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## History

The journal Rendiconti dell'Istituto di Matematica dell'Università di Trieste was founded in 1969 by Arno Predonzan, with the aim of publishing original research articles in all fields of mathematics.

Rendiconti dell'Istituto di Matematica dell'Università di Trieste has been the first Italian mathematical journal to be published also on-line. The access to the electronic version of the journal is free. All published articles are available on-line.
In 2008 the Dipartimento di Matematica e Informatica, the owner of the journal, decided to renew it. The name of the journal however remained unchanged, but the subtitle An International Journal of Mathematics was added. The journal can be obtained by subscription, or by reciprocity with other similar journals. Currently more than 100 exchange agreements with mathematics departments and institutes around the world have been entered in.

The articles published by Rendiconti dell'Istituto di Matematica dell'Università di Trieste are reviewed/indexed by MathSciNet, Zentralblatt Math, Scopus, OpenStarTs.

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## Foreword

This issue of Rendiconti dell'Istituto di Matematica dell'Università di Trieste is dedicated to our friend and colleague Jean Mawhin, on the occasion of his 75 th birthday.

Jean Mawhin is a world-wide recognized master in the development and the application of functional analytic methods, mainly topological and variational, in the study of boundary value problems for various classes of nonlinear differential equations, while he has also given relevant contributions to the history of Mathematics. He mentored and inspired many young mathematicians, who now have well established positions in and outside Europe.

Jean Mawhin was born in Heusy, Belgium, where he still lives with his wife Margaret, who gave him three children (and he is now the grandfather of six grandsons). He pursued his studies in Liège, and obtained the title of Docteur en Sciences Mathématiques, avec la plus grande distinction in 1969. Strangely enough, his PhD supervisor was a professor in Astrophysics, Paul Ledoux, who by the way gave important contributions to the theory of stellar oscillations.

Soon after, Jean became a full professor at the Université Catholique de Louvain-la-Neuve, where he was able to create a strong group of mathematicians working in nonlinear analysis.

He retired from his University at the age of 65 , becoming professeur émérite, but since then the rhythm of his scientific activity has been maintained, if not even increased. He is still a member of the Académie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique, since 1992, having been its President in 2002. The same year he received the Bolzano medal, the highest recognition of achievements in the mathematical sciences awarded by the Czech Academy of Sciences. Then, in 2012, he was the first winner of the Schauder medal, which is now awarded every two years for scientific achievements and contributions to nonlinear analysis and its applications.

Those who have the chance to know Jean Mawhin all appreciate his warm and sincere personality, always intruded with some irony and a fine sense of humor.
Now that Jean Mawhin is completing his 75th turn around the Sun, we are
looking forward to see, after his exploits during these first three, what his beautiful human and scientific activity will reserve us in the next quarter of a century.

To conclude, let us also mention that Jean Mawhin has been in the Editorial Board of our journal since 2005. It has been an honour for us, and we are sure that his collaboration will continue helping us improving the reputation of the journal.

Happy Birthday Jean!
Bon Anniversaire!

On the 22th of July 2017, Professor Russell Johnson suddenly passed away. We will always remember Russell with friendship and admiration. And we are greatly honoured for his contribution to this volume.

# Hardy inequality, compact embeddings and properties of certain eigenvalue problems 

Pavel Drábek and Alois Kufner<br>Dedicated to Jean Mawhin on the occasion of his 75th birthday


#### Abstract

We point out the connection between the Hardy inequality, compact embedding of weighted function spaces and the properties of the spectra of certain eigenvalue problems. Necessary and sufficient conditions in terms of the Muckenhoupt function are formulated.


Keywords: degenerate and singular eigenvalue problem, Hardy's inequality, Muckenhoupt function, BD-property, Sturm-Liouville problem, compact embeddings, weighted function spaces.
MS Classification 2010: 34L30, 34B24, 34B40, 35J92.

## 1. Introduction

In 1958 the authors of [8] studied the spectrum of the initial value problem

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=\lambda \sigma(x) u(x), \quad x \in(0, \infty)  \tag{1}\\
u(0)=0, u^{\prime}(0)=1
\end{array}\right.
$$

Their result can be stressed as follows:
(i) The spectrum of (1) is bounded from below provided there exist a constant $c>0$ such that for all $x \in(0, \infty)$,

$$
x \int_{x}^{\infty} \sigma(\tau) \mathrm{d} \tau \leq c .
$$

Moreover, the spectrum is bounded from below by $\frac{1}{4 c}$.
(ii) The spectrum of (1) is discrete if and only if

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} x \int_{x}^{\infty} \sigma(\tau) \mathrm{d} \tau=0 \tag{2}
\end{equation*}
$$

On the other hand, the equation of order $2 k, k \in \mathbb{N}$,

$$
\begin{equation*}
(-1)^{k}\left(\rho(x) u^{(k)}(x)\right)^{(k)}=\lambda u(x), x \in(0,+\infty) \tag{3}
\end{equation*}
$$

was investigated in $[7,9,11]$. More precisely, it was shown that the spectrum of the minimal selfadjoint extension of the formal differential operator on the left-hand side in (3) is bounded from below and discrete if and only if

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} x^{2 k-1} \int_{x}^{\infty} \frac{1}{\rho(\tau)} \mathrm{d} \tau=0 \tag{4}
\end{equation*}
$$

(see [7] for sufficiency of (4) and [9, 11] for necessity of (4)).
Both expressions in (2) and (4) are closely related to the Muckenhoupt function which plays a key role in the theory of the Hardy inequality (see e.g. [13]). In particular, certain properties of the Muckenhoupt function provide necessary and sufficient conditions for the Hardy inequality as well as for the compact embedding of certain weighted Sobolev and Lebesgue spaces to hold. Making use of these properties of the Muckenhoupt function combined with some results from the oscillation theory of ODEs we formulate necessary and sufficient conditions for the boundedness from below and the discreteness of the spectrum of equations which generalize both (1) and (3). We also show that these conditions are equivalent with the compactness of the embedding of a weighted Sobolev space into a weighted Lebesgue space with weights which appear as nonconstant coefficients in the equation.

In Section 2 we consider quasilinear problems on both bounded and/or unbounded interval. Section 3 deals with the higher order quasilinear equations. We give some examples in Section 4 with the emphasis on the consequences of our general estimates to the decay of radial solutions of certain quasilinear PDEs.

## 2. Second order equations

Let us consider the Sturm-Liouville boundary value problem

$$
\left\{\begin{array}{l}
-\left(\rho(x) u^{\prime}\right)^{\prime}+q(x) u=\lambda \sigma(x) u, a<x<b  \tag{5}\\
\alpha u(a)+\beta u^{\prime}(a)=0 \\
\gamma u(b)+\delta u^{\prime}(b)=0
\end{array}\right.
$$

where $\alpha^{2}+\beta^{2}>0, \gamma^{2}+\delta^{2}>0, \rho, \rho^{\prime}, q$ and $\sigma$ are continuous real functions on $[a, b]$, and $\rho(x)>0, \sigma(x)>0$ for $a \leq x \leq b$. Any value of the parameter $\lambda \in \mathbb{R}$ for which a nontrivial solution of (5) exists is called an eigenvalue. The corresponding nontrivial solution is called an eigenfunction related to the eigenvalue $\lambda$.

The following Sturm-Liouville property of (5) (SL-property for short) is well known:
"The eigenvalues of the problem (5) form an increasing sequence

$$
\lambda_{1}<\lambda_{2}<\lambda_{3}<\cdots<\lambda_{n}<\cdots \rightarrow+\infty
$$

To each eigenvalue $\lambda_{n}$ there corresponds a unique (up to a nonzero multiple) eigenfunction $u_{n}(x)$, which has exactly $n-1$ zeros in $(a, b)$. Moreover, between two consecutive zeros of $u_{n}$ there is exactly one zero of $u_{n+1}$."

In particular, the spectrum of (5) is bounded from below and discrete. For this reason, in the literature, such eigenvalue problems are said to have the $B D$-property (see e.g. [7, 9, 11]).

The purpose of this paper is to show that both $B D$-property and $S L$ property hold true also for more general equations

$$
\begin{equation*}
\left(-\rho(x)\left|u^{\prime}\right|^{p-2} u^{\prime}\right)^{\prime}=\lambda \sigma(x)|u|^{q-2} u \tag{6}
\end{equation*}
$$

on ( $a, b$ ) with $-\infty \leq a<b \leq+\infty$ and with $\rho$ and $\sigma$ positive measurable functions in $(a, b)$. Here, $1<p \leq q$, and equation (6) is complemented by the boundary conditions

$$
\begin{equation*}
\lim _{x \rightarrow a+} \rho(x)\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x)=\lim _{x \rightarrow b-} u(x)=0 . \tag{7}
\end{equation*}
$$

The boundedness from below of the set of all eigenvalues of (6), (7) follows from Hardy's inequality. Indeed, let $u$ be a nonzero solution of (6), (7). Multiplying (6) by $u$, integrating formally by parts and taking into account (7), we get

$$
\begin{equation*}
\int_{a}^{b} \rho(x)\left|u^{\prime}\right|^{p} \mathrm{~d} x=\lambda \int_{a}^{b} \sigma(x)|u|^{q} \mathrm{~d} x \tag{8}
\end{equation*}
$$

Since Hardy's inequality is of the form

$$
\begin{equation*}
\left(\int_{a}^{b} \sigma(x)|u|^{q} \mathrm{~d} x\right)^{\frac{1}{q}} \leq C\left(\int_{a}^{b} \rho(x)\left|u^{\prime}\right|^{p} \mathrm{~d} x\right)^{\frac{1}{p}} \tag{9}
\end{equation*}
$$

with a suitable constant $C>0$, after normalization, we obtain from (8) and (9) that

$$
\lambda \geq \frac{1}{C^{q}}
$$

holds for any eigenvalue of (6), (7).
To be more specific, let $W_{b}^{1, p}(\rho)$ be the weighted Sobolev space of all functions $u$ which are absolutely continuous on every compact subinterval of $(a, b)$, such that $\lim _{x \rightarrow b-} u(x)=0$ and

$$
\|u\|_{1, p ; \rho}:=\left(\int_{a}^{b} \rho(x)\left|u^{\prime}(x)\right|^{p} \mathrm{~d} x\right)^{1 / p}<+\infty
$$

Let $L^{q}(\sigma)$ be the weighted Lebesgue space of all measurable functions $u$ defined on $(a, b)$, for which

$$
\|u\|_{q ; \sigma}:=\left(\int_{a}^{b} \sigma(x)|u(x)|^{q} \mathrm{~d} x\right)^{1 / q}<+\infty
$$

Inequality (9) actually means that the embedding of $W_{b}^{1, p}(\rho)$ into $L^{q}(\sigma)$ is continuous ( $W_{b}^{1, p}(\rho) \hookrightarrow L^{q}(\sigma)$ for short).

Next we assume that for any $x \in(a, b)$ we have $\sigma \in L^{1}(a, x)$ and $\rho^{1-p^{\prime}} \in$ $L^{q}(x, b)$, where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$.

The expression

$$
\begin{equation*}
A_{M}(x):=\left(\int_{a}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / q}\left(\int_{x}^{b} \rho(\tau)^{1-p^{\prime}} \mathrm{d} \tau\right)^{1 / p^{\prime}} \tag{10}
\end{equation*}
$$

defines the so-called Muckenhoupt function. It is proved in [13] that (9) holds for all $u \in W_{b}^{1, p}(\rho)$ (i.e. $\left.W_{b}^{1, p}(\rho) \hookrightarrow L^{q}(\sigma)\right)$ if and only if

$$
\begin{equation*}
\sup _{x \in(a, b)} A_{M}(x)<+\infty \tag{11}
\end{equation*}
$$

Moreover, it is proved in [13] that the embedding of $W_{b}^{1, p}(\rho)$ into $L^{q}(\sigma)$ is compact ( $W_{b}^{1, p}(\rho) \hookrightarrow \hookrightarrow L^{q}(\sigma)$ for short) if and only if

$$
\begin{equation*}
\lim _{x \rightarrow a+} A_{M}(x)=\lim _{x \rightarrow b-} A_{M}(x)=0 \tag{12}
\end{equation*}
$$

Expressions of type (10) appear in the literature in connection with the $B D$ property and oscillation properties of differential operator of the second order (see e.g. [1, 2, 3, 4, 5]).

With the compactness of the above embedding in hands, we can prove the following assertion.
THEOREM 2.1. Assume that (12) holds true. Then there exists minimal value of $\lambda:=\lambda_{1}>0$ such that (6), (7) has a nontrivial solution $u_{1} \in W_{b}^{1, p}(\rho)$ normalized by $\left\|u_{1}\right\|_{q ; \sigma}=1$.

The proof of this assertion follows from minimization of the Rayleigh type quotient

$$
R(u)=\frac{\int_{a}^{b} \rho(x)\left|u^{\prime}\right|^{p} \mathrm{~d} x}{\int_{a}^{b} \sigma(x)|u|^{q} \mathrm{~d} x}
$$

on $W_{b}^{1, p}(\rho)$ subject to the constraint $\int_{a}^{b} \sigma(x)|u|^{q} \mathrm{~d} x=1$. The compact embedding $W_{b}^{1, p}(\rho) \hookrightarrow \hookrightarrow L^{q}(\sigma)$ implies that $\lambda_{1}=\min R(u)$ is achieved at $u_{1} \in$ $W_{b}^{1, p}(\rho)$ satisfying $\int_{a}^{b} \sigma(x)\left|u_{1}\right|^{q} \mathrm{~d} x=1$. Application of the Lagrange multiplier method then yields that

$$
\int_{a}^{b} \rho(x)\left|u_{1}^{\prime}\right|^{p-2} u_{1}^{\prime} v^{\prime} \mathrm{d} x=\lambda_{1} \int_{a}^{b} \sigma(x)\left|u_{1}\right|^{q-2} u_{1} v \mathrm{~d} x
$$

holds for any $v \in W_{b}^{1, p}(\rho)$. In other words, $u_{1}$ is a weak solution of (6), (7). Standard regularity argument for the second order ODEs then implies that
$u_{1} \in C^{1}(a, b) \rho\left|u^{\prime}\right|^{p-2} u^{\prime} \in C^{1}(a, b)$, the equation (6) holds at every point in ( $a, b$ ), boundary conditions (7) hold true and $\left\|u_{1}\right\|_{1, p ; \rho}<+\infty$. Hence, $u_{1}$ is a classical solution to (6), (7), as well.

Remark 2.2. Note that the weaker condition (11) is sufficient for the boundedness from below of any possible eigenvalue of (6), (7). However, without compactness of the embedding $W_{b}^{1, p}(\rho) \hookrightarrow \hookrightarrow L^{q}(\sigma)$ (which is equivalent to (12)) it is not clear whether (6), (7) has any eigenvalues and eigenfunctions at all.

Actually, with compactness of $W_{b}^{1, p}(\rho) \hookrightarrow \hookrightarrow L^{p}(\sigma)$ in hands we can get more precise information about the spectrum of (6), (7) in case of homogeneous equation when $p=q$. In particular, we can generalize the Sturm-Liouville theory for the half-linear problem

$$
\left\{\begin{array}{l}
\left(\rho(x)\left|u^{\prime}\right|^{p-2} u^{\prime}\right)^{\prime}+\lambda \sigma(x)|u|^{p-2} u=0 \quad \text { in }(a, b)  \tag{13}\\
\lim _{x \rightarrow a+} \rho(x)\left|u^{\prime}(x)\right|^{p-2} u^{\prime}(x)=\lim _{x \rightarrow b-} u(x)=0
\end{array}\right.
$$

Theorem 2.3 (see [5] and cf. [2, 3]). The SL-property for (13) is satisfied if and only if the following two conditions hold:

$$
\begin{align*}
& \lim _{x \rightarrow a+}\left(\int_{a}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / p}\left(\int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{1 / p^{\prime}}=0  \tag{14}\\
& \lim _{x \rightarrow b-}\left(\int_{a}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / \rho}\left(\int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{1 / p^{\prime}}=0 \tag{15}
\end{align*}
$$

Remark 2.4. Note that (14), (15) are equivalent to (12) where $q=p$. Note also that (14), (15) are equivalent with the compact embedding

$$
\begin{equation*}
W_{b}^{1, p}(\rho) \hookrightarrow \hookrightarrow L^{p}(\sigma) . \tag{16}
\end{equation*}
$$

This fact implies the following "round about" assertion.
Theorem 2.5 (see $[3,5]$ ). The following statements are equivalent:
(i) The SL-property for (13) is satisfied.
(ii) Conditions (14), (15) hold.
(iii) The compact embedding (16) holds.

If we know the asymptotics of the limit in (15), we get an asymptotic estimate for the behavior of eigenfunctions of (13) as $x \rightarrow b-$. Namely, assume that there exist $\varepsilon \in(0, p-1)$ and $C>0$ such that for all $x \in(a, b)$ we have

$$
\begin{equation*}
\left(\int_{a}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / p}\left(\int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{1 / p^{\prime}} \leq C\left(\int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{\varepsilon / p} \tag{17}
\end{equation*}
$$

Theorem 2.6 (see [4]). Let (14) and (17) hold. Then for any eigenfunction $u$ of (13) there exist $\bar{b} \in(a, b)$ and $0<C_{1}<C_{2}$ such that for all $x \in(\bar{b}, b)$ we have

$$
C_{1} \int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau \leq|u(x)| \leq C_{2} \int_{x}^{b} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau
$$

Remark 2.7. We would like to mention also the pioneering work [12] where a different approach than that of ours was used to prove the discreteness of the spectrum of the second order quasilinear Sturm-Liouville problem. The method of [12] had been extended to the fourth order problem in [10] and became a motivation for our research mentioned in the next section.

## 3. Higher order equations

Let us consider the eigenvalue problem:

$$
\left\{\begin{array}{l}
\left(\rho(x)\left|u^{\prime \prime}(x)\right|^{p-2} u^{\prime \prime}(x)\right)^{\prime \prime}-\lambda \sigma(x)|u(x)|^{p-2} u(x)=0, \quad x>0  \tag{18}\\
u^{\prime}(0)=\lim _{x \rightarrow 0+}\left(\rho(x)\left|u^{\prime \prime}(x)\right|^{p-2} u^{\prime \prime}(x)\right)^{\prime}=0 \\
\lim _{x \rightarrow+\infty} u(x)=\lim _{x \rightarrow+\infty} u^{\prime}(x)=0
\end{array}\right.
$$

We assume that $\rho$ and $\sigma$ are continuous and positive in $[0,+\infty)$, and the function $x^{p^{\prime}} \rho^{1-p^{\prime}}(x)$ belongs to $L^{1}(0,+\infty)$. By a solution of (18) we understand a function $u \in C^{2}(0,+\infty)$ such that $\rho\left|u^{\prime \prime}\right|^{p-2} u^{\prime \prime} \in C^{2}(0,+\infty)$, the equation in (18) holds at every point in $(0,+\infty)$, the boundary conditions are satisfied and the Dirichlet integral $\int_{0}^{\infty} \rho(x)\left|u^{\prime \prime}(x)\right|^{p} \mathrm{~d} x$ is finite.

We say that the $D$-property for (18) is satisfied if the set of all eigenvalues of (18) forms on increasing sequence $\left\{\lambda_{n}\right\}_{n=1}^{\infty}$ such that $\lambda_{1}>0$ and $\lim _{n \rightarrow \infty} \lambda_{n}=$ $\infty$. Moreover, the set of all normalized eigenfunctions associated with a given eigenvalue is finite and every eigenfunction has a finite number of nodes.

Theorem 3.1 (see [6]). The D-property for (18) is satisfied if and only if the following two conditions hold

$$
\left\{\begin{array}{l}
\lim _{x \rightarrow+\infty}\left(\int_{0}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / p}\left(\int_{x}^{\infty}(\tau-x)^{p^{\prime}} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{1 / p^{\prime}}=0  \tag{19}\\
\lim _{x \rightarrow+\infty}\left(\int_{0}^{x}(x-\tau)^{p} \sigma(\tau) \mathrm{d} \tau\right)^{1 / p}\left(\int_{x}^{\infty} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{1 / p^{\prime}}=0
\end{array}\right.
$$

REmark 3.2. The conditions (19) are equivalent to the compact embedding

$$
\begin{equation*}
W_{\infty}^{2, p}(\rho) \hookrightarrow \hookrightarrow L^{p}(\sigma), \tag{20}
\end{equation*}
$$

where $W_{\infty}^{2, p}(\rho)$ is the weighted Sobolev space of all functions $u \in C^{1}[0,+\infty)$, $u^{\prime}$ is absolutely continuous on every compact subinterval of $(0,+\infty), u^{\prime}(0)=$

$$
\begin{aligned}
\lim _{x \rightarrow+\infty} u(x)=\lim _{x \rightarrow+\infty} u^{\prime}(x) & =0, \text { and } \\
\|u\|_{2, p ; \rho}: & =\left(\int_{0}^{\infty} \rho(x)\left|u^{\prime \prime}(x)\right|^{p} \mathrm{~d} x\right)^{1 / p}<+\infty,
\end{aligned}
$$

see [6] for details.
Hence, the following analogue of Theorem 2.5 holds also for the fourth order problem.

Theorem 3.3. The following statements are equivalent:
(i) The D-property for (18) is satisfied.
(ii) Conditions (19) hold.
(iii) The compact embedding (20) holds.

Remark 3.4. The reader is invited to compare $S L$-property for the second order problem and $D$-property for the fourth order problem. The former one is stronger than the latter one. One of the reasons consists in the fact that in the fourth order case it is substantially more difficult to establish that all eigenfunctions have finitely many nodes in $(0,+\infty)$.

Let $k \in \mathbb{N}$. Consider the quasilinear equation of order $2 k$,

$$
\begin{equation*}
(-1)^{k}\left(\rho(x)\left|u^{(k)}(x)\right|^{p-2} u^{(k)}(x)\right)^{(k)}=\lambda \sigma(x)|u(x)|^{q-2} u(x), x \in(0, \infty) \tag{21}
\end{equation*}
$$

together with boundary conditions

$$
\begin{align*}
& u^{\prime}(0)=\cdots=u^{(k-1)}(0)=\lim _{x \rightarrow 0+}\left(\rho(x) u^{(k)}(x)\right)^{\prime}=0,  \tag{22}\\
& \lim _{x \rightarrow+\infty} u(x)=\lim _{x \rightarrow+\infty} u^{\prime}(x)=\cdots=\lim _{x \rightarrow+\infty} u^{(k-1)}(x)=0 . \tag{23}
\end{align*}
$$

This problem was considered in [1].
Let $W_{\infty}^{k, p}(\rho)$ be the weighted Sobolev space of functions $u \in C^{k-1}[0,+\infty)$, $u^{(k-1)}$ be absolutely continuous on every compact subinterval of $(0,+\infty), u$ satisfy (23) and

$$
\|u\|_{k, p ; \rho}=\left(\int_{0}^{\infty} \rho(x)\left|u^{(k)}(x)\right|^{p} \mathrm{~d} x\right)^{1 / p}<+\infty
$$

Let us introduce functions

$$
\begin{aligned}
& B_{1}(x):=\left(\int_{0}^{x}(x-\tau)^{q(k-1)} \sigma(\tau) \mathrm{d} \tau\right)^{\frac{1}{q}}\left(\int_{x}^{\infty} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{\frac{1}{p^{\prime}}} \\
& B_{2}(x):=\left(\int_{0}^{x} \sigma(\tau) \mathrm{d} \tau\right)^{1 / q}\left(\int_{x}^{\infty}(\tau-x)^{p^{\prime}(k-1)} \rho^{1-p^{\prime}}(\tau) \mathrm{d} \tau\right)^{\frac{1}{p^{\prime}}}
\end{aligned}
$$

The following assertions can be found in [13].

LEMMA 3.5. The embedding $W_{\infty}^{k, p}(\rho) \hookrightarrow L^{q}(\sigma)$ is continuous if and only if $B_{1}(x)$ and $B_{2}(x)$ are bounded on $(0,+\infty)$.

Lemma 3.6. The embedding $W_{\infty}^{k, p}(\rho) \hookrightarrow \hookrightarrow L^{q}(\sigma)$ is compact if and only if

$$
\begin{equation*}
\lim _{x \rightarrow 0+} B_{i}(x)=\lim _{x \rightarrow+\infty} B_{i}(x)=0, \quad i=1,2 . \tag{24}
\end{equation*}
$$

Using the compactness argument and Lemma 3.6, as in the proof of Theorem 2.1, we can prove the following assertion.

Theorem 3.7. Assume that (24) holds true. Then there exists the minimal value $\lambda:=\lambda_{1}>0$ such that (21)-(23) has a nontrivial solution $u_{1} \in W_{\infty}^{k, p}(\rho)$ normalized by $\left\|u_{1}\right\|_{q ; \sigma}=1$.

Remark 3.8. The fact that all possible eigenvalues of (21)-(23) are bounded from below follows just from the boundedness of $B_{1}$ and $B_{2}$ combined with Lemma 3.5. On the other hand, Theorem 3.7 guarantees that there exists the least eigenvalue and the corresponding eigenfunction of (21)-(23). However, the discreteness of the entire spectrum remains an open question:

Conjecture 3.9. Assume that (24) holds true. Then (21)-(23) has the BDproperty.

## 4. Applications

In this section we present applications of our general estimates to some concrete boundary value problems. In particular, the asymptotic properties of radial solutions to quasilinear eigenvalue problems for PDEs with degenerated and/or singular coefficients are new results.

Example 4.1 (cf. [5]). Let us consider the radial eigenvalue problem for the $p$-Laplacian $\Delta_{p}$ on $\mathbb{R}^{N}$ :

$$
\left\{\begin{array}{l}
-\Delta_{p} u=\frac{\lambda}{1+|x| \gamma}|u|^{p-2} u \quad \text { in } \mathbb{R}^{N}  \tag{25}\\
\lim _{|x| \rightarrow+\infty} u(x)=0
\end{array}\right.
$$

This problem reduces to the one-dimensional equation

$$
\begin{equation*}
-\left(r^{N-1}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)\right)^{\prime}=\lambda \frac{r^{N-1}}{1+r^{\gamma}}|u(r)|^{p-2} u(r), r \in(0,+\infty), \tag{26}
\end{equation*}
$$

where $r=|x|$. For $1<p<N$ and $\gamma>p$ the weights

$$
\rho(r)=r^{N-1} \text { and } \sigma(r)=\frac{r^{N-1}}{1+r^{\gamma}}
$$

satisfy (14) and (15). Moreover, the solution of (26) is also forced to satisfy the so-called Neumann-Dirichlet boundary conditions

$$
\begin{equation*}
\lim _{r \rightarrow 0+} r^{N-1}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)=\lim _{r \rightarrow+\infty} u(r)=0 \tag{27}
\end{equation*}
$$

Hence, Theorem 2.3 applies to (26), (27).
In particular, we have the following assertion for the original problem (25):
Theorem 4.2. Let $1<p<N$ and $\gamma>p$. Then the eigenvalues of the radial eigenvalue problem (25) exhaust the sequence $\left\{\lambda_{n}\right\}_{n=1}^{\infty}, 0<\lambda_{1}<\lambda_{2}<\cdots \rightarrow$ $+\infty$ with all $\lambda_{n}$ being simple. A normalized eigenfunction $u_{\lambda_{n}}$ associated with $\lambda_{n}, n \geq 1$, has precisely $n$ nodal domains in $\mathbb{R}^{N}$. The nodal "lines" of $u_{\lambda_{n}}$ are concentric spheres in $\mathbb{R}^{N}$ centered at the origin. The nodal "lines" of $u_{\lambda_{n-1}}$ separate those of $u_{\lambda_{u}}$.

Example 4.3 (cf. [4]). Let us consider the radial eigenvalue problem for the weighted $p$-Laplacian

$$
\left\{\begin{array}{l}
\quad-\operatorname{div}\left(\frac{1}{(1+|x|)^{\alpha}}|\nabla u(x)|^{p-2} \nabla u(x)\right)  \tag{28}\\
\quad=\lambda \frac{1}{(1+|x|)^{\beta}}|u(x)|^{p-2} u(x), \quad x \in \mathbb{R}^{N} \\
\lim _{|x| \rightarrow+\infty} u(x)=0
\end{array}\right.
$$

This problem reduces to the equation

$$
\begin{equation*}
-\left(\frac{r^{N-1}}{(1+r)^{\alpha}}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)\right)^{\prime}=\lambda \frac{r^{N-1}}{(1+r)^{\beta}}|u(r)|^{p-2} u(r), \quad r \in(0, \infty) \tag{29}
\end{equation*}
$$

with boundary conditions (27). Let $\alpha+p<N, \alpha+p<\beta$ and $\varepsilon=$ $(p-1)(\alpha+p-\beta) /(\alpha+p-N)$. Then (14) and (17) hold. Hence, Theorems 2.3 and 2.6 apply to (29).

In particular, we have the following assertion:
Theorem 4.4. Let $\alpha+p<\min \{N, \beta\}$. Then the conclusions of Theorem 4.2 hold also for the boundary value problem (28). Moreover, there exist $r_{0}>0$ and $0<C_{1}<C_{2}$ such that

$$
\frac{C_{1}}{|x|^{\frac{N-(\alpha+p)}{p-1}}} \leq|u(x)| \leq \frac{C_{2}}{|x|^{\frac{N-(\alpha+p)}{p-1}}}
$$

for any $x \in \mathbb{R}^{N}$ satisfying $|x| \geq r_{0}$.

Example 4.5. Let us consider the radial eigenvalue problem for the $p$-Laplacian on the ball:

$$
\begin{cases}-\operatorname{div}\left((R-|x|)^{\alpha}|\nabla u|^{p-2} \nabla u\right)=\lambda(R-|x|)^{\beta}|u|^{p-2} u & \text { in } B_{R}(0)  \tag{30}\\ 7 u=0 & \text { on } \partial B_{R}(0) .\end{cases}
$$

Here $1<p<N$ and $B_{R}(0)$ is a ball centered at the origin with radius $R>0$. The weight functions $x \mapsto(R-|x|)^{\alpha}, x \mapsto(R-|x|)^{\beta}$ are just power of the distance from the boundary. Obviously, this problem reduces to

$$
\left\{\begin{array}{l}
-\left(r^{N-1}(R-r)^{\alpha}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)\right)^{\prime}  \tag{31}\\
\quad=\lambda r^{N-1}(R-r)^{\beta}|u(r)|^{p-2} u(r), \quad r \in(0, R), \\
\lim _{r \rightarrow 0+} r^{N-1}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)=\lim _{r \rightarrow R-} u(r)=0 .
\end{array}\right.
$$

For

$$
\begin{equation*}
\beta<-1 \text { and } \alpha-\beta<p \text { or } \beta \geq-1 \text { and } \alpha<p-1 \tag{32}
\end{equation*}
$$

the weights

$$
\rho(r)=r^{N-1}(R-r)^{\alpha} \text { and } \sigma(r)=r^{N-1}(R-r)^{\beta}
$$

satisfy (14) and (17). Hence Theorems 2.3 and 2.6 apply to (31).
In particular, we have the following assertion:
TheOrem 4.6. Let us assume (32). Then the eigenvalues of the radial eigenvalue problem (30) exhaust the sequence $\left\{\lambda_{n}\right\}_{n=1}^{\infty}, 0<\lambda_{1}<\lambda_{2}<\cdots \rightarrow \infty$ with all $\lambda_{n}$ being simple. A normalized eigenfunction $u_{\lambda_{n}}$ associated with $\lambda_{n}$, $n \geq 1$, has precisely $n$ nodal domains in $B_{R}(0)$. The nodal "lines" of $u_{\lambda_{n}}$ are concentric spheres contained in $B_{R}(0)$ centered at the origin. The nodal "lines" of $u_{\lambda_{n-1}}$, separate those of $u_{\lambda_{n}}$. Moreover, there exist $\bar{R} \in(0, R), C_{1}, C_{2}>0$ such that for all $x \in B_{R}(0) \backslash B_{\bar{R}}(0)$ we have

$$
\begin{equation*}
C_{1}(R-|x|)^{1-\frac{\alpha}{p-1}} \leq|u(x)| \leq C_{2}(R-|x|)^{1-\frac{\alpha}{p-1}} . \tag{33}
\end{equation*}
$$

Remark 4.7. Let $\frac{\partial u}{\partial \nu}(x)$ denote the derivative of an eigenfunction $u$ with respect to the external normal at the point $x \in \partial B_{R}(0)$. Let an eigenfunction $u$ be positive in the neighborhood of $\partial B_{R}(0)$. Then
(i) For $\alpha=0$ we have $\frac{\partial u}{\partial \nu}(x)<0, x \in \partial B_{R}(0)$, due to well-known Hopf's (for $p=2$, see [14]) and Vázquez's (for $p \neq 2$, see [15]) maximum principle.
(ii) For $\alpha>0$ we have $\frac{\partial u}{\partial \nu}(x)=-\infty, x \in \partial B_{R}(0)$ by (33).
(iii) For $\alpha<0$ we have $\frac{\partial u}{\partial \nu}(x)=0, x \in \partial B_{R}(0)$ by (33).

Example 4.8. Let $1<p<N, q \geq p$. Consider the radial problem

$$
\begin{cases}-\Delta_{p} u=\lambda|u|^{q-2} u & \text { in } B_{R}(0)  \tag{34}\\ u=0 & \text { on } \partial B_{R}(0)\end{cases}
$$

This problem reduces to

$$
\left\{\begin{array}{l}
-\left(r^{N-1}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)\right)^{\prime}=\lambda r^{N-1}|u(r)|^{q-2} u(r), \quad r \in(0, R),  \tag{35}\\
\lim _{r \rightarrow 0+} r^{N-1}\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)=\lim _{r \rightarrow R_{-}} u(r)=0
\end{array}\right.
$$

In particular, this corresponds to (6) with $\rho(r)=\sigma(r)=r^{N-1}$ and

$$
\begin{aligned}
A_{M}(x) & =\left(\int_{0}^{x} \tau^{N-1} \mathrm{~d} \tau\right)^{\frac{1}{q}}\left(\int_{x}^{R} \tau^{\frac{1-N}{p-1}} \mathrm{~d} \tau\right)^{\frac{1}{p^{\prime}}} \\
& =\left(\frac{p-1}{N-p}\right)^{\frac{1}{p^{\prime}}}\left(\frac{x^{N}}{N}\right)^{\frac{1}{q}}\left(x^{\frac{p-N}{p-1}}-R^{\frac{p-N}{p-1}}\right)^{\frac{1}{p^{\prime}}}
\end{aligned}
$$

Consequently,

$$
\lim _{x \rightarrow 0+} A_{M}(x)=\lim _{x \rightarrow R_{-}} A_{M}(x)=0
$$

if and only if $1<q<p^{*}:=\frac{N p}{N-p}$ (critical Sobolev exponent). Applying Theorem 2.1 to (35), we get the existence of a value $\lambda>0$ and of the corresponding normalized solution $u \in W_{0}^{1, p}\left(B_{R}(0)\right)$ of (34). It is possible to show that this solution is $C^{1, \alpha}$-regular and positive in $B_{R}(0)$, with some $\alpha \in(0,1)$.

On the other hand, using the well-known Pohozaev identity, one can prove that no such solution exists for $q \geq p^{*}$.
Example 4.9. Let us consider the boundary value problem

$$
\left\{\begin{array}{l}
\left((x+1)^{2} u^{\prime}(x)\right)^{\prime}+\lambda u(x)=0, \quad x \in(0,+\infty)  \tag{36}\\
u^{\prime}(0)=u(+\infty)=0
\end{array}\right.
$$

Notice that (36) is a special case of (13) with $a=0, b=+\infty, p=2, \rho(x)=$ $(x+1)^{2}, \sigma(x) \equiv 1, x \in(0,+\infty)$. That is

$$
A_{M}(x)=\left(\int_{0}^{x} \mathrm{~d} \tau\right)^{1 / 2}\left(\int_{x}^{+\infty} \frac{1}{(\tau+1)^{2}} \mathrm{~d} \tau\right)^{1 / 2}=\left(\frac{x}{1+x}\right)^{1 / 2}, x \in(0,+\infty)
$$

satisfies (11) but violates (12).
Elementary calculation yields that the initial value problem

$$
\left\{\begin{array}{l}
\left((x+1)^{2} u^{\prime}(x)\right)^{\prime}+\lambda u(x)=0, \quad x \in(0,+\infty) \\
u(0)=1, u^{\prime}(0)=0
\end{array}\right.
$$

has the following unique solutions:
(i) for $\lambda=\frac{1}{4}, u(x)=\frac{1}{\sqrt{x+1}}(1+\ln \sqrt{x+1})$;
(ii) for $\lambda<\frac{1}{4}$,

$$
u(x)=\frac{1}{\sqrt{x+1}}\left[\left(\frac{1}{2}-\frac{1}{2 \sqrt{1-4 \lambda}}\right)(x+1)^{\frac{1}{2} \sqrt{1-4 \lambda}}+\left(\frac{1}{2}-\frac{1}{2 \sqrt{1-4 \lambda}}\right)(x+1)^{-\frac{1}{2} \sqrt{1-4 \lambda}}\right]
$$

(iii) for $\lambda>\frac{1}{4}$,

$$
u(x)=\frac{1}{\sqrt{x+1}}\left[\cos \left(\frac{1}{2} \sqrt{4 \lambda-1} \ln (x+1)\right)-\frac{1}{\sqrt{4 \lambda-1}} \sin \left(\frac{1}{2} \sqrt{4 \lambda-1} \ln (x+1)\right)\right]
$$

Thus (36) has no solution $u \in W_{\infty}^{1,2}(\rho)$ for any $\lambda \in \mathbb{R}$.

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# The prescribed mean curvature problem with Neumann boundary conditions in FLRW spacetimes 

Pedro J. Torres

Dedicated to Jean Mawhin on the occasion of his 75th anniversary


#### Abstract

We provide sufficient conditions for the existence of solution of the radially symmetric prescribed curvature problem with Neumann boundary condition on a general Friedmann-Lemaître-RobertsonWalker (FLRW) spacetime.

Keywords: Neumann boundary condition, radially symmetric solutions, singular $\phi$ Laplacian, prescribed mean curvature function, Friedmann-Lemaître-Robertson-Walker spacetime. MS Classification 2010: 35J93, 35J25, 35A01, 35A16, 53C42, 53C50.


## 1. Introduction

A Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime is a metric space given by the cartesian product $I \times \mathbb{R}^{n}$ of an open interval $\left.I=\right] a, b[$ with the $n$-dimensional Euclidean space endowed with the Lorentzian metric

$$
d s^{2}=-d t^{2}+f(t) d x^{2}
$$

where $f(t)$ is a positive function of time known as the scale factor or warping function. In Cosmology, the FLRW space is the accepted model for a spatially homogeneous and isotropic Universe. In this context, the scaling factor $f(t)$ represents the size of the Universe at time $t$ and must be determined as an exact solution of Einstein's field equations under the assumptions of isotropy and homogeneity. Observe that for the particular case $f(t) \equiv 1$ we recover the Lorentz-Minkowski spacetime. Other relevant examples are

- Einstein-De Sitter spacetime: $\left.f(t)=\left(t+t_{0}\right)^{2 / 3}, \quad I=\right]-t_{0},+\infty[$
- Steady state spacetime: $f(t)=e^{t}, \quad I=\mathbb{R}$
- Lambda-CDM model: $\left.f(t)=A \sinh ^{2 / 3}\left(t+t_{0}\right), \quad I=\right]-t_{0},+\infty[$
- Cycloid model: $\left.f(\theta)=\frac{R}{2}(1-\cos \theta), \quad t(\theta)=\theta-\sin \theta, \quad I=\right]-\pi / 2, \pi / 2[$

We refer to the monograph [8] for more details on the derivation and physical interpretation of these cosmologies.

We are interested on the problem of the existence of spacelike graphs with a prescribed the mean curvature function. For a FLRW spacetime, the curvature operator is given by the expression

$$
\begin{equation*}
Q[u]:=\frac{1}{n}\left\{\operatorname{div}\left(\frac{\nabla u}{f(u) \sqrt{f(u)^{2}-|\nabla u|^{2}}}\right)+\frac{f^{\prime}(u)}{\sqrt{f(u)^{2}-|\nabla u|^{2}}}\left(n+\frac{|\nabla u|^{2}}{f(u)^{2}}\right)\right\} . \tag{1}
\end{equation*}
$$

Then, the general problem of the curvature prescription is, given a function $H: I \times \mathbb{R}^{n} \rightarrow \mathbb{R}$, to obtain solutions of the quasilinear elliptic problem

$$
Q[u]=H(u, x), \quad|\nabla u|<f(u) .
$$

Here, $|\nabla u|<f(u)$ means that $|\nabla u(x)|<f(u(x))$ for all $x$. The prescription of curvature has a physical meaning. Intuitively, a spacelike hypersurface is the spatial universe at one instant of proper time of a family of normal observers. Then, the mean curvature function measures how these observers spread away $(H>0)$ or come together $(H<0)$ with respect to the surrounding observers. In this sense, the problem may be seen as a local prescription of the behaviour of normal observers.

The consideration of this problem is rather new on the literature. Up to now, most of the efforts have been directed to the curvature prescription on the Lorentz-Minkowski spacetime $(f(t) \equiv 1)$, see for instance $[1,3,6]$. For more general FLRW spacetimes, up to our knowledge the first contributions to the literature are $[2,4]$, where it is studied the problem with radial symmetry and Dirichlet conditions on a ball for a family of expanding FLRW spacetimes, including the Einstein-de Sitter, steady state and Lambda-CDM models. A first approach to the problem with Neumann conditions has been done in the recent paper [7], where a kind of universal result is proved for big bang-big crunch models that includes the cycloid as a particular case. Our purpose is to revise the proof employed there and state a result applicable to any example of FLRW spacetime.

## 2. Main result

Let us state precisely the mathematical problem under study. Let $B(R)$ be the Euclidean ball of $\mathbb{R}^{n}$ centered at 0 with radius $R$. Let $\left.I=\right] a, b[\subseteq \mathbb{R},-\infty \leq a<$ $0<b \leq+\infty$ be an open interval, and let $f \in C^{1}(I)$ a positive function. For a given continuous function $H: \mathbb{R} \times[0,+\infty) \rightarrow \mathbb{R}$, we look for radially symmetric
solutions of the problem

$$
\begin{gather*}
Q[u]=H(u,|x|) \\
|\nabla u|<f(u) \text { in } B(R),  \tag{2}\\
\frac{\partial u}{\partial \nu}=0 \quad \text { in } \quad \partial B(R),
\end{gather*}
$$

where the operator $Q$ is defined by (1) and $\frac{\partial u}{\partial \nu}$ denotes the outward normal derivative of $u$.

Our main result is as follows.
Theorem 2.1. Let us assume that

$$
\begin{equation*}
\limsup _{t \rightarrow a^{+}}\left\{H(t, r)-\frac{f^{\prime}(t)}{f(t)}\right\}<0<\liminf _{t \rightarrow b^{-}}\left\{H(t, r)-\frac{f^{\prime}(t)}{f(t)}\right\}, \text { for all } r>0 \tag{3}
\end{equation*}
$$

Then, there exists $R_{0}>0$ (depending on $f, H$ ) such that if $0<R<R_{0}$, problem (2) has at least one radially symmetric solution $u(|x|)$.

It is worth to note that for the Lorentz-Minkowski spacetime $f(t) \equiv 1$, condition (3) is known in the literature as a Landesman-Lazer condition, in fact for this case $R_{0}$ can be taken as $+\infty$ and Theorem 2.1 is just a particular case of [3, Theorem 3.1]. On the other hand, taking the family of warping functions considered in [7] we recover the main result therein. Furthermore, Theorem 2.1 admits any general warping function with the minimal conditions of being positive and regular. For example, for the Einstein-de Sitter spacetime $f(t)=\left(t+t_{0}\right)^{2 / 3}$, this result is applicable to any curvature function $H(t, r)$ taking positive values for large times. This condition is natural in some way, because if $H(t, r) \leq 0$ for every $(t, r)$, a simple integration (see equation (5) below) proves that the problem has no solution. This occurs in general for any expanding cosmology (that is, any strictly increasing scale factor $f(t)$ ).

## 3. Preliminaries

We follow the main ideas of [7]. Let us define the function $\varphi: I \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\varphi(s):=\int_{0}^{s} \frac{d t}{f(t)} \tag{4}
\end{equation*}
$$

Note that $\varphi$ is an increasing diffeomorphism from $I$ onto $J:=\varphi(I)$ and $\varphi(0)=0$.
Doing the change $v=\varphi(u)$ and taking radial coordinates, problem (2) is equivalent to the boundary value problem

$$
\begin{gather*}
\left(r^{n-1} \frac{v^{\prime}}{\sqrt{1-v^{\prime 2}}}\right)^{\prime}=n r^{n-1}\left[-\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{\sqrt{1-v^{\prime 2}}}+f\left(\varphi^{-1}(v)\right) H\left(\varphi^{-1}(v), r\right)\right]  \tag{5}\\
v^{\prime}(0)=0=v^{\prime}(R)
\end{gather*}
$$

Let $\phi(s)=\frac{s}{\sqrt{1-s^{2}}}$. The proof relies on a Leray-Schauder degree argument. We introduce the homotopy

$$
\begin{align*}
& \left(r^{n-1} \phi\left(v^{\prime}\right)\right)^{\prime} \\
& \quad=\lambda n r^{n-1}\left[-\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{\sqrt{1-v^{\prime 2}}}+f\left(\varphi^{-1}(v)\right) H\left(\varphi^{-1}(v), r\right)\right], r \in(0, R),  \tag{6}\\
& v^{\prime}(0)=0=v^{\prime}(R)
\end{align*}
$$

where $\lambda \in[0,1]$.
Let us define the operator

$$
F[v](t)=\int_{0}^{t} n r^{n-1}\left[-\frac{f^{\prime}\left(\varphi^{-1}(v(r))\right)}{\sqrt{1-v^{\prime}(r)^{2}}}+f\left(\varphi^{-1}(v(r))\right) H\left(\varphi^{-1}(v(r)), r\right)\right] d r
$$

Then, taking some $\Gamma<1$, let us consider the family of operators $\mathcal{G}:\{v \in$ $\left.C^{1}([0, R], J):\left\|v^{\prime}\right\|_{\infty} \leq \Gamma\right\} \times[0,1] \rightarrow C^{1}([0, R])$ defined as

$$
\mathcal{G}(v, \lambda)(r)=v(0)+\frac{1}{R} F[v](R)+\int_{0}^{r} \phi^{-1}\left(\frac{\lambda}{t^{n-1}} F[v](t)\right) d t .
$$

It is not hard to prove that $v \in C^{1}([0, R], J)$ is a fixed point of $\mathcal{G}(\cdot, \lambda)$ if and only if $v$ is a solution of (6) (see [7, Lemma 1]). Then, by the basic properties of topological degree, the proof is reduced to the estimation of some a priori bounds.

## 4. Proof of the main result

The key point is to obtain a proper bound for the fixed points of operator $\mathcal{G}(v, \lambda)$ in the uniform norm. To this aim, we are going to use our main hypothesis (3). Using that $\varphi^{-1}: J \rightarrow I$ is an increasing homeomorphism and (3), there exist $\rho_{*}, \rho^{*} \in J$ such that

$$
\begin{equation*}
\left.\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{f\left(\varphi^{-1}(v)\right)}-H\left(\varphi^{-1}(v), r\right)<0, \quad v \in\right] \rho^{*}, \varphi^{-1}(b)[, r>0 \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{f\left(\varphi^{-1}(v)\right)}-H\left(\varphi^{-1}(v), r\right)>0, \quad v \in\right] \varphi^{-1}(a), \rho^{*}[, r>0 \tag{9}
\end{equation*}
$$

The first lemma proves the every solution must lie between these two values.
Lemma 4.1. Let $v$ be a fixed point of $\mathcal{G}(\cdot, \lambda)$ for some $\lambda \in[0,1]$. Then,

$$
\begin{equation*}
\rho_{*} \leq v(r) \leq \rho^{*}, \quad \text { for all } r \in[0, R] \tag{10}
\end{equation*}
$$

Proof. First, we consider the case $\lambda=0$. A fixed point $v=\mathcal{G}(v, 0)$ takes the constant value

$$
v(r)=v(0)+\frac{1}{R} F[v](R)
$$

Evaluating at $r=0$ one has

$$
v(0)=v(0)+F[v](R)
$$

and therefore

$$
F[v](R)=0,
$$

and considering that $v$ is constant, then

$$
F(v)(R)=\left[-\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{f\left(\varphi^{-1}(v)\right)}+H\left(\varphi^{-1}(v(r)), r\right)\right] f\left(\varphi^{-1}(v)\right) R^{n}=0
$$

From this last equation and (8) - (9), one deduces that $\rho_{*} \leq v(r) \leq \rho^{*}$.
From now on, we can assume that $\lambda>0$. Let $v$ a fixed point of $\mathcal{G}(\cdot, \lambda)$. Let $r^{*} \in[0, R]$ such that $v\left(r^{*}\right)=\max _{[0, R]} v(r)$. Our aim is to prove that $v\left(r_{*}\right) \leq \rho^{*}$ by contradiction. Suppose that $v\left(r_{*}\right)>\rho^{*}$. We consider first the case $r^{*}>0$. Observe that developing the derivative of the left-hand side term of (6) and dividing by $r^{n-1}$ we have

$$
\begin{equation*}
\frac{v^{\prime \prime}}{\left(1-v^{\prime 2}\right)^{3 / 2}}+\frac{v^{\prime}}{r \sqrt{1-v^{\prime 2}}}=\lambda n\left[-\frac{f^{\prime}\left(\varphi^{-1}(v)\right)}{\sqrt{1-v^{\prime 2}}}+f\left(\varphi^{-1}(v)\right) H\left(\varphi^{-1}(v), r\right)\right], \tag{11}
\end{equation*}
$$

then, evaluating at $r^{*}$ and using that $v^{\prime}\left(r^{*}\right)=0, v\left(r_{*}\right)>\rho^{*}$, one has

$$
v^{\prime \prime}\left(r^{*}\right)=\lambda n\left[-f^{\prime}\left(\varphi^{-1}\left(v\left(r^{*}\right)\right)+f\left(\varphi^{-1}(v)\right) H\left(\varphi^{-1}\left(v\left(r^{*}\right), r^{*}\right)\right]>0\right.\right.
$$

as a consequence of (8). But then $v\left(r^{*}\right)$ can not be the global maximum, this is a contradiction. The case $r^{*}=0$ is studied analogously, with the difference that the second term of the left-hand side of (11) presents the indeterminate limit $0 / 0$ when $r \rightarrow 0^{+}$. We can solve it easily by L'Hôpital rule and the limit is $v^{\prime \prime}\left(0^{+}\right)$, and we conclude as before.

Hence, we have proved that $v\left(r_{*}\right) \leq \rho^{*}$. A totally analogous argument shows that $v(r) \geq \rho_{*}$, using now (9).

Finally, we derive a bound for the derivative of the fixed points, by using the same idea of [7, Lemma 3].

Lemma 4.2. There exists $R_{0}>0$ (depending on $f, H$ ) such that if $0<R<R_{0}$, there exists $\gamma^{*}<1$ such that for each $\lambda \in[0,1]$ and each possible fixed point $v$ of $\mathcal{G}(\cdot, \lambda)$, one has

$$
\max _{r \in[0, R]}\left|v^{\prime}(r)\right| \leq \gamma^{*}
$$

Proof. Let us define

$$
\begin{gathered}
M=\max \left\{\left|f^{\prime}\left(\varphi^{-1}(v)\right)\right|: v \in\left[\rho_{*}, \rho^{*}\right]\right\}, \\
N_{R}=\max \left\{f\left(\varphi^{-1}(v)\right)\left|H\left(\varphi^{-1}(v), r\right)\right|: v \in\left[\rho_{*}, \rho^{*}\right], r \in[0, R]\right\} .
\end{gathered}
$$

We fix $R_{0}=1 / M$.
Now, recall that a fixed point of $\mathcal{G}(\cdot, \lambda)$ verifies (6), then integrating both members from 0 to $r$ and using the boundary conditions, we get

$$
r^{n-1} \phi\left(v^{\prime}(r)\right)=\lambda F[v](r) .
$$

If $\left|v^{\prime}(\rho)\right|=\max _{r \in[0, R]}\left|v^{\prime}(r)\right|=\gamma<1$, we get,

$$
\rho^{n-1} \frac{\left|v^{\prime}(\rho)\right|}{\sqrt{1-\left|v^{\prime}(\rho)\right|^{2}}} \leq\left[\frac{M}{\sqrt{1-\left|v^{\prime}(\rho)\right|^{2}}}+N_{R}\right] \rho^{n} .
$$

As we can assume, without loss of generality, that $\rho \in(0, R)$, we obtain

$$
\gamma<R\left[M+N_{R} \sqrt{1-\gamma^{2}}\right] .
$$

Since $R<R_{0}$ means $R M<1$, solving this inequality we obtain a fixed $\gamma^{*}<1$ such that $\gamma<\gamma^{*}$. The result is proved with $R_{0}=1 / M$.

Now that some a priori bounds are stated, the proof of Theorem 2.1 follows from a standard degree computation. The argument is completely analogous to the one exposed in [7], so we just include here an outline for completeness. The homotophy $\mathcal{G}(\cdot, \lambda)$ is well-defined on the domain

$$
\Omega=\left\{v \in C^{1}([0, R]): \rho_{*}<v<\rho^{*},\left\|v^{\prime}\right\|_{\infty}<\gamma^{*}\right\}
$$

and by the homotopy invariance of Leray-Schauder degree

$$
d_{L S}[I-\mathcal{G}(\cdot, 1), \Omega, 0]=d_{L S}[I-\mathcal{G}(\cdot, 0), \Omega, 0] .
$$

Now, the reduction theorem of Leray-Schauder degree (see for instance [5, Proposition II.12], with $L=I$ ) implies that

$$
d_{L S}[I-\mathcal{G}(\cdot, 0), \Omega, 0]= \pm d_{B}\left[g,\left(\rho_{*}, \rho^{*}\right), 0\right]
$$

where $d_{B}$ is the Bouwer degree and $g: J \rightarrow \mathbb{R}$ is the continuous mapping defined by

$$
g(c)=\int_{0}^{R} n r^{n-1}\left[-f^{\prime}\left(\varphi^{-1}(c)\right)+f\left(\varphi^{-1}(c)\right) H\left(\varphi^{-1}(c), r\right)\right] d r .
$$

Noting that $g\left(\left(\rho_{*}\right)<0<g\left(\rho^{*}\right)\right.$, then $d_{B}\left[g,\left(\rho_{*}, \rho^{*}\right), 0\right]=1$ and the proof is done.

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# On the existence of nontrivial solutions of differential equations subject to linear constraints 

Klaus Schmitt

This paper is a birthday present for Jean Mawhin, my dear friend and valued collaborator of many years. Greetings and all the best wishes from afar


#### Abstract

The purpose of this paper is to consider boundary value problems for second order ordinary differential equations where the solutions sought are subject to a host of linear constraints (such as multipoint constraints) and to present a unifying framework for studying such. We show how Leray-Schauder continuation techniques may be used to obtain existence results for nontrivial solutions of a variety of nonlinear second order differential equations. A typical example may be found in studies of the four-point boundary value problem for the differential equation $y^{\prime \prime}(t)+a(t) f(y(t))=0$ on $[0,1]$, where the values of $y$ at 0 and 1 are each some multiple of $y(t)$ at two interior points of $(0,1)$. The techniques most often used in such studies have their origins in fixed point theory. By embedding such problems into parameter dependent ones, we show that detailed information may be obtained via global bifurcation theory. Of course, such techniques, as they are consequences of properties of the topological degree, are similar in nature.


Keywords: second order ode's; nonlinear multi-point boundary value problem; linear constraints; global bifurcation. MS Classification 2010: 34B10, 34B15, 34B18.

## 1. Introduction

This paper is motivated by the paper [15] and several related ones (e.g. [7, $8,16,21,42,43,45]$ ), where the authors were interested in the existence of positive solutions of second-order nonlinear differential equations

$$
\begin{equation*}
y^{\prime \prime}(t)+a(t) f(y(t))=0, \quad 0<t<1 \tag{1}
\end{equation*}
$$

subject to the four-point boundary conditions

$$
\begin{equation*}
y(0)=\alpha y(\xi), \quad y(1)=\beta y(\eta) \tag{2}
\end{equation*}
$$

where $0<\xi \leq \eta<1, a(t)$ is a nonzero continuous, and nonnegative function on $(0,1)$ and

$$
f: \mathbb{R} \rightarrow \mathbb{R}, f:[0, \infty) \rightarrow[0, \infty)
$$

is continuous, or other similar multi-point boundary value problems. In case $\xi=\eta$ and $\alpha+\beta \neq 2$, boundary conditions (2) were already considered Loud in [22], where Green's functions and their properties of such multi-point boundary value problems and their adjoints were discussed in great detail.

Under the assumption that the limits

$$
\begin{equation*}
f_{0}=\lim _{u \rightarrow 0} \frac{f(u)}{u}, \quad f_{\infty}=\lim _{u \rightarrow \infty} \frac{f(u)}{u} . \tag{3}
\end{equation*}
$$

exist and satisfy certain inequalities, it was proved [15] that (1), (2) has a positive solution. The proof was based on a use of the Krasnosel'skii compression and expansion theorems for positive completely continuous operators on a Banach space [14]. Results for the existence of solutions of nonlinear boundary value problems where the nonlinear terms behave as in (3) have a long history and such results (usually for boundary value problems subject to homogeneous end point boundary conditions, but also valid for nonlinear elliptic partial differential equations) may be found in [1, 2, 6, 9, 26, 27, 28, 44]. While the boundary conditions (2) are very much much different from those usually employed, such as Dirichlet, Neumann, Robin, or periodic ones, it is still straight forward to transform the problems into equivalent integral equations (cf. $[7,8,13,15,16,21,23,24,39,40,45]$ ) and thus employ fixed point theory for completely continuous operators on a Banach space of continuous functions. Further studies are also available for problems defined on time scales, see e.g. [3, 12, 46], among others.

Since the approach used here is variational and uses global bifurcation theory, the results and approach discussed here for the semilinear case should be extendable to problems of a nonlinear nature for both ordinary and elliptic partial differential equations, such as problems involving the p-Laplacian, and obtain results as in [17, 18, 25, 33].

In this paper we shall discuss a class of nonlinear boundary value problems and show, using global bifurcation techniques ([4, 30, 31, 32]), how solutions may be obtained as part of a continuum of solutions of a problem which depends upon a parameter into which the given problem has been imbedded. We shall adhere here to a prototypical example motivated by (1), (2) but want to point out that similar arguments may be used to obtain results of this type for semilinear and nonlinear elliptic problems in higher dimensions using, see e.g. [17]. We shall not attempt to consider these more general situations here, but remark that some of the work cited here will provide the tools for studying such problems.

## 2. Notation, assumptions, and preliminaries

We let $V$ be a closed subspace of $H^{1}(0,1)$ which has the property that 0 is the only constant function that belongs to $V$ and in addition that there exists an open set

$$
\Omega \subset(0,1), \text { such that } \bar{\Omega}=[0,1], m(\Omega)=1, C_{0}^{\infty}(\Omega) \subset V,
$$

(here $m(\cdot)$ denotes Lebesgue measure).
For example, if $L: H^{1}(0,1) \rightarrow \mathbb{R}^{2}$ is defined by the boundary conditions (2) as

$$
L y:=(y(0)-\alpha y(\xi), y(1)-\beta y(\eta)), 0<\xi \leq \eta<1, \alpha \neq 1
$$

then

$$
V:=\left\{u \in H^{1}(0,1): L u=0\right\}
$$

is such a subspace with

$$
\Omega:=(0, \xi) \cup(\xi, \eta) \cup(\eta, 1) .
$$

For other examples of operators $L$ defined by multipoint boundary conditions, we refer the interested reader to $[7,8,15,16]$, and the references in these papers and those in the other references given above. Of course, homogeneous Dirichlet and anti periodic boundary conditions $(y(0)=-y(1))$ yield such examples, as do the boundary conditions

$$
u(0)=0
$$

or

$$
u(0)=0, u(\eta)=\alpha u(1), \eta \in(0,1)
$$

or

$$
\alpha u(\eta)+\beta u(\mu)=u(1), 0<\eta<\mu<1, \alpha, \beta \geq 0, \alpha+\beta<1
$$

whereas classical Neumann and periodic boundary conditions do not (note that these boundary conditions are natural ones imposed by minimization problems in $H^{1}(0,1)$, respectively in $\left.\left\{u \in H^{1}(0,1): u(0)=u(1)\right\}\right)$.

The norm of $H^{1}(0,1)$ is given by

$$
\|u\|_{H^{1}}^{2}=\int_{0}^{1}\left(u^{\prime}\right)^{2} d t+\int_{0}^{1} u^{2} d t
$$

and it is the case that

$$
\|u\|^{2}:=\int_{0}^{1}\left(u^{\prime}\right)^{2} d t
$$

defines an equivalent norm on such subspaces $V$, i.e., there exists a positive constant $c$ such that

$$
\|u\|_{L^{2}(0,1)} \leq c\left\|u^{\prime}\right\|_{L^{2}(0,1)}, \forall u \in V .
$$

To see this, one may use an often employed argument of Nečas [20], and assume there exists a sequence $\left\{u_{n}\right\} \subset V$ such that

$$
\begin{equation*}
\left\|u_{n}\right\|_{L^{2}(0,1)} \geq n\left\|u_{n}^{\prime}\right\|_{L^{2}(0,1)}, n=1,2, \cdots \tag{4}
\end{equation*}
$$

Then we may assume that $\left\|u_{n}\right\|_{L^{2}(0,1)}=1, n=1,2, \cdots$. So $\left\{u_{n}\right\}$ is bounded in $H^{1}(0,1)$, hence may assumed to converge weakly to say $u$. Hence it will converge strongly to $u$ in $L^{2}(0,1)$. So, by (4) $u_{n}^{\prime} \rightarrow 0$ in $L^{2}(0,1)$, which implies that $u^{\prime}=0$, i.e. $u$ must be piecewise constant, but since $u$ is continuous, it must be a constant throughout. On the other hand, $V$ is closed and hence, since $u \in V, u$ must equal 0 , a contradiction.

Definition 2.1. For given $V$, as above, we let $V^{\prime}$ denote its topological dual and for $h \in V^{\prime}$, we call $u \in V$ a weak solution of the boundary value problem

$$
\begin{equation*}
-u^{\prime \prime}=h, u \in V, \tag{5}
\end{equation*}
$$

provided that

$$
\begin{equation*}
\int_{0}^{1} u^{\prime} v^{\prime} d t=(h, v), \forall v \in V \tag{6}
\end{equation*}
$$

where

$$
(\cdot, \cdot): V^{\prime} \times V \rightarrow \mathbb{R}
$$

is the pairing between $V^{\prime}$ and $V$.
The above considerations have the following immediate consequence, whose proof follows from the Lax-Milgram theorem (see [38]) and the fact that $V$ is a Hilbert space with respect to the inner product

$$
(u, v)_{V}:=\int_{0}^{1} u^{\prime} v^{\prime} d t
$$

Lemma 2.2. Let $V$ be as above, and let $V^{\prime}$ be its topological dual, then for every $h \in V^{\prime}$ there exists a unique $u \in V$ which is a unique weak solution of (5). Further

$$
\|u\| \leq\|h\|_{V^{\prime}}
$$

Remark 2.3. If $h \in L^{2}(0,1)$, we may deduce that the weak solution $u$, given above, since $C_{0}^{\infty}(\Omega) \subset V$, must satisfy

$$
u^{\prime \prime}=h, \text { on } \Omega
$$

in the sense of distributions and thus $u$ is a solution of the differential equation on $\Omega$, further, since $m(\Omega)=1$, it follows that $u$ is a solution on $(0,1)$, as well.

Let $a:[0,1] \rightarrow[0, \infty)$ be a continuous nontrivial function, then, via this lemma, we may define the mapping

$$
T: L^{2}(0,1) \rightarrow V \subset H^{1}(0,1) \hookrightarrow L^{2}(0,1)
$$

by

$$
T h:=u
$$

where $u$ is the unique weak solution of

$$
\begin{equation*}
-u^{\prime \prime}=a h, u \in V \tag{7}
\end{equation*}
$$

and hence $u$ solves the differential equation (7) on $\Omega$ in a classical sense (viz. $\left.C_{0}^{\infty}(\Omega) \subset V\right)$. We note that the last inclusion is compact. Thus,

$$
T: L^{2}(0,1) \rightarrow L^{2}(0,1)
$$

is compact linear mapping. Thus, we have that

$$
T: C[0,1] \rightarrow H^{2}(0,1) \hookrightarrow C^{1}[0,1] \hookrightarrow C[0,1]
$$

i.e., we may even view $T$ as a compact linear mapping

$$
T: C[0,1] \rightarrow C[0,1]
$$

and we may apply the Riesz theory for compact linear operators to obtain the spectral properties of this operator. For general multi-point boundary value problems, the study of the spectrum of the associated integral operator, has a long history, with notable contributions in [22], and recently in [5]. In fact, since the problems, in general are not self-adjoint, complex eigenvalues may exist. In the case at hand, we shall not be concerned with such complications but rather concentrate on boundary conditions (subspaces $V$ ) which have one distinguished positive eigenvalue (see below), namely a smallest positive one, called $\lambda_{1}$.
Remark 2.4. Since there exists $u \in V \backslash\{0\}$, such that

$$
T u=\frac{1}{\lambda_{1}} u
$$

we have that

$$
\int_{0}^{1} u^{\prime} v^{\prime} d t=\lambda_{1} \int_{0}^{1} a u v d t, \forall v \in V
$$

we obtain that (by normalizing)

$$
0<\lambda_{1}=\inf _{V}\left\{\int_{0}^{1}\left(v^{\prime}\right)^{2} d t: \int_{0}^{1} a v^{2} d t=1\right\}
$$

In the given generality not much else may be asserted concerning the spectrum of $T$. In fact, the first example below shows that the principal eigenvalue may be of multiplicity 2 .

Example 2.5. a. Let the space $V$ be defined by

$$
V:=\left\{u \in H^{1}(0,1): u(0)=u(1), \int_{0}^{1} u d t=0\right\} .
$$

Then $V$ is a closed subspace with 0 the only constant function. In the case that $a \equiv 1$, the eigenfunctions of the operator $T$ satisfy

$$
\int_{0}^{1} u^{\prime} v^{\prime} d t=\lambda_{1} \int_{0}^{1} u v d t, \forall v \in V
$$

and, since $H_{0}^{1}(0,1) \subset V$ we have that

$$
-u^{\prime \prime}=\lambda_{1} u
$$

in the sense of distributions. Integrating the last equality we obtain that (since $\left.u \in H^{2}(0,1)\right)$

$$
u^{\prime}(0)=u^{\prime}(1),
$$

and so $u$ is an eigenfunction of

$$
-u^{\prime \prime}=\lambda_{1} u, u(0)=u(1), u^{\prime}(0)=u^{\prime}(1)
$$

i.e. $\lambda_{1}=4 \pi^{2}$, with an associated 2-dimensional eigenspace.
b. Let the space $V$ be defined by

$$
V:=\left\{u \in H^{1}(0,1): \int_{0}^{1} u d t=0\right\} .
$$

Then, again, $V$ is a closed subspace with 0 the only constant function. With $a \equiv 1$, the eigenfunctions of the operator $T$ satisfy

$$
\int_{0}^{1} u^{\prime} v^{\prime} d t=\lambda_{1} \int_{0}^{1} u v d t, \forall v \in V
$$

and, since $H_{0}^{1}(0,1) \subset V$ we have that

$$
\begin{equation*}
-u^{\prime \prime}=\lambda_{1} u \tag{8}
\end{equation*}
$$

in the sense of distributions. Multiplying the equality (8) by $v \in V$ and integrating, we obtain that

$$
-u^{\prime}(1) v(1)+u^{\prime}(0) v(0)+\int_{0}^{1} u^{\prime} v^{\prime} d t=\lambda_{1} \int_{0}^{1} u v d t
$$

and hence, choosing $v$ such that $v(0)=v(1) \neq 0$ we obtain

$$
u^{\prime}(0)=u^{\prime}(1) .
$$

Further, choosing $v(0)=0, v(1) \neq 0$, we must have $u^{\prime}(0)=0$. Hence $u$ is an eigenfunction of the Neumann problem

$$
-u^{\prime \prime}=\lambda_{1} u, u^{\prime}(0)=u^{\prime}(1)=0
$$

i.e. $\lambda_{1}=\pi^{2}$, the second eigenvalue of the Neumann problem with an associated 1 -dimensional eigenspace, spanned by $u(t)=\cos \pi t$.

