$$



The corresponding Kronecker module can be taken to be

$$
\Phi=\left(\begin{array}{ccc}
x_{2}+2 s t x_{0} & x_{1}-s x_{2} & t x_{0} \\
x_{1}+s x_{2} & 0 & x_{2}
\end{array}\right)
$$

The fibre of $\nu$ over the class of $\Phi$ in $N_{0}$ consists of the orbits of the matrices

$$
A=\left(\begin{array}{ccc}
q_{0}\left(x_{0}, x_{1}, x_{2}\right) & q_{1}\left(x_{0}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}\right)  \tag{13}\\
x_{2}+2 s t x_{0} & x_{1}-s x_{2} & t x_{0} \\
x_{1}+s x_{2} & 0 & x_{2}
\end{array}\right)
$$

The coefficients of $q_{0}, q_{1}, q_{2}$ as in (12) can be seen as the projective coordinates of the fibre $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$.

The fibre of $M_{0}^{\prime}$ over $[\Phi] \in N$ is given by the ideal whose radical is

$$
\left(b_{0}, a_{0}-2 s c_{0}\right),
$$

which means that the fibre consists of one component which is a projective subspace of codimension 2 in $\nu^{-1}([\Phi]) \cong \mathbb{P}_{11}$. For computations see A.3.

### 4.3.4. Fibres of type (4)

Let $Z$ be a non-curvilinear triple point. After a change of coordinates we may assume that $Z$ is supported at $\mathrm{pt}=\langle 1,0,0\rangle$. Since there is only one noncurvilinear triple point at a given point of a smooth surface, the ideal of $Z$ equals ( $x_{1}^{2}, x_{1} x_{2}, x_{2}^{2}$ ), the corresponding Kronecker module can be taken to be

$$
\begin{gathered}
\Phi=\left(\begin{array}{ccc}
x_{2} & x_{1} & 0 \\
x_{1} & 0 & x_{2}
\end{array}\right) . \\
x_{2}=0
\end{gathered}
$$

The fibre of $\nu$ over the class of $\Phi$ in $N_{0}$ consists of the orbits of the matrices

$$
A=\left(\begin{array}{ccc}
q_{0}\left(x_{0}, x_{1}, x_{2}\right) & q_{1}\left(x_{0}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}\right) \\
x_{2} & x_{1} & 0 \\
x_{1} & 0 & x_{2}
\end{array}\right)
$$

By Lemma 4.1 all such matrices define singular sheaves since all $(2 \times 2)$-minors vanish at pt. Therefore, $M_{0}^{\prime}$ is a $\mathbb{P}_{11}$-bundle over the locus of non-curvilinear triple points.

### 4.3.5. Fibres of type (5)

Let $[\Phi] \in N^{\prime}$, then without loss of generality

$$
\Phi=\left(\begin{array}{ccc}
x_{0} & 0 & x_{1} \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

and the fibre of $\nu$ over [ $\Phi$ ] consists of the orbits of the matrices (10). By Lemma 4.1 the sheaf defined by

$$
\left(\begin{array}{ccc}
q_{0}\left(x_{1}, x_{2}\right) & q_{1}\left(x_{1}, x_{2}\right) & q_{2}\left(x_{0}, x_{1}, x_{2}\right) \\
x_{0} & 0 & x_{1} \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

is singular if and only if the quadratic forms
$q_{0}\left(x_{1}, x_{2}\right)=a_{3} x_{1}^{2}+a_{4} x_{1} x_{2}+a_{5} x_{2}^{2} \quad$ and $\quad q_{1}\left(x_{1}, x_{2}\right)=b_{3} x_{1}^{2}+b_{4} x_{1} x_{2}+b_{5} x_{2}^{2}$
have a common zero. The latter holds if and only if the resultant of $q_{0}$ and $q_{1}$

$$
R=R\left(q_{0}, q_{1}\right)\left(a_{3}, a_{4}, a_{5}, b_{3}, b_{4}, b_{5}\right)
$$

vanishes. Since $R$ is an irreducible homogeneous polynomial of degree 4 in variables $a_{3}, a_{4}, a_{5}, b_{3}, b_{4}, b_{5}$, the fibres over $N^{\prime}$ are open subsets of irreducible hyper-surfaces of degree 4 in $\mathbb{P}_{11}$. These subsets are obtained by throwing away the points corresponding to matrices with zero determinant, i. e., the line $L^{*}$ (cf. 3.2.3), which is contained in the hypersurface.

## 5. Main result

The information about the fibres of $M_{0}^{\prime}$ over $N$ obtained in the previous section allows to prove Proposition 1.1.

### 5.1. Dimension

We showed that the fibres of $M_{0}^{\prime}$ over $N$ are generically 9-dimensional, the fibres are more than 9 -dimensional only over a subvariety of $N$ of dimension 2. Therefore, the dimension of $M_{0}^{\prime}$ (and thus of $M^{\prime}$ ) is 15 , i. e., $M^{\prime}$ has codimension 2 in $M$.

### 5.2. Singularities

Notice that the computation from 4.3 .1 works also locally over the base. Let us make this clear in the case of $\mathbb{k}=\mathbb{C}$, i. e., in the analytic category with analytic topology.

Let us vary the points

$$
p_{0}=\left\langle 1, p_{1}^{(0)}, p_{2}^{(0)}\right\rangle, \quad p_{1}=\left\langle p_{0}^{(1)}, 1, p_{2}^{(1)}\right\rangle, \quad p_{2}=\left\langle p_{0}^{(2)}, p_{1}^{(2)}, 1\right\rangle
$$

in disjoint neighborhoods in $\mathbb{P}_{2}$ of points $\langle 1,0,0\rangle,\langle 0,1,0\rangle,\langle 0,0,1\rangle$ respectively. Assume moreover that $p_{0}, p_{1}, p_{2}$ are always non-collinear. Then $p_{1}^{(0)}, p_{2}^{(0)}, p_{0}^{(1)}$, $p_{2}^{(1)}, p_{0}^{(2)}, p_{1}^{(2)}$ are local coordinates of $N$ around the class of the Kronecker module

$$
\Phi=\left(\begin{array}{ccc}
x_{0} & x_{1} & 0 \\
x_{0} & 0 & x_{2}
\end{array}\right)
$$

Denote by $U_{p_{0}, p_{1}, p_{2}}$ the corresponding neighborhood of $[\Phi]$.
Let $\bar{x}_{i}, i=0,1,2$, be a linear form that defines the line not passing through $p_{i}$ and passing through the other two points.

The fibre of $\nu$ over the class of

$$
\bar{\Phi}=\left(\begin{array}{ccc}
\bar{x}_{0} & \bar{x}_{1} & 0 \\
\bar{x}_{0} & 0 & \bar{x}_{2}
\end{array}\right)
$$

consists of the orbits of the matrices

$$
A=\left(\begin{array}{ccc}
\bar{q}_{0}\left(\bar{x}_{0}, \bar{x}_{1}, \bar{x}_{2}\right) & \bar{q}_{1}\left(\bar{x}_{0}, \bar{x}_{2}\right) & \bar{q}_{2}\left(\bar{x}_{0}, \bar{x}_{1}\right) \\
\bar{x}_{0} & \bar{x}_{1} & 0 \\
\bar{x}_{0} & 0 & \bar{x}_{2}
\end{array}\right) .
$$

The coefficients of

$$
\begin{aligned}
& \bar{q}_{0}=a_{0} \bar{x}_{0}^{2}+a_{1} \bar{x}_{0} \bar{x}_{1}+a_{2} \bar{x}_{0} \bar{x}_{2}+a_{3} \bar{x}_{1}^{2}+a_{4} \bar{x}_{1} \bar{x}_{2}+a_{5} \bar{x}_{2}^{2}, \\
& \bar{q}_{1}=b_{0} \bar{x}_{0}^{2}+b_{2} \bar{x}_{0} \bar{x}_{2}+b_{5} \bar{x}_{2}^{2}, \\
& \bar{q}_{2}=c_{0} \bar{x}_{0}^{2}+c_{1} \bar{x}_{0} \bar{x}_{1}+c_{3} \bar{x}_{1}^{2}
\end{aligned}
$$

can be seen as the projective coordinates of the fibre $\nu^{-1}([\bar{\Phi}]) \cong \mathbb{P}_{11}$, this gives a trivialization of $\mathbb{B}$ around $[\Phi]$. As in 4.3.1 we conclude that $M^{\prime}$ over $U_{p_{0}, p_{2}, p_{3}}$ is a trivial bundle with the fibre computed in 4.3.1. Therefore, $M_{c}^{\prime}=M^{\prime} \cap M_{c}$ is a bundle over $N_{c}$ with this singular fibre, which shows that $M^{\prime}$ is singular.

REmark 5.1. Our argument shows that the singularities of $M_{0}^{\prime}$ over $N_{c}$ lie in codimension 2.

Remark 5.2. Notice that in the algebraic category a modification of the argument above would lead to a local triviality of $M^{\prime}$ over $N_{c}$ only in étale topology. This would not affect however our conclusions.

### 5.3. Connectedness

As shown in 4.3, every fibre of $M_{0}^{\prime}$ over $N$ is (path-)connected. Therefore, since $N$ is (path-)connected, $M_{0}^{\prime}$ is (path-)connected. Since $M_{1}^{\prime}$, which is isomorphic to the universal singular locus of plane quartic curves, is (path-)connected, it remains to connect $M_{1}^{\prime}$ with $M_{0}^{\prime}$.

The latter can be done, for example, as follows. Let $C$ be a quartic curve with a simple double point singularity $p_{0} \in C$. Fix a line through $p_{0}$ that is not a component of $C$ and intersects $C$ at 3 different points $p_{0}, p_{1}, p_{2}$.

Consider a degeneration $Z_{t}=\left\{p_{0}, p_{1}, p(t)\right\}$ of a configuration of 3 noncollinear points on $C$ to the configuration $Z_{0}=\left\{p_{0}, p_{1}, p_{2}\right\}$, i. e., $p(t) \rightarrow p_{2}$, $t \rightarrow 0$.


This gives a degeneration of the twisted ideal sheaf $\mathcal{E}_{t}=\mathcal{I}_{Z_{t}}(1)$ of $Z_{t}$ in $C$ to the twisted ideal sheaf $\mathcal{I}_{Z_{0}}(1)$ of $Z_{0}$.

Notice that $\mathcal{E}_{0}=\mathcal{I}_{Z_{0}}(1)$ is a non-trivial extension (2) with $p=p_{0}$. Therefore, $\mathcal{E}_{0}$ defines a point in $M_{1}$. Since $p_{0}$ is a singular point of $C$, as mentioned
in 4.1.1, $\mathcal{E}$ must be a singular sheaf. On the other hand, as we shall show in Proposition 6.1, $\mathcal{E}_{t}$ is a singular sheaf for $t \neq 0$ as $p_{0}$ is a singular point of $C$. This gives a path connecting $M_{0}^{\prime}$ with $M_{1}^{\prime}$.

## 6. Singular sheaves and singularities of their support

Let $[\mathcal{E}] \in M_{00}=\mathbb{B}_{0}$. Let $C=\operatorname{Supp} \mathcal{E}$ be its support, which is a quartic curve in $\mathbb{P}_{2}$. As $\mathcal{E}$ is a part of an exact sequence (5), it is a subsheaf of $\mathcal{O}_{C}(1)$, hence a torsion free sheaf on $C$. Since torsion free sheaves on smooth curves are locally free (see e.g. [10, Lemma 5.2.1]), we conclude that $\mathcal{E}$ is non-singular if $C$ is smooth at all points of $Z$. So $\mathcal{E}$ can only be singular if $C$ is singular at some points of $Z$. This demonstrates that the image of $M_{00}^{\prime}=M^{\prime} \cap M_{00}$ under (9) is included in the subvariety of pairs $(C, Z)$ such that $Z$ contains a singular point of $C$. We shall demonstrate that $M_{00}^{\prime}$ generically coincides with this variety. More precisely, the image of $M_{c}^{\prime}=M^{\prime} \cap M_{c}$ under the morphism

$$
M_{c} \xrightarrow{\mu \times \nu} \mathbb{P} S^{4} V^{*} \times H_{0}
$$

consists of the pairs $(C, Z), Z \subseteq C$, such that $C$ is a singular plane curve of degree 4 whose singular locus contains at least one of the points of $Z$.

Proposition 6.1. Let $[\mathcal{E}]$ as above belong to $M_{c}$, then

1) $\mathcal{E}$ is singular if and only if $\operatorname{Sing} C \cap Z \neq \emptyset$;
2) the fibre of $M_{0}^{\prime}$ over $Z \in H_{c}, Z=\left\{\mathrm{pt}_{0}, \mathrm{pt}_{1}, \mathrm{pt}_{2}\right\}$, under the morphism $M_{0}^{\prime} \xrightarrow{\nu} N_{c} \cong H_{c}$ corresponds to the variety of plane quartic curves through $Z$ such that one of the points of $Z$ is a singular point of $C$;
3) for each $i=0,1,2$, the variety of quartics through $Z$ such that $\mathrm{pt}_{i}$ is a singular point of $C$ coincides with one of three different irreducible components of the fibre.

Proof. Follows from the computations given in A.1.
Remark 6.2. Since $\mathcal{E}$ is a twisted ideal sheaf of 3 different points on a quartic curve (cf. (5)), the statement 1) of Proposition 6.1 immediately follows from Lemma 6.3 below.

### 6.1. An observation from commutative algebra

Let $R=\mathcal{O}_{C, p}$ be a local $\mathbb{k}$-algebra of a curve $C$ at point $p \in C$. Let $\mathfrak{m}=\mathfrak{m}_{C, p}$ be its maximal ideal and let $\mathbb{k}_{p}=R / \mathfrak{m}$ be the local ring of the structure sheaf of the one point subscheme $\{p\} \subseteq C$. An $R$-module homomorphism $R \xrightarrow{\varphi} \mathbb{k}_{p}$ is uniquely defined by $\varphi(1)=\lambda \in \mathbb{k}_{p}$. Then $\varphi(s)=\bar{s} \cdot \lambda$. If $\varphi$ is different from zero, then the kernel of $\varphi$ coincides with $\mathfrak{m}$.

Lemma 6.3. Consider an exact sequence of $R$-modules.

$$
0 \rightarrow M \rightarrow R \rightarrow \mathbb{k}_{p} \rightarrow 0
$$

with a non-zero $R$-module $M$. Then $M$ is free if and only if $R$ is regular.
Proof. If $M$ is free, then $M \cong R$ (otherwise $M \rightarrow R$ would not be injective) and we obtain an exact sequence of $R$-modules

$$
0 \rightarrow R \rightarrow R \rightarrow \mathbb{k}_{p} \rightarrow 0
$$

which means that the maximal ideal $\mathfrak{m}$ of $p$ is generated by one element. Therefore, $R$ is regular in this case.

Vice versa, assume $R$ is regular. Notice that $M$ is always a torsion free $R$ module as a submodule of $R$. Therefore, if $R$ is regular, $M$ is free as a torsion free module over a regular one-dimensional local ring.

Remark 6.4. Notice that Proposition 6.1 does not hold over $N_{0} \backslash N_{c}$. Indeed, take $\left[\mathcal{E}_{A}\right] \in M_{00} \backslash M_{c}$ with

$$
A=\left(\begin{array}{ccc}
x_{2}^{2} & 0 & x_{1}^{2} \\
x_{0} & x_{1} & 0 \\
0 & x_{0} & x_{2}
\end{array}\right)
$$

Then the support $C$ of $\mathcal{E}_{A}$ is given by $x_{1}\left(x_{2}^{3}+x_{0}^{2} x_{1}\right)=0$, one obtains an exact sequence

$$
0 \rightarrow \mathcal{E}_{A} \rightarrow \mathcal{O}_{C}(1) \rightarrow \mathcal{O}_{Z} \rightarrow 0
$$

such that $Z$ consists of the simple point $\langle 0,0,1\rangle$ and the double point $\langle 0,1,0\rangle$. In this case $\mathcal{E}_{A}$ is a non-singular sheaf but $\langle 0,1,0\rangle \in Z \cap \operatorname{Sing} C$.

In A. 2 we compute that every matrix $A$ as in 4.3 .2 with $a_{3}=0, a_{5} \neq 0$ defines a non-singular sheaf, however the intersection of $Z$ with the singular locus of the supporting curve $C$ is non-empty.

## A. Computations of the fibres of $M_{0}^{\prime}$ over $N$ with Singular

## A.1. Fibres of type (1)

```
t1.sng 1> LIB "elim.lib";
t1.sng 2> ring r=0, (x(0..2), a(0..5), b(0..5), c(0..5)), dp;
t1.sng 3> ideal maxm=x(0..2);
t1.sng 4> poly X=x(0)*x(1)*x(2);
t1.sng 5> poly q(0..2);
l1.sng 6> q(0)=a(0)*x(0)~2+a(1)*x(0)*x(1) +a(2)*x(0)*x(2)+a(3)*x(1)~2+a(4)*x(1)*x(2) +a(5)*x(2)~2;
t1.sng 7> q(1)=b(0)*x(0)~2 + b(1)*x(0)*x(1) +b(2)*x(0)*x(2) +b(3)*x(1) -2 +b(4)*x(1)*x(2) +b(5)*x(2)~2;
t1.sng 8> q(2)=c(0)*x(0) ~2+c(1)*x(0)*x(1) +c(2)*x(0)*x(2) +c(3)*x(1) -2 +c(4)*x(1)*x(2) +c(5)*x(2) -2;
t1.sng 9> q(1)=subst(q(1), x(1), 0);
t1.sng 11> // general form of matrices representing the points in the fibre
t1.sng 12. matrix A[3][3] =q(0),q(1),q(2), x(0), x(1), 0, x(0), 0, x(2);
```

```
t1.sng 13> print(A)
A[1,1],A[1,2],A[1,3]
x(0), x(1), 0,
t1.sng 14> // the Kronecker module corresponding to 3 non-collinear points
t1.sng 15. matrix Phi=submat(A, 2..3, 1..3);
t1.sng 16> print(Phi);
x(0),x(1),0,
t1.sng 17> // the ideal of 2x2 minors of A
t1.sng 17> // the ideal of 2x2 minors
t1.sng 19> minm = sat(minm, maxm) [1]; // compute its saturation
t1.sng 20> minm = elim(minm, X); // eliminate the variables x(0), x(1), x(2)
t1.sng 21> print(minm);
b(5)*c(0)*c(3),
a(5)*c(0)*c(3),
b(0)*b(5)*c(3),
a(5)*b(0)*c(3),
a(3)*b(5)*c(0),
a(3)*b(0)*b(5),
a(3)*a(5)*b(0)
t1.sng 22> // look at the primary decomposition of the result
t1.sng 23. primdecGTZ(minm);
[1]:
        -[1]=c(3)
    [2]:
    -[1]=c(3)
[2]:
    _[1]=c(0)
    [2]:
        -[1]=c(0)
        -[2]=b(0)
[3]:
    -[1]=b(5)
    [2]:
    -[1]=b(5)
    -[1]=b(5)
t1.sng 24> // let us establish a link between the singular sheaves
t1.sng 25. // and the singularities of their support
t1.sng 26. poly f=\operatorname{det}(A); // determinant of A
t1.sng 27> // ideal of partial derivatives of }
t1.sng 28. // with respect to x(0), x(1), x(2)
t1.sng 29.// together with the 2x2-minors of Phi,
t1.sng 30. // its zeroes are exactly the singular points of C contained in Z
t1.sng 31. ideal D = diff(f,x(0)), diff(f, x(1)), diff(f, x(2)), minor(Phi, 2);
t1.sng 32> // look at the equations of the subvariety of the fibre defining such sheaves
t1.sng 32> // look at the equat
t1.sng 33. D = sat(D, maxm)
t1.sng 35> // the result coincides with the ideal for singular sheaves
t1.sng 36. primdecGTZ(D);
[1]:
        -[1]=c(3)
    [2]:
    -[1]=c(3)
[2]:
    _[1]=c(0)
        [2]:
        -[1]=c(0)
        -[2]=b(0)
[3]:
    -[1]=b(5)
    [2]:-[2]=a(5)
    C}\begin{array}{l}{[1]=\textrm{b}(5)}\\{[2]=a(5)}
t1.sng 37> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng 38. D= diff(f,x(0)), diff(f,x(1)), diff(f,x(2)), x(1), x(2);
t1.sng 39> D = sat(D, maxm)[1];
t1.sng 40> D = elim(D, X);
t1.sng 41> // the result coincides with one of the components
```

```
t1.sng 42. // of the fibre of singular sheaves computed above
t1.sng 43. primdecGTZ(D);
[1]:
    _[1]=c(0)
    <-[2]=b(0)
    -[1]=c(0)
sng 44> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng 44> // the ideal of the intersection of pt0 with the singular loc
t1.sng 45. D = diff(f, x(0), d
*1.sng 46> D = sat(D,maxm;
t1.sng 48> // the result coincides with one of the components
t1.sng 49.// of the fibre of singular sheaves computed above
t1.sng 50. primdecGTZ(D);
[1]:
    -[1]=c(3)
    [2]:[2]=a(3)
    [2]: [1]=c(3)
    [2]=a(3)
t1.sng 51> // the ideal of the intersection of pt0 with the singular locus of C
t1.sng 52. D = diff(f,x(0)), diff(f,x(1)), diff(f,x(2)), x(0), x(1);
t1.sng 53> D = sat(D, maxm) [1];
t1.sng 54> D = elim(D, X);
t1.sng 55> // the result coincides with one of the components
t1.sng 56. // of the fibre of singular sheaves computed above
t1.sng 57. primdecGTZ(D)
[1]:
    _[1]=b(5)
    -[2]=a(5)
    _[1]=b(5)
    [2]=a(5)
t1.sng 58> - $
```