Both of the above examples, of course, are examples of classical SturmLiouville boundary value problems, where, because of the constraints built into the space $V$, the eigenvalue $\lambda_{1}$ is actually the second eigenvalue of the problem (8) with respect to either periodic or Neumann boundary conditions in the space $H^{1}(0,1)$.

Next let us consider the example, related to (1)

$$
\begin{equation*}
-u^{\prime \prime}(t)=\lambda a(t) u, \quad 0<t<1 \tag{9}
\end{equation*}
$$

subject to the four-point boundary conditions

$$
\begin{equation*}
u(0)=\alpha u(\xi), \quad u(1)=\beta u(\eta), 0<\xi<\eta<1 \tag{10}
\end{equation*}
$$

where, as above, $a:[0,1] \rightarrow[0, \infty)$ is a continuous function assuming positive values.

Proposition 2.6. Assuming that

$$
0<\alpha, \beta<1
$$

then the principal (weak) eigenvalue of (9), (10) is positive, simple, and has an associated eigenfunction which is positive in [0,1]. All other eigenvalues are simple, as well, and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the $L^{2}$ inner product with weight function $a$.

Proof. In this case we define

$$
V=\left\{u \in H^{1}(0,1): u(0)=\alpha u(\xi), \quad u(1)=\beta u(\eta)\right\}
$$

Then $V$ is a closed subspace of $H^{1}(0,1)$ with $C_{0}^{\infty}((0, \xi) \cup(\xi, \eta) \cup(\eta, 1))$ dense in $V$. The principal (weak) eigenvalue is characterized by

$$
0<\lambda_{1}=\inf _{v \in V}\left\{\int_{0}^{1}\left(v^{\prime}\right)^{2} d t: \int_{0}^{1} a v^{2} d t=1\right\}
$$

furthermore this infimum is assumed, by, say $u \in V$, and $u$ satisfies

$$
\int_{0}^{1} u^{\prime} v^{\prime} d t=\lambda_{1} \int_{0}^{1} a u v d t, \forall v \in V
$$

Since, for $v \in V$, we have that $|v| \in V$ and since

$$
0<\lambda_{1}=\inf _{v \in V}\left\{\int_{0}^{1}|v|^{\prime 2} d t: \int_{0}^{1} a|v|^{2} d t=1\right\}
$$

we may assume that the eigenfunction $u$ is one signed, say $u \geq 0$, which implies, because of the boundary conditions that $u>0$ in $[0,1]$. Hence, again because of the boundary conditions, and, since

$$
-u^{\prime \prime}=\lambda_{1} a u
$$

$u$ will assume its maximum in the interval $[\xi, \eta]$. If $v$ is any other eigenfunction corresponding to $\lambda_{1}$, we may assume $v(0) \geq 0$. If $v(0)>0$, we may let $w(t)=$ $\mu v(t)$, where $\mu=\frac{u(0)}{v(0)}$. Then $w$ is an eigenfunction with

$$
w(0)=u(0)
$$

and hence

$$
z(t):=u(t)-w(t)
$$

is an eigenfunction having zeros at 0 and $\xi$, which by the Sturm Separation Theorem [11] implies that $u$ must vanish in $(0, \xi)$. Thus it must be the case that $w(t) \equiv u(t)$. If, on the other hand, $v(0)=0$, then $v(\xi)=0$, then we again obtain a contradiction by use of the Sturm Separation Theorem.

Next, let $u_{i}$ and $u_{j}$ be eigenfunctions corresponding to the eigenvalues $\lambda_{i}$ and $\lambda_{j}, i \neq j$. Then

$$
\int_{0}^{1} u_{l}^{\prime} v^{\prime} d t=\lambda_{l} \int_{0}^{1} a u_{l} v d t, \quad \forall v \in V, l=i, j
$$

and hence

$$
\int_{0}^{1} u_{i}^{\prime} u_{j}^{\prime} d t=\lambda_{i} \int_{0}^{1} a u_{i} u_{j} d t=\lambda_{j} \int_{0}^{1} a u_{i} u_{j} d t
$$

thus

$$
\left(\lambda_{j}-\lambda_{i}\right) \int_{0}^{1} a u_{i} u_{j} d t=0
$$

## 3. Bifurcating continua

We shall assume that

$$
a:[0,1] \rightarrow[0, \infty), f: \mathbb{R} \rightarrow \mathbb{R}, f:(0, \infty) \rightarrow(0, \infty)
$$

are continuous functions such that $a$ is nontrivial. $V \subset H^{1}(0,1)$ is a subspace with the property that the only constant function in $V$ is the zero function and that the smallest positive eigenvalue $\lambda_{1}$ of

$$
\begin{equation*}
-u^{\prime \prime}(t)=\lambda a(t) u, \quad 0<t<1, u \in V \tag{11}
\end{equation*}
$$

is simple and has an associated eigenfunction which is positive in $(0,1)$. This assumption holds, for example (among others), in the cases of the boundary conditions imposed in the various papers cited and related work (cf. for example Proposition 2.6).

We now consider the nonlinear problem (1). This problem we shall embed into the problem

$$
\begin{equation*}
-y^{\prime \prime}(t)=\mu a(t) f(y(t)), \quad 0<t<1, y \in V \tag{12}
\end{equation*}
$$

We shall prove that, under assumptions on $f$, spelled out below, a continuum of positive solutions (in the space $\mathbb{R} \times C[0,1]$ ) exists which crosses the hyperplane $\{1\} \times C[0,1]$ and thus conclude that the problem

$$
\begin{equation*}
-y^{\prime \prime}(t)=a(t) f(y(t)), \quad 0<t<1, y \in V \tag{13}
\end{equation*}
$$

has a nontrivial solution. To this end, let

$$
\begin{equation*}
f_{0}=\lim _{u \rightarrow 0} \frac{f(u)}{u}, \quad f_{\infty}=\lim _{u \rightarrow \infty} \frac{f(u)}{u} \tag{14}
\end{equation*}
$$

We have the following theorem.
Theorem 3.1. Let $V$ be as above and assume that the limits in (14) exist and satisfy

$$
\begin{equation*}
0<f_{0}<\lambda_{1}<f_{\infty} \tag{15}
\end{equation*}
$$

or

$$
\begin{equation*}
0<f_{\infty}<\lambda_{1}<f_{0} \tag{16}
\end{equation*}
$$

Then the boundary value problem (13) has a solution $y$ which is positive in $(0,1)$.

Proof. We consider the problem (12) and apply the global bifurcation theorem of Krasnosels'kii-Rabinowitz, see [30, 32], which guarantees the existence of an unbounded continuum $\mathbb{C}:=\{(\mu, y)\} \subset \mathbb{R} \times C[0,1]$ with the solution component $y$ such that $y(t)>0, t \in(0,1)$, which bifurcates from the trivial solution at the bifurcation point $\left(\mu f_{0}, 0\right)=\left(\lambda_{1}, 0\right)$ (while the application of the global bifurcation theorem also allows for the alternative that the continuum might bifurcate from another eigenvalue, this alternative may be quickly ruled out by refering to Proposition 2.6). One may further show (using arguments as in $[26,27]$ ) that the continuum $\mathbb{C}$ is bounded in the $\mu$ - direction and hence
must become unbounded in some bounded $\mu$-interval, i.e., it will bifurcate from infinity in that interval. Using results about bifurcation from infinity as in $[31,34,35,37]$, we deduce that bifurcation from infinity will take place at $\mu f_{\infty}=\lambda_{1}$. Therefore the continuum $\mathbb{C}$, projected onto the $\mu$-axis $=\mathbb{R}$ will include the open interval determined by the values $\frac{\lambda_{1}}{f_{0}}$ and $\frac{\lambda_{1}}{f_{\infty}}$. This open interval will contain the value $\mu=1$, if either (15) or (16) hold.

The above result and its proof may be extended to the following:
THEOREM 3.2. Under the same assumptions on the subspace $V$, assume that

$$
\begin{equation*}
0=f_{0}<\lambda_{1}<f_{\infty} \tag{17}
\end{equation*}
$$

or

$$
\begin{equation*}
0=f_{\infty}<\lambda_{1}<f_{0} \tag{18}
\end{equation*}
$$

Then the boundary value problem (13) has a solution $y$ which is positive in $(0,1)$.

Proof. In the case of (17) there will be no bifurcation from the trivial solution, however, bifurcation from infinity will take place at $\mu=\frac{\lambda_{1}}{f_{\infty}}$ with the corresponding continuum existing for all values of $\mu>\frac{\lambda_{1}}{f_{\infty}}$, and hence (13) will have a positive solution, whereas in the case (18), bifurcation from the trivial solution occurs at $\mu=\frac{\lambda_{1}}{f_{0}}$, with the continuum existing for all values $\mu>\frac{\lambda_{1}}{f_{0}}$.

Global bifurcation theory may also be applied at simple eigenvalues $\lambda_{j}>\lambda_{1}$, and various results may be formulated using the ideas used above; here it will be important again that bifurcating continua are global, which will follow from nodal properties of solutions inherited by the nodal properties of the eigenfunctions of the associated linearized problems.

## 4. Concluding Remarks

Remark 4.1. The methods developed in [26, 27, 28] may be employed to study various multi-point and nonlocal boundary value problems involving nonlinear terms $f$ different from those considered above, as long as solution branches of positive solutions may be found which exist globally and can be shown to cross the appropriate parameter hyperplane. To this end we refer to [40, 41], where fixed point techniques have been used.

REmARK 4.2. If we replace, in (2), one of boundary conditions by, say, the following

$$
\begin{equation*}
y(0)=\alpha y(\xi)+b \tag{19}
\end{equation*}
$$

one obtains a problem from a class of problems studied in [43]. Here one may view $b$ as a parameter and then employ homotopy continuation techniques,
as done in [10], to obtain parameter intervals for the parameter $b$, for which solutions may be shown to exist.

Remark 4.3. The interested reader might wish to revisit the example (1), i.e.

$$
y^{\prime \prime}(t)+a(t) f(y(t))=0, \quad 0<t<1
$$

subject to the three-point boundary conditions

$$
y(0)=\alpha y\left(\frac{1}{2}\right), \quad y(1)=\beta y\left(\frac{1}{2}\right)
$$

in case $a \equiv 1$ and do the necessary computations to find that if $|\alpha+\beta|<2$, then positive real eigenvalues exist having the properties required above, whereas if $\alpha+\beta=2$, the problem is in fact in resonance (c.f. also [22], where it has been shown that only if $\alpha+\beta \neq 2$, a Green's function may be computed) and if $|\alpha+\beta|>2$, no real eigenvalues exist. In the case that real eigenvalues exist, the principal eigenvalue $\lambda_{1}$ is given by

$$
\lambda_{1}=4 \mu_{1}^{2},
$$

where $\mu_{1}$ is the smallest positive solution of

$$
\cos \mu=\frac{\alpha+\beta}{2}
$$

Another interesting example is obtained for the same nonlinear differential equation which is subject to boundary conditions such as

$$
u(0)=\int_{0}^{\frac{1}{2}} u(s) d s
$$

(see also [41], where similar boundary conditions are considered).
Remark 4.4. For problems at resonance, such as the example in the previous remark, when $\alpha+\beta=2$, continuation arguments based on Mawhin's continuation theorem, as was done in [29], may be used to establish existence criteria for such multi-point boundary value problems.

REmark 4.5. A useful tool to study boundary value problems for nonlinear elliptic equations has been the method of sub-supersolutions. In this regard we refer to [19], where such a theory has been developed for general variational inequalities, and hence may be applied to multi-point and nonlocal boundary value problems of the types discussed here. These methods not only apply for semilinear but nonlinear problems, as well. Here also the variational eigenvalue theory as presented in [17] may be useful.

REMARK 4.6. In the case of multi-point or nonlocal boundary value problems for elliptic partial differential equations, these problems may be formulated as variational inequalities (actually equalities, since $V$ is a subspace). Problems involving nonlinear terms $f$, as above, may then be analyzed using bifurcation techniques as presented in [18].

Remark 4.7. If it is the case that either of the the limits (3) does not exist, but the quotients lie asymptotically in certain non overlapping intervals, ideas, as developed in [36], may be used to develop analogous existence results for such nondifferentiable nonlinear problems.

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# Flat solutions of the 1-Laplacian equation 

Luigi Orsina and Augusto C. Ponce

$\grave{A}$ Jean Mawhin, dont l'enthousiasme et l'amour pour les mathématiques font encore rêver les nouvelles générations


#### Abstract

For every $f \in L^{N}(\Omega)$ defined in an open bounded subset $\Omega$ of $\mathbb{R}^{N}$, we prove that a solution $u \in W_{0}^{1,1}(\Omega)$ of the 1-Laplacian equation $-\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)=f$ in $\Omega$ satisfies $\nabla u=0$ on a set of positive Lebesgue measure. The same property holds if $f \notin L^{N}(\Omega)$ has small norm in the Marcinkiewicz space of weak- $L^{N}$ functions or if $u$ is a $B V$ minimizer of the associated energy functional. The proofs rely on Stampacchia's truncation method.


Keywords: 1-Laplacian, degenerate elliptic equations, nonlinear elliptic equation, nonexistence of solution. MS Classification 2010: 35J70, 35J25, 35J62, 35J92.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^{N}$ be a smooth bounded open subset. Given a convex function $\Phi: \mathbb{R}^{N} \rightarrow \mathbb{R}$ and $f \in L^{1}(\Omega)$, consider the energy functional

$$
E_{\Phi}(u)=\int_{\Omega} \Phi(\nabla u)-\int_{\Omega} f u
$$

defined on some class of functions $u: \Omega \rightarrow \mathbb{R}$ for which the integrands are summable. Although $\Phi$ need not be smooth, one can express the EulerLagrange equation of $E_{\Phi}$ using the subdifferential of $\Phi$. Indeed, by convexity of $\Phi$, at each point $x \in \mathbb{R}^{N}$ there exists a subgradient $\xi \in \mathbb{R}^{N}$ such that

$$
\Phi(y) \geq \Phi(x)+\xi \cdot(y-x)
$$

for every $y \in \mathbb{R}^{N}$; see [18, Chapter 2]. Denoting the collection of all subgradients $\xi$ at $x$ by $\partial \Phi(x)$, one can then write the Euler-Lagrange equation of $E_{\Phi}$ at some function $u$ as (see [12, Chapter IV] and [22])
$-\operatorname{div} Z=f \quad$ in the sense of distributions in $\Omega$,
where $Z$ is a summable function with values in $\mathbb{R}^{N}$ such that

$$
\begin{equation*}
Z \in \partial \Phi(\nabla u) \quad \text { almost everywhere in } \Omega . \tag{2}
\end{equation*}
$$

For example, if $\Phi_{p}(x)=|x|^{p} / p$ for some exponent $p>1$, then $\Phi_{p}$ is differentiable pointwise. Thus, $\partial \Phi_{p}(x)=\left\{|x|^{p-2} x\right\}$, and one recovers an equation involving the $p$-Laplace operator:

$$
-\Delta_{p} u=-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)=f .
$$

When $p=1$, the function $\Phi_{1}$ is not differentiable at 0 , and one should be careful about the meaning of the quotient $\nabla u /|\nabla u|$ that appears in the formal notation of the 1-Laplacian. The correct interpretation is based on the formalism of subdifferentials above. Indeed, for $\Phi_{1}(x)=|x|$, one has

$$
\partial \Phi_{1}(x)= \begin{cases}\overline{B_{1}(0)} & \text { if } x=0,  \tag{3}\\ \{x /|x|\} & \text { if } x \neq 0,\end{cases}
$$

where $B_{1}(0)$ denotes the unit open ball in $\mathbb{R}^{N}$.
The vector field $Z$ in the Euler-Lagrange equation now satisfies the conditions

$$
|Z| \leq 1 \quad \text { and } \quad Z|\nabla u|=\nabla u
$$

almost everywhere in $\Omega$. Observe that, in dimension 1, equation (3) provides one with the maximal monotone graph associated to the sign function.

Assuming that $f \in L^{N}(\Omega)$, the functional $E_{\Phi_{1}}$ associated to $\Phi_{1}$ is welldefined in $W_{0}^{1,1}(\Omega)$, and the Euler-Lagrange equation (1)-(2) is indeed satisfied by a minimizer. The goal of this paper is to show that one cannot abandon the vector field $Z$ and replace it by the quotient $\nabla u /|\nabla u|$ since the gradient $\nabla u$ must vanish on a set of positive Lebesgue measure.

Every function $u \in W^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in $\Omega$ has a legitimate 1-Laplacian $\Delta_{1} u$ defined in the sense of distributions as

$$
\left\langle\Delta_{1} u, \varphi\right\rangle:=-\int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi,
$$

for every test function $\varphi \in C_{c}^{\infty}(\Omega)$ with compact support in $\Omega$, but even for smooth functions $u$ something strange happens near an interior extremum point:
Example 1.1. For every $N \geq 1$, let $u: B_{1}(0) \rightarrow \mathbb{R}$ be the function defined by $u(x)=1-|x|^{2}$. In the sense of distributions we have, for $N=1$,

$$
-\Delta_{1} u=2 \delta_{0},
$$

while for $N \geq 2$,

$$
-\Delta_{1} u=\frac{N-1}{|x|} .
$$

In the previous example, the topological singularity of the vector field $-x /|x|$ is detected by its divergence, and the 1-Laplacian does not belong to $L^{N}(\Omega)$. We show that this is a general fact that holds for Sobolev functions, not necessarily smooth:
Theorem 1.2. There exists no function $u \in W_{0}^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in $\Omega$ and

$$
\Delta_{1} u \in L^{N}(\Omega)
$$

In Example 1.1 above for $N \geq 2$, one sees that the right-hand side belongs to the Marcinkiewicz space $\mathcal{M}^{N}(\Omega)$ of weak- $L^{N}$ functions $f$ in $\Omega$ equipped with the seminorm

$$
\|f\|_{\mathcal{M}^{N}(\Omega)}=\sup _{A \subset \Omega} \frac{1}{|A|^{\frac{N-1}{N}}} \int_{A}|f|
$$

where $|A|$ denotes the Lebesgue measure of $A$ and the supremum is computed with respect to every Borel subset of $\Omega$. In the case of the example, the function $f=(N-1) /|x|$ satisfies

$$
\begin{equation*}
\|f\|_{\mathcal{M}^{N}(B(0 ; 1))}=N \omega_{N}^{1 / N} \tag{4}
\end{equation*}
$$

where $\omega_{N}$ denotes the volume of the unit ball in $\mathbb{R}^{N}$.
A variant of the proof of Theorem 1.2 based on Peetre-Alvino's imbedding of $W^{1,1}\left(\mathbb{R}^{N}\right)$ in the Lorentz space $L^{\frac{N}{N-1}, 1}\left(\mathbb{R}^{N}\right)$ shows that this quantity (4) is critical for the existence of flat levels of solutions involving the 1-Laplacian:

Theorem 1.3. Let $N \geq 2$. There exists no function $u \in W_{0}^{1,1}(\Omega)$ such that $\nabla u \neq 0$ a.e. in $\Omega$,

$$
\Delta_{1} u \in \mathcal{M}^{N}(\Omega) \quad \text { and } \quad\left\|\Delta_{1} u\right\|_{\mathcal{M}^{N}(\Omega)}<N \omega_{N}^{1 / N}
$$

Theorems 1.2 and 1.3 are related to the degenerate limit behavior of solutions of the $p$-Laplacian equation as $p$ tends to 1 that has been studied by several authors; see e.g. [9, 20, 21], starting with the pioneering work of Kawohl [15], and also clarify the need for relying on the vector field $Z$ in replacement of $\nabla u /|\nabla u|$.

Example 1.4. For any $0<r<1$, let $u: B_{1}(0) \rightarrow \mathbb{R}$ be the function defined by

$$
u(x)= \begin{cases}1-|x|^{2} & \text { if }|x| \geq r \\ 1-r^{2} & \text { if }|x|<r\end{cases}
$$

Then, $u \in W_{0}^{1,1}\left(B_{1}(0)\right)$. If $Z: B_{1}(0) \rightarrow \overline{B_{1}(0)}$ is any smooth extension of the function

$$
x \in B_{1}(0) \backslash B_{r}(0) \longmapsto-\frac{x}{|x|} \in \mathbb{R}^{N}
$$

then $u$ and $Z$ satisfy the Euler-Lagrange equation (1)-(2) for some function $f \in L^{\infty}\left(B_{1}(0)\right)$.

Observe that the Sobolev space $W_{0}^{1,1}(\Omega)$ is not the natural setting for looking for minimizers of $E_{\Phi_{1}}$, due to the lack of reflexivity of $L^{1}\left(\Omega ; \mathbb{R}^{N}\right)$. This is in contrast to minimization problems in $W^{1, p}(\Omega)$ for $1<p<+\infty$ which can be investigated using techniques based on the uniform convexity of the space; see [11].

Let us assume that $E_{\Phi_{1}}$ is bounded from below for some given $f \in L^{N}(\Omega)$. This is the case for example if the norm $\|f\|_{L^{N}(\Omega)}$ is small, depending on the Sobolev constant; see e.g. [16]. One can now take a minimizing sequence $\left(u_{n}\right)_{n \in \mathbb{N}}$ in $W_{0}^{1,1}(\Omega)$ such that

$$
\lim _{n \rightarrow \infty} E_{\Phi_{1}}\left(u_{n}\right)=\inf _{W_{0}^{1,1}(\Omega)} E_{\Phi_{1}} .
$$

Each function $u_{n}$, extended by zero to $\mathbb{R}^{N}$, is an element of $W^{1,1}\left(\mathbb{R}^{N}\right)$. Since the sequence $\left(\nabla u_{n}\right)_{n \in \mathbb{N}}$ is bounded in $L^{1}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right)$, we can extract a subsequence $\left(\nabla u_{n_{k}}\right)_{k \in \mathbb{N}}$ converging weakly to some finite vector-valued measure in $\mathbb{R}^{N}$ supported in $\bar{\Omega}$. Applying the Rellich-Kondrashov compactness theorem, we deduce that there exists $u \in B V\left(\mathbb{R}^{N}\right)$ such that $u=0$ in $\mathbb{R}^{N} \backslash \Omega$, and

$$
\lim _{k \rightarrow \infty} E_{\Phi_{1}}\left(u_{n_{k}}\right) \geq \int_{\mathbb{R}^{N}}|D u|-\int_{\Omega} f u .
$$

The limit function $u$ is a minimizer of the extended functional

$$
\begin{equation*}
\bar{E}_{\Phi_{1}}(v):=\int_{\mathbb{R}^{N}}|D v|-\int_{\Omega} f v \tag{5}
\end{equation*}
$$

over the class of functions $v \in B V\left(\mathbb{R}^{N}\right)$ such that $v=0$ in $\mathbb{R}^{N} \backslash \Omega$. Such a functional provides a relaxed formulation of the minimization problem for which a solution exists; see [14]. In the spirit of Theorems 1.2 and 1.3, minimizers of (5) must have flat level sets:

Theorem 1.5. Let $f \in L^{N}(\Omega)$ and let $u \in B V\left(\mathbb{R}^{N}\right)$ with $u=0$ in $\mathbb{R}^{N} \backslash \Omega$ be a minimizer of the extended functional $\bar{E}_{\Phi_{1}}$. Then, $u \in L^{\infty}\left(\mathbb{R}^{N}\right)$ and the set of extremal points

$$
\left\{x \in \mathbb{R}^{N}:|u(x)|=\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}\right\}
$$

has positive Lebesgue measure.
We deduce in this case that the absolute continuous part $D^{\text {a }} u$ of the measure $D u$ vanishes a.e. on a set of positive measure since $D^{\mathrm{a}} u=0$ a.e. on level sets $\{u=\alpha\}$ for every $\alpha \in \mathbb{R}$ [4, Proposition 3.73]. The counterpart of Theorem 1.5 involving the condition $\|f\|_{\mathcal{M}^{N}(\Omega)}<N \omega_{N}^{1 / N}$ is true but uninteresting since $\bar{E}_{\Phi_{1}}$
is nonnegative and 0 is the unique minimizer. This follows from a standard application of Alvino's version of the Sobolev inequality in Lorentz spaces.

Renormalized solutions to equations involving the 1-Laplacian have been introduced in the spirit of the relaxed minimization problem above, but in general such solutions merely have bounded variation or do not satisfy the homogeneous Dirichlet boundary condition [1, 5, $6,8,10,19]$.

Example 1.6 (Remark 3.10 in [19]). For every $N<r \leq R$, the function $u=(1-N / r) \chi_{B_{r}(0)}$ is a renormalized solution of the Dirichlet problem

$$
\left\{\begin{array}{cl}
-\Delta_{1} v=h-v & \text { in } B_{R}(0) \\
v=0 & \text { on } \partial B_{R}(0)
\end{array}\right.
$$

with bounded datum $h=\chi_{B_{r}(0)}$. Note that if $r<R$ then $u_{r}$ is a BV function with compact support in $B_{R}(0)$, while if $r=R$ then $u_{r}$ is a $W^{1,1}$ function which does not vanish on the boundary.

In the next section, we prove Theorems 1.2, 1.3 and 1.5. This paper is a revised and extended version of a note written by the authors in 2012 that was only available at the arxiv.org website.

## 2. Proofs of the main results

Proof of Theorem 1.2. Assume by contradiction that there exists a function $u \in W_{0}^{1,1}(\Omega)$ such that $\nabla u \neq 0$ almost everywhere in $\Omega$ and $f:=\Delta_{1} u \in L^{N}(\Omega)$. Then,

$$
\int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla \varphi=\int_{\Omega} f \varphi
$$

for every $\varphi \in C_{c}^{\infty}(\Omega)$. Note that $\nabla u /|\nabla u| \in L^{\infty}(\Omega)$ and $u \in L^{\frac{N}{N-1}}(\Omega)$ by the Gagliardo-Nirenberg-Sobolev imbedding. By density of $C_{c}^{\infty}(\Omega)$ in $W_{0}^{1,1}(\Omega)$ we deduce that

$$
\begin{equation*}
\int_{\Omega} \frac{\nabla u}{|\nabla u|} \cdot \nabla v=\int_{\Omega} f v \tag{6}
\end{equation*}
$$

for every $v \in W_{0}^{1,1}(\Omega)$.
We proceed using Stampacchia's truncation method. For this purpose, for every $\kappa>0$ let $G_{\kappa}: \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by

$$
G_{\kappa}(t)= \begin{cases}t+\kappa & \text { if } t<-\kappa  \tag{7}\\ 0 & \text { if }-\kappa \leq t \leq \kappa \\ t-\kappa & \text { if } t>\kappa\end{cases}
$$

Since $u \in W_{0}^{1,1}(\Omega)$, we have $G_{\kappa}(u) \in W_{0}^{1,1}(\Omega)$. Hence,

$$
\frac{\nabla u}{|\nabla u|} \cdot \nabla G_{\kappa}(u)=G_{\kappa}^{\prime}(u)|\nabla u|=\left|\nabla G_{\kappa}(u)\right| .
$$

Applying identity (6) with test function $G_{\kappa}(u)$, we get

$$
\int_{\Omega}\left|\nabla G_{\kappa}(u)\right|=\int_{\Omega} f G_{\kappa}(u) .
$$

Since $G_{\kappa}$ vanishes on the interval $[-\kappa, \kappa]$, by the Hölder inequality we have

$$
\int_{\Omega} f G_{\kappa}(u)=\int_{\{|u|>\kappa\}} f G_{\kappa}(u) \leq\|f\|_{L^{N}(\{|u|>\kappa\})}\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N-1}}(\Omega)}
$$

Thus,

$$
\int_{\Omega}\left|\nabla G_{\kappa}(u)\right| \leq\|f\|_{L^{N}(\{|u|>\kappa\})}\left\|G_{\kappa}(u)\right\|_{L^{N-1}(\Omega)}
$$

By the Gagliardo-Nirenberg-Sobolev inequality,

$$
\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N-1}}(\Omega)} \leq C \int_{\Omega}\left|\nabla G_{\kappa}(u)\right|
$$

for some constant $C>0$ depending only on the dimension $N$. Hence,

$$
\begin{equation*}
\left(1-C\|f\|_{L^{N}(\{|u|>\kappa\})}\right)\left\|G_{\kappa}(u)\right\|_{L^{N^{N}-1}(\Omega)} \leq 0 . \tag{8}
\end{equation*}
$$

Let $T:=\|u\|_{L^{\infty}(\Omega)}$ if $u$ is essentially bounded, or $T:=+\infty$ otherwise. We have

$$
\lim _{\kappa \nearrow T}\|f\|_{L^{N}(\{|u|>\kappa\})}=\|f\|_{L^{N}(\{|u|=T\})} .
$$

We observe that the set $\{|u|=T\}$ has zero Lebesgue measure. This is indeed the case when $T=+\infty$ since $u$ is finite a.e. When $T<+\infty$, we observe that $\nabla u=0$ a.e. on the level set $\{|u|=T\}$; since by assumption $\nabla u \neq 0$ a.e. in $\Omega$, the set $\{u=T\}$ must have zero Lebesgue measure. This implies that

$$
\lim _{\kappa \nearrow T}\|f\|_{L^{N}(\{|u|>\kappa\})}=\|f\|_{L^{N}(\{|u|=T\})}=0 .
$$

In particular, there exists $0<\kappa<T$ such that $C\|f\|_{L^{N}(\{|u|>\kappa\})}<1$. We deduce from (8) that

$$
\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N-1}}(\Omega)} \leq 0
$$

Therefore, $|u| \leq \kappa$ a.e. in $\Omega$. Hence, $T=\|u\|_{L^{\infty}(\Omega)} \leq \kappa$, and this contradicts the choice of $\kappa$. The proof of the theorem is complete.

To prove Theorem 1.3, we rely on Peetre's imbedding of Sobolev functions in Lorentz spaces, with the best constant computed by Alvino. We recall that the Lorentz space $L^{p, 1}\left(\mathbb{R}^{N}\right)$ for $1 \leq p<\infty$ can be defined as the vector space of measurable functions $g$ in $\mathbb{R}^{N}$ such that

$$
\|g\|_{L^{p, 1}\left(\mathbb{R}^{N}\right)}:=\int_{0}^{\infty}|\{|g|>t\}|^{1 / p} \mathrm{~d} t<+\infty
$$

Using an equivalent definition to this one, Lorentz [17] established the duality between $L^{p, 1}\left(\mathbb{R}^{N}\right)$ and $\mathcal{M}^{\frac{p}{p-1}}\left(\mathbb{R}^{N}\right)$ for $p>1$ by proving an estimate which amounts to

$$
\int_{\mathbb{R}^{d}}|f g| \leq\|f\|_{\mathcal{M}^{\frac{p}{p-1}}\left(\mathbb{R}^{N}\right)}\|g\|_{L^{p, 1}\left(\mathbb{R}^{N}\right)}
$$

for every $g \in L^{p, 1}\left(\mathbb{R}^{N}\right)$ and $f \in \mathcal{M}^{\frac{p}{p-1}}\left(\mathbb{R}^{N}\right)$, where

$$
\|f\|_{\mathcal{M}^{\frac{p}{p-1}}\left(\mathbb{R}^{N}\right)}:=\sup _{A \subset \Omega} \frac{1}{|A|^{\frac{1}{p}}} \int_{A}|f|
$$

see [17, Theorem 5] and the computation of the Lorentz norm in [7, Section 2]. Here one should not rely on the quasi-norm $\sup _{t>0}\left\{t|\{|f|>t\}|^{\frac{p-1}{p}}\right\}$, which gives a quantity that is only equivalent to $\|f\|_{\mathcal{M}^{\frac{p}{p-1}}\left(\mathbb{R}^{N}\right)}$.

Peetre [23] proved by interpolation that $W^{1,1}\left(\mathbb{R}^{N}\right) \subset L^{\frac{N}{N-1}, 1}\left(\mathbb{R}^{N}\right)$ and Alvino [2] later showed using rearrangements that the inequality

$$
\|v\|_{L^{\frac{N}{N-1}, 1}\left(\mathbb{R}^{N}\right)} \leq \gamma_{1}\|\nabla v\|_{L^{1}\left(\mathbb{R}^{N}\right)}
$$

holds with the best constant given by $\gamma_{1}:=1 /\left(N \omega_{N}^{1 / N}\right)$.
Proof of Theorem 1.3. Proceeding as in the previous proof, by the duality between $L^{\frac{N}{N-1}, 1}$ and $\mathcal{M}^{N}$ one gets

$$
\int_{\mathbb{R}^{N}}\left|\nabla G_{\kappa}(u)\right|=\int_{\Omega} f G_{\kappa}(u) \leq\|f\|_{\mathcal{M}^{N}\left(\mathbb{R}^{N}\right)}\left\|G_{\kappa}(u)\right\|_{L^{N^{N-1}, 1}\left(\mathbb{R}^{N}\right)}
$$

where the functions $f$ and $u$ have been extended by zero to $\mathbb{R}^{N}$; this does not change their seminorms. Using Alvino's improvement of the Sobolev inequality with $v=G_{\kappa}(u)$, it follows that

$$
\left(1-\gamma_{1}\|f\|_{\mathcal{M}^{N}\left(\mathbb{R}^{N}\right)}\right)\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N^{N}}, 1}\left(\mathbb{R}^{N}\right)} \leq 0
$$

Under the assumption of the theorem we have $\|f\|_{\mathcal{M}^{N}\left(\mathbb{R}^{N}\right)}<1 / \gamma_{1}$, hence the quantity in parenthesis is positive. We deduce that $\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N-1}, 1}\left(\mathbb{R}^{N}\right)}=0$ for every $\kappa>0$, and then $u=0$ a.e. in $\Omega$, but this is not possible.

The proof of Theorem 1.5 relies on a property of BV function related to the chain rule. For this purpose, given $\kappa>0$ denote by $T_{\kappa}: \mathbb{R} \rightarrow \mathbb{R}$ the truncation function at levels $\pm \kappa$ :

$$
T_{\kappa}(t)= \begin{cases}-\kappa & \text { if } t<-\kappa \\ t & \text { if }-\kappa \leq t \leq \kappa \\ \kappa & \text { if } t>\kappa\end{cases}
$$

Observe that, for every $t \in \mathbb{R}$,

$$
\begin{equation*}
t=T_{\kappa}(t)+G_{\kappa}(t) \tag{9}
\end{equation*}
$$

where $G_{\kappa}$ is the function defined by (7). Since $T_{\kappa}$ and $G_{\kappa}$ are Lipschitz continuous, it is straightforward to verify using an approximation argument that $T_{\kappa}(u)$ and $G_{\kappa}(u)$ both belong to $B V\left(\mathbb{R}^{N}\right)$ for every $u \in B V\left(\mathbb{R}^{N}\right)$. In addition, by the identity above we have

$$
D u=D\left(T_{\kappa}(u)\right)+D\left(G_{\kappa}(u)\right)
$$

One then verifies that

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}|D u|=\int_{\mathbb{R}^{N}}\left|D\left(T_{\kappa}(u)\right)\right|+\int_{\mathbb{R}^{N}}\left|D\left(G_{\kappa}(u)\right)\right|, \tag{10}
\end{equation*}
$$

where, for a given vector-valued measure $\mu$,

$$
\int_{\mathbb{R}^{N}}|\mu|=\sup \left\{\int_{\mathbb{R}^{N}} \Phi \cdot \mu: \Phi \in C_{c}^{\infty}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right) \text { and }|\Phi| \leq 1 \text { in } \mathbb{R}^{N}\right\}
$$

Indeed, the inequality $\leq$ in (10) follows from the triangle inequality in $\mathbb{R}^{N}$. The reverse inequality $\geq$ can be deduced from Vol'pert's chain rule for $B V$ functions [3]. A more elementary approach is based on an approximation of $u$ using the sequence of smooth functions $\left(\rho_{n} * u\right)_{n \in \mathbb{N}}$, where $\left(\rho_{n}\right)_{n \in \mathbb{N}}$ is a sequence of mollifiers in $C_{c}^{\infty}\left(\mathbb{R}^{N}\right)$. In this case, one observes that

$$
\int_{\mathbb{R}^{N}}\left|D\left(\rho_{n} * u\right)\right| \rightarrow \int_{\mathbb{R}^{N}}|D u|
$$

as $n \rightarrow \infty$; see [13, Theorem 5.3]. On the other hand, there exist a subsequence $\left(\rho_{n_{j}} * u\right)_{j \in \mathbb{N}}$ and finite positive measures $\sigma_{1}$ and $\sigma_{2}$ such that

$$
\begin{array}{ll}
\left|D\left(T_{\kappa}\left(\rho_{n_{j}} * u\right)\right)\right| \stackrel{*}{\rightharpoonup} \sigma_{1} & \text { in } \mathcal{M}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right), \\
\left|D\left(G_{\kappa}\left(\rho_{n_{j}} * u\right)\right)\right| \stackrel{*}{\rightharpoonup} \sigma_{2} & \text { in } \mathcal{M}\left(\mathbb{R}^{N} ; \mathbb{R}^{N}\right),
\end{array}
$$

as $j \rightarrow \infty$, where $\sigma_{1} \geq\left|D\left(T_{\kappa}(u)\right)\right|$ and $\sigma_{2} \geq\left|D\left(G_{\kappa}(u)\right)\right|$. This implies the reverse inequality in (10).

Proof of Theorem 1.5. Since $u$ minimizes $\bar{E}_{\Phi_{1}}$, and $T_{\kappa}(u)$ is also an admissible function in the minimization class, we have

$$
\bar{E}_{\Phi_{1}}(u) \leq \bar{E}_{\Phi_{1}}\left(T_{\kappa}(u)\right) .
$$

Thus,

$$
\int_{\mathbb{R}^{N}}\left[|D u|-\left|D\left(T_{\kappa}(u)\right)\right|\right] \leq \int_{\mathbb{R}^{N}} f\left(u-T_{\kappa}(u)\right)
$$

We deduce from (10) and (9) that

$$
\int_{\mathbb{R}^{N}}\left|D\left(G_{\kappa}(u)\right)\right| \leq \int_{\mathbb{R}^{N}} f G_{\kappa}(u)
$$

We can now pursue the strategy of the proof of Theorem 1.2 to get the conclusion. Indeed, the Sobolev and Hölder inequalities imply that

$$
\left(1-C\|f\|_{L^{N}(\{|u|>\kappa\})}\right)\left\|G_{\kappa}(u)\right\|_{L^{\frac{N}{N^{N}}}(\Omega)} \leq 0 .
$$

For every $0<\kappa<\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}$, where we do not exclude the possibility that $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}=+\infty$, we have $\left\|G_{\kappa}(u)\right\|_{L^{N-1}(\Omega)}>0$. Thus,

$$
\|f\|_{L^{N}(\{|u|>\kappa\})} \geq \frac{1}{C}
$$

Since $u$ is finite a.e., this inequality cannot hold for every $\kappa>0$. Therefore, we must have $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}<\infty$ and so $u$ is essentially bounded. Letting $\kappa \rightarrow$ $\|u\|_{L^{\infty}\left(\mathbb{R}^{N}\right)}$, we deduce that $\{|u|>\kappa\}$ has positive measure.

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# Positive and nodal single-layered solutions to supercritical elliptic problems above the higher critical exponents 

Mónica Clapp and Matteo Rizzi

## To Jean Mawhin on his 75th birthday, with great appreciation

Abstract. We study the problem

$$
-\Delta v+\lambda v=|v|^{p-2} v \text { in } \Omega, \quad v=0 \text { on } \partial \Omega,
$$

for $\lambda \in \mathbb{R}$ and supercritical exponents $p$, in domains of the form

$$
\Omega:=\left\{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1}:(y,|z|) \in \Theta\right\}
$$

where $m \geq 1, N-m \geq 3$, and $\Theta$ is a bounded domain in $\mathbb{R}^{N-m}$ whose closure is contained in $\mathbb{R}^{N-m-1} \times(0, \infty)$. Under some symmetry assumptions on $\Theta$, we show that this problem has infinitely many solutions for every $\lambda$ in an interval which contains $[0, \infty)$ and $p>2$ up to some number which is larger than the $(m+1)^{\text {st }}$ critical exponent $2_{N, m}^{*}:=\frac{2(N-m)}{N-m-2}$. We also exhibit domains with a shrinking hole, in which there are a positive and a nodal solution which concentrate on a sphere, developing a single layer that blows up at an m-dimensional sphere contained in the boundary of $\Omega$, as the hole shrinks and $p \rightarrow 2_{N, m}^{*}$ from above. The limit profile of the positive solution, in the transversal direction to the sphere of concentration, is a rescaling of the standard bubble, whereas that of the nodal solution is a rescaling of a nonradial sign-changing solution to the problem

$$
-\Delta u=|u|^{2_{n}^{*}-2} u, \quad u \in D^{1,2}\left(\mathbb{R}^{n}\right)
$$

where $2_{n}^{*}:=\frac{2 n}{n-2}$ is the critical exponent in dimension $n$.

Keywords: Supercritical elliptic problem, positive solutions, nodal solutions, blow up, higher critical exponents.
MS Classification 2010: 35J61, 35B33, 35B44.

## 1. Introduction

We study the existence and concentration behavior of solutions to the problem

$$
\begin{cases}-\Delta v+\lambda v=|v|^{p-2} v & \text { in } \Omega  \tag{p}\\ v=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}, \lambda \in \mathbb{R}$, and $p$ is supercritical, i.e., it is larger than the critical Sobolev exponent $2_{N}^{*}:=\frac{2 N}{N-2}$ for $N \geq 3$. We shall consider domains of the form

$$
\begin{equation*}
\Omega:=\left\{(y, z) \in \mathbb{R}^{N-m-1} \times \mathbb{R}^{m+1}:(y,|z|) \in \Theta\right\} \tag{1}
\end{equation*}
$$

where $m \geq 1, N-m \geq 3$, and $\Theta$ is a bounded domain in $\mathbb{R}^{N-m}$ whose closure is contained in $\mathbb{R}^{N-m-1} \times(0, \infty)$.

In domains of this type, the true critical exponent is $2_{N, m}^{*}:=\frac{2(N-m)}{N-m-2}$, which is the critical Sobolev exponent in the dimension of $\Theta$ and is larger than $2_{N}^{*}$. Indeed, one can easily verify that the solutions to the problem ( $\wp_{p}$ ) which are radial in the variable $z$, correspond to the solutions of the problem

$$
\begin{cases}-\operatorname{div}(f(x) u)+\lambda f(x) u=f(x)|u|^{p-2} u & \text { in } \Theta  \tag{2}\\ u=0 & \text { on } \partial \Theta\end{cases}
$$

where $f\left(x_{1}, \ldots, x_{N-m}\right)=x_{N-m}^{m}$. Standard variational methods show that this last problem has infinitely many solutions for $p \in\left(2,2_{N-m}^{*}\right)$, hence, also does the problem $\left(\wp_{p}\right)$. On the other hand, Passaseo showed in [18, 19] that, if $\lambda=0$ and $\Theta$ is a ball centered on the half-line $\{0\} \times(0, \infty)$, then the problem $\left(\wp_{p}\right)$ does not have a nontrivial solution for $p \geq 2_{N-m}^{*}=2_{N, m}^{*}$. The number $2_{N, m}^{*}$ has been called the $(m+1)^{\text {st }}$ critical exponent in dimension $N$.

The concentration behavior of solutions to the problem $\left(\wp_{p}\right)$ for $\lambda=0$ and $p \in\left(2,2_{N, m}^{*}\right)$, as $p \rightarrow 2_{N, m}^{*}$ from below, has been investigated in several papers. In [10], del Pino, Musso and Pacard exhibited positive solutions which concentrate and blow up at a nondegenerate closed geodesic in $\partial \Omega$, as $p$ approaches the second critical exponent $2_{N, 1}^{*}$ from below. For any $m \geq 1$, positive and signchanging solutions in domains of the form (1) were constructed in [1, 13]. These solutions concentrate and blow up at one or several $m$-dimensional spheres, as $p \rightarrow 2_{N, m}^{*}$ from below. In all of these cases the limit profile of the solutions, in the transversal direction to each sphere of concentration, is a sum of rescalings of $\pm U$, where

$$
U(x):=[n(n-2)]^{(n-2) / 4}\left(\frac{1}{1+|x|^{2}}\right)^{(n-2) / 2}
$$

is the standard bubble in dimension $n:=N-m$, which is the only positive solution to the limit problem

$$
\begin{equation*}
-\Delta u=|u|^{2_{n}^{*}-2} u, \quad u \in D^{1,2}\left(\mathbb{R}^{n}\right) \tag{3}
\end{equation*}
$$

up to translation and dilation.
It was recently shown in [4] that there exist nonradial sign-changing solutions to the problem (3), that do not resemble a sum of rescaled positive and negative standard bubbles, which occur as limit profiles for concentration of sign-changing solutions to the problem $\left(\wp_{p}\right)$ that blow up at a single point, as $p \rightarrow 2_{N}^{*}$ from below. For the higher critical exponents $2_{N, m}^{*}$ with $m \geq 1$, it was shown in [5] that for every $\lambda$ in some interval which contains $[0, \infty)$ there are sign-changing solutions to the problem ( $\wp_{p}$ ), in domains of the form (1), which concentrate and blow up at an $m$-dimensional sphere, as $p \rightarrow 2_{N, m}^{*}$ from below, whose limit profile in the transversal direction to the sphere of concentration is a nonradial sign-changing solution to (3), like those found in [4].

The study of concentration phenomena for $p$ approaching $2_{N}^{*}$ from above, is a much more delicate issue, beginning with the fact that solutions to $\left(\wp_{p}\right)$ for $p>2_{N}^{*}$ do not always exist. For $\lambda=0$, standard bubbles were used as basic cells in $[8,9,16,20]$ to construct positive solutions to the slightly supercritical problem $\left(\wp_{p}\right)$ with $p=2_{N}^{*}+\varepsilon$, for small enough $\varepsilon>0$, in domains with a hole, using the Lyapunov-Schmidt reduction method. These solutions blow up, as $\varepsilon \rightarrow 0$, and their limit profile at each blow-up point is a rescaling of the standard bubble. Solutions in some contractible domains were constructed in $[14,15]$.

Quite recently, sign-changing solutions to the slightly supercritical problem $\left(\wp_{p}\right)$ with $p=2_{N}^{*}+\varepsilon, \varepsilon>0$, were exhibited by Musso and Wei [17] in domains with a small fixed hole, and by Clapp and Pacella [6] in domains with a shrinking hole. The solutions obtained in [17] concentrate at two different points in the domain, as $\varepsilon \rightarrow 0$, and their limit profile at each of them is a rescaling of one of the sign-changing solutions to the limit problem (3) in $\mathbb{R}^{N}$ constructed by del Pino, Musso, Pacard and Pistoia in [11], which resemble a large number of negative bubbles, placed evenly along a circle, surrounding a positive bubble, placed at its center. On the other hand, the sign-changing solutions exhibited in [6] concentrate at a single point in the interior of the shrinking hole, as the hole shrinks and $\varepsilon \rightarrow 0$, and their limit profile is a rescaling of a nonradial sign-changing solution to (3) like those found in [4].

For $m \geq 1$, the existence of solutions for $p=2_{N, m}^{*}+\varepsilon$ and their concentration behavior seems to be, so far, an open question; see Problem 4 in [7]. In this paper we will show that, under some symmetry assumptions, the problem $\left(\wp_{p}\right)$ has infinitely many solutions in domains of the form (1) for $p>2_{N, m}^{*}$, up to some value which depends on the symmetries; see Theorem 2.3. We will also exhibit domains with a shrinking hole, in which there are positive and
sign-changing solutions which concentrate and blow up at an $m$-dimensional sphere contained in the boundary of $\Omega$, as the hole shrinks and $p \rightarrow 2_{N, m}^{*}$ from above. The limit profile of the positive solutions, in the direction transversal to the sphere of concentration, will be a rescaling of the standard bubble, whereas that of the sign-changing ones will resemble one of the solutions to (3) that were found in [4].

We give, next, some examples of our results. For $n:=N-m$, let $B$ be an $n$-dimensional ball of radius $\delta_{0}$, centered on the half-line $\{0\} \times(0, \infty)$, whose closure is contained in the half-space $\mathbb{R}^{n-1} \times(0, \infty)$. We write the points in $\mathbb{R}^{n-1} \times(0, \infty)$ as $(y, t)$ with $y \in \mathbb{R}^{n-1}, t \in(0, \infty)$ and we set

$$
\begin{aligned}
& B_{\delta}:=\{(y, t) \in B:|y|>\delta\} \quad \text { if } \delta \in\left(0, \delta_{0}\right), \quad B_{0}:=B, \\
& \Omega_{\delta}:=\left\{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1}:(y,|z|) \in B_{\delta}\right\}, \quad \Omega:=\Omega_{0}
\end{aligned}
$$

We denote by $O(k)$ the group of all linear isometries of $\mathbb{R}^{k}$ and, for $v \in$ $D^{1,2}\left(\mathbb{R}^{N}\right)$, we write

$$
\|v\|:=\left(\int_{\mathbb{R}^{N}}|\nabla v|^{2}\right)^{1 / 2}
$$

The following results establish the existence of positive and sign-changing solutions to the problem $\left(\wp_{p}\right)$ in $\Omega_{\delta}$ and describe their limit profile as $\delta \rightarrow 0$ and $p \rightarrow 2_{N, m}^{*}$ from above. They are special cases of Theorems 2.3 and 4.4, which apply to more general domains, and are stated and proved in Sections 2 and 4, respectively.
THEOREM 1.1. There exists $\lambda_{*} \leq 0$ such that, for each $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$, $\delta \in\left(0, \delta_{0}\right)$ and $p \in(2, \infty)$, the problem $\left(\wp_{p}\right)$ has a positive solution $v_{\delta, p}$ in $\Omega_{\delta}$ which satisfies

$$
v_{\delta, p}(\gamma y, \varrho z)=v_{\delta, p}(y, z) \quad \forall \gamma \in O(n-1), \varrho \in O(m+1), \quad(y, z) \in \Omega_{\delta},
$$

and has minimal energy among all nontrivial solutions to $\left(\wp_{p}\right)$ in $\Omega_{\delta}$ with these symmetries.

Moreover, there exist sequences $\left(\delta_{k}\right)$ in $\left(0, \delta_{0}\right),\left(p_{k}\right)$ in $\left(2_{N, m}^{*}, \infty\right),\left(\varepsilon_{k}\right)$ in $(0, \infty)$ and $\left(\zeta_{k}\right)$ in $B \cap[\{0\} \times(0, \infty)]$ such that
(i) $\delta_{k} \rightarrow 0, p_{k} \rightarrow 2_{N, m}^{*}, \varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty$, and $\zeta_{k} \rightarrow \zeta$ with

$$
\operatorname{dist}\left(\zeta, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(B, \mathbb{R}^{n-1} \times\{0\}\right)
$$

(ii) $\lim _{k \rightarrow \infty}\left\|v_{\delta_{k}, p_{k}}-\widetilde{U}_{\varepsilon_{k}, \zeta_{k}}\right\|=0$, where

$$
\widetilde{U}_{\varepsilon_{k}, \zeta_{k}}(y, z):=\varepsilon_{k}^{(2-n) / 2} U\left(\frac{(y,|z|)-\zeta_{k}}{\varepsilon_{k}}\right)
$$

and $U$ is the standard bubble in dimension $n$.

The number $\lambda_{*}$ is negative if $m \geq 2$.
The solutions given by the previous theorem concentrate on an $m$-dimensional sphere, developing a positive layer which blows up at an $m$-dimensional sphere contained in the boundary of $\Omega$ and located at minimal distance to the plane of rotation $\mathbb{R}^{n-1} \times\{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of the standard bubble.

The next theorem gives sign-changing solutions to the problem $\left(\wp_{p}\right)$ with a different type of asymptotic profile. For $n \geq 5$, we write $\mathbb{R}^{n-1} \equiv \mathbb{C}^{2} \times \mathbb{R}^{n-5}$, and the points in $\mathbb{R}^{n-1}$ as $y=(\eta, \xi)$, with $\eta=\left(\eta_{1}, \eta_{2}\right) \in \mathbb{C}^{2}, \xi \in \mathbb{R}^{n-5}$.

Theorem 1.2. Assume that $n=5$ or $n \geq 7$. Then, there exists $\lambda_{*} \leq 0$ such that, for each $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}, \delta \in\left(0, \delta_{0}\right)$ and $p \in\left(2,2_{N, m+1}^{*}\right)$, the problem $\left(\wp_{p}\right)$ has a nontrivial sign-changing solution $w_{\delta, p}$ in $\Omega_{\delta}$ which satisfies

$$
w_{\delta, p}(\eta, \xi, z)=w_{\delta, p}\left(\mathrm{e}^{\mathrm{i} \vartheta} \eta, \alpha \xi, \varrho z\right), \quad w_{\delta, p}\left(\eta_{1}, \eta_{2}, \xi, z\right)=-w_{\delta, p}\left(-\bar{\eta}_{2}, \bar{\eta}_{1}, \xi, z\right)
$$

for every $\vartheta \in[0, \pi), \alpha \in O(n-5), \varrho \in O(m+1)$ and $(y, z) \in \Omega_{\delta}$, and which has minimal energy among all nontrivial solutions to $\left(\wp_{p}\right)$ in $\Omega_{\delta}$ with these symmetry properties.

Moreover, there exist sequences $\left(\delta_{k}\right)$ in $\left(0, \delta_{0}\right),\left(p_{k}\right)$ in $\left(2_{N, m}^{*}, 2_{N, m+1}^{*}\right),\left(\varepsilon_{k}\right)$ in $(0, \infty)$ and $\left(\zeta_{k}\right)$ in $B \cap[\{0\} \times(0, \infty)]$, and a nontrivial sign-changing solution $W$ to the limit problem (3), such that
(i) $\delta_{k} \rightarrow 0, p_{k} \rightarrow 2_{N, m}^{*}, \varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty$, and $\zeta_{k} \rightarrow \zeta$ with

$$
\operatorname{dist}\left(\zeta, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(B, \mathbb{R}^{n-1} \times\{0\}\right)
$$

(ii) $W(\eta, \xi, t)=W\left(\mathrm{e}^{\mathrm{i} \vartheta} \eta, \alpha \xi, t\right)$ and $W\left(\eta_{1}, \eta_{2}, \xi, t\right)=-W\left(-\bar{\eta}_{2}, \bar{\eta}_{1}, \xi, t\right)$ for every $\vartheta \in[0, \pi), \alpha \in O(n-5)$ and $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \equiv \mathbb{R}^{n}$, and $W$ has minimal energy among all nontrivial solutions to (3) with these symmetry properties,
(iii) $\lim _{k \rightarrow \infty}\left\|w_{\delta_{k}, p_{k}}-\widetilde{W}_{\varepsilon_{k}, \zeta_{k}}\right\|=0$, where

$$
\widetilde{W}_{\varepsilon_{k}, \zeta_{k}}(y, z):=\varepsilon_{k}^{(2-n) / 2} W\left(\frac{(y,|z|)-\zeta_{k}}{\varepsilon_{k}}\right) .
$$

The number $\lambda_{*}$ is negative if $m \geq 2$.
The solutions given by the previous theorem concentrate on an $m$-dimensional sphere, developing a sign-changing layer which blows up at an $m$-dimensional sphere contained in the boundary of $\Omega$ and located at minimal distance
to the plane of rotation $\mathbb{R}^{n-1} \times\{0\}$. The asymptotic profile of each layer in the transversal direction to its sphere of concentration is a rescaling of a nonradial sign-changing solution to the limit problem (3), like those found in [4].

As we mentioned before, the solutions to the anisotropic problem (2) give rise to solutions of the problem $\left(\wp_{p}\right)$ in domains of the form (1). In Section 2 we will study a general anisotropic problem in an $n$-dimensional domain $\Theta$. We will assume that $\Theta$ has some symmetries and we will establish the existence of infinitely many positive and sign-changing solutions to the anisotropic problem for supercritical exponents $p>2_{n}^{*}$, up to some value which depends on the symmetries. These results extend those obtained in [6] for the problem with constant coefficients. In Section 3 we will describe the behavior of the minimizing sequences for the variational functional associated to the anisotropic problem for $p=2_{n}^{*}$. These sequences, either converge to a solution, or they blow up. We will provide information on the location of the blow-up points and on the symmetries of the solutions to the limit problem (3) which occur as limit profiles. This will be used in Section 4 to obtain information on the concentration behavior and the limit profile of positive and sign-changing solutions to the problem $\left(\wp_{p}\right)$ in domains with a shrinking hole, as the hole shrinks and $p \rightarrow 2_{N, m}^{*}$ from above.

## 2. Symmetries and existence for supercritical problems

Let $\Gamma$ be a closed subgroup of $O(n)$ and $\phi: \Gamma \rightarrow \mathbb{Z}_{2}$ be a continuous homomorphism of groups. A function $u: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is said to be $\phi$-equivariant if

$$
\begin{equation*}
u(\gamma x)=\phi(\gamma) u(x) \quad \forall \gamma \in \Gamma, x \in \mathbb{R}^{n} \tag{4}
\end{equation*}
$$

If $\phi$ is the trivial homomorphism, then (4) simply says that $u$ is a $\Gamma$-invariant function, whereas, if $\phi$ is surjective and $u$ is not trivial, then (4) implies that $u$ is sign-changing, nonradial and $G$-invariant, where $G:=\operatorname{ker} \phi$.

Let $\Theta$ be a bounded $\Gamma$-invariant domain in $\mathbb{R}^{n}, n \geq 3$, and $a \in \mathcal{C}^{1}(\bar{\Theta})$, $b, c \in \mathcal{C}^{0}(\bar{\Theta})$ be $\Gamma$-invariant functions satisfying $a, c>0$ on $\bar{\Theta}$. We assume that

$$
\begin{equation*}
\text { there exists } x_{0} \in \Theta \text { such that }\left\{\gamma \in \Gamma: \gamma x_{0}=x_{0}\right\} \subset \operatorname{ker} \phi \tag{5}
\end{equation*}
$$

This assumption guarantees that the space

$$
D_{0}^{1,2}(\Theta)^{\phi}:=\left\{u \in D_{0}^{1,2}(\Theta): u \text { is } \phi \text {-equivariant }\right\}
$$

is infinite dimensional; see [3]. As usual, $D_{0}^{1,2}(\Theta)$ denotes the closure of $\mathcal{C}_{c}^{\infty}(\Theta)$ in the Hilbert space

$$
D^{1,2}\left(\mathbb{R}^{n}\right):=\left\{u \in L^{2_{n}^{*}}\left(\mathbb{R}^{n}\right): \nabla u \in L^{2}\left(\mathbb{R}^{n}, \mathbb{R}^{n}\right)\right\}
$$

equiped with the norm

$$
\|u\|:=\left(\int_{\Theta}|\nabla u|^{2}\right)^{1 / 2}
$$

We shall also assume that the operator $-\operatorname{div}(a \nabla)+b$ is coercive in $D_{0}^{1,2}(\Theta)^{\phi}$, i.e., that

$$
\begin{equation*}
\inf _{\substack{u \in D_{0}^{1,2}(\Theta)^{\phi} \\ u \neq 0}} \frac{\int_{\Theta}\left(a(x)|\nabla u|^{2}+b(x) u^{2}\right) \mathrm{d} x}{\int_{\Theta}|\nabla u|^{2}}>0 \tag{6}
\end{equation*}
$$

We set

$$
\|u\|_{a, b}^{2}:=\int_{\Theta}\left(a(x)|\nabla u|^{2}+b(x) u^{2}\right) \mathrm{d} x, \quad|u|_{c ; p}^{p}:=\int_{\Theta} c(x)|u|^{p} \mathrm{~d} x
$$

Assumption (6) implies that $\|\cdot\|_{a, b}$ is a norm in $D_{0}^{1,2}(\Theta)^{\phi}$ which is equivalent to $\|\cdot\|$. Note that, as $c>0,|\cdot|_{c ; p}$ is equivalent to the standard norm in $L^{p}(\Theta)$, which we denote by $|\cdot|_{p}$.

Our aim is to establish the existence of solutions to the problem

$$
\begin{cases}-\operatorname{div}(a(x) \nabla u)+b(x) u=c(x)|u|^{p-2} u & \text { in } \Theta,  \tag{7}\\ u=0 & \text { on } \partial \Theta . \\ u(\gamma x)=\phi(\gamma) u(x), & \forall \gamma \in \Gamma, x \in \Theta\end{cases}
$$

for every $2<p<2_{n-d}^{*}$, where

$$
d:=\min \{\operatorname{dim}(\Gamma x): x \in \bar{\Theta}\},
$$

$\Gamma x:=\{\gamma x: \gamma \in \Gamma\}$ is the $\Gamma$-orbit of $x, 2_{k}^{*}:=\frac{2 k}{k-2}$ if $k \geq 3$ and $2_{k}^{*}:=\infty$ if $k=1,2$. Note that $2_{n-d}^{*}>2_{n}^{*}$ if $d>0$.

A (weak) solution to the problem (7) is a function $u \in D_{0}^{1,2}(\Theta)^{\phi} \cap L^{p}(\Theta)$ such that

$$
\int_{\Theta}(a(x) \nabla u \cdot \nabla \psi+b(x) u \psi) \mathrm{d} x-\int_{\Theta} c(x)|u|^{p-2} u \psi \mathrm{~d} x=0 \quad \forall \psi \in \mathcal{C}_{c}^{\infty}(\Theta) .
$$

Proposition 2.1 of [6] asserts that $D_{0}^{1,2}(\Theta)^{\phi}$ is continuously embedded in $L^{p}(\Theta)$ for any real number $p \in\left[1,2_{n-d}^{*}\right]$, and that the embedding is compact for $p \in\left[1,2_{n-d}^{*}\right)$. The proof relies on a result by Hebey and Vaugon [12] which establishes these facts for $\Gamma$-invariant functions. Therefore, the functional

$$
J_{p}(u):=\frac{1}{2}\|u\|_{a, b}^{2}-\frac{1}{p}|u|_{c ; p}^{p}
$$

is well defined in the space $D_{0}^{1,2}(\Theta)^{\phi}$ if $p \in\left(2,2_{n-d}^{*}\right]$.

Lemma 2.1. For any real number $p \in\left(2,2_{n-d}^{*}\right]$ the critical points of the functional $J_{p}$ in the space $D_{0}^{1,2}(\Theta)^{\phi}$ are the solutions to the problem (7).

Proof. Let $u \in D_{0}^{1,2}(\Theta)^{\phi}$ be a critical point of $J_{p}$ in $D_{0}^{1,2}(\Theta)^{\phi}$. Then,

$$
J_{p}^{\prime}(u) \vartheta=\int_{\Theta}\left(a(x) \nabla u \cdot \nabla \vartheta+b(x) u \vartheta-c(x)|u|^{p-2} u \vartheta\right) \mathrm{d} x=0 \quad \forall \vartheta \in D_{0}^{1,2}(\Theta)^{\phi} .
$$

As $D_{0}^{1,2}(\Theta)^{\phi} \subset L^{p}(\Theta)$ for $1 \leq p \leq 2_{n-d}^{*}$ we need only to prove that $u$ satisfies (8). Let $\psi \in \mathcal{C}_{c}^{\infty}(\Theta)$, and define

$$
\widetilde{\psi}(x):=\frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi(\gamma) \psi(\gamma x) \mathrm{d} \mu
$$

where $\mu$ is the Haar measure on $\Gamma$. Note that $\widetilde{\psi} \in D_{0}^{1,2}(\Theta)^{\phi}$. Observe also that, as $u$ is $\phi$-equivariant, we have that

$$
\phi(\gamma) \nabla u(x)=\nabla(u \circ \gamma)(x)=\gamma^{-1} \nabla u(\gamma x) \quad \forall \gamma \in \Gamma, x \in \Theta .
$$

Since $J_{p}^{\prime}(u) \widetilde{\psi}=0$, and $a, b, c$ are $\Gamma$-invariant, using Fubini's theorem and performing a change of variable, we get

$$
\begin{aligned}
& 0=\int_{\Theta}\left(a(x) \nabla u(x) \cdot \nabla \widetilde{\psi}(x)+b(x) u(x) \widetilde{\psi}(x)-c(x)|u(x)|^{p-2} u(x) \widetilde{\psi}(x)\right) \mathrm{d} x \\
& =\frac{1}{\mu(\Gamma)} \int_{\Theta} \int_{\Gamma}\left[a(x) \phi(\gamma) \nabla u(x) \cdot \gamma^{-1} \nabla \psi(\gamma x)+b(x) \phi(\gamma) u(x) \psi(\gamma x)\right. \\
& \left.-c(x)|\phi(\gamma) u(x)|^{p-2} \phi(\gamma) u(x) \psi(\gamma x)\right] \mathrm{d} \mu \mathrm{~d} x \\
& =\frac{1}{\mu(\Gamma)} \int_{\Gamma} \int_{\Theta}\left[a(x) \gamma^{-1} \nabla u(\gamma x) \cdot \gamma^{-1} \nabla \psi(\gamma x)+b(x) u(\gamma x) \psi(\gamma x)\right. \\
& \left.-c(x)|u(\gamma x)|^{p-2} u(\gamma x) \psi(\gamma x)\right] \mathrm{d} x \mathrm{~d} \mu \\
& =\frac{1}{\mu(\Gamma)} \int_{\Gamma} \int_{\Theta}[a(\gamma x) \nabla u(\gamma x) \cdot \nabla \psi(\gamma x)+b(\gamma x) u(\gamma x) \psi(\gamma x) \\
& \left.-c(\gamma x)|u(\gamma x)|^{p-2} u(\gamma x) \psi(\gamma x)\right] \mathrm{d} x \mathrm{~d} \mu \\
& =\frac{1}{\mu(\Gamma)} \int_{\Gamma} \mathrm{d} \mu \int_{\Theta}[a(\xi) \nabla u(\xi) \cdot \nabla \psi(\xi)+b(\xi) u(\xi) \psi(\xi) \\
& \left.-c(\xi)|u(x)|^{p-2} u(\xi) \psi(\xi)\right] \mathrm{d} \xi \\
& =\int_{\Theta}\left[a(\xi) \nabla u(\xi) \cdot \nabla \psi(\xi)+b(\xi) u(\xi) \psi(\xi)-c(\xi)|u(x)|^{p-2} u(\xi) \psi(\xi)\right] \mathrm{d} \xi .
\end{aligned}
$$

Therefore $u$ is a solution to the problem (7).

The nontrivial critical points of the functional $J_{p}: D_{0}^{1,2}(\Theta)^{\phi} \rightarrow \mathbb{R}$ lie on the Nehari manifold

$$
\mathcal{N}_{p}^{\phi}:=\left\{u \in D_{0}^{1,2}(\Theta)^{\phi}:\|u\|_{a, b}^{2}=|u|_{c ; p}^{p}, u \neq 0\right\}
$$

which is a $\mathcal{C}^{2}$-Hilbert manifold, radially diffeomorphic to the unit sphere in $D_{0}^{1,2}(\Theta)^{\phi}$, and a natural constraint for this functional. Set

$$
\ell_{p}^{\phi}:=\inf \left\{J_{p}(u): u \in \mathcal{N}_{p}^{\phi}\right\} .
$$

Then, $\ell_{p}^{\phi}>0$. A least energy solution to the problem (7) is a minimizer for $J_{p}$ on $\mathcal{N}_{p}^{\phi}$. The following result extends Theorem 2.3 in [6].

Theorem 2.2. If $p \in\left(2,2_{n-d}^{*}\right)$ then the problem (7) has a least energy solution, and an unbounded sequence of solutions.

Proof. By Lemma 2.1, the critical points of the functional $J_{p}$ in the space $D_{0}^{1,2}(\Theta)^{\phi}$ are the solutions to the problem (7). Proposition 2.1 of [6] asserts that $D_{0}^{1,2}(\Theta)^{\phi}$ is compactly embedded in $L^{p}(\Theta)$ for $p \in\left(2,2_{n-d}^{*}\right)$, hence, a standard argument shows that the functional $J_{p}: D_{0}^{1,2}(\Theta)^{\phi} \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition. Therefore, $J_{p}$ attains its minimum on $\mathcal{N}_{p}^{\phi}$. Moreover, as the functional is even and has the mountain pass geometry, the symmetric mountain pass theorem [2] yields the existence of an unbounded sequence of critical values for $J_{p}$ in $D_{0}^{1,2}(\Theta)^{\phi}$.

We now derive a multiplicity result for the supercritical problem ($\left(\wp_{p}\right)$. Assume that the closure of $\Theta$ is contained in $\mathbb{R}^{n-1} \times(0, \infty)$ and, for $m \geq 1$, let

$$
\begin{equation*}
\lambda_{1}^{\phi}:=\inf _{\substack{u \in D_{0}^{1,2}(\Theta)^{\phi} \\ u \neq 0}} \frac{\int_{\Theta} x_{n}^{m}|\nabla u|^{2}}{\int_{\Theta} x_{n}^{m} u^{2}} . \tag{9}
\end{equation*}
$$

As the $n$-th coordinate $x_{n}$ of $x$ is positive for every $x \in \bar{\Theta}$, from the Poincaré inequality we obtain that $\lambda_{1}^{\phi}>0$.

Theorem 2.3. If $\lambda \in\left(-\lambda_{1}^{\phi}, \infty\right)$ and $p \in\left(2,2_{n-d}^{*}\right)$, then the problem $\left(\wp_{p}\right)$ has a least energy solution and an unbounded sequence of solutions in

$$
\Omega:=\left\{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1}:(y,|z|) \in \Theta\right\}
$$

which satisfy

$$
\begin{equation*}
v(\gamma y, \varrho z)=\phi(\gamma) v(y, z) \quad \forall \gamma \in \Gamma, \varrho \in O(m+1),(y, z) \in \Omega \tag{10}
\end{equation*}
$$

Proof. A straightforward computation shows that $v$ is a solution to the problem $\left(\wp_{p}\right)$ in $\Omega$ which satisfies (10) if and only if the function $u$ given by $v(y, z)=$ $u(y,|z|)$ is a solution to the problem (7) with $a(x):=x_{n}^{m}=: c(x)$ and $b(x):=$ $\lambda x_{n}^{m}$. Moreover, $v$ has minimal energy if and only if $u$ does. Note that (6) is satisfied if $\lambda \in\left(-\lambda_{1}^{\phi}, \infty\right)$. So this result follows from Theorem 2.2.

For $p \in\left(2,2_{n-d}^{*}\right)$ let $u_{p}$ be a least energy solution to the problem (7). Fix $q \in\left(2,2_{n-d}^{*}\right)$ and let $t_{q, p} \in(0, \infty)$ be such that $\widetilde{u}_{p}:=t_{q, p} u_{p} \in \mathcal{N}_{q}^{\phi}$, i.e.,

$$
\begin{equation*}
t_{q, p}=\left(\frac{\left\|u_{p}\right\|_{a, b}^{2}}{\left|u_{p}\right|_{c ; q}^{q}}\right)^{\frac{1}{q-2}}=\left(\frac{\left|u_{p}\right|_{c ; p}^{p}}{\left|u_{p}\right|_{c ; q}^{q}}\right)^{\frac{1}{q-2}} \tag{11}
\end{equation*}
$$

We will show that $\lim _{p \rightarrow q} J_{q}\left(\widetilde{u}_{p}\right)=\ell_{q}^{\phi}$. The proof is similar to that of Proposition 2.5 in [6]. We give the details for the reader's convenience, starting with the following lemma.

Lemma 2.4. If $p_{k}, q \in\left(2,2_{n-d}^{*}\right), p_{k} \rightarrow q$, and $\left(u_{k}\right)$ is a bounded sequence in $D_{0}^{1,2}(\Theta)^{\phi}$, then

$$
\lim _{k \rightarrow \infty} \int_{\Theta}\left(c(x)\left|u_{k}\right|^{p_{k}}-c(x)\left|u_{k}\right|^{q}\right) \mathrm{d} x=0
$$

Proof. By the mean value theorem, for each $x \in \Theta$, there exists $q_{k}(x)$ between $p_{k}$ and $q$ such that

$$
\left\|\left.u_{k}(x)\right|^{p_{k}}-\left|u_{k}(x)\right|^{q}\left|=|\ln | u_{k}(x) \|\left|u_{k}(x)\right|^{q_{k}(x)}\right| p_{k}-q \mid .\right.
$$

Fix $r>0$ such that $[q-r, q+r] \subset\left(2,2_{n-d}^{*}\right)$. Then, for some positive constant $C$ and $k$ large enough,

$$
|\ln | u_{k}| |\left|u_{k}\right|^{q_{k}} \leq\left\{\begin{array}{lll}
\ln \left|u_{k}\right|\left|u_{k}\right|^{q+r} & \leq C\left|u_{k}\right|^{2_{n-d}^{*}} & \text { if }\left|u_{k}\right| \geq 1 \\
\left(\ln \frac{1}{\left|u_{k}\right|}\right)\left|u_{k}\right|^{q-r} & \leq C\left|u_{k}\right|^{2} & \text { if }\left|u_{k}\right| \leq 1
\end{array}\right.
$$

As $D_{0}^{1,2}(\Theta)^{\phi}$ is continuously embedded in $L^{p}(\Theta)$ for $p \in\left[2,2_{n-d}^{*}\right]$, we obtain

$$
\begin{aligned}
\left|\int_{\Theta} c\left(\left|u_{k}\right|^{p_{k}}-\left|u_{k}\right|^{q}\right)\right| & \leq|c|_{\infty}\left(\int_{\left|u_{k}\right| \leq 1}\left\|\left.u_{k}\right|^{p_{k}}-\left|u_{k}\right|^{q}\left|+\int_{\left|u_{k}\right|>1} \| u_{k}\right|^{p_{k}}-\left|u_{k}\right|^{q} \mid\right)\right. \\
& \leq|c|_{\infty} C\left|p_{k}-q\right| \int_{\Theta}\left(\left|u_{k}\right|^{2}+\left|u_{k}\right|^{2_{n-d}^{*}}\right) \\
& \leq \bar{C}\left|p_{k}-q\right|\left\|u_{k}\right\|^{2_{n-d}^{*}}
\end{aligned}
$$

for some positive constant $\bar{C}$, where $|c|_{\infty}:=\sup _{x \in \Theta}|c(x)|$. Since $\left(u_{k}\right)$ is bounded in $D_{0}^{1,2}(\Theta)$, our claim follows.

Proposition 2.5. For $q \in\left(2,2_{n-d}^{*}\right)$ we have that

$$
\lim _{p \rightarrow q} \ell_{p}^{\phi}=\ell_{q}^{\phi}, \quad \lim _{p \rightarrow q} t_{q, p}=1, \quad \lim _{p \rightarrow q} J_{q}\left(\widetilde{u}_{p}\right)=\ell_{q}^{\phi}
$$

Proof. Set

$$
S_{p}^{\phi}:=\inf _{u \in D_{0}^{1,2}(\Omega)^{\phi} \backslash\{0\}} \frac{\|u\|_{a, b}^{2}}{|u|_{c ; p}^{2}} .
$$

It is easy to see that $\ell_{p}^{\phi}=\frac{p-2}{2 p}\left(S_{p}^{\phi}\right)^{\frac{p}{p-2}}$. So, to prove the first identity, it suffices to show that $\lim _{p \rightarrow q} S_{p}^{\phi}=S_{q}^{\phi}$. From Hölder's inequality we get that $|u|_{c ; q} \leq|c|_{1}^{(p-q) / p q}|u|_{c ; p}$ if $p>q$. Hence, $S_{q}^{\phi} \geq|c|_{1}^{2(q-p) / p q} S_{p}^{\phi}$ if $p>q$. So, as $p$ approaches $q$ from the right, we have that

$$
\limsup _{p \rightarrow q^{+}} S_{p}^{\phi} \leq S_{q}^{\phi}
$$

Assume that $\liminf _{p \rightarrow q^{+}} S_{p}^{\phi}<S_{q}^{\phi}$. Then, there exist $\varepsilon>0$ and sequences $\left(p_{k}\right)$ in $\left(q, 2_{n-d}^{*}\right)$ and $\left(u_{k}\right)$ in $D_{0}^{1,2}(\Omega)^{\phi}$ with $\left|u_{k}\right|_{c ; p_{k}}=1$ such that $\left\|u_{k}\right\|_{a, b}^{2}<S_{q}^{\phi}-\varepsilon$. Lemma 2.4 implies that $\frac{\left\|u_{k}\right\|_{\text {a,b }}^{2}}{\left|u_{k}\right|_{c ; q}}<S_{q}^{\phi}$ for $k$ large enough, contradicting the definition of $S_{q}^{\phi}$. This proves that

$$
\lim _{p \rightarrow q^{+}} S_{p}^{\phi}=S_{q}^{\phi}
$$

The corresponding statement when $p$ approaches $q$ from the left is proved in a similar way. Since $J_{p}\left(u_{p}\right)=\frac{p-2}{2 p}\left\|u_{p}\right\|_{a, b}^{2}=\ell_{p}^{\phi}$ we have that $\left(u_{p}\right)$ is bounded in $D_{0}^{1,2}(\Omega)^{\phi}$ for $p$ close to $q$. Lemma 2.4 applied to (11) yields $\lim _{p \rightarrow q} t_{q, p}=1$. It follows that $\lim _{p \rightarrow q} J_{q}\left(\widetilde{u}_{p}\right)=\lim _{p \rightarrow q} \frac{q-2}{2 q}\left\|t_{q, p} u_{p}\right\|_{a, b}^{2}=\ell_{q}^{\phi}$, as claimed.

## 3. Minimizing sequences for the critical problem

In this section we analize the behavior of the minimizing sequences for the problem (7) when $p$ is the critical exponent $2_{n}^{*}=\frac{2 n}{n-2}$. The solutions to the limit problem (3) will play a crucial role in this analysis. We denote the energy functional associated to (3) by

$$
J_{\infty}(u):=\frac{1}{2}\|u\|^{2}-\frac{1}{2^{*}}|u|_{2^{*}}^{2^{*}}
$$

and, for any closed subgroup $K$ of $\Gamma$, we set

$$
\begin{aligned}
D^{1,2}\left(\mathbb{R}^{n}\right)^{\phi \mid K} & :=\left\{u \in D^{1,2}\left(\mathbb{R}^{n}\right): u(\gamma z)=\phi(\gamma) u(z) \forall \gamma \in K, z \in \mathbb{R}^{n}\right\}, \\
\mathcal{N}_{\infty}^{\phi \mid K} & :=\left\{u \in D^{1,2}\left(\mathbb{R}^{n}\right)^{\phi \mid K}: u \neq 0,\|u\|^{2}=|u|_{2^{*}}^{2^{*}}\right\}, \\
\ell_{\infty}^{\phi \mid K} & :=\inf _{u \in \mathcal{N}_{\infty}^{\phi \mid K}} J_{\infty}(u) .
\end{aligned}
$$

If $K=\Gamma$ we write $\mathcal{N}_{\infty}^{\phi}$ and $\ell_{\infty}^{\phi}$ instead of $\mathcal{N}_{\infty}^{\phi \mid K}$ and $\ell_{\infty}^{\phi \mid K}$.
Recall that the $\Gamma$-orbit of a point $x \in \mathbb{R}^{n}$ is the set $\Gamma x:=\{\gamma x: \gamma \in \Gamma\}$, and its isotropy group is $\Gamma_{x}:=\{\gamma \in \Gamma: \gamma x=x\}$. Then, $\Gamma x$ is $\Gamma$-homeomorphic to the homogeneous space $\Gamma / \Gamma_{x}$. In particular, the cardinality of $\Gamma x$ is the index of $\Gamma_{x}$ in $\Gamma$, which is usually denoted by $\left|\Gamma / \Gamma_{x}\right|$. If $\Gamma x=\{x\}$ then $x$ is said to be a fixed point of $\Gamma$. We denote

$$
\Theta^{\Gamma}:=\{x \in \Theta: x \text { is a fixed point of } \Gamma\} .
$$

For simplicity, we will write $J_{*}, \mathcal{N}_{*}^{\phi}$ and $\ell_{*}^{\phi}$ instead of $J_{2_{n}^{*}}, \mathcal{N}_{2_{n}^{*}}^{\phi}$ and $\ell_{2_{n}^{*}}^{\phi}$.
THEOREM 3.1. Let $\left(u_{k}\right)$ be a sequence in $\mathcal{N}_{*}^{\phi}$ such that $J_{*}\left(u_{k}\right) \rightarrow \ell_{*}^{\phi}$. Then, after passing to a subsequence, one of the following two possibilities occurs:

1. $\left(u_{k}\right)$ converges strongly in $D_{0}^{1,2}(\Theta)$ to a minimizer of $J_{*}$ on $\mathcal{N}_{*}^{\phi}$.
2. There exist a closed subgroup $K$ of finite index in $\Gamma$, a sequence $\left(\zeta_{k}\right)$ in $\Theta$, a sequence $\left(\varepsilon_{k}\right)$ in $(0, \infty)$ and a nontrivial solution $\omega$ to the problem (3) with the following properties:
(a) $\Gamma_{\zeta_{k}}=K$ for all $k \in \mathbb{N}$, and $\zeta_{k} \rightarrow \zeta$,
(b) $\varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty$ and $\varepsilon_{k}^{-1}\left|\alpha \zeta_{k}-\beta \zeta_{k}\right| \rightarrow \infty$ for all $\alpha, \beta \in \Gamma$ with $\alpha^{-1} \beta \notin K$,
(c) $\omega(\gamma z)=\phi(\gamma) \omega(z)$ for all $\gamma \in K, z \in \mathbb{R}^{n}$, and $J_{\infty}(\omega)=\ell_{\infty}^{\phi \mid K}$,
(d) $\lim _{k \rightarrow \infty}\left\|u_{k}-\sum_{[\gamma] \in \Gamma / K} \phi(\gamma)\left(\frac{a(\zeta)}{c(\zeta)}\right)^{\frac{n-2}{4}} \varepsilon_{k}^{\frac{2-n}{2}}\left(\omega \circ \gamma^{-1}\right)\left(\frac{-\gamma \zeta_{k}}{\varepsilon_{k}}\right)\right\|=0$,
(e) $\ell_{*}^{\phi}=\min _{x \in \Theta} \frac{a(x)^{n / 2}}{c(x)^{(n-2) / 2}}\left|\Gamma / \Gamma_{x}\right| \ell_{\infty}^{\phi \mid \Gamma_{x}}=\frac{a(\zeta)^{n / 2}}{c(\zeta)^{(n-2) / 2}}|\Gamma / K| J_{\infty}(\omega)$.

Proof. The proof is exactly the same as that of Theorem 2.5 in [5], omitting the first two lines.

Let us state an interesting special case of Theorem 3.1.
Corollary 3.2. Assume that every $\Gamma$-orbit in $\Theta$ is either infinite or a fixed point. Let $\left(u_{k}\right)$ be a sequence in $\mathcal{N}_{*}^{\phi}$ such that $J_{*}\left(u_{k}\right) \rightarrow \ell_{*}^{\phi}$. Then, after passing to a subsequence, one of the following statements holds true:

1. $\left(u_{k}\right)$ converges strongly in $D_{0}^{1,2}(\Theta)$ to a minimizer of $J_{*}$ on $\mathcal{N}_{*}^{\phi}$.
2. There exist a sequence $\left(\zeta_{k}\right)$ in $\Theta^{\Gamma}$, a sequence $\left(\varepsilon_{k}\right)$ in $(0, \infty)$ and a nontrivial $\phi$-equivariant solution $\omega$ to the limit problem (3) such that $\zeta_{k} \rightarrow \zeta \in \bar{\Theta}$, $\varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty, J_{\infty}(\omega)=\ell_{\infty}^{\phi}$,

$$
\lim _{k \rightarrow \infty}\left\|u_{k}-\left(\frac{a(\zeta)}{c(\zeta)}\right)^{\frac{n-2}{4}} \varepsilon_{k}^{\frac{2-n}{2}} \omega\left(\frac{\cdot-\zeta_{k}}{\varepsilon_{k}}\right)\right\|=0
$$

and

$$
\frac{a(\zeta)^{n / 2}}{c(\zeta)^{(n-2) / 2}}=\min _{x \in \Theta^{\Gamma}} \frac{a(x)^{n / 2}}{c(x)^{(n-2) / 2}}
$$

In particular, if every $\Gamma$-orbit in $\Theta$ has positive dimension, then (1) must hold true.

Proof. Since the group $K=\Gamma_{\zeta_{k}}$, given by case 2 of Theorem 3.1, has finite index in $\Gamma$ and this index is the cardinality of the $\Gamma$-orbit of $\zeta_{k}$, our assumption implies that $K=\Gamma$ and $\zeta_{k}$ is a fixed point. So case 2 of Theorem 3.1 reduces to case 2 of this corollary.

Note that the functions $a$ and $c$ determine the location of the concentration point $\zeta$.

It was shown in [4, Theorem 2.3] that, if $a=c=1, b=0$ and $\Theta^{\Gamma} \neq \emptyset$, then $\ell_{*}^{\phi}$ is not attained by $J_{*}$ on $\mathcal{N}_{*}^{\phi}$. So, if every $\Gamma$-orbit in $\Theta \backslash \Theta^{\Gamma}$ has positive dimension, statement 2 of Corollary 3.2 must hold true.

In the following section we will state a nonexistence result which allows us to obtain information on the limit profile of solutions to the problem $\left(\wp_{p}\right)$.

## 4. Blow-up at the higher critical exponents

Throughout this section we will assume that $\Theta$ is a $\Gamma$-invariant bounded smooth domain in $\mathbb{R}^{n}$ whose closure is contained in $\mathbb{R}^{n-1} \times(0, \infty)$. Then, the points in $\{0\} \times(0, \infty)$ must be fixed points of $\Gamma$, so $\mathbb{R}^{n-1} \times\{0\}$ is $\Gamma$-invariant and we may regard $\Gamma$ as a subgroup of $O(n-1)$. We will also assume that $\Theta \backslash \Theta^{\Gamma}$ and $\Theta^{\Gamma}$ are nonempty, and that every $\Gamma$-orbit in $\Theta \backslash \Theta^{\Gamma}$ has positive dimension. As before, $\phi: \Gamma \rightarrow \mathbb{Z}_{2}$ will be a continuous homomorphism which satisfies assumption (5).

We set

$$
\Theta_{\delta}:=\left\{x \in \Theta: \operatorname{dist}\left(x, \Theta^{\Gamma}\right)>\delta\right\} \quad \text { if } \delta>0, \quad \text { and } \quad \Theta_{0}:=\Theta
$$

and we fix $\delta_{0}>0$ such that $\Theta_{\delta_{0}} \neq \emptyset$. For $m \geq 1$ and $\delta \in\left[0, \delta_{0}\right)$, we consider the problem

$$
\left(\wp_{\delta, p}^{\#}\right) \quad \begin{cases}-\operatorname{div}\left(x_{n}^{m} \nabla u\right)+\lambda x_{n}^{m} u=x_{n}^{m}|u|^{p-2} u & \text { in } \Theta_{\delta}, \\ u=0 & \text { on } \partial \Theta_{\delta} . \\ u(\gamma x)=\phi(\gamma) u(x), & \forall \gamma \in \Gamma, x \in \Theta_{\delta},\end{cases}
$$

where $x_{n}^{m}$ denotes the function $x=\left(x_{1}, \ldots, x_{n}\right) \mapsto x_{n}^{m}$, and $\lambda \in\left(-\lambda_{1}^{\phi}, \infty\right)$, with $\lambda_{1}^{\phi}$ as defined in (9). Then, the operator $-\operatorname{div}\left(x_{n}^{m} \nabla\right)+\lambda x_{n}^{m}$ is coercive in $D_{0}^{1,2}(\Theta)^{\phi}$. So the data of this problem satisfy all assumptions stated at the beginning of Section 2.

Theorem 2.2 asserts that the problem $\left(\wp_{\delta, p}^{\#}\right)$ has a least energy solution $u_{\delta, p}$ if $\delta \in\left(0, \delta_{0}\right)$ and $p \in\left(2,2_{n-\mathfrak{d}}^{*}\right)$, where

$$
\mathfrak{d}:=\min \left\{\operatorname{dim}(\Gamma x): x \in \Theta \backslash \Theta^{\Gamma}\right\} .
$$

Note that, by assumption, $\mathfrak{d}>0$. On the other hand, for $\delta=0$ and $p=2_{n}^{*}$, the following nonexistence result was proved in [5].
Theorem 4.1. If $\operatorname{dist}\left(\Theta^{\Gamma}, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(\Theta, \mathbb{R}^{n-1} \times\{0\}\right)$, then there exists $\lambda_{*} \in\left(-\lambda_{1}^{\phi}, 0\right]$ such that, if $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$, the critical problem $\left(\wp_{0,2_{n}^{*}}^{\#}\right)$ does not have a least energy solution.

Moreover, $\lambda_{*}<0$ if $m \geq 2$.
Proof. See Theorem 3.2 in [5].
For $\delta \in\left(0, \delta_{0}\right)$ and $p \in\left(2,2_{n-\mathfrak{d}}^{*}\right)$, let $J_{\delta, p}: D_{0}^{1,2}\left(\Theta_{\delta}\right)^{\phi} \rightarrow \mathbb{R}$ be the variational funcional and $\mathcal{N}_{\delta, p}^{\phi}$ be the Nehari manifold associated to the problem ( $\wp_{\delta, p}^{\#}$ ), and set

$$
\ell_{\delta, p}^{\phi}:=\inf \left\{J_{\delta, p}(u): u \in \mathcal{N}_{\delta, p}^{\phi}\right\} .
$$

We write $J_{*}, \mathcal{N}_{*}^{\phi}$ and $\ell_{*}^{\phi}$ for the variational functional, the Nehari manifold and the infimum associated to the critical problem $\left(\wp_{0,2_{n}^{*}}^{\#}\right)$ in the whole domain $\Theta$. Extending each function in $\mathcal{N}_{\delta, 2_{n}^{*}}^{\phi}$ by 0 outside of $\Theta_{\delta}$, we have that $\mathcal{N}_{\delta, 2_{n}^{*}}^{\phi} \subset \mathcal{N}_{*}^{\phi}$ and $J_{\delta, 2_{n}^{*}}(u)=J_{*}(u)$ for every $u \in \mathcal{N}_{\delta, 2_{n}^{*}}^{\phi}$. Hence, $\ell_{*}^{\phi} \leq \ell_{\delta, 2_{n}^{*}}^{\phi}$.

LEMMA 4.2. $\ell_{\delta, 2_{n}^{*}}^{\phi} \rightarrow \ell_{*}^{\phi}$ as $\delta \rightarrow 0$.
Proof. Let $X:=\left(\mathbb{R}^{n}\right)^{\Gamma}$ and $Y$ be its orthogonal complement in $\mathbb{R}^{n}$. Since $\Theta \backslash \Theta^{\Gamma} \neq \emptyset$ and every $\Gamma$-orbit in $\Theta \backslash \Theta^{\Gamma}$ has positive dimension, we have that $\operatorname{dim}(Y) \geq 2$.

We claim that there are radial functions $\chi_{k} \in \mathcal{C}_{c}^{\infty}(Y)$ such that $\chi_{k}(y)=1$ if $|y| \leq \frac{1}{k}$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \int_{Y}\left|\chi_{k}\right|^{2}=0 \quad \text { and } \quad \lim _{k \rightarrow \infty} \int_{Y}\left|\nabla \chi_{k}\right|^{2}=0 \tag{12}
\end{equation*}
$$

To show this, we choose a radial function $g \in \mathcal{C}_{c}^{\infty}(Y)$ such that $g(y)=1$ if $|y| \leq 1$ and $g(y)=0$ if $|y| \geq 2$, and we set $g_{k}(y):=g(k y)$. Define

$$
\chi_{k}(y):=\frac{1}{\sigma_{k}} \sum_{j=1}^{k} \frac{g_{j}(y)}{j}, \quad \text { where } \sigma_{k}:=\sum_{j=1}^{k} \frac{1}{j}
$$

Clearly, $\chi_{k}(y)=1$ if $|y| \leq \frac{1}{k}$ and $\chi_{k}(y)=0$ if $|y| \geq 2$. As $\operatorname{dim}(Y) \geq 2$, we have that $\int_{Y}\left|\nabla g_{k}\right|^{2} \leq \int_{Y}|\nabla g|^{2}$. Hence, for some positive constant $C$,

$$
\int_{Y}\left|\nabla \chi_{k}\right|^{2} \leq \frac{C}{\sigma_{k}^{2}} \sum_{j=1}^{k} \frac{1}{j^{2}} \rightarrow 0 \quad \text { as } k \rightarrow \infty
$$

Finally, as all functions $\chi_{k}$ are supported in the closed ball of radius 2 in $Y$, the Poincaré inequality yields

$$
\int_{Y}\left|\chi_{k}\right|^{2} \leq C \int_{Y}\left|\nabla \chi_{k}\right|^{2} \rightarrow 0
$$

and our claim is proved.
Given $\varepsilon>0$ we choose $\psi \in \mathcal{N}_{*}^{\phi}$ such that $J_{*}(\psi)<\ell_{*}^{\phi}+\frac{\varepsilon}{2}$. For $(x, y) \in X \times Y$, we define $\psi_{k}(x, y):=\left(1-\chi_{k}(y)\right) \psi(x, y)$. Note that, as $\chi_{k}$ is radial and $\psi$ is is $\phi$-equivariant, $\psi_{k}$ is also $\phi$-equivariant. Moreover, the identities (12) easily imply that $\psi_{k} \rightarrow \psi$ in $D_{0}^{1,2}(\Theta)$. So, for $k$ large enough, there exists $t_{k} \in(0, \infty)$ such that $\widetilde{\psi}_{k}:=t_{k} \psi_{k} \in \mathcal{N}_{*}^{\phi}$ and $t_{k} \rightarrow 1$. Hence, $\widetilde{\psi}_{k} \rightarrow \psi$ in $D_{0}^{1,2}(\Theta)$, and we may choose $k_{0} \in \mathbb{N}$ such that $J_{*}\left(\widetilde{\psi}_{k_{0}}\right)<\ell_{*}^{\phi}+\varepsilon$. Observe that $\operatorname{supp}\left(\widetilde{\psi}_{k}\right)=$ $\operatorname{supp}\left(\psi_{k}\right) \subset \Theta_{\delta}$ if $\delta<\frac{1}{k}$. So $\widetilde{\psi}_{k} \in \mathcal{N}_{\delta, 2_{n}^{*}}^{\phi}$ if $\delta<\frac{1}{k}$. It follows that

$$
\ell_{*}^{\phi} \leq \ell_{\delta, 2_{n}^{*}}^{\phi} \leq J_{\delta, 2_{n}^{*}}\left(\widetilde{\psi}_{k_{0}}\right)=J_{*}\left(\widetilde{\psi}_{k_{0}}\right)<\ell_{*}^{\phi}+\varepsilon \quad \forall \delta \in\left(0, \frac{1}{k_{0}}\right)
$$

This finishes the proof.

Set $N:=n+m$ and

$$
\Omega_{\delta}:=\left\{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}^{m+1}:(y,|z|) \in \Theta_{\delta}\right\}, \quad \delta \in\left[0, \delta_{0}\right)
$$

Note that $\Omega_{\delta}$ is $[\Gamma \times O(m+1)]$-invariant, i.e., $(\gamma y, \varrho z) \in \Omega_{\delta}$ for every $(y, z) \in$ $\Omega_{\delta}, \gamma \in \Gamma, \varrho \in O(m+1)$. A straightforward computation shows that $u_{\delta, p}$ is a least energy solution to the problem $\left(\wp_{\delta, p}^{\#}\right)$ if and only if $v_{\delta, p}(y, z):=u_{\delta, p}(y,|z|)$ is a least energy solution to the problem

$$
(\wp \delta, p) \quad \begin{cases}-\Delta v+\lambda v=|v|^{p-2} v & \text { in } \Omega_{\delta}, \\ v=0 & \text { on } \partial \Omega_{\delta}, \\ v(\gamma y, \varrho z)=\phi(\gamma) v(y, z), & \forall \gamma \in \Gamma, \varrho \in O(m+1),(y, z) \in \Omega_{\delta}\end{cases}
$$

Therefore, for every $\lambda \in\left(-\lambda_{1}^{\phi}, \infty\right), \delta \in\left(0, \delta_{0}\right)$ and $p \in\left(2,2_{n-\mathfrak{o}}^{*}\right)$, the problem $\left(\wp_{\delta, p}\right)$ has a least energy solution. The following results describe its limit profile.

THEOREM 4.3. For $\delta \in\left(0, \delta_{0}\right)$ let $v_{\delta, *}$ be a least energy solution to the problem $\left(\wp_{\delta, 2_{N, m}^{*}}\right)$. Assume that

$$
\operatorname{dist}\left(\Theta^{\Gamma}, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(\Theta, \mathbb{R}^{n-1} \times\{0\}\right)
$$

Then, there exists $\lambda_{*} \leq 0$ such that, if $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$, there exist sequences $\left(\delta_{k}\right)$ in $\left(0, \delta_{0}\right),\left(\varepsilon_{k}\right)$ in $(0, \infty),\left(\zeta_{k}\right)$ in $\Theta^{\Gamma}$, and a nontrivial solution $\omega$ to the limit problem (3) such that
(i) $\delta_{k} \rightarrow 0, \varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty$, and $\zeta_{k} \rightarrow \zeta$ with

$$
\operatorname{dist}\left(\zeta, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(\Theta, \mathbb{R}^{n-1} \times\{0\}\right)
$$

(ii) $\omega$ is $\phi$-equivariant and has minimal energy among all nontrivial $\phi$-equivariant solutions to the problem (3),
(iii) $v_{\delta_{k}, *}=\widetilde{\omega}_{\varepsilon_{k}, \zeta_{k}}+o(1)$ in $D^{1,2}\left(\mathbb{R}^{N}\right)$, where

$$
\widetilde{\omega}_{\varepsilon_{k}, \zeta_{k}}(y, z):=\varepsilon_{k}^{(2-n) / 2} \omega\left(\frac{(y,|z|)-\zeta_{k}}{\varepsilon_{k}}\right)
$$

Moreover, $\lambda_{*}<0$ if $m \geq 2$.
Proof. Let $\lambda_{*}$ be the number given by Theorem 4.1. Fix $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$, and let $u_{\delta, *}$ be the least energy solution to the problem $\left(\wp_{\delta 2_{n}^{*}}^{\#}\right)$ given by $v_{\delta, *}(y, z)=$ $u_{\delta, *}(y,|z|)$. Choose a sequence $\delta_{k} \rightarrow 0$ and set $u_{k}:=u_{\delta_{k}, *}$. Then, $u_{k} \in \mathcal{N}_{*}^{\phi}$ and, by Lemma 4.2, $J_{*}\left(u_{k}\right) \rightarrow \ell_{*}^{\phi}$. It follows from Corollary 3.2 and Theorem 4.1 that, after passing to a subsequence, there exist sequences $\left(\varepsilon_{k}\right)$ in $(0, \infty)$ and $\left(\zeta_{k}\right)$ in $\Theta^{\Gamma}$, and a nontrivial $\phi$-equivariant solution $\omega$ to the limit problem (3) such that $\zeta_{k} \rightarrow \zeta, \varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty, J_{\infty}(\omega)=\ell_{\infty}^{\phi}$,

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|u_{k}-\varepsilon_{k}^{\frac{2-n}{2}} \omega\left(\frac{\cdot-\zeta_{k}}{\varepsilon_{k}}\right)\right\|=0 \tag{13}
\end{equation*}
$$

and

$$
\left[\operatorname{dist}\left(\zeta, \mathbb{R}^{n-1} \times\{0\}\right)\right]=\min _{x \in \bar{\Theta}}\left[\operatorname{dist}\left(x, \mathbb{R}^{n-1} \times\{0\}\right)\right]
$$

Equation (13) implies that $v_{\delta_{k}, *}$ satisfies (3). This concludes the proof.
Theorem 4.4. For $\delta \in\left(0, \delta_{0}\right)$ and $p \in\left(2_{N, m}^{*}, 2_{N, m+\mathfrak{d}}^{*}\right)$ let $v_{\delta, p}$ be a least energy solution to the problem $\left(\wp_{\delta, p}\right)$. Assume that

$$
\operatorname{dist}\left(\Theta^{\Gamma}, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(\Theta, \mathbb{R}^{n-1} \times\{0\}\right)
$$

Then, there exists $\lambda_{*} \leq 0$ such that, if $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$, there exist sequences $\left(\delta_{k}\right)$ in $\left(0, \delta_{0}\right),\left(\varepsilon_{k}\right)$ in $(0, \infty),\left(p_{k}\right)$ in $\left(2_{N, m}^{*}, 2_{N, m+\mathfrak{d}}^{*}\right)$, and $\left(\zeta_{k}\right)$ in $\Theta^{\Gamma}$, and a nontrivial solution $\omega$ to the limit problem (3) such that
(i) $\delta_{k} \rightarrow 0, p_{k} \rightarrow 2_{N, m}^{*}, \varepsilon_{k}^{-1} \operatorname{dist}\left(\zeta_{k}, \partial \Theta\right) \rightarrow \infty$, and $\zeta_{k} \rightarrow \zeta$ with

$$
\operatorname{dist}\left(\zeta, \mathbb{R}^{n-1} \times\{0\}\right)=\operatorname{dist}\left(\Theta, \mathbb{R}^{n-1} \times\{0\}\right)
$$

(ii) $\omega$ is $\phi$-equivariant and has minimal energy among all nontrivial $\phi$-equivariant solutions to the problem (3),
(iii) $v_{\delta_{k}, p_{k}}=\widetilde{\omega}_{\varepsilon_{k}, \zeta_{k}}+o(1)$ in $D^{1,2}\left(\mathbb{R}^{N}\right)$, where

$$
\widetilde{\omega}_{\varepsilon_{k}, \zeta_{k}}(y, z):=\varepsilon_{k}^{(2-n) / 2} \omega\left(\frac{(y,|z|)-\zeta_{k}}{\varepsilon_{k}}\right) .
$$

Moreover, $\lambda_{*}<0$ if $m \geq 2$.
Proof. Let $\lambda_{*}$ be the number given by Theorem 4.1. Fix $\lambda \in\left(\lambda_{*}, \infty\right) \cup\{0\}$. Let $u_{\delta, p}$ be the least energy solution to the problem $\left(\wp_{\delta, p}^{\#}\right)$ given by $v_{\delta, p}(y, z)=$ $u_{\delta, p}(y,|z|)$ and let $t_{\delta, p} \in(0, \infty)$ be such that $\widetilde{u}_{\delta, p}:=t_{\delta, p} u_{\delta, p} \in \mathcal{N}_{\delta, 2_{n}^{*}}^{\phi} \subset \mathcal{N}_{*}^{\phi}$. Proposition 2.5 and Lemma 4.2 allow us to choose $\delta_{k} \in\left(0, \delta_{0}\right)$ and $p_{k} \in$ $\left(2_{n}^{*}, 2_{n-\mathfrak{d}}^{*}\right)$ such that $\delta_{k} \rightarrow 0, p_{k} \rightarrow 2_{n}^{*}$, and $J_{*}\left(\widetilde{u}_{k}\right) \rightarrow \ell_{*}^{\phi}$, where $\widetilde{u}_{k}:=\widetilde{u}_{\delta_{k}, p_{k}}$. The rest of the proof is the same as that of Theorem 4.3

Finally, we derive Theorems 1.1 and 1.2 from Theorems 2.3 and 4.4.
Proof of Theorem 1.1. Let $\Gamma:=O(n-1)$ and $\phi$ be the trivial homomorphism $\phi \equiv 1$. Then, $B^{\Gamma}=B \cap[\{0\} \times(0, \infty)]$. A $\phi$-equivariant function is simply a $\Gamma$-invariant function and, as the standard bubble is radial, it is the least energy $\Gamma$-invariant solution to the problem (3), which is unique up to translations and dilations. Since $\operatorname{dim}(\Gamma x)=n-2 \geq 1$ for every $x \in B \backslash B^{\Gamma}$, applying Theorems 2.3 and 4.4 to $\Theta:=B$ with this group action we obtain Theorem 1.1.

Proof of Theorem 1.2. For $n \geq 5$, let $\Gamma$ be the subgroup of $O(n-1)$ generated by $\left\{\mathrm{e}^{\mathrm{i} \vartheta}, \alpha, \tau: \vartheta \in[0,2 \pi), \alpha \in O(n-5)\right\}$ acting on a point $y=(\eta, \xi) \in$ $\mathbb{C}^{2} \times \mathbb{R}^{n-5} \equiv \mathbb{R}^{n-1}, \eta=\left(\eta_{1}, \eta_{2}\right) \in \mathbb{C} \times \mathbb{C}$, as

$$
\mathrm{e}^{\mathrm{i} \vartheta} y:=\left(\mathrm{e}^{\mathrm{i} \vartheta} \eta, \xi\right), \quad \alpha y:=(\eta, \alpha \xi), \quad \tau y:=\left(-\bar{\eta}_{2}, \bar{\eta}_{1}, \xi\right),
$$

and let $\phi$ be the homomorphism given by $\phi\left(\mathrm{e}^{\mathrm{i} \vartheta}\right)=1=\phi(\alpha)$ and $\phi(\tau)=-1$. Then, $B^{\Gamma}=B \cap[\{0\} \times(0, \infty)]$. If $n=5$ then $\operatorname{dim}(\Gamma y)=1$ for every $y \in$ $\mathbb{R}^{n-1} \backslash\{0\}$, whereas for $n \geq 6$ we have that

$$
\operatorname{dim}(\Gamma y)= \begin{cases}n-5 & \text { if } \eta \neq 0 \text { and } \xi \neq 0 \\ 1 & \text { if } \xi=0, \\ n-6 & \text { if } \eta=0\end{cases}
$$

Therefore, if $n=5$ or $n \geq 7$, we have that $\operatorname{dim}(\Gamma x) \geq 1$ for every $x \in B \backslash B^{\Gamma}$. Notice that any point $x_{0}=(\eta, \xi) \in B$ with $\eta \neq 0$ satisfies condition (5). Hence, Theorem 1.2 follows from Theorems 2.3 and 4.4.

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# On general properties of $n$-th order retarded functional differential equations 

Pierluigi Benevieri, Alessandro Calamai, Massimo Furi and Maria Patrizia Pera<br>"Dedicated to the outstanding mathematician Jean Mawhin"


#### Abstract

Consider the second order RFDE (retarded functional differential equation) $x^{\prime \prime}(t)=f\left(t, x_{t}\right)$, where $f$ is a continuous realvalued function defined on the Banach space $\mathbb{R} \times C^{1}([-r, 0], \mathbb{R})$. The weak assumption of continuity on $f$ (due to the strong topology of $\left.C^{1}([-r, 0], \mathbb{R})\right)$ makes not convenient to transform this equation into a first order RFDE of the type $z^{\prime}(t)=g\left(t, z_{t}\right)$. In fact, in this case, the associated $\mathbb{R}^{2}$-valued function $g$ could be discontinuous (with the $C^{0}$ topology) and, in addition, not necessarily defined on the whole space $\mathbb{R} \times C\left([-r, 0], \mathbb{R}^{2}\right)$. Consequently, in spite of what happens for ODEs, the classical results regarding existence, uniqueness, and continuous dependence on data for first order RFDEs could not apply. Motivated by this obstruction, we provide results regarding general properties, such as existence, uniqueness, continuous dependence on data and continuation of solutions of RFDEs of the type $x^{(n)}(t)=f\left(t, x_{t}\right)$, where $f$ is an $\mathbb{R}^{k}$-valued continuous function on the Banach space $\mathbb{R} \times C^{(n-1)}\left([-r, 0], \mathbb{R}^{k}\right)$. Actually, for the sake of generality, our investigation will be carried out in the case of infinite delay.


Keywords: Retarded functional differential equations (RFDEs), RFDEs with infinite delay, initial value problems, properties of solutions.
MS Classification 2010: 34K05, 34C40.

## 1. Introduction

Delay differential equations and retarded functional differential equations (for short RFDEs) represent a well-studied subject in view of many applications (see e.g. $[1,9,11])$. Recently, we devoted a series of papers to first and second order RFDEs on possibly noncompact manifolds, allowing also the case of infinite
delay (see $[2,3,4,5,6,7,8]$ ). We mostly focused on the problem of existence of periodic solutions, as well as on the structure of the set of solutions of parameterized RFDEs. For such equations, we obtained global continuation results by means of topological methods. In this framework, we also performed a preliminary study in the paper [7] in which we investigated general properties of RFDEs with infinite delay on differentiable manifolds.

Here we settle in the context of Euclidean spaces, and we tackle a different but related problem regarding higher order RFDEs whose reduction to first order equations is not convenient, in spite of what happens for ODEs.

Consider, for example, the second order RFDE

$$
\begin{equation*}
x^{\prime \prime}(t)=-\varepsilon x^{\prime}(t)+g\left(x_{t}\right), \tag{1}
\end{equation*}
$$

where $\varepsilon>0$ and $\left.g: C^{0}[-r, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}(r>0)$ is a continuous function. Here, as usual when dealing with RFDEs, if $x: J \rightarrow \mathbb{R}^{k}$ is a function defined on an interval, given $t \in J, x_{t}$ denotes the map $\theta \in[-r, 0] \mapsto x(t+\theta)$, whenever it makes sense, that is, whenever $[t-r, t] \subseteq J$.

Obviously, an equation (no matter how it is written) is well-defined if it is clear what is a solution. For the equation (1), as well as for a broader class of second order RFDEs, two different notions are prominent. The first one is the following.

Definition $1.1\left(C^{0}\right.$-solution of (1)). A function $x: J \rightarrow \mathbb{R}^{k}$, defined on an interval $J$, is a $C^{0}$-solution of (1) if it is continuous and satisfies eventually the equality

$$
x^{\prime \prime}(t)=-\varepsilon x^{\prime}(t)+g\left(x_{t}\right),
$$

meaning that there exists $\tau \in J(\tau<\sup J)$ such that $[\tau-r, \tau] \subseteq J$ and the equality is verified for each $t \in(\tau,+\infty) \cap J$.

The second definition of solution is given by modifying the previous one just by additionally requiring $x$ to be of class $C^{1}$.

Definition $1.2\left(C^{1}\right.$-solution of (1)). A function $x: J \rightarrow \mathbb{R}^{k}$, defined on an interval $J$, is a $C^{1}$-solution of (1) if it is of class $C^{1}$ and satisfies eventually the equality

$$
x^{\prime \prime}(t)=-\varepsilon x^{\prime}(t)+g\left(x_{t}\right) .
$$

Obviously, with any one of these notions, a solution turns out to be eventually of class $C^{2}$.

In spite of the similarity of the two notions of solution, when dealing with an initial value problem such as

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t)=-\varepsilon x^{\prime}(t)+g\left(x_{t}\right), \quad t>\tau  \tag{2}\\
x_{\tau}=\eta,
\end{array}\right.
$$

with $\eta \in C^{1}\left([-r, 0], \mathbb{R}^{k}\right)$, the above definitions yield very divergent consequences.

If one seeks for a $C^{1}$-solution, the problem is, in some sense, well-posed, since any $C^{1}$-solution must satisfy the additional initial condition $x^{\prime}(\tau)=\eta^{\prime}(0)$. Therefore, under suitable assumptions on $g$ (such as Lipschitz continuity), one gets the uniqueness in the future (i.e. for $t \geq \tau$ ). To see this, it is sufficient to transform the above problem into the following first order initial value problem in $\mathbb{R}^{k} \times \mathbb{R}^{k}$ :

$$
\left\{\begin{array}{l}
x^{\prime}(t)=y(t), \quad t>\tau  \tag{3}\\
y^{\prime}(t)=-\varepsilon y(t)+g\left(x_{t}\right), \quad t>\tau \\
x_{\tau}=\eta, \\
y(\tau)=\eta^{\prime}(0)
\end{array}\right.
$$

For existence, as well as uniqueness, results regarding initial value problems such as (3) we suggest [14].

On the other hand, if one seeks for solutions of (2) according to the first definition, the problem is under-determined: it becomes well-posed for any additional condition $x^{\prime}\left(\tau^{+}\right)=c$, with $c \in \mathbb{R}^{k}$ (where $x^{\prime}\left(\tau^{+}\right)$denotes the right derivative of $x$ at $\tau$ ), so that the uniqueness of the solution of problem (2) is never obtained.

Contrary to the above first notion of solution (the $C^{0}$ one), the second one is suitable for the following general second order RFDE in $\mathbb{R}^{k}$ :

$$
\begin{equation*}
x^{\prime \prime}(t)=h\left(t, x_{t}, x_{t}^{\prime}\right), \tag{4}
\end{equation*}
$$

where $\left.\left.h: \mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k}\right) \times C^{0}[-r, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ is a continuous function, and $x_{t}^{\prime}$ is a shortened form of $\left(x^{\prime}\right)_{t}$. Of course, (4) includes as a particular case the equation (1). To see this, put $h(t, \varphi, \psi)=-\varepsilon \psi(0)+g(\varphi)$.

Anyhow, if one is interested in the $C^{1}$-solutions of (4), it is convenient to consider the graphically simpler and more general equation

$$
\begin{equation*}
x^{\prime \prime}(t)=f\left(t, x_{t}\right), \tag{5}
\end{equation*}
$$

where $f$ is an $\mathbb{R}^{k}$-valued continuous function defined on $\mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k}\right)$. To see that (5) is, in fact, more general than (4), it is sufficient to define $f(t, \varphi)=h\left(t, \varphi, \varphi^{\prime}\right)$.

Apart its graphic simplicity, the equation (5) has two advantages. Firstly, $f$ may be defined on $\mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k}\right)$ and not necessarily on $\left.\mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k}\right)$. Secondly, the assumption of continuity of $f$ on $\mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k}\right)$ is a milder condition than the one we would get by requiring $f$ continuous with the topology induced by $\left.\mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k}\right)$, consequence of the fact that the $C^{1}$ topology is stronger than the $C^{0}$ one.

Of course if $f$ is defined and continuous on $\left.\mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k}\right)$, it is, in particular, defined and continuous on the Banach space $\mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k}\right)$.

However, dealing with the equation (5) has a disadvantage: when (5) is converted into the first order equation $z^{\prime}(t)=g\left(t, z_{t}\right)$ by putting $z(t)=(x(t), y(t))$ and $g\left(t, z_{t}\right)=\left(y(t), f\left(t, x_{t}\right)\right)$, the associated continuous function

$$
g: \mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k} \times \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k} \times \mathbb{R}^{k}, \text { given by }(t, \varphi, \psi) \mapsto(\psi(0), f(t, \varphi))
$$

could not be compatible with any $\left(\mathbb{R}^{k} \times \mathbb{R}^{k}\right)$-valued continuous function defined on $\mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k} \times \mathbb{R}^{k}$ ). In other words, $g$ could be discontinuous with the coarse topology induced on $\mathbb{R} \times C^{1}\left([-r, 0], \mathbb{R}^{k} \times \mathbb{R}^{k}\right)$ by the containing Banach space $\left.\mathbb{R} \times C^{0}[-r, 0], \mathbb{R}^{k} \times \mathbb{R}^{k}\right)$. Thus, the classical existence, as well as uniqueness, results regarding initial value problems for first order RFDEs could not apply.

Our purpose is to alleviate this disadvantage by proving general properties of the equation (5), as well as higher order equations of the type

$$
\begin{equation*}
x^{(n)}(t)=f\left(t, x_{t}\right), \tag{6}
\end{equation*}
$$

with $f: \mathbb{R} \times C^{(n-1)}\left([-r, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ continuous. These equations will sometimes be associated with an initial condition of Cauchy type as $x_{\tau}=\eta$, with $\tau \in \mathbb{R}$ and $\eta \in C^{(n-1)}\left([-r, 0], \mathbb{R}^{k}\right)$, obtaining results regarding existence, uniqueness and continuous dependence on data.

Actually, for the sake of generality, we will investigate the case of infinite delay, which includes the equation (6) as a special case.

As already pointed out, delay equations and RFDEs in Euclidean spaces have been studied by many authors from different points of view. For a general reference about RFDEs with finite delay, we suggest the monograph by Hale and Verduyn Lunel [14]. Among others, we refer also to the works of Gaines and Mawhin [12], Nussbaum [17, 18] and Mallet-Paret, Nussbaum and Paraskevopoulos [16]. For RFDEs with infinite delay we recommend the articles of Hale and Kato [13] and, more recently, of Oliva and Rocha [19], and the book by Hino, Murakami and Naito [15].

In the above papers and books, the basic properties of RFDEs in $\mathbb{R}^{k}$ have been investigated, as well as other related issues (e.g. characterizing the space of initial functions of RFDEs with infinite delays, see [15]). In spite of this, to the best of our knowledge, our particular point of view on higher order RFDEs has been never pursued. The main purpose of this paper is to fill this gap.

## 2. Preliminaries

Given $m \in\{0,1,2, \ldots\}$ and $b \in \mathbb{R}$, we will denote by $B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ the space of all functions $x:(-\infty, b] \rightarrow \mathbb{R}^{k}$ which are bounded and uniformly continuous with their derivatives up to the order $m$. This is a Banach space, being a closed subset of the space $B C^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ of the $C^{m}$-functions which are
bounded with their derivatives up to the order $m$. As usual, $B U\left((-\infty, b], \mathbb{R}^{k}\right)$ and $B C\left((-\infty, b], \mathbb{R}^{k}\right)$ stand for $B U^{0}\left((-\infty, b], \mathbb{R}^{k}\right)$ and $B C\left((-\infty, b], \mathbb{R}^{k}\right)$, respectively.

When $m>0$, in $B C^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$, and consequently in $B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$, among the many equivalent Banach norms we consider the following:

$$
\|x\|=\sup _{t \in(-\infty, b]}|x(t)|+\sup _{t \in(-\infty, b]}\left|x^{(m)}(t)\right|
$$

where here, and throughout the paper, $|\cdot|$ is the Euclidean norm of $\mathbb{R}^{k}$.
For simplicity's sake, the norm of any infinite-dimensional Banach space will be denoted uniquely by $\|\cdot\|$. No confusion should arise: the space whose norm is considered will be apparent from the context.

We recall that a subset $Q$ of $B C\left((-\infty, b], \mathbb{R}^{k}\right)$ is precompact (i.e. totally bounded) if and only if it is bounded and given any $\varepsilon>0$ there exists a finite covering $\mathcal{F}$ of arbitrary subsets of $(-\infty, b]$ such that the oscillation of any $\varphi \in Q$ in each $S \in \mathcal{F}$ is less than $\varepsilon$ (see e.g. [10, Part 1, IV.6.5]). Of course, the same holds true for the subspace $B U\left((-\infty, b], \mathbb{R}^{k}\right)$ of $B C\left((-\infty, b], \mathbb{R}^{k}\right)$. Consequently, a subset $Q$ of $B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ is precompact if and only if it is bounded and given any $\varepsilon>0$ there exists a finite covering $\mathcal{F}$ of $(-\infty, b]$ such that for any $\varphi \in Q$, the oscillation of $\varphi^{(m)}$ in each $S \in \mathcal{F}$ is less than $\varepsilon$. Clearly, the space being complete, any precompact subset of $B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ is relatively compact.

Remark 2.1. Due to the fact that in $B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ the derivative of order $m$ of any function is uniformly continuous, a subset of this space is totally bounded only if it is bounded and made up of functions whose $m$-th derivatives are equi-uniformly continuous. The converse is not true even when $m=0$ : think about a traveling wave with compact support that goes to $-\infty$ and preserves its shape.

Let $x: J \rightarrow \mathbb{R}^{k}$ be a continuous function defined on an unbounded below real interval (that is, $J$ is either a left, open or closed, half-line, or it coincides with $\mathbb{R}$ ). As usual, given any $t \in J$, by $x_{t}:(-\infty, 0] \rightarrow \mathbb{R}^{k}$ we mean the function $\theta \mapsto x(t+\theta)$.
Remark 2.2. One can easily check that the function that associates to any $(t, x) \in(-\infty, b] \times B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$ the element $x_{t} \in B U^{m}\left((-\infty, 0], \mathbb{R}^{k}\right)$ is continuous. Thus, in particular, given $x \in B U^{m}\left((-\infty, b], \mathbb{R}^{k}\right)$, the map $t \in$ $(-\infty, b] \mapsto x_{t}$ is a continuous curve in $B U^{m}\left((-\infty, 0], \mathbb{R}^{k}\right)$.

Let $n$ be a positive integer and $f: \Omega \rightarrow \mathbb{R}^{k}$ a continuous function defined on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Let us consider in $\mathbb{R}^{k}$ a retarded functional differential equation of order $n$ of the type

$$
\begin{equation*}
x^{(n)}(t)=f\left(t, x_{t}\right) \tag{7}
\end{equation*}
$$

Definition 2.3 (Solution of the equation (7)). A solution of (7) is a function $x: J \rightarrow \mathbb{R}^{k}$, defined on an unbounded below interval, such that

$$
x_{t} \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

for all $t \in J$, which verifies eventually the equality $x^{(n)}(t)=f\left(t, x_{t}\right)$. That is, $x$ is a solution of (7) if there exists $\tau<\sup J$, such that, for each $t \in(\tau,+\infty) \cap J$, one has $\left(t, x_{t}\right) \in \Omega$ and $x^{(n)}(t)=f\left(t, x_{t}\right)$.

Obviously, according to Remark 2.2, any solution of the equaltion (7) is eventually of class $C^{n}$.

A solution of (7) is said to be maximal if it is not a proper restriction of another solution. As in the case of ODEs, Zorn's lemma implies that any solution is the restriction of a maximal solution.

Given an initial function $\eta \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and an instant $\tau$ such that $(\tau, \eta) \in \Omega$, we will be interested in the following initial value problem:

$$
\left\{\begin{array}{l}
x^{(n)}(t)=f\left(t, x_{t}\right), \quad t>\tau  \tag{8}\\
x_{\tau}=\eta
\end{array}\right.
$$

Definition 2.4 (Solution of the initial value problem (8)). $A$ solution of (8) is a solution $x: J \rightarrow \mathbb{R}^{k}$ of the equation (7) such that $\sup J>\tau, x^{(n)}(t)=f\left(t, x_{t}\right)$ for all $t \in(\tau,+\infty) \cap J$, and $x_{\tau}=\eta$.

Clearly, a function $x: J \rightarrow \mathbb{R}^{k}$, defined on an unbounded below interval, is a solution of (8) if and only if $\sup J>\tau$ and for all $t \in J$ one has

$$
x(t)= \begin{cases}\sum_{j=0}^{n-1} \frac{(t-\tau)^{j}}{j!} \eta^{(j)}(0)+\int_{\tau}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f\left(s, x_{s}\right) d s, & \text { if } t \geq \tau  \tag{9}\\ \eta(t-\tau), & \text { if } t \leq \tau,\end{cases}
$$

where $\eta^{(0)}(0):=\eta(0)$.
Remark 2.5. In spite of the fact that the $n$-th order derivative of a solution $x: J \rightarrow \mathbb{R}^{k}$ of (8) may not exist at $t=\tau$, the right $n$-th derivative $x^{(n)}\left(\tau_{+}\right)$of $x$ at $\tau$ always exists and is equal to $f(\tau, \eta)$. In fact, by definition, $x^{(n)}\left(\tau_{+}\right)$is the $n$-th derivative at $\tau$ of the restriction $x_{+}$of $x$ to the interval $[\tau,+\infty) \cap J$, and this restriction, because of (9), is a $C^{n}$ function such that $x_{+}^{(n)}(t)=f\left(t, x_{t}\right)$ for all $t \in[\tau,+\infty) \cap J$.

## 3. Existence

Here, our attention is devoted to the existence, global or local, of the solutions of the initial value problem (8).

A continuous function $f: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ will be called strictly retarded if there exists $\varepsilon>0$ such that the value $f(t, \varphi)$ depends only on $t$ and the restriction of $\varphi$ to $(-\infty,-\varepsilon]$. That is, $\varphi_{1}(\theta)=\varphi_{2}(\theta)$ for all $\theta \in(-\infty,-\varepsilon]$ implies $f\left(t, \varphi_{1}\right)=f\left(t, \varphi_{2}\right)$ for all $t \in \mathbb{R}$.
LEMMA 3.1. Let $f: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ be a strictly retarded function and choose $\tau \in \mathbb{R}$. Then,
(1) any $\xi \in B U^{n-1}\left((-\infty, \tau], \mathbb{R}^{k}\right)$ is the restriction of a unique maximal solution of equation (7);
(2) any maximal solution of equation (7) is defined on the whole real line.

Proof. (1) Choose any $\xi \in B U^{n-1}\left((-\infty, \tau], \mathbb{R}^{k}\right)$ and let $\hat{\xi}$ be the $C^{n-1}$ extension of $\xi$ to the whole real line given by

$$
\hat{\xi}(t)=\sum_{j=0}^{n-1} \frac{(t-\tau)^{j}}{j!} \xi^{(j)}(\tau), \quad \text { for } t \geq \tau
$$

and, of course, $\hat{\xi}(t)=\xi(t)$ for $t \leq \tau$. Clearly $\hat{\xi}_{s} \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$, for any $s \in \mathbb{R}$, and the map $s \mapsto \hat{\xi}_{s}$ is continuous (see Remark 2.2). Thus, we may define the $C^{n-1}$ function

$$
\hat{x}(t)= \begin{cases}\hat{\xi}(t)+\int_{\tau}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f\left(s, \hat{\xi}_{s}\right) d s, & \text { if } t \geq \tau \\ \hat{\xi}(t), & \text { if } t \leq \tau\end{cases}
$$

Since $f$ is strictly retarded, there exists $\varepsilon>0$ such that $f(t, \varphi)$ depends only on $t$ and the restriction of $\varphi$ to $(-\infty,-\varepsilon]$. Hence, for any $t$ in the interval $(\tau, \tau+\varepsilon]$, one has $f\left(t, \hat{\xi}_{t}\right)=f\left(t, \hat{x}_{t}\right)$ and, thus, $\hat{x}^{(n)}(t)=f\left(t, \hat{x}_{t}\right)$. This proves that the restriction $x$ of $\hat{x}$ to $(-\infty, \tau+\varepsilon]$ is a solution of (7). Therefore, by Zorn's lemma, $x$ is the restriction of a maximal solution of (7), still denoted for simplicity by $x$, and defined on an interval $J$ containing $(-\infty, \tau+\varepsilon]$. Now, taking into account (9), again from the fact that $f$ is strictly retarded it follows that, for any $t \in J$, the value $x(t)$ depends only on the restriction of $x$ to $(-\infty, t-\varepsilon]$. This implies the uniqueness of the maximal solution of (7).
(2) Let $x: J \rightarrow \mathbb{R}^{k}$ be a maximal solution of (7) and, by contradiction, assume that $\sup J<+\infty$. Due to the fact that $f$ is strictly retarded, the same argument used in the proof of (1) shows that $x$ can be extended to a solution defined on $(-\infty, \sup J+\varepsilon)$, contradicting the maximality of $x$.

Notice that, as a consequence of both the assertions of Lemma 3.1, if $f$ is a strictly retarded function, then the initial value problem (8) admits exactly one global solution (i.e. a solution defined on the whole real line). This fact will be applied in the proof of the following lemma, which is crucial in our proof of Theorem 3.3 below.

LEmma 3.2 (Global existence). Let $f: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ be a continuous function with bounded image. Then, problem (8) has a global solution.

Proof. Let $\left\{\varepsilon_{j}\right\}$ be a sequence of positive numbers converging to 0 and consider the following auxiliary problem depending on $j \in \mathbb{N}$ :

$$
\left\{\begin{array}{l}
x^{(n)}(t)=f\left(t, x_{t-\varepsilon_{j}}\right), \quad t>\tau  \tag{10}\\
x_{\tau}=\eta
\end{array}\right.
$$

Let $f_{j}: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ be defined by $f_{j}(t, \varphi)=f\left(t, \varphi_{-\varepsilon_{j}}\right)$; so that in problem (10) the expression $f\left(t, x_{t-\varepsilon_{j}}\right)$ can be replaced by $f_{j}\left(t, x_{t}\right)$. Clearly, $f_{j}$ is a strictly retarded function. Hence, because of Lemma 3.1, problem (10) has a unique solution $x^{j}$ defined on the whole real line. Now, observe that the restrictions to the half line $[\tau,+\infty)$ of the functions $x^{j}$ are all of class $C^{n}$ (see Remark 2.5) and, $f$ having bounded image, these restrictions have equibounded $n$-th derivatives. Consequently, in any compact interval $[\tau, b]$ these functions are also equibounded, due to the fact that their values at $\tau$ are all equal to $\eta(0)$. Thus, taking into account that in the left half line $(-\infty, \tau]$ the functions $x^{j}$ do not depend on $j$, applying Ascoli's Theorem to any compact interval $[\tau, b]$, and by using a standard diagonal procedure, we may assume, without loss of generality, that the sequence $\left\{x^{j}\right\}$ has the following properties:

- there exists a function $x: \mathbb{R} \rightarrow \mathbb{R}^{k}$ such that $x^{j}(t) \rightarrow x(t)$, for any $t \in \mathbb{R}$;
- $x_{s} \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$, for all $s \in \mathbb{R}$;
- $x_{s}^{j} \rightarrow x_{s}$ in the space $B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$, for all $s \in \mathbb{R}$.

Now observe that, as $j \rightarrow+\infty$, one actually has $x_{s-\varepsilon_{j}}^{j} \rightarrow x_{s}$, for any $s \in \mathbb{R}$ (see Remark 2.2). Consequently, by applying Lebesgue's Dominated Convergence Theorem in equality (9), with $f_{j}\left(t, x_{t}\right)$ in place of $f\left(t, x_{t}\right)$, we get that $x$ is a solution of the initial value problem (8), proving the assertion.

Theorem 3.3 (Local existence). Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be a continuous function on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and $(\tau, \eta) \in \Omega$. Then, the initial value problem (8) admits at least one solution. In particular, any maximal solution is defined on an open interval.

Proof. Let $N \subseteq \Omega$ be a closed neighborhood of $(\tau, \eta)$ whose image under $f$ is bounded. Due to the Tietze extension Theorem, the restriction $\left.f\right|_{N}$ has a continuous extension $\hat{f}: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ with bounded image. By applying Lemma 3.2 to the problem

$$
\left\{\begin{aligned}
x^{(n)}(t) & =\hat{f}\left(t, x_{t}\right), \quad t>\tau \\
x_{\tau} & =\eta
\end{aligned}\right.
$$

we get the existence of a solution $\hat{x}$ defined on the whole real line. Because of the continuity of the map $t \mapsto\left(t, \hat{x}_{t}\right)$, and taking into account that $\left(\tau, \hat{x}_{\tau}\right)=$ $(\tau, \eta) \in N$, one can find $\delta>0$ such that $\left(t, \hat{x}_{t}\right) \in N$ for all $t \in[\tau, \tau+\delta)$. Since $f=\hat{f}$ in $N$, the restriction of $\hat{x}$ to the half line $(-\infty, \tau+\delta)$ is a solution of problem (8).

It remains to show that the domain of a maximal solution, call it $x$, cannot be a closed interval of the type $(-\infty, b]$. In fact, if this were the case, by applying the above argument to problem (8) with initial data $(\tau, \eta)=\left(b, x_{b}\right)$ we would get a contradiction.

## 4. Uniqueness

In this section we will give conditions ensuring the unique dependence on the past of the solutions of equation (7). We need the following folk result, whose proof is given here for the sake of completeness.

LEmma 4.1. Let $\alpha:[\tau, \tau+h) \rightarrow \mathbb{R}^{k}(0<h \leq+\infty)$ be a $C^{1}$ function such that $\alpha(\tau)=0$ and $\left|\alpha^{\prime}(t)\right| \leq c \sup _{s \in[\tau, t]}|\alpha(s)|$ for some constant $c \geq 0$ and all $t \in[\tau, \tau+h)$. Then, $\alpha(t) \equiv 0$ in $[\tau, \tau+h)$.

Proof. Assume the contrary. Then, without loss of generality, we may suppose that $\tau$ is such that $\alpha$ is nontrivial in any interval $[\tau, \tau+\delta]$, with $0<\delta<h$. Take $\delta$ such that $\delta c<1$ and let $t_{0} \in[\tau, \tau+\delta]$ satisfy the condition $\left|\alpha\left(t_{0}\right)\right|=$ $\max _{s \in[\tau, \tau+\delta]}|\alpha(s)|>0$. We have

$$
\left|\alpha\left(t_{0}\right)\right|=\left|\alpha\left(t_{0}\right)-\alpha(\tau)\right| \leq\left(t_{0}-\tau\right) \sup _{s \in\left[\tau, t_{0}\right]}\left|\alpha^{\prime}(s)\right| \leq \delta c\left|\alpha\left(t_{0}\right)\right| .
$$

Being $\delta c<1$, the above inequality implies $\alpha\left(t_{0}\right)=0$, and this is a contradiction.

Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be continuous on an open subset of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Given an open set $U \subseteq \Omega$, we will say that $f$ is compactly Lipschitz in $U$ with respect to the second variable or, for short, $c$-Lipschitz in $U$, if, given any compact subset $Q$ of $U$, there exists $L \geq 0$ such that

$$
|f(t, \varphi)-f(t, \psi)| \leq L\|\varphi-\psi\|
$$

for all $(t, \varphi),(t, \psi) \in Q$.
Moreover, we will say that $f$ is locally $c$-Lipschitz in $\Omega$ if for any $(\tau, \eta) \in \Omega$ there exists an open neighborhood of $(\tau, \eta)$ in $\Omega$ in which $f$ is c-Lipschitz. In spite of the fact that a locally Lipschitz function is not necessarily (globally) Lipschitz, one could actually show that if $f$ is locally c-Lipschitz in $\Omega$, then it is
also (globally) c-Lipschitz in $\Omega$. As a consequence, if $f$ is $C^{1}$ or, more generally, locally Lipschitz in the second variable, then it is additionally c-Lipschitz.

Roughly speaking, Theorem 4.2 below shows that, if $f$ is c-Lipschitz in $\Omega$, then the future of the solutions of equation (7) is uniquely determined by the past. In particular, under this assumption, the initial value problem (8) has a unique maximal solution, which is necessarily defined on an open (unbounded below) interval, as stated in Theorem 3.3.

Theorem 4.2 (Uniqueness). Let $\Omega$ be an open subset of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and let $f: \Omega \rightarrow \mathbb{R}^{k}$ be c-Lipschitz. Let $x^{1}: J_{1} \rightarrow \mathbb{R}^{k}, x^{2}: J_{2} \rightarrow \mathbb{R}^{k}$ be two maximal solutions of equation (7). If there exists $\tau \in J_{1} \cap J_{2}$ such that $x^{1}(t)=$ $x^{2}(t)$ for $t \leq \tau$ and $\left(x^{i}\right)^{(n)}(t)=f\left(t, x_{t}^{i}\right)$ for $t \in J_{i}(i=1,2), t>\tau$, then $J_{1}=J_{2}$ and $x^{1}=x^{2}$.