## A.2. Fibres of type (2)

```
t2.sng 1> LIB "elim.lib";
t2.sng 2> ring r=0, (x(0..2), a(0..5), b(0..5), c(0..5)), dp;
t2.sng 3> ideal maxm = x(0..2);
t2.sng 4> poly X = x(0)*
t2.sng 5> poly q(0..2); 
t2.sng 6> q(0)=a(0)*x(0)~2+a(1)*x(0)*x(1)+a(2)*x(0)*x(2)+a(3)*x(1)~2+a(4)*x(1)*x(2) +a(5)*x(2)~2;
```



```
t2.sng 9> q(1) = subst(q(1),x(1),0);
t2.sng 10> q(2)=subst(q(2),x(2),0);
t2.sng 11> // general form of matrices representing the points in the fibre
t2.sng 12. matrix A[3][3] = q(0), q(1),q(2), x(0), x(1), 0, 0, x(0), x(2);
t2.sng 13> print(A);
A[1,1],A[1,2],A[1,3]
lll
t2.sng 14> // the Kronecker module corresponding to 3 non-collinear points
t2.sng 14> matrix Phi = submat(A, 2..3,1..3);
t2.sng 16> print(Phi);
x(0),x(1),0,
0, x(0),x(2)
t2.sng 17> // the ideal of 2x2 minors
t2.sng 18. ideal minm = minor(A, 2);
t2.sng 19> minm = sat(minm, maxm) [1]; // compute its saturation
t2.sng 20> minm=elim(minm, x); // eliminate the variables x(0), x(1), x(2)
t2.sng 21> minm;
minm[1]=b(5)*c(3)^2
minm[2]=a(5)*c(3)*
minm
minm[5]=a(3)*b(5)*c(1)-a(1)*b(5)*c(3)
minm[6]=a(3)*a(5)*c(1)-a(1)*a(5)*c(3)
minm[7]=a(3)-2*b(5)
minm[8]=a(3)}-2*\textrm{a}(5
t2.sng 22> // look at the primary decomposition of the result
t2.sng 22> // look at the prim
[1]:
    _[1]=c(3)~2
    -[2]=a(3)*c(3)
```

```
        [3]=a(3)~2
        -[4]=-a(3)*c(1)+a(1)*c(3)
    [2]:
        -[1]=c(3)
        -[2]=a(3)
[2]:
    -[1]=b(5)
    [2]:-[2]=a(5)
    -[1]=b(5)
t2.sng 24> // polynomial defining the quartic curve C
t2.sng 25. poly f=det(A);
t2.sng 26> // ideal of singularities of the curve C lying on }
t2.sng 27. ideal D = diff(f,x(0)), diff(f,x(1)), diff(f, x(2)), minor(Phi, 2);
t2.sng 28> // compute the equations of the subvariety of the corresponding sheaves
t2.sng 29. D = sat(D, maxm)[1];
t2.sng 30> D = elim(D, X);
t2.sng 31> D;
利[1]=a(3)*b(5)*c(3)
D[2]=a(3)*a(5)*c(3)
D[4]=a(3)~2**(5)
t2.sng 32> // look at its primary decomposition
t2.sng 33. // the corresponding variety has an extra component
t2.sng 34. // whose points do not define singular sheaves
t2.sng 35. primdecGTZ(D);
[1]:
    [2]:
    [[1]=a(3)
[2]:
    -[1]=b(5)
    2]:-[2]=a(5)
    [2]:
        -[1]=b(5)
[3]:
    -[1]=c(3)
    _-[1]=c(3)
    [2]:
    [2]=a(3)
t2.sng (2] 36> $
```


## A.3. Fibres of type (3)

t3.sng $1>$ LIB "elim.lib";
t3.sng 2> ring $r=(0, s, t),(x(0 . .2), a(0 . .5), b(0 . .5), c(0 . .5)), d p ;$
$\begin{array}{ll}\text { t3.sng } & \text { 3> ideal } I=x(2)-3, x(1) * x(0)-s * x(2) * x(0)-(1 / t) * x(2)-2 \text {; } \\ \text { t3.sng } & 4> \\ I=s a t(I, ~ & x(0))[1] ;\end{array}$
t3.sng 4> $I=\operatorname{sat}(I, x(0))[1]$;
t3.sng 5> I;
I[1] $=\mathrm{x}(1) * \mathrm{x}(2)+(-\mathrm{s}) * \mathrm{x}(2)-2$
$1[2]=x(1)-2+(-2 * s) * x(1) * x(2)+\left(s^{\wedge} 2\right) * x(2)-2$
$I[3]=(t) * x(0) * x(1)+(-s * t) * x(0) * x(2)-x(2) \sim 2$
$\mathrm{I}[4]=\mathrm{x}(2)-3$
t3.sng 6> ideal $\mathrm{J}=\mathrm{I}[1 . .3]$;
t3.sng $7>\operatorname{std}(\mathrm{J})$;
$-[1]=x(1) * x(2)+(-s) * x(2)-2$
$-[2]=\mathrm{x}(1)-2+(-2 * \mathrm{~s}) * \mathrm{x}(1) * \mathrm{x}(2)+\left(\mathrm{s}^{\wedge} 2\right) * \mathrm{x}(2) \sim 2$
$-[3]=(t) * x(0) * x(1)+(-s * t) * x(0) * x(2)-x(2) \sim 2$
$-[4]=x(2)-3$
$-[4]=x(2)-3$
t3.sng $8>/ /$ thus $I=J$
$\mathrm{t} 3 . \mathrm{sng} \quad$ 9. $\mathrm{I}=\mathrm{J} ;$
$\mathrm{t} 3 . \mathrm{sng} \quad 10>$ matrix $\mathrm{S}[2][3]=(2 * \mathrm{~s} * \mathrm{t}) * \mathrm{x}(0)+\mathrm{x}(2), \mathrm{x}(1)-(\mathrm{s}) * \mathrm{x}(2),(\mathrm{t}) * \mathrm{x}(0), \mathrm{x}(1)+(\mathrm{s}) * \mathrm{x}(2), 0, \mathrm{x}(2)$;
t3.sng 11> print (S);
$(2 * \mathrm{~s} * \mathrm{t}) * \mathrm{x}(0)+\mathrm{x}(2), \mathrm{x}(1)+(-\mathrm{s}) * \mathrm{x}(2),(\mathrm{t}) * \mathrm{x}(0)$,
$x(1)+(s) * x(2), \quad 0, \quad x(2)$
t3.sng 12> // the ideal of maximal minors coincides with I
t3.sng 13. minor (S, 2);
$[1]=x(1) * x(2)+(-s) * x(2)-2$
$-[2]=(-\mathrm{t}) * \mathrm{x}(0) * \mathrm{x}(1)+(\mathrm{s} * \mathrm{t}) * \mathrm{x}(0) * \mathrm{x}(2)+\mathrm{x}(2)-2$
$-[3]=-x(1)-2+\left(s^{-} 2\right) * x(2)-2$
$\begin{array}{ll}\text { t3.sng } & 14> \\ \text { t3.sng } & 15> \\ \text { poleal } \mathrm{maxm}=\mathrm{x}(0 . .2) \text {; } \\ \mathrm{x}(0) * \mathrm{x}(1) * \mathrm{x}(2) \text {; }\end{array}$
t3.sng 16> poly q(0..2);
t3.sng 17> $q(0)=a(0) * x(0) \wedge 2+a(1) * x(0) * x(1)+a(2) * x(0) * x(2)+a(3) * x(1) \sim 2+a(4) * x(1) * x(2)+a(5) * x(2) \sim 2$;

```
t3.sng 18> q(1) = b(0)*x(0)^2 + b(1)*x(0)*x(1) +b(2)*x(0)*x(2) +b(3)*x(1)^2 +b(4)*x(1)*x(2) +b(5)*x(2) 2;
t3.sng 19> q(2) = c(0)*x(0)~2 + c(1)*x(0)*x(1) + c(2)*x(0)*x(2) + c(3)*x(1) ~2 + c(4)*x (1)*x(2) +c(5)*x (2) -2;
t3.sng 20> q(1) = subst(q(1), x(1), 0);
t3.sng 21> q(2) = subst(q(2),x(2),0);
t3.sng 22> // the linear part is S
t3.sng 23. matrix A[3] [3];
t3.sng 24> A = q(0),q(1),q(2),(2*s*t)*x(0)+x(2), x(1)-(s)*x(2),(t)*x(0), x(1)+(s)*x(2),0,x(2);
t3.sng 25> print(A);
A[1,1], A[1,2], A[1,3],
(2*s*t)*x(0)+x(2),x(1)+(-s)*x(2),(t)*x(0),
ex(1)+(s)*x(2), 0, 0, (-)
t3.sng 26> // the linear part Phi
t3.sng 27. matrix Phi = submat(A, 2..3, 1..3);
t3.sng 28> //ideal of 2x2 minor sof A
t3.sng 29. ideal minm = minor(A, 2);
t3.sng 30> // compute the ideal of the subvariety of singular sheaves in the fibre
t3.sng 31.minm = sat(minm, maxm)[1];
t3.sng 32> minm = elim(minm, X);
t3.sng 33> // look at its primary decomposition
t3.sng 34. list PD = primdecGTZ (minm);
t3.sng 35> // it has only one component
t3.sng 36. size(primdecGTZ(minm));
t3.sng 37> // the corresponding prime ideal is
t3.sng 38. PD[1][2];
-[1]=b(0)
-[2]=a(0)+(-2*s)*c(0)
l
```


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# Multiply warped products as quasi-Einstein manifolds with a quarter-symmetric connection 

Sampa Pahan, Buddhadev Pal and Arindam Bhattacharyya


#### Abstract

In this paper we study warped products and multiply warped products on quasi-Einstein manifolds with a quarter-symmetric connection. Then we apply our results to generalize Robertson-Walker spacetime with a quarter-symmetric connection.


Keywords: Quasi-Einstein manifold, warped product, multiply warped product, quartersymmetric connection.
MS Classification 2010: 53C25.

## 1. Introduction

A Riemannian manifold $\left(M^{n}, g\right), n \geq 2$, is said to be an Einstein manifold if its Ricci tensor $S$ satisfies the condition $S=\frac{r}{n} g$, where $r$ denotes the scalar curvature of $M$. M. C. Chaki and R. K. Maity introduced the notion of quasiEinstein manifold in [2]. A non-flat Riemannian manifold $(M, g), n \geq 2$, is said to be a quasi-Einstein manifold if the condition

$$
S(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y)
$$

is fulfilled on $M$, where $\alpha$ and $\beta$ are scalars of which $\beta \neq 0$ and $\eta$ is a non-zero 1-form such that $g(X, U)=\eta(X)$, for all vector field $X$ and $U$, a unit vector field.

Let $\left(B, g_{B}\right)$ and $\left(F, g_{F}\right)$ be two Riemannian manifolds and $f>0$ be a differential function on $B$. Consider the product manifold $B \times F$ with its projections $\pi: B \times F \rightarrow B$ and $\sigma: B \times F \rightarrow F$. The warped product $B \times_{f} F$ is the manifold $B \times F$ with the Riemannian structure such that $\|X\|^{2}=\left\|\pi^{*}(X)\right\|^{2}+f^{2}(\pi(p))\left\|\sigma^{*}(X)\right\|^{2}$, for any vector field $X$ on $M$. Thus we have that $g_{M}=g_{B}+f^{2} g_{F}$ holds on $M$. Here $B$ is called the base of $M$ and $F$ is called the fiber. The function $f$ is called the warping function of the warped product [7]. The concept of warped product was first introduced by Bishop
and O'Neill [1] to construct examples of Riemannian manifolds with negative curvature.

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold $M=B \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ with the metric $g=g_{B} \oplus b_{1}^{2} g_{F_{1}} \oplus b_{2}^{2} g_{F_{2}} \oplus b_{3}^{2} g_{F_{3}} \ldots \oplus b_{m}^{2} g_{F_{m}}$, where for each $i \in\{1,2, \ldots m\}, b_{i}: B \rightarrow(0, \infty)$ is smooth and $\left(F_{i}, g_{F_{i}}\right)$ is a pseudo-Riemannian manifold. In particular, when $B=(c, d)$, the metric $g_{B}=-d t^{2}$ is negative and $\left(F_{i}, g_{F_{i}}\right)$ is a Riemannian manifold. We call $M$ the multiply generalized Robertson-Walker spacetime.