Proof. Since, according to Theorem 3.3, $J_{1} \cap J_{2}$ is an open interval, there exists $h>0$ such that $[\tau, \tau+h] \subseteq J_{1} \cap J_{2}$. Then, each one of the sets

$$
Q_{i}=\left\{\left(t, x_{t}^{i}\right) \in \Omega \subseteq \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right): t \in[\tau, \tau+h]\right\}, \quad i=1,2
$$

is compact, as the image of the continuous curve $t \mapsto\left(t, x_{t}^{i}\right) \in \Omega$ defined on $[\tau, \tau+h]$. Since $f$ is c-Lipschitz in $\Omega$, there exists $L \geq 0$, corresponding to the compact set $Q=Q_{1} \cup Q_{2}$, such that for any $t \in[\tau, \tau+h]$ we have

$$
\left|f\left(t, x_{t}^{2}\right)-f\left(t, x_{t}^{1}\right)\right| \leq L\left\|x_{t}^{2}-x_{t}^{1}\right\|
$$

Set $\beta(t)=x^{2}(t)-x^{1}(t)$, for $t \in J_{1} \cap J_{2}$. Then, choosing any $t \in[\tau, \tau+h]$, we get

$$
\left|\beta^{(n)}(t)\right| \leq L\left\|\beta_{t}\right\|=L\left(\sup _{\theta \leq 0}|\beta(t+\theta)|+\sup _{\theta \leq 0}\left|\beta^{(n-1)}(t+\theta)\right|\right) .
$$

Consequently, since $\beta(s)=0$ for $s \leq \tau$, one has

$$
\begin{equation*}
\left|\beta^{(n)}(t)\right| \leq L\left(\sup _{s \in[\tau, t]}|\beta(s)|+\sup _{s \in[\tau, t]}\left|\beta^{(n-1)}(s)\right|\right) . \tag{11}
\end{equation*}
$$

Moreover, the fact that $\beta(\tau)=\beta^{\prime}(\tau)=\cdots=\beta^{(n-1)}(\tau)=0$ implies

$$
\begin{equation*}
|\beta(s)|=\left|\int_{\tau}^{s} \frac{(s-\sigma)^{n-2}}{(n-2)!} \beta^{(n-1)}(\sigma) d \sigma\right| \leq M \sup _{\sigma \in[\tau, t]}\left|\beta^{(n-1)}(\sigma)\right|, \text { for } s \in[\tau, t], \tag{12}
\end{equation*}
$$

where

$$
M:=\int_{\tau}^{\tau+h} \frac{(\tau+h-\sigma)^{n-2}}{(n-2)!} d \sigma=\frac{h^{n-1}}{(n-1)!}
$$

Hence, by (11) and (12), we get

$$
\left|\beta^{(n)}(t)\right| \leq L(M+1) \sup _{s \in[\tau, t]}\left|\beta^{(n-1)}(s)\right| .
$$

Now, by applying Lemma 4.1 with $\alpha=\beta^{(n-1)}$, we obtain $\beta^{(n-1)}(t)=0$ for all $t \in[\tau, \tau+h)$ and, thus, again by (12), $\beta(t)=0$ for all $t \in[\tau, \tau+h)$.

This shows that $x^{1}$ and $x^{2}$ coincide in any right neighborhood of $\tau$ contained in the open interval $J_{1} \cap J_{2}$. This implies $J_{1}=J_{2}$ and, consequently, $x_{1}=x_{2}$. In fact, if one of the intervals were strictly contained in the other, the corresponding solution would admit an extension to the bigger interval, contradicting its maximality.

## 5. Continuous dependence on data

Below we will be concerned with upper semicontinuous multivalued maps. We recall that a multivalued map $\Psi$ between two metric spaces $\mathcal{X}$ and $\mathcal{Y}$ is said to be upper semicontinuous if it is compact valued and for any open subset $U$ of $\mathcal{Y}$ the upper inverse image of $U$, i.e. the set $\Psi^{-1}(U)=\{x \in \mathcal{X}: \Psi(x) \subseteq U\}$, is an open subset of $\mathcal{X}$. Equivalently, $\Psi$ is upper semicontinuous if and only if it has closed graph and sends compact sets into relatively compact sets.

The next lemma regards the continuous dependence on the initial data of the set of solutions of problem (8) in the case when $f$ is globally defined with bounded image.

LEMMA 5.1. Let $f: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ be a continuous function with bounded image. Then, for any $b \in \mathbb{R}$, the multivalued map

$$
\Sigma_{b}^{f}:(-\infty, b) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \multimap B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right)
$$

that associates to any $(\tau, \eta)$ the set

$$
\Sigma_{b}^{f}(\tau, \eta)=\left\{x \in B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right): x \text { is a solution of problem }(8)\right\}
$$

is upper semicontinuous.
Proof. According to the characterization stated above, it is enough to show that $\Sigma_{b}^{f}$ has closed graph and sends compact sets into relatively compact sets.

Let us prove first that $\Sigma_{b}^{f}$ has closed graph. To this end, take $(\tau, \eta, x)$ in the graph $G$ of $\Sigma_{b}^{f}$. This means that $x$ belongs to $\Sigma_{b}^{f}(\tau, \eta)$ and, by (9), satisfies

$$
x(t)= \begin{cases}\sum_{j=0}^{n-1} \frac{(t-\tau)^{j}}{j!} \eta^{(j)}(0)+\int_{\tau}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f\left(s, x_{s}\right) d s, & \text { if } \tau \leq t \leq b  \tag{13}\\ \eta(t-\tau), & \text { if } t \leq \tau\end{cases}
$$

Define the subset $F$ of the space

$$
(-\infty, b] \times(-\infty, b) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \times B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right)
$$

consisting of the quadruples $(t, \tau, \eta, x)$ which satisfy (13). Notice that $F$ is closed, because of the continuity of the following four $\mathbb{R}^{k}$-valued functions involved in (13):

$$
\begin{gathered}
(t, \tau, \eta, x) \mapsto x(t) \\
(t, \tau, \eta, x) \mapsto \eta(t-\tau) \\
(t, \tau, \eta, x) \mapsto \sum_{j=0}^{n-1} \frac{(t-\tau)^{j}}{j!} \eta^{(j)}(0) \\
(t, \tau, \eta, x) \mapsto \int_{\tau}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f\left(s, x_{s}\right) d s
\end{gathered}
$$

The continuity of the last one, the integral function, can be deduced from the Dominated Convergence Theorem. Thus, the slices $F_{t}=\{(\tau, \eta, x):(t, \tau, \eta, x) \in$ $F\}$ of $F$ are all closed. Consequently, so is the graph $G=\bigcap_{t \leq b} F_{t}$ of $\Sigma_{b}^{f}$.

It remains to show that $\Sigma_{b}^{f}$ sends compact sets into relatively compact sets. Take a compact set of $(-\infty, b) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and observe that it is contained in a set of the type $[\alpha, \beta] \times A$, with $\beta<b$ and $A$ a compact subset of $B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Thus, it is enough to show that the subset $K=\Sigma_{b}^{f}([\alpha, \beta] \times A)$ of $B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right)$ is relatively compact. To this end, observe that $K$ is relatively compact if and only if so are both $T_{1}(K)$ and $T_{2}(K)$, where

$$
T_{1}: B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right) \rightarrow B U^{n-1}\left((-\infty, \alpha], \mathbb{R}^{k}\right)
$$

and

$$
T_{2}: B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right) \rightarrow C^{n-1}\left([\alpha, b], \mathbb{R}^{k}\right)
$$

denote the restriction operators to the intervals $(-\infty, \alpha]$ and $[\alpha, b]$, respectively.
Let us consider first $T_{1}(K)$. According to Remark 2.2, the map

$$
(\tau, \eta) \in[\alpha,+\infty) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \mapsto \eta_{\alpha-\tau} \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

is continuous. Therefore, $A$ being compact, so is the set

$$
\left\{\eta_{\alpha-\tau}: \tau \in[\alpha, \beta], \eta \in A\right\}
$$

which, up to the isometry

$$
x \in B U^{n-1}\left((-\infty, \alpha], \mathbb{R}^{k}\right) \mapsto x_{\alpha} \in B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

can be identified with $T_{1}(K)$. Thus, $T_{1}(K)$ is compact.
To complete the proof, let us show that $T_{2}(K)$ is relatively compact. To this end, consider in $C^{n-1}\left([\alpha, b], \mathbb{R}^{k}\right)$ the Banach norm

$$
\|x\|=|x(\alpha)|+\max _{t \in[\alpha, b]}\left|x^{(n-1)}(t)\right| .
$$

Observe first that, because of the continuity of the evaluation map $x \mapsto x(\alpha)$, there exists $C>0$ such that $|x(\alpha)| \leq C$ for all $x \in K$. According to Ascoli's theorem, it remains to prove that the functions of $T_{2}(K)$ have equicontinuous derivatives of order $(n-1)$ on $[\alpha, b]$.

To this purpose, choose $\varepsilon>0$ and take any $x \in K$. According to Remark 2.1, the compactness of $A$ implies that $A$ is bounded and its elements have equi-uniformly continuous derivatives of order $n-1$. Hence, there exists $\sigma>0$ such that, if $\theta_{1}, \theta_{2} \in(-\infty, 0]$ with $\left|\theta_{1}-\theta_{2}\right|<\sigma$ and $\eta \in A$, then $\left|\eta^{(n-1)}\left(\theta_{1}\right)-\eta^{(n-1)}\left(\theta_{2}\right)\right|<\varepsilon$. Therefore, recalling that $x(t)=\eta(t-\tau)$ for some $\eta \in A$ and all $t \leq \tau$, one has

$$
\left|x^{(n-1)}\left(t_{1}\right)-x^{(n-1)}\left(t_{2}\right)\right|<\varepsilon
$$

for any $t_{1}, t_{2} \in[\alpha, \tau]$, with $\left|t_{1}-t_{2}\right|<\sigma$.
Now, let us consider the function $x$ in the interval $[\tau, b]$. Since $f$ is bounded, there exists $L>0$ such that $\left|x^{(n)}(t)\right| \leq L$ for all $t \in[\tau, b]$, and, consequently, $x^{(n-1)}$ is Lipschitz continuous on $[\tau, b]$, with constant $L$.

Now, by taking $\delta=\min \{\sigma, \varepsilon / L\}$, we obtain

$$
\left|x^{(n-1)}\left(t_{1}\right)-x^{(n-1)}\left(t_{2}\right)\right|<2 \varepsilon
$$

for any $t_{1}, t_{2} \in[\alpha, b]$ with $\left|t_{1}-t_{2}\right|<\delta$. Since $x \in K$ is arbitrary, this proves that $T_{2}(K)$ is relatively compact in $C^{n-1}\left([\alpha, b], \mathbb{R}^{k}\right)$.

Our purpose now is to remove the assumptions, in Lemma 5.1, that the function $f$ is defined on the whole space $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and has bounded image. More precisely, we will prove a result concerning the continuous dependence of the solutions of problem (8) on the initial data, in the general case in which the function $f$ is merely continuous on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. To this end we will previously extend the validity of the notation $\Sigma_{b}^{f}$ introduced in Lemma 5.1.

Take $b \in \mathbb{R}$ and assume that $(\tau, \eta) \in \Omega$, with $\tau<b$, is such that any maximal solution of (8) is defined up to $b$. In this case, and only in this case, we define the set
$\Sigma_{b}^{f}(\tau, \eta)=\left\{x \in B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right): x\right.$ is a solution of problem (8) $\}$.
Notice that $\Sigma_{b}^{f}(\tau, \eta)$ is nonempty, whenever it is defined. In fact, in this case, the vacuous truth does not apply, due to the existence of at least one maximal solution of the initial value problem (8) ensured by Theorem 3.3.

As a special example, we observe that, under the assumptions of Lemma 5.1, the set $\Sigma_{b}^{f}(\tau, \eta)$ is defined for any $(\tau, \eta)$ in the open subset

$$
\mathcal{D}_{b}^{f}=(-\infty, b) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Moreover, in this case, $\Sigma_{b}^{f}(\tau, \eta)$ is always compact.

Theorem 5.2 (Continuous dependence). Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be a continuous function on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Then, given $b \in \mathbb{R}$, the set

$$
\mathcal{D}_{b}^{f}=\left\{(\tau, \eta) \in \Omega: \Sigma_{b}^{f}(\tau, \eta) \text { is defined and compact }\right\}
$$

is open and the multivalued map $(\tau, \eta) \in \mathcal{D}_{b}^{f} \longmapsto \Sigma_{b}^{f}(\tau, \eta)$ is upper semicontinuous.

Proof. Let us show first that $\mathcal{D}_{b}^{f}$ is open. To this end, consider the multivalued map

$$
\Gamma:(-\infty, b) \times B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right) \multimap \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

defined by $(a, x) \longmapsto\left\{\left(t, x_{t}\right): a \leq t \leq b\right\}$. Notice that $\Gamma$ is compact valued, since, given $(a, x)$ in its domain, the curve $t \in[a, b] \mapsto\left(t, x_{t}\right)$ is continuous, according to Remark 2.2. We claim that $\Gamma$ is actually upper semicontinuous. To this purpose, let us prove that the graph of $\Gamma$ is a closed subset of

$$
\mathcal{T}=(-\infty, b) \times B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right) \times \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

and that $\Gamma$ maps compact sets into relatively compact sets.
Observe that the graph of $\Gamma$ is equal to

$$
\left\{(a, x, t, \varphi) \in \mathcal{T}: a \leq t \leq b, \varphi=x_{t}\right\}
$$

which is closed because of the continuity of the maps

$$
(a, x, t, \varphi) \mapsto \varphi \quad \text { and } \quad(a, x, t, \varphi) \mapsto x_{t}
$$

Now, any compact set in the domain of $\Gamma$ is contained in another compact set of the type $[c, b] \times K$, and $\Gamma([c, b] \times K)=\left\{\left(t, x_{t}\right): t \in[c, b], x \in K\right\}$ is as well compact, being the image of $[c, b] \times K$ under the continuous map $(t, x) \mapsto\left(t, x_{t}\right)$. Thus, $\Gamma$ is upper semicontinuous, as claimed.

As a consequence of this, we get that the multivalued map

$$
P_{b}^{f}: \mathcal{D}_{b}^{f} \multimap \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

that to any $(\tau, \eta) \in \mathcal{D}_{b}^{f}$ associates the brush of $f$ starting at $(\tau, \eta)$,

$$
P_{b}^{f}(\tau, \eta):=\Gamma\left(\{\tau\} \times \Sigma_{b}^{f}(\tau, \eta)\right),
$$

is compact valued (recall that an upper semicontinuous multivalued map transforms compact sets into compact sets).

Observe that the initial point $(\tau, \eta)$ belongs to $P_{b}^{f}(\tau, \eta)$ whatever is $(\tau, \eta) \in$ $\mathcal{D}_{b}^{f}$. Moreover, one has

$$
P_{b}^{f}(\tau, \eta)=\left\{\left(t, x_{t}\right): \tau \leq t \leq b, x \text { is a solution in }(-\infty, b] \text { of problem }(8)\right\}
$$

Thus, recalling that any solution $x \in \Sigma_{b}^{f}(\tau, \eta)$ must satisfy the condition $\left(t, x_{t}\right) \in \Omega$ for all $t \in[\tau, b]$, one gets $P_{b}^{f}(\tau, \eta) \subseteq \Omega$ for all $(\tau, \eta) \in \mathcal{D}_{b}^{f}$. Notice also that any brush $P_{b}^{f}(\tau, \eta)$ is a connected set. In fact, it is actually path connected, since any element in it can be joined with the initial point $(\tau, \eta)$. This fact will be useful later.

In order to show that $\mathcal{D}_{b}^{f}$ is open, take an arbitrary element $(\check{\tau}, \check{\eta}) \in \mathcal{D}_{b}^{f}$. We need to find a neighborhood of $(\check{\tau}, \check{\eta})$ which is contained in $\mathcal{D}_{b}^{f}$. To this purpose consider the compact set $P_{b}^{f}(\check{\tau}, \check{\eta})$ and let $V$ be an open neighborhood of $P_{b}^{f}(\check{\tau}, \check{\eta})$ whose closure $\bar{V}$ is contained in $\Omega$ and whose image $f(V)$ is bounded. Because of the Tietze Extension Theorem, the restriction $\left.f\right|_{\bar{V}}$ of $f$ to the closure of $V$ admits a continuous extension $g: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ with bounded image.

Now, we can apply Lemma 5.1 with the function $f$ replaced by $g$, getting the upper semicontinuous multivalued map

$$
\Sigma_{b}^{g}: \mathcal{D}_{b}^{g} \multimap B U^{n-1}\left((-\infty, b], \mathbb{R}^{k}\right)
$$

defined on $\mathcal{D}_{b}^{g}=(-\infty, b) \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Thus, taking into account that the composition of upper semicontinuous maps is upper semicontinuous, the multivalued map

$$
P_{b}^{g}: \mathcal{D}_{b}^{f} \multimap \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)
$$

given by $P_{b}^{g}(\tau, \eta):=\Gamma\left(\{\tau\} \times \Sigma_{b}^{g}(\tau, \eta)\right)$, is as well upper semicontinuous.
Let $(\tau, \eta) \in V$. We claim that, whenever one of the two brushes, $P_{b}^{f}(\tau, \eta)$ and $P_{b}^{g}(\tau, \eta)$, is contained in $V$, then so is the other one and $P_{b}^{f}(\tau, \eta)=P_{b}^{g}(\tau, \eta)$.

For sure we have $P_{b}^{f}(\tau, \eta) \cap V=P_{b}^{g}(\tau, \eta) \cap V$, since the two functions $f$ and $g$ agree in $V$. Therefore, assume, for example, that $P_{b}^{f}(\tau, \eta)$ is contained in $V$. Then $P_{b}^{g}(\tau, \eta) \cap V$ is a compact set, since so is $P_{b}^{f}(\tau, \eta)$. Consequently $P_{b}^{g}(\tau, \eta) \cap V=P_{b}^{g}(\tau, \eta)$, since otherwise it would be disconnected. This proves our claim.

As a first consequence of this we get that the set $P_{b}^{g}(\check{\tau}, \check{\eta})$ is equal to $P_{b}^{f}(\check{\tau}, \check{\eta})$ and is contained in $V$. Due to the upper semicontinuity of $P_{b}^{g}$, there exists a neighborhood $U$ in $\mathcal{D}_{b}^{g}$ such that, for any $(\tau, \eta) \in U$, one has $P_{b}^{g}(\tau, \eta) \subseteq V$ and, consequently, $P_{b}^{f}(\tau, \eta)=P_{b}^{g}(\tau, \eta)$.

This implies that, whenever $(\tau, \eta) \in U$, any solution in $(-\infty, b]$ of problem (8) is as well a solution of the same problem with $f$ replaced by $g$, and viceversa. Thus, because of the arbitrariness of $(\check{\tau}, \check{\eta})$ and taking into account of Lemma 5.1, one gets both that $\mathcal{D}_{b}^{f}$ is open and the multivalued map $\Sigma_{b}^{f}$ is upper semicontinuous.

REmark 5.3. In the case when the uniqueness of the solution of problem (8)
is assumed (as e.g. in [7]), the map $\Sigma_{b}^{f}$ turns out to be single valued and, by the above theorem, it is in fact continuous.

## 6. Continuation property

Here we are interested in the continuation property of the solutions of equation (7). That is, the property of the solutions of being continuable (not maximal). Our first result states that, given a solution $x: J \rightarrow \mathbb{R}^{k}$, if the curve $t \mapsto\left(t, x_{t}\right)$ lies eventually in a complete subset $\mathcal{C}$ of $\Omega$ and $f(\mathcal{C})$ is bounded, then $x$ is not maximal, unless $J=\mathbb{R}$. In this way we include Lemma 3.2 as well as some special cases, stated for $n=1$, that can be found in [14, Chapter 12] and [15, Chapter 2].

THEOREM 6.1 (Continuation of solutions). Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be a continuous function on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Let $x: J \rightarrow \mathbb{R}^{k}$ be a solution of equation (7) such that $\left(t, x_{t}\right)$ belongs eventually to a complete subset $\mathcal{C}$ of $\Omega$. If $f(\mathcal{C})$ is bounded and $\sup J<+\infty$, then $x$ is continuable. In particular, if $f$ is defined on the whole space $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and has bounded image, then any maximal solution is defined on the whole real line.

Proof. If $J=(-\infty, b]$, then $x$ is continuable, since, according to Theorem 3.3, any maximal solution is defined on an open interval. Therefore, we may assume $J=(-\infty, b)$.

Since $\left(t, x_{t}\right)$ belongs eventually to $\mathcal{C}, f\left(t, x_{t}\right)$ is eventually bounded, and so is $x^{(n)}(t)$. Thus,

$$
\lim _{t \rightarrow b^{-}} x^{(n-1)}(t)
$$

exists and is finite. This implies that all the functions $x^{(j)}(t), j=1, \ldots, n-1$, are eventually bounded as well, so that the limits

$$
\lim _{t \rightarrow b^{-}} x^{(j)}(t), \quad j=0, \ldots, n-2
$$

exist and are finite. Therefore, $x$ admits a $C^{n-1}$ extension, call it $\bar{x}$, to the closed interval $(-\infty, b]$. Clearly, $\bar{x}_{b}$ belongs to $B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ and

$$
\lim _{t \rightarrow b^{-}}\left(t, x_{t}\right)=\left(b, \bar{x}_{b}\right) .
$$

Since $\mathcal{C}$ is complete, $\left(b, \bar{x}_{b}\right)$ belongs to $\mathcal{C}$ and, thus, to $\Omega$. Consequently, because of the continuity of $f$, we get $\bar{x}^{(n)}(b)=f\left(b, \bar{x}_{b}\right)$, so that $\bar{x}$ is a solution of (7), which cannot be maximal according to Theorem 3.3.

The last assertion follows from the previous one.

The following two corollaries are straightforward consequences of Theorem 6.1. Therefore, their proofs will be omitted. We only point out that, in both the corollaries, the condition $\sup J<+\infty$, required in Theorem 6.1, is trivially satisfied.
Corollary 6.2. Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be a continuous function on an open subset $\Omega$ of $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Let $x: J \rightarrow \mathbb{R}^{k}$ be a solution of equation (7) and assume that $\left(t, x_{t}\right)$ belongs eventually to a compact subset of $\Omega$. Then $x$ is continuable.
Corollary 6.3. Let $f: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R}^{k}$ be a continuous function sending bounded sets into bounded sets. If $x: J \rightarrow \mathbb{R}^{k}$ is a solution of equation (7) such that $\left(t, x_{t}\right)$ is eventually bounded, then $x$ is continuable.

The continuation property of the solutions may fail if, in Corollary 6.3, the assumption that $f$ sends bounded sets into bounded sets is removed (see [7] for an example of a first order RFDE).

The following consequence of Theorem 6.1 is an extension of Corollary 6.3 and can be regarded as a Kamke-type result for RFDEs.
Corollary 6.4. Let $W$ be an open subset of $\mathbb{R} \times \mathbb{R}^{k} \times \mathbb{R}^{k}$ and set

$$
\Omega=\left\{(s, \varphi) \in \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right):\left(s, \varphi(0), \varphi^{(n-1)}(0)\right) \in W\right\}
$$

Let $f: \Omega \rightarrow \mathbb{R}^{k}$ be a continuous function sending bounded sets into bounded sets. If $x:(-\infty, b) \rightarrow \mathbb{R}^{k}$ is a solution of equation (7) such that $\left(t, x(t), x^{(n-1)}(t)\right)$ belongs eventually to a compact subset of $W$, then $x$ is continuable.

Proof. Denote by $\Phi: \mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right) \rightarrow \mathbb{R} \times \mathbb{R}^{k} \times \mathbb{R}^{k}$ the continuous map $(s, \varphi) \mapsto\left(s, \varphi(0), \varphi^{(n-1)}(0)\right)$, and observe that $\Omega=\Phi^{-1}(W)$, so that $\Omega$ is an open set. By assumption, there exist $\tau<b$ and a compact subset $K$ of $W$ such that $\left(t, x(t), x^{(n-1)}(t)\right) \in K$ when $\tau \leq t<b$.

Let $\mathcal{C}$ denote the closure in $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ of the set

$$
\left\{\left(t, x_{t}\right), \tau \leq t<b\right\}
$$

Notice that $\mathcal{C}$ is bounded, because of the assumption $\left(t, x(t), x^{(n-1)}(t)\right) \in K$ for $t$ in a left neighborhood of $b$, and it is complete, being contained in the closed subset $\Phi^{-1}(K)$ of the Banach space $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Moreover, $\mathcal{C}$ is contained in $\Omega$, since so is $\Phi^{-1}(K)$, and $f(\mathcal{C})$ is bounded, since $f$ maps bounded sets into bounded sets. Therefore, all the assumptions in Theorem 6.1 are satisfied, and $x$ is continuable.

We point out that, if in Corollary 6.4 we take $W=\mathbb{R} \times \mathbb{R}^{k} \times \mathbb{R}^{k}$, then $\Omega$ turns out to be the entire space $\mathbb{R} \times B U^{n-1}\left((-\infty, 0], \mathbb{R}^{k}\right)$. Consequently, in this special case, given a solution $x:(-\infty, b) \rightarrow \mathbb{R}^{k}$ of (7), the following two assumptions are equivalent:

- $\left(t, x_{t}\right)$ is eventually bounded;
- $\left(t, x(t), x^{(n-1)}(t)\right)$ belongs eventually to a compact set.

This shows that Corollary 6.3 is a special case of Corollary 6.4, as claimed above.

## 7. Examples

Here we give two examples showing how some initial value problems for higher order ODEs, as well as higher order RFDEs with finite delay, can be interpreted in the framework of RFDEs with infinite delay.

Example 7.1 (From ODEs to RFDEs). Let $g: W \rightarrow \mathbb{R}^{k}$ be a continuous function defined on an open subset $W$ of $\mathbb{R} \times \mathbb{R}^{k} \times \mathbb{R}^{k}$ and consider the initial value problem

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t)=g\left(t, x(t), x^{\prime}(t)\right), \quad t \geq \tau  \tag{14}\\
x(\tau)=a \\
x^{\prime}(\tau)=b,
\end{array}\right.
$$

where $(\tau, a, b)$ is a given element of $W$.
Let us show how this problem can be interpreted as an initial value problem of a second order RFDE with infinite delay defined on an open subset of the space $\mathbb{R} \times B U^{1}\left((-\infty, 0], \mathbb{R}^{k}\right)$.

To this end, consider the open set

$$
\Omega=\left\{(t, \varphi) \in \mathbb{R} \times B U^{1}\left((-\infty, 0], \mathbb{R}^{k}\right):\left(t, \varphi(0), \varphi^{\prime}(0)\right) \in W\right\}
$$

and define $f: \Omega \rightarrow \mathbb{R}^{k}$ by $f(t, \varphi)=g\left(t, \varphi(0), \varphi^{\prime}(0)\right)$. Choose any function $\eta$ in $B U^{1}\left((-\infty, 0], \mathbb{R}^{k}\right)$ such that $\eta(0)=a$ and $\eta^{\prime}(0)=b$. For example, take $\eta(\theta)=(a+\theta b) \exp \left(-\theta^{2}\right)$. Then, any solution $x: J \rightarrow \mathbb{R}^{k}$ of the system

$$
\left\{\begin{array}{l}
x^{\prime \prime}(t)=f\left(t, x_{t}\right), \quad t>\tau  \tag{15}\\
x_{\tau}=\eta,
\end{array}\right.
$$

if restricted to the interval $J \cap[\tau,+\infty)$, is as well a solution of the initial value problem (14). In fact, for $t>\tau$ one has

$$
x^{\prime \prime}(t)=f\left(t, x_{t}\right)=g\left(t, x_{t}(0), x_{t}^{\prime}(0)\right)=g\left(t, x(t), x^{\prime}(t)\right),
$$

and for $t=\tau$ we get $x(\tau)=x_{\tau}(0)=\eta(0)=a$ and $x^{\prime}(\tau)=x_{\tau}^{\prime}(0)=\eta^{\prime}(0)=b$.
The same argument shows that, in some sense, the converse is also true. More precisely, if $x: I \rightarrow \mathbb{R}^{k}$ is a solution of (14), then $x$ can be extended to a solution of $(15)$ defined on the interval $(-\infty, \tau] \cup I$. Thus, the two problems, (14) and (15), may be regarded as equivalent.

Example 7.2 (From finite to infinite delay). Let $g: W \rightarrow \mathbb{R}^{k}$ be a continuous function defined on an open subset $W$ of $\mathbb{R} \times C^{(n-1)}\left([-r, 0], \mathbb{R}^{k}\right), r>0$, and consider, in $W$, the following initial value problem with finite delay:

$$
\left\{\begin{align*}
x^{(n)}(t) & =g\left(t, x_{t}\right), \quad t>\tau  \tag{16}\\
x_{\tau} & =\psi
\end{align*}\right.
$$

where $\tau \in \mathbb{R}$ and $\psi \in C^{(n-1)}\left([-r, 0], \mathbb{R}^{k}\right)$ are given.
The above system can also be viewed as an initial value problem with infinite delay. To see this, consider the subset of $\mathbb{R} \times B U^{(n-1)}\left((-\infty, 0], \mathbb{R}^{k}\right)$ given by

$$
\Omega=\left\{(t, \varphi):\left(t,\left.\varphi\right|_{[-r, 0]}\right) \in W\right\}
$$

where $\left.\varphi\right|_{[-r, 0]}$ denotes the restriction of $\varphi$ to the interval $[-r, 0]$. The continuity of the map $(t, \varphi) \mapsto\left(t,\left.\varphi\right|_{[-r, 0]}\right)$ implies that $\Omega$ is an open set.

Now, define $f: \Omega \rightarrow \mathbb{R}^{k}$ by $f(t, \varphi)=g\left(t,\left.\varphi\right|_{[-r, 0]}\right)$ and choose any function $\eta \in B U^{(n-1)}\left((-\infty, 0], \mathbb{R}^{k}\right)$ such that $\left.\eta\right|_{[-r, 0]}=\psi$. Then, as one can easily check, problem (16) and

$$
\begin{cases}x^{(n)}(t) & =f\left(t, x_{t}\right), \quad t>\tau \\ x_{\tau} & =\eta\end{cases}
$$

may be regarded as equivalent, in the sense that any solution $x: J \rightarrow \mathbb{R}^{k}$ of one of them coincides, for $t \in[\tau-r,+\infty) \cap J$, with a solution of the other.

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# Smoothness issues in differential equations with state-dependent delay 

Tibor Krisztin and Hans-Otto Walther

Dedicated to Jean Mawhin on the occasion of his 75th birthday


#### Abstract

For differential equations with state-dependent delays a satisfactory theory is developed by the second author [6] on the solution manifold to guarantee $C^{1}$-smoothness for the solution operators. We present examples showing that better than $C^{1}$-smoothness cannot be expected in general for the solution manifold and for local stable manifolds at stationary points on the solution manifold. Then we propose a new approach to overcome the difficulties caused by the lack of smoothness. The mollification technique is used to approximate the nonsmooth evaluation map with smooth maps. Several examples show that the mollified systems can have nicer smoothness properties than the original equation. Examples are also given where better smoothness than $C^{1}$ can be obtained on the solution manifold.


Keywords: Delay differential equation, state-dependent delay, solution manifold, stable manifold, solution operator, smoothness, mollification, threshold delay.
MS Classification 2010: 34K05, 34K19.

## 1. Introduction

Let $h>0$, a subset $U \subset\left(\mathbf{R}^{n}\right)^{[-h, 0]}$ and a map $f: U \rightarrow \mathbf{R}^{n}$ be given. Under additional conditions on $U$ and $f$, we consider solutions of the initial value problem (IVP)

$$
\begin{equation*}
x^{\prime}(t)=f\left(x_{t}\right) \quad \text { for } \quad t>0, \quad x_{0}=\phi \in U \tag{1}
\end{equation*}
$$

which are $C^{1}$-maps $x:\left[-h, t_{e}\right) \rightarrow \mathbf{R}^{n}, 0<t_{e} \leq \infty$, with all segments $x_{t}$ : $[-h, 0] \ni s \mapsto x(t+s) \in \mathbf{R}^{n}, 0 \leq t<t_{e}$, in $U$ so that $x^{\prime}(t)=f\left(x_{t}\right)$ holds for all $t \in\left(0, t_{e}\right)$, and $x_{0}=\phi$.

For $k \in \mathbf{N}_{0}$, let $X^{k}=C^{k}\left([-h, 0], \mathbf{R}^{n}\right)$ denote the Banach spaces of the $k$-times continuously differentiable functions $\phi:[-h, 0] \rightarrow \mathbf{R}^{n}$ equipped with the norm $|\phi|_{k}=\sum_{j=0}^{k}\left|\phi^{(j)}\right|_{0}$ where $|\phi|_{0}=\max _{-h \leq s \leq 0}|\phi(s)|$ with a fixed norm $|\cdot|$ in $\mathbf{R}^{n}$. We use $X=X^{0}$.

If $U \subset X$ is open and if $f: U \rightarrow \mathbf{R}^{n}$ is $C^{p}$-smooth for some integer $p \geq 1$ then each $\phi \in U$ uniquely determines a maximal solution $x^{\phi}:\left[-h, t_{\phi}\right) \rightarrow$ $\mathbf{R}^{n}$ of the IVP (1). Then $(t, \phi) \mapsto x_{t}^{\phi}, \phi \in U$ and $0 \leq t<t_{\phi}$, defines a continuous semiflow on $U$. The solution operators $\phi \mapsto x_{t}^{\phi}, 0 \leq t$, on nonempty domains are $C^{1}$-smooth, see [3, 2]. It is stated without proof on page 51 in [3] that $C^{p}$-smoothness holds as well. The construction of the semiflow and $C^{1}$-smoothness of solution operators are also given in [2, Chapter VII]. A proof that the solution operators are in fact $C^{p}$-smooth requires appropriate modifications of the arguments in [2, Chapter VII]. The necessary modifications are similar to those which are sketched in Section 5 for $C^{p}$-smoothness in a different framework used for equations with state-dependent delays. In the sequel, we refer to the case where $f: U \rightarrow \mathbf{R}^{n}$ is $C^{p}$-smooth on an open $U \subset X$ as the classical situation where the solution operators are $C^{p}$-smooth. This framework is satisfactory for differential equations with constant delays, but not for equations with state-dependent delays.

A large class of differential equations with state-dependent delays can effectively be handled within the following framework developed by the second author [6]. Let $U$ be an open subset of $X^{1}$, and consider a $C^{1}$-smooth map $f: U \rightarrow \mathbf{R}^{n}$ with the following extension property:
(e) each $D f(\phi): X^{1} \rightarrow \mathbf{R}^{n}$ has a linear extension $D_{e} f(\phi) \in L_{c}\left(X, \mathbf{R}^{n}\right)$ so that the map

$$
U \times X \ni(\phi, \chi) \mapsto D_{e} f(\phi) \chi \in \mathbf{R}^{n}
$$

is continuous.
Suppose $\phi^{\prime}(0)=f(\phi)$ for some $\phi \in U$. Then the set

$$
X_{f}^{1}=\left\{\phi \in U: \phi^{\prime}(0)=f(\phi)\right\} \neq \emptyset
$$

is a $C^{1}$-submanifold of $X^{1}$ with codimension $n$, each $\phi \in X_{f}^{1}$ uniquely determines a maximal solution $x^{\phi}:\left[-h, t_{\phi}\right) \rightarrow \mathbf{R}^{n}$ of the IVP (1) so that any other solution of the same initial value problem is a restriction of $x^{\phi}$. The relations

$$
S(t, \phi)=x_{t}^{\phi}, \quad 0 \leq t<t_{\phi}, \quad \phi \in X_{f}^{1}
$$

define a continuous semiflow $S$ on $X_{f}^{1}$ such that all solution operators

$$
S(t, \cdot):\left\{\phi \in X_{f}^{1}: t<t_{\phi}\right\} \rightarrow X_{f}^{1}, \quad t \geq 0
$$

on non-empty domains are $C^{1}$-smooth.
Let a stationary point $\phi_{0} \in X_{f}^{1}$ of $S$ be given. The continuous solutions $[-h, \infty) \rightarrow \mathbf{R}^{n}$ of the IVP

$$
\begin{equation*}
v^{\prime}(t)=D_{e} f\left(\phi_{0}\right) v_{t} \quad \text { for } \quad t>0, \quad v_{0}=\chi \in X \tag{2}
\end{equation*}
$$

tell us about the nature of the dynamics near $\phi_{0}$ : If all of these (whose restrictions $\left.v\right|_{[0, \infty)}$ are differentiable and satisfy the differential equation in (2)) tend to 0 as $t \rightarrow \infty$ then $\phi_{0}$ is a stable and attractive stationary point of $S$, and any local stable manifold is a neighbourhood of $\phi_{0}$ in $X_{f}^{1}$. If $t \mapsto 0$ is the only bounded solution $\mathbf{R} \rightarrow \mathbf{R}^{n}$ of the differential equation in (2) then $\phi_{0}$ is hyperbolic, and we have the decomposition

$$
T_{\phi_{0}} X_{f}^{1}=L_{s}\left(\phi_{0}\right) \oplus L_{u}\left(\phi_{0}\right)
$$

into the closed stable and unstable spaces $L_{s}\left(\phi_{0}\right)$ and $L_{u}\left(\phi_{0}\right)$, respectively.
$L_{s}\left(\phi_{0}\right)$ consists of all segments of all continuously differentiable solutions $[-h, \infty) \rightarrow \mathbf{R}^{n}$ of the IVP

$$
\begin{equation*}
v^{\prime}(t)=D f\left(\phi_{0}\right) v_{t} \quad \text { for } \quad t>0, \quad v_{0}=\chi \in T_{\phi_{0}} X_{f}^{1} \tag{3}
\end{equation*}
$$

which tend to 0 as $t \rightarrow \infty$. For any local stable manifold $W^{s}\left(\phi_{0}\right) \subset X_{f}^{1}$ of $S$ at $\phi_{0}$,

$$
T_{\phi_{0}} W^{s}\left(\phi_{0}\right)=L_{s}\left(\phi_{0}\right)
$$

For example, the above framework works for the equation

$$
\begin{equation*}
x^{\prime}(t)=g\left(x\left(t-r\left(x_{t}\right)\right)\right) \tag{4}
\end{equation*}
$$

with a given map $g: \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ and a given delay functional $r: U \rightarrow[0, h]$, $U \subset\left(\mathbf{R}^{n}\right)^{[-h, 0]}$. Equation (4) has the form (1) with

$$
f=g \circ e v \circ(i d \times(-r))
$$

where the evaluation map ev : $\left(\mathbf{R}^{n}\right)^{[-h, 0]} \times[-h, 0] \rightarrow \mathbf{R}^{n}$ is given by

$$
e v(\phi, s)=\phi(s)
$$

Let $e v_{k}$ denote the restriction of $e v$ to $X^{k} \times[-h, 0], k \in \mathbf{N}_{0}$. The smoothness properties of the evaluation map and its restrictions play a crucial role in the theory. The map $e v_{0}$ is continuous (but not locally Lipschitz continuous). Therefore a map $f$ involving the evaluation map - like in equation (4) above - in general is not locally Lipschitz continuous on open subsets of $X$, and uniqueness of solutions with respect to only continuous initial data may fail, which is indeed the case for certain examples, see [4].

The restrictions $e v_{k}, k \in \mathbf{N}$, of $e v$ have nice smoothness properties. In particular the map $e v_{1}$ is $C^{1}$-smooth on $X^{1} \times[-h, 0]$, with

$$
D e v_{1}(\phi, t)(\chi, s)=\chi(t)+s \phi^{\prime}(t) .
$$

Lemma 4.2 below states that for each integer $k \geq 2$, the map $e v_{k}$ is $C^{k}{ }_{-}$ smooth. $C^{k}$-smoothness of these maps, which are not defined on open subsets
of $X^{k} \times \mathbf{R}$, means that they have extensions to open subsets of $X^{k} \times \mathbf{R}$ which are $C^{k}$-smooth in the usual sense.

It is an open problem whether, for equations with state-dependent delays, better than $C^{1}$-smoothness ( $C^{p}$-smoothness with $p>1$ ) can be obtained for the solution operators, either on the solution manifold $X_{f}^{1} \subset X^{1}$ or on other phase spaces. The first step towards an affirmative answer would be to prove that the solution manifold $X_{f}^{1} \subset X^{1}$ is $C^{p}$-smooth for some $p \geq 2$. In Section 2 we give an example showing that in general, for a $C^{p}$-map $f: U \rightarrow \mathbf{R}^{n}$ on an open subset $U$ of $X^{1}$ with the extension property (e), the solution manifold $X_{f}^{1} \subset X^{1}$ is only $C^{1}$-smooth, not twice continuously differentiable, no matter how large $p$ is. The example has the form

$$
\begin{equation*}
x^{\prime}(t)=-\alpha x(t-d(x(t))), \tag{5}
\end{equation*}
$$

and it is crucial that $e v_{1}$ is not $C^{2}$-smooth.
In spite of the lack of results on better than $C^{1}$-smoothness for the solution operators generated by equations with state-dependent delays, the paper [5], for each $k \in \mathbf{N}$, gave conditions for the $C^{k}$-smoothness of local unstable manifolds $W^{u}\left(\phi_{0}\right)$ at stationary points. For example, the required conditions hold for equations (4) and (5) with at least $C^{k}$-smooth $g, r, d$. Therefore, within the $C^{1}$-smooth solution manifold $X_{f}^{1}$ it is possible to find certain invariant manifolds with better smoothness properties. This is known for the local unstable manifolds [5], and it is expected for the local center and center-unstable manifolds at stationary points. Does an analogous result exist for local stable manifolds $W^{s}\left(\phi_{0}\right)$ ? In the example of Section 2 the stationary point is attracting, and the local stable manifold $W^{s}\left(\phi_{0}\right)$ is an open neighbourhood in $X_{f}^{1}$ of the stationary point which is not a $C^{2}$-smooth submanifold of $X^{1}$. Thus, the answer is in general negative for local stable manifolds at stationary points. Section 3 contains another example in this direction where the stationary point is unstable.

In Section 4 we propose a new approach to overcome the difficulties caused by the lack of smoothness. We use the convolution and mollification to approximate the non-smooth map $e v$ with smooth maps. Let $\eta: \mathbf{R} \rightarrow \mathbf{R}$ be a $C^{\infty}$-smooth function so that $\operatorname{supp} \eta \subset[-1,1]$, and $\int_{\mathbf{R}} \eta(s) d s=1$. For $\epsilon>0$ set $\eta_{\epsilon}(t)=(1 / \epsilon) \eta(t / \epsilon), t \in \mathbf{R}$. The idea is the following for equation (4) provided that $g: \mathbf{R}^{n} \rightarrow \mathbf{R}^{n}$ is $C^{k}$-smooth, $r: X \rightarrow \mathbf{R}$ is $C^{k}$-smooth, and $r(X) \subset(\delta, h-\delta)$ for some $\delta>0$. We choose $\epsilon \in(0, \delta)$ and in equation (4) replace the term $x\left(t-r\left(x_{t}\right)\right)$ by

$$
\int_{-h}^{0} \eta_{\epsilon}\left(-r\left(x_{t}\right)-s\right) x(t+s) d s=-\int_{-r\left(x_{t}\right)-\epsilon}^{-r\left(x_{t}\right)+\epsilon} x(t+u) \eta_{\epsilon}(u) d u
$$

That is, the map

$$
\begin{equation*}
X \ni \phi \mapsto g(e v(\phi,-r(\phi))) \in \mathbf{R}^{n} \tag{6}
\end{equation*}
$$

on the right hand side of (4) is changed to the map

$$
\begin{equation*}
X \ni \phi \mapsto g\left(\int_{-h}^{0} \eta_{\epsilon}(-r(\phi)-s) \phi(s) d s\right) \in \mathbf{R}^{n} \tag{7}
\end{equation*}
$$

Thus the discrete state-dependent delay is changed to a distributed delay term expressed by the convolution of the solution and a smooth function with compact support. We show that for the modified equation

$$
\begin{equation*}
x^{\prime}(t)=g\left(\int_{-h}^{0} \eta_{\epsilon}\left(-r\left(x_{t}\right)-s\right) x(t+s) d s\right) \tag{8}
\end{equation*}
$$

the solutions of the corresponding IVP define $C^{k}$-smooth solution operators on the phase space $X$. It turns out that (7) defines a $C^{k}$-map on $X$, and the classical theory developed for constant delays works.

In several models involving state-dependent delays the delay functional is not given explicitly, and its smoothness properties are not obvious. We consider an example of the form (4) in which the delay functional $r$ is given by a threshold condition.

In Section 5 we explain how to get $C^{k}$-smoothness with $k>1$ for solution operators on solution manifolds in $X^{1}$, in certain particular cases.

## 2. An example with an attracting stationary point

Take $h=2, n=1, U=X^{1}$, and $f(\phi)=-\alpha \phi(-d(\phi(0)))$ with $0<\alpha<\frac{\pi}{2}$ and $d: \mathbf{R} \rightarrow(0,2)$ at least $C^{2}$-smooth with

$$
d(\xi)=1+\xi \quad \text { for } \quad|\xi|<\frac{1}{2}
$$

Then $f$ is $C^{1}$-smooth with

$$
D f(\phi) \chi=-\alpha\left[\chi(-d(\phi(0)))-\phi^{\prime}(-d(\phi(0))) d^{\prime}(\phi(0)) \chi(0)\right]
$$

see for example Chapter 3 in [4]. The extension property (e) holds. We have

$$
X_{f}^{1}=\left\{\phi \in X^{1}: \phi^{\prime}(0)=-\alpha \phi(-d(\phi(0)))\right\}
$$

$0 \in X_{f}^{1}$ is a stationary point of the semiflow on $X_{f}^{1}$, and

$$
D_{e} f(0) \chi=-\alpha \chi(-1)
$$

so that the linear differential delay equation of the IVP (2) becomes

$$
v^{\prime}(t)=-\alpha v(t-1),
$$

for which all maximal solutions tend to 0 as $t \rightarrow \infty$ because of $\alpha<\frac{\pi}{2}[8,3,2]$.
So any local stable manifold $W^{s}$ of the stationary point $0 \in X_{f}^{1}$ is given by $W^{s}=X_{f}^{1} \cap N$ with some open neighbourhood $N$ of 0 in $X^{1}$.

We shall show that $X_{f}^{1} \cap N$ is not a $C^{2}$-submanifold of $X^{1}$. We begin with a graph representation of $X_{f}^{1}$. Notice that the tangent space $Y=T_{0} X_{f}^{1}$ is the closed hyperplane

$$
\left\{\eta \in X^{1}: \eta^{\prime}(0)=-\alpha \eta(-1)\right\} .
$$

Choose a $C^{2}$-function $\psi \in X^{1} \backslash Y$ with

$$
\psi(0)=0=\psi^{\prime}(0), \quad \psi(-1)=1, \quad \text { and } \quad \psi(t) \neq 0 \quad \text { for all } \quad t \in[-2,0)
$$

for example, $\psi(t)=t^{2}$. Then

$$
X^{1}=\mathbf{R} \psi \oplus Y
$$

Proposition 2.1.

$$
X_{f}^{1}=\{a(\eta) \psi+\eta: \eta \in Y\}
$$

with the map $a: Y \rightarrow \mathbf{R}$ given by

$$
\begin{aligned}
a(\eta) & =\frac{1}{\psi(-d(\eta(0)))}[\eta(-1)-\eta(-d(\eta(0)))] \\
& =\frac{1}{\psi(-d(\eta(0)))}\left[\eta(-1)+\frac{1}{\alpha} f(\eta)\right]
\end{aligned}
$$

Proof. For $A \in \mathbf{R}$ and $\eta \in Y$ the relation $A \psi+\eta \in X_{f}^{1}$ is equivalent to

$$
(A \psi+\eta)^{\prime}(0)=-\alpha[(A \psi+\eta)(-d((A \psi+\eta)(0)))]
$$

or

$$
\begin{aligned}
(-\alpha \eta(-1)=) \quad \eta^{\prime}(0) & =-\alpha[A \psi(-d(A \psi(0)+\eta(0)))+\eta(-d(A \psi(0)+\eta(0)))] \\
= & -\alpha[A \psi(-d(\eta(0)))+\eta(-d(\eta(0)))]
\end{aligned}
$$

or

$$
A=\frac{1}{\psi(-d(\eta(0)))}[\eta(-1)-\eta(-d(\eta(0)))] .
$$

The map $a$ is $C^{1}$-smooth. The linear continuous projection $P: X^{1} \rightarrow X^{1}$ along $\mathbf{R} \psi$ onto $Y$ maps $X_{f}^{1}$ one-to-one onto the hyperplane $Y$, and the inverse of $\left.P\right|_{X_{f}^{1}}$ is the map

$$
Y \ni \eta \mapsto a(\eta) \psi+\eta \in X_{f}^{1} \subset X^{1}
$$

Suppose now that $X_{f}^{1} \cap N$ is a $C^{2}$-submanifold of $X^{1}$. Then $P$ defines a $C^{2}$-diffeomorphism from $X_{f}^{1} \cap N$ onto the open neighbourhood $P\left(X_{f}^{1} \cap N\right)$ of 0 in $Y$. For some open neighbourhood $V$ of 0 in $X^{1}, P\left(X_{f}^{1} \cap N\right)=Y \cap V$. It follows that the inverse

$$
Y \cap V \ni \eta \mapsto a(\eta) \psi+\eta \in X_{f}^{1} \cap N \subset X^{1}
$$

is $C^{2}$-smooth. Using the projection $i d-P$ and the topological isomorphism $\mathbf{R} \psi \ni s \psi \mapsto s \in \mathbf{R}$ we obtain that also the restriction of $a$ to $Y \cap V$ is $C^{2}$ smooth.

It follows that the map

$$
Y \cap V \ni \eta \mapsto \psi(-d(\eta(0))) a(\eta)-\eta(-1) \in \mathbf{R}
$$

is $C^{2}$-smooth. It equals $\frac{1}{\alpha} f$, and we obtain that the map

$$
g: Y \cap V \ni \eta \mapsto \eta(-d(\eta(0))) \in \mathbf{R}
$$

is $C^{2}$-smooth. A look at the formula for the derivative of $f$ in case $\alpha=1$ and an application of the chain rule to the composition of $f$ with the embedding $Y \rightarrow X^{1}$ shows that for every $\eta \in Y \cap V$ and for each $\hat{\eta} \in Y$ we have

$$
D g(\eta) \hat{\eta}=\hat{\eta}(-d(\eta(0)))-d^{\prime}(\eta(0)) \eta^{\prime}(-d(\eta(0))) \hat{\eta}(0)
$$

Fix some $\hat{\eta} \in Y$ with $\hat{\eta}(0)=1$. The evaluation map

$$
E v: L_{c}(Y, \mathbf{R}) \ni \lambda \mapsto \lambda(\hat{\eta}) \in \mathbf{R}
$$

is linear and continuous. An application of the chain rule yields that

$$
E v \circ D g: Y \cap V \ni \eta \mapsto D g(\eta) \hat{\eta} \in \mathbf{R}
$$

is $C^{1}$-smooth. Notice that for every $\eta \in Y \cap V$,

$$
(E v \circ D g)(\eta)=\hat{\eta}(-d(\eta(0)))-d^{\prime}(\eta(0)) \eta^{\prime}(-d(\eta(0))) \cdot 1
$$

The maps

$$
Y \cap V \ni \eta \mapsto \hat{\eta}(-d(\eta(0))) \in \mathbf{R}
$$

and

$$
Y \cap V \ni \eta \mapsto d^{\prime}(\eta(0)) \in \mathbf{R}
$$

are $C^{1}$-smooth.
Now choose $\eta_{0} \in Y \cap V$ with

$$
0<\eta_{0}(0)<\frac{1}{2}
$$

which has no second derivative at $-1-\eta_{0}(0)$. Notice that $d^{\prime}\left(\eta_{0}(0)\right)=1$. There is an open neighbourhood $U$ of $\eta_{0}$ in $X^{1}, U \subset V$, with

$$
d^{\prime}(\eta(0))>0 \quad \text { for all } \quad \eta \in U
$$

This implies that the map

$$
H: Y \cap U \ni \eta \mapsto \frac{1}{-d^{\prime}(\eta(0))}[(E v \circ D g)(\eta)-\hat{\eta}(-d(\eta(0)))] \in \mathbf{R}
$$

is $C^{1}$-smooth. Notice that

$$
H(\eta)=\eta^{\prime}(-d(\eta(0))) \quad \text { for all } \quad \eta \in Y \cap U
$$

Choose $\eta \in Y$ with $\eta(0)=1$. There exists $\epsilon \in\left(0, \frac{1}{2}\right)$ such that for all $s \in(-\epsilon, \epsilon)$ we have

$$
0<\eta_{0}(0)+s<\frac{1}{2} \quad \text { and } \quad \eta_{0}+s \eta \in U .
$$

As the curve $\mathbf{R} \ni s \mapsto \eta_{0}+s \eta \in Y$ is affine linear and continuous the chain rule applies and yields that the map

$$
j:(-\epsilon, \epsilon) \ni s \mapsto H\left(\eta_{0}+s \eta\right) \in \mathbf{R}
$$

is $C^{1}$-smooth. For $0<|s|<\epsilon$ we have

$$
\begin{aligned}
\frac{1}{s}[j(s)-j(0)] & =\frac{1}{s}\left[H\left(\eta_{0}+s \eta\right)-H\left(\eta_{0}\right)\right] \\
& =\frac{1}{s}\left[\left(\eta_{0}+s \eta\right)^{\prime}\left(-d\left(\left(\eta_{0}+s \eta\right)(0)\right)\right)-\eta_{0}^{\prime}\left(-d\left(\eta_{0}(0)\right)\right)\right] \\
& =\frac{1}{s}\left[\eta_{0}^{\prime}\left(-d\left(\eta_{0}(0)+s\right)\right)+s \eta^{\prime}\left(-d\left(\eta_{0}(0)+s\right)\right)-\eta_{0}^{\prime}\left(-d\left(\eta_{0}(0)\right)\right)\right] \\
& =\frac{1}{s}\left[\eta_{0}^{\prime}\left(-1-\eta_{0}(0)-s\right)+s \eta^{\prime}\left(-1-\eta_{0}(0)-s\right)-\eta_{0}^{\prime}\left(-1-\eta_{0}(0)\right)\right] \\
& =\frac{1}{s}\left[\eta_{0}^{\prime}\left(-1-\eta_{0}(0)-s\right)-\eta_{0}^{\prime}\left(-1-\eta_{0}(0)\right)\right]+\eta^{\prime}\left(-1-\eta_{0}(0)-s\right)
\end{aligned}
$$

This shows that for $0 \neq s \rightarrow 0$ the quotient

$$
\frac{1}{s}\left[\eta_{0}^{\prime}\left(-1-\eta_{0}(0)-s\right)-\eta_{0}^{\prime}\left(-1-\eta_{0}(0)\right)\right]
$$

converges to $j^{\prime}(0)-\eta^{\prime}\left(-1-\eta_{0}(0)\right)$, in contradiction to the choice of $\eta_{0}$ without a second derivative at $-1-\eta_{0}(0)$.

## 3. An example with an unstable stationary point

In this section $X^{1}=C^{1}\left([-h, 0], \mathbf{R}^{n}\right)$ will appear with $n=1$ and $n=2$. In order to avoid confusion we introduce $X_{1}^{1}=C^{1}([-h, 0], \mathbf{R})$ and $X_{2}^{1}=$ $C^{1}\left([-h, 0], \mathbf{R}^{2}\right)$.

Take $h=2$ and $\alpha$ and $d$ as in Section 2, but now $n=2$, and consider

$$
g: X_{2}^{1} \rightarrow \mathbf{R}^{2}, \quad g(\phi, \eta)=(f(\phi), \eta(0)),
$$

with $f$ from Section 2. The map $g$ is $C^{1}$-smooth with

$$
D g(\phi, \eta)(\hat{\phi}, \hat{\eta})=(D f(\phi) \hat{\phi}, \hat{\eta}(0))
$$

The extension property (e) holds. The solution manifold

$$
X_{g}^{1}=\left\{(\phi, \eta) \in X_{2}^{1}: \phi^{\prime}(0)=-\alpha \phi\left(-(d(\phi(0))), \eta^{\prime}(0)=\eta(0)\right\}\right.
$$

has codimension 2. The semiflow $S_{g}$ on $X_{g}^{1}$ given by the $C^{1}$-solutions of the system

$$
\begin{align*}
x^{\prime}(t) & =-\alpha x(t-d(x(t))),  \tag{9}\\
y^{\prime}(t) & =y(t), \tag{10}
\end{align*}
$$

satisfies $S_{g}(t,(0,0))=(0,0)$ for all $t \geq 0$, and

$$
T_{(0,0)} X_{g}^{1}=\left\{(\xi, \eta) \in X_{2}^{1}: \xi^{\prime}(0)=-\alpha \xi(-1), \eta^{\prime}(0)=\eta(0)\right\}
$$

The linear system $z^{\prime}(t)=D_{e} g(0,0) z_{t}$, or

$$
\begin{align*}
u^{\prime}(t) & =-\alpha u(t-1),  \tag{11}\\
v^{\prime}(t) & =v(t) \tag{12}
\end{align*}
$$

has no nontrivial bounded solution $\mathbf{R} \rightarrow \mathbf{R}^{2}$, so the stationary point $(0,0) \in X_{g}^{1}$ of $S_{g}$ is hyperbolic. The solution $\mathbf{R} \ni t \mapsto\left(0, e^{t}\right) \in \mathbf{R}^{2}$ of both systems shows that $(0,0)$ is unstable, and we have the decomposition

$$
T_{(0,0)} X_{g}^{1}=L_{s} \oplus L_{u}
$$

with the stable and unstable linear spaces $L_{s}=L_{s}(0,0) \neq T_{(0,0)} X_{g}^{1}$ and $L_{u}=$ $L_{u}(0,0) \neq\{0\}$. The facts that all solutions $[-2, \infty) \rightarrow \mathbf{R}$ of equation (11) tend to 0 as $t \rightarrow \infty$ and $v(t)=0$ on $[0, \infty)$ for any solution $[-2, \infty) \rightarrow \mathbf{R}$ of equation (12) with $v(0)=0$ combined imply

$$
\begin{equation*}
\left\{(\xi, \eta) \in T_{(0,0)} X_{g}^{1}: \eta(0)=0\right\} \subset L_{s} . \tag{13}
\end{equation*}
$$

As $\mathbf{R} \ni t \mapsto\left(0, e^{t}\right) \in \mathbf{R}^{2}$ is a solution of the system (11)-(12) we also get

$$
\left(0, \eta_{u}\right) \notin L_{s} .
$$

for $\eta_{u}=\left.\exp \right|_{[-2,0]}$. Notice that we have

$$
\begin{equation*}
T_{(0,0)} X_{g}^{1}=\left\{(\xi, \eta) \in T_{(0,0)} X_{g}^{1}: \eta(0)=0\right\} \oplus \mathbf{R}\left(0, \eta_{u}\right) \tag{14}
\end{equation*}
$$

Corollary 3.1.

$$
L_{s}=\left\{(\xi, \eta) \in T_{(0,0)} X_{g}^{1}: \eta(0)=0\right\}
$$

Proof. Due to instability the codimension of $L_{s}$ in the tangent space is at least 1. By (14) the codimension of $\left\{(\xi, \eta) \in T_{(0,0)} X_{g}^{1}: \eta(0)=0\right\}$ in the tangent space is 1 . Use the inclusion (13).

We proceed to a complement of $L_{s}$ in $X_{2}^{1}$. Choose $\psi \in C^{2}([-2,0], \mathbf{R}) \backslash T_{0} X_{f}^{1}$ as in Section 2 (for example, $\psi(t)=t^{2}$ ). Then $\psi^{\prime}(0) \neq-\alpha \psi(-1)$. The constant function 1: $[-2,0] \ni t \mapsto 1 \in \mathbf{R}$ does not satisfy $\eta^{\prime}(0)=\eta(0)$. Both facts combined imply

$$
X_{2}^{1}=T_{(0,0)} X_{g}^{1} \oplus \mathbf{R}(\psi, 0) \oplus \mathbf{R}(0, \mathbf{1})
$$

Using (14) and Corollary 3.1 we arrive at $X_{2}^{1}=L_{s} \oplus Q$ with

$$
Q=\mathbf{R}\left(0, \eta_{u}\right) \oplus \mathbf{R}(\psi, 0) \oplus \mathbf{R}(0, \mathbf{1})
$$

A local stable manifold $W^{s} \subset X_{g}^{1}$ of the semiflow $S_{g}$ at the stationary point $(0,0)$ is given by a map

$$
w^{s}: L_{s} \supset O_{s} \rightarrow Q
$$

on an open neighbourhood $O_{s}$ of $(0,0)$ in $L_{s}$, and every solution of the system (9)-(10) starting from a point $(\phi, \eta) \in W^{s} \subset X_{g}^{1}$ tends to $(0,0)$ as $t \rightarrow \infty$. Notice that for such a solution, necessarily $\eta(0)=0$. We infer

$$
\begin{aligned}
W^{s} \subset & \left\{(\phi, \eta) \in X_{2}^{1}: \phi^{\prime}(0)=-\alpha \phi(-d(\phi(0))), \eta^{\prime}(0)=\eta(0), \eta(0)=0\right\} \\
= & \left\{(\phi, \eta) \in X_{2}^{1}: \phi \in X_{f}^{1} \subset X_{1}^{1}, \eta \in X_{1}^{1}, \eta^{\prime}(0)=\eta(0), \eta(0)=0\right\} \\
= & \left\{(a(\xi) \psi+\xi, \eta+0) \in X_{2}^{1}: \xi \in T_{0} X_{f}^{1} \subset X_{1}^{1}, \eta \in X_{1}^{1},\right. \\
& \left.\quad \eta^{\prime}(0)=\eta(0), \eta(0)=0\right\} \quad \text { (with Proposition 2.1) } \\
= & \left\{(a(\xi) \psi+\xi, \eta+0) \in X_{2}^{1}: \xi \in X_{1}^{1}, \xi^{\prime}(0)=-\alpha \xi(-1), \eta \in X_{1}^{1},\right. \\
& \left.\eta^{\prime}(0)=\eta(0), \eta(0)=0\right\} \\
= & \left\{(\xi+a(\xi) \psi, \eta+0) \in X_{2}^{1}:(\xi, \eta) \in T_{(0,0)} X_{g}^{1}, \eta(0)=0\right\} \\
= & \left\{(\xi+a(\xi) \psi, \eta+0) \in X_{2}^{1}:(\xi, \eta) \in L_{s}\right\} \quad \text { (see Corollary 3.1). }
\end{aligned}
$$

The last set is given by a map $\gamma: L_{s} \rightarrow Q$. It follows that

$$
w^{s}=\left.\gamma\right|_{O_{s}}
$$

Now it becomes easy to show that $W^{s}$ is not a $C^{2}$-submanifold of $X_{2}^{1}$. Indeed, if it were a $C^{2}$ submanifold then the projection along $Q$ onto $L_{s}$ would define a $C^{2}$-diffeomorphism from $W^{s}$ onto $O_{s}$ whose inverse

$$
O_{s} \ni(\xi, \eta) \mapsto(\xi+a(\xi) \psi, \eta) \in X_{2}^{1}
$$

would be $C^{2}$-smooth, too. The restriction of $a$ to the open neighbourhood

$$
O_{Y}=\left\{\eta \in Y:(\eta, 0) \in O_{s}\right\}
$$

of 0 in $Y$ can be written as a composition, beginning with the restricted continuous linear map

$$
O_{Y} \ni \xi \mapsto(\xi, 0) \in O_{s},
$$

followed by the previous inverse, and upon that followed by further continuous linear maps. This implies that $a$ is $C^{2}$-smooth, which leads to a contradiction, see Section 2.

## 4. Smooth functionals involving state-dependent delay

Let $\chi: \mathbf{R} \rightarrow \mathbf{R}^{n}, \eta: \mathbf{R} \rightarrow \mathbf{R}$ be two continuous functions, $\eta$ is assumed to have compact support. The convolution $\chi * \eta: \mathbf{R} \rightarrow \mathbf{R}^{n}$ is defined by

$$
\chi * \eta(t)=\int_{\mathbf{R}} \chi(t-s) \eta(s) d s=\int_{\mathbf{R}} \chi(s) \eta(t-s) d s=\eta * \chi(t) .
$$

In particular, suppose $\eta$ is $C^{\infty}$-smooth, $\operatorname{supp} \eta \subset[-1,1]$, and $\int_{\mathbf{R}} \eta(s) d s=1$. For $\epsilon>0$ set $\eta_{\epsilon}(t)=(1 / \epsilon) \eta(t / \epsilon), t \in \mathbf{R}$. Then $\int_{\mathbf{R}} \eta_{\epsilon}(s) d s=1$. Moreover, $\chi * \eta_{\epsilon}(t) \rightarrow \chi(t)$ uniformly on compact subsets of $\mathbf{R}$ as $\epsilon \rightarrow 0$. For $\phi \in X$ let $\hat{\phi}: \mathbf{R} \rightarrow \mathbf{R}^{n}$ be the extension of $\phi$ so that $\hat{\phi}(t)=\phi(-h)$ for $t<-h$, and $\hat{\phi}(t)=\phi(0)$ for $t>0$. The restriction of $\hat{\phi} * \eta_{\epsilon}$ to $[-h, 0]$ is called the mollification $m_{\epsilon}(\phi)$ of $\phi$. The map $m_{\epsilon}: X \rightarrow X$ is called a mollifier. The function $m_{\epsilon}(\phi):[-h, 0] \rightarrow \mathbf{R}^{n}$ is $C^{\infty}$-smooth, and, for every $k \in \mathbf{N}, t \in[-h, 0]$,

$$
\frac{d^{k}}{d t^{k}} m_{\epsilon}(\phi)(t)=\frac{d^{k}}{d t^{k}}\left(\hat{\phi} * \eta_{\epsilon}\right)(t)=\left(\hat{\phi} * \frac{d^{k}}{d t^{k}} \eta_{\epsilon}\right)(t)
$$

It follows that each linear map

$$
m_{\epsilon, j}: X \ni \phi \mapsto m_{\epsilon}(\phi) \in X^{j}, \quad j \in \mathbf{N}_{0},
$$

is continuous.
Proposition 4.1. Let $m_{\epsilon}: X \rightarrow X$ be a mollifier. Assume that $f: X \rightarrow \mathbf{R}^{n}$ is a map such that its restriction $f_{k}: X^{k} \rightarrow \mathbf{R}^{n}$ is $C^{k}$-smooth. Then the map

$$
f_{\epsilon}: X \ni \phi \mapsto f\left(m_{\epsilon}(\phi)\right) \in \mathbf{R}^{n}
$$

is $C^{k}$-smooth.
Proof. We have

$$
f_{\epsilon}(\phi)=f\left(m_{\epsilon}(\phi)\right)=f_{k}\left(m_{\epsilon, k}(\phi)\right),
$$

and $m_{\epsilon, k}: X \rightarrow X^{k}, f_{k}: X^{k} \rightarrow \mathbf{R}^{n}$ are $C^{k}$-smooth.

Recall the following result on the restrictions of the evaluation map ev.
Lemma 4.2. For each $k \in \mathbf{N}$, the restricted evaluation map

$$
e v_{k}: X^{k} \times[-h, 0] \ni(\phi, t) \mapsto \phi(t) \in \mathbf{R}^{n}
$$

is $C^{k}$-smooth with

$$
D^{j} e v_{k}(\phi, t)\left(\chi_{1}, s_{1} ; \ldots ; \chi_{j}, s_{j}\right)=\phi^{(j)}(t) \prod_{l=1}^{j} s_{l}+\sum_{l=1}^{j} \chi_{l}^{(j-1)}(t) \prod_{m \neq l} s_{m}
$$

$j \in\{1, \ldots, k\}, \chi_{1}, \ldots, \chi_{j} \in X^{k}, s_{1}, \ldots, s_{j} \in \mathbf{R}$. In addition, ev $v_{k}$ is not $C^{k+1}-$ smooth.

Proof. This follows from results in [5, Section 4]). It can also be shown by induction following the technique of [2, Appendix IV]. The partial derivative of $D^{k} e v_{k}$ with respect to its second variable $t$ requires $C^{k+1}$-smoothness of $\phi$. Therefore $e v_{k}$ is not $k+1$-times differentiable.

The above facts suggest that if the term

$$
x\left(t-r\left(x_{t}\right)\right)=e v\left(x_{t},-r\left(x_{t}\right)\right)
$$

in equation (4) is replaced with

$$
e v\left(m_{\epsilon}\left(x_{t}\right),-r\left(x_{t}\right)\right)
$$

or with

$$
\operatorname{ev}\left(m_{\epsilon}\left(x_{t}\right),-r\left(m_{\epsilon}\left(x_{t}\right)\right)\right)
$$

then we may get better smoothness properties for the semiflow. However, it is still a nontrivial problem to find the appropriate phase spaces where smoother solution operators can be obtained. Below we consider several versions of this mollification technique for equation (4).

Of course, the mollification $m_{\epsilon}\left(x_{t}\right)$ of the term $x_{t}$ in equation (4) changes the original equation. So, the smoothness is obtained for a modified equation, not for the original one. It is an interesting question - which is not studied here - how the modified equation can be used to get information on the original one.

Example 4.3. Let $n=1, k \in \mathbf{N}$, and let $g: \mathbf{R} \rightarrow \mathbf{R}$ and $r: X \rightarrow \mathbf{R}$ be $C^{k}{ }_{-}$ smooth functions, and assume that there exist $\delta>0$ so that $r(X) \subset(\delta, h-\delta)$. An example for $r$ is

$$
r(\phi)=\frac{a+b(\phi(0))^{2}}{c+d(\phi(0))^{2}}
$$

with positive reals $a, b, c, d$ and $\delta<\frac{a}{c}<\frac{b}{d}<h-\delta$. It is the composition of the continuous linear functional $\phi \mapsto \phi(0)$ with an analytical real function which strictly increases on $[0, \infty)$.

Consider the equation

$$
\begin{equation*}
x^{\prime}(t)=g\left(x\left(t-r\left(x_{t}\right)\right)\right) . \tag{15}
\end{equation*}
$$

For this equation a $C^{1}$-smooth solution manifold $X_{f}^{1}$ exists with $f(\phi)=g \circ$ $e v_{1}(\phi,-r(\phi))$, and the solution operators are $C^{1}$-smooth. For the mollified equation we can get better smoothness.

Let $\epsilon \in(0, \delta)$, define

$$
F_{\epsilon}: X \ni \phi \mapsto g \circ e v\left(m_{\epsilon}(\phi),-r(\phi)\right) \in \mathbf{R}^{n},
$$

and consider the equation

$$
\begin{equation*}
x^{\prime}(t)=F_{\epsilon}\left(x_{t}\right), \tag{16}
\end{equation*}
$$

or equivalently

$$
x^{\prime}(t)=g\left(-\int_{-r\left(x_{t}\right)-\epsilon}^{-r\left(x_{t}\right)+\epsilon} x(t+u) \eta_{\epsilon}\left(-r\left(x_{t}\right)-u\right) d u\right) .
$$

The assumptions on $g, r$, the continuity of $m_{\epsilon, k}: X \rightarrow X^{k}$, Lemma 4.2 and

$$
F_{\epsilon}(\phi)=g \circ e v\left(m_{\epsilon}(\phi),-r(\phi)\right)=g \circ e v_{k}\left(m_{\epsilon, k}(\phi),-r(\phi)\right)
$$

imply that $F_{\epsilon}: X \rightarrow \mathbf{R}$ is $C^{k}$-smooth. It follows that equation (16), the mollified version of (15), can be studied in the phase space $X$, and classical results show that there is a continuous semiflow with $C^{k}$-smooth solution operators.

Example 4.4. Consider equation (15) with the same condition on $g$ as in Example 4.3. On the delay functional $r$ we assume that its restriction $r_{k}: X^{k} \rightarrow \mathbf{R}$ is $C^{k}$-smooth, and $r_{k}\left(X^{k}\right) \subset(\delta, h-\delta)$ for some $\delta>0$. For example, the threshold delay in the next example has this property with $k=1$.

Let $\epsilon \in(0, \delta)$, define

$$
f_{\epsilon}: X \ni \phi \mapsto g \circ e v_{k} \circ\left(i d,-r_{k}\right)\left(m_{\epsilon, k}(\phi)\right) \in \mathbf{R},
$$

and consider the equation

$$
\begin{equation*}
x^{\prime}(t)=f_{\epsilon}\left(x_{t}\right) \tag{17}
\end{equation*}
$$

on the phase space $X$. Proposition 4.1 gives that $f_{\epsilon}: X \rightarrow \mathbf{R}$ is $C^{k}$-smooth, and again the classical theory implies the $C^{k}$-smoothness of the solution operators.

Example 4.5. Let $n=1, k \in \mathbf{N}$, and suppose that $g: \mathbf{R} \rightarrow \mathbf{R}$ and $a: \mathbf{R} \rightarrow \mathbf{R}$ are $C^{k}$-smooth. In addition assume that $a(\mathbf{R}) \subset\left(a_{0}, a_{1}\right)$ with constants $0<$ $a_{0}<a_{1}$. We consider Equation (15) so that the delay $\tau\left(x_{t}\right)$ is defined by the threshold condition

$$
\begin{equation*}
\int_{-\tau\left(x_{t}\right)}^{0} a(x(t+s)) d s=1 . \tag{18}
\end{equation*}
$$

From $a(\mathbf{R}) \subset\left(a_{0}, a_{1}\right)$ it follows that $\tau\left(x_{t}\right) \in\left(\frac{1}{a_{1}}, \frac{1}{a_{0}}\right)$ provided it exists. Choose $h>0$ and $\delta>0$ so that $h>\frac{1}{a_{0}}$ and $\delta<\min \left\{\frac{1}{a_{1}}, h-\frac{1}{a_{0}}\right\}$.

Let $\epsilon \in(0, \delta)$. We want to define $f_{\epsilon}$ or $F_{\epsilon}$ analogously to Examples 4.3-4.4. For the smoothness properties of $f_{\epsilon}$ and $F_{\epsilon}$ we need more information on the threshold delay $\tau$.

Define the substitution operator $A: X \rightarrow X$ by

$$
(A \phi)(s)=a(\phi(s)), \quad \phi \in X, \quad s \in[-h, 0] .
$$

Let the integral operator $I: X \rightarrow X$ be given by

$$
(I \phi)(s)=\int_{s}^{0} \phi(u) d u, \quad \phi \in X, \quad s \in[-h, 0]
$$

Define

$$
G: X \times(0, h) \ni(\phi, u) \mapsto e v(I \circ A(\phi),-u)-1 \in \mathbf{R} .
$$

Then the threshold condition

$$
\int_{-\tau(\phi)}^{0} a(\phi(u)) d u=1, \quad \phi \in X
$$

is equivalent to the equation

$$
G(\phi, \tau(\phi))=0, \quad \phi \in X
$$

The following smoothness properties of $A$ and $I$ can be easily shown or obtained from [2, Appendix IV]. The restrictions of $A$ and $I$ to $X^{j}$ are denoted by $A_{j}$ and $I_{j}$, respectively, with $A_{0}=A, I_{0}=I$.

Lemma 4.6. Let $j \in \mathbf{N}_{0}, p \in \mathbf{N}$.

1. If $a$ is $C^{p+j}$-smooth then the restriction $A_{j}$ of $A$ to $X^{j}$ is $C^{p}$-smooth.
2. The restriction $I_{j}$ of $I$ to $X^{j}$ is a bounded linear map into $X^{j+1}$.

It is obvious that for each $\phi \in X$ there is a unique $u^{*}=u^{*}(\phi) \in(0, h)$ such that $G\left(\phi, u^{*}(\phi)\right)=0$. Define $\tau: X \rightarrow(0, h)$ by $\tau(\phi)=u^{*}(\phi)$.

For $k \in \mathbf{N}$, let $G_{k-1}$ denote the restriction of $G$ to $X^{k-1} \times(0, h)$. As $I_{k-1}$ maps into $X^{k}$ we have

$$
G_{k-1}(\phi, u)=e v_{k}\left(I_{k-1} \circ A_{k-1}(\phi),-u\right)-1, \quad \phi \in X^{k-1}, u \in(0, h)
$$

By Lemma 4.6, $I_{k-1} \circ A_{k-1}: X^{k-1} \rightarrow X^{k}$ is $C^{k}$-smooth provided $a$ is $C^{2 k-1}$ smooth, and by Lemma $4.2 \mathrm{ev}_{k}: X^{k} \times(0, h) \rightarrow \mathbf{R}$ is also $C^{k}$-smooth. Therefore, $G_{k-1}: X^{k-1} \times(0, h) \rightarrow \mathbf{R}$ is $C^{k}$-smooth. It is easy to see that

$$
D G_{k-1}(\phi, u)(\chi, t)=\int_{-u}^{0} a^{\prime}(\phi(s)) \chi(s) d s-a(\phi(-u)) t, \quad \chi \in X^{k-1}, t \in \mathbf{R}
$$

and

$$
D_{2} G_{k-1}(\phi, u) 1=-a(\phi(-u)) \neq 0
$$

The Implicit Function Theorem yields that the restriction $\tau_{k-1}: X^{k-1} \rightarrow(0, h)$ of the map $\tau: X \rightarrow(0, h)$ is $C^{k}$-smooth. For later use in Section 5 we now show that $\tau_{1}$ has the extension property (e): Differentiation of the equation $G_{k-1}\left(\phi, \tau_{k-1}(\phi)\right)=0, \phi \in X^{k-1}$, yields

$$
D \tau_{k-1}(\phi) \chi=\left(a\left(\phi\left(-\tau_{k-1}(\phi)\right)\right)\right)^{-1} \int_{-\tau_{k-1}(\phi)}^{0} a^{\prime}(\phi(s)) \chi(s) d s, \quad \chi \in X^{k-1}
$$

It follows that, in case $k>1, D \tau_{k-1}(\phi) \in L_{c}\left(X^{k-1}, \mathbf{R}\right)$ can be extended to a bounded linear operator $D_{e} \tau_{k-1}(\phi): X \rightarrow \mathbf{R}$ such that

$$
X^{k-1} \times X \ni(\phi, \chi) \mapsto D_{e} \tau_{k-1}(\phi) \chi \in \mathbf{R}
$$

is continuous. In particular, $\tau_{1}$ has the extension property (e) of Section 1. If $k=1$ and if $a$ is $C^{1}$-smooth then we are in the situation of Example 4.3 with $k=1$, and for the mollified equation

$$
x^{\prime}(t)=F_{\epsilon}\left(x_{t}\right)
$$

in the phase space $X$, the solution operators are $C^{1}$-smooth.
We can apply the mollification also in the threshold equation (18). This means that, for a fixed $\epsilon \in(0, \delta)$, the delay $\tau_{\epsilon}(\phi)$ is defined from the equation

$$
\int_{-\tau_{\epsilon}(\phi)}^{0} a\left(m_{\epsilon}(\phi)(s)\right) d s=1, \quad \phi \in X .
$$

That is $\tau_{\epsilon}(\phi)$, for a given $\phi \in X$, is the zero of the map

$$
G(\phi, \cdot):(0, h) \ni u \mapsto e v_{k}\left(I_{k-1} \circ A_{k-1} \circ m_{\epsilon, k-1}(\phi),-u\right)-1 \in \mathbf{R} .
$$

Clearly, the unique zero is $\tau_{\epsilon}(\phi)=\tau_{k-1}\left(m_{\epsilon, k-1}(\phi)\right)$, and the map $X \ni \phi \mapsto$ $\tau_{k-1}\left(m_{\epsilon, k-1}(\phi)\right) \in(0, h)$ is $C^{k}$-smooth provided $a$ is $C^{2 k-1}$-smooth. Observe
that $\tau_{k-1}\left(m_{\epsilon, k-1}(\phi)\right)=\tau_{k-1} \circ i_{k-1, k}\left(m_{\epsilon, k}(\phi)\right), \phi \in X$, with the inclusion map $i_{k-1, k}: X^{k} \rightarrow X^{k-1}$.

Therefore, the equation

$$
\begin{equation*}
x^{\prime}(t)=f_{\epsilon}\left(x_{t}\right) \tag{19}
\end{equation*}
$$

with the $C^{k}$-smooth map

$$
f_{\epsilon}: X \ni \phi \mapsto g \circ e v_{k}\left(i d,-\tau_{k-1} \circ i_{k-1, k}\right) \circ m_{\epsilon, k}(\phi) \in \mathbf{R}
$$

can be handled in the phase space $X$ by the classical theory to get $C^{k}$-smoothness of the solution operators. Equation (19) is the mollified version of the equation (15) with the threshold condition (18).

## 5. $C^{k}$-smoothness of solution manifolds and solution operators

Suppose $U \subset X^{1}$ is open and $f: U \rightarrow \mathbf{R}^{n}$ is $C^{k}$-smooth, $1 \leq k<\infty, f$ has property (e), and $X_{f}^{1} \neq \emptyset$. Then the solution manifold $X_{f}^{1}$ is a $C^{k}$-submanifold of the space $X^{1}$, and all solution operators $S(t, \cdot), t \geq 0$, on non-empty domains are $C^{k}$-smooth. This follows by means of appropriate modifications in the proofs from [6]. First, the present hypothesis on $f$ implies that the hypotheses (P1) and (P2) from [6, Section 1] are satisfied, see for example [7, Corollary 1] and [4, Section 3.2]. In order to obtain $C^{k}$-smoothness of $X_{f}^{1}$ proceed exactly as in the proof of [6, Proposition 1] and use the Implicit Function Theorem for zerosets of $C^{k}$-maps, for example, Theorem 2.3 in [1, Chapter 2, Section 2.2].
$C^{k}$-smoothness of solution operators follows as in [6, Section 2] provided the map $R_{T r}$ in [6, Proposition 5] is $C^{k}$-smooth, and in the paragraph following the proof of [6, Proposition 5] a uniform contraction principle is applied which yields that fixed points are $C^{k}$-smooth with respect to the parameters. Such a uniform contraction principle is Theorem 2.2 in [1, Chapter 2, Section 2.2], for example.

In the proof of $\left[6\right.$, Proposition 5] it is shown that the map $R_{T r}$ is a composition of continuous linear maps between Banach spaces and of restrictions of such maps to open sets with the map

$$
f_{T} \times i d: C\left([0, T], C^{1}([-h, 0])\right) \times \mathbf{R}^{n} \rightarrow C([0, T]) \times \mathbf{R}^{n}
$$

given by $\left(f_{T} \times i d\right)(\eta, \xi)=(f \circ \eta, \xi)$. Here, $T>0$ is some constant, the set $C^{1}([-h, 0])$ equals $X^{1}$ in our notation, and $C\left([0, T], C^{1}([-h, 0])\right)$ is the Banach space of continuous maps $[0, T] \rightarrow C^{1}([-h, 0])$ with the norm given by $|\eta|=\max _{0 \leq t \leq T}|\eta(t)|_{1} . C([0, T])$ denotes the Banach space of continuous maps $[0, T] \rightarrow \mathbf{R}^{\bar{n}}$ with the norm given by $|\xi|=\max _{0 \leq t \leq T}|\xi(t)|$.

We infer that $R_{T r}$ is $C^{k}$-smooth provided the substitution operator

$$
f_{T}: C\left([0, T], C^{1}([-h, 0])\right) \ni \eta \mapsto f \circ \eta \in C([0, T])
$$

is $C^{k}$-smooth, which is true, see [2, Appendix IV,Lemma 1.5], for example.
Example 5.1. For a map $f: X \rightarrow \mathbf{R}^{n}$ define the restriction $f_{1}=\left.f\right|_{X^{1}}=f \circ i_{01}$, with the inclusion map $i_{01}: X^{1} \rightarrow X$. If $f: X \rightarrow \mathbf{R}^{n}$ is $C^{k}$-smooth then $f_{1}$ is also $C^{k}$-smooth. For $k \in \mathbf{N}$ the initial value problem (1) with $f=\left(f_{\epsilon}\right)_{1}$ or with $f=\left(F_{\epsilon}\right)_{1}$, where $f_{\epsilon}$ is given in Proposition 4.1, $F_{\epsilon}$ is given in Example 4.3, defines a continuous semiflow on the $C^{k}$-smooth submanifold $X_{f}^{1}$ of the space $X^{1}$, with all solution operators on non-empty domains $C^{k}$-smooth.

Example 5.2. Let $h>0, \delta \in(0, h / 2), \epsilon \in(0, \delta)$. Assume that $g: \mathbf{R} \rightarrow \mathbf{R}$ is $C^{2}$-smooth. Let $m_{\epsilon}$ be a mollifier given by the $C^{2}$-function $\eta: \mathbf{R} \rightarrow \mathbf{R}$. Consider the equation

$$
\begin{equation*}
x^{\prime}(t)=g\left(\int_{-h}^{0} \eta_{\epsilon}\left(-r\left(x_{t}\right)-s\right) x(t+s) d s\right)=g\left(m_{\epsilon}\left(x_{t}\right)\left(-\tau\left(x_{t}\right)\right)\right) \tag{20}
\end{equation*}
$$

where $\tau\left(x_{t}\right)$ is defined by the threshold condition (18). We suppose that $a$ : $\mathbf{R} \rightarrow \mathbf{R}$ is $C^{3}$-smooth with $a(\mathbf{R}) \subset\left(a_{0}, a_{1}\right)$ for positive reals $a_{0}<a_{1}$ satisfying $\frac{1}{a_{0}}<h$ and $\delta<\min \left\{\frac{1}{a_{1}}, h-\frac{1}{a_{0}}\right\}$.

Example 4.5 in case $k=1$ shows that, for each $\phi \in X^{1}$, the threshold equation

$$
\int_{-\tau}^{0} a(\phi(s)) d s=1
$$

has a unique solution $\tau_{1}(\phi)$, and $\tau_{1}: X^{1} \rightarrow(0, h)$ is $C^{2}$-smooth. In addition, $D \tau_{1}$ has the extension property (e).

On the space $X^{1}$, the right hand side of equation (20) is given by the $C^{2}$ map

$$
f: X^{1} \ni \phi \mapsto g \circ e v_{2}\left(m_{\epsilon, 2}(\phi),-\tau_{1}(\phi)\right) \in \mathbf{R} .
$$

From the fact that $D \tau_{1}$ has the extension property (e) it is easy to check that $D f$ also has property (e).

Therefore, the initial value problem of (20) together with the threshold condition (18) defines a continuous semiflow on the $C^{2}$-submanifold $X_{f}^{1}$ of the space $X^{1}$, with all solution operators on non-empty domains $C^{2}$-smooth.

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# Gauge-measurable functions 

Augusto C. Ponce and Jean Van Schaftingen

To our master and friend Jean Mawhin


#### Abstract

In 1973, E. J. McShane introduced an alternative definition of the Lebesgue integral based on Riemann sums, where gauges are used to decide what tagged partitions are allowed. Such an approach does not require any preliminary knowledge of Measure Theory. We investigate in this paper a definition of measurable functions also based on gauges. Its relation to the gauge-integrable functions that satisfy McShane's definition is obtained using elementary tools from Real Analysis. We show in particular a dominated integration property of gauge-measurable functions.


Keywords: gauge integral, Kurzweil-Henstock integral, Lebesgue integral, generalized Riemann integral, measurable function, gauge.
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## 1. Introduction

In its classical original setting, the Lebesgue integral of a function $f$ is defined in terms of the outer Lebesgue measure of the measurable sublevel sets of $f[2,12,16]$. Compared to the definition of the integral by Cauchy or Riemann as a limit of Riemann sums [4,26], the definition of the Lebesgue integral seems somehow indirect: it is a limit of a sum of measures, where these measures are themselves computed as the infima or suprema of volumes.

This issue has led to the definition of gauge integrals as a way of recovering the original approach based on Riemann sums, without the defects associated to the Riemann integral of Riemann-integrable functions [21]. Around 1960, Kurzweil and Henstock independently defined a gauge integral which allows one to integrate more functions than the Lebesgue-integrable ones [11, 15]. A few years later, in 1973, McShane presented the Lebesgue integral itself as a gauge integral [23,24]. We can rephrase McShane's definition as follows:

Definition 1.1 (Gauge integrability). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gaugeintegrable whenever there exists $I \in \mathbb{R}^{p}$ verifying the following property: for every $\varepsilon>0$, there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ and a compact set $K \subset \mathbb{R}^{d}$ such that, for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ that covers $K$ and
every finite set of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ satisfying

$$
R_{i} \subset \gamma\left(c_{i}\right) \quad \text { for every } i \in\{1, \ldots, k\}
$$

one has

$$
\left|\sum_{i=1}^{k} f\left(c_{i}\right) \operatorname{vol}\left(R_{i}\right)-I\right| \leq \varepsilon
$$

In this definition, a gauge $\gamma$ on $\mathbb{R}^{d}$ is a function mapping each point of $x \in \mathbb{R}^{d}$ to an open set $\gamma(x) \subset \mathbb{R}^{d}$ such that $x \in \gamma(x)$; for example $\gamma(x)$ might be taken to be a non-empty open ball centered at $x$. A rectangle $R \subset \mathbb{R}^{d}$ is a set that can be written as $R=\left[a_{1}, b_{1}\right) \times \cdots \times\left[a_{d}, b_{d}\right)$, where $a_{1}<b_{1}, \ldots, a_{d}<b_{d}$ are all real numbers; its volume is the positive number $\operatorname{vol}(R)=\left(b_{1}-a_{1}\right) \cdots\left(b_{d}-a_{d}\right)$. Rectangles are disjoint whenever their intersection is empty, and the family $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ covers $K$ if

$$
\bigcup_{i=1}^{k} R_{i} \supset K
$$

The compact set $K$ corresponds in McShane's original definition to the complement of his gauge at infinity; the equivalent formulation above avoids compactifying the Euclidean space $\mathbb{R}^{d}$ and considering unbounded rectangles.

By Cousin's lemma, which is a variant of the Heine-Borel theorem, for any gauge $\gamma$ on $\mathbb{R}^{d}$ and any compact set $K \subset \mathbb{R}^{d}$, there always exists some finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ that covers $K$ and points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ such that $R_{i} \subset \gamma\left(c_{i}\right)$ for every $i$ [24, Theorem IV-3-1]. This fact ensures the uniqueness of the integral $I$ of $f$, which entitles one to adopt the usual notation

$$
\int_{\mathbb{R}^{d}} f:=I .
$$

A non-intuitive feature of the definition of the gauge integral above is that each tag $c_{i}$ need not belong to the rectangle $R_{i}$. Adding this restriction gives the broader definition of integral of Kurzweil and Henstock, which is a gauge definition of the Denjoy-Perron integral for which all derivatives of one-dimensional functions are integrable on bounded intervals [9, 18, 25]. This Kurzweil-Henstock integral has been taught by Jean Mawhin at the Université catholique de Louvain (UCL) for thirty years [19, 20], continuing the Louvain tradition of cutting-edge lectures on integration theory initiated by Ch.-J. de la Vallée Poussin with the Lebesgue integral at the beginning of the 20th century [5-7,22]. The further restriction that the gauge $\gamma(x)$ contain some uniform ball $B_{\delta}(x)$ for some radius $\delta>0$ independent of $x \in \mathbb{R}^{d}$ yields the classical Riemann integral.

Measurability of functions is not a prerequisite of McShane's definition of gauge integrability. This is an important aspect one should not neglect about
the gauge integral that makes the Lebesgue integral readily available, without the need of any preliminary development of tools from Measure Theory. This is an approach we have been pursuing at UCL since 2009.

When measurability is needed to state some integrability condition, measurable functions have been defined as pointwise limits of integrable functions [24, Definition III-10-1] or almost everywhere limits of locally integrable step functions (see [1, §19] and [17, Definition 3.5.3]), or in terms of measurable sets whose characteristic functions are locally integrable (see [19, §6.B] and [20, §13.7]). It thus seems that the straightforwardness of McShane's definition of the integral is lost in an ad hoc indirect definition of measurability based on the integral itself.

In order to remedy to this issue, we introduce here a direct definition of measurability of functions in terms of gauges inspired by Lusin's property for Lebesgue-measurable functions.

Definition 1.2 (Gauge measurability). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gaugemeasurable whenever, for every $\varepsilon>0$ and every $\eta>0$, there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ such that, for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ and every finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ satisfying

$$
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \eta \quad \text { and } \quad R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right) \quad \text { for every } i \in\{1, \ldots, k\}
$$

one has

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

The goal of this paper is to provide various properties of gauge-measurable functions that can be deduced using elementary ideas of Real Analysis. These are well-known properties of Lebesgue-measurable functions, and both notions of measurability are equivalent, but the main message we want to emphasize is that one can obtain these properties in a self-contained approach based on gauge integrability and gauge measurability. As an example, we show in Section 5 below that these two concepts are related through the following dominatedintegrability characterization of gauge-integrable functions:

Theorem 1.3. A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-integrable if and only if $f$ is gauge-measurable and there exists a gauge-integrable function $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $|f| \leq h$ in $\mathbb{R}^{d}$.

The paper is organized as follows. In Sections 2 and 3, we prove properties of gauge-measurable functions that can be straightforwardly obtained from the definition. Some of them will be superseded in later sections using two important properties of the gauge integral: the Absolute Cauchy criterion and the Dominated convergence theorem. In Section 4, we prove Lusin's theorem for
gauge-measurable functions using an alternative formulation of the gauge measurability based on the inner measure of open sets in $\mathbb{R}^{d}$. We then prove Theorem 1.3 in Section 5. In Section 6, we prove the stability of gauge measurability under pointwise convergence. In Sections 7 and 8, we define gauge-measurable sets in the same spirit as for functions, and then we prove that every gaugemeasurable function is the pointwise limit of gauge-measurable step functions. We thus recover the approach which leads to the Lebesgue integral.

## 2. Elementary properties

The goal of this section is to present some properties of gauge measurability that readily follow from its definition. We begin by noting that every continuous function is gauge-measurable.

Proposition 2.1 (Gauge measurability of continuous functions). If the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is continuous, then $f$ is gauge-measurable.

Proof. Given a pair of points $c_{i}, c_{i}^{\prime} \in \mathbb{R}^{d}$, by the triangle inequality for every $z \in \mathbb{R}^{d}$ we have

$$
\begin{equation*}
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \leq\left|f(z)-f\left(c_{i}\right)\right|+\left|f(z)-f\left(c_{i}^{\prime}\right)\right| \tag{1}
\end{equation*}
$$

Using the continuity of $f$, we choose a gauge $\gamma$ in such a way that the righthand side is always less than $\eta>0$ provided that $\gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right) \neq \emptyset$. Indeed, given $\eta>0$, for every $x \in \mathbb{R}^{d}$ we define

$$
\gamma(x)=\left\{z \in \mathbb{R}^{d}| | f(z)-f(x) \left\lvert\,<\frac{\eta}{2}\right.\right\} .
$$

In particular, $x \in \gamma(x)$; since the function $f$ is continuous, the set $\gamma(x)$ is open. If there exists $z \in \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$, then by the choice of $\gamma$ we have simultaneously

$$
\left|f(z)-f\left(c_{i}\right)\right|<\frac{\eta}{2} \quad \text { and } \quad\left|f(z)-f\left(c_{i}^{\prime}\right)\right|<\frac{\eta}{2}
$$

In view of (1), we then have

$$
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|<\eta
$$

Therefore, no matter what $\varepsilon>0$ we take, there is no finite family of rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ that needs to be checked in Definition 1.2, so the latter is automatically satisfied by the continuous function $f$.

Proposition 2.2 (Composition with uniformly continuous functions). If the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable and the function $\Phi: \mathbb{R}^{p} \rightarrow \mathbb{R}^{\ell}$ is uniformly continuous, then the composition $\Phi \circ f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{\ell}$ is gauge-measurable.

This property is reminiscent of the integrability of compositions with Lipschitz functions for the gauge integral [24, Theorem II-3-1]; the class of admissible functions is larger here because gauge measurability is more qualitative than gauge integrability. As for the gauge integrability, Proposition 2.2 is not the end of the story: we prove in Section 6 using more elaborate tools that the proposition remains true when the function $\Phi$ is merely continuous; see Proposition 6.5 below.

Proof of Proposition 2.2. Given $\eta>0$, by definition of uniform continuity there exists $\delta>0$ such that, for every $y, z \in \mathbb{R}^{d}$ satisfying $|y-z|<\delta$, one has $|\Phi(y)-\Phi(z)|<\eta$. This is equivalent to saying that if $|\Phi(y)-\Phi(z)| \geq \eta$, then $|y-z| \geq \delta$. Hence, for every pair of points $c_{i}, c_{i}^{\prime} \in \mathbb{R}^{d}$ such that

$$
\begin{equation*}
\left|(\Phi \circ f)\left(c_{i}\right)-(\Phi \circ f)\left(c_{i}^{\prime}\right)\right| \geq \eta \tag{2}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \delta \tag{3}
\end{equation*}
$$

Given $\varepsilon>0$, by Definition 1.2 of gauge measurability of $f$ with parameter $\eta=\delta$ there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ such that, for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ and finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ satisfying (2) and $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$ for every $i$, we have that (3) also holds for every $i$, and then by the choice of the gauge $\gamma$,

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

The function $\Phi \circ f$ is thus gauge-measurable.
An interesting consequence of Proposition 2.2 is that the family of gaugemeasurable functions forms a vector space, and the product of two bounded gauge-measurable functions is also gauge-measurable. We provide an independent proof of these facts in the next section for the sake of clarity. The latter property concerning the product will be superseded later on by using the fact that measurability is stable under pointwise convergence, which allows one to remove the boundedness assumption of the functions; see Corollary 6.4. For the moment, we restrict ourselves to the case of uniform limits of gauge-measurable functions:

Proposition 2.3 (Uniform limit). Let $\left(f_{n}\right)_{n \in \mathbb{N}}$ be a sequence of gauge-measurable functions from $\mathbb{R}^{d}$ to $\mathbb{R}^{p}$. If the sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges uniformly to the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$, then $f$ is gauge-measurable.

Proof. For every pair of points $c_{i}, c_{i}^{\prime} \in \mathbb{R}^{d}$ and every $n \in \mathbb{N}$, by the triangle inequality we have

$$
\left|f_{n}\left(c_{i}\right)-f_{n}\left(c_{i}^{\prime}\right)\right| \geq\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|-\left|f_{n}\left(c_{i}\right)-f\left(c_{i}\right)\right|-\left|f_{n}\left(c_{i}^{\prime}\right)-f\left(c_{i}^{\prime}\right)\right|
$$

Given $\eta>0$, by the definition of uniform convergence there exists $n \in \mathbb{N}$ such that, for every $x \in \mathbb{R}^{d},\left|f_{n}(x)-f(x)\right| \leq \eta / 4$. Hence, assuming that

$$
\begin{equation*}
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \eta \tag{4}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|f_{n}\left(c_{i}\right)-f_{n}\left(c_{i}^{\prime}\right)\right| \geq \frac{\eta}{2} \tag{5}
\end{equation*}
$$

Given $\varepsilon>0$, let $\gamma$ be a gauge on $\mathbb{R}^{d}$ given by the definition of gauge measurability of $f_{n}$ with parameter $\eta / 2$. For every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ and every sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ satisfying (4) and $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$ for every $i$, we then have that (5) is satisfied by $f_{n}$ for every $i$, and then, by the choice of $\gamma$,

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

The function $f$ is thus gauge-measurable.

## 3. Algebraic stability

We show that the class of gauge-measurable functions forms a vector space:
Proposition 3.1 (Linearity). If the functions $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ and $g: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ are gauge-measurable and $\lambda \in \mathbb{R}$, then $f+g$ and $\lambda f$ are gauge-measurable.

Proof. We focus on the proof that $f+g$ is gauge measurable; the case of $\lambda f$ is left as an exercise (see also Proposition 2.2). For every pair of points $c_{i}, c_{i}^{\prime} \in \mathbb{R}^{d}$, by the triangle inequality we have

$$
\left|(f+g)\left(c_{i}\right)-(f+g)\left(c_{i}^{\prime}\right)\right| \leq\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|+\left|g\left(c_{i}\right)-g\left(c_{i}^{\prime}\right)\right|
$$

Given $\eta>0$, and assuming that

$$
\begin{equation*}
\left|(f+g)\left(c_{i}\right)-(f+g)\left(c_{i}^{\prime}\right)\right| \geq \eta \tag{6}
\end{equation*}
$$

then we necessarily have

$$
\begin{equation*}
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \frac{\eta}{2} \quad \text { or } \quad\left|g\left(c_{i}\right)-g\left(c_{i}^{\prime}\right)\right| \geq \frac{\eta}{2} \tag{7}
\end{equation*}
$$

Given $\varepsilon>0$, let $\gamma_{1}$ and $\gamma_{2}$ be two gauges on $\mathbb{R}^{d}$ arising from the definitions of gauge measurability of $f$ and $g$, respectively, with parameters $\varepsilon / 2$ and $\eta / 2$. Consider the gauge $\gamma$ defined for $x \in \mathbb{R}^{d}$ by $\gamma(x)=\gamma_{1}(x) \cap \gamma_{2}(x)$. For a finite collection of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ and finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ verifying (6) and $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$ for every $i \in\{1, \ldots, k\}$, let us denote by $I_{1}$ the set of indices $i$ for which the first inequality in (7) holds for $f$ and by $I_{2}$ the set of indices $i$ for which the second inequality in (7) holds for $g$. We can thus assert that

$$
\begin{equation*}
\{1, \ldots, k\}=I_{1} \cup I_{2} \tag{8}
\end{equation*}
$$

We have in particular $R_{i} \subset \gamma_{1}\left(c_{i}\right) \cap \gamma_{1}\left(c_{i}^{\prime}\right)$ for every $i \in I_{1}$, and thus by the choice of $\gamma_{1}$,

$$
\sum_{i \in I_{1}} \operatorname{vol}\left(R_{i}\right) \leq \frac{\varepsilon}{2}
$$

We also have $R_{i} \subset \gamma_{2}\left(c_{i}\right) \cap \gamma_{2}\left(c_{i}^{\prime}\right)$ for every $i \in I_{2}$, and thus by the choice of $\gamma_{2}$,

$$
\sum_{i \in I_{2}} \operatorname{vol}\left(R_{i}\right) \leq \frac{\varepsilon}{2}
$$

Since the sets $I_{1}$ and $I_{2}$ cover $\{1, \ldots, k\}$, we deduce that

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

Therefore, the function $f+g$ is gauge-measurable.
Using a similar idea, one shows that the product of bounded gauge-measurable functions is also gauge-measurable. The conclusion is still true without assuming the functions are bounded, but the proof is more subtle; see Section 6.
Proposition 3.2 (Product of bounded functions). If the functions $f: \mathbb{R}^{d} \rightarrow$ $\mathbb{R}^{p}$ and $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ are gauge-measurable and bounded, then $f g$ is also gaugemeasurable.

Proof. Take $M>0$ and $N>0$ such that $|f| \leq M$ and $|g| \leq N$ in $\mathbb{R}^{d}$. Given finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$, by the triangle inequality for every $x \in \mathbb{R}^{d}$ we have

$$
\begin{aligned}
\left|(f g)\left(c_{i}\right)-(f g)\left(c_{i}^{\prime}\right)\right| & \leq\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|\left|g\left(c_{i}\right)\right|+\left|f\left(c_{i}^{\prime}\right)\right|\left|g\left(c_{i}\right)-g\left(c_{i}^{\prime}\right)\right| \\
& \leq N\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|+M\left|g\left(c_{i}\right)-g\left(c_{i}^{\prime}\right)\right|
\end{aligned}
$$

Given $\eta>0$, if for every $i \in\{1, \ldots, k\}$ we have

$$
\left|(f g)\left(c_{i}\right)-(f g)\left(c_{i}^{\prime}\right)\right| \geq \eta
$$

then necessarily

$$
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \frac{\eta}{2 N} \quad \text { or } \quad\left|g\left(c_{i}\right)-g\left(c_{i}^{\prime}\right)\right| \geq \frac{\eta}{2 M}
$$

As in the previous proof, one defines the subsets of indices $I_{1}$ and $I_{2}$ accordingly, so that the counterpart of (8) also holds in this case. One can now proceed along the lines of the proof of Proposition 3.1 to deduce that $f g$ is gaugemeasurable.

## 4. Lusin's theorem

We now relate the notion of gauge measurability with Lusin's theorem, which trivially extends Proposition 2.1 that is valid for continuous functions:

Proposition 4.1 (Lusin's theorem). A function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable if and only if, for every $\varepsilon>0$, there exists a closed set $C \subset \mathbb{R}^{d}$ such that the restriction $\left.f\right|_{C}$ is continuous and the inner measure of the open set $\mathbb{R}^{d} \backslash C$ satisfies $\mu\left(\mathbb{R}^{d} \backslash C\right) \leq \varepsilon$.

We recall the notion of inner measure of an open set $U \subset \mathbb{R}^{d}$ :

$$
\begin{array}{r}
\mu(U):=\sup \left\{\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \mid\left(R_{i}\right)_{i \in\{1, \ldots, k\}}\right. \text { is a family of disjoint rectangles } \\
\text { contained in } U\} .
\end{array}
$$

Observe that $\mu$ is nondecreasing and countably subadditive. The quantity $\mu(U)$ is unchanged if the supremum is computed over the smaller class of disjoint rectangles $\left(S_{i}\right)_{i \in\{1, \ldots, k\}}$ such that $\bar{S}_{i} \subset U$ for every $i \in\{1, \ldots, k\}$. The reason is that for any number $0<\theta<1$ one can construct a rectangle $S_{i}$ such that $\bar{S}_{i} \subset R_{i}$ and $\operatorname{vol}\left(S_{i}\right) \geq \theta \operatorname{vol}\left(R_{i}\right)$, which gives

$$
\theta \sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \sum_{i=1}^{k} \operatorname{vol}\left(S_{i}\right) \leq \sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right)
$$

Lusin's theorem above gives the equivalence between gauge measurability and the measurability in the sense of Bourbaki, defined in terms of Lusin's property [3, Definition IV- $\S 5-1]$. To prove Proposition 4.1 above, we rely on the following lemma which reformulates Definition 1.2 without relying on tagged partitions:

LEMMA 4.2 (Gauge-intersection characterization). The function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable if and only if, for every $\varepsilon>0$ and every $\eta>0$, there exists
a gauge $\gamma$ on $\mathbb{R}^{d}$ such that the open set

$$
U_{\gamma, \eta}:=\bigcup_{\substack{x, z \in \mathbb{R}^{d} \\|f(x)-f(z)| \geq \eta}}(\gamma(x) \cap \gamma(z))
$$

satisfies $\mu\left(U_{\gamma, \eta}\right) \leq \varepsilon$.
A byproduct of Lemma 4.2 is the invariance of gauge measurability under bi-Lipschitz homeomorphisms of $\mathbb{R}^{d}$, which include isometries.

Proof of Lemma 4.2. " $\Longleftarrow "$. Given $\eta>0$ and a gauge $\gamma$ on $\mathbb{R}^{d}$, take a finite disjoint family of rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ and finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ such that

$$
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \eta \quad \text { and } \quad R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right) \quad \text { for every } i
$$

In particular, $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right) \subset U_{\gamma, \eta}$, hence by definition of the inner measure $\mu\left(U_{\gamma, \eta}\right)$ we have

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \mu\left(U_{\gamma, \eta}\right)
$$

To conclude it suffices to choose the gauge $\gamma$ so that, for any given $\varepsilon>0$, we have $\mu\left(U_{\gamma, \eta}\right) \leq \varepsilon$.
" $\Longrightarrow$ ". Assume that the function $f$ is gauge-measurable, and let $\gamma$ be a gauge on $\mathbb{R}^{d}$ given by Definition 1.2 for some $\varepsilon>0$ and $\eta>0$. Let $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ be a finite family of disjoint rectangles contained in $U_{\gamma, \eta}$. By the remark following the definition of the inner measure $\mu$, we may restrict our attention to the case where $\bar{R}_{i} \subset U_{\gamma, \eta}$ for every $i$. Then, by compactness of $\bar{R}_{i}$, the rectangle $R_{i}$ can be covered by a finite collection of sets of the form $\gamma(x) \cap \gamma(z)$ such that $x, z \in \mathbb{R}^{d}$ and $|f(x)-f(z)| \geq \eta$. By a suitable subdivision of the rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ into smaller rectangles, which does not change their total volume, we can thus assume without loss of generality that, for every $i \in\{1, \ldots, k\}$, there exist points $x, z \in \mathbb{R}^{d}$ such that

$$
R_{i} \subset \gamma(x) \cap \gamma(z) \quad \text { and } \quad|f(x)-f(z)| \geq \eta
$$

[Such a subdivision is allowed since the points $x$ and $z$ are not required to belong to $R_{i}$.] We then choose $c_{i}=x$ and $c_{i}^{\prime}=z$. The finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ satisfy the conditions of Definition 1.2, and we deduce that

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

Since the family of rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ is chosen arbitrarily, we thus have that $\mu\left(U_{\gamma, \eta}\right) \leq \varepsilon$.

Proof of Proposition 4.1. We first observe that given $\eta>0$, a gauge $\gamma$ on $\mathbb{R}^{d}$, and $z \in \mathbb{R}^{d}$, then for every $x \in \gamma(z) \backslash U_{\gamma, \eta}$ we have

$$
\begin{equation*}
|f(x)-f(z)|<\eta \tag{9}
\end{equation*}
$$

Indeed, since $x \in \gamma(x) \cap \gamma(z)$ and $x \notin U_{\gamma, \eta}$, the set $\gamma(x) \cap \gamma(z)$ is not contained in $U_{\gamma, \eta}$, hence $x$ and $z$ are not admissible indices in the union that defines the set $U_{\gamma, \eta}$. We deduce that (9) holds.

Proceeding with the proof of the proposition, we now assume that the function $f$ is gauge-measurable and let $\varepsilon>0$. For each $n \in \mathbb{N}$, by Lemma 4.2 there exists a gauge $\gamma_{n}$ on $\mathbb{R}^{d}$ such that

$$
\mu\left(U_{\gamma_{n}, 1 / 2^{n}}\right) \leq \frac{\varepsilon}{2^{n+1}}
$$

We set $C=\mathbb{R}^{d} \backslash \bigcup_{n \in \mathbb{N}} U_{\gamma_{n}, 1 / 2^{n}}$. By countable subadditivity of $\mu$, we have

$$
\mu\left(\mathbb{R}^{d} \backslash C\right) \leq \sum_{n \in \mathbb{N}} \mu\left(U_{\gamma_{n}, 1 / 2^{n}}\right) \leq \sum_{n \in \mathbb{N}} \frac{\varepsilon}{2^{n+1}}=\varepsilon
$$

It remains to prove that the restricted function $\left.f\right|_{C}$ is continuous at any point $z \in C$. For every $x \in \gamma_{n}(z) \cap C \subset \gamma_{n}(z) \backslash U_{\gamma_{n}, 1 / 2^{n}}$, we deduce from estimate (9) above that

$$
|f(x)-f(z)|<\frac{1}{2^{n}}
$$

Since this estimate holds on the relatively open subset $\gamma_{n}(z) \cap C$ of $C$ and $n \in \mathbb{N}$ is arbitrary, we deduce that the function $\left.f\right|_{C}$ is continuous at $z$.

Conversely, we take a closed set $C$ such that the restriction $\left.f\right|_{C}$ is continuous. For every $\eta>0$, the set

$$
\gamma(x)=\mathbb{R}^{d} \backslash\left\{w \in C| | f(x)-f(w) \left\lvert\, \geq \frac{\eta}{2}\right.\right\}
$$

contains $x$ and is open in $\mathbb{R}^{d}$, since the function $\left.f\right|_{C}$ is continuous and the set $C$ is closed. Hence, $\gamma$ is a gauge on $\mathbb{R}^{d}$. We now observe that if $x, z \in \mathbb{R}^{d}$ and $|f(x)-f(z)| \geq \eta$, then

$$
\gamma(x) \cap \gamma(z) \cap C=\emptyset
$$

Indeed, if this were not true, there would exist a point $w \in \gamma(x) \cap \gamma(z) \cap C$. Since $w \in C$, we would have, by definition of $\gamma,|f(x)-f(w)|<\eta / 2$ and $|f(z)-f(w)|<\eta / 2$ and thus by the triangle inequality $|f(x)-f(z)|<\eta$, which would be a contradiction.

We thus have $U_{\gamma, \eta} \subset \mathbb{R}^{d} \backslash C$, and then by monotonicity of the inner measure $\mu$,

$$
\mu\left(U_{\gamma, \eta}\right) \leq \mu\left(\mathbb{R}^{d} \backslash C\right)
$$

Given $\varepsilon>0$, by the Lusin property satisfied by the function $f$, we may choose the closed set $C$ so as to have $\mu\left(\mathbb{R}^{d} \backslash C\right) \leq \varepsilon$. We conclude from Lemma 4.2 that the function $f$ is gauge-measurable.

## 5. Gauge measurability and integrability

The goal of this section is to establish Theorem 1.3. The relationship between gauge measurability and gauge integrability relies on the following Absolute Cauchy criterion for gauge-integrable functions [24, Theorem II-2-4] (see also [14, Lemma 5.13]).
Proposition 5.1 (Absolute Cauchy criterion). The function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-integrable if and only if, for every $\varepsilon>0$, there exist a gauge $\gamma$ on $\mathbb{R}^{d}$ and a compact subset $K \subset \mathbb{R}^{d}$ such that the following properties hold:
(i) for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ and every finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ satisfying $R_{i} \subset \gamma\left(c_{i}\right) \cap$ $\gamma\left(c_{i}^{\prime}\right)$ for every $i$, one has

$$
\sum_{i=1}^{k}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

(ii) for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d} \backslash K$ and every finite set of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ such that $R_{i} \subset \gamma\left(c_{i}\right)$ for every $i$, one has

$$
\sum_{i=1}^{k}\left|f\left(c_{i}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

This condition is a Cauchy criterion because it does not require nor gives the value of the integral of $f$. It is an absolute Cauchy condition because the norm is taken inside the Riemann sum. An important consequence of Proposition 5.1 is the fact that if $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-integrable and if $\Phi$ is a Lipschitzcontinuous function such that $\Phi(0)=0$, then the composite function $\Phi \circ f$ is also gauge-integrable [24, Theorem II-3-1]. In particular, $|f|$ is gauge-integrable whenever $f$ is gauge-integrable.

We first consider the question of gauge measurability of gauge-integrable functions.

Proposition 5.2 (Gauge measurability). If $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-integrable, then $f$ is gauge-measurable.
Proof. Let $\eta>0$ and take a finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ and finite sets of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ in $\mathbb{R}^{d}$ such that

$$
\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \eta \quad \text { for every } i
$$

Then, we have

$$
\begin{equation*}
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \frac{1}{\eta} \sum_{i=1}^{k}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \tag{10}
\end{equation*}
$$

Applying Property ( $i$ ) of the Absolute Cauchy criterion with parameter $\eta \varepsilon$, there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ such that if $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$, then the sum in the right-hand side of (10) is smaller than $\eta \varepsilon$, and we get

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \frac{1}{\eta} \cdot \eta \varepsilon=\varepsilon
$$

We deduce that the function $f$ is gauge-measurable in view of Definition 1.2.
We now handle the reverse implication of Theorem 1.3 under the additional assumption that $f$ is a bounded function.
Proposition 5.3 (Dominated integrability for bounded functions). If $f: \mathbb{R}^{d} \rightarrow$ $\mathbb{R}^{p}$ is gauge-measurable and bounded and if $|f| \leq h$ in $\mathbb{R}^{d}$ for some gaugeintegrable function $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$, then $f$ is gauge-integrable.
Proof. Property (ii) of the Absolute Cauchy criterion is satisfied by h, hence also by $f$. We now focus on Property $(i)$. For this purpose, let $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$ be a finite collection of disjoint rectangles, and let $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ and $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ be finitely many points in $\mathbb{R}^{d}$. Given $\eta>0$ and a compact subset $K \subset \mathbb{R}^{d}$, we can relabel the rectangles and points simultaneously so as to have
(a) for every $i \in\{1, \ldots, m\},\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \geq \eta$,
(b) for every $i \in\{m+1, \ldots, l\},\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|<\eta$ and $R_{i} \cap K \neq \emptyset$,
(c) for every $i \in\{l+1, \ldots, k\},\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right|<\eta$ and $R_{i} \cap K=\emptyset$,
for some integers $0 \leq m \leq l \leq k$; some of these conditions might be empty, and in this case one simply ignores them.

By the assumption of boundedness of $f$, there exists $M>0$ such that, for every $x \in \mathbb{R}^{d},|f(x)| \leq M$. By the triangle inequality, we then have

$$
\sum_{i=1}^{m}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq 2 M \sum_{i=1}^{m} \operatorname{vol}\left(R_{i}\right),
$$

and, by (b),

$$
\sum_{i=m+1}^{l}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \eta \sum_{i=m+1}^{l} \operatorname{vol}\left(R_{i}\right)
$$

Since $|f| \leq h$ in $\mathbb{R}^{d}$, we also have

$$
\begin{aligned}
\sum_{i=l+1}^{k}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) & \leq \sum_{i=l+1}^{k}\left(\left|f\left(c_{i}\right)\right|+\left|f\left(c_{i}^{\prime}\right)\right|\right) \operatorname{vol}\left(R_{i}\right) \\
& \leq \sum_{i=l+1}^{k}\left(h\left(c_{i}\right)+h\left(c_{i}^{\prime}\right)\right) \operatorname{vol}\left(R_{i}\right)
\end{aligned}
$$

These are the three main estimates that we need in the sequel. We now proceed to choose the gauge $\gamma$ that yields the Absolute Cauchy criterion for $f$.

Given $\varepsilon>0$, by Property (ii) of the Absolute Cauchy criterion satisfied by $h$ with parameter $\varepsilon / 6$, we can take the compact set $K \subset \mathbb{R}^{d}$ and a gauge $\gamma_{1}$ on $\mathbb{R}^{d}$ such that if $R_{i} \subset \gamma_{1}\left(c_{i}\right) \cap \gamma_{1}\left(c_{i}^{\prime}\right)$ for every $i \in\{l+1, \ldots, k\}$, then we have

$$
\sum_{i=l+1}^{k}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \sum_{i=l+1}^{k}\left(h\left(c_{i}\right)+h\left(c_{i}^{\prime}\right)\right) \operatorname{vol}\left(R_{i}\right) \leq \frac{\varepsilon}{6}+\frac{\varepsilon}{6}=\frac{\varepsilon}{3}
$$

Fix a bounded open set $U \subset \mathbb{R}^{d}$ that contains $K$, and take the gauge $\gamma_{2}$ on $\mathbb{R}^{d}$ defined by $\gamma_{2}(x)=U$ if $x \in U$ and $\gamma_{2}(x)=\mathbb{R}^{d} \backslash K$ if $x \notin U$. Observe that if $R_{i} \subset \gamma_{2}\left(c_{i}\right) \cap \gamma_{2}\left(c_{i}^{\prime}\right)$ for every $i \in\{m+1, \ldots, l\}$, then since $R_{i} \cap K \neq \emptyset$, we necessarily have $\gamma_{2}\left(c_{i}\right)=\gamma_{2}\left(c_{i}^{\prime}\right)=U$, and thus $R_{i} \subset U$. By definition of the inner measure $\mu$, and choosing $\eta>0$ so as to have $\eta \mu(U) \leq \varepsilon / 3$, we then get

$$
\sum_{i=m+1}^{l}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \eta \sum_{i=m+1}^{l} \operatorname{vol}\left(R_{i}\right) \leq \eta \mu(U) \leq \frac{\varepsilon}{3}
$$

By definition of gauge measurability of $f$ with $\varepsilon / 6 M$ and $\eta$ chosen as above, there exists a gauge $\gamma_{3}$ on $\mathbb{R}^{d}$ such that if $R_{i} \subset \gamma_{3}\left(c_{i}\right) \cap \gamma_{3}\left(c_{i}^{\prime}\right)$ for every $i \in$ $\{1, \ldots, m\}$, then we have

$$
\sum_{i=1}^{m}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq 2 M \sum_{i=1}^{m} \operatorname{vol}\left(R_{i}\right) \leq 2 M \cdot \frac{\varepsilon}{6 M}=\frac{\varepsilon}{3}
$$

Combining these three estimates, we get

$$
\sum_{i=1}^{k}\left|f\left(c_{i}\right)-f\left(c_{i}^{\prime}\right)\right| \operatorname{vol}\left(R_{i}\right) \leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon
$$

and thus $f$ satisfies the Absolute Cauchy criterion with the gauge $\gamma$ defined for $x \in \mathbb{R}^{d}$ by $\gamma(x)=\gamma_{1}(x) \cap \gamma_{2}(x) \cap \gamma_{3}(x)$. Hence, $f$ is gauge-integrable by Proposition 5.1.

The boundedness assumption of $f$ can be removed using the Dominated convergence theorem for gauge-integrable functions [24, Theorem II-10-1]:
Proposition 5.4 (Dominated convergence). Let $\left(f_{n}\right)_{n \in \mathbb{N}}$ be a sequence of gaugeintegrable functions from $\mathbb{R}^{d}$ to $\mathbb{R}^{p}$. If $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges pointwise to the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$, and if there exists a gauge-integrable function $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $\left|f_{n}\right| \leq h$ in $\mathbb{R}^{d}$ for every $n \in \mathbb{N}$, then $f$ is gauge-integrable and

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{d}} f_{n}=\int_{\mathbb{R}^{d}} f
$$

Proof of Theorem 1.3. If $f$ is gauge-integrable, then $f$ is gauge-measurable by Proposition 5.2, and it follows from the Absolute Cauchy criterion above that the function $h:=|f|$ is gauge-integrable.

Conversely, if $f$ is gauge-measurable, then by Proposition 2.2, for every $n \in$ $\mathbb{N}$ the truncated function $T_{n} \circ f$ is also gauge-measurable, where $T_{n}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ is the truncation function defined for $w \in \mathbb{R}^{p}$ by

$$
T_{n}(w)= \begin{cases}w & \text { if }|w| \leq n \\ n w /|w| & \text { if }|w|>n\end{cases}
$$

Since the function $T_{n} \circ f$ is bounded and satisfies $\left|T_{n} \circ f\right| \leq|f| \leq h$ in $\mathbb{R}^{d}$, it follows from Proposition 5.3 that $T_{n} \circ f$ is gauge-integrable, and we conclude applying the Dominated convergence theorem for gauge integrals as $n$ tends to infinity.

## 6. Pointwise limit

A crucial feature of Lebesgue-measurable functions is their stability under pointwise convergence. Up to now, we only have proved that gauge measurability is stable under uniform convergence, see Proposition 2.3. Thanks to the relationship that we have established between gauge measurability and gauge integrability, we now obtain a pointwise-convergence property in full generality.

Proposition 6.1 (Pointwise limit). Let $\left(f_{n}\right)_{n \in \mathbb{N}}$ be a sequence of gauge-measurable functions from $\mathbb{R}^{d}$ to $\mathbb{R}^{p}$. If $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges pointwise to the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$, then $f$ is gauge-measurable.

We first prove two particular cases of this proposition, which as we shall see yield the general case. We denote the characteristic function of a set $A \subset \mathbb{R}^{d}$ by $\chi_{A}$, that is $\chi_{A}: \mathbb{R}^{d} \rightarrow \mathbb{R}$ is the function defined for each $x \in \mathbb{R}^{d}$ by

$$
\chi_{A}(x)= \begin{cases}1 & \text { if } x \in A \\ 0 & \text { if } x \notin A\end{cases}
$$

LEMMA 6.2. Let $\left(A_{l}\right)_{l \in \mathbb{N}}$ be an increasing sequence of open subsets which cover $\mathbb{R}^{d}$ with $\overline{A_{l-1}} \subset A_{l}$ for every $l \in \mathbb{N}_{*}$. If a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is such that $f \chi_{A_{l}}$ is gauge-measurable for every $l \in \mathbb{N}$, then $f$ is also gauge-measurable.

Proof. Given $l, m \in \mathbb{N}_{*}$ with $l \leq m$, we first observe that if

$$
\begin{equation*}
\left(A_{l} \backslash A_{l-2}\right) \cap\left(A_{m} \backslash A_{m-2}\right) \neq \emptyset, \tag{11}
\end{equation*}
$$

then by monotonicity of the sequence $\left(A_{l}\right)_{l \in \mathbb{N}}$ we have $m=l$ or $m=l+1$. Here, we use the convention that $A_{-1}=\emptyset$. Now let $\left(\gamma_{l}\right)_{l \in \mathbb{N} \backslash\{0,1\}}$ be a sequence
of gauges on $\mathbb{R}^{d}$ to be chosen later on. We define a new gauge $\gamma$ on $\mathbb{R}^{d}$ as follows: for every $x \in \mathbb{R}^{d}$, denote by $l$ the smallest integer in $\mathbb{N}_{*}$ such that $x \in A_{l}$ and let

$$
\gamma(x)=\gamma_{l}(x) \cap \gamma_{l+1}(x) \cap\left(A_{l} \backslash \overline{A_{l-2}}\right)
$$

Since $\overline{A_{l-2}} \subset A_{l-1}$ by assumption, we have $x \notin \overline{A_{l-2}}$, and then the open set $A_{l} \backslash \overline{A_{l-2}}$ contains $x$. Thus, $\gamma$ is a well-defined gauge on $\mathbb{R}^{d}$.

Given $\eta>0$, we claim that

$$
\begin{equation*}
U_{\gamma, \eta} \subset \bigcup_{l \in \mathbb{N}_{*}} V_{l+1} \tag{12}
\end{equation*}
$$

where

$$
V_{l+1}:=\bigcup_{\substack{x, z \in \mathbb{R}^{d} \\\left|f \chi_{A_{l+1}}(x)-f \chi_{A_{l+1}}(z)\right| \geq \eta}}\left(\gamma_{l+1}(x) \cap \gamma_{l+1}(z)\right)
$$

Indeed, assume that $x, z \in \mathbb{R}^{d}$ are such that $|f(x)-f(z)| \geq \eta$. Let $l$ and $m$ be the smallest integers in $\mathbb{N}_{*}$ such that $x \in A_{l}$ and $z \in A_{m}$; we may assume without loss of generality that $l \leq m$. If $\gamma(x) \cap \gamma(z) \neq \emptyset$, then (11) holds, and thus $m=l$ or $m=l+1$. Hence, we have

$$
\left|f \chi_{A_{l+1}}(x)-f \chi_{A_{l+1}}(z)\right|=|f(x)-f(z)| \geq \eta
$$

and

$$
\gamma(x) \cap \gamma(z) \subset \gamma_{l+1}(x) \cap \gamma_{l+1}(z) \subset V_{l+1}
$$

which implies (12).
Let $\varepsilon>0$. Since the function $f \chi_{A_{l+1}}$ is gauge-measurable, by Lemma 4.2 we can choose the gauge $\gamma_{l+1}$ on $\mathbb{R}^{d}$ such that $\mu\left(V_{l+1}\right) \leq \varepsilon / 2^{l}$. Thus, by the inclusion (12) and the countable subadditivity of the inner measure $\mu$ we get

$$
\mu\left(U_{\gamma, \eta}\right) \leq \sum_{l \in \mathbb{N}_{*}} \mu\left(V_{l+1}\right) \leq \sum_{l \in \mathbb{N}_{*}} \frac{\varepsilon}{2^{l}}=\varepsilon
$$

By Lemma 4.2, we deduce that $f$ is gauge-measurable.
Lemma 6.3. If the function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is such that the truncation $T_{j} \circ f$ is gauge-measurable for every $j \in \mathbb{N}$, then $f$ is gauge-measurable.

Proof. Given a sequence of gauges $\left(\gamma_{j}\right)_{j \in \mathbb{N}_{*}}$ on $\mathbb{R}^{d}$, consider the gauge $\gamma$ defined for $x \in \mathbb{R}^{d}$ by

$$
\gamma(x)=\gamma_{0}(x) \cap \cdots \cap \gamma_{j+1}(x)
$$

where $j \in \mathbb{N}$ is the smallest integer such that $|f(x)| \leq j$. For every $0<\eta \leq 1$, we claim that

$$
\begin{equation*}
U_{\gamma, \eta} \subset \bigcup_{j \in \mathbb{N}} W_{j+1} \tag{13}
\end{equation*}
$$

where

$$
W_{j+1}:=\bigcup_{\substack{x, z \in \mathbb{R}^{d} \\\left|T_{j+1} \circ f(x)-T_{j+1} \circ f(z)\right| \geq \eta}}\left(\gamma_{j+1}(x) \cap \gamma_{j+1}(z)\right)
$$

For this purpose, for every $x, z \in \mathbb{R}^{d}$ such that $|f(x)-f(z)| \geq \eta$, which we may assume that $|f(z)| \geq|f(x)|$, let $j \in \mathbb{N}$ be the smallest integer such that $|f(x)| \leq j$. Since $\eta \leq 1$, we also have

$$
\left|T_{j+1} \circ f(x)-T_{j+1} \circ f(z)\right| \geq \eta .
$$

From the choice of the gauge $\gamma$, we deduce that

$$
\gamma(x) \cap \gamma(z) \subset \gamma_{j+1}(x) \cap \gamma_{j+1}(z) \subset W_{j+1}
$$

and the inclusion (13) follows.
Let $\varepsilon>0$. Since the function $T_{j+1} \circ f$ is gauge-measurable, by Lemma 4.2 we can choose the gauge $\gamma_{j+1}$ on $\mathbb{R}^{d}$ such that $\mu\left(W_{j+1}\right) \leq \varepsilon / 2^{j+1}$. Proceeding as in the previous lemma, we have $\mu\left(U_{\gamma, \eta}\right) \leq \varepsilon$, hence $f$ is gauge-measurable.

Proof of Proposition 6.1. We first assume that there exists a gauge-integrable function $h: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $\left|f_{n}\right| \leq h$ in $\mathbb{R}^{d}$ for every $n \in \mathbb{N}$. By Theorem 1.3, each function $f_{n}$ is gauge-integrable, and it then follows from the Dominated convergence theorem that $f$ is gauge-integrable, hence also gauge-measurable.

In the general case where the sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ need not be bounded by an integrable function, for every $n, l, j \in \mathbb{N}$ we consider the function

$$
g_{n, l, j}=\left(T_{j} \circ f_{n}\right) \chi_{B_{l+1}(0)} .
$$

These functions are all gauge-measurable. Indeed, $T_{j} \circ f_{n}$ is gauge-measurable by composition with the uniformly continuous function $T_{j}$ (Proposition 2.2), and thus $g_{n, l, j}$ is gauge-measurable as the product of bounded gauge-measurable functions (Proposition 3.2).

Since $\left|g_{n, l, j}\right| \leq j \chi_{B_{l+1}(0)}$ in $\mathbb{R}^{d}$ and the characteristic function $\chi_{B_{l+1}(0)}$ is gauge-integrable, as $n$ tends to infinity it follows from the first case we considered above that the functions $\left(T_{j} \circ f\right) \chi_{B_{l+1}(0)}$ are gauge-measurable for every $l, j \in \mathbb{N}$. By Lemma 6.2, as $l$ tends to infinity we deduce that $T_{j} \circ f$ is gaugemeasurable for every $j \in \mathbb{N}$. The conclusion then follows from Lemma 6.3 as $j$ tends to infinity.

A consequence of Propositions 3.2 and 6.1 is that the product of gaugemeasurable functions is also gauge-measurable:

Proposition 6.4 (Product). If the functions $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ and $g: \mathbb{R}^{d} \rightarrow \mathbb{R}$ are gauge-measurable, then their product $f g$ is also gauge-measurable.

More generally, we can weaken the assumptions of Proposition 2.2 on the gauge measurability of composite functions:

Proposition 6.5 (Composition with continuous functions). If the function $f$ : $\mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable and the function $\Phi: \mathbb{R}^{p} \rightarrow \mathbb{R}^{\ell}$ is continuous, then the composition $\Phi \circ f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{\ell}$ is gauge-measurable.

Proof. We consider a continuous function $\varphi: \mathbb{R}^{p} \rightarrow \mathbb{R}$ with compact support, and we define $\Phi_{n}: \mathbb{R}^{p} \rightarrow \mathbb{R}^{\ell}$ for each $n \in \mathbb{N}_{*}$ and $y \in \mathbb{R}^{p}$ by $\Phi_{n}(y)=$ $\varphi(y / n) \Phi(y)$. Since the function $\Phi_{n}$ is continuous and has compact support, $\Phi_{n}$ is uniformly continuous, and thus, in view of Proposition 2.2, the function $\Phi_{n} \circ f$ is gauge-measurable. We conclude by observing that, for every $x \in \mathbb{R}^{n}$, the sequence $\left(\Phi_{n}(f(x))\right)_{n \in \mathbb{N}_{*}}$ converges to $\Phi(f(x))$ provided that $\varphi(0)=1$, and thus by Proposition 6.1 the function $\Phi \circ f$ is gauge-measurable.

The proof of Proposition 6.5 shows that the class of functions $\Phi: \mathbb{R}^{p} \rightarrow \mathbb{R}^{\ell}$ such that, for every gauge-measurable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$, the composition $\Phi \circ f$ is measurable is stable under pointwise convergence. This class thus forms a Baire system and contains in particular all Baire (or analytic representable) functions, which coincide by the Lebesgue-Hausdorff theorem with all Borelmeasurable functions, see [10, Theorem 43.IV] and [13, §31].

Another consequence of Proposition 6.1 combined with the gauge measurability of gauge-integrable functions (Proposition 5.2) is that the pointwise limit of a sequence of gauge-integrable functions is always gauge-measurable. This implies in particular that measurable functions in the sense of McShane [24, Definition III-10-1] are indeed gauge-measurable. Conversely, every gaugemeasurable function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ in the sense of Definition 1.2 is the limit of a sequence of gauge-integrable functions. This assertion follows from a diagonalization procedure using the functions $g_{n, l, j}$ which are used in the proof of Proposition 6.1 above. For example, the sequence of gauge-integrable functions $\left(g_{n, n, n}\right)_{n \in \mathbb{N}}$ converges pointwise to the gauge-measurable function $f$. Another pointwise approximation of $f$ in terms of gauge-measurable step functions is pursued in Section 8.

## 7. Gauge-measurable sets

We define gauge measurability of a set in the spirit of its counterpart for functions:

Definition 7.1. $A$ set $A \subset \mathbb{R}^{d}$ is gauge-measurable whenever, for every $\varepsilon>0$, there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ such that, for every finite set of disjoint rectangles $\left(R_{i}\right)_{i \in\{1, \ldots, k\}}$, every finite set of points $\left(c_{i}\right)_{i \in\{1, \ldots, k\}}$ contained in $A$, and every finite set of points $\left(c_{i}^{\prime}\right)_{i \in\{1, \ldots, k\}}$ contained in $\mathbb{R}^{d} \backslash A$ that satisfy $R_{i} \subset \gamma\left(c_{i}\right) \cap \gamma\left(c_{i}^{\prime}\right)$
for every $i \in\{1, \ldots, k\}$, one has

$$
\sum_{i=1}^{k} \operatorname{vol}\left(R_{i}\right) \leq \varepsilon
$$

It follows from this definition that $A$ is gauge-measurable if and only if its complement $\mathbb{R}^{d} \backslash A$ is gauge-measurable. Also observe that for any $0<\eta<1$ we have

$$
\left|\chi_{A}(x)-\chi_{A}(z)\right| \geq \eta
$$

if and only if $x \in A$ and $z \in \mathbb{R}^{d} \backslash A$, or $x \in \mathbb{R}^{d} \backslash A$ and $z \in A$. In view of Definitions 1.2 and 7.1, it thus follows that the set $A \subset \mathbb{R}^{d}$ is gauge-measurable if and only if the characteristic function $\chi_{A}$ is gauge-measurable.