A multiply twisted product $(M, g)$ is a product manifold of the form $M=$ $B \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ with the metric $g=g_{B} \oplus b_{1}^{2} g_{F_{1}} \oplus b_{2}^{2} g_{F_{2}} \oplus b_{3}^{2} g_{F_{3}} \ldots \oplus$ $b_{m}^{2} g_{F_{m}}$, where for each $i \in\{1,2, \ldots m\}, b_{i}: B \times F_{i} \rightarrow(0, \infty)$ is smooth.

In 1924, Friedmann and Schouten introduced the notion of a semi-symmetric linear connection on a differentiable manifold [3]. The definition of metric connection with torsion on a Riemannian manifold, was given by Hayden (1932) in [5]. In 1970, K. Yano [10] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [4] introduced the definition of a quarter-symmetric linear connection on a differentiable manifold, which is a generalization of semi-symmetric connection. Later in [8], Q. Qu and Y. Wang generalized the results to warped product and multiply warped product with a quarter-symmetric connection.

In this paper we consider multiply warped products as quasi-Einstein manifolds endowed with a quarter-symmetric connection. In section 2 and 3, we discuss some preliminary concepts and results which are useful for proving our main results in the next sections 4 and 5 . In Theorem 4.1, we obtain a necessary and sufficient condition for the warped product manifold to be a quasi-Einstein manifold with respect to a quarter-symmetric connection. Then in Theorem 4.2, under some assumptions on base and fiber we study quasiEinstein manifold with respect to a quarter-symmetric connection. Next in Theorem 4.3, we establish that if $(M, g)$ admits a metric for Robertson-Walker spacetime then it is a quasi-Einstein manifold with respect to the above mentioned connection under certain conditions. Then in Theorem 4.5, we characterize the warping function for a warped product space $(M, g)$ with a quartersymmetric connection. Later in Theorem 4.5, we show that for quasi-Einstein warped product with respect to a quarter-symmetric connection the complete connected ( $\bar{n}-1$ )-dimensional base is isometric to a ( $\bar{n}-1$ )-dimensional sphere. In the last section, we study special multiply warped product manifold with respect to a quarter-symmetric connection.

## 2. Preliminaries

Let $\left(M^{n}, g\right)$ be a Riemannian manifold with the Levi-Civita connection $\nabla$. A linear connection $\breve{\nabla}$ on $\left(M^{n}, g\right)$ is said to be a quarter-symmetric connection if its torsion tensor $T$ with respect to the connection $\breve{\nabla}$ defined by

$$
T(X, Y)=\breve{\nabla}_{X} Y-\breve{\nabla}_{Y} X-[X, Y],
$$

satisfies

$$
T(X, Y)=\omega(Y) \phi X-\omega(X) \phi Y
$$

where $\omega$ is a 1 -form on $M^{n}$ with the associated vector field $P$ defined by $\omega(X)=g(X, P)$, for all vector field $X$, and $\phi$ is a $(1,1)$ tensor field.

A quarter-symmetric connection $\nabla$ is called a quarter-symmetric metric connection if $\breve{\nabla} g=0 . \breve{\nabla}$ is called a quarter-symmetric non-metric connection if $\nabla \vec{\nabla} \neq 0$.

The relation between a quarter-symmetric connection $\breve{\nabla}$ and the Levi-Civita connection $\nabla$ of $M^{n}$ is given by [9]

$$
\begin{equation*}
\breve{\nabla}_{X} Y=\nabla_{X} Y+\lambda_{1} \omega(Y) X-\lambda_{2} g(X, Y) P, \tag{1}
\end{equation*}
$$

where $g(X, P)=\omega(X)$ and $\lambda_{1} \neq 0, \lambda_{2} \neq 0$ are scalar functions.
We can easily see that:
when $\lambda_{1}=\lambda_{2}=1, \breve{\nabla}$ is a semi-symmetric metric connection,
when $\lambda_{1}=\lambda_{2} \neq 1, \breve{\nabla}$ is a quarter-symmetric metric connection,
when $\lambda_{1} \neq \lambda_{2}, \breve{\nabla}$ is a quarter-symmetric non-metric connection.
Further, a relation between the curvature tensors $R$ and $\breve{R}$ of type $(1,3)$ of the connections $\nabla$ and $\nabla$ respectively is given by [9],

$$
\begin{align*}
& \breve{R}(X, Y) Z=R(X, Y) Z+\lambda_{1} g\left(Z, \nabla_{X} P\right) Y-\lambda_{2} g\left(Z, \nabla_{Y} P\right) X, \\
& \quad+\lambda_{2}\left[g(X, Z) \nabla_{Y} P-g(Y, Z) \nabla_{X} P\right]+\lambda_{1} \lambda_{2} \omega(P)[g(X, Z) Y-g(Y, Z) X] \\
& \quad+\lambda_{2}^{2}[g(Y, Z) \omega(X)-g(X, Z) \omega(Y)] P+\lambda_{1}^{2} \omega(Z)[\omega(Y) X-\omega(X) Y] \tag{2}
\end{align*}
$$

for vector fields $X, Y, Z$ on $M$.

## 3. Warped Product Manifolds with Quarter-Symmetric Connection

In this section we consider the following propositions from Propositions 3.5, $3.6,3.7$ and 3.8 of [8], which will be helpful to prove our main results of next section.

Proposition 3.1. Let $M=B \times{ }_{f} F$ be a warped product. Let $S$ and $\breve{S}$ denote the Ricci tensors of $M$ with respect to the Levi-Civita connection and a quartersymmetric connection respectively. Let $\operatorname{dim} B=n_{1}, \operatorname{dimF}=n_{2}, \operatorname{dim} M=\bar{n}=$ $n_{1}+n_{2}$. If $X, Y \in \chi(B), V, W \in \chi(F)$ and $P \in \chi(B)$, then
(i) $\breve{S}(X, Y)=\breve{S}^{B}(X, Y)+n_{2}\left[\frac{H_{B}^{f}(X, Y)}{f}+\lambda_{2} \frac{P f}{f} g(X, Y)+\lambda_{1} \lambda_{2} \omega(P) g(X, Y)+\right.$ $\left.\lambda_{1} g\left(Y, \nabla_{X} P\right)-\lambda_{1}^{2} \omega(X) \omega(Y)\right]$,
(ii) $\breve{S}(X, V)=\breve{S}(V, X)=0$,
(iii) $\breve{S}(V, W)=S^{F}(V, W)+\left\{\lambda_{2} \operatorname{div}_{B} P+\left(n_{2}-1\right) \frac{\left|\operatorname{grad}_{B} f\right|_{B}^{2}}{f^{2}}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\right.\right.$ $\left.\left.\lambda_{2}^{2}\right] \omega(P)+\left[(\bar{n}-1) \lambda_{1}+\left(n_{2}-1\right) \lambda_{2}\right] \frac{P f}{f}+\frac{\Delta_{B} f}{f}\right\} g(V, W)$, where div ${ }_{B} P=$ $\sum_{k=1}^{n_{1}} \varepsilon_{k}\left\langle\nabla_{E_{k}} P, E_{k}\right\rangle$ and $E_{k}, 1 \leq k \leq n_{1}$, is an orthonormal basis of $B$ with $\varepsilon_{k}=g\left(E_{k}, E_{k}\right)$.

Proposition 3.2. Let $M=B \times_{f} F$ be a warped product, $\operatorname{dim} B=n_{1}$, $\operatorname{dimF}=$ $n_{2}, \operatorname{dim} M=\bar{n}=n_{1}+n_{2}$. If $X, Y \in \chi(B), V, W \in \chi(F)$ and $P \in \chi(F)$, then
(i) $\breve{S}(X, Y)=S^{B}(X, Y)+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P) g(X, Y)+n_{2} \frac{H_{B}^{f}(X, Y)}{f}+$ $\lambda_{2} g(X, Y) \operatorname{div}_{F} P$,
(ii) $\breve{S}(X, V)=\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] \omega(V) \frac{X f}{f}$,
(iii) $\breve{S}(V, X)=\left[\lambda_{2}-(\bar{n}-1) \lambda_{1}\right] \omega(V) \frac{X f}{f}$,
(iv) $\breve{S}(V, W)=S^{F}(V, W)+g(V, W)\left\{\left(n_{2}-1\right) \frac{\left|\operatorname{grad}_{B} f\right|_{B}^{2}}{f^{2}}+\frac{\Delta_{B} f}{f}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\right.\right.$ $\left.\left.\lambda_{2}^{2}\right] \omega(P)+\lambda_{2} \operatorname{div}_{F} P\right\}+\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] g\left(W, \nabla_{V} P\right)+\left[\lambda_{2}^{2}+(1-\right.$ $\left.\bar{n}) \lambda_{1}^{2}\right] \omega(V) \omega(W)$.

By Proposition 3.1 and Proposition 3.2 and by the definition of the scalar curvature, we have the following propositions.

Proposition 3.3. Let $M=B \times_{f} F$ be a warped product, $\operatorname{dim} B=n_{1}, \operatorname{dimF}=$ $n_{2}, \operatorname{dim} M=\bar{n}=n_{1}+n_{2}$. If $P \in \chi(B)$, then

$$
\begin{aligned}
\breve{r}^{M}=\breve{r}^{B} & +\frac{r^{F}}{f^{2}}+n_{2}\left(n_{2}-1\right) \frac{\left|\operatorname{grad}_{B} f\right|_{B}^{2}}{f^{2}}+n_{2}(\bar{n}-1)\left(\lambda_{1}+\lambda_{2}\right) \frac{P f}{f}+2 n_{2} \frac{\Delta_{B} f}{f} \\
& +\left[n_{2}\left(\bar{n}+n_{1}-1\right) \lambda_{1} \lambda_{2}-n_{2}\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)\right] \omega(P)+n_{2}\left(\lambda_{1}+\lambda_{2}\right) d i v_{B} P .
\end{aligned}
$$

Proposition 3.4. Let $M=B \times{ }_{f} F$ be a warped product, $\operatorname{dim} B=n_{1}, \operatorname{dimF}=$ $n_{2}, \operatorname{dim} M=\bar{n}=n_{1}+n_{2}$. If $P \in \chi(F)$, then

$$
\begin{aligned}
\breve{r}^{M}=r^{B}+\frac{r^{F}}{f^{2}}+(\bar{n}-1)\left(\lambda_{1}+\lambda_{2}\right) \operatorname{div}_{F} P & +\left[\bar{n}(\bar{n}-1) \lambda_{1} \lambda_{2}+(1-\bar{n})\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)\right] \omega(P) \\
& +n_{2}\left(n_{2}-1\right) \frac{\left|\operatorname{grad}_{B} f\right|_{B}^{2}}{f^{2}}+2 n_{2} \frac{\Delta_{B} f}{f}
\end{aligned}
$$

## 4. Generalized Robertson-Walker Spacetime with a Quarter-Symmetric Connection

In this section we consider a quasi-Einstein warped product manifold with respect to a quarter-symmetric connection. We prove the following theorem.