As in Lemma 4.2, the definition above can be reformulated by replacing the tagged partitions with the inner measure of an open set:

Lemma 7.2 (Gauge-intersection characterization). The set $A \subset \mathbb{R}^{d}$ is gaugemeasurable if and only if, for every $\varepsilon>0$, there exists a gauge $\gamma$ on $\mathbb{R}^{d}$ such that the open set

$$
U_{A, \gamma}:=\bigcup_{\substack{x \in A, z \in \mathbb{R}^{d} \backslash A}}(\gamma(x) \cap \gamma(z))
$$

satisfies $\mu\left(U_{A, \gamma}\right) \leq \varepsilon$.
This characterization can be established along the lines of the proof of Lemma 4.2 and is left as an exercise. The family of gauge-measurable sets forms an algebra:

Proposition 7.3. If the sets $A_{1}, A_{2} \subset \mathbb{R}^{d}$ are gauge-measurable, then $A_{1} \cup A_{2}$, $A_{1} \cap A_{2}$, and $A_{1} \backslash A_{2}$ are also gauge-measurable.

Proof. We prove that $A_{1} \cup A_{2}$ is gauge-measurable. For this purpose, observe that every $z \in \mathbb{R}^{d} \backslash\left(A_{1} \cup A_{2}\right)$ satisfies $z \in \mathbb{R}^{d} \backslash A_{1}$ and $z \in \mathbb{R}^{d} \backslash A_{2}$. Thus, for any gauge $\gamma$ on $\mathbb{R}^{d}$ we have

$$
U_{A_{1} \cup A_{2}, \gamma} \subset U_{A_{1}, \gamma} \cup U_{A_{2}, \gamma}
$$

Given $\varepsilon>0$, let $\gamma_{1}$ and $\gamma_{2}$ be two gauges on $\mathbb{R}^{d}$ satisfying the conclusion of Lemma 7.2 for $A_{1}$ and $A_{2}$, respectively, with parameter $\varepsilon / 2$. Take the gauge $\gamma$ defined for $x \in \mathbb{R}^{d}$ by $\gamma(x)=\gamma_{1}(x) \cap \gamma_{2}(x)$. Thus,

$$
U_{A_{1} \cup A_{2}, \gamma} \subset U_{A_{1}, \gamma} \cup U_{A_{2}, \gamma} \subset U_{A_{1}, \gamma_{1}} \cup U_{A_{2}, \gamma_{2}}
$$

and by the monotonicity and subadditivity of $\mu$ we then get

$$
\mu\left(U_{A_{1} \cup A_{2}, \gamma}\right) \leq \mu\left(U_{A_{1}, \gamma_{1}}\right)+\mu\left(U_{A_{2}, \gamma_{2}}\right) \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon
$$

Hence, $A_{1} \cup A_{2}$ is gauge-measurable by Lemma 7.2.
Since we have

$$
\mathbb{R}^{d} \backslash\left(A_{1} \cap A_{2}\right)=\left(\mathbb{R}^{d} \backslash A_{1}\right) \cup\left(\mathbb{R}^{d} \backslash A_{2}\right)
$$

and both sets $\mathbb{R}^{d} \backslash A_{1}$ and $\mathbb{R}^{d} \backslash A_{2}$ are gauge-measurable, we deduce that $\mathbb{R}^{d} \backslash\left(A_{1} \cap A_{2}\right)$ is gauge-measurable, and thus the intersection $A_{1} \cap A_{2}$ is also gauge-measurable. Finally, since

$$
A_{1} \backslash A_{2}=A_{1} \cap\left(\mathbb{R}^{d} \backslash A_{2}\right)
$$

is the intersection of two gauge-measurable sets, $A_{1} \backslash A_{2}$ is also gauge-measurable.

Using the equivalence between the gauge measurability of the set $A$ and the gauge-measurability of the characteristic function $\chi_{A}$, we deduce that the family of gauge-measurable sets forms a $\sigma$-algebra:
Proposition 7.4 (Countable union). If $\left(A_{n}\right)_{n \in \mathbb{N}}$ is a sequence of gauge-measurable sets in $\mathbb{R}^{d}$, then the set $\bigcup_{k \in \mathbb{N}} A_{k}$ is also gauge-measurable.

Proof. The sequence of characteristic functions $\left(f_{n}\right)_{n \in \mathbb{N}}$ defined for each $n \in \mathbb{N}$ by $f_{n}=\chi_{\bigcup_{k=0}^{n} A_{k}}$ converges pointwise to the characteristic function $\chi_{\bigcup_{k \in \mathbb{N}} A_{k}}$ in $\mathbb{R}^{d}$. By induction using Proposition 7.3, each set $\bigcup_{k=0}^{n} A_{k}$ is gauge-measurable and thus each function $f_{n}$ is gauge-measurable. From the stability of gauge measurability under pointwise convergence (Proposition 6.1), we deduce that the function $\chi_{\bigcup_{k \in \mathbb{N}} A_{k}}$ is also gauge-measurable, hence the set $\bigcup_{k \in \mathbb{N}} A_{k}$ is gaugemeasurable.

Let us now prove Lebesgue's regularity property, which yields the equivalence between gauge measurability and Lebesgue measurability; see [8, §17], [27, Lemma 3.22], and also [28].
Proposition 7.5 (Regularity). The set $A \subset \mathbb{R}^{d}$ is gauge-measurable if and only if, for every $\varepsilon>0$, there exist an open set $V \subset \mathbb{R}^{d}$ and a closed set $C \subset \mathbb{R}^{d}$ such that $C \subset A \subset V$ and $\mu(V \backslash C) \leq \varepsilon$.
Proof. Given a gauge $\gamma$ on $\mathbb{R}^{d}$, set

$$
V=\bigcup_{x \in A} \gamma(x) \quad \text { and } \quad C=\bigcap_{z \in \mathbb{R}^{d} \backslash A}\left(\mathbb{R}^{d} \backslash \gamma(z)\right)
$$

Observe that $V$ is open, $C$ is closed, and $V \backslash C=U_{A, \gamma}$. Thus, given $\varepsilon>0$, if the set $A$ is gauge-measurable and one takes a gauge $\gamma$ such that $\mu\left(U_{A, \gamma}\right) \leq \varepsilon$, then the sets $V$ and $C$ above satisfy the requirements.

Conversely, if the set $A$ satisfies the regularity condition, then given $\varepsilon>0$ we take the sets $V$ and $C$ as in the statement. The gauge $\gamma$ defined on $\mathbb{R}^{d}$ by setting $\gamma(x)=V$ if $x \in A$ and $\gamma(x)=\mathbb{R}^{d} \backslash C$ if $x \in \mathbb{R}^{d} \backslash A$ satisfies $U_{A, \gamma}=V \backslash C$, and thus $\mu\left(U_{A, \gamma}\right) \leq \varepsilon$. Hence, the set $A$ is gauge-measurable by Lemma 7.2.

## 8. Pointwise approximation

We conclude with the pointwise approximation of a gauge-measurable function by step functions. We recall that $g: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is a step function if the image $g\left(\mathbb{R}^{d}\right)$ is a finite set.

Proposition 8.1. If a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable, then there exists a sequence of gauge-measurable step functions $\left(f_{n}\right)_{n \in \mathbb{N}}$ from $\mathbb{R}^{d}$ to $\mathbb{R}^{p}$ which converges pointwise to $f$ in $\mathbb{R}^{d}$ and satisfies $\left|f_{n}\right| \leq|f|$ in $\mathbb{R}^{d}$ for every $n \in \mathbb{N}$.

This statement allows one to recover a widespread strategy to define the Lebesgue integral via measurable step functions. In our case, if $f$ is gaugeintegrable, and thus $|f|$ is also gauge-integrable by the Absolute Cauchy criterion, then from the Dominated convergence theorem (Proposition 5.4) we indeed have that

$$
\int_{\mathbb{R}^{d}} f=\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{d}} f_{n}
$$

The difference here is that this is a property of the gauge integral, rather than a definition.

Before proving Proposition 8.1, we first study the inverse image of rectangles by gauge-measurable functions:

Proposition 8.2. If a function $f: \mathbb{R}^{d} \rightarrow \mathbb{R}^{p}$ is gauge-measurable, then, for every rectangle $R \subset \mathbb{R}^{p}$, the set $f^{-1}(R)$ is gauge-measurable.
Proof. Observe that $\chi_{R} \circ f=\chi_{f^{-1}(R)}$. To prove the proposition, it thus suffices to prove that the function $\chi_{R} \circ f$ is gauge-measurable. For this purpose, take a sequence of uniformly continuous functions $\left(\Phi_{n}\right)_{n \in \mathbb{N}}$ from $\mathbb{R}^{p}$ to $\mathbb{R}$ which converges pointwise to $\chi_{R}$. Then, by Proposition 2.2 the function $\Phi_{n} \circ f$ is gauge-measurable for every $n \in \mathbb{N}$, and the sequence $\left(\Phi_{n} \circ f\right)_{n \in \mathbb{N}}$ converges pointwise to $\chi_{R} \circ f$. By the stability property of sequences of gauge-measurable functions (Proposition 6.1), we deduce that $\chi_{R} \circ f$ is gauge-measurable, and the conclusion follows.

The converse of Proposition 8.2 is also true: if $f^{-1}(R)$ is gauge-measurable for every rectangle $R \subset \mathbb{R}^{p}$, then $f$ is gauge-measurable. This assertion can be deduced from the proof of Proposition 8.1 below, since under such an assumption the functions $f_{n}$ which are defined in (14) below are all gauge-measurable and the function $f$ is the pointwise limit of the sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$.

Proof of Proposition 8.1. Take a sequence of positive numbers $\left(\varepsilon_{n}\right)_{n \in \mathbb{N}}$ that converges to 0 . For each $n \in \mathbb{N}$, let $\left(R_{i, n}\right)_{i \in\left\{1, \ldots, k_{n}\right\}}$ be a finite family of disjoint rectangles whose diameters do not exceed $\varepsilon_{n}$ that covers the ball $B_{n+1}(0)$ in $\mathbb{R}^{p}$. For each $i \in\left\{1, \ldots, k_{n}\right\}$, let $a_{i}$ be a point with smallest norm in $\overline{R_{i, n}}$, and define

$$
\begin{equation*}
f_{n}:=\sum_{i=1}^{k_{n}} a_{i} \chi_{f^{-1}\left(R_{i, n}\right)} \tag{14}
\end{equation*}
$$

For every $x \in f^{-1}\left(B_{n+1}(0)\right)$, we then have

$$
\left|f_{n}(x)-f(x)\right| \leq \varepsilon_{n}
$$

hence the sequence $\left(f_{n}\right)_{n \in \mathbb{N}}$ converges pointwise to $f$ in $\mathbb{R}^{d}$. [The convergence is uniform when $f$ is a bounded function.] By the choice of the point $a_{i}$, we also have

$$
\left|f_{n}(x)\right| \leq|f(x)|
$$

if $f(x) \in \bigcup_{i=1}^{k_{n}} R_{i, n}$, while the left-hand side vanishes otherwise. This estimate thus holds for every $x \in \mathbb{R}^{d}$.

Assuming that the set $f^{-1}(R)$ is gauge-measurable for every rectangle $R \subset \mathbb{R}^{p}$, which by Proposition 8.2 is the case when the function $f$ is gaugemeasurable, it follows from the linear stability of gauge-measurable functions (Proposition 3.1) that $f_{n}$ is a gauge-measurable step function, and this gives the conclusion.

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# Hamilton-Jacobi on the symplectic group 

Ivar Ekeland<br>Dedicated to Jean Mawhin and his students


#### Abstract

The classical Hamilton-Jacobi-Bellman theory in the calculus of variations, which is associated with the Bolza problem, is extended to other kinds of boundary-value problems, such as periodicity. By using the dual action principle of Clarke and earlier results by the author, one can establish the analogue of HJB on the symplectic group and show that it has a solution


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## 1. Introduction

It is great pleasure to do mathematics, and I have enjoyed it for more than fifty years now. Part of that pleasure comes from meeting other mathematicians, and there are very few, if any, that I have enjoyed more than Jean Mawhin. His sense of humour of course is part of the enjoyment, but so are his mathematics, always deeply rooted in classical problems, and bringing to them modern methods and deceptively simple solutions. He belongs to a long and distinguished Belgian tradition of the calculus of variations, starting with de la Vallée-Poussin and de Donder [2], continued by Jean himself and his students, such as Michel Willem. I wish to point out that their teaching is no less remarkable than their research. The treatise of de la Vallée-Poussin, in its second edition [6], was the first one to introduce the Lebesgue integral, and Jean's treatise [4] , is no less revolutionary and a pleasure to read. I cannot resist the opportunity of commending Michel's expository talent as well, in $[7,8,9]$ : short books, which contain an amazing amount of well-digested material.

All these people have been a great inspiration to me, and I dedicate this work to them. I will try to fit into the same tradition by describing a research program which starts in the classical calculus of variations and ends in the symplectic group. I have long asked myself whether the Hamilton-Jacobi equation, nowadays known as Hamilton-Jacobi-Bellman because of the latter's important contribution, and shortened to HJB, can be extended to other set-
tings, for instance to periodic boundary conditions. It turns out that they can. I will describe how to proceed, and leave the detailed study of the equation to further studies, preferably by younger people.

## 2. The classical situation

Let me first summarize the classical theory (see [1] or [7]). Consider the classical Bolza problem in the one-dimensional calculus of variations:

$$
\begin{gather*}
\inf \int_{0}^{T} f\left(q, \frac{d q}{d t}\right) d t  \tag{1}\\
q(0)=q_{0}, \quad q(T)=q_{1}, \quad \frac{d q}{d t} \in L^{1}, \tag{2}
\end{gather*}
$$

where $T>0, q_{0}$ and $q_{1} \in R^{n}$ are prescribed. Suppose $f(q, \chi)$ is continuously differentiable, convex wrt $\chi$ and coercive, meaning that we have $f(q, \chi) \geq$ $\Phi(|\chi|)$ where $\Phi$ is bounded from below and $\Phi(t) t^{-1} \rightarrow \infty$ when $t \rightarrow \infty$. Then it can be shown that the minimizer $q$ exists, and that $\left|\frac{d q}{d t}\right| \in L^{\infty}$, so that it satisfies the Euler-Lagrange equation:

$$
\frac{d}{d t} \frac{\partial f}{\partial \chi}=\frac{\partial f}{\partial q}
$$

Since the function $f$ has been assumed to be convex wrt the second variable, it has a Legendre transform:

$$
\begin{equation*}
H(p, q)=\max \{p \chi-f(q, \chi)\} \tag{3}
\end{equation*}
$$

and it is well-known that the Euler-Lagrange equation (1), which is a secondorder equation in $R^{n}$, can be rewritten as a Hamiltonian system, which is a first-order equation in $R^{2 n}$ :

$$
\begin{align*}
\frac{d p}{d t} & =-\frac{\partial H}{\partial q}  \tag{4}\\
\frac{d q}{d t} & =\frac{\partial H}{\partial p} \tag{5}
\end{align*}
$$

If now we fix $q_{0}$ and introduce the so-called value function $V\left(q_{1}, T\right)$ associated with the optimization problem (1)-(2), namely:

$$
\begin{equation*}
V\left(q_{1}, T\right):=\inf \left\{\left.\int_{0}^{T} f\left(q, \frac{d q}{d t}\right) d t \right\rvert\, q(0)=q_{0}, \quad q(T)=q_{1}\right\} \tag{6}
\end{equation*}
$$

we find that it satisfies a first-order PDE on $R^{n} \times[0, T]$ :

$$
\begin{equation*}
\frac{\partial V}{\partial T}+H\left(q_{1}, \frac{\partial V}{\partial q_{1}}\right)=0 \tag{7}
\end{equation*}
$$

This is the HJB equation. We have approached it through the value function, but the same equation can also be obtained by trying to find a change of variable which would simplify the problem (i.e. generating functions), and this is how it appears in the work of Hamilton and Jacobi.

Note that $q_{0}$ and $q_{1}$ play symmetric roles, so that a similar equation exists for $q_{0}$.

## 3. Other boundary conditions

From now on we shall simplify notations by writing $x=(p, q)$ and:

$$
J=\left(\begin{array}{cc}
0 & -I_{n} \\
I_{n} & 0
\end{array}\right) .
$$

It follows that the system (4)-(5) can be rewritten compactly as $\frac{d x}{d t}=$ $J H^{\prime}(x)$.

Recall that a matrix $M$ is symplectic if $M^{*} J M=J$. The group of symplectic matrices in $R^{n} \times R^{n}$ will be denoted by $\mathrm{Sp}(n)$. It has dimension $n(2 n+1)$. It is a compact Lie group, and the tangent space at $I_{2 n}$ is given by:

$$
\begin{equation*}
T_{I_{2 n}} \operatorname{Sp}(n)=\left\{m \mid m^{*} J+J m=0\right\} \tag{8}
\end{equation*}
$$

In other words, $J m$ is symmetric.
The Bolza problem is a natural one when $f$ is coercive, for instance when $f(q, \chi)=\frac{1}{2} q^{2}+\frac{1}{2} \chi^{2}$. Note that in that case, by formula (3), we have $H(p, q)=$ $\frac{1}{2} p^{2}-\frac{1}{2} q^{2}$, which is neither convex nor coercive. For convex and coercive Hamiltonians, such as $H(p, q)=\frac{1}{2} p^{2}+\frac{1}{2} q^{2}$, the natural one is to look for periodic solutions, that is, to investigate the problem:

$$
\begin{aligned}
\frac{d x}{d t} & =J H^{\prime}(x), \\
x(T) & =x(0) .
\end{aligned}
$$

This problem can be imbedded in a family of problems indexed by $M \in \operatorname{Sp}(n)$

$$
\begin{align*}
\frac{d x}{d t} & =J H^{\prime}(x),  \tag{9}\\
x(T) & =M x(0) \tag{10}
\end{align*}
$$

Here $M$ will play the role devoted to $\left(q_{0}, q_{1}\right)$ in the Bolza problem: the value function will be $V(M, T)$ instead of $V\left(q_{0}, q_{1}, T\right)$. Note that we cannot define it by using the least action principle:

$$
V(M, T)=\inf \left\{\left.\int_{0}^{T}\left[\frac{1}{2}\left(J \frac{d x}{d t}, x\right)+H(x)\right] d t \right\rvert\, x(T)=M x(0)\right\}
$$

because the right-hand side takes the value $+\infty$ (and would take the value $-\infty$ if we tried to minimize). In fact, a solution to problem (9)-(10) is neither a minimizer nor a maximizer, but a critical point of the right-hand side. So we have to define the value in another way, and for this reason we make some additional assumptions on $H$

Definition 3.1. Suppose $H: R^{2 n} \rightarrow R$ is convex, with $H(0)=0$. It is called subquadratic near infinity if $H(x)|x|^{-2} \rightarrow 0$ when $|x| \rightarrow \infty$, and subquadratic near 0 if $H(x)|x|^{-2} \rightarrow \infty$ when $|x| \rightarrow 0$.

We use the results in [3] (see Chapter II.4, notably Proposition 6, Chapter III.3, notably Corollary 6; see also [5])

Theorem 3.2. Suppose $H$ is convex and subquadratic near 0 , and consider the problem:

$$
\begin{align*}
& \inf \int_{0}^{T}\left[\left(J \frac{d y}{d t}, y\right)+H^{*}\left(-J \frac{d y}{d t}\right)\right] d t  \tag{11}\\
& y(T)=M y(0) \tag{12}
\end{align*}
$$

This problem has a solution $y(t)$ for any $M \in \operatorname{Sp}(n)$, and there is a constant $y_{0} \in R^{2 n}$ such that $y(t)+y_{0}$ solves problem (9)-(10). If in addition $H$ is subquadratic near 0 , this solution is not constant.

To understand the theorem, note that $y(t)=0$ is always a solution. Note also that, for any $y(t)$, adding a constant $y_{0}$ changes the value of the integral by

$$
\int_{0}^{T}\left(J \frac{d y}{d t}, y_{0}\right) d t=\left(J(y(T)-y(0)), y_{0}\right)=-\left(y(0),\left(M^{*}-I_{2 n}\right) y_{0}\right)
$$

If $M$ has 1 as an eigenvalue, and if $y_{0}$ is an eigenvector, the right-hand side vanishes. So the solution $y(t)$ of (11)-(12) is defined modulo a 1-eigenvector $y_{0}$ of $M$, and the latter can be chosen so that $y(t)+y_{0}$ solves (9)-(10).

We are now in a position to define the function $V(M, T)$ in a proper way:

$$
\begin{equation*}
V(M, T)=\inf \left\{\left.\int_{0}^{T}\left[\frac{1}{2}\left(J \frac{d y}{d t}, y\right)+H^{*}\left(-J \frac{d y}{d t}\right)\right] d t \right\rvert\, y(T)=M y(0)\right\} \tag{13}
\end{equation*}
$$

The right-hand side is the dual action functional. Note that setting $y=0$ gives the value 0 to the integral, so that $V(M, T)<0$ for $T>0$. The function $V: \operatorname{Sp}(n) \times[0, T] \rightarrow R$ is well-defined and does not vanish. Let us show that it satisfies a PDE system of the first order.

For the sake of simplicity, assume that $M$ does not have the eigenvalue 1 . Set

$$
\begin{aligned}
u & =\frac{d x}{d t} \\
x(t) & =x(0)+\int_{0}^{t} u(s) d s
\end{aligned}
$$

The boundary condition $x(T)=M x(0)$ becomes:

$$
\begin{align*}
x(0)+\int_{0}^{T} u d t & =M x(0)  \tag{14}\\
x(0) & =\left(M-I_{2 n}\right)^{-1} \int_{0}^{T} u d t \tag{15}
\end{align*}
$$

Set $\Pi u(t):=\int_{0}^{t} u(s) d s$. We have:

$$
\begin{equation*}
x(t)=x(0)+\int_{0}^{t} u(s) d s=\left(M-I_{2 n}\right)^{-1} \Pi u(T)+\Pi u(t) . \tag{16}
\end{equation*}
$$

Writing this into the right-hand side of (13), we get:

$$
\begin{equation*}
V(M, T)=\inf _{u} \int_{0}^{T}\left[\frac{1}{2}\left(J u,\left(M-I_{2 n}\right)^{-1} \Pi u(T)+\Pi u(t)\right)+H^{*}(-J u)\right] d t . \tag{17}
\end{equation*}
$$

Let us now compute the partial derivatives wrt $M$ and to $T$.
If $m \in T_{I_{2 n}} \operatorname{Sp}(n)$, we have $m M \in T_{M} \operatorname{Sp}(n)$. Hence, for every $m$ satisfying (8), we have, by the envelope theorem:

$$
\begin{aligned}
\frac{\partial V}{\partial M}(M, T) m M & =-\int_{0}^{T} \frac{1}{2}\left(J u,\left(M-I_{2 n}\right)^{-1} m M\left(M-I_{2 n}\right)^{-1} \Pi u(T)\right) d t \\
& =-\int_{0}^{T} \frac{1}{2}\left(J u,\left(M-I_{2 n}\right)^{-1} m M x(0)\right) d t \\
& =-\frac{1}{2}\left(\int_{0}^{T} J u d t,\left(M-I_{2 n}\right)^{-1} m x(T)\right) \\
& =-\frac{1}{2}\left(J\left(M-I_{2 n}\right) x(0),\left(M-I_{2 n}\right)^{-1} m x(T)\right) \\
& =-\frac{1}{2}\left(\left(M^{*}-I_{2 n}\right)^{-1} J\left(M-I_{2 n}\right) x(0), m x(T)\right)
\end{aligned}
$$

where $u(t)$ is a minimizer in (17) and $x(t)$ is given by formula (16).

Lemma 3.3. If $M$ is symplectic, we have:

$$
\left(M^{*}-I_{2 n}\right)^{-1} J\left(M-I_{2 n}\right)=-J M
$$

Proof. Multiply both sides by $M^{*}-I_{2 n}$. We get:

$$
J M-J=-M^{*} J M+J M
$$

which is true since $M^{*} J M=J$.

Finally, we find:

$$
\begin{equation*}
\frac{\partial V}{\partial M}(M, T) m M=\frac{1}{2}(J M x(0), m x(T))=\frac{1}{2}(J x(T), m x(T)) . \tag{18}
\end{equation*}
$$

To find the partial derivative wrt $T$, we rewrite the right-hand side of (17) as follows:

$$
\int_{0}^{1}\left[\frac{1}{2} \frac{1}{T}\left(J \frac{d x}{d t}, x\right)+H^{*}\left(-J \frac{1}{T} \frac{d x}{d t}\right)\right] T d t
$$

The envelope theorem then yields:

$$
\frac{\partial V}{\partial T}(M, T)=\int_{0}^{1}\left[H^{*}\left(-J \frac{1}{T} \frac{d x}{d t}\right)+\left(\nabla H^{*}\left(-J \frac{1}{T} \frac{d x}{d t}\right), J \frac{1}{T^{2}} \frac{d x}{d t}\right) T\right] d t
$$

By the Fenchel identity, $H(x)=\left(\nabla H^{*}(y), y\right)-H^{*}(y)$ for $x=\nabla H^{*}(y)$, so the integrand is just:

$$
-H\left(\nabla H^{*}\left(-J \frac{1}{T} \frac{d x}{d t}\right)\right)
$$

Inverting the equation $\frac{d x}{d t}=J H^{\prime}(x)$, we have $x=\nabla H^{*}\left(-J \frac{d x}{d t}\right)$. Finally, we get

$$
\frac{\partial V}{\partial T}(M, T)=-\int_{0}^{T} H(x) \frac{d t}{T}
$$

Bearing in mind that $H(x(t))$ is constant along trajectories of the Hamiltonian system, we get:

$$
\begin{equation*}
\frac{\partial V}{\partial T}(M, T)=-H(x(0))=-H(x(T)) \tag{19}
\end{equation*}
$$

## 4. HJB

We have found the partial derivatives of $V(M, T)$ at any point $(M, T) \in$ $\operatorname{Sp}(n) \times R_{+}$such that $M$ does not have the eigenvalue 1 . If $u(t)$ is the corresponding minimizer in (8), and $x(t)$ is the corresponding solution of (9)-(10) given by formula (15), so that $\frac{d x}{d t}=u$, we have:

$$
\begin{align*}
\frac{\partial V}{\partial M}(M, T) m M & =\frac{1}{2}(J x(T), m x(T)), \quad \forall m \in T_{I_{2 n}} \operatorname{Sp}(n)  \tag{20}\\
\frac{\partial V}{\partial T}(M, T) & =-H(x(T)) \tag{21}
\end{align*}
$$

This is the HJB equation we are seeking. Indeed, we can invert the first equation to express $x(T)$ in terms of $\frac{\partial V}{\partial M}$, say $x(T)=\varphi\left(\frac{\partial V}{\partial M}\right)$ and write the result in the second, getting $\frac{\partial V}{\partial T}=-H \circ \varphi\left(\frac{\partial V}{\partial M}\right)$. Note that equation (20) is in reality a system of $2 n^{2}+n$ equation (one for each $m \in T_{I_{2 n}} \operatorname{Sp}(n)$ ) in $2 n$ variables, so that it is overdetermined. However, by the preceding analysis, we have shown that formula (17) gives a solution. Let us summarize:

Theorem 4.1. Suppose $H: R^{2 n} \rightarrow R$ is convex, $H(0)=0$, and subquadratic near 0 and infinity. Then the function $V: \operatorname{Sp}(n) \times[0, T] \rightarrow R$ defined by formula (17) is negative for $T>0$. If $M$ does not have 1 as an eigenvalue, and $V$ is differentiable at $(M, T)$, the HJB relations (20) and (21) hold.

Note that there are two terms in the HJB system: the first one, (20), does not depend on $H$, which appears only in the second, (21).

Let us give an example. Take $n=1$, so that $M$ is symplectic if and only if it preserves volume:

$$
\begin{aligned}
M & =\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \operatorname{Sp}(2) \Longleftrightarrow a d-b c=1 \\
m & =\left(\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right) \in T_{I_{2 n}} \mathrm{Sp}(2) \Longleftrightarrow \alpha+\delta=0
\end{aligned}
$$

Then, for $\xi=\left(\xi_{1}, \xi_{2}\right)$ :

$$
\frac{1}{2}(J \xi, m \xi)=\gamma \xi_{1}^{2}-\beta \xi_{2}^{2}
$$

Let us now take local coordinates in $\operatorname{Sp}(2)$. If $c \neq 0$, for instance, we can
take $a, b, c$ and set $d=(a b-1) c^{-1}$. Then $V(M, T)$ becomes $V(a, b, c, T)$ and:

$$
\begin{aligned}
\frac{\partial V}{\partial M} m M & =\frac{\partial V}{\partial a}(\alpha a+\beta c)+\frac{\partial V}{\partial b}(\alpha b+\beta d)+\frac{\partial V}{\partial c}(\gamma a+\delta c) \\
& =\alpha\left(a \frac{\partial V}{\partial a}+b \frac{\partial V}{\partial b}\right)+\beta\left(c \frac{\partial V}{\partial a}+d \frac{\partial V}{\partial b}\right)+\gamma\left(a \frac{\partial V}{\partial c}\right)+\delta\left(c \frac{\partial V}{\partial c}\right) \\
& =\alpha\left(a \frac{\partial V}{\partial a}+b \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial c}\right)+\beta\left(c \frac{\partial V}{\partial a}+\frac{a b-1}{c} \frac{\partial V}{\partial b}\right)+\gamma\left(a \frac{\partial V}{\partial c}\right)
\end{aligned}
$$

Equation (20) becomes:

$$
\begin{align*}
a \frac{\partial V}{\partial a}+b \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial c} & =0  \tag{22}\\
c \frac{\partial V}{\partial a}+\frac{a b-1}{c} \frac{\partial V}{\partial b} & =-x_{2}(T)^{2}  \tag{23}\\
a \frac{\partial V}{\partial c} & =x_{1}(T)^{2} \tag{24}
\end{align*}
$$

We can derive $x_{1}(T)$ and $x_{2}(T)$ from the last two equations, and plug into second HJB relation (21), getting:

$$
\frac{\partial V}{\partial T}=-H\left( \pm \sqrt{a \frac{\partial V}{\partial c}}, \pm \sqrt{\frac{1-a b}{c} \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial a}}\right)
$$

We still have to satisfy equation (22). Finally, we get an overdetermined system for $V(a, b, c, T)$ :

$$
\begin{aligned}
0 & =a \frac{\partial V}{\partial a}+b \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial c} \\
\frac{\partial V}{\partial T} & =-H\left( \pm \sqrt{a \frac{\partial V}{\partial c}}, \pm \sqrt{\frac{1-a b}{c} \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial a}}\right)
\end{aligned}
$$

The sign indeterminacy in the second equation arises also in the Bolza problem. For instance, it is found in the classical eikonal equation.

If one takes $H(x)=|x|^{\alpha}$ with $0<\alpha<2$, which is convex and subquadratic near 0 and infinity, the system becomes:

$$
\begin{aligned}
0 & =a \frac{\partial V}{\partial a}+b \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial c} \\
0 & =\frac{\partial V}{\partial T}+\left|a \frac{\partial V}{\partial c}+\frac{1-a b}{c} \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial a}\right|^{\frac{\alpha}{2}}, \\
a \frac{\partial V}{\partial c} & >0, \quad \frac{1-a b}{c} \frac{\partial V}{\partial b}-c \frac{\partial V}{\partial a}>0
\end{aligned}
$$

and the problem (9)-(10) can be solved explicitly, yielding a solution to this system.

## 5. Conclusion

This aim of this paper is to open up a problem. There are many questions to be answered:

1. We have investigated only points where $M$ does not have the eigenvalue 1 and $V(M, T)$ is differentiable. What happens at other points? Is it true that the value function $V(M, T)$ provides a viscosity solution of the system (20)-(21) over $\operatorname{Sp}(n) \times R_{+}$?
2. What is the geometry of the solution? What is the meaning of the indeterminacy which arises when solving (20), and which appears as $\pm$ in the example? Does it mean that the graph of the value function can be extended to a sheet which covers $\operatorname{Sp}(n) \times R_{+}$several times, in the manner of a Riemann surface?
3. What happens when the Hamiltonian $H$ is no longer subquadratic? If it is superquadratic, for instance, the dual action principle still holds, and can be used to prove the existence of a solution to the problem (9)(10), but the minimum on the right-hand side of (13) is not attained. The solution is a saddle-point, and defines a critical value rather than a minimum. However, the system (20)-(21) is still valid. Does it have a solution, and is it provided by the analogue of formula (17), where one seeks a critical value of the right-hand side?
4. Finally, what about general Hamiltonians $H$, assuming simply $H(0)=0$ and $H(x) \rightarrow \infty$ when $|x| \rightarrow \infty$, so that energy surfaces $H(x)=h$ are bounded, and the boundary-value problem (9)-(10) is reasonable?

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# Positive decaying solutions to BVPs with mean curvature operator 

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#### Abstract

A boundary value problem on the whole half-closed interval $[1, \infty)$, associated to differential equations with the Euclidean mean curvature operator or with the Minkowski mean curvature operator is here considered. By using a new approach, based on a linearization device and some properties of principal solutions of certain disconjugate second-order linear equations, the existence of global positive decaying solutions is examined.

Keywords: Second order nonlinear differential equation; Euclidean mean curvature operator, Minkowski mean curvature operator, Radial solution, Principal solution, Disconjugacy. MS Classification 2010: 34B18, 34B40, 34C10.


## 1. Introduction

In this paper we deal with the following boundary value problems (BVPs) on the half-line for equations with the Euclidean mean curvature operator

$$
\left\{\begin{array}{l}
\left(a(t) \frac{x^{\prime}}{\sqrt{1+x^{\prime 2}}}\right)^{\prime}+b(t) F(x)=0, \quad t \in[1, \infty)  \tag{1}\\
x(1)=1, x(t)>0, x^{\prime}(t) \leq 0 \text { for } t \geq 1, \lim _{t \rightarrow \infty} x(t)=0
\end{array}\right.
$$

and with the Minkowski mean curvature operator

$$
\left\{\begin{array}{l}
\left(a(t) \frac{x^{\prime}}{\sqrt{1-x^{\prime 2}}}\right)^{\prime}+b(t) F(x)=0, \quad t \in[1, \infty)  \tag{2}\\
x(1)=1, x(t)>0, x^{\prime}(t) \leq 0 \text { for } t \geq 1, \lim _{t \rightarrow \infty} x(t)=0
\end{array}\right.
$$

Troughout the paper the following conditions are assumed:
$\left(\mathrm{H}_{1}\right)$ The function $a$ is continuous on $[1, \infty), a(t)>0$ in $[1, \infty)$, and

$$
\begin{equation*}
\int_{1}^{\infty} \frac{1}{a(t)} d t<\infty \tag{3}
\end{equation*}
$$

$\left(\mathrm{H}_{2}\right)$ The function $b$ is continuous on $[1, \infty), b(t) \geq 0$ and

$$
\begin{equation*}
\int_{1}^{\infty} b(t) \int_{t}^{\infty} \frac{1}{a(s)} d s d t<\infty \tag{4}
\end{equation*}
$$

$\left(\mathrm{H}_{3}\right)$ The function $F$ is continuous on $\mathbb{R}, F(u) u>0$ for $u \neq 0$, and

$$
\limsup _{u \rightarrow 0^{+}} \frac{F(u)}{u}<\infty
$$

Define

$$
\Phi_{E}(v)=\frac{v}{\sqrt{1+v^{2}}}, \quad \Phi_{M}(v)=\frac{v}{\sqrt{1-v^{2}}}
$$

The operator $\Phi_{E}$ arises in the search for radial solutions to partial differential equations which model fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids. The operator $\Phi_{M}$ originates from studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory. Therefore, it is called also the relativity operator.

For instance, the study of radial solutions for the problem

$$
\begin{gathered}
\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 \pm|\nabla u|^{2}}}\right)+f(|x|, u)=0, \quad x \in \Omega \subset \mathbb{R}^{N} \\
u(x)>0 \text { in } \Omega, \quad \lim _{|x| \rightarrow \infty} u(|x|)=0
\end{gathered}
$$

where $\Omega$ is the exterior of a ball of radius $R>0$, leads to the boundary value problem on the half-line

$$
\begin{aligned}
\left(r^{N-1} \frac{v^{\prime}}{\sqrt{1 \pm v^{\prime 2}}}\right)^{\prime}+r^{N-1} f(r, v) & =0, \quad r \in[R, \infty) \\
v(r)>0, \lim _{r \rightarrow \infty} v(r) & =0
\end{aligned}
$$

where $r=|x|$ and $v(r)=u(|x|)$. If $N>2, f(r, v)=\hat{b}(r) F(v)$, with $\hat{b}(r) \geq 0$ in $[R, \infty)$ and $\int_{R}^{\infty} r \hat{b}(r) d r<\infty$, then assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ are satisfied. In particular, if $\hat{b}(r) \approx r^{\delta}$, then $\left(\mathrm{H}_{2}\right)$ reads as $\delta<-2$.

Boundary value problems associated to equations with the curvature operator in compact intervals are widely considered in the literature. We refer, in particular, to $[3,4,5,6,7,8,11,25,27]$, and references therein. In unbounded domains, these equations have been considered in [14, 15], in which some asymptotic BVPs are studied, and in $[1,2,13,19]$, in which the search of ground state solutions, that is solutions which are globally positive on the whole half-line and tend to zero as $t \rightarrow \infty$, is examined.

Finally, equations with sign-changing coefficients are recently considered when the differential operator is the $p$-Laplacian, see, e.g. $[9,10,22,23]$ and references therein.

Here, our main aim is to study the solvability of the BVPs (1) and (2). As claimed, these BVPs originate from the search of ground state solutions for PDE with Euclidean or Minkowski mean curvature operator. Our approach is based on a fixed point theorem for operators defined in a Fréchet space by a Schauder's linearization device, see [16, Theorem 1.1]. This tool does not require the explicit form of the fixed point operator $\mathcal{T}$. Moreover, it simplifies the check of the topological properties of $\mathcal{T}$ in the noncompact interval $[1, \infty)$, since these properties become an immediate consequence of $a$-priori bounds for an associated linear equation. These bounds are obtained in an implicit form by means of the concepts of disconjugacy and principal solutions for second order linear equations. The main properties on this topic, needed in our arguments, are presented in Section 2. In Section 3 the solvability of (1) and (2) is given, by assuming some implicit conditions on functions $a$ and $b$. Explicit conditions for the solvability of these BVPs, are derived in Section 4. Observe that also the BVP for equations with the Sturm-Liouville operator

$$
\left\{\begin{array}{l}
\left(a(t) x^{\prime}\right)^{\prime}+b(t) F(x)=0, \quad t \in[1, \infty)  \tag{5}\\
x(1)=1, x(t)>0, \text { for } t \geq 1, \lim _{t \rightarrow \infty} x(t)=0
\end{array}\right.
$$

can been treated by a similar method. Some examples and a discussion on these topics complete the paper.

## 2. Auxiliary results

To obtain a-priori bounds for solutions of BVPs (1) and (2), we employ a linearization method. Therefore, in this section we consider linear equations, we point out some properties of principal solutions, and we state new comparison results.

Consider the linear equation

$$
\begin{equation*}
\left(r(t) y^{\prime}\right)^{\prime}+q(t) y=0, \quad t \in[1, \infty) \tag{6}
\end{equation*}
$$

where $r, q$ are continuous functions, $r(t)>0, q(t) \geq 0$ for $t \geq 1$.
The equation (6) is called nonoscillatory if all its solutions are nonoscillatory. In view of the Sturm theorem, see, e.g., [24, Chap. XI, Section 3], the existence of a nonoscillatory solution implies the nonoscillation of (6). When (6) is nonoscillatory, a powerful tool for studying the qualitative behavior of its solutions is based on the analysis of the corresponding Riccati equation

$$
\begin{equation*}
\xi^{\prime}+q(t)+\frac{\xi^{2}}{r(t)}=0 \tag{7}
\end{equation*}
$$

see, e.g., $[18,24]$. More precisely, if $y$ is a non-vanishing solution of (6), then

$$
\xi(t)=\frac{r(t) y^{\prime}(t)}{y(t)}
$$

is a solution of (7). Conversely, if $\xi$ is a solution of (7), then any nontrivial solution $y$ of the first order linear equation

$$
y^{\prime}=\frac{\xi(t)}{r(t)} y
$$

is also a non-vanishing solution of (6). If (6) is nonoscillatory, then the corresponding Riccati equation (7) has a solution $\xi_{0}$, defined for large $t$, such that for any other solution $\xi$ of (7), defined in a neighborhood $I_{\xi}$ of infinity, we have $\xi_{0}(t)<\xi(t)$ for $t \in I_{\xi}$. The solution $\xi_{0}$ is called the minimal solution of (7) and any solution $y_{0}$ of

$$
\begin{equation*}
y^{\prime}=\frac{\xi_{0}(t)}{r(t)} y \tag{8}
\end{equation*}
$$

is called principal solution of (6). Clearly, $y_{0}$ is uniquely determined up to a constant factor and so by the principal solution of (6) we mean any solution of (8) which is eventually positive. The principal solution is, roughly speaking, the smallest solution of (6) near infinity. Indeed it holds

$$
\lim _{t \rightarrow \infty} \frac{y_{0}(t)}{y(t)}=0
$$

where $y$ denotes any linearly independent solution of (6).
We recall that (6) is said to be disconjugate on an interval $I \subset[1, \infty)$, if any nontrivial solution of (6) has at most one zero on $I$. Equation (6) is disconjugate on $[1, \infty)$, if and only if it is disconjugate on $(1, \infty)$, see, e.g., [18, Theorem 2, Chap.1]. The relation between the notions of disconjugacy and principal solution is given by the following, see, e.g., [18, Chap. 1] or [24, Chap. XI, Section 6].

Lemma 2.1. The following statements are equivalent.
( $i_{1}$ ) Equation (6) is disconjugate on $[1, \infty)$.
( $i_{2}$ ) The principal solution $y_{0}$ of (6) does not have zeros on $(1, \infty)$.
( $i_{3}$ ) The Riccati equation (7) has a solution defined throughout $(1, \infty)$.
The following characterization of principal solution of (6) holds, see [24, Chap. XI, Theorem 6.4].
Lemma 2.2. Let (6) be nonoscillatory. Then a nontrivial solution $y_{0}$ of (6) is the principal solution if and only if we have for large $T$

$$
\int_{T}^{\infty} \frac{1}{r(s) y_{0}^{2}(s)} d s=\infty
$$

Some asymptotic properties for solutions of (6) are summarized in the next lemma.

Lemma 2.3. Assume

$$
\int_{1}^{\infty} \frac{1}{r(t)} d t<\infty, \quad \int_{1}^{\infty} q(t) R(t) d t<\infty
$$

where

$$
R(t)=\int_{t}^{\infty} \frac{1}{r(s)} d s
$$

Then (6) is nonoscillatory, and the set of eventually nonincreasing positive solutions, with zero limit at infinity, is nonempty. Further, any such solution y satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{y(t)}{R(t)}=c_{y} \tag{9}
\end{equation*}
$$

where $0<c_{y}<\infty$ is a suitable constant.
Proof. From [17, Theorem 1], see also [17, Lemma 2], we have the existence of eventually nonincreasing positive solutions, with zero limit at infinity. The asymptotic estimate (9) follows from [17, Theorem 2] and the l'Hopital rule.

Under the assumptions of Lemma 2.3, the principal solution $y_{0}$ of (6) is nonincreasing for large $t$. However, $y_{0}^{\prime}$ can change sign on $[1, \infty)$, even if (6) is disconjugate on $[1, \infty)$, see, e.g., [20, Example 1]. Now, the question under what assumptions the principal solution is monotone on the whole interval $[1, \infty)$ arises. In the following we give conditions ensuring that $y_{0}(t) y_{0}^{\prime}(t) \leq 0$ on the whole interval $[1, \infty)$. To this end the following comparison criterion between two Riccati equations plays a crucial role, see [24, Chap. XI, Corollary 6.5].

Consider the linear equations

$$
\begin{equation*}
\left(r_{2}(t) y^{\prime}\right)^{\prime}+q_{2}(t) y=0, \quad t \geq 1 \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(r_{1}(t) w^{\prime}\right)^{\prime}+q_{1}(t) w=0, \quad t \geq 1 \tag{11}
\end{equation*}
$$

where $r_{i}, q_{i}$ are continuous functions on $[1, \infty), r_{i}(t)>0, q_{i}(t) \geq 0$ for $t \geq 1, i=$ $1,2$.
Lemma 2.4. Let (10) be a Sturm majorant of (11), that is, for $t \geq 1$

$$
\begin{equation*}
r_{1}(t) \geq r_{2}(t), \quad q_{1}(t) \leq q_{2}(t) . \tag{12}
\end{equation*}
$$

Let (10) be disconjugate on $[T, \infty), T \geq 1$, and assume that a solution $y$ of (10) exists, without zeros on $[T, \infty)$. Then (11) is disconjugate on $[T, \infty)$ and its principal solution $w_{0}$ satisfies for $t \geq T$

$$
\frac{r_{1}(t) w_{0}^{\prime}(t)}{w_{0}(t)} \leq \frac{r_{2}(t) y^{\prime}(t)}{y(t)}
$$

Using Lemma 2.4, we get the following comparison result, which will play a crucial role in the sequel.

Lemma 2.5. Let (10) be a majorant of (11), that is (12) holds for $t \geq 1$ and at least one of the inequalities in (12) is strict on a subinterval of $[1, \infty)$ of positive measure. If the principal solution of (10) is positive nonincreasing on $[1, \infty)$, then (11) has the principal solution which is positive nonincreasing on $[1, \infty)$.

Proof. The assertion is an easy consequence of a well-known result on conjugate points for linear equations, see, e.g., [21, Theorem 4.2.3]. Since (10) is disconjugate on $[1, \infty)$, by Lemma 2.4 also (11) is disconjugate on the same interval. By Lemma 2.1 the principal solution $w_{0}$ of (11) is positive for $t>1$. If $w_{0}(1)=0$, using [21, Theorem 4.2.3], every solution of (10) should have a zero point on $(1, \infty)$, which contradicts the fact that the principal solution of $(10)$ is positive on $(1, \infty)$. Thus $w_{0}(t)>0$ on $[1, \infty)$. Using Lemma 2.4 we get $w_{0}^{\prime}(t) \leq 0$ for $t \geq 1$, and the assertion follows.

## 3. The existence results

Define

$$
\begin{equation*}
\bar{F}=\sup _{u \in(0,1]} \frac{F(u)}{u} . \tag{13}
\end{equation*}
$$

We start by considering the BVP associated to the equation with the Euclidean mean curvature operator. The following holds.

Theorem 3.1. Let $\left(H_{i}\right), i=1,2,3$, be verified. Assume

$$
\begin{equation*}
\alpha=\inf _{t \geq 1} a(t) A(t)>1, \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
A(t)=\int_{t}^{\infty} \frac{1}{a(s)} d s \tag{15}
\end{equation*}
$$

If the principal solution $z_{0}$ of the linear equation

$$
\begin{equation*}
\left(a(t) z^{\prime}\right)^{\prime}+\frac{\alpha}{\sqrt{\alpha^{2}-1}} \bar{F} b(t) z=0, \quad t \geq 1 \tag{16}
\end{equation*}
$$

is positive and nonincreasing on $[1, \infty)$, then the $B V P$ (1) has at least one solution.

To prove this result, we use a general fixed point theorem for operators defined in the Fréchet space $C\left([1, \infty), \mathbb{R}^{2}\right)$, based on $[16$, Theorem 1.1]. We state the result in the form that will be used.

Theorem 3.2. Let $S$ be a nonempty subset of the Fréchet space $C\left([1, \infty), \mathbb{R}^{2}\right)$. Assume that there exists a nonempty, closed, convex and bounded subset $\Omega \subset$ $C\left([1, \infty), \mathbb{R}^{2}\right)$ such that, for any $(u, v) \in \Omega$, the linear equation

$$
\begin{equation*}
\left(\sqrt{a^{2}(t)-v^{2}(t)} x^{\prime}\right)^{\prime}+b(t) \frac{F(u(t))}{u(t)} x=0 \tag{17}
\end{equation*}
$$

admits a unique solution $x_{u v}$, such that $\left(x_{u v}, x_{u v}^{[1]}\right) \in S$, where

$$
x_{u v}^{[1]}=\sqrt{a^{2}(t)-v^{2}(t)} x_{u v}^{\prime}
$$

is the quasiderivative of $x_{u v}$.
Let $\mathcal{T}$ be the operator $\Omega \rightarrow S$, given by

$$
\mathcal{T}(u, v)=\left(x_{u v}, x_{u v}^{[1]}\right) .
$$

Assume:
(i $\left.i_{1}\right) \mathcal{T}(\Omega) \subset \Omega ;$
(i $i_{2}$ ) if $\left\{\left(u_{n}, v_{n}\right)\right\} \subset \Omega$ is a sequence converging in $\Omega$ and $\mathcal{T}\left(\left(u_{n}, v_{n}\right)\right) \rightarrow$ $\left(x_{1}, x_{2}\right)$, then $\left(x_{1}, x_{2}\right) \in S$.

Then the operator $\mathcal{T}$ has a fixed point $(\bar{x}, \bar{y}) \in \Omega \cap S$ and $\bar{x}$ is a solution of

$$
\begin{equation*}
\left(a(t) \frac{x^{\prime}}{\sqrt{1+x^{\prime 2}}}\right)^{\prime}+b(t) F(x)=0 \tag{18}
\end{equation*}
$$

If the equation (17) is replaced by

$$
\begin{equation*}
\left(\sqrt{a^{2}(t)+v^{2}(t)} x^{\prime}\right)^{\prime}+b(t) \frac{F(u(t))}{u(t)} x=0 \tag{19}
\end{equation*}
$$

and ( $i_{1}$ ), ( $i_{2}$ ) are verified, then $\mathcal{T}$ has a fixed point $(\widetilde{x}, \widetilde{y}) \in \Omega \cap S$ and $\widetilde{x}$ is a solution of

$$
\left(a(t) \frac{x^{\prime}}{\sqrt{1-x^{\prime 2}}}\right)^{\prime}+b(t) F(x)=0
$$

Proof. Equation (17) can be written as the linear system

$$
\begin{equation*}
x_{1}^{\prime}=\frac{1}{\sqrt{a^{2}(t)-v^{2}(t)}} x_{2}, \quad x_{2}^{\prime}=-b(t) \frac{F(u(t))}{u(t)} x_{1}, \tag{20}
\end{equation*}
$$

where $x_{1}=x$ and $x_{2}=x^{[1]}$. Hence, from [16, Theorem 1.1], the set $\mathcal{T}(\Omega)$ is relatively compact and $\mathcal{T}$ is continuous on $\Omega$. The Schauder-Tychonoff fixed point theorem can now be applied to the operator $\mathcal{T}: \Omega \rightarrow \mathcal{T}(\Omega)$, since $\Omega$
is bounded, closed, convex, $\mathcal{T}(\Omega)$ is relatively compact and $\mathcal{T}$ is continuous on $\Omega$. Thus, $\mathcal{T}$ has a fixed point in $\Omega$, say $(\bar{x}, \bar{y})$, and $(\bar{x}, \bar{y})=\mathcal{T}(\bar{x}, \bar{y})$. Since $T(\Omega) \subset S$ and $T(\Omega) \subset \Omega$, we get $(\bar{x}, \bar{y}) \in \Omega \cap S$. From (20) we have

$$
\bar{x}^{\prime}(t)=\frac{\bar{y}(t)}{\sqrt{a^{2}(t)-\bar{y}^{2}(t)}}, \quad \bar{y}^{\prime}(t)=-b(t) F(\bar{x}(t))
$$

Since

$$
\bar{x}^{\prime}(t)=\frac{\bar{y}(t)}{\sqrt{a^{2}(t)-\bar{y}^{2}(t)}}=\Phi_{M}\left(\frac{\bar{y}(t)}{a(t)}\right)
$$

or

$$
\Phi_{E}\left(\bar{x}^{\prime}(t)\right)=\Phi_{E}\left(\Phi_{M}\left(\frac{\bar{y}(t)}{a(t)}\right)\right)
$$

using the fact that $\Phi_{E}\left(\Phi_{M}(d)\right)=d$, we obtain

$$
a(t) \frac{\bar{x}^{\prime}(t)}{\sqrt{1+\left(\bar{x}^{\prime}(t)\right)^{2}}}=\bar{y}(t), \quad \bar{y}^{\prime}(t)=-b(t) F(\bar{x}(t)) .
$$

Then $\bar{x}$ is a solution of (18). A similar argument holds when the operator $\mathcal{T}$ is defined via the linear equation (19).

Proof of Theorem 3.1. In view of assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$, Lemma 2.3 is applicable and (16) is nonoscillatory. Since the principal solution $z_{0}$ of (16) is positive nonincreasing on $[1, \infty)$, we can suppose also $z_{0}(1)=1$. Using Lemma 2.3 we have $\lim _{t \rightarrow \infty} z_{0}(t)=0$. From Lemma 2.1, equation (16) is disconjugate on $[1, \infty)$. Moreover, (16) is equivalent to

$$
\begin{equation*}
\left(\frac{\sqrt{\alpha^{2}-1}}{\alpha} a(t) z^{\prime}\right)^{\prime}+\bar{F} b(t) z=0, \quad t \geq 1 \tag{21}
\end{equation*}
$$

which is a Sturm majorant of

$$
\begin{equation*}
\left(a(t) w^{\prime}\right)^{\prime}=0, \quad t \geq 1 \tag{22}
\end{equation*}
$$

whose principal solution is

$$
\begin{equation*}
w_{0}(t)=\frac{1}{A(1)} A(t), \tag{23}
\end{equation*}
$$

where $A$ is given in (15). Clearly, $w_{0}$ satisfies the boundary conditions:

$$
w_{0}(1)=1, w_{0}(t)>0, w_{0}^{\prime}(t)<0 \text { on }[1, \infty), \lim _{t \rightarrow \infty} w_{0}(t)=0
$$

Put

$$
\begin{equation*}
\beta=\alpha \Phi_{M}(1 / \alpha)=\frac{\alpha}{\sqrt{\alpha^{2}-1}} . \tag{24}
\end{equation*}
$$

By applying Lemma 2.4, we get for $t \in[1, \infty)$

$$
\frac{w_{0}^{\prime}(t)}{w_{0}(t)} \leq \frac{1}{\beta} \frac{z_{0}^{\prime}(t)}{z_{0}(t)} \leq 0
$$

or, taking into account that $0<w_{0}(t) \leq 1$,

$$
w_{0}(t)^{\beta} \leq w_{0}(t) \leq z_{0}(t)^{1 / \beta}
$$

In the Fréchet space $C\left([1, \infty), \mathbb{R}^{2}\right)$, consider the subsets given by

$$
\Omega=\left\{(u, v) \in C\left([1, \infty), \mathbb{R}^{2}\right):\left(w_{0}(t)\right)^{\beta} \leq u(t) \leq\left(z_{0}(t)\right)^{1 / \beta},|v(t)| \leq \frac{1}{\alpha} a(t)\right\}
$$

and

$$
\begin{equation*}
S=\left\{(x, y) \in C\left([1, \infty), \mathbb{R}^{2}\right): x(1)=1, x(t)>0, \int_{1}^{\infty} \frac{1}{a(t) x^{2}(t)} d t=\infty\right\} \tag{25}
\end{equation*}
$$

Since $w_{0}(1)=z_{0}(1)=1$ and $z_{0}(t) \leq 1$, for any $(u, v) \in \Omega$ we get $u(1)=$ $1, u(t) \leq 1$.

For any $(u, v) \in \Omega$, consider the linear equation

$$
\begin{equation*}
\left(\sqrt{a^{2}(t)-v^{2}(t)} x^{\prime}\right)^{\prime}+b(t) \frac{F(u(t))}{u(t)} x=0 \tag{26}
\end{equation*}
$$

Since

$$
\begin{equation*}
a(t) \geq \sqrt{a^{2}(t)-v^{2}(t)} \geq \frac{\sqrt{\alpha^{2}-1}}{\alpha} a(t) \tag{27}
\end{equation*}
$$

equation (21) is a majorant of (26), and, by Lemma 2.4, (26) is disconjugate on $[1, \infty)$. Let $x_{u v}$ be the principal solution of $(26)$, such that $x_{u v}(1)=1$. In virtue of Lemma 2.5, $x_{u v}$ is positive nonincreasing on $[1, \infty)$. Put

$$
\begin{equation*}
x_{u v}^{[1]}=\sqrt{a^{2}(t)-v^{2}(t)} x_{u v}^{\prime}, \tag{28}
\end{equation*}
$$

and let $\mathcal{T}$ be the operator which associates to any $(u, v) \in \Omega$ the vector $\left(x_{u v}, x_{u v}^{[1]}\right)$, that is

$$
\mathcal{T}(u, v)(t)=\left(x_{u v}(t), x_{u v}^{[1]}(t)\right) .
$$

In view of Lemma 2.2 and (27), we have $\mathcal{T}(u, v) \in S$.
Equations (21) and (22) are a majorant and a minorant of (26), respectively. Applying Lemma 2.4 to (21) and (26), from (27), we obtain

$$
a(t) \frac{x_{u v}^{\prime}(t)}{x_{u v}(t)} \leq \sqrt{a^{2}(t)-v^{2}(t)} \frac{x_{u v}^{\prime}(t)}{x_{u v}(t)} \leq \frac{\sqrt{\alpha^{2}-1}}{\alpha} a(t) \frac{z_{0}^{\prime}(t)}{z_{0}(t)} \leq 0 .
$$

Thus

$$
x_{u v}(t) \leq\left(z_{0}(t)\right)^{1 / \beta} .
$$

Similarly, applying Lemma 2.4 to equations (22) and (26), we obtain

$$
\begin{equation*}
a(t) \frac{w_{0}^{\prime}(t)}{w_{0}(t)} \leq \sqrt{a^{2}(t)-v^{2}(t)} \frac{x_{u v}^{\prime}(t)}{x_{u v}(t)} \leq \frac{\sqrt{\alpha^{2}-1}}{\alpha} a(t) \frac{x_{u v}^{\prime}(t)}{x_{u v}(t)} . \tag{29}
\end{equation*}
$$

Hence

$$
\left(w_{0}(t)\right)^{\beta} \leq x_{u v}(t)
$$

where $\beta$ is given in (24).
To prove that $\mathcal{T}$ maps $\Omega$ into itself, we have to show that

$$
\begin{equation*}
\left|x_{u v}^{[1]}(t)\right| \leq \frac{1}{\alpha} a(t) . \tag{30}
\end{equation*}
$$

From (28) and (29) we obtain

$$
\begin{equation*}
\frac{\left|x_{u v}^{[1]}(t)\right|}{x_{u v}(t)}=\sqrt{a^{2}(t)-v^{2}(t)} \frac{\left|x_{u v}^{\prime}(t)\right|}{x_{u v}(t)} \leq a(t) \frac{\left|w_{0}^{\prime}(t)\right|}{w_{0}(t)} . \tag{31}
\end{equation*}
$$

In view of (23) we get

$$
\frac{\left|w_{0}^{\prime}(t)\right|}{w_{0}(t)}=\frac{1}{a(t) A(t)} .
$$

Thus, from (31), since $0<x_{u v}(t) \leq 1$, we have

$$
\left|x_{u v}^{[1]}(t)\right| \leq \frac{1}{A(t)} x_{u v}(t) \leq \frac{1}{A(t)}
$$

and, in virtue of (14), the inequality (30) follows.
In order to apply Theorem 3.2 , let us show that, if $\left\{\left(u_{n}, v_{n}\right)\right\}$ converges in $\Omega$ and $\left\{\mathcal{T}\left(u_{n}, v_{n}\right)\right\}$ converges to $(\bar{x}, \bar{y}) \in \Omega$, then $(\bar{x}, \bar{y}) \in S$. Clearly, $\bar{x}$ is positive for $t \geq 1$ and $\bar{x}(1)=1$. Thus, it remains to prove that

$$
\begin{equation*}
\int_{1}^{\infty} \frac{1}{a(t) \bar{x}^{2}(t)} d t=\infty \tag{32}
\end{equation*}
$$

Since $\overline{\mathcal{T}(\Omega)} \subset \bar{\Omega}=\Omega$, we have $0<\bar{x}(t) \leq\left(z_{0}(t)\right)^{1 / \beta}$, and $\lim _{t \rightarrow \infty} \bar{x}(t)=0$. Further, since $\left\{\mathcal{T}\left(u_{n}, v_{n}\right)\right\}$ converges to $(\bar{x}, \bar{y})$ uniformly in every compact of $[1, \infty)$, the function $\bar{x}$ is a solution of (26) for some $u=\bar{u}, v=\bar{v}$ such that $(\bar{u}, \bar{v}) \in \Omega$. Applying (27) and Lemma 2.3, there exist $T \geq 1$ and a constant $k>0$ such that $\bar{x}(t) \leq k A(t)$ on $[T, \infty)$, where $A$ is given in (15). Thus

$$
\int_{T}^{t} \frac{1}{a(s) \bar{x}^{2}(s)} d s \geq \frac{1}{k^{2}}\left(\frac{1}{A(T)}-\frac{1}{A(t)}\right)
$$

and (32) is satisfied. Applying Theorem 3.2, the operator $\mathcal{T}$ has a fixed point $(\bar{x}, \bar{y}) \in \Omega \cap S$ and $\bar{x}$ is a solution of (1).

Now, we consider the case of the Minkowski curvature operator. The following holds.

Theorem 3.3. Assume that $\left(H_{i}\right), i=1,2,3$, are verified and let (14) be satisfied. If the linear equation

$$
\begin{equation*}
\left(a(t) z^{\prime}\right)^{\prime}+\bar{F} b(t) z=0, \quad t \geq 1 \tag{33}
\end{equation*}
$$

has the principal solution $z_{0}$ positive nonincreasing on $[1, \infty)$, then the $B V P$ (2) has at least one solution.

Proof. The proof is similar to the one given in Theorem 3.1. Jointly with (33), consider the equation (22). Reasoning as in the proof of Theorem 3.1, we obtain $w_{0}(t) \leq z_{0}(t)$, where $w_{0}$ and $z_{0}$ are the principal solutions of (22) and (33), respectively, such that $w_{0}(1)=z_{0}(1)=1$. Since $z_{0}$ is positive nonincreasing on $[1, \infty)$, we obtain

$$
\left(w_{0}(t)\right)^{\beta} \leq\left(z_{0}(t)\right)^{1 / \beta}
$$

where $\beta=\alpha / \sqrt{\alpha^{2}-1}>1$. Let $\Omega_{1} \subset C\left([1, \infty), \mathbb{R}^{2}\right)$ be the set
$\Omega_{1}=\left\{(u, v) \in C\left([1, \infty), \mathbb{R}^{2}\right):\left(w_{0}(t)\right)^{\beta} \leq u(t) \leq\left(z_{0}(t)\right)^{1 / \beta},|v(t)| \leq \frac{\beta}{\alpha} a(t)\right\}$,
and for any $(u, v) \in \Omega_{1}$, consider the linear equation

$$
\begin{equation*}
\left(\sqrt{a^{2}(t)+v^{2}(t)} x^{\prime}\right)^{\prime}+b(t) \frac{F(u(t))}{u(t)} x=0 . \tag{34}
\end{equation*}
$$

Let $x_{u v}$ be the principal solution of (34) such that $x_{u v}(1)=1$. Then $\left(x_{u v}, x_{u v}^{[1]}\right) \in$ $S$, where $S$ is given in (25). Since equation (22) is equivalent to

$$
\left(\beta a(t) w^{\prime}\right)^{\prime}=0
$$

which is a minorant of (34), the assertion follows by using a similar argument to the one in the proof of Theorem 3.1, with minor changes. The details are left to the reader.

## 4. Applications and examples

Theorem 3.1 requires that the principal solution of (16) is positive nonincreasing on the whole half-line $[1, \infty)$. Lemma 2.5 can be used to assure this property if a majorant of (16) exists, whose principal solution is known. A similar argument holds for the conditions which are required in Theorem 3.3 for (33). In the following, some applications in this direction are presented.

Prototypes of a Sturmian majorant equation, for which the principal solution is positive nonincreasing on the whole interval $[1, \infty)$, can be obtained from the Riemann-Weber equation

$$
\begin{equation*}
v^{\prime \prime}+\frac{1}{4(t+1)^{2}}\left(1+\frac{1}{\log ^{2}(t+1)}\right) v=0 \tag{35}
\end{equation*}
$$

or from the Euler equation

$$
\begin{equation*}
v^{\prime \prime}+\frac{1}{4 t^{2}} v=0 . \tag{36}
\end{equation*}
$$

Indeed, equation (35) is disconjugate on $(0, \infty)$, see [18, page 20]. Thus, from Lemma 2.1, the principal solution $v_{0}$ of (35) is positive on $[1, \infty)$. Since $v_{0}$ is concave for any $t \geq 1$, then $v_{0}^{\prime}(t)>0$ on $[1, \infty)$. Set

$$
y_{0}(t)=v_{0}^{\prime}(t),
$$

a standard calculation shows that $y_{0}$ is solution of the linear equation

$$
\begin{equation*}
\left(\frac{4(t+1)^{2} \log ^{2}(t+1)}{1+\log ^{2}(t+1)} y^{\prime}\right)^{\prime}+y=0 \tag{37}
\end{equation*}
$$

Moreover, in view of [12, Theorem 1], $y_{0}$ is the principal solution of (37) and $y_{0}^{\prime}(t)=v_{0}^{\prime \prime}(t)<0$.

A similar argument holds for (36). Equation (36) is nonoscillatory and the principal solution is

$$
v_{0}(t)=\sqrt{t}
$$

see, e.g., [26, Chap. 2.1]. Hence, the function

$$
y_{0}(t)=\frac{1}{2} \frac{1}{\sqrt{t}}
$$

is the principal solution of the linear equation

$$
\begin{equation*}
\left(4 t^{2} y^{\prime}\right)^{\prime}+y=0 \tag{38}
\end{equation*}
$$

and $y_{0}^{\prime}(t)<0$.
Fix $\lambda>0$. Equations (37) and (38) are equivalent to

$$
\left(\lambda \frac{4(t+1)^{2} \log ^{2}(t+1)}{1+\log ^{2}(t+1)} y^{\prime}\right)^{\prime}+\lambda y=0
$$

and

$$
\left(4 \lambda t^{2} y^{\prime}\right)^{\prime}+\lambda y=0
$$

respectively. Now, from Lemma 2.5 and Theorem 3.1, we obtain the following.

Corollary 4.1. Let $\left(H_{i}\right), i=1,2,3$, be verified. Assume that there exists $\lambda>0$ such that for $t \geq 1$

$$
\begin{equation*}
a(t) \geq \min \left\{\lambda \frac{4(t+1)^{2} \log ^{2}(t+1)}{1+\log ^{2}(t+1)}, \quad 4 \lambda t^{2}\right\}, \quad \frac{\alpha}{\sqrt{\alpha^{2}-1}} \bar{F} b(t) \leq \lambda \tag{39}
\end{equation*}
$$

where $\bar{F}$ and $\alpha$ are defined in (13) and (14), respectively. If at least one of the inequalities in (39) is strict on a subinterval of $[1, \infty)$ of positive measure and (14) is verified, then the BVP (1) has at least one solution.

A similar result can be formulated for the problem (2).
Corollary 4.2. Let $\left(H_{i}\right), i=1,2,3$, be verified. Assume that there exists $\lambda>0$ such that for $t \geq 1$

$$
\begin{equation*}
a(t) \geq \min \left\{\lambda \frac{4(t+1)^{2} \log ^{2}(t+1)}{1+\log ^{2}(t+1)}, \quad 4 \lambda t^{2}\right\}, \quad \bar{F} b(t) \leq \lambda \tag{40}
\end{equation*}
$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (40) is strict on a subinterval of $[1, \infty)$ of positive measure and (14) is verified, then the BVP (2) has at least one solution.

Corollary 4.1 and Corollary 4.2 require the boundedness of $b$. Nevertheless, our results can be applied also when $\limsup _{t \rightarrow \infty} b(t)=\infty$, as the following shows.

Corollary 4.3. Let $\left(H_{i}\right), i=1,2,3$, be verified.
( $i_{1}$ ) Assume that (14) holds, and that there exists $\lambda>0$ such that for every $t \geq 1$ and some $n \geq 1$

$$
\begin{equation*}
a(t) \geq \lambda t^{n+2}, \quad \frac{\alpha}{\sqrt{\alpha^{2}-1}} \bar{F} b(t) \leq n \lambda t^{n}, \tag{41}
\end{equation*}
$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (41) is strict on a subinterval of $[1, \infty)$ of positive measure, then the BVP (1) has at least one solution.
( $i_{2}$ ) Assume that (14) holds, and that there exists $\lambda>0$ such that for every $t \geq 1$ and some $n \geq 1$

$$
\begin{equation*}
a(t) \geq \lambda t^{n+2}, \quad \bar{F} b(t) \leq n \lambda t^{n} \tag{42}
\end{equation*}
$$

where $\bar{F}$ is defined in (13). If at least one of the inequalities in (42) is strict on a subinterval of $[1, \infty)$ of positive measure, then the BVP (2) has at least one solution.

Proof. Claim $\left(i_{1}\right)$. For any $\lambda>0$ the function $v_{0}(t)=t^{-n}$ is a solution of the linear equation

$$
\begin{equation*}
\left(\lambda t^{n+2} v^{\prime}\right)^{\prime}+n \lambda t^{n} v=0, t \geq 1 \tag{43}
\end{equation*}
$$

Moreover, in view of Lemma 2.2, $v_{0}$ is the principal solution. Since, in view of (41), equation (43) is a Sturmian majorant of (16), from Lemma 2.5 the principal solution of (16) is positive nonincreasing on $[1, \infty)$.Thus, the assertion follows by Theorem 3.1. The proof of Claim ( $i_{2}$ ) follows in the same way from Theorem 3.3.

The following examples illustrate our results.
Example 4.4. Consider the equation with the Minkowski mean curvature operator

$$
\begin{equation*}
\left(2 \pi(t+2)^{2} \log ^{2}(t+4) \Phi_{M}\left(x^{\prime}\right)\right)^{\prime}+\frac{|\sin t|}{t} x^{3}=0, \quad t \geq 1 \tag{44}
\end{equation*}
$$

It is easy to show that assumptions (3) and (4) are satisfied. Moreover, we have

$$
\int_{t}^{\infty} \frac{1}{(s+2)^{2} \log ^{2}(s+4)} d s \geq \int_{t}^{\infty} \frac{1}{(s+2)^{3}} d s=\frac{1}{2(t+2)^{2}}
$$

Then

$$
a(t) A(t) \geq \frac{1}{2} \log ^{2}(t+4) \geq \frac{\log ^{2} 5}{2} \simeq 1.2951
$$

and (14) holds. Since

$$
2 \pi(t+2)^{2} \log ^{2}(t+4) \geq \frac{4(t+1)^{2} \log ^{2}(t+1)}{1+\log ^{2}(t+1)}, \quad b(t) \leq \frac{1}{t} \leq 1
$$

conditions (40) hold with $\lambda=1$. Thus, by Corollary 4.2, equation (44) has at least one solution $x$ which satisfies the boundary conditions

$$
\begin{equation*}
x(1)=1, x(t)>0, x^{\prime}(t) \leq 0, \quad \lim _{t \rightarrow \infty} x(t)=0 . \tag{45}
\end{equation*}
$$

Example 4.5. Consider the equation with the Euclidean mean curvature operator

$$
\begin{equation*}
\left(6(t+1)^{2} \Phi_{E}\left(x^{\prime}\right)\right)^{\prime}+\frac{|\sin t|}{t} x^{3}=0, \quad t \geq 1 \tag{46}
\end{equation*}
$$

Assumptions (3) and (4) are satisfied. Further, we have $\bar{F}=1$ and

$$
a(t) A(t)=t+1 \geq 2
$$

Thus, $\alpha=2$ and (14) holds. Moreover, since $\beta=\alpha / \sqrt{\alpha^{2}-1}=2 / \sqrt{3}$, conditions (39) hold with $\lambda=3 / 2$. Using Corollary 4.1 we get that the equation (46) has at least one solution $x$ which satisfies the boundary conditions (45).

Example 4.6. Consider the equation with the Minkowski mean curvature operator

$$
\begin{equation*}
\left(3(t+3)^{4} \Phi_{M}\left(x^{\prime}\right)\right)^{\prime}+2 t|\sin t+\cos t| x^{2 n+1}=0, \quad t \geq 1 \tag{47}
\end{equation*}
$$

Similarly to Example 2, also for (47) assumptions (3) and (4) are satisfied. Further, we have $\bar{F}=1$. Moreover

$$
a(t) A(t)=\frac{t+3}{3} \geq \frac{4}{3}
$$

and so (14) holds. Since

$$
3(t+3)^{4} \geq 3 t^{3}, \quad 2 t|\sin t+\cos t| \leq 2 \sqrt{2} t<3 t
$$

and these inequalities are strict on a subinterval of $[1, \infty)$ of positive measure, then by Corollary $4.3-\left(i_{2}\right)$ with $n=1$ and $\lambda=3$, the equation (47) has at least one solution $x$ which satisfies the boundary conditions (45). Observe that in equation (47) the function $b$ is unbounded.

We close the section with some remarks concerning our assumptions.
Remark 4.7. If

$$
\begin{equation*}
\liminf _{t \rightarrow \infty} a(t)=0 \tag{48}
\end{equation*}
$$

then the BVP (1) is not solvable. Indeed, let $x$ be a nonoscillatory solution of (18), $x(t)>0$ for $t \geq t_{0} \geq 1$. Then the function $a(t) \Phi_{E}\left(x^{\prime}(t)\right)$ is nonincreasing on $\left[t_{0}, \infty\right)$ and the limit

$$
\lim _{t \rightarrow \infty} a(t) \Phi_{E}\left(x^{\prime}(t)\right)
$$

exists. In virtue of (48), since $\Phi_{E}$ is bounded, we get

$$
\lim _{t \rightarrow \infty} a(t) \Phi_{E}\left(x^{\prime}(t)\right)=0
$$

which implies $x^{\prime}(t)>0$ in a neighborhood of infinity. Thus, the BVP (1) is not solvable.

REmark 4.8. The assumption (4) guarantees that the principal solution $y_{0}$ of the majorant equation (16) satisfies

$$
\lim _{t \rightarrow \infty} \frac{y_{0}(t)}{A(t)}=c, \quad 0<c<\infty
$$

see Lemma 2.3. This property is needed for obtaining the continuity of the fixed point operator, see [16, Theorem 1]. If (3) holds and

$$
\int_{1}^{\infty} b(t) \int_{t}^{\infty} \frac{1}{a(s)} d s d t=\infty
$$

then all solutions of (16) tend to zero as $t \rightarrow \infty$. In this situation, it seems hard to obtain the continuity of $\mathcal{T}$, since the solutions $\mathcal{T}\left(x_{n}\right)$ are principal solutions, but the sequence $\left\{\mathcal{T}\left(x_{n}\right)\right\}$ could converge to a nonprincipal solution.

Remark 4.9. BVPs on the half-line for equations involving the operator $\Phi_{E}$ or $\Phi_{M}$ with sign-changing coefficient have attracted very minor attention, especially when the boundary conditions concern the behavior of solutions on the whole half-line $[1, \infty)$. According to our knowledge, the only paper in this direction is [19], in which the existence of a global positive solution, bounded away from zero, is obtained. It should be interesting to extend Theorem 3.1 and Theorem 3.3 for obtaining the solvability of (1) and (2) when the function $b$ does not have fixed sign.

Remark 4.10. Analougous results to the ones obtained in Theorems 3.1 and 3.3 can be formulated also for the BVP (5). Nevertheless the existence of solutions of (5) requires weaker assumptions than those in Theorems 3.1 or 3.3. Indeed, in this situation the operator $\mathcal{T}$ is defined via the linear equation

$$
\begin{equation*}
\left(a(t) x^{\prime}\right)^{\prime}+b(t) \frac{F(u(t))}{u(t)} x=0 . \tag{49}
\end{equation*}
$$

This fact permit us to simplify the above argument, by considering the set $\Omega$ as a subset of $C([1, \infty), \mathbb{R})$ instead of $C\left([1, \infty), \mathbb{R}^{2}\right)$, because a-priori bounds for the quasiderivative are not necessary. In addition, no assumptions on $\alpha$ are needed. The details are left to the reader.

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# Bifurcation from infinity and multiplicity of solutions for nonlinear periodic boundary value problems 

Nsoki Mavinga and Mubenga N. Nkashama<br>Dedicated to Professor Jean Mawhin for his birthday


#### Abstract

We are concerned with multiplicity and bifurcation results for solutions of nonlinear second order differential equations with general linear part and periodic boundary conditions. We impose asymptotic conditions on the nonlinearity and let the parameter vary. We then proceed to establish a priori estimates and prove multiplicity results (for large-norm solutions) when the parameter belongs to a (nontrivial) continuum of real numbers. Our results extend and complement those in the literature. The proofs are based on degree theory, continuation methods, and bifurcation from infinity techniques.


Keywords: Nonlinear periodic bvp, maximum principles, principal eigenvalue, resonance, multiplicity, bifurcation from infinity, oscillatory conditions, a-priori estimates. MS Classification 2010: 34B08, 34B15, 34C23, 34C25.

## 1. Introduction

We consider nonlinear second order differential equations with general linear part and periodic boundary conditions

$$
\begin{gather*}
u^{\prime \prime}+b(x) u^{\prime}+c(x) u+\lambda u+g(x, u)=h(x) \quad \text { a.e. in }(0,2 \pi),  \tag{1}\\
u(0)-u(2 \pi)=u^{\prime}(0)-u^{\prime}(2 \pi)=0,
\end{gather*}
$$

where the coefficients $b, c \in L^{1}(0,2 \pi)$ with $c$ bounded from above; i.e., $c(x) \leq c_{0}$ for a.e. $x \in(0,2 \pi)$ for some (fixed) constant $c_{0} \in \mathbb{R}$. The non-homogeneous term $h \in L^{1}(0,2 \pi)$, and the nonlinearity $g:(0,2 \pi) \times \mathbb{R} \rightarrow \mathbb{R}$ (which may be unbounded) is an $L^{1}(0,2 \pi)$-Carathéodory function which is sublinear in $u$ at infinity (i.e., $g(x, u)=\mathrm{o}(|u|)$ as $|u| \rightarrow \infty$ ), uniformly for a.e. $x \in(0,2 \pi)$ (see conditions (C1) and (C2) below). The (real) parameter $\lambda$ varies in some neighborhood of $\lambda_{1}$, where $\lambda_{1} \in \mathbb{R}$ is the principal eigenvalue (see below) of the
second order linear periodic boundary value problem

$$
\begin{gather*}
-u^{\prime \prime}-b(x) u^{\prime}-c(x) u=\lambda u, \quad \text { a.e. on }(0,2 \pi), \\
u(0)-u(2 \pi)=u^{\prime}(0)-u^{\prime}(2 \pi)=0, \tag{2}
\end{gather*}
$$

where $\lambda$ is a real spectral parameter.
Throughout this paper, we shall use standard notations for Lebesgue spaces $L^{p}(0,2 \pi)$, Sobolev spaces $W^{k, p}(0,2 \pi)$ (with $W^{k, 2}(0,2 \pi)$ denoted by $\left.H^{k}(0,2 \pi)\right)$, and spaces of continuous functions $C^{k}([0,2 \pi])$, where $k$ is a non-negative integer and $p \in \mathbb{R}$ with $p \geq 1$ (see e.g. $[1,6])$ ).

It should be pointed out that all functions defined on $(0,2 \pi)$ are understood to be appropriately extended to the entire real line as $2 \pi$-periodic functions (possibly in a discontinuous fashion or in the a.e. sense if only Lebesgue measurable, for e.g., so as to agree at 0 and $2 \pi$, if need be). Also the period $2 \pi$ is used only as a placeholder for convenience, any fixed period $T>0$ will work.

By a solution to Eq.(1) we mean a function $u \in W_{P}^{2,1}(0,2 \pi)$ which satisfies the first equation in (1) a.e., where

$$
W_{P}^{2,1}(0,2 \pi):=\left\{u \in W^{2,1}(0,2 \pi): u(0)-u(2 \pi)=u^{\prime}(0)-u^{\prime}(2 \pi)=0\right\} .
$$

(Observe that by the Fundamental Theorem of Calculus the space $W^{2,1}(0,2 \pi)$ is equivalent to $A C^{1}([0,2 \pi])$; i.e., the collection of absolutely continuous $u$ such that $u^{\prime}$ is also absolutely continuous on $[0,2 \pi]$, see e.g. [1].)

Periodic solutions of nonlinear second order ordinary differential equations have been studied extensively. For a more recent account of the progress in this area (in the framework of resonance and nonresonance problems), we refer to the excellent monograph by A. Fonda [6]. Let us mention that when the function $g \equiv 0$, then the Fredholm Alternative type arguments describe completely the structure of the solution-set for Eq.(1) once the existence and isolation of the eigenvalue $\lambda_{1}$ are shown. That is, if $\lambda \neq \lambda_{1}\left(\right.$ near $\left.\lambda_{1}\right)$, then Eq.(1) is uniquely solvable for every $h \in L^{1}(0,2 \pi)$. Otherwise, it is solvable only for those $h \in L^{1}(0,2 \pi)$ that are orthogonal (in the sense of 'duality pairing') to the eigenspace associated with $\lambda_{1}$, and the associated solutions can be taken as large (in an appropriate norm) as one would like since solutions are (uniquely) determined 'modulo' the associated eigenspace.

However, when $g \not \equiv 0$ is a (genuine) nonlinearity, the structure of the solution-set may be quite different from that of the linear problem. Therefore, we are interested in the solution-set structure for the nonlinear problem (1) for $\lambda$ in a neighborhood of $\lambda_{1}$, and the nonlinearity $g$ (which may be unbounded) satisfies some asymptotic conditions. In particular, we are concerned with the existence of multiple large-norm solutions.

Roughly speaking, in addition to a (fairly) general existence result (see Theorem 3.1), our results state that as long as the nonlinearity $g$ satisfies
(asymptotically) a 'sign-like' condition, then when $\lambda$ is in an interval on one side of the principal eigenvalue $\lambda_{1}$ (see Section 2 below), Eq.(1) has at least two solutions, provided $h$ is in an appropriate range (using the duality pairing) which includes orthogonality. Moreover, as $\lambda \rightarrow \lambda_{1}$ (strictly from one side), the norm of these solutions become infinitely large, whereas all solutions with $\lambda$ on the other side ( of $\lambda_{1}$ ) are uniformly bounded. In this way, we locate the solution-set and describe its behavior in terms of bifurcation from infinity as the parameter $\lambda$ varies. Our asymptotic conditions include (very) 'strong resonances' (see Theorem 3.2); i.e., $g \rightarrow 0$ as $|u| \rightarrow \infty$ at $\lambda=\lambda_{1}$, and no 'decay-rate' at infinity is required; 'weaker resonances' (see Theorem 3.4) such as the so-called Landesman-Lazer type conditions (i.e., $g \nrightarrow 0$ as $|u| \rightarrow \infty$ ); as well as an asymptotic ('one-sided') oscillatory behavior (see Theorem 3.5); i.e., asymptotically $g$ has infinitely many discrete-countable 'bounce-off' zeros in $u$. We point out that the case when the nonlinearity $g$ is unbounded is included in our results as well.

We use an abstract set up on appropriate spaces, establish a priori estimates, and use a combination of degree theory (see e.g. Mawhin [10]), continuation methods and Rabinowitz bifurcation from infinity techniques ( $[7,12,17$, $18,19,20]$ ) to prove our results. An important ingredient in obtaining the necessary estimates is the use of comparison principles and estimates for the linear problem obtained in Section 2 below (under somewhat weaker conditions than those usually considered in the literature; particularly in the one-dimensional case (see e.g. [3, 4, 16])).

Let us recall that some results on multiplicity or bifurcation from infinity for nonlinear problems with periodic boundary conditions have been obtained before under a different set of conditions (see e.g. [5, 6, 8, 12] and references therein). However, our results are more in line with those in [12, 13] and references therein; herein we consider a more general linear part and more general nonlinearities.

We wish to mention that a systematic study of periodic solutions of (autonomous) nonlinear differential equations with small parameters was initiated by H. Poincaré in his celebrated treatise on celestial mechanics ([14]) in connection with the three body problem (also see [11, 15]). Since then, a great deal of work has been devoted to the study of periodic solutions of nonlinear differential equations depending on parameters in many different directions; especially using homotopy, continuation, as well as global methods (see e.g. $[6,7,10,17,18])$. In the last fifty years Professor Mawhin has tremendously contributed in an unparalleled way to the development of the theory of periodic solutions of nonlinear differential equations; which most likely served as a catalyst to his introducing the coincidence degree theory ([10]); an extension of Leray-Schauder degree to nonlinear problems which cannot necessarily be written as compact perturbations of the identity. It is with an immense gratitude
that we write this paper on periodic solutions in his honor.
This paper is organized as follows. In Section 2, we consider the linear problem and obtain the necessary comparison principles and estimates that will be needed for the nonlinear problem. As indicated above, these results are of independent interest in their own right. In Section 3, we give the general assumptions on the data, state our main results for nonlinear problems, and give some simple illustrative examples (for the reader's convenience) along the way. In Section 4, we cast the problem in an abstract setting and establish the necessary a priori estimates for possible solutions. Finally, Section 5 is devoted to the proofs of our main results. Remarks are included throughout as appropriate, and a visual rendition sketch of a bifurcation diagram for a 'bounce-off' oscillatory nonlinearity is given in Section 3.

## 2. A general periodic linear eigenproblem and estimates

In this section, we consider the issue of comparison principle(s) and the existence of a (unique) principle eigenvalue for a general (i.e., not necessarily symmetric) linear periodic problems with (possibly) unbounded coefficients. We also obtain some estimates on the linear problem that will prove useful when considering nonlinear problems.

Pick $\mu \in \mathbb{R}$ be such that $\mu>c_{0}$; which implies that $\mu-c(x) \geq \mu-c_{0}>0$ for a.e. $x \in(0,2 \pi)$. Consider the ('augmented') linear differential operator defined on $W_{P}^{2,1}(0,2 \pi)$ by

$$
\begin{equation*}
L_{\mu} u=-u^{\prime \prime}-b(x) u^{\prime}-c(x) u+\mu u . \tag{3}
\end{equation*}
$$

We first set $a(x):=e^{\int_{0}^{x} b(s) d s}$, and multiply $L_{\mu} u$ by the 'integrating factor' $a(x)$. It follows that the operator $L_{\mu}$ is transformed into the linear differential operator

$$
\begin{equation*}
S_{\mu} u:=-\left(a(x) u^{\prime}\right)^{\prime}+a(x)(\mu-c(x)) u ; \tag{4}
\end{equation*}
$$

which (despite its appearance) is not necessarily symmetric on $W_{P}^{2,1}(0,2 \pi)$.
Observe that a pair $(\lambda, \varphi)$ with $\varphi \in W_{P}^{2,1}(0,2 \pi) \backslash\{0\}$ is an eigenpair for the eigenvalue problem

$$
\begin{equation*}
L_{\mu} u=\lambda u \tag{5}
\end{equation*}
$$

if and only if it is also an eigenpair for the eigenvalue problem with weight

$$
\begin{equation*}
S_{\mu} u=\lambda a(x) u \tag{6}
\end{equation*}
$$

We shall show that the eigenvalue problem (5) has a (real) positive principal eigenvalue with a positive (i.e., bounded away from zero) eigenfunction on the closed interval $[0,2 \pi]$, even when $b, c \in L^{1}(0,2 \pi)$ are not necessarily locally bounded (with $c$ bounded from above only), as indicated. We first investigate
some properties of the linear differential operator $L_{\mu}$ on the space $W_{P}^{2,1}(0,2 \pi)$. As a first result in that direction, we have the following order-preserving or weak minimum/comparison principle.
Proposition 2.1. Suppose that $u \in W_{P}^{2,1}(0,2 \pi)$ satisfies the differential inequality $L_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$. Then $u \geq 0$ on $[0,2 \pi]$.
Proof. Let $u \in W_{P}^{2,1}(0,2 \pi)$ be such that $L_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$, by using the 'integrating factor' $a(x):=e^{\int_{0}^{x} b(s) d s}$, it follows immediately that $S_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$; which implies that $\left(a(x) u^{\prime}\right)^{\prime} \leq a(x)(\mu-c(x)) u$ for a.e. $x \in(0,2 \pi)$.

Now, suppose that $u(x)<0$ for some $x \in[0,2 \pi]$, then $u$ has a negative minimum value in this interval, say at $x_{0} \in[0,2 \pi]$. Therefore, there is a neighborhood $I_{\delta}:=\left(x_{0}-\delta, x_{0}+\delta\right)$ such that $u\left(x_{0}\right) \leq u(x)<0$ for all $x \in I_{\delta}$ and $u^{\prime}\left(x_{0}\right)=0$, where we have used the continuity of $u(x)$ and (possibly) the $2 \pi$-periodic extension of $u$ if $x_{0}$ is an end-point of the interval [ $0,2 \pi$ ]. It follows that $\left(a(x) u^{\prime}\right)^{\prime} \leq a(x)(\mu-c(x)) u<a(x)\left(\mu-c_{0}\right) u<0$ for a.e. $x \in I_{\delta}$. The Fundamental Theorem of Calculus immediately implies that $a(x) u^{\prime}$ is (strictly) decreasing in $I_{\delta}$. Since $u^{\prime}\left(x_{0}\right)=0$ (i.e., $a\left(x_{0}\right) u^{\prime}\left(x_{0}\right)=0$ ), we obtain that $a(x) u^{\prime}(x)>0$ for $x \in\left(x_{0}-\delta, x_{0}\right)$ and $a(x) u^{\prime}(x)<0$ for $x \in\left(x_{0}, x_{0}+\delta\right)$; that is, $u^{\prime}(x)>0$ for $x \in\left(x_{0}-\delta, x_{0}\right)$; which implies that $u(x)$ is (strictly) increasing in $\left(x_{0}-\delta, x_{0}\right)$. This is a contradiction with the fact that $u\left(x_{0}\right)$ is a (negative) minimum value of the function $u$. Therefore, $u(x) \geq 0$ on $[0,2 \pi]$, and the proof is complete.

This proposition immediately implies that $\lambda=0$ is not an eigenvalue of the differential operator $L_{\mu}$ in Eq.(5), since any possible eigenfunction would be identically zero in this case. We now want to show that $\lambda=0$ is actually in the 'resolvent' of $L_{\mu}$; that is; to show that the equation $L_{\mu} u=e(x)$ has a (unique) solution $u \in W_{P}^{2,1}(0,2 \pi)$ for every $e \in L^{1}(0,2 \pi)$. For that purpose, we need the following a priori estimate; which will be also useful in studying nonlinear problems.

Lemma 2.2. There exists a constant $\alpha:=\alpha(b, c, \mu)>0$ such that

$$
\begin{equation*}
\left|L_{\mu} u\right|_{L^{1}(0,2 \pi)} \geq \alpha|u|_{W_{P}^{2,1}(0,2 \pi)} \quad \text { for all } u \in W_{P}^{2,1}(0,2 \pi) . \tag{7}
\end{equation*}
$$

Proof. Suppose the conclusion doe not hold. Then, there is a sequence $\left(u_{n}\right) \subset$ $W_{P}^{2,1}(0,2 \pi) \backslash\{0\}$ such that for all $n \in \mathbb{N}$ one has that

$$
\left|L_{\mu} u_{n}\right|_{L^{1}(0,2 \pi)} \leq \frac{1}{n}\left|u_{n}\right|_{W_{P}^{2,1}(0,2 \pi)} .
$$

Setting $v_{n}:=u_{n} /\left|u_{n}\right|_{W_{P}^{2,1}(0,2 \pi)}$ and $L_{\mu} v_{n}=h_{n}$, we get that $\left|v_{n}\right|_{W_{P}^{2,1}(0,2 \pi)}=1$ for all $n \in \mathbb{N}$, and that $h_{n} \rightarrow 0$ in $L^{1}(0,2 \pi)$ as $n \rightarrow \infty$. By the continuous imbedding of $W_{P}^{2,1}(0,2 \pi)$ into $C_{P}^{1}[0,2 \pi]$, one has that there exist a constant $C_{1}>0$
(independent of $n$ ) such that $\left|v_{n}\right|_{C_{P}^{1}[0,2 \pi]} \leq C_{1}$. Moreover, since $W_{P}^{2,1}(0,2 \pi)$ is compactly imbedded into $W_{P}^{1,1}(0,2 \pi)$, one has (by going to a subsequence relabeled $\left(v_{n}\right)$, if need be) that there is a function $v \in W_{P}^{1,1}(0,2 \pi)$ such that $v_{n} \rightarrow v$ in $W_{P}^{1,1}(0,2 \pi)$ as $n \rightarrow \infty$; which implies (for a subsequence similarly relabeled if need be) that $v_{n}(x) \rightarrow v(x)$ and $v_{n}^{\prime}(x) \rightarrow v^{\prime}(x)$ for a.e. $x \in(0,2 \pi)$ (see e.g. [1, Theorem 4.9]). Since $\left|b(x) v_{n}^{\prime}(x)\right| \leq C_{1}|b(x)|$ and $\left|c(x) v_{n}(x)\right| \leq C_{1}|c(x)|$ for a.e. $x \in(0,2 \pi)$, it follows from the Lebesgue Dominated Convergence Theorem that $b(\cdot) v_{n}^{\prime} \rightarrow b(\cdot) v^{\prime}$ and $c(\cdot) v \rightarrow c(\cdot) v$ in $L^{1}(0,2 \pi)$ as $n \rightarrow \infty$. This and the fact that $v_{n}^{\prime \prime}=-h_{n}-b(x) v_{n}^{\prime}-c(x) v_{n}+\mu v_{n}$ imply that $v_{n}^{\prime \prime} \rightarrow-b(x) v^{\prime}-c(x) v+\mu v$ in $L^{1}(0,2 \pi)$ with $v_{n} \rightarrow v$ in $W_{P}^{1,1}(0,2 \pi)$ as $n \rightarrow \infty$. The (strong) closedness (see e.g. [1, p. 204, Remark4]) of the differentiation-operator from $W_{P}^{1,1}(0,2 \pi)$ into $L^{1}(0,2 \pi)$ implies that $v \in W_{P}^{2,1}(0,2 \pi)$ and that $v_{n} \rightarrow v$ in $W_{P}^{2,1}(0,2 \pi)$ as $n \rightarrow \infty$ with $v^{\prime \prime}=-b(x) v^{\prime}-c(x) v+\mu v$ for a.e. $x \in(0,2 \pi)$; that is, $L_{\mu} v=0$ for a.e. $x \in(0,2 \pi)$. It follows immediately from Proposition 2.1 that $v \equiv 0$. This is a contradiction with the fact that $\left|v_{n}\right|_{W_{P}^{2,1}(0,2 \pi)}=1$ for all $n \in \mathbb{N}$ and $v_{n} \rightarrow v$ in $W_{P}^{2,1}(0,2 \pi)$ as $n \rightarrow \infty$. The proof is complete.

Since the linear operator $L_{\mu}: W_{P}^{2,1} \Subset L^{1}(0,2 \pi) \rightarrow L^{1}(0,2 \pi)$ is compactly and densely defined, takes bounded sets in $W_{P}^{2,1}(0,2 \pi)$ into bounded sets in $L^{1}(0,2 \pi)$ and is one-to-one (see Lemma 2.2), we claim that it is onto $L^{1}(0,2 \pi)$; i.e., $L_{\mu}$ is invertible on $L^{1}(0,2 \pi)$. In fact, one has the following existence (and uniqueness) result.
Lemma 2.3. For every $e \in L^{1}(0,2 \pi)$, the equation $L_{\mu} u=e(x)$ a.e. in $(0,2 \pi)$ has a (unique) $2 \pi$-periodic solution $u \in W_{P}^{2,1}(0,2 \pi)$.

Proof. Uniqueness follows from Proposition 2.1 or Lemma 2.2. To prove existence, we use the topological degree theory by considering the homotopy

$$
-u^{\prime \prime}+\theta\left(-b(x) u^{\prime}+(\mu-c(x)) u\right)+(1-\theta)\left(\mu-c_{0}\right) u=\theta e(x) \quad \text { a.e. in }(0,2 \pi)
$$

where $\theta \in[0,1]$. Notice that the homotopy reduces to the equation $L_{\mu} u=e(x)$ when $\theta=1$, and when $\theta=0$ it reduces to the periodic linear differential equation with constant coefficients $-u^{\prime \prime}+\left(\mu-c_{0}\right) u=0$ on $[0,2 \pi]$, where $\mu-c_{0}>$ 0 . It therefore suffices to show that all possible solutions to the homotopy are (uniformly) bounded in $W_{P}^{2,1}(0,2 \pi)$ independently of $\theta \in[0,1]$. Indeed, suppose that this is not the case, then one can find sequences $\left(u_{n}\right) \subset W_{P}^{2,1}(0,2 \pi) \backslash\{0\}$ and $\left(\theta_{n}\right) \subset[0,1]$ such that for all $n \in \mathbb{N},\left|u_{n}\right|_{W_{P}^{2,1}(0,2 \pi)} \geq n$ and
$u_{n}^{\prime \prime}=\theta_{n}\left(-b(x) u_{n}^{\prime}+(\mu-c(x)) u_{n}\right)+\left(1-\theta_{n}\right)\left(\mu-c_{0}\right) u_{n}-\theta_{n} e(x)$ a.e. in $(0,2 \pi)$. Setting $v_{n}:=u_{n} /\left|u_{n}\right|_{W_{P}^{2,1}(0,2 \pi)}$ and using the fact that $W_{P}^{2,1}(0,2 \pi)$ is continuously imbedded into $C_{P}^{1}[0,2 \pi]$ and compactly imbedded into $W_{P}^{1,1}(0,2 \pi)$, the

Lebesgue Dominated Convergence Theorem, the closedness of the differentiation operator, and arguments similar to those used in the proof of Lemma 2.2, it follows that there exist $v \in W_{P}^{2,1}(0,2 \pi)$ and $\theta_{0} \in[0,1]$ such that (by going if necessary to subsequences similarly relabeled) $v_{n} \rightarrow v$ in $W_{P}^{2,1}(0,2 \pi), \theta_{n} \rightarrow \theta_{0}$ as $n \rightarrow \infty$, and $v$ satisfies the homogeneous linear equation

$$
-v^{\prime \prime}-\theta_{0} b(x) v^{\prime}+\theta_{0}(\mu-c(x)) v+\left(1-\theta_{0}\right)\left(\mu-c_{0}\right) v=0 \quad \text { for a.e. in }(0,2 \pi)
$$

Since $\theta_{0} b \in L^{1}(0,2 \pi)$ and $\theta_{0}(\mu-c(x))+\left(1-\theta_{0}\right)\left(\mu-c_{0}\right) \geq \mu-c_{0}>0$ for a.e. $x \in(0,2 \pi)$, it follows from arguments used in the proof of Proposition 2.1 that $v \geq 0$ and $v \leq 0$; that is, $v=0$. This is a contradiction with the fact that $\left|v_{n}\right|_{W_{P}^{2,1}(0,2 \pi)}=1$ for all $n \in \mathbb{N}$ and $v_{n} \rightarrow v$ in $W_{P}^{2,1}(0,2 \pi)$ as $n \rightarrow \infty$. The proof is complete.

Now, we wish to show that a strong minimum/comparison principle also holds for the differential operator $L_{\mu}$ under the weak assumptions imposed on the coefficient-functions $b$ and $c$. That is, a strong positivity or strong order preserving property holds for the second order differential operator $L_{\mu}$. (Some techniques from [21] and periodicity prove useful here.)

Proposition 2.4. Suppose that $u \in W_{P}^{2,1}(0,2 \pi)$ satisfies the differential inequality $L_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$ with $u \not \equiv 0$, then $u>0$ on the closed interval $[0,2 \pi]$; that is $u$ is positive and bounded away from zero on the whole closed interval $[0,2 \pi]$, unless it is identically zero.

Proof. Since $u \in W_{P}^{2,1}(0,2 \pi)$ satisfies the differential inequality $L_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$, one has immediately that $S_{\mu} u \geq 0$ for a.e. $x \in(0,2 \pi)$. Moreover, it follows from Proposition 2.1 that $u(x) \geq 0$ for all $x \in[0,2 \pi]$. Since $u \not \equiv 0$ is $2 \pi$-periodic, one has that either $u>0$ on $[0,2 \pi]$ (in which case the conclusion holds), or otherwise, one may assume (without loss of generality) that there is a point $x_{0} \in(0,2 \pi]$ such that $u\left(x_{0}\right)=0$ and $u(x)>0$ for all $x \in\left(x_{0}-\delta, x_{0}\right)$, where $\delta \in(0,2 \pi)$ is a (fixed) constant; that is, the function $u$ has a strict local minimum at a point $x_{0}$ in a (deleted) left-neighborhood of $x_{0}$; which implies that $u^{\prime}\left(x_{0}\right) \leq 0$. Actually, the $2 \pi$-periodicity of $u$ implies that $u^{\prime}\left(x_{0}\right)=0$ for otherwise one reaches a contradiction in the light of Proposition 2.1 (by possibly extending the function $u$ periodically if $x_{0}=2 \pi$, and hence $x_{0}=0$ as well). It follows that $a\left(x_{0}\right) u^{\prime}\left(x_{0}\right)=0$, and by using the Fundamental Theorem of Calculus and (4), one has that $u(x)=\int_{x_{0}}^{x} u^{\prime}(s) d s$ and that $-a(x) u^{\prime}(x) \leq$ $\int_{x}^{x_{0}} a(s)(\mu-c(s)) u(s) d s$ for all $x \in\left(x_{0}-\delta, x_{0}\right)$. This implies that $-u^{\prime}(x) \leq$ $v(x)\left(a_{0} \int_{0}^{2 \pi} a(s)(\mu-c(s)) d s\right)$, where $v(x):=\max _{s \in\left[x, x_{0}\right]} u(s)>0$ and $a_{0}^{-1}=$ $\min _{s \in[0,2 \pi]} a(s)$. Therefore, by the Fundamental Theorem of Calculus again, one
has that

$$
u(x) \leq\left(x_{0}-x\right) v(x)\left(a_{0} \int_{0}^{2 \pi} a(s)(\mu-c(s)) d s\right) \text { for all } x \in\left(x_{0}-\delta, x_{0}\right)
$$

For every $n \in \mathbb{N}$ such that $n>1 / \delta$, let $x_{n} \in\left[x_{0}-\frac{1}{n}, x_{0}\right)$ be a point such that $\max _{\left.x_{0}-\frac{1}{n}, x_{0}\right]} u(s):=u\left(x_{n}\right)$; which exists since the function $u$ is continuous on the compact interval $\left[x_{0}-\frac{1}{n}, x_{0}\right]$. Given that $\left[x_{n}, x_{0}\right] \subset\left[x_{0}-\frac{1}{n}, x_{0}\right]$, it follows that $v\left(x_{n}\right):=\max _{\left[x_{n}, x_{0}\right]} u(s)=u\left(x_{n}\right)$, and $0<x_{0}-x_{n} \leq 1 / n$ for all $n \in \mathbb{N}$ such that $n>1 / \delta$. Therefore, by setting $A:=a_{0} \int_{0}^{2 \pi} a(s)(\mu-c(s)) d s$, one has that

$$
0<u\left(x_{n}\right) \leq\left(x_{0}-x_{n}\right) v\left(x_{n}\right) A \leq \frac{A}{n} v\left(x_{n}\right)=\frac{A}{n} u\left(x_{n}\right)<u\left(x_{n}\right)
$$

for all $n \in \mathbb{N}$ such that $n>\max (A, 1 / \delta)$. This is a contradiction. Thus, $u(x)>0$ on the closed interval $[0,2 \pi]$, and the proof is complete.

Now, we let $K:=\left\{u \in H_{P}^{1}(0,2 \pi): u \geq 0\right\} \subset H_{P}^{1}(0,2 \pi)$ be the (solid) cone with non-empty interior. Setting $T_{\mu}:=L_{\mu}^{-1}: L^{1}(0,2 \pi) \rightarrow W_{P}^{2,1}(0,2 \pi) \Subset$ $L^{1}(0,2 \pi)$, it follows from Lemma 2.3 that (the scalar) zero is not an eigenvalue of the compact linear operator $T_{\mu}: L^{1}(0,2 \pi) \rightarrow L^{1}(0,2 \pi)$; although, it is always in the spectrum of $T_{\mu}$ (see e.g. [1, p. 164, Theorem 6.8]). Moreover, due to Proposition 2.4, one can show that $T_{\mu}$ has a positive spectral radius $r:=r\left(T_{\mu}\right)>0$. By Proposition 2.1, one has that $T_{\mu}(K) \subset K$. Since (the restriction) $T_{\mu}: H_{P}^{1}(0,2 \pi) \rightarrow H_{P}^{1}(0,2 \pi)$ satisfies all the assumptions of the Krein-Rutman Theorem, it follows that $r\left(T_{\mu}\right)$ is a (real) positive eigenvalue of $T_{\mu}$ with an eigenfunction $\phi_{1} \in K, \phi_{1} \not \equiv 0$. In addition, $r\left(T_{\mu}^{*}\right)=r\left(T_{\mu}\right)$ is also an eigenvalue of the adjoint $T_{\mu}^{*}$ with an eigenfunction $\phi_{1}^{*} \in K^{*}:=$ $\left\{f \in\left(H_{P}^{1}(0,2 \pi)\right)^{*}: f(x) \geq 0\right.$ for all $\left.x \in K\right\}$ called the dual cone of $K$; which in this instance is also a cone in $\left(H_{P}^{1}(0,2 \pi)\right)^{*}$ since one can easily show that $K^{*} \cap\left(-K^{*}\right)=\{0\}$ by using the definition of $K^{*}$ and the fact that $H_{P}^{1}(0,2 \pi)=$ $K-K$ (i.e., the cone $K$ "reproduces" the space $H_{P}^{1}(0,2 \pi)$ ).

Before proceeding, we want to make a few observations that will be needed later on. First observe that $\phi_{1} \in W_{P}^{2,1}(0,2 \pi)$ since it is in the range of $T_{\mu}$ (i.e., regularity of solutions). Also, notice that by using the (equivalent) inner product $(u, v):=\int_{0}^{2 \pi} u^{\prime} v^{\prime} d x+\int_{0}^{2 \pi}(\mu-c(x)) u v d x$ for all $u, v \in H_{P}^{1}(0,2 \pi)$ (or simply the standard inner product), it follows from the Riesz-Fréchet Representation Theorem (see e.g. [1, p. 135, Theorem 5.5]) that the Hilbert space $H_{P}^{1}(0,2 \pi)$ may be (isometrically) identified with its dual; i.e., $\left(H_{P}^{1}(0,2 \pi)\right)^{*} \cong$ $\left.H_{P}^{1}(0,2 \pi)\right)$, and hence $\phi_{1}^{*}$ may be identified with an element of $H_{P}^{1}(0,2 \pi)$, still denoted by $\phi_{1}^{*} \in H_{P}^{1}(0,2 \pi) \subset L^{\infty}(0,2 \pi)$. Furthermore, using the fact
that the dual $\left(L^{1}(0,2 \pi)\right)^{*}=L^{\infty}(0,2 \pi)$ by the Riesz Representation Theorem (see e.g. [1, p. 99, Theorem 4.11]), one has that the duality pairing $\langle\cdot, \cdot\rangle:=\langle\cdot, \cdot\rangle_{\left.\left(L^{\infty}(0,2 \pi)\right), L^{1}(0,2 \pi)\right)}$ implies that

$$
\left\langle L_{\mu}^{*}\left(\phi_{1}^{*}\right), u\right\rangle \stackrel{\text { def }}{=}\left\langle\phi_{1}^{*}, L_{\mu}(u)\right\rangle_{\left(L^{\infty}(0,2 \pi), L^{1}(0,2 \pi)\right)}=\left(\phi_{1}^{*}, u\right)-\int_{0}^{2 \pi} b(x) \phi_{1}^{*} u^{\prime} d x
$$

for all $u \in \operatorname{Dom}\left(L_{\mu}\right)=W_{P}^{2,1}(0,2 \pi) \subset L^{1}(0,2 \pi)$ (see e.g. [1, p. 44]); that is,

$$
\begin{aligned}
\left\langle L_{\mu}^{*}\left(\phi_{1}^{*}\right), u\right\rangle=\left\langle\phi_{1}^{*}, L_{\mu}(u)\right\rangle=\int_{0}^{2 \pi} \phi_{1}^{* \prime} u^{\prime} d x & +\int_{0}^{2 \pi}(\mu-c(x)) \phi_{1}^{*} u d x \\
& -\int_{0}^{2 \pi} b(x) \phi_{1}^{*} u^{\prime} d x
\end{aligned}
$$

for all $u \in \operatorname{Dom}\left(L_{\mu}\right)=W_{P}^{2,1}(0,2 \pi)$. This type of identity holds true for $L_{0}$ and $L_{0}^{*}$ as well (i.e., when $\mu=0$ ); it boils down to multiplying $\phi_{1}^{*} \in H_{P}^{1}(0,2 \pi) \subset$ $L^{\infty}(0,2 \pi)$ by $L_{0}(u)$ for any $u \in W_{P}^{2,1}(0,2 \pi)$ and integrating over $[0,2 \pi]$. (It will be used repeatedly in the sequel.)

Now, under the weaker conditions imposed on the coefficients of the linear operator $L_{\mu}$, it follows from Proposition 2.4 above and the (stronger version of) the Krein-Rutman Theorem that $\phi_{1}$ is in the interior of the cone $K$ and that the corresponding eigenvalue is simple. However, by using the periodicity of $\phi_{1}$ and the uniqueness of solutions to linear initial value problems, we present below a shorter and simpler proof adapted to our specific situation since it also allows us to get more information on the ('dual') eigenfunction $\phi_{1}^{*}$. Indeed, we have the following result.

Proposition 2.5. The linear spectral problem

$$
\begin{equation*}
L_{0} u:=-u^{\prime \prime}-b(x) u^{\prime}-c(x) u=\lambda u, \quad u \in W_{P}^{2,1}(0,2 \pi) \tag{8}
\end{equation*}
$$

has a real simple eigenvalue $\lambda_{1}$ with nonnegative eigenfunction $\phi_{1} \in W_{P}^{2,1}(0,2 \pi)$ which is actually positive; i.e., bounded away from zero on the whole closed interval $[0,2 \pi]$. Moreover, $\lambda_{1}$ is also a real eigenvalue of the adjoint operator $L_{0}^{*}$ of $L_{0}$ with a nonnegative eigenfunction $\phi_{1}^{*}$.

If, in addition, the coefficient $b \in A C_{P}([0,2 \pi])=W_{P}^{1,1}(0,2 \pi)$, then $\phi_{1}^{*}$ is also positive on the closed interval $[0,2 \pi]$.

Proof. As above, we first consider the ('augmented') invertible linear operator $L_{\mu}$ given by $L_{\mu} u=-u^{\prime \prime}-b(x) u^{\prime}+(\mu-c(x)) u$ whose inverse is denoted by $T_{\mu}$. Then, by the Krein-Rutman Theorem, the spectral problem $T_{\mu} \phi=\lambda \phi$ has a (real) eigenvalue $\lambda:=r\left(T_{\mu}\right)>0$ with a nonnegative eigenfunction $\phi_{1}$ as indicated above. Applying $L_{\mu}$ on both sides, one deduces that $L_{\mu} \phi_{1}=\left(r\left(T_{\mu}\right)\right)^{-1} \phi_{1}$;
which implies immediately that $\lambda_{1}:=\left(r\left(T_{\mu}\right)\right)^{-1}-\mu$ is an eigenvalue of the operator $L_{0} u:=-u^{\prime \prime}-b(x) u^{\prime}-c(x) u$ with nonnegative eigenfunction $\phi_{1}$, and that it is also an eigenvalue of the operator $L_{0}^{*}$ with a nonnegative eigenfunction $\phi_{1}^{*}$. Now, if there is $x_{0} \in[0,2 \pi]$ such that $\phi_{1}\left(x_{0}\right)=0$, then $x_{0}$ is a minimum point for $\phi_{1}$, and hence (extending $\phi_{1}$ by $2 \pi$-periodicity if $x_{0}$ is a boundary point $) \phi_{1}^{\prime}\left(x_{0}\right)=0$ as well since $\phi_{1} \in W_{P}^{2,1}(0,2 \pi) \subset C_{P}^{1}([0,2 \pi])$. Therefore, uniqueness results for (Carathéodory) solutions (see e.g. [21]) to initial value problems for second order homogeneous linear ordinary differential equations with $L^{1}(0,2 \pi)$-coefficients (written as integral solutions to a first order system and use of generalized Gronwall's inequality on their norm) would imply that the only solution to $L_{0} \phi_{1}-\lambda_{1} \phi_{1}=0$ a.e. is given by $\phi_{1} \equiv 0$ on $[0,2 \pi]$; which would contradict the fact that $\phi_{1}$ is an eigenfunction. Thus $\phi_{1}$ is positive (and hence bounded away from zero) on $[0,2 \pi]$ as needed.

To show that $\lambda_{1}$ is simple, let $w \in W_{P}^{2,1}(0,2 \pi)$ be an eigenfunction associated with $\lambda_{1}$. Then, one has that $L_{0}\left(\phi_{1}+t w\right)=\lambda_{1}\left(\phi_{1}+t w\right)$ for all $t \in \mathbb{R}$. Since $\phi_{1}$ is positive on $[0,2 \pi]$ and $w$ is continuous, it follows that for $|t|$ small one has that $\phi_{1}+t w$ remains positive on $[0,2 \pi]$, and that for some $t \in \mathbb{R}$ with $|t|$ large, $\phi_{1}+t w$ does not remain positive on $[0,2 \pi]$ since $w \not \equiv 0$. Therefore, by continuity (and connectedness), one has that there is $t_{0} \in \mathbb{R}$ such that $\left(\phi_{1}+t_{0} w\right)(x) \geq 0$ on $[0,2 \pi]$, and $\left(\phi_{1}+t_{0} w\right)\left(x_{0}\right)=0$ for some $x_{0} \in[0,2 \pi]$ with $L_{0}\left(\phi_{1}+t_{0} w\right)-\lambda_{1}\left(\phi_{1}+t_{0} w\right)=0$ a.e. on $(0,2 \pi)$. The above uniqueness argument implies that $\left(\phi_{1}+t_{0} w\right) \equiv 0$ on $[0,2 \pi]$; that is, $w=-t_{0}^{-1} \phi_{1}$, and the simplicity of $\lambda_{1}$ follows.

If in addition $b \in A C_{P}([0,2 \pi])=W_{P}^{1,1}(0,2 \pi)$, then one has that $\left(b \phi_{1}^{*}\right) \in$ $W_{P}^{1,1}(0,2 \pi)$. Using integration by parts in the pairing, one has that $\phi_{1}^{*} \in$ $H_{P}^{1}(0,2 \pi)$ satisfies

$$
\int_{0}^{2 \pi} \phi_{1}^{* \prime} u^{\prime} d x=-\int_{0}^{2 \pi}\left(b \phi_{1}^{*}\right)^{\prime} u d x+\int_{0}^{2 \pi} c(x) \phi_{1}^{*} u d x+\lambda_{1} \int_{0}^{2 \pi} \phi_{1}^{*} u d x
$$

for every $u \in \operatorname{Dom}\left(L_{0}\right)=W_{P}^{2,1}(0,2 \pi)$, and hence in particular for every $u \in$ $C_{0}^{\infty}(0,2 \pi)$; which implies that $\phi_{1}^{*} \in W^{2,1}(0,2 \pi)$ by the definition of the Sobolev space $W^{1,1}(0,2 \pi)$ (see e.g. [1, p. 202]). Since $(b v)^{\prime}=b^{\prime} v+b v^{\prime} \in L^{1}(0,2 \pi)$ for every $v \in W^{1,1}(0,2 \pi)=A C([0,2 \pi])$, and the (formal) adjoint linear operator $L_{0}^{*}$ is explicitly given by

$$
L_{0}^{*} v=-v^{\prime \prime}+(b(x) v)^{\prime}-c(x) v=-v^{\prime \prime}+b(x) v^{\prime}-\left(c(x)-b^{\prime}(x)\right) v
$$

it follows that $L_{0}^{*}\left(\phi_{1}^{*}\right)-\lambda_{1} \phi_{1}^{*}=0$ a.e. on $(0,2 \pi)$ with $\phi_{1}^{*} \in W_{P}^{2,1}(0,2 \pi)$. The nonnegativity of $\phi_{1}^{*}$ and the above uniqueness arguments can now be used to show that $\phi_{1}^{*}$ is positive on the closed interval $[0,2 \pi]$. The proof is complete.

The following result will prove useful in obtaining a priori estimates for possible solutions to some nonlinear periodic problems in subsequent sections.

Proposition 2.6. There exists a constant $\lambda_{0}>0$ such that for all $p \in L^{1}(0,2 \pi)$ with $0 \leq p(x) \leq \lambda_{0}$ and all $u \in W_{P}^{2,1}(0,2 \pi)$ satisfying a.e. the equation

$$
u^{\prime \prime}+b(x) u^{\prime}+c(x) u+\lambda_{1} u+p(x) u=0
$$

one has that either $u=0$ on $[0,2 \pi]$ or $\min _{[0,2 \pi]}|u(x)|>0$ (i.e., $u$ is either positive or negative on $[0,2 \pi]$ ).

Proof. Since $u \equiv 0$ is a solution to the (homogeneous linear periodic) equation, we may suppose without loss of generality that $u \in W_{P}^{2,1}(0,2 \pi) \backslash\{0\}$, and we claim that under the above assumptions one must have that $\min _{[0,2 \pi]}|u(x)|>0$. Indeed, assume that the conclusion of the proposition does not hold. Then, for every $n \in \mathbb{N}$ there exist $p_{n} \in L^{1}(0,2 \pi)$ with $0 \leq p_{n}(x) \leq 1 / n$ a.e. and $u_{n} \in W_{P}^{2,1}(0,2 \pi)$ with $\left|u_{n}\right|_{W^{2,1}(0,2 \pi)}=1$ such that $\min _{[0,2 \pi]}\left|u_{n}(x)\right|=0$ and for a.e. $x \in(0,2 \pi)$ one has that

$$
u_{n}^{\prime \prime}+b(x) u_{n}^{\prime}+c(x) u_{n}+\lambda_{1} u_{n}+p_{n}(x) u_{n}=0
$$

Using the fact that $W_{P}^{2,1}(0,2 \pi)$ is continuously imbedded into $C_{P}^{1}[0,2 \pi]$ and compactly imbedded into $W_{P}^{1,1}(0,2 \pi)$, the Lebesgue Dominated Convergence Theorem, the closedness of the differentiation operator, and arguments similar to those used in the proof of Lemma 2.2, it follows (by going if necessary to subsequence relabeled $\left.\left(u_{n}\right)\right)$ that there exist $u \in W_{P}^{2,1}(0,2 \pi) \backslash\{0\}$ such that $u_{n} \rightarrow u$ in $W^{2,1}(0,2 \pi),|u|_{W^{2,1}(0,2 \pi)}=1$, and $u^{\prime \prime}+b(x) u^{\prime}+c(x) u+\lambda_{1} u=0$. Therefore, $u$ is an eigenfunction associated with the simple eigenvalue $\lambda_{1}$, and hence is proportional to $\phi_{1}$. Thus, it has one sign and is bounded away from zero by Proposition 2.5; i.e., $\min _{[0,2 \pi]}|u(x)|>0$. This fact and the uniform convergence of $u_{n}$ to $u$ in $C_{P}^{0}[0,2 \pi]$ imply that there is $n_{0} \in \mathbb{N}$ such that for all $n \geq n_{0}$ one has that $\min _{[0,2 \pi]}\left|u_{n}(x)\right|>0$. This is a contradiction, and the proof is complete.

Remark 2.7. Propositions 2.4 and 2.5 may be used (in conjunction with the Krein-Rutman Theorem) to show that the eigenvalue $\lambda_{1}$ is principal and unique; i.e., it is the only (real) eigenvalue with a positive eigenfunction $\phi_{1}$ and onedimensional eigenspace (see e.g. [1]). Moreover, an analysis of the proof of Proposition 2.1 and the result in Proposition 2.5 show that if $\lambda \neq \lambda_{1}$ is a real eigenvalue of the spectral problem (8), then $\lambda>\lambda_{1}$. Indeed, if $\lambda<\lambda_{1}$ is an eigenvalue of Eq.(8) with eigenfunction $u \in W_{P}^{2,1}(0,2 \pi)$; i.e., $L_{0} u+\lambda u=0$ a.e. on $[0,2 \pi]$, then using the fact that $\phi_{1}$ is positive on $[0,2 \pi]$ and setting $v:=u / \phi_{1}$, one has that $-\lambda v=\left[\phi_{1}(\cdot)\right]^{-1} L_{0}\left(v \phi_{1}\right)$. Using direct calculations of $L_{0}\left(v \phi_{1}\right)$ through the product rule for derivatives and collecting terms, it follows easily that $v \in W_{P}^{2,1}(0,2 \pi)$ satisfies a.e. the homogeneous linear differential equation
$v^{\prime \prime}+d(x) v^{\prime}+\left(\lambda-\lambda_{1}\right) v=0$ with $\lambda-\lambda_{1}<0$, where $d(x)=b(x)+2 \phi_{1}^{\prime}(x) / \phi_{1}(x)$. Now, arguments similar to those used in the proof of Proposition 2.1 imply that $v \geq 0$ and $-v \geq 0$ for all $x \in[0,2 \pi]$. That is, $v \equiv 0$, and hence $u \equiv 0$; contradicting the fact that $u \neq 0$ is an eigenfunction.

## 3. Main results

From now on, we shall write the nonlinear equation (1) in the equivalent form

$$
\begin{gather*}
u^{\prime \prime}+b(x) u^{\prime}+c(x) u+\lambda_{1} u+\lambda u+g(x, u)=h(x) \quad \text { a.e. in }(0,2 \pi) \\
u(0)-u(2 \pi)=u^{\prime}(0)-u^{\prime}(2 \pi)=0, \tag{9}
\end{gather*}
$$

where $\lambda_{1} \in \mathbb{R}$ is the principal eigenvalue obtained in Proposition 2.5, and the parameter $\lambda \in \mathbb{R}$ will vary in a neighborhood of zero. Therefore, Eq.(1) is equivalent to

$$
\begin{gather*}
L u+\lambda u+g(x, u)=h(x) \quad \text { a.e. in }(0,2 \pi), \\
u(0)-u(2 \pi)=u^{\prime}(0)-u^{\prime}(2 \pi)=0 \tag{10}
\end{gather*}
$$

where the linear operator $L: W_{P}^{2,1}(0,2 \pi) \rightarrow L^{1}(0,2 \pi)$ is defined by

$$
L u:=u^{\prime \prime}+b(x) u^{\prime}+c(x) u+\lambda_{1} u
$$

for which the scalar $\lambda=0$ is the principal eigenvalue with associated (positive) eigenfunction $\phi_{1}$. (Notice that $\lambda=0$ is also a principal eigenvalue of the adjoint $L^{*}$ of $L$ with associated nonnegative eigenfunction $\phi_{1}^{*} \neq 0$.)

In this section we state our general assumptions on the nonlinearity $g$ and the function $h$. We assume that $g:(0,2 \pi) \times \mathbb{R} \rightarrow \mathbb{R}$ is an $L^{1}(0,2 \pi)$ Carathéodory function which is sublinear at infinity in $u$, uniformly a.e. in $x$, and satisfies 'sign-like' conditions. We also impose asymptotic conditions on $g$ and their relationship with the forcing term $h$. These conditions include, among others, strong resonance conditions, Landesman-Lazer type conditions, as well as oscillatory conditions. (Some results herein were motivated by [9].)

In addition to a (fairly) general existence result, we state our main results on multiplicity of solutions (with large norms for $\lambda$ 'small') when $\lambda$ is in an interval on one side of the first eigenvalue, and the existence of (at least) one solution for $\lambda$ on the other side. The existence of a third solution (with a somewhat 'smaller norm') is also discussed. Simple examples are provided to motivate and illustrate the results.

As mentioned above, we specifically assume the following general conditions; the first three of which refer to the nonlinearity $g$, whereas the last one relate the nonhomogeneous term $h$ to the asymptotic behavior of $g$ and the null-space associated with the eigenvalue $\lambda_{1}$.
(C1) $g(\cdot, u)$ is measurable for all $u \in \mathbb{R}, g(x, \cdot)$ is continuous for a.e. $x \in(0,2 \pi)$, and for every $r>0$ there is a function $\gamma_{r} \in L^{1}(0,2 \pi)$ such that

$$
\begin{equation*}
|g(x, u)| \leq \gamma_{r}(x) \tag{11}
\end{equation*}
$$

for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$ with $|u| \leq r$.
(C2) $\lim _{|u| \rightarrow \infty} \frac{g(x, u)}{u}=0$ uniformly a.e. in $x$; that is, for every $\varepsilon>0$ there is a constant $r_{\varepsilon}>0$ such that

$$
\begin{equation*}
|g(x, u)| \leq \varepsilon|u| \quad \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with }|u| \geq r_{\varepsilon} \tag{12}
\end{equation*}
$$

(C3) $g$ satisfies 'sign-like' conditions, i.e., there are functions $A, B \in L^{1}(0,2 \pi)$ and constants $r<0<R$ such that

$$
\begin{array}{ll}
g(x, u) \geq A(x) & \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } u \geq R \\
g(x, u) \leq B(x) & \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } u \leq r
\end{array}
$$

(C4) Moreover, we assume that the non-homogeneous term $h \in L^{1}(0,2 \pi)$ satisfies the 'orthogonality-like' conditions

$$
\begin{equation*}
\int_{0}^{2 \pi} B(x) \phi_{1}^{*} d x \leq \int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x \leq \int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x \tag{13}
\end{equation*}
$$

where as aforementioned $\phi_{1}^{*}$ is the eigenfunction associated with the (principal) eigenvalue $\lambda_{1}$ through the dual linear operator.

Before taking up the issue of multiplicity of solutions and the behavior of the solution-set, we first state an existence result for all $\lambda \leq \lambda_{0}$ (where $\lambda_{0}$ is given by Proposition 2.6), and establish uniform a priori bounds when the parameter $\lambda$ lies in appropriate intervals around zero.

Theorem 3.1. Assume that the assumptions (C1)-(C4) hold, then Eq.(9) (or equivalently Eq.(10)) has at least one solution for every $\lambda \in \mathbb{R}$ with $\lambda \leq \lambda_{0}$. Moreover, for $0<\lambda \leq \lambda_{0}$, all solutions are uniformly bounded in $W^{2,1}(0,2 \pi)$, independently of $\lambda$.

Recall that no multiplicity results occur for Eq.(9) when $g \equiv 0$ and either $\lambda<0$ or $0<\lambda \leq \lambda_{0}$, since the Fredholm alternative argument guarantees uniqueness in this case. We claim that, by somewhat strengthening either (C3) or (C4), we obtain multiplicity results and more importantly describe the behavior of the solution-set. The first result is motivated by the fact that one may allow the equality $A(x)=B(x)$ for a.e. $x \in[0,2 \pi]$ in the assumption (C3). We would like to point that, in this instance, multiplicity may occur only for one
value of $\lambda$; more precisely at $\lambda=0$ (even if $g \not \equiv 0$ ), with the bifurcation branches in the $\left(\lambda,|u|_{\infty}\right)$-plane being only (semi-infinite) straight line rays located on the vertical $|u|_{\infty}$-axis. It suffices to consider any (nonlinearity) $g$ such that $g(x, u)=0$ outside a rectangular region $[0,2 \pi] \times[-R, R]$. Indeed, for $\lambda=0$, it is easily seen that the function defined by $u_{t}:=t \phi_{1}$ is a solution to Eq.(9) for every $t \in \mathbb{R}$ that is such that $|t| \min _{[0,2 \pi]}\left\{\phi_{1}(x)\right\} \geq R$; provided $h \equiv 0$ of course. Actually an analysis of the proof of the above existence result (or more precisely, the multiplicity results obtained below) indicates that, provided $h$ is such that $\int_{0}^{2 \pi} h \phi_{1}^{*} d x=0, \lambda=0$ is the only parameter-value for which large solutions exist, and the bifurcation from infinity branches are (semi-infinite) straight line rays on the $|u|_{\infty}$-axis in the $\left(\lambda,|u|_{C^{0}([0,2 \pi])}\right)$-plane, as described above. Therefore, the bifurcation from infinity parameter-interval collapses to just one-point interval $\{\lambda\}=\{0\}$.

For the rest of the paper, we will be interested in nonlinearities $g$ that satisfy a sign-like condition and that are not identically null outside a compact $u$-interval in $\mathbb{R}$. In the following result we strengthen somewhat the condition (C3) by requiring strict inequalities (on subsets of $\partial \Omega$ of positive measure) while still retaining the condition ( C 4 ).

A simple example to keep in mind here is the (continuous) function $g$ given by $g(x, u):=\eta_{+}(x)\left(1+u^{2}\right)^{-1}$ for $u \geq R>0$ and $g(x, u):=-\eta_{-}(x)\left(1+u^{2}\right)^{-1}$ for $u \leq-r<0$, where $\eta_{ \pm} \in C_{P}^{0}[0,2 \pi]$ are nonnegative functions which are positive on subsets of $[0,2 \pi]$ of positive measure, or a non-bounded counterpart $g(x, u):=\sqrt[3]{u} \sin ^{2}(u) \pm \eta_{ \pm}(x)\left(1+u^{2}\right)^{-1}$. Here, $A=B=0$ and $\int_{0}^{2 \pi} h \phi_{1}^{*} d x=0$ by (C4). Notice that for the bounded case $\lim _{|u| \rightarrow \infty} g(x, u)=$ 0 and $\lim _{|u| \rightarrow \infty} u g(x, u)=0$ on $\partial \Omega$, whereas for the unbounded counterpart $\liminf _{u \rightarrow \infty} g(x, u)=0=\limsup _{u \rightarrow-\infty} g(x, u)$ and $\liminf _{u \rightarrow \infty} u g(x, u)=0=\limsup _{u \rightarrow-\infty} u g(x, u)$; that is, no (linear) decay 'rate' at infinity is required. Thus, the terminology (asymptotic) 'very' strong resonance. Observe also that the so-called Landesman-Lazer condition (see below) fails since one has equality in (C4); however, we are able to 'locate' and 'describe' the solution-branches. The following result is an extension of the main result in [13] to more general linear operators and more general nonlinearities (also see Remark 3.3 below).
Theorem 3.2. Assume that conditions (C1)-(C2) are met, and that (C3) holds with strict inequalities on subsets of $[0,2 \pi]$ of positive measure; that is, there are functions $A, B \in L^{1}(0,2 \pi)$ and constants $r<0<R$ such that

$$
\begin{array}{ll}
g(x, u)>A(x) & \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } u \geq R, \\
g(x, u)<B(x) & \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } u \leq r,
\end{array}
$$

Then, provided (C4) holds, there is a constant $\lambda_{-}<0$ such that, for every $\varepsilon \in\left(0,\left|\lambda_{-}\right|\right)$, Eq.(9) has at least two solutions, denoted $\left(\lambda_{\varepsilon}^{+}, u_{\varepsilon}\right)$ and $\left(\lambda_{\varepsilon}^{-}, v_{\varepsilon}\right)$,
with $-\varepsilon<\lambda_{\varepsilon}^{ \pm}<0$ and

$$
\lim _{\varepsilon \rightarrow 0^{+}} \min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\}=\infty ;
$$

that is, they bifurcate from infinity since $\lambda_{\varepsilon}^{ \pm} \rightarrow 0$ as $\varepsilon \rightarrow 0^{+}$.
Moreover, for $0 \leq \lambda \leq \lambda_{0}$, all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of $\lambda$. Therefore, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue $\lambda_{1}$. (In some sense, the 'strong resonance' conditions 'bend' the bifurcation branches.)
Remark 3.3. An analysis of the proof of this result will show that the conditions on the nonlinearity $g$ may be replaced by the (slightly) more general (integral) conditions

$$
\begin{array}{ll}
\int_{0}^{2 \pi} g(x, u) \phi_{1}^{*} d x>\int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x & \text { for all } u \in \mathbb{R} \text { with } u \geq R \\
\int_{0}^{2 \pi} g(x, u) \phi_{1}^{*} d x>\int_{0}^{2 \pi} B(x) \phi_{1}^{*} d x & \text { for all } u \in \mathbb{R} \text { with } u \leq R
\end{array}
$$

which are in particular fulfilled if the coefficient $b \in A C_{P}([0,2 \pi])=W_{P}^{1,1}(0,2 \pi)$, and
$g(x, u) \geq A(x)$ for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$ with $u \geq R$, with strict inequality on a subset of $(0,2 \pi)$ of positive measure,
$g(x, u) \leq B(x) \quad$ for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$ with $u \leq r$, with strict inequality on a subset of $(0,2 \pi)$ of positive measure,
since, in this instance, the conditions on the coefficient $b$ imply that the eigenfunction $\phi_{1}^{*}$ is (strictly) positive on the interval $[0,2 \pi]$ by Proposition 2.5.

In the following result we strengthen a little bit the condition (C4) by requiring strict inequalities while keeping (C3) as given. This is the so-called Landsman-Lazer type condition; which has been widely considered in the literature (see e.g. [6]). Again, a simple example to keep in mind here is the (continuous) function $g$ (independent of $x$ ) given by $g(u):=\sqrt[3]{u} \sin ^{2}(u)+\eta_{ \pm} \tanh (u)$ for $|u| \geq R>0$ where $\eta_{ \pm}$are positive numbers with $\eta_{-}<\eta_{+}$. Notice that $\liminf _{u \rightarrow \infty} g(u)=\eta_{+}$and $\lim \sup _{u \rightarrow-\infty} g(u)=-\eta_{-}$. The following result is an extension of the main result in [12] to more general linear operators and more general nonlinearities (at least as far as periodic solutions are concerned).

Theorem 3.4. Assume that (C1)-(C3) hold and that

$$
\begin{equation*}
\int_{0}^{2 \pi} g_{-}(x) \phi_{1}^{*} d x<\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x<\int_{0}^{2 \pi} g_{+}(x) \phi_{1}^{*} d x \tag{14}
\end{equation*}
$$

where $g_{+}(x):=\liminf _{u \rightarrow \infty} g(x, u)$ and $g_{-}(x):=\limsup _{u \rightarrow-\infty} g(x, u)$.
Then there is a constant $\lambda_{-}<0$ such that, for every $\varepsilon \in\left(0,\left|\lambda_{-}\right|\right)$, Eq.(9) has at least two solutions, denoted $\left(\lambda_{\varepsilon}^{+}, u_{\varepsilon}\right)$ and $\left(\lambda_{\varepsilon}^{-}, v_{\varepsilon}\right)$, with $-\varepsilon<\lambda_{\varepsilon}^{ \pm}<0$ and

$$
\lim _{\varepsilon \rightarrow 0^{+}} \min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\}=\infty ;
$$

that is, they bifurcate from infinity since $\lambda_{\varepsilon}^{ \pm} \rightarrow 0$ as $\varepsilon \rightarrow 0^{+}$.
Moreover, for $0 \leq \lambda \leq \lambda_{0}$, all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of $\lambda$. Again, bifurcation from infinity occurs only (strictly) to the left of the eigenvalue $\lambda_{1}$.