Theorem 4.1. Let $(M, g)$ be a warped product $I \times{ }_{f} F$ where $I$ is an open interval in $\mathbb{R}$, $\operatorname{dim} I=1$ and $\operatorname{dimF}=\bar{n}-1, \bar{n} \geq 3$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if $F$ is a quasi-Einstein manifold for $P=\frac{\partial}{\partial t}$ with respect to the Levi-Civita connection or the warping function $f$ is a constant on $I$ for $P \in \chi(F), \lambda_{2} \neq(\bar{n}-1) \lambda_{1}$.
Proof. Assume that $P \in \chi(B)$ and let $g_{I}$ be the metric on $I$. Taking $f=e^{\frac{q}{2}}$ and using the Proposition 3.1, we get

$$
\begin{gather*}
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=(1-\bar{n})\left[\frac{1}{2} q^{\prime \prime}+\frac{1}{4} q^{\prime^{2}}-\frac{1}{2} \lambda_{2} q^{\prime}+\lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] g_{I}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right),  \tag{3}\\
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=0  \tag{4}\\
\breve{S}(V, W)=S^{F}(V, W)+e^{q}\left[\frac{\bar{n}-1}{4}\left(q^{\prime}\right)^{2}+\frac{1}{2}\left[(\bar{n}-1) \lambda_{1}+(\bar{n}-2) \lambda_{2}\right] q^{\prime}\right. \\
\left.\quad+\lambda_{2}^{2}+\frac{1}{2} q^{\prime \prime}+(1-\bar{n}) \lambda_{1} \lambda_{2}\right] g_{F}(V, W) \tag{5}
\end{gather*}
$$

for vector fields $V, W$ on $F$.
Since $M$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection, we have

$$
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)+\beta \eta\left(\frac{\partial}{\partial t}\right) \eta\left(\frac{\partial}{\partial t}\right)
$$

and

$$
\breve{S}(V, W)=\alpha g(V, W)+\beta \eta(V) \eta(W)
$$

Then the last two equations reduce to

$$
\begin{equation*}
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\alpha g_{I}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)+\beta \eta\left(\frac{\partial}{\partial t}\right) \eta\left(\frac{\partial}{\partial t}\right) \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\breve{S}(V, W)=\alpha e^{q} g_{F}(V, W)+\beta \eta(V) \eta(W) . \tag{7}
\end{equation*}
$$

Decomposing the vector field $U$ uniquely into its components $U_{I}$ and $U_{F}$ on $I$ and $F$, respectively, we have $U=U_{I}+U_{F}$. Since $\operatorname{dim} I=1$, we can take $U_{I}=v \frac{\partial}{\partial t}$ which gives $U=v \frac{\partial}{\partial t}+U_{F}$, where $v$ is a function on $M$. Thus, we can write

$$
\begin{equation*}
\eta\left(\frac{\partial}{\partial t}\right)=g\left(U, \frac{\partial}{\partial t}\right)=v \tag{8}
\end{equation*}
$$

Using equations (3) and (5), equations (6), (7) reduce to

$$
\begin{equation*}
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\alpha+\beta v^{2} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\breve{S}(V, W)=\alpha e^{q} g_{F}(V, W)+\beta \eta(V) \eta(W) \tag{10}
\end{equation*}
$$

Comparing the right hand sides of (3) and (9), we get

$$
\begin{equation*}
\alpha+\beta v^{2}=(1-\bar{n})\left[\frac{1}{2} q^{\prime \prime}+\frac{1}{4} q^{\prime^{2}}-\frac{\lambda_{2} q^{\prime}}{2}+\lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] . \tag{11}
\end{equation*}
$$

Similarly, comparing the right hand sides of (5) and (10), we obtain

$$
\begin{align*}
S^{F}(V, W)=e^{q}[\alpha & +\frac{1-\bar{n}}{4}\left(q^{\prime}\right)^{2}-\frac{1}{2}\left[(\bar{n}-1) \lambda_{1}+(\bar{n}-2) \lambda_{2}\right] q^{\prime} \\
& \left.-\lambda_{2}^{2}-\frac{1}{2} q^{\prime \prime}+(\bar{n}-1) \lambda_{1} \lambda_{2}\right] g_{F}(V, W)+\beta \eta(V) \eta(W) \tag{12}
\end{align*}
$$

which gives that $F$ is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(B)$.

Taking $P \in \chi(F)$ and by the use of Proposition 3.2, we get

$$
\begin{equation*}
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=\frac{q^{\prime}}{2}\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] \omega(V) \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\breve{S}\left(V, \frac{\partial}{\partial t}\right)=\frac{q^{\prime}}{2}\left[\lambda_{2}-(\bar{n}-1) \lambda_{1}\right] \omega(V) \tag{14}
\end{equation*}
$$

for any vector field $V \in \chi(F)$.

Since $M$ is a quasi-Einstein manifold, we have

$$
\begin{equation*}
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=\tilde{S}\left(V, \frac{\partial}{\partial t}\right)=\alpha g\left(V, \frac{\partial}{\partial t}\right)+\beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right) . \tag{15}
\end{equation*}
$$

Now $g\left(V, \frac{\partial}{\partial t}\right)=0$ as $\frac{\partial}{\partial t} \in \chi(B)$ and $V \in \chi(F)$.
Hence, from the last equation, we get

$$
\begin{equation*}
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=\breve{S}\left(V, \frac{\partial}{\partial t}\right)=\beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right) \tag{16}
\end{equation*}
$$

Therefore, we have

$$
\begin{align*}
& \beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right)=\frac{q^{\prime}}{2}\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] \omega(V),  \tag{17}\\
& \beta \eta(V) \eta\left(\frac{\partial}{\partial t}\right)=\frac{q^{\prime}}{2}\left[\lambda_{2}-(\bar{n}-1) \lambda_{1}\right] \omega(V) . \tag{18}
\end{align*}
$$

From equations (17) and (18), we get

$$
q^{\prime}=0
$$

when $\lambda_{2}-(\bar{n}-1) \lambda_{1} \neq 0$. It follows that $q$ is a constant on $I$. Then $f$ is constant on $I$. This completes the proof.

Now, we consider the warped product $M=B \times_{f} I$ with $\operatorname{dim} B=\bar{n}-1$, $\operatorname{dim} I=1, \bar{n} \geq 3$. Under this assumption, we obtain the following theorem.

Theorem 4.2. Let $(M, g)$ be a warped product $B \times_{f} I$, where $\operatorname{dim} I=1$ and $\operatorname{dim} B=\bar{n}-1, \bar{n} \geq 3$, then
i) if $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection, $P \in \chi(B)$ is parallel on $B$ with respect to the Levi-Civita connection on $B$ and $f$ is a constant on $B$, then,

$$
\left.\alpha=\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right)\right] \omega(P) .
$$

ii) If $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection for $P \in \chi(I)$, and $\lambda_{2} \neq(\bar{n}-1) \lambda_{1}$ then $f$ is a constant on $B$.
iii) If $f$ is a constant on $B$ and $B$ is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(I)$, then $M$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection.

Proof. Assume that $(M, g)$ is a quasi-Einstein manifold with respect to a quar-ter-symmetric connection. Then we write

$$
\begin{equation*}
\breve{S}(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y) \tag{19}
\end{equation*}
$$

Decomposing the vector field $U$ uniquely into its components $U_{B}$ and $U_{I}$ on $B$ and $I$, respectively, we have

$$
\begin{equation*}
U=U_{B}+U_{I} \tag{20}
\end{equation*}
$$

Since $\operatorname{dim} I=1$, we can take $U_{I}=v \frac{\partial}{\partial t}$ which gives $U=U_{B}+v \frac{\partial}{\partial t}$, where $v$ is a function on $M$. From (19), (20) and Proposition 3.1, we have

$$
\begin{align*}
& \breve{S}^{B}(X, Y)=\alpha g_{B}(X, Y)+\beta g_{B}\left(X, U_{B}\right) g_{B}\left(Y, U_{B}\right)-\left[\frac{H^{f}(X, Y)}{f}\right. \\
& \left.+\lambda_{2} \frac{P f}{f} g(X, Y)+\lambda_{1} \lambda_{2} \omega(P) g(X, Y)+\lambda_{1} g\left(Y, \nabla_{X} P\right)-\lambda_{1}^{2} \omega(X) \omega(Y)\right] \tag{21}
\end{align*}
$$

By contraction over $X$ and $Y$, we get

$$
\begin{align*}
\breve{r}^{B}=\alpha(\bar{n}-1)+\beta g_{B}\left(U_{B}, U_{B}\right)- & {\left[\frac{\Delta_{B} f}{f}+\lambda_{2}(\bar{n}-1) \frac{P f}{f}\right.} \\
& \left.+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] \omega(P)+\lambda_{1} \sum_{i=1}^{\bar{n}-1} g\left(e_{i}, \nabla_{e_{i}} P\right)\right] \tag{22}
\end{align*}
$$

Also from (19), we have

$$
\begin{equation*}
\breve{r}^{M}=\alpha \bar{n}+\beta g_{B}\left(U_{B}, U_{B}\right) \tag{23}
\end{equation*}
$$

Now, putting the value of (23) in (22), we get

$$
\begin{align*}
\breve{r}^{B}=\breve{r}^{M}-\alpha-\frac{\Delta_{B} f}{f} & -\lambda_{2}(\bar{n}-1) \frac{P f}{f} \\
& -\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] \omega(P)-\lambda_{1} \sum_{i=1}^{\bar{n}-1} g\left(e_{i}, \nabla_{e_{i}} P\right) \tag{24}
\end{align*}
$$

On the other hand, from Proposition 3.3, we get

$$
\begin{aligned}
\breve{r}^{M}=\breve{r}^{B}+ & (\bar{n}-1)\left(\lambda_{1}+\lambda_{2}\right) \frac{P f}{f}+2 \frac{\Delta_{B} f}{f} \\
& +\left[2(\bar{n}-1) \lambda_{1} \lambda_{2}-\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)\right] \omega(P)+\left(\lambda_{1}+\lambda_{2}\right) \sum_{i=1}^{\bar{n}-1} g\left(\nabla_{e_{i}} P, e_{i}\right)
\end{aligned}
$$

Then from the above two relations, we get

$$
\begin{array}{r}
\alpha+\frac{\Delta_{B} f}{f}+\lambda_{2}(\bar{n}-1) \frac{P f}{f}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] \omega(P)+\lambda_{1} \sum_{i=1}^{\bar{n}-1} g\left(e_{i}, \nabla_{e_{i}} P\right) \\
=(\bar{n}-1)\left(\lambda_{1}+\lambda_{2}\right) \frac{P f}{f}+2 \frac{\Delta f}{f}+\left[2(\bar{n}-1) \lambda_{1} \lambda_{2}-\left(\lambda_{1}^{2}+\lambda_{2}^{2}\right)\right] \omega(P) \\
+\left(\lambda_{1}+\lambda_{2}\right) \sum_{i=1}^{\bar{n}-1} g\left(\nabla_{e_{i}} P, e_{i}\right) .
\end{array}
$$

Since $P \in \chi(B)$ is parallel and $f$ is a constant on $B$, then we get

$$
\alpha=\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P) .
$$

ii) Let $P \in \chi(I)$. By the use of Proposition 3.2, we get

$$
\begin{equation*}
\breve{S}(X, P)=\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] \omega(P) \frac{X f}{f}, \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
\breve{S}(P, X)=\left[\lambda_{2}-(\bar{n}-1) \lambda_{1}\right] \omega(P) \frac{X f}{f} \tag{26}
\end{equation*}
$$

Since $M$ is a quasi-Einstein manifold, we have

$$
\breve{S}(X, P)=\breve{S}(P, X)=\alpha g(P, X)+\beta \eta(P) \eta(X)
$$

Again, we have $g(P, X)=0$ for $X \in \chi(B)$ and $P \in \chi(I)$.
Hence, we have

$$
X f=0
$$

where $\lambda_{2} \neq(\bar{n}-1) \lambda_{1}$. This implies that $f$ is a constant on $B$.
iii) Assume that $B$ is a quasi-Einstein manifold with respect to the LeviCivita connection. Then we have

$$
\begin{equation*}
S^{B}(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y) \tag{27}
\end{equation*}
$$

for vector fields $X, Y$ tangent to $B$.
From Proposition 3.2, we get

$$
\breve{S}^{M}(X, Y)=S^{B}(X, Y)+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P) g(X, Y)+\frac{H^{f}(X, Y)}{f}
$$

for any vector field $P \in \chi(I)$. Since $f$ is a constant, $H^{f}(X, Y)=0$ for all $X, Y \in \chi(B)$.

The above equation reduces to

$$
\begin{equation*}
\breve{S}^{M}(X, Y)=S^{B}(X, Y)+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P) g(X, Y) \tag{28}
\end{equation*}
$$

Using the value of (27) in (28), we get

$$
\begin{equation*}
\breve{S}^{M}(X, Y)=\left\{\alpha+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P)\right\} g(X, Y)+\beta \eta(X) \eta(Y) \tag{29}
\end{equation*}
$$

which shows that $M$ is a quasi-Einstein manifold with respect to a quartersymmetric connection.

Next, we study $M=I \times_{f} F$ with metric $-d t^{2}+f(t)^{2} g_{F}$, where $I$ is an open interval in $\mathbb{R}$, and we prove the following theorem.

Theorem 4.3. Let $(M, g)$ be a warped product $I \times_{f} F$ with the metric tensor $-d t^{2}+f(t)^{2} g_{F}, P=\frac{\partial}{\partial t}, \operatorname{dimF}=l$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection $\breve{\nabla}$ with constant associated scalars $\alpha$ and $\beta$ if and only if the following conditions are satisfied:
i) $\left(F, g_{F}\right)$ is a quasi-Einstein manifold with scalar $\alpha_{F}, \beta_{F}$;
ii) $-l\left(\lambda_{2} \frac{f^{\prime}}{f}-\frac{f^{\prime \prime}}{f}+\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right)=-\alpha+v^{2} \beta$;
iii) $\alpha_{F}-f f^{\prime \prime}-(l-1) f^{\prime 2}+\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}-\alpha\right) f^{2}+\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}=0$ and $\beta=\beta_{F}$.

Proof. By Proposition 3.1, we have

$$
\begin{gathered}
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=-l\left(\lambda_{2} \frac{f^{\prime}}{f}-\frac{f^{\prime \prime}}{f}+\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right) \\
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=\breve{S}\left(V, \frac{\partial}{\partial t}\right)=0 \\
\breve{S}(V, W)=S^{F}(V, W)+g_{F}(V, W)\left\{-f f^{\prime \prime}-(l-1) f^{\prime^{2}}\right. \\
\\
\left.+\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}\right) f^{2}+\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}\right\}
\end{gathered}
$$

Since $M$ is a quasi-Einstein manifold, we have

$$
\breve{S}(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y) .
$$

Now,

$$
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)+\beta \eta\left(\frac{\partial}{\partial t}\right) \eta\left(\frac{\partial}{\partial t}\right) .
$$

We can decompose the vector field $U$ uniquely into its components $U_{I}$ and $U_{F}$ on $I$ and $F$, respectively. Then we have $U=U_{I}+U_{F}$. Since $\operatorname{dim} I=1$, we can take $U_{I}=v \frac{\partial}{\partial t}$ which gives $U=v \frac{\partial}{\partial t}+U_{F}$, where $v$ is a function on $M$. Thus, we can write

$$
\begin{equation*}
\eta\left(\frac{\partial}{\partial t}\right)=g\left(U, \frac{\partial}{\partial t}\right)=v \tag{30}
\end{equation*}
$$

Therefore, we get

$$
-l\left(\lambda_{2} \frac{f^{\prime}}{f}-\frac{f^{\prime \prime}}{f}+\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right)=-\alpha+v^{2} \beta
$$

Again, $\breve{S}(V, W)=\alpha g(V, W)+\beta \eta(V) \eta(W)$.
Also, we have

$$
\begin{aligned}
\breve{S}(V, W)=S^{F}(V, W)+g_{F}(V, W) & \left\{-f f^{\prime \prime}-(l-1) f^{\prime 2}\right. \\
& \left.+\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}\right) f^{2}+\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}\right\}
\end{aligned}
$$

From the above two equations, we get

$$
\begin{aligned}
& S^{F}(V, W)=\left\{f f^{\prime \prime}+(l-1) f^{\prime^{2}}-\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}-\alpha\right) f^{2}\right. \\
&\left.-\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}\right\} g_{F}(V, W)+\beta \eta(V) \eta(W)
\end{aligned}
$$

Hence, $\left(F, g_{F}\right)$ is a quasi-Einstein manifold.
Also, we have

$$
\begin{aligned}
\breve{S}(V, W)=S^{F}(V, W)+g_{F}(V, W) & \left\{-f f^{\prime \prime}-(l-1) f^{\prime 2}\right. \\
+ & \left.\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}\right) f^{2}+\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}\right\}
\end{aligned}
$$

After some calculations, we show that

$$
\alpha_{F}-f f^{\prime \prime}-(l-1) f^{\prime^{2}}+\left(\lambda_{2}^{2}-l \lambda_{1} \lambda_{2}-\alpha\right) f^{2}+\left[l \lambda_{1}+(l-1) \lambda_{2}\right] f f^{\prime}=0
$$

and $\beta=\beta_{F}$. Thus, the proof is completed.
Putting $\operatorname{dim} F=1$ in Theorem 4.3, we get the following corollary.
Corollary 4.4. Let $(M, g)$ be a warped product $I \times_{f} F$ with the metric tensor $-d t^{2}+f(t)^{2} g_{F}, P=\frac{\partial}{\partial t}, \operatorname{dimF}=1$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if

$$
f^{\prime \prime}-\lambda_{2} f^{\prime}+\left[\left(\alpha-v^{2} \beta\right)-\left(\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right)\right] f=0
$$

By using Corollary 4.4 and elementary methods for ordinary differential equations, we obtain the following theorem.

Theorem 4.5. Let $(M, g)$ be a warped product $I \times_{f} F$ with the metric tensor $-d t^{2}+f(t)^{2} g_{F}, P=\frac{\partial}{\partial t}, \operatorname{dimF}=1$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection if and only if
i) $\alpha-v^{2} \beta<\left(\lambda_{1}-\frac{\lambda_{2}}{2}\right)^{2}$,
$f(t)=c_{1} e^{\left(\frac{\lambda_{2}+\sqrt{\left(2 \lambda_{1}-\lambda_{2}\right)^{2}-4\left(\alpha-v^{2} \beta\right)}}{2}\right) t}+c_{2} e^{\left(\frac{\lambda_{2}-\sqrt{\left(2 \lambda_{1}-\lambda_{2}\right)^{2}-4\left(\alpha-v^{2} \beta\right)}}{2}\right) t}$,
ii) $\alpha-v^{2} \beta=\left(\lambda_{1}-\frac{\lambda_{2}}{2}\right)^{2}, f(t)=c_{1} e^{\left(\frac{\lambda_{2}}{2}\right) t}+c_{2} t e^{\left(\frac{\lambda_{2}}{2}\right) t}$,
iii) $\alpha-v^{2} \beta>\left(\lambda_{1}-\frac{\lambda_{2}}{2}\right)^{2}, f(t)=c_{1} e^{\left(\frac{\lambda_{2}}{2}\right) t} c_{1} \cos \left(\left(\frac{\sqrt{4\left(\alpha-v^{2} \beta\right)-\left(2 \lambda_{1}-\lambda_{2}\right)^{2}}}{2}\right) t\right)+$ $c_{2} e^{\left(\frac{\lambda_{2}}{2}\right) t} \sin \left(\left(\frac{\sqrt{4\left(\alpha-v^{2} \beta\right)-\left(2 \lambda_{1}-\lambda_{2}\right)^{2}}}{2}\right) t\right)$.

Corollary 4.6. Let $(M, g)$ be a warped product $I \times_{f} F$ with the metric tensor $-d t^{2}+f(t)^{2} g_{F}, P=\frac{\partial}{\partial t}$, $\operatorname{dimF}=1$, and $\lambda_{2}=2 \lambda_{1}$. Then $(M, g)$ is a quasiEinstein manifold with respect to a quarter-symmetric connection if and only if
i) $\alpha-v^{2} \beta<0, f(t)=c_{1} e^{\left(\lambda_{1}+\sqrt{-\left(\alpha-v^{2} \beta\right)}\right) t}+c_{2} e^{\left(\lambda_{1}-\sqrt{-\left(\alpha-v^{2} \beta\right)}\right) t}$,
ii) $\alpha-v^{2} \beta=0, f(t)=c_{1} e^{\lambda_{1} t}+c_{2} t e^{\lambda_{1} t}$,
iii) $\alpha-v^{2} \beta>0, f(t)=c_{1} e^{\lambda_{1} t} \cos \left(\left(\sqrt{\alpha-v^{2} \beta}\right) t\right)+c_{2} e^{\lambda_{1} t} \sin \left(\left(\sqrt{\alpha-v^{2} \beta}\right) t\right)$.

Next, the following theorem shows when the base of a quasi-Einstein warped product manifold is isometric to a sphere of a particular radius.

Theorem 4.7. Let $(M, g)$ be a warped product $B \times{ }_{f} I$ of a complete connected ( $\bar{n}-1$ )-dimensional Riemannian manifold $B$ where $\bar{n} \geq 3$ and one-dimensional Riemannian manifold I. If $(M, g)$ is a quasi-Einstein manifold with constant associated scalars $\alpha$ and $\beta, U \in \chi(M)$ with respect to a quarter-symmetric connection, $P \in \chi(B)$ and the Hessian of $f$ is proportional to the metric tensor $g_{B}$, then $\left(B, g_{B}\right)$ is a $(\bar{n}-1)$-dimensional sphere of radius $\rho=\frac{\bar{n}-1}{\sqrt{\bar{r}^{B}+\alpha}}$.

Proof. Let $M$ be a connected warped product manifold. Then from Proposition 3.1, we have

$$
\begin{align*}
& \breve{S}^{M}(X, Y)=\breve{S}^{B}(X, Y)+\frac{H_{B}^{f}(X, Y)}{f}+\lambda_{2} \frac{P f}{f} g(X, Y) \\
&+\lambda_{1} \lambda_{2} \omega(P) g(X, Y)+\lambda_{1} g\left(Y, \nabla_{X} P\right)-\lambda_{1}^{2} \omega(X) \omega(Y) \tag{31}
\end{align*}
$$

for any vector field $X, Y$ on $B$. Since $M$ is a quasi-Einstein manifold with respect to a quarter-symmetric metric connection, we have

$$
\begin{equation*}
\breve{S}^{M}(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y) . \tag{32}
\end{equation*}
$$

Decomposing the vector field $U$ uniquely into its components $U_{B}$ and $U_{I}$ on $B$ and $I$, respectively, we have

$$
\begin{equation*}
U=U_{B}+U_{I} \tag{33}
\end{equation*}
$$

Putting the values of (32), (33) in (31), we get

$$
\begin{align*}
& \breve{S}^{B}(X, Y)=\alpha g_{B}(X, Y)+\beta g_{B}\left(X, U_{B}\right) g_{B}\left(Y, U_{B}\right)-\left[\frac{H_{B}^{f}(X, Y)}{f}\right. \\
& \left.+\lambda_{2} \frac{P f}{f} g(X, Y)+\lambda_{1} \lambda_{2} \omega(P) g(X, Y)+\lambda_{1} g\left(Y, \nabla_{X} P\right)-\lambda_{1}^{2} \omega(X) \omega(Y)\right] \tag{34}
\end{align*}
$$

By contraction over $X$ and $Y$, we get

$$
\begin{align*}
\breve{r}^{B}=\breve{r}^{M}-\alpha-\frac{\Delta_{B} f}{f} & -(\bar{n}-1) \lambda_{2} \frac{P f}{f} \\
& -\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right] \pi(P)-\lambda_{1} \sum_{i=1}^{\bar{n}-1} g\left(e_{i}, \nabla_{e_{i}} P\right) . \tag{35}
\end{align*}
$$

Again from Proposition 3.1, we obtain

$$
\begin{equation*}
\frac{\breve{r}^{M}}{\bar{n}}=\lambda_{2} \sum_{i=1}^{\bar{n}-1} g\left(e_{i}, \nabla_{e_{i}} P\right)+(\bar{n}-1) \lambda_{1} \frac{P f}{f}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P)+\frac{\Delta_{B} f}{f} \tag{36}
\end{equation*}
$$

From the last two equations, it follows that

$$
\begin{align*}
\left(\breve{r}^{B}+\alpha\right) f=\left(\bar{n} \lambda_{2}-\right. & \left.\lambda_{1}\right) \sum_{i=1}^{\bar{n}-1} f g\left(e_{i}, \nabla_{e_{i}} P\right)+(\bar{n}-1)\left[\bar{n} \lambda_{1}-\lambda_{2}\right] P f \\
& +\left[(\bar{n}-1)^{2} \lambda_{1} \lambda_{2}+\lambda_{1}^{2}-\bar{n} \lambda_{2}^{2}\right] f \omega(P)+(\bar{n}-1) \Delta_{B} f \tag{37}
\end{align*}
$$

Since the Hessian of $f$ is proportional to the metric tensor $g_{B}$, then we have

$$
\begin{aligned}
H^{f}(X, Y)= & \frac{1}{(\bar{n}-1)^{2}}\left[\left(\lambda_{1}-\bar{n} \lambda_{2}\right) \sum_{i=1}^{\bar{n}-1} f g\left(e_{i}, \nabla_{e_{i}} P\right)+(\bar{n}-1)\left[\lambda_{2}-\bar{n} \lambda_{1}\right] P f\right. \\
& \left.+\left(\bar{n} \lambda_{2}^{2}-(\bar{n}-1)^{2} \lambda_{1} \lambda_{2}-\lambda_{1}^{2}\right) f \omega(P)+(1-\bar{n}) \Delta_{B} f\right] g_{B}(X, Y)
\end{aligned}
$$

Hence, from the above equation, we obtain

$$
\begin{equation*}
H^{f}(X, Y)+\frac{\breve{r}^{B}+\alpha}{(\bar{n}-1)^{2}} f g_{B}(X, Y)=0 \tag{38}
\end{equation*}
$$

So $B$ is isometric to the $(\bar{n}-1)$-dimensional sphere of radius $\frac{\bar{n}-1}{\sqrt{\bar{r}^{B}+\alpha}}[6]$. Thus, the theorem is proved.

## 5. Multiply Twisted Product Manifold with Quarter-Symmetric Connection

Now, we have the following propositions from Propositions 4.5 and 4.7 of [8], for later use.

Proposition 5.1. Let $M=B \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ be a multiply twisted product manifold with $\operatorname{dim} B=n, \operatorname{dim} F_{i}=l_{i}, \operatorname{dim} M=\bar{n}$. If $X, Y \in \chi(B)$, $V \in \chi\left(F_{i}\right), W \in \chi\left(F_{j}\right)$ and $P \in \chi(B)$, then
(i) $\breve{S}(X, Y)=\breve{S}^{B}(X, Y)+\sum_{i=1}^{m} l_{i}\left[\lambda_{1} \lambda_{2} \omega(P) g(X, Y)+\frac{H_{B}^{b_{i}}(X, Y)}{b_{i}}+\right.$ $\left.\lambda_{2} \frac{P b_{i}}{b_{i}} g(X, Y)+\lambda_{1} g\left(Y, \nabla_{X} P\right)-\lambda_{1}^{2} \omega(X) \omega(Y)\right]$,
(ii) $\breve{S}(X, V)=\breve{S}(V, X)=\left(l_{i}-1\right)\left[V X\left(\ln b_{i}\right)\right]$,
(iii) $\breve{S}(V, W)=0$ if $i \neq j$,
(iv) $\breve{S}(V, W)=S^{F_{i}}(V, W)+g(V, W)\left\{\left(l_{i}-1\right) \frac{\left|\operatorname{grad}_{B} b_{i}\right|_{B}^{2}}{b_{i}^{2}}+\frac{\Delta_{B} b_{i}}{b_{i}}+\right.$ $\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P)+\lambda_{2} \operatorname{div}_{F} P+\left[(\bar{n}-1) \lambda_{1}+\left(l_{i}-1\right) \lambda_{2}\right] \frac{P b_{i}}{b_{i}}+$ $\left.\sum_{s \neq i} l_{s} \frac{g_{B}\left(\operatorname{grad}_{B} b_{i}, \operatorname{grad}_{B} b_{s}\right)}{b_{i} b_{s}}+\lambda_{2} \sum_{s \neq i} l_{s} \frac{P b_{s}}{b_{s}}\right\}$ if $i=j$, where $\operatorname{div}_{B} P=$ $\sum_{k=1}^{n} \varepsilon_{k}\left\langle\nabla_{E_{k}} P, E_{k}\right\rangle$ and $E_{k}, 1 \leq k \leq n$, is an orthonormal basis of $B$ with $\varepsilon_{k}=g\left(E_{k}, E_{k}\right)$.
Proposition 5.2. Let $M=B \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ be a multiply twisted product, $\operatorname{dim} B=n$, $\operatorname{dim} F_{i}=l_{i}, \operatorname{dim} M=\bar{n}$. If $X, Y \in \chi(B), V \in \chi\left(F_{i}\right)$, $W \in \chi\left(F_{j}\right)$ and $P \in \chi\left(F_{r}\right)$ for a fixed $r$, then
(i) $\breve{S}(X, Y)=S^{B}(X, Y)+\sum_{i=1}^{m} l_{i} \frac{H_{B}^{b_{i}}(X, Y)}{b_{i}}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P) g(X, Y)+$
$\lambda_{2} g(X, Y) d i v_{F} P$ $\lambda_{2} g(X, Y) d i v_{F_{r}} P$,
(ii) $\breve{S}(X, V)=\left(l_{i}-1\right)\left[V X\left(\ln b_{i}\right)\right]+\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] \omega(V) \frac{X b_{r}}{b_{r}}$,
(iii) $\breve{S}(V, X)=\left(l_{i}-1\right)\left[V X\left(\ln b_{i}\right)\right]+\left[\lambda_{2}-(\bar{n}-1) \lambda_{1}\right] \omega(V) \frac{X b_{r}}{b_{r}}$,
(iv) $\breve{S}(V, W)=0$ if $i \neq j$,
(v) $\breve{S}(V, W)=S^{F_{i}}(V, W)+g(V, W)\left\{\left(l_{i}-1\right) \frac{\left|g r a d_{B} b_{i}\right|_{B}^{2}}{b_{i}^{2}}+\frac{\Delta_{B} b_{i}}{b_{i}}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\right.\right.$ $\left.\left.\lambda_{2}^{2}\right] \pi(P)+\sum_{s \neq i} l_{s} \frac{g_{B}\left(\operatorname{grad}_{B} b_{i}, \operatorname{grad}_{B} b_{s}\right)}{b_{i} b_{s}}\right\}+\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] g\left(W, \nabla_{V} P\right)+$ $\left[\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1}^{2}\right] \omega(V) \omega(W)+\lambda_{2} g(V, W) d i v_{F_{r}} P$ if $i=j$.

Let $M=B \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ be a multiply warped product with the metric tensor $-d t^{2} \oplus b_{1}^{2} g_{F_{1}} \oplus \ldots \oplus b_{m}^{2} g_{F_{m}}$, and let $I$ be an open interval in $\mathbb{R}$ and $b_{i} \in C^{\infty}(I)$.

Now, we prove the following theorem for multiply generalized RobertsonWalker spacetime.

ThEOREM 5.3. Let $M=I \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ be a multiply warped product with the metric tensor $-d t^{2} \oplus b_{1}^{2} g_{F_{1}} \oplus \ldots \oplus b_{m}^{2} g_{F_{m}}$ and $P=\frac{\partial}{\partial t}$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection $\breve{\nabla}$ with constant associated scalars $\alpha$ and $\beta$, if and only if the following conditions are satisfied:
i) $\left(F_{i}, g_{F_{i}}\right)$ are quasi-Einstein manifolds with scalars $\alpha_{F_{i}}, \beta_{F_{i}}, i \in\{1,2, \ldots m\}$;
ii) $\sum_{i=1}^{m} l_{i}\left(\lambda_{2} \frac{b_{i}^{\prime}}{b_{i}}-\frac{b_{i}^{\prime \prime}}{b_{i}}+\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right)=\alpha-v^{2} \beta$;
iii) $\alpha_{F_{i}}-b_{i} b_{i}^{\prime \prime}-\left(l_{i}-1\right) b_{i}^{\prime^{2}}+\left(\lambda_{2} b_{i}^{2}-b_{i} b_{i}^{\prime}\right) \sum_{s \neq i} l_{s}\left(\frac{b_{s}^{\prime}}{b_{s}}\right)+\left(\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1} \lambda_{2}-\right.$ $\alpha) b_{i}^{2}+\left((\bar{n}-1) \lambda_{1}+\left(l_{i}-1\right) \lambda_{2}\right) b_{i} b_{i}^{\prime}=0$ and $\beta=\beta_{F_{i}}$.

Proof. By Proposition 5.1, we have

$$
\begin{gather*}
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\sum_{i=1}^{m} l_{i}\left(-\lambda_{2} \frac{b_{i}^{\prime}}{b_{i}}+\frac{b_{i}^{\prime \prime}}{b_{i}}-\lambda_{1}^{2}+\lambda_{1} \lambda_{2}\right),  \tag{39}\\
\breve{S}\left(\frac{\partial}{\partial t}, V\right)=\breve{S}\left(V, \frac{\partial}{\partial t}\right)=\left(l_{i}-1\right) V\left(\frac{b_{i}^{\prime}}{b_{i}}\right),  \tag{40}\\
\breve{S}(V, W)=0, \text { if } i \neq j \tag{41}
\end{gather*}
$$

$$
\begin{align*}
& \breve{S}(V, W)=S^{F_{i}}(V, W)+g_{F_{i}}(V, W)\left\{-\left(l_{i}-1\right) b_{i}^{2^{2}}-b_{i}^{\prime \prime} b_{i}+\left[(\bar{n}-1) \lambda_{1}\right.\right. \\
& \left.\left.\quad+\left(l_{i}-1\right) \lambda_{2}\right] b_{i}^{\prime} b_{i}+\left(\lambda_{2} b_{i}^{2}-b_{i}^{\prime} b_{i}\right) \sum_{s \neq i} l_{s} \frac{b_{s}^{\prime}}{b_{s}}+\left(\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1} \lambda_{2}\right) b_{i}^{2}\right\} . \tag{42}
\end{align*}
$$

Since $M$ is a quasi-Einstein manifold, we have

$$
\breve{S}(X, Y)=\alpha g(X, Y)+\beta \eta(X) \eta(Y) .
$$

Now,

$$
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\alpha g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)+\beta \eta\left(\frac{\partial}{\partial t}\right) \eta\left(\frac{\partial}{\partial t}\right) .
$$

Decomposing the vector field $U$ uniquely into its components $U_{I}$ and $U_{F}$ on $I$ and $F$, respectively, we have $U=U_{I}+U_{F}$. Since $\operatorname{dim} I=1$, we can take $U_{I}=v \frac{\partial}{\partial t}$ which gives $U=v \frac{\partial}{\partial t}+U_{F}$, where $v$ is a function on $M$. Then we can write

$$
\begin{equation*}
\eta\left(\frac{\partial}{\partial t}\right)=g\left(U, \frac{\partial}{\partial t}\right)=v \tag{43}
\end{equation*}
$$

Hence, we get

$$
\sum_{i=1}^{m} l_{i}\left(\lambda_{2} \frac{b_{i}^{\prime}}{b_{i}}-\frac{b_{i}^{\prime \prime}}{b_{i}}+\lambda_{1}^{2}-\lambda_{1} \lambda_{2}\right)=\alpha-v^{2} \beta
$$

Again, $\breve{S}(V, W)=\alpha g(V, W)+\beta \eta(V) \eta(W)$.
From Proposition 5.1 and equation (42), we obtain that $\left(F_{i}, g_{F_{i}}\right)$ are quasiEinstein manifolds.

After a brief calculation, we can easily prove that

$$
\begin{aligned}
\alpha_{F_{i}}-b_{i} b_{i}^{\prime \prime}- & \left(l_{i}-1\right) b_{i}^{\prime 2}+\left(\lambda_{2} b_{i}^{2}-b_{i} b_{i}^{\prime}\right) \sum_{s \neq i} l_{s}\left(\frac{b_{s}^{\prime}}{b_{s}}\right) \\
& +\left[\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1} \lambda_{2}-\alpha\right] b_{i}^{2}+\left[(\bar{n}-1) \lambda_{1}+\left(l_{i}-1\right) \lambda_{2}\right] b_{i} b_{i}^{\prime}=0
\end{aligned}
$$

and $\beta=\beta_{F_{i}}$.
Thus, the proof of the theorem is completed.
Next, the following theorem establishes the necessary and sufficient conditions on a multiply warped product to be a quasi-Einstein manifold with a quarter-symmetric connection whenever $P \in \chi\left(F_{r}\right)$.

THEOREM 5.4. Let $M=I \times_{b_{1}} F_{1} \times_{b_{2}} F_{2} \ldots \times_{b_{m}} F_{m}$ be a multiply warped product with the metric tensor $-d t^{2} \oplus b_{1}^{2} g_{F_{1}} \oplus \ldots \oplus b_{m}^{2} g_{F_{m}}$ with $P \in \chi\left(F_{r}\right)$ and $g_{F_{r}}(P, P)=1$ and $\bar{n} \geq 2$. Then $(M, g)$ is a quasi-Einstein manifold with respect to a quarter-symmetric connection $\breve{\nabla}$ with constant associated scalars $\alpha$ and $\beta$, if and only if the following conditions are satisfied:
i) $\left(F_{i}, g_{F_{i}}\right)(i \neq r)$ are quasi-Einstein manifolds with scalars $\alpha_{i}, \beta_{i}, i \in$ $\{1,2, \ldots m\} ;$
ii) $b_{r}$ is constant and $\sum_{i=1}^{m} l_{i} \frac{b_{i}^{\prime \prime}}{b_{i}}=\mu_{0}, \operatorname{div}_{F_{r}} P=\mu_{1}, \mu_{0}-\lambda_{2} \mu_{1}+\alpha-v^{2} \beta=$ $\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{r}^{2}$, where $\mu_{0}, \mu_{1}$ are constants;
iii) $S^{F_{r}}(V, W)+\bar{\alpha} g_{F_{r}}(V, W)+\beta \eta(V) \eta(W)=\left[(\bar{n}-1) \lambda_{1}^{2}-\lambda_{2}^{2}\right] \omega(V) \omega(W)-$ $\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] g\left(W, \nabla_{V} P\right)$, for $V, W \in \chi\left(F_{r}\right)$, where $\bar{\alpha}=b_{r}^{2}\{[(\bar{n}-$ 1) $\left.\left.\lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{r}^{2}+\lambda_{2} \mu_{1}-\alpha\right\}$.
iv) $\alpha_{F_{i}}-b_{i} b_{i}^{\prime \prime}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{i}^{2} b_{r}^{2}-b_{i} b_{i}^{\prime} \sum_{s \neq i} l_{s} \frac{b_{s}^{\prime}}{b_{s}}-\left(l_{i}-1\right)\left(b_{i}^{\prime}\right)^{2}=(\alpha-$ $\left.\lambda_{2} \mu_{1}\right) b_{i}^{2}$ and $\beta=\beta_{F_{i}}$.
Proof. By Proposition 5.2 (ii) and $g_{F_{r}}(P, P)=1$, it follows that $b_{r}$ is a constant. By Proposition 5.2 (i), we obtain

$$
\breve{S}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)=\sum_{i=1}^{m} l_{i} \frac{b_{i}^{\prime \prime}}{b_{i}}+\left[\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1} \lambda_{2}\right] b_{r}^{2}-\lambda_{2} \operatorname{div}_{F_{r}} P=-\alpha+v^{2} \beta
$$

By separation of variables, we have

$$
\sum_{i=1}^{m} l_{i} \frac{b_{i}^{\prime \prime}}{b_{i}}=\mu_{0}, \operatorname{div}_{F_{r}} P=\mu_{1}, \mu_{0}-\lambda_{2} \mu_{1}+\alpha-v^{2} \beta=\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{r}^{2}
$$

Then we get $i i)$. By proposition $5.2(v)$, we have

$$
\begin{aligned}
& \breve{S}(V, W)=S^{F_{i}}(V, W)+b_{i}^{2} g_{F_{i}}(V, W)\left\{\left(l_{i}-1\right) \frac{-\left(b_{i}^{\prime}\right)^{2}}{b_{i}^{2}}+\frac{-b_{i}^{\prime \prime}}{b_{i}}\right. \\
& \left.+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P)+\sum_{s \neq i} l_{s} \frac{-b_{i}^{\prime} b_{s}^{\prime}}{b_{i} b_{s}}\right\}+\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] g\left(W, \nabla_{V} P\right) \\
& +\left[\lambda_{2}^{2}+(1-\bar{n}) \lambda_{1}^{2}\right] \omega(V) \omega(W)+\lambda_{2} g(V, W) \operatorname{div}_{F_{r}} P, \quad \text { if } i=j
\end{aligned}
$$

When $i \neq r$, then $\nabla_{V} P=\omega(V)=0$, so,

$$
\begin{aligned}
& \breve{S}(V, W)=S^{F_{i}}(V, W)+b_{i}^{2} g_{F_{i}}(V, W)\left\{\left(l_{i}-1\right) \frac{-\left(b_{i}^{\prime}\right)^{2}}{b_{i}^{2}}+\frac{-b_{i}^{\prime \prime}}{b_{i}}\right. \\
& \left.+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] \omega(P)+\sum_{s \neq i} l_{s} \frac{-b_{i}^{\prime} b_{s}^{\prime}}{b_{i} b_{s}}\right\}+\lambda_{2} \mu_{1} b_{i}^{2} g_{F_{i}}(V, W) \\
& \quad=\alpha b_{i}^{2} g_{F_{i}}(V, W)+\beta \eta(V) \eta(W)
\end{aligned}
$$

By separation of variables, it follows that $\left(F_{i}, g_{F_{i}}\right)(i \neq r)$ are quasi-Einstein manifolds with scalars $\alpha_{i}, \beta_{i}, i \in\{1,2, \ldots m\}$, and

$$
\begin{aligned}
\alpha_{F_{i}}-b_{i} b_{i}^{\prime \prime}+\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{i}^{2} b_{r}^{2}-b_{i} b_{i}^{\prime} \sum_{s \neq i} l_{s} \frac{b_{s}^{\prime}}{b_{s}}-\left(l_{i}-1\right)\left(b_{i}^{\prime}\right)^{2} & \\
& =\left(\alpha-\lambda_{2} \mu_{1}\right) b_{i}^{2}
\end{aligned}
$$

and $\beta=\beta_{F_{i}}$. Then we have $i$ ) and $i v$ ).
When $i=r$ and $b_{r}$ is a constant, then we get

$$
\begin{aligned}
& S^{F_{r}}(V, W)+\bar{\alpha} g_{F_{r}}(V, W)+\beta \eta(V) \eta(W) \\
& \quad=\left[(\bar{n}-1) \lambda_{1}^{2}-\lambda_{2}^{2}\right] \omega(V) \omega(W)-\left[(\bar{n}-1) \lambda_{1}-\lambda_{2}\right] g\left(W, \nabla_{V} P\right), \\
& \\
& \text { for } V, W \in \chi\left(F_{r}\right),
\end{aligned}
$$

where $\bar{\alpha}=b_{r}^{2}\left\{\left[(\bar{n}-1) \lambda_{1} \lambda_{2}-\lambda_{2}^{2}\right] b_{r}^{2}+\lambda_{2} \mu_{1}-\alpha\right\}$, and thus we obtain $\left.i i i\right)$.

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[^0]:    ${ }^{1}$ surely, we mean the mathematically rigorous approaches

[^1]:    ${ }^{2}$ another jumps also do occur but are beyond our interest

[^2]:    ${ }^{3}$ this property can be derived from Theorem 3.3 of [19].

[^3]:    ${ }^{4}$ Actually, a long list of the characterization conditions is not something unusual: see, e.g., the conditions on a spectral triple corresponding to a Riemannian manifold in [12].

[^4]:    ${ }^{5}$ Such an over-determinacy does not occur for $n=1$, where $r=r(t)$. In this case, a positive definiteness of a relevant $C^{T}$ exhausts the characterization [11].

[^5]:    ${ }^{1}$ This assumption can be removed when $\Omega$ is simply connected, by using the analyticity of $u$ (see [4])

[^6]:    ${ }^{2} \Omega(\Omega)$ has also been considered in [72] under the name of minimal unfolded region.

[^7]:    ${ }^{1}$ See http://www-sop.inria.fr/apics/FindSources3D/.