Now, we take up the case when the nonlinearity $g$ may have (asymptotically) infinitely many (discrete-countable) zeros (i.e. a sign-like condition with 'oscillation'). In this instance, we strengthen a little bit the condition on the coefficient function $b$. Therefore, for the sake of clarity, we first state the result for the case when $A=B=0$; which again implies that the condition (C4) is equivalent to saying that $\int_{0}^{2 \pi} h \phi_{1}^{*} d x=0$. The function to keep in mind here is for instance $g(x, u)=\eta_{ \pm} u^{-1} \sin ^{2}(u)$ for $|u| \geq R>0$ where $\eta_{ \pm}$are positive numbers, or an unbounded counterpart $g(x, u)=\eta_{+} \sqrt[3]{u} \sin ^{2}(u)$ for $|u| \geq R>0$. Therefore, we consider functions which satisfy a sign condition, vanish asymptotically at discrete-countably many points going to infinity, and have a strict sign in-between them.
THEOREM 3.5. Let the coefficient $b$ be such that $b \in A C_{P}([0,2 \pi])=W_{P}^{1,1}(0,2 \pi)$. Assume that conditions (C1) and (C2) are met. Suppose there are sequences of real numbers $0 \gg r_{k}>r_{k+1} \rightarrow-\infty$ and $0 \ll R_{k}<R_{k+1} \rightarrow \infty$ as $k \rightarrow \infty$ such that for all $k \in \mathbb{N}$,

$$
\begin{aligned}
& g\left(x, r_{k}\right)=0 \quad \text { and } \quad g\left(x, R_{k}\right)=0 \quad \text { for a.e. } x \in(0,2 \pi) \text { and } \\
& g(x, u)>0 \quad \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } R_{k}<u<R_{k+1} \text {, } \\
& g(x, u)<0 \quad \text { for a.e. } x \in(0,2 \pi) \text { and all } u \in \mathbb{R} \text { with } r_{k+1}<u<r_{k} \text {. }
\end{aligned}
$$

Then, provided $h$ is $L^{1}(0,2 \pi)$ with $\int_{0}^{2 \pi} h \phi_{1}^{*} d x=0$, there is a constant $\lambda_{-}<0$ such that, for every $\varepsilon \in\left(0,\left|\lambda_{-}\right|\right)$, Eq.(9) has at least two solutions, denoted $\left(\lambda_{\varepsilon}^{+}, u_{\varepsilon}\right)$ and $\left(\lambda_{\varepsilon}^{-}, v_{\varepsilon}\right)$, with $-\varepsilon<\lambda_{\varepsilon}^{ \pm}<0$ and

$$
\lim _{\varepsilon \rightarrow 0^{+}} \min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\}=\infty ;
$$

that is, they bifurcate from infinity since $\lambda_{\varepsilon}^{ \pm} \rightarrow 0$ as $\varepsilon \rightarrow 0^{+}$.
Moreover, for $0<\lambda \leq \lambda_{0}$, all solutions (which exist by Theorem 3.1) are uniformly bounded, independently of $\lambda$. Therefore, bifurcation continua from infinity occur to the left of the eigenvalue $\lambda_{1}$.

A visual rendition sketch for the case when $g$ is as in the above example is given below. (For example, $g(x, u)=\eta_{ \pm} u^{-1} \sin ^{2}(u)$ when $|u| \geq R$ with $c=0$ and $h=0$.)


Figure 1: Bifurcation diagram in the case of a 'bounce-off' oscillatory nonlinearity.

REmARK 3.6. (Existence of a third solution) Let us mention that by using a consequence of the Leray-Schauder Homotopy Continuation Theorem or the so-called Wyburn Lemma (see e.g. $[8,12,13,10]$ ), one can show that there is $\lambda_{-}^{*}<0$ with $\lambda_{-}<\lambda_{-}^{*}$ such that for every $\varepsilon \in\left(0,\left|\lambda_{-}^{*}\right|\right)$, one has a third solution $w_{\varepsilon}$ in Theorems 3.2 and 3.4. (The (uniform) bound of these third solutions could for instance be twice the uniform a-priori bound obtained for all solutions in the homotopy.)

Remark 3.7. Let us finally point out that one can reverse the inequalities in the conditions (C3)-(C4) appropriately to get results similar to all the ones above. In which case, multiplicity and bifurcation from infinity occur (for $\lambda$ in a nontrivial interval) to the right of $\lambda_{1}$ only, whereas solutions are uniformly bounded on bounded $\lambda$-intervals to the left of $\lambda_{1}$. The reader can easily carry out the details.

## 4. Abstract Setting and a priori Bounds

In this section we formulate the problem (9) in an abstract setting. We then proceed to establish a priori bounds in $W^{2,1}(0,2 \pi)$ for all possible solutions. For that purpose we define the linear operator

$$
\begin{gathered}
L: W_{P}^{2,1}(0,2 \pi) \\
L u:=C^{0}([0,2 \pi]) \subset L^{1}(0,2 \pi) \rightarrow L^{1}(0,2 \pi) \quad \text { by } \\
L x) u^{\prime}+c(x) u+\lambda_{1} u,
\end{gathered}
$$

where $W_{P}^{2,1}(0,2 \pi) \Subset C^{0}([0,2 \pi])$ denotes the compact imbedding of $W_{P}^{2,1}(0,2 \pi)$ in $C^{0}([0,2 \pi])$ (see e.g. [1]). Next, we define the nonlinear (Nemytskǐi) superposition operator

$$
\mathcal{N}: C^{0}([0,2 \pi]) \rightarrow L^{1}(0,2 \pi) \quad \text { by } \quad \mathcal{N} u=g(\cdot, u(\cdot))
$$

Eq.(9) is then equivalent to

$$
\begin{equation*}
L u+\lambda u+\mathcal{N} u=h, \quad u \in \operatorname{Dom}(L):=W_{P}^{2,1}(0,2 \pi) . \tag{15}
\end{equation*}
$$

Now, we shall establish an a priori bound for all possible solutions of Eq.(9) or equivalently Eq.(15).
Proposition 4.1. Assume that the assumptions (C1)-(C4) hold true. Let $\lambda_{0} \in$ $\mathbb{R}$ with $\lambda_{0}>0$ be a fixed constant given in Proposition 2.6. Then, there is a constant $R_{0}:=R_{0}\left(\lambda_{0}\right)>0$ such that all possible solutions of Eq.(9) (or equivalently Eq.(15)) with $0<\lambda \leq \lambda_{0}$ satisfy

$$
|u|_{W^{2,1}(0,2 \pi)} \leq R_{0} .
$$

That is, all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ independently of $\lambda$, provided $0<\lambda \leq \lambda_{0}$.

Proof. Suppose that all (possible) solutions in $W_{P}^{2,1}(0,2 \pi)$ are not uniformly bounded in $W^{2,1}(0,2 \pi)$. Then, there are sequences $\left\{\lambda_{n}\right\} \subset\left(0, \lambda_{0}\right]$ and $\left\{u_{n}\right\} \subset$ $W_{P}^{2,1}(0,2 \pi)$ with $\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \geq n$ for all $n \in \mathbb{N}$ such that

$$
\begin{equation*}
u_{n}^{\prime \prime}+b(x) u_{n}^{\prime}+c(x) u_{n}+\lambda_{1} u_{n}+\lambda_{n} u_{n}+g\left(x, u_{n}\right)=h(x) \quad \text { a.e. in }(0,2 \pi) . \tag{16}
\end{equation*}
$$

Letting $v_{n}:=u_{n} /\left|u_{n}\right|_{W^{2,1(0,2 \pi)}}$, one has that $\left|v_{n}\right|_{W^{2,1(0,2 \pi)}}=1$, and $v_{n} \in$ $W_{P}^{2,1}(0,2 \pi)$ satisfies

$$
\begin{align*}
v_{n}^{\prime \prime}+b(x) v_{n}^{\prime}+c(x) v_{n}+\lambda_{1} v_{n}+\lambda_{n} v_{n}+ & \frac{g\left(x, u_{n}\right)}{\left|u_{n}\right|_{W^{2,1}(0,2 \pi)}} \\
& =\frac{h(x)}{\left|u_{n}\right|_{W^{2,1}(0,2 \pi)}} \text { a.e. in }(0,2 \pi) \tag{17}
\end{align*}
$$

Notice that, by the fact that the function $g$ is $L^{1}(0,2 \pi)$-Carathéodory and the sublinear growth condition (12) with $\varepsilon=1$ e.g., one has that the sequence $\left\{g\left(\cdot, u_{n}(\cdot)\right) /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)}\right\}$ is bounded in $L^{1}(0,2 \pi)$ since there is a function $\gamma_{1} \in L^{1}(0,2 \pi)$ such that $|g(x, u)| \leq|u|+\gamma_{1}(x)$ for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$. Therefore, since $W_{P}^{2,1}(0,2 \pi)$ is continuously imbedded into $C_{P}^{1}([0,2 \pi])$, there is a constant $C_{1}>0$ (independent of $n$ ) such that

$$
\begin{equation*}
\left|g\left(x, u_{n}(x)\right)\right| /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \leq\left|v_{n}(x)\right|+\left|\gamma_{1}(x)\right| /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \leq C_{1}+\left|\gamma_{1}(x)\right| \tag{18}
\end{equation*}
$$

$\left|b(x) v_{n}^{\prime}(x)\right| \leq C_{1}|b(x)|$, and $\left|c(x) v_{n}(x)\right| \leq C_{1}|c(x)|$ for a.e. $x \in(0,2 \pi)$ and all $n \in \mathbb{N}$. Moreover, since $\lambda_{n} \in\left(0, \lambda_{0}\right]$ and $W_{P}^{2,1}(0,2 \pi)$ is compactly imbedded into $W_{P}^{1,1}(0,2 \pi)$, one has (by going to subsequences relabeled $\left(\left\{\lambda_{n}\right\}\right.$ and $\left\{v_{n}\right\}$, if need be) that there exist a number $\mu_{0} \in\left[0, \lambda_{0}\right]$ and a function $v \in W_{P}^{1,1}(0,2 \pi)$ such that $\lambda_{n} \rightarrow \mu_{0}$ and $v_{n} \rightarrow v$ in $W_{P}^{1,1}(0,2 \pi)$ as $n \rightarrow \infty$; which implies (for a subsequence similarly relabeled if need be) that $v_{n}(x) \rightarrow v(x)$ and $v_{n}^{\prime}(x) \rightarrow$ $v^{\prime}(x)$ for a.e. $x \in(0,2 \pi)$ (see e.g. [1, Theorem 4.9]). By using the first inequality in (18), we deduce that $g\left(x, u_{n}(x)\right) /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $x \in$ $(0,2 \pi)$ where $v(x)=0$. Observe that $u_{n}(x) \rightarrow \infty$ if $v(x)>0$ and $u_{n}(x) \rightarrow-\infty$ if $v(x)<0$. Therefore, for a.e. $x \in(0,2 \pi)$ such that $v(x) \neq 0$, (considering $n$ sufficiently large if need be) we write the quotient $g\left(x, u_{n}(x)\right) /\left|u_{n}\right|_{W^{2,1(0,2 \pi)}}$ in the form

$$
\frac{g\left(x, u_{n}(x)\right)}{\left|u_{n}\right|_{W^{2,1}(0,2 \pi)}}=\left(\frac{g\left(x, u_{n}(x)\right)}{u_{n}(x)}\right) v_{n}(x) \rightarrow 0 \cdot v(x)=0 \text { as } n \rightarrow \infty
$$

by the sublinear condition (C2). Thus, in either case one has that the sequence $g\left(x, u_{n}(x)\right) /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $x \in(0,2 \pi)$. By the Lebesgue Dominated Convergence Theorem, it follows that $b(\cdot) v_{n}^{\prime} \rightarrow b(\cdot) v, c(\cdot) v_{n} \rightarrow c(\cdot) v$ and $g\left(\cdot, u_{n}(\cdot)\right) /\left|u_{n}\right|_{W^{2,1}(0,2 \pi)} \rightarrow 0$ in $L^{1}(0,2 \pi)$ as $n \rightarrow \infty$.

Now, by using Eq.(17), we deduce that $v_{n}^{\prime \prime} \rightarrow-b(x) v-c(x) v-\lambda_{1} v-$ $\mu_{0} v$ in $L^{1}(0,2 \pi)$ with $v_{n} \rightarrow v$ in $W_{P}^{1,1}(0,2 \pi)$ as $n \rightarrow \infty$ and $\mu_{0} \in\left[0, \lambda_{0}\right]$. The (strong) closedness of the differentiation-operator from $W_{P}^{1,1}(0,2 \pi)$ into $L^{1}(0,2 \pi)$ implies that $v \in W_{P}^{2,1}(0,2 \pi)$ and that $v_{n} \rightarrow v$ in $W_{P}^{2,1}(0,2 \pi)$ as $n \rightarrow \infty$ with $v^{\prime \prime}=-b(x) v^{\prime}-c(x) v-\lambda_{1} v-\mu_{0} v$ for a.e. $x \in(0,2 \pi)$; that is,

$$
\begin{equation*}
L v+\mu_{0} v=0 \tag{19}
\end{equation*}
$$

It follows from Proposition 2.6 that either $v(x)>0$ on $[0,2 \pi]$ or $v(x)<0$ on $[0,2 \pi]$ since $|v|_{W^{2,1}(0,2 \pi)}=1$. Using the duality pairing (see e.g. [1]), we get that

$$
0=\left\langle L v+\mu_{0} v, \phi_{1}^{*}\right\rangle=\left\langle v, L^{*}\left(\phi_{1}^{*}\right)\right\rangle+\mu_{0} \int_{0}^{2 \pi} v \phi_{1}^{*} d x=\mu_{0} \int_{0}^{2 \pi} v \phi_{1}^{*} d x
$$

since $\phi_{1}^{*}$ is an eigenfunction of the adjoint $L^{*}$ associated with the eigenvalue zero. This implies that $\mu_{0}=0$ since $\phi_{1}^{*}$ is a nonnegative eigenfunction and $|v(x)|>0$ on $[0,2 \pi]$. Therefore, $L v=0$; i.e., $v=t \phi_{1}$ for some real constant $t \neq 0$ since $\lambda_{1}$ is simple.

In what follows, we assume without loss of generality that $v(x)>0$ on $[0,2 \pi]$; i.e., $t>0$ (the case $v(x)<0$ can be treated in a similar way). This implies that there is a constant $\epsilon_{0}>0$ such that $v(x)=t \phi_{1}(x) \geq \epsilon_{0}$ for all $x \in[0,2 \pi]$ since the eigenfunction $\phi_{1}$ of $L$ is (strictly) positive on $[0,2 \pi]$.

Since $v_{n} \rightarrow v$ uniformly on $[0,2 \pi]$, one has that $u_{n}(\cdot)=v_{n}(\cdot)\left|u_{n}\right|_{W^{2,1(0,2 \pi)}} \rightarrow$ $\infty$ uniformly on $[0,2 \pi]$. Therefore, there exists $n_{0} \in \mathbb{N}$ such that for all $n \geq n_{0}$ one has that

$$
\begin{equation*}
u_{n}(x) \geq R \quad \text { for all } x \in[0,2 \pi] \tag{20}
\end{equation*}
$$

where $R>0$ is the constant given in the assumption (C3). Now, using again the duality pairing in Eq.(16), we deduce that $\left\langle L u_{n}+\lambda_{n} u_{n}+\mathcal{N} u_{n}, \phi_{1}^{*}\right\rangle=$ $\int_{0}^{2 \pi} h \phi_{1}^{*} d x$; i.e., $\left\langle u_{n}, L^{*}\left(\phi_{1}^{*}\right)\right\rangle+\lambda_{n} \int_{0}^{2 \pi} u_{n} \phi_{1}^{*} d x+\int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x=\int_{0}^{2 \pi} h \phi_{1}^{*} d x$.

Since $0<\lambda_{n} \leq \lambda_{0}$, it follows from Eq.(16), the inequality (20) and the assumption (C3) that for each $n \geq n_{0}$,

$$
\begin{aligned}
0>-\lambda_{n} \int_{0}^{2 \pi} u_{n} \phi_{1}^{*} d x & =\int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x \\
& \geq \int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x
\end{aligned}
$$

that is,

$$
\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x>\int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x
$$

which is a contradiction with the second inequality in the assumption ( C 4 ). Therefore, all possible solutions of Eq.(9) (or equivalently Eq.(15)) are (uniformly) bounded in $W^{2,1}(0,2 \pi) \subset C^{0}([0,2 \pi])$ independently of $\lambda$, provided that $0<\lambda \leq \lambda_{0}$. The proof is complete.

Let us mention that a similar result holds for all $\lambda$ negative (and bounded away from zero). More precisely, we have the following uniform a priori bound.

Proposition 4.2. Let $\alpha_{0}, \alpha_{1} \in \mathbb{R}$ be (fixed negative) constants such that $-\infty<$ $\alpha_{0}<\alpha_{1}<0$. Suppose that the assumptions (C1)-(C2) hold. Then, there exists a constant $R_{0}:=R_{0}\left(\alpha_{0}, \alpha_{1}\right)>0$ such that all possible solutions of Eq.(9), with $\alpha_{0} \leq \lambda \leq \alpha_{1}$, satisfy

$$
|u|_{W^{2,1}(0,2 \pi)} \leq R_{0}
$$

That is, all possible solutions of Eq.(9) (or equivalently Eq.(15)) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ independently of $\lambda$, provided that $\alpha_{0} \leq \lambda \leq \alpha_{1}<0$.

The proof is similar to the one above up to Eq.(19) where now $\mu_{0} \in\left[\alpha_{0}, \alpha_{1}\right]$. However, since $\alpha_{1}<0$, it follows that $\mu_{0} \leq \alpha_{1}<0$ is in the resolvent of $L$ (see e.g. the second part of Remark 2.7), and hence $v \equiv 0$ on $[0,2 \pi]$. This is a contradiction with the fact that $|v|_{W^{2,1}(0,2 \pi)}=1$. Therefore, all possible solutions of Eq.(15) (or equivalently Eq.(9)) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ independently of $\lambda$, provided that $\alpha_{0} \leq \lambda \leq \alpha_{1}$. The proof is complete.

## 5. Proofs of main results

In this section we prove the main results by using the topological degree theory, continuation methods and bifurcation from infinity techniques. We first prove the existence part of the results, and then proceed to show multiplicity and bifurcation.

Proof of Theorem 3.1. First we consider the case when $\lambda \geq 0$ is fixed. Picking $\delta \in \mathbb{R}$ such that $0<\delta<\lambda_{0}$, and following the notation of the previous section, we consider the homotopy

$$
\begin{equation*}
L u+\delta u+\theta[(\lambda-\delta) u+\mathcal{N} u]=\theta h, \quad u \in \operatorname{Dom}(L) \tag{21}
\end{equation*}
$$

where $\theta \in[0,1)$; which, when $\theta=0$, reduces to the homogeneous linear problem $L u+\delta u=0$ that has only the trivial solution; for otherwise, Proposition 2.6 and an argument similar to that used after Eq.(19) would imply that $\delta=0$. Since the linear operator $L+\delta I$ defined by $L+\delta I: W_{P}^{2,1}(0,2 \pi) \rightarrow L^{1}(0,2 \pi)$ is bounded, one-to-one and onto (see e.g. the arguments used in the proof of Lemma 2.3), it follows that (21) is equivalent to the fixed point homotopy

$$
\begin{equation*}
u=\theta(L+\delta I)^{-1}((\delta-\lambda) I u-\mathcal{N} u+h), \quad u \in \operatorname{Dom}(L) \tag{22}
\end{equation*}
$$

Therefore, by the compactness of the imbedding $W_{P}^{2,1}(0,2 \pi)$ into $L^{1}(0,2 \pi)$ and the topological degree theory (see e.g. [10]), it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0,2 \pi)$, independently of $\theta \in[0,1)$, in order to conclude that Eq.(22) has at least one solution for $\theta=1$ as well.

Indeed, observing that $0<(1-\theta) \delta+\theta \lambda \leq \max \{\lambda, \delta\} \leq \lambda_{0}$ for $0 \leq \theta<$ 1, it follows from Proposition 4.1 that all possible solutions of Eq.(21) (or equivalently Eq.(22)) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ independently of $\theta \in[0,1)$. This proves the first part of Theorem 3.1. The second part of Theorem 3.1 follows readily from Proposition 4.1.

To prove the existence of at least one solution for $\lambda<0$ (fixed), we consider the homotopy (21) where $\delta<0$ and now $\theta \in[0,1]$. (Notice that $\theta=1$ is included here.) Observing that $\alpha_{0}:=\min \{\lambda, \delta\} \leq(1-\theta) \delta+\theta \lambda \leq \max \{\lambda, \delta\}:=$ $\alpha_{1}<0$ for $0 \leq \theta \leq 1$, it follows from Proposition 4.2 that all possible solutions
of Eq.(21) (or equivalently Eq.(22)) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ independently of $\theta \in[0,1]$. The existence of at least one solution for each $\theta \in[0,1]$ follows from topological degree arguments as above. (It should be noted that Assumptions (C3)-(C4) do not matter when $\lambda<0$, at least as far as the existence of at least one solution is concerned.) The proof is complete.

Now, we take up the issue of multiplicity and bifurcation (from infinity) of solutions for $\lambda$ "near" zero; actually $\lambda$ to the left of zero as it will be seen.

Proof of Theorem 3.2. We first show that all possible solutions of Eq.(15) are (uniformly) bounded in $W_{P}^{2,1}(0,2 \pi)$ when $\lambda=0$ as well; that is, the conclusion of Theorem 3.1 actually holds true for all $\lambda \in\left[0, \lambda_{0}\right]$. Indeed the proof is similar to that of Theorem 3.1 except that we consider the homotopy-parameter $\theta \in[0,1]$. Therefore, it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0,2 \pi)$ for $\theta=1$ and $\lambda=0$ as well. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda=0$ up to the inequality (20). Now, using the duality pairing with the eigenfunction $\phi_{1}^{*}$ in Eq.(16) (recall that $\theta=1$ and $\lambda=0$ ), and the fact that $\phi_{1}^{*}$ is an eigenfunction of $L^{*}$, it follows from Eq.(16), the inequality (20) and the (stronger) assumption on the functions $g$ and $A$ in Theorem 3.2 that for each $n \geq n_{0}$,

$$
0=\int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h \phi_{1}^{*} d x>\int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x
$$

that is,

$$
\int_{0}^{2 \pi} h(x) \phi_{1}^{*} d x>\int_{0}^{2 \pi} A(x) \phi_{1}^{*} d x
$$

This is a contradiction with the second inequality in the assumption (C4). Hence, all possible solutions of Eq.(15) are (uniformly) bounded in $W_{P}^{2,1}(0,2 \pi)$ for $\lambda=0$ as well. Thus, in this case, one gets the boundedness of all possible solutions in $W_{P}^{2,1}(0,2 \pi)$ as in Proposition 4.1.

Now, we proceed to look into the situation regarding multiplicity and bifurcation from infinity. As in the proof of Theorem 3.1, we let $\delta \in \mathbb{R}$ be sufficiently small such that $0<\delta<\lambda_{0}$, and observe that Eq.(15) is equivalent to the fixed point equation

$$
u=(\delta-\lambda)(L+\delta I)^{-1} u-(L+\delta I)^{-1}(\mathcal{N} u-h)
$$

Setting

$$
\mu:=\delta-\lambda, H u:=(L+\delta I)^{-1} u \text { and } K u:=-(L+\delta I)^{-1}(\mathcal{N} u-h),
$$

it follows that the above fixed point equation is equivalent to the equation

$$
\begin{equation*}
u=\mu H u+K(u), u \in C_{P}^{0}([0,2 \pi]) . \tag{23}
\end{equation*}
$$

Notice that Eq.(23) has now an abstract form considered e.g. in [17] for bifurcation from infinity purposes. From this setup, it follows that, when $\lambda=0$, the constant $\mu^{-1}=\delta^{-1}$ is the principal eigenvalue of $H$ and that, by the compact imbedding of $W_{P}^{2,1}(0,2 \pi)$ into $C_{P}^{0}([0,2 \pi])$, the solution-map

$$
H:=(L+\delta I)^{-1}: C_{P}^{0}([0,2 \pi]) \rightarrow W_{P}^{2,1}(0,2 \pi) \stackrel{\mathrm{c}}{\hookrightarrow} C_{P}^{0}([0,2 \pi])
$$

is a compact linear operator when considered as an operator from $C_{P}^{0}([0,2 \pi])$ into $C_{P}^{0}([0,2 \pi])$. Since by the Carathédory condition (C1) the superposition operator $\mathcal{N}: C_{P}^{0}([0,2 \pi]) \rightarrow L^{1}(0,2 \pi)$ (defined by $\left.\mathcal{N}(u):=g(\cdot, u(\cdot))\right)$ is continuous (by e.g. using Lebesgue Dominated Convergence Theorem) and $h \in L^{1}(0,2 \pi)$, one has that $\mathcal{N}(\cdot)+h$ maps $C_{P}^{0}([0,2 \pi])$ continuously into $L^{1}(0,2 \pi)$. Therefore

$$
K: C_{P}^{0}([0,2 \pi]) \rightarrow W_{P}^{2,1}(0,2 \pi) \stackrel{\mathrm{c}}{\hookrightarrow} C_{P}^{0}([0,2 \pi])
$$

is a completely continuous mapping when viewed as a nonlinear operator from $C_{P}^{0}([0,2 \pi])$ into $C_{P}^{0}([0,2 \pi])$.

Now, we wish to show that $K(u)=\mathrm{o}\left(|u|_{C_{P}^{0}([0,2 \pi])}\right)$ as $|u|_{C_{P}^{0}([0,2 \pi])} \rightarrow \infty$. Let us set $w=K(u)$ for $u \in C_{P}^{0}([0,2 \pi])$; that is, $w \in W_{P}^{2,1}(0,2 \pi)$ satisfies the operator equation $(L+\delta I) w=-\mathcal{N}(u)+h$ for $u \in C_{P}^{0}([0,2 \pi])$. By the arguments similar to those used in the proof of Lemma 2.2, there is a constant $C_{1}>0$ (independent of $u$ ) such that

$$
\begin{equation*}
|w|_{W_{P}^{2,1}(0,2 \pi)} \leq C_{1}\left(|g(\cdot, u(\cdot))|_{L^{1}(0,2 \pi)}+|h|_{L^{1}(0,2 \pi)}\right) \tag{24}
\end{equation*}
$$

Using the sublinear growth condition (C2), we first proceed to show that the real-valued function $|g(\cdot, u(\cdot))|_{L^{1}(0,2 \pi)}$ is a o $\left(|u|_{C_{P}^{0}([0,2 \pi])}\right)$ as $|u|_{C_{P}^{0}([0,2 \pi])} \rightarrow \infty$. Indeed, let $\varepsilon>0$ be given, it follows from the Carathéodory condition (C1) and the sublinearity assumption ( C 2 ) that there exist a constant $r_{\varepsilon}>0$ and a function $a_{\varepsilon} \in L^{1}(0,2 \pi) \backslash\{0\}$ such that for every $u \in C_{P}^{0}([0,2 \pi])$ one has

$$
|g(x, u(x))| \leq \frac{\varepsilon}{2}|u(x)| \leq \frac{\varepsilon}{2}|u|_{C_{P}^{0}([0,2 \pi])} \quad \text { a.e. where } \quad|u(x)| \geq r_{\varepsilon}
$$

and

$$
|g(x, u(x))| \leq\left|a_{\varepsilon}(x)\right| \quad \text { a.e. where } \quad|u(x)| \leq r_{\varepsilon}
$$

Picking $R_{\varepsilon}:=R(\varepsilon) \geq 2\left|a_{\varepsilon}\right|_{L^{1}(0,2 \pi)} / \varepsilon$, it follows that for $|u|_{C_{P}^{0}([0,2 \pi])} \geq R_{\varepsilon}$ one has

$$
\begin{equation*}
|g(\cdot, u(\cdot))|_{L^{1}(0,2 \pi)} /|u|_{C^{0}([0,2 \pi])} \leq \varepsilon \tag{25}
\end{equation*}
$$

This shows that for every $\varepsilon>0$ there is a constant $R_{\varepsilon}>0$ such that the inequality (25) holds provided $|u|_{C_{P}^{0}([0,2 \pi])} \geq R_{\varepsilon}$; that is, $|g(\cdot, u(\cdot))|_{L^{1}(0,2 \pi)}=$ $\mathrm{o}\left(|u|_{C_{P}^{0}([0,2 \pi])}\right)$ as $|u|_{C_{P}^{0}([0,2 \pi])} \rightarrow \infty$; which by using the inequality (24) implies that $|w|_{W^{2,1}(0,2 \pi)}=\mathrm{o}\left(|u|_{C_{P}^{0}([0,2 \pi])}\right)$ as $|u|_{C_{P}^{0}([0,2 \pi])} \rightarrow \infty$. Since $W_{P}^{2,1}(0,2 \pi)$
is continuously imbedded into $C_{P}^{0}([0,2 \pi])$ and $w=K(u)$, it follows that $|K(u)|_{C_{P}^{0}([0,2 \pi])}=\mathrm{o}\left(|u|_{C_{P}^{0}([0,2 \pi])}\right)$ for $|u|_{C_{P}^{0}([0,2 \pi])} \rightarrow \infty$; as needed.

Therefore, $\lambda=0$ is a bifurcation point from infinity since all assumptions of the bifurcation from infinity result are fulfilled (see e.g. [17, p. 465, Theorem 1.6 and Corollary 1.8], also see [20, 2]); that is, there exist two connected sets of solutions $\mathcal{C}^{+}, \mathcal{C}^{-} \subset \mathbb{R} \times C_{P}^{0}([0,2 \pi])$ with $\mathcal{C}^{+} \cap \mathcal{C}^{-}=\emptyset$ which are such that for every (sufficiently) small $\varepsilon>0, \mathcal{C}^{+} \cap U_{\varepsilon} \neq \emptyset, \mathcal{C}^{-} \cap U_{\varepsilon} \neq \emptyset$ where $U_{\varepsilon}:=\left\{(\lambda, u) \in \mathbb{R} \times C_{P}^{0}([0,2 \pi]):|\lambda|<\varepsilon,|u|_{C_{P}^{0}([0,2 \pi])}>1 / \varepsilon\right\}$. (Observe that, by the regularity of solutions, $u \in W_{P}^{2,1}(0,2 \pi)$ since it is a solution of the fixed point equation (23).)

Now, since all $2 \pi$-periodic solutions are uniformly bounded in $W^{2,1}(0,2 \pi)$ for all $\lambda \in\left[0, \lambda_{0}\right]$ (see Proposition 4.1 and the above bound in the case $\lambda=0$ ) and for all $\lambda \in\left[\alpha_{0}, \alpha_{1}\right]$ with $\alpha_{1}<0$ (see Proposition 4.2), there then exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_{-}<0$, such that for every $\varepsilon>$ 0 with $\varepsilon<\left|\lambda_{-}\right|$, there are two distinct solutions $\left(\lambda_{\varepsilon}^{+}, u_{\varepsilon}\right) \in C^{+}$and $\left(\lambda_{\varepsilon}^{-}, v_{\varepsilon}\right) \in$ $C^{-}$with $-\varepsilon<\lambda_{\varepsilon}^{ \pm}<0, u_{\varepsilon} \neq v_{\varepsilon}$ and $\min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\}>1 / \varepsilon$. Letting $\varepsilon \rightarrow 0^{+}$, it follows that $\lambda_{\varepsilon}^{ \pm} \rightarrow 0$ and $\min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\} \rightarrow$ $\infty$. The proof is complete.

Proof of Theorem 3.4. As in the proof of Theorem 3.2, we first show that all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ when $\lambda=$ 0 as well; that is, the conclusion of Theorem 3.1 actually holds true for all $\lambda \in\left[0, \lambda_{0}\right]$. As before, the proof is similar to that of Theorem 3.1 except that we consider the homotopy-parameter $\theta \in[0,1]$. Therefore, it suffices to show that all possible solutions of the homotopy (22) are bounded in $W^{2,1}(0,2 \pi)$ for $\theta=1$ and $\lambda=0$ as well. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda=0$ up to the inequality (20). Now, using the duality pairing with the eigenfunction $\phi_{1}^{*}$ in Eq.(16) and the fact that $\phi_{1}^{*}$ is an eigenfunction of $L^{*}$, it follows from Eq.(16) that for each $n \geq n_{0}$, $0=\int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h \phi_{1}^{*} d x$. The inequality (20), the assumption (C3), and Fatou's lemma imply that

$$
\begin{aligned}
0 & =\liminf _{n \rightarrow \infty} \int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h \phi_{1}^{*} d x \\
& \geq \int_{0}^{2 \pi} \liminf _{n \rightarrow \infty} g\left(x, u_{n}\right) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h \phi_{1}^{*} d x=\int_{0}^{2 \pi} g_{+}(x) \phi_{1}^{*} d x-\int_{0}^{2 \pi} h \phi_{1}^{*} d x
\end{aligned}
$$

that is, $\int_{0}^{2 \pi} h \phi_{1}^{*} d x \geq \int_{0}^{2 \pi} g_{+}(x) \phi_{1}^{*} d x$. This is a contradiction with the second inequality in the assumption (14) of Theorem 3.4. Therefore, all possible solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ for $\lambda=0$ as well. One
can now can proceed as in the proof of Theorem 3.2 to establish multiplicity and bifurcation from infinity. The proof is complete.

Proof of Theorem 3.5. As in the above proofs, we analyse more carefully the behavior of all possible solutions of Eq.(15) (or equivalently Eq.(9)) when $\lambda=0$. We first show that all possible non-constant solutions of Eq.(15) are (uniformly) bounded in $W^{2,1}(0,2 \pi)$ when $\lambda=0$. For that purpose, we follow the arguments in the proof of Proposition 4.1 with $\lambda=0$ up to the inequality (20) with $u_{n} \neq$ cst for all $n \geq n_{0}$ and $R=R_{1}$ is the first element of the sequence $\left\{R_{k}\right\}_{k>1}$ given in the statement of the theorem. Now, using the duality pairing with the eigenfunction $\phi_{1}^{*}$ in Eq.(16) and the fact that $\phi_{1}^{*}$ is an eigenfunction of $L^{*}$, it follows from Eq.(16) that for each $n \geq n_{0}, 0=\int_{0}^{2 \pi} g\left(x, u_{n}\right) \phi_{1}^{*} d x$. By using the (strict) positivity of the eigenfunction $\phi_{1}^{*}$ (see Proposition 2.5), the inequality (20) which implies the non-negativity of $g\left(\cdot, u_{n}(\cdot)\right)$ by the assumption in the theorem, we get that $g\left(\cdot, u_{n}(\cdot)\right) \equiv 0$ a.e. on $[0,2 \pi]$. This is a contradiction with the positivity assumption on $g$ in the theorem since $u_{n} \neq$ constant for all $n \geq n_{0}$ (i.e., $u_{n} \not \equiv R_{k}$ for some $k$ ). Thus, all possible non-constant solutions of Eq.(15) where $\lambda=0$ are (uniformly) bounded in $W^{2,1}(0,2 \pi)$. However, in this instance, large (in norm) constant solutions might occur in Eq.(15) when $\lambda=0$. The above argument shows that if they do occur, then they must necessarily be elements of the sequences $\left\{R_{k}\right\}$ or $\left\{r_{k}\right\}$ of real numbers given in the statement of the theorem (for $k$ large enough).

Since the sequences $\left\{R_{k}\right\}$ and $\left\{r_{k}\right\}$ are discrete sets, and the continua $\mathcal{C}^{+}$ and $\mathcal{C}^{+}$are connected, we deduce as in the proof of Theorem 3.2 that there exists a deleted left-neighborhood of 0 in $\mathbb{R}$; i.e., there is $\lambda_{-}<0$, such that for every $\varepsilon>0$ with $\varepsilon<\left|\lambda_{-}\right|$, there are two distinct solutions $\left(\lambda_{\varepsilon}^{+}, u_{\varepsilon}\right) \in C^{+}$and $\left(\lambda_{\varepsilon}^{-}, v_{\varepsilon}\right) \in C^{-}$with $-\varepsilon<\lambda_{\varepsilon}^{ \pm}<0, u_{\varepsilon} \neq v_{\varepsilon}, \min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\}>$ $1 / \varepsilon$. It follows that $\lambda_{\varepsilon}^{ \pm} \rightarrow 0$ and $\min \left\{\left|u_{\varepsilon}\right|_{C^{0}([0,2 \pi])},\left|v_{\varepsilon}\right|_{C^{0}([0,2 \pi])}\right\} \rightarrow \infty$ as $\varepsilon \rightarrow 0^{+}$. (Notice that these continua could 'connect' to the discrete set of large constant solutions, if any; i.e, oscillate on the left of $\lambda=0$ and 'bounce-off' theses discrete constant solutions as $\varepsilon \rightarrow 0$ !) The proof is complete.

REMARK 5.1. As indicated above, with the coefficient $b \in A C_{P}([0,2 \pi])=$ $W_{P}^{1,1}(0,2 \pi)$, we may replace the condition $A=B=0$ in Theorem 3.5 by a (slightly) more general condition where $B \leq A$ are possibly nonzero constants. In this case, in addition to assuming that the conditions (C1), (C2) and (C4) are met, we suppose that there exist sequences of real numbers $0 \gg r_{k}>$ $r_{k+1} \rightarrow-\infty$ and $0 \ll R_{k}<R_{k+1} \rightarrow \infty$ as $k \rightarrow \infty$ such that for all $k \in \mathbb{N}$,

$$
g\left(x, r_{k}\right)=B \quad \text { and } \quad g\left(x, R_{k}\right)=A \quad \text { for a.e. } x \in(0,2 \pi) \text { and }
$$

$g(x, u)>A \quad$ for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$ with $R_{k}<u<R_{k+1}$,
$g(x, u)<B \quad$ for a.e. $x \in(0,2 \pi)$ and all $u \in \mathbb{R}$ with $r_{k+1}<u<r_{k}$.

That is, the (bounce-off) oscillations of the nonlinearity occur with respect to the constants $A$ and $B$. Observe that the condition $\int_{0}^{2 \pi} h \phi_{1}^{*} d x=0$ is now replaced by the more general condition ( C 4$)$. The proof is similar to that of Theorem 3.5.

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# Global stability, or instability, of positive equilibria of $p$-Laplacian boundary value problems with $p$-convex nonlinearities 

Bryan P. Rynne

Dedicated to Jean Mawhin on the occasion of his 75th birthday
Abstract. We consider the parabolic, initial value problem

$$
\begin{align*}
v_{t} & =\Delta_{p}(v)+\lambda g(x, v) \phi_{p}(v), & & \text { in } \Omega \times(0, \infty), \\
v & =0, & & \text { in } \partial \Omega \times(0, \infty),  \tag{IVP}\\
v & =v_{0} \geqslant 0, & & \text { in } \Omega \times\{0\},
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$, for some integer $N \geqslant 1$, with smooth boundary $\partial \Omega, \phi_{p}(s):=|s|^{p-1} \operatorname{sgn} s, s \in \mathbb{R}$, and $\Delta_{p}$ denotes the $p$-Laplacian, with $p>\max \{2, N\}, v_{0} \in C^{0}(\bar{\Omega})$, and $\lambda>0$. The function $g: \bar{\Omega} \times[0, \infty) \rightarrow(0, \infty)$ is $C^{0}$ and, for each $x \in \bar{\Omega}$, the function $g(x, \cdot):[0, \infty) \rightarrow(0, \infty)$ is Lipschitz continuous and strictly increasing.

Clearly, (IVP) has the trivial solution $v \equiv 0$, for all $\lambda>0$. In addition, there exists $0<\lambda_{\min }(g)<\lambda_{\max }(g)$ such that:

- if $\lambda \notin\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$ then (IVP) has no non-trivial, positive equilibrium;
- there exists a closed, connected set of positive equilibria bifurcating from ( $\left.\lambda_{\max }(g), 0\right)$ and 'meeting infinity' at $\lambda=\lambda_{\min }(g)$.

We prove the following results on the positive solutions of (IVP):

- if $0<\lambda<\lambda_{\min }(g)$ then the trivial solution is globally asymptotically stable;
- if $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$ then the trivial solution is locally asymptotically stable and all non-trivial, positive equilibria are unstable;
- if $\lambda_{\max }(g)<\lambda$ then any non-trivial solution blows up in finite time.

Keywords: Global stability, positive equilibria, p-Laplacian. MS Classification 2010: 35K92.

## 1. Introduction

We consider the parabolic, initial-boundary value problem

$$
\begin{align*}
v_{t} & =\Delta_{p}(v)+\lambda g(x, v) \phi_{p}(v), & & \text { in } \Omega \times(0, \infty) \\
v & =0, & & \text { in } \partial \Omega \times(0, \infty)  \tag{1}\\
v & =v_{0} \geqslant 0, & & \text { in } \Omega \times\{0\},
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}$, for some integer $N \geqslant 1$, with smooth boundary $\partial \Omega, \phi_{p}(s):=|s|^{p-1} \operatorname{sgn} s, s \in \mathbb{R}$, and $\Delta_{p}$ denotes the $p$-Laplacian, with $p>\max \{2, N\}, v_{0} \in C^{0}(\bar{\Omega})$, and $\lambda>0$.

We suppose that $g: \bar{\Omega} \times[0, \infty) \rightarrow(0, \infty)$ is $C^{0}$ and, for each $x \in \bar{\Omega}$,

$$
\begin{align*}
g(x, \cdot) & :[0, \infty) \rightarrow(0, \infty) \text { is strictly increasing, }  \tag{2}\\
0<g_{0}(x) & :=g(x, 0)<g_{\infty}(x):=\lim _{\xi \rightarrow \infty} g(x, \xi), \text { and } g_{\infty} \in L^{\infty}(\Omega) \tag{3}
\end{align*}
$$

We also suppose that $g$ is Lipschitz with respect to $\xi$, in the following sense: for any $K>0$ there exists $L_{K}$ such that

$$
\begin{equation*}
\left|g\left(x, \xi_{1}\right)-g\left(x, \xi_{2}\right)\right| \leqslant L_{K}\left|\xi_{1}-\xi_{2}\right|, \quad x \in \bar{\Omega}, 0 \leqslant \xi_{1}, \xi_{2} \leqslant K \tag{4}
\end{equation*}
$$

We are interested in positive solutions of (1), so we introduce the following notation: $C_{+}^{0}(\bar{\Omega})$ (respectively $W_{0,+}^{1, p}(\Omega)$ ) denotes the set of $\omega \in C^{0}(\bar{\Omega})$ (respectively $\left.\omega \in W_{0}^{1, p}(\Omega)\right)$ with $\omega \geqslant 0$ on $\Omega$.

It is known that for any $v_{0} \in C_{+}^{0}(\bar{\Omega})$ and fixed $\lambda>0$ the problem (1) has a unique, positive solution $t \rightarrow v_{\lambda g, v_{0}}(t) \in W_{0,+}^{1, p}(\Omega)$, on some maximal interval $(0, T)$, where we may have $T<\infty$ or $T=\infty$ (what we mean by a solution will be made precise in Theorem 4.1 below). We are interested in the asymptotic behaviour of these solutions. This asymptotic behaviour is determined by the structure of the set of positive equilibria of (1), so we first describe this.

For a given $\lambda>0$, a positive equilibrium is a time-independent solution $u \in W_{0,+}^{1, p}(\Omega)$ of (1), that is, $u$ satisfies $\Delta_{p}(u)+\lambda g(u) \phi_{p}(u)=0$ (this will be made precise in Section 3 below). For convenience, we also call $(\lambda, u)$ an equilibrium. For any $\lambda>0$ the function $v \equiv 0($ or $(\lambda, v)=(\lambda, 0))$ is a (trivial) equilibrium. Regarding non-trivial equilibria, we have the following results (see Theorem 3.1 below for a more precise description). There exists $0<\lambda_{\min }(g)<\lambda_{\max }(g)<\infty$ such that:

- if $\lambda \notin\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$ then (1) has no non-trivial, positive equilibrium in $W_{0,+}^{1, p}(\Omega)$;
- there exists a closed, connected set of positive equilibria $(\lambda, e)$ bifurcating from $\left(\lambda_{\max }(g), 0\right)$ in $\mathbb{R} \times W_{0,+}^{1, p}(\Omega)$ and 'meeting infinity' at $\lambda=\lambda_{\min }(g)$.

In some radially symmetric cases, when $\Omega$ is a ball, it is known that when $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$ there is a unique, non-trivial equilibrium $e_{\lambda} \in W_{0,+}^{1, p}(\Omega)$. This is discussed briefly in Section 6 below.

We will prove the following results on the asymptotic behaviour of the positive solutions of (1). For any $0 \neq v_{0} \in C_{+}^{0}(\bar{\Omega})$ :

- if $0<\lambda<\lambda_{\min }(g)$ then $\lim _{t \rightarrow \infty}\left\|v_{\lambda g, v_{0}}(t)\right\|_{0, p}=0$
(so the trivial solution is globally asymptotically stable);
- if $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$ then:
- if $v_{0}$ is 'small' then $\lim _{t \rightarrow \infty}\left\|v_{\lambda g, v_{0}}(t)\right\|_{0, p}=0$
(so the trivial solution is locally asymptotically stable);
- if $v_{0}$ is 'large' then $\lim _{t \rightarrow \infty}\left|v_{\lambda g, v_{0}}(t)\right|_{0}=\infty$;
- all the non-trivial, positive equilibria are unstable;
- if $\lambda_{\max }(g)<\lambda$ then there exists $T<\infty$ such that $\lim _{t \nmid T}\left|v_{\lambda g, v_{0}}(\cdot)\right|_{0}=\infty$.

These results are consistent with a bifurcation analysis of the corresponding semilinear ( $p=2$ ) problem, using the 'principle of linearised stability' to obtain local stability. Such problems have been extensively investigated, see [9] and the references therein for a summary of the main results. However, we do not use bifurcation theory to obtain our results, which usually yields local stability results. Instead, we use a mixture of comparison and compactness arguments to obtain the above results.

For the quasilinear problem involving the $p$-Laplacian operator considered here, these results are consistent with the results on 'linearised stability' in the ' $p$-convex' case in [10] (condition (2) is termed ' $p$-convex' in [10]; this terminology has been used in other publication for very similar, but slightly different, conditions). However, the term 'linearised stability' in [10] refers to the sign of the principal eigenvalue of the linearisation of the problem at an equilibrium solution, not to the dynamic (time-dependent) stability that we consider. For the quasilinear problem considered here it is not clear that 'linearised stability', in this sense, implies stability in the usual dynamic sense. Even if such a result could be proved, it would give local rather than global stability.

Similar results to those obtained here have been obtained in $[3,4]$ for a quasilinear problem involving the mean-curvature operator in 1-dimension. The mean-curvature operator is significantly different to the $p$-Laplacian operator considered here, so our results do not follow from those of $[3,4]$, even in 1 dimension.

## 2. Preliminaries

### 2.1. Notation

We let $C^{0}(\bar{\Omega})$ denote the standard space of real valued, continuous functions defined on $\bar{\Omega}$, with the standard sup-norm on $|\cdot|_{0}$ (throughout, all function spaces will be real); $L^{q}(\Omega), q>1$, denotes the standard space of functions on $\Omega$ whose $q$ th power is integrable, with norm $\|\cdot\|_{q} ; W_{0}^{1, p}(\Omega)$ denotes the standard, first order Sobolev space of functions on $\bar{\Omega}$ which are zero on $\partial \Omega$, with norm $\|\cdot\|_{1, p}$, and its dual space is denoted by $W^{-1, p^{\prime}}(\Omega)$, where $p^{\prime}:=p /(p-1)$ is the conjugate exponent of $p$. By our assumption that $p>N$, the space $W_{0}^{1, p}(\Omega)$ is compactly embedded into $C^{0}(\bar{\Omega})$.

If $h: \bar{\Omega} \times[0, \infty) \rightarrow \mathbb{R}$ is continuous then, for any $\omega \in C_{+}^{0}(\bar{\Omega})$, we define $h(\omega) \in C_{+}^{0}(\bar{\Omega})$ by

$$
h(\omega)(x):=h(x, \omega(x)), \quad x \in \bar{\Omega} .
$$

Clearly, the 'Nemitskii' mapping $\omega \rightarrow h(\omega): C_{+}^{0}(\bar{\Omega}) \rightarrow C_{+}^{0}(\bar{\Omega})$ is continuous. In particular, we repeatedly use the Nemitskii mapping $\phi_{p}: \omega \rightarrow \phi_{p}(\omega)$ : $C_{+}^{0}(\bar{\Omega}) \rightarrow C_{+}^{0}(\bar{\Omega})$.

### 2.2. The $p$-Laplacian

Formally, the $p$-Laplacian is defined by

$$
\Delta_{p} \omega:=\nabla \cdot\left(|\nabla \omega|^{p-2} \nabla \omega\right),
$$

for suitable $\omega$, where $|\boldsymbol{v}|:=\left(v_{1}^{2}+\cdots+v_{N}^{2}\right)^{1 / 2}$ for $\boldsymbol{v} \in \mathbb{R}^{N}$. More precisely, for any $\omega \in W_{0}^{1, p}(\Omega)$, we define $\Delta_{p}(\omega) \in W^{-1, p^{\prime}}(\Omega)$ by

$$
\begin{equation*}
\int_{\Omega} \Delta_{p}(\omega) \varphi:=-\int_{\Omega}|\nabla \omega|^{p-2} \nabla \omega \cdot \nabla \varphi, \quad \forall \varphi \in W_{0}^{1, p}(\Omega) \tag{5}
\end{equation*}
$$

A precise definition of what is meant by a solution of (1) will be given in Section 4.1 below.

### 2.3. Principal eigenvalues of the $p$-Laplacian

We briefly consider the weighted, nonlinear eigenvalue problem

$$
\begin{equation*}
-\Delta_{p}(\psi)=\mu \rho \phi_{p}(\psi), \quad \psi \in W_{0}^{1, p}(\Omega) \tag{6}
\end{equation*}
$$

where $\mu \in \mathbb{R}$ and the weight function $\rho \in L^{1}(\Omega)$. We say that $\mu$ is an eigenvalue of (6), with eigenfunction $\psi \in W_{0}^{1, p}(\Omega) \backslash\{0\}$, if the following weak formulation
of (6) holds

$$
\begin{equation*}
\int_{\Omega}|\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \varphi=\mu \int_{\Omega} \rho \phi_{p}(\psi) \varphi, \quad \forall \varphi \in W_{0}^{1, p}(\Omega) \tag{7}
\end{equation*}
$$

A principal eigenvalue of (6) is an eigenvalue $\mu_{0}$ which has a positive eigenfunction $\psi_{0} \in W_{0,+}^{1, p}(\Omega)$ (which we will normalise by, say, $\left|\psi_{0}\right|_{0}=1$ ). The following result is well known - see, for example, [6, Sections 3-4].
Lemma 2.1. Suppose that the weight function $\rho$ satisfies: $\rho \geqslant 0$ on $\Omega$, with $\rho>0$ on a set of positive Lebesgue measure. Then the eigenvalue problem (6) has a unique principal eigenvalue $\mu_{0}(\rho)$. This eigenvalue has the properties, $\mu_{0}(\rho)>0, \psi_{0}(\rho)>0$ on $\Omega$, and

$$
\begin{equation*}
\int_{\Omega}|\nabla \omega|^{p} \geqslant \mu_{0}(\rho) \int_{\Omega} \rho|\omega|^{p}, \quad \forall \omega \in W_{0}^{1, p}(\Omega) \tag{8}
\end{equation*}
$$

In addition, if $\rho_{1}, \rho_{2}$ are two such weight functions, then

$$
\begin{aligned}
\rho_{1} \leqslant \rho_{2} \text { on } \Omega \text { and } \rho_{1} & <\rho_{2} \text { on a set of positive Lebesgue measure } \\
& \Longrightarrow \mu_{0}\left(\rho_{1}\right)>\mu_{0}\left(\rho_{2}\right) .
\end{aligned}
$$

Now, since $g_{\infty} \in L^{\infty}(\Omega)$, we may define

$$
0<\lambda_{\min }(g):=\mu_{0}\left(g_{\infty}\right)<\lambda_{\max }(g):=\mu_{0}\left(g_{0}\right)
$$

and we denote the corresponding eigenfunctions by $\psi_{\min }(g), \psi_{\max }(g)$.

## 3. Non-trivial, positive equilibria of (1)

A positive equilibrium of (1) is a solution of the problem

$$
\begin{equation*}
-\Delta_{p}(u)=\lambda g(u) \phi_{p}(u), \quad u \in W_{0,+}^{1, p}(\Omega) \tag{9}
\end{equation*}
$$

More precisely, a solution of (9) is defined to be a function $u \in W_{0,+}^{1, p}(\Omega)$ which satisfies the following weak formulation of (9),

$$
\begin{equation*}
\int_{\Omega}|\nabla u|^{p-2} \nabla u \cdot \nabla \varphi=\lambda \int_{\Omega} g(u) \phi_{p}(u) \varphi, \quad \forall \varphi \in W_{0}^{1, p}(\Omega) \tag{10}
\end{equation*}
$$

For convenience, we also call $(\lambda, u)$ an equilibrium.
Clearly, for any $\lambda \in \mathbb{R}$, the function $u=0$ is a (trivial) positive equilibrium. We denote the set of non-trivial, positive equilibria by

$$
\mathcal{E}^{+}:=\left\{(\lambda, u): \lambda \in(0, \infty), 0 \neq u \in W_{0,+}^{1, p}(\Omega) \text { satisfies }(9)\right\}
$$

We can say somewhat more about the overall structure of the set $\mathcal{E}^{+}$. In fact, we have the following global-bifurcation-type description of $\mathcal{E}^{+}$.

THEOREM 3.1. $(a)(\lambda, u) \in \mathcal{E}^{+} \Longrightarrow \lambda \in\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$ and $u>0$ on $\Omega$.
(b) If $\left(\lambda_{n}, u_{n}\right) \in \mathcal{E}^{+}, n=1,2, \ldots$, then

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \lambda_{n}=\lambda_{\min }(g) \Longleftrightarrow \lim _{n \rightarrow \infty}\left\|u_{n}\right\|_{1, p}=\infty \\
& \lim _{n \rightarrow \infty} \lambda_{n}=\lambda_{\max }(g) \Longleftrightarrow \lim _{n \rightarrow \infty}\left\|u_{n}\right\|_{1, p}=0 .
\end{aligned}
$$

(c) There exists a set $\mathcal{S}^{+} \subset \mathcal{E}^{+}$such that $\mathcal{S}^{+} \cup\left(\lambda_{\max }(g), 0\right)$ is closed and connected, and

$$
\begin{align*}
P_{\mathbb{R}} \mathcal{S}^{+} & :=\left\{\lambda:(\lambda, u) \in \mathcal{S}^{+}, \text {for some } 0 \neq u \in W_{0,+}^{1, p}(\Omega)\right\}  \tag{11}\\
& =\left(\lambda_{\min }(g), \lambda_{\max }(g)\right) .
\end{align*}
$$

Proof. (a) These results follow immediately from (2), (3), Lemma 2.1 and the definitions of $\lambda_{\min }(g)$ and $\lambda_{\max }(g)$, together with the form of equation (9).
(b) Consider a sequence $\left(\lambda_{n}, u_{n}\right) \in \mathcal{E}^{+}, n=1,2, \ldots$, such that

$$
\lim _{n \rightarrow \infty} \lambda_{n}=\lambda_{\infty} \in\left[\lambda_{\min }(g), \lambda_{\max }(g)\right] \quad \text { and } \quad \lim _{n \rightarrow \infty}\left\|u_{n}\right\|_{1, p}=N_{\infty}
$$

(i) Suppose that $0<N_{\infty}<\infty$. Then, by the compactness properties described on p. 299 of [7], we may suppose that there exists $0 \neq u_{\infty} \in$ $W_{0,+}^{1, p}(\Omega)$ such that $\left\|u_{n}-u_{\infty}\right\|_{1, p} \rightarrow 0$ and $\left(\lambda_{\infty}, u_{\infty}\right) \in \mathcal{E}^{+}$. Part (a) now implies that $\lambda_{\min }(g)<\lambda_{\infty}<\lambda_{\max }(g)$.
(ii) Suppose that $N_{\infty}=\infty$. By defining $w_{n}:=u_{n} /\left\|u_{n}\right\|_{1, p}, n=1,2, \ldots$, we may suppose (by compactness and our assumption that $g_{\infty} \in L^{\infty}(\Omega)$ ) that there exists $0 \neq w_{\infty} \in W_{0,+}^{1, p}(\Omega)$ such that $\left\|w_{n}-w_{\infty}\right\|_{1, p} \rightarrow 0$ and

$$
\begin{align*}
-\Delta_{p}\left(w_{\infty}\right) & =\lambda_{\infty} \bar{g} \phi_{p}\left(w_{\infty}\right) \\
\bar{g}(x) & =\lim _{n \rightarrow \infty} g\left(x, u_{n}(x)\right), \quad x \in \Omega \tag{12}
\end{align*}
$$

By (2) and (3), $0<\bar{g} \leqslant g_{\infty} \in L^{\infty}(\Omega)$, so by Lemma 2.1 and (12), $w_{\infty}(x)>0$ for each $x \in \Omega$, so that $u_{n}(x) \rightarrow \infty$, and $\bar{g}(x)=g_{\infty}(x)$. Hence, $\lambda_{\infty}=\lambda_{\text {min }}(g)$.
(iii) Suppose that $0=N_{\infty}$. A similar (slightly simpler) argument to that of part ( $i i$ ) shows that in this case $\lambda_{\infty}=\lambda_{\max }(g)$.

Combining the results of $(i)$-(iii) now proves part (b) of the theorem.
(c) We will use the Rabinowitz-type global bifurcation results in [7] to prove this. To do this it is convenient to extend the domain of $g$ in (9) to $\Omega \times \mathbb{R}$, by setting $g(x,-\xi)=-g(x, \xi), x \in \Omega, \xi>0$. Clearly, this has no effect on the positive solutions of (9).

Let $\mathcal{N} \subset \mathbb{R} \times W_{0}^{1, p}(\Omega)$ denote the set of non-trivial solutions of (9), with $\overline{\mathcal{N}}$ its closure, and let $\mathcal{S}$ denote the (maximal) connected component of $\overline{\mathcal{N}}$ containing $\left(\lambda_{\max }(g), 0\right)$. The results in [7] (in particular, [7, Theorem 1.1] and [7, Lemma 3.1]) show that $\mathcal{S}$ is unbounded in $\mathbb{R} \times W_{0}^{1, p}(\Omega)$ and has the decomposition

$$
\mathcal{S}=\left\{\left(\lambda_{\max }(g), 0\right)\right\} \cup \mathcal{S}^{+} \cup \mathcal{S}^{-}
$$

where

$$
\mathcal{S}^{ \pm}:=\{(\lambda, w) \in \mathcal{S}: \pm w(x)>0 \text { for all } x \in \Omega\}
$$

We note that there are some very minor differences between equation (9) and the problem discussed in [7]. For instance our $g_{0}$ depends on $x$ but the corresponding term in [7] is constant. However, it can be seen that the results from [7] that we use are still valid in our case.

Clearly, $\mathcal{S}^{+} \subset \mathcal{E}^{+}$. Furthermore, it follows from the form of our extended function $g$ in (9) that $\mathcal{S}^{+}=-\mathcal{S}^{-}$, so both the sets $\mathcal{S}^{ \pm}$must be unbounded, and the sets $\left\{\left(\lambda_{\max }(g), 0\right)\right\} \cup \mathcal{S}^{ \pm}$are connected. The relation (11) now follows from the connectedness and unboundedness of $\mathcal{S}^{+}$, together with the results of parts $(a),(b)$ of the theorem. This proves part (c), and so completes the proof of Theorem 3.1.

## 4. Time-dependent solutions of (1)

In Section 3 we discussed equilibrium (time-independent) solutions of (1). In this section we will discuss time-dependent solutions of (1). We first describe an existence and uniqueness result, and then a comparison result, which will be used to determine the long-time behaviour of the solutions.

### 4.1. Existence and uniqueness of positive solutions

Existence and uniqueness properties of solutions of the time-dependent problem (1) are known, and the results that we require were summarised in [14, Section 3]. We will briefly restate these results here - for further details see [14], and the references therein.

To state precisely what we mean by a solution of (1) we define the spaces $\Sigma(T):=C\left([0, T), L^{2}(\Omega)\right) \cap C\left((0, T), W_{0}^{1, p}(\Omega)\right) \cap W_{\mathrm{loc}}^{1,2}\left((0, T), L^{2}(\Omega)\right), \quad T>0$
(we allow $T=\infty$ here, and likewise for other such numbers below). The space $W^{1,2}\left((0, T), L^{2}(\Omega)\right)$ is defined on p. 378 of [13], using the notation $H^{1}\left((0, T), L^{2}(\Omega)\right)$; the loc version can be defined by a simple adaptation of this definition. We will search for a solution of (1) in $\Sigma(T)$, for some $T>0$. Thus, in this setting, a solution $v$ will be regarded as a time-dependent mapping $t \rightarrow v(t):(0, T) \rightarrow W_{0}^{1, p}(\Omega)$, with $\Delta_{p}(v(t)) \in W^{-1, p^{\prime}}(\Omega)$ defined in a
weak sense, for each $t \in(0, T)$ (see [14]), and satisfying the initial condition at $t=0$ as a limit in $L^{2}(\Omega)$. More (or less) regularity at $t=0$ can be attained, depending on the regularity of $v_{0}$ (for example if $v_{0} \in W_{0,+}^{1, p}(\Omega)$ then the solution will belong to $C\left([0, T), W_{0}^{1, p}(\Omega)\right)$ ), but the above setting will suffice here.

In view of this, we will rewrite (1) in the form

$$
\begin{equation*}
\frac{d v}{d t}=\Delta_{p}(v)+\lambda g(v) \phi_{p}(v), \quad v(0)=v_{0} \in C_{+}^{0}(\bar{\Omega}) \tag{13}
\end{equation*}
$$

The following theorem summarises known results on the existence and uniqueness of solutions of (13), together with various additional properties which will be required below. For details and references, see the proofs of Theorem 3.1 and Corollary 3.4 in [14], together with the discussion in [5], which also describes most of these results, with further explanations. We note that the theorem does not require $g$ to satisfy the monotonicity condition (2).

THEOREM 4.1. Suppose that $g$ satisfies conditions (3) and (4) on $\bar{\Omega} \times[0, \infty$ ), and $\lambda>0, v_{0} \in C_{+}^{0}(\bar{\Omega})$. Then (13) has a unique solution $v_{\lambda g, v_{0}} \in \Sigma\left(T_{\lambda g, v_{0}}\right)$, defined on a maximal interval $\left[0, T_{\lambda g, v_{0}}\right)$, for some $T_{\lambda g, v_{0}}>0$, having the following properties:
(a) $v_{\lambda g, v_{0}}(0)=v_{0}$ and $v_{\lambda g, v_{0}}(t) \in W_{0,+}^{1, p}(\Omega)$ for all $t \in\left(0, T_{\lambda g, v_{0}}\right)$;
(b) the function $v_{\lambda g, v_{0}}:\left[0, T_{\lambda g, v_{0}}\right) \rightarrow L^{2}(\Omega)$ is differentiable at almost all $t \in\left[0, T_{\lambda g, v_{0}}\right)$, and at such $t$,

$$
\frac{d v_{\lambda g, v_{0}}}{d t}(t), \Delta_{p}\left(v_{\lambda g, v_{0}}(t)\right) \in L^{2}(\Omega)
$$

and

$$
\frac{d v_{\lambda g, v_{0}}}{d t}(t)=\Delta_{p}\left(v_{\lambda g, v_{0}}(t)\right)+\lambda g\left(v_{\lambda g, v_{0}}(t)\right) \phi_{p}\left(v_{\lambda g, v_{0}}(t)\right), \quad \text { in } L^{2}(\Omega) ;
$$

(c) the interval $\left[0, T_{\lambda g, v_{0}}\right.$ ) on which the solution $v_{\lambda g, v_{0}}$ exists is maximal, in the sense that

$$
\begin{equation*}
T_{\lambda g, v_{0}}<\infty \Longrightarrow \lim _{t \succ T_{\lambda g, v_{0}}}\left|v_{\lambda g, v_{0}}(t)\right|_{0}=\infty \tag{14}
\end{equation*}
$$

If $T_{\lambda g, v_{0}}<\infty$ then the solution $v_{\lambda g, v_{0}}$ is said to blow up in finite time.

### 4.2. Comparison results

We now consider the auxiliary problem

$$
\begin{equation*}
\frac{d w}{d t}=\Delta_{p}(w)+\lambda \gamma \phi_{p}(w), \quad w(0)=w_{0} \in C_{+}^{0}(\bar{\Omega}) \tag{15}
\end{equation*}
$$

where $\gamma \in L^{\infty}(\Omega)$ is independent of $w$, and $\gamma \geqslant 0$ on $\Omega$. This is a special case of (13) (with $g(x, \xi)$ having the form $\gamma(x)$ ) so, by Theorem 4.1, the problem (15) has a unique solution $w_{\lambda \gamma, w_{0}}$ defined on a maximal interval $\left[0, T_{\lambda \gamma, w_{0}}\right)$.
Remark 4.2. Theorem 4.1 was stated for continuous functions $g$ depending on $(x, \xi)$ (and Lipschitz with respect to $\xi$ ), but as noted in [14, Remark 3.3], the result is valid for the problem (15), containing an $x$-dependent function $\gamma \in L^{\infty}(\Omega)$.

We now describe a 'comparison' result for solutions of (13) and (15). For any $T>0$ and functions $\omega_{1}, \omega_{2} \in \Sigma(T)$, we write $\omega_{1} \geqslant \omega_{2}$ on $[0, T)$ if $\omega_{1}(t) \geqslant \omega_{2}(t)$, on $\bar{\Omega}$, for each $t \in[0, T)$. Also, in inequalities involving $\gamma$, we may regard $\gamma$ as a function on $\bar{\Omega} \times[0, \infty)$ which is constant with respect to $\xi \in[0, \infty)$.
LEMmA 4.3. (a) If $g \geqslant \gamma \geqslant 0$ on $\bar{\Omega} \times[0, \infty)$ and $v_{0} \geqslant w_{0} \geqslant 0$ on $\bar{\Omega}$, then

$$
T_{\lambda g, v_{0}} \leqslant T_{\lambda \gamma, w_{0}} \quad \text { and } \quad v_{\lambda g, v_{0}} \geqslant w_{\lambda \gamma, w_{0}} \quad \text { on }\left[0, T_{\lambda g, v_{0}}\right) .
$$

(b) If $0 \leqslant g \leqslant \gamma$ on $\bar{\Omega} \times[0, \infty)$ and $v_{0} \leqslant w_{0}$ on $\bar{\Omega}$, then

$$
T_{\lambda g, v_{0}} \geqslant T_{\lambda \gamma, w_{0}} \quad \text { and } \quad v_{\lambda g, v_{0}} \leqslant w_{\lambda \gamma, w_{0}} \quad \text { on }\left[0, T_{\lambda \gamma, w_{0}}\right)
$$

Proof. The proof follows, with minor modifications, the proof of [12, Theorem 2.5]. We omit the details. However, we note that [12, Theorem 2.5] considers equations of the form $v_{t}=\Delta_{p}(v)+\lambda \phi_{p}(v)$, but the proof can be adapted to give the above result; the argument in [12] is based on the proof of [8, Lemma 3.1, Ch. VI], which considered the equation $v_{t}=\Delta_{p}(v)$.

In the next section we will use the comparison result Lemma 4.3 to describe the behaviour of solutions of (13). The following results will be useful for this.
Lemma 4.4. Suppose that $0 \neq w_{0} \in C_{+}^{0}(\bar{\Omega})$.
(a) If $\lambda<\mu_{0}(\gamma)$ then $T_{\lambda \gamma, w_{0}}=\infty$ and $\lim _{t \rightarrow \infty}\left\|w_{\lambda \gamma, w_{0}}(t)\right\|_{1, p}=0$.
(b) If $\lambda>\mu_{0}(\gamma)$, then $T_{\lambda \gamma, w_{0}}<\infty$.

Proof. (a) By following the proof of [12, Theorem 3.1], it can be shown that $T_{\lambda \gamma, w_{0}}=\infty$ and $\left|w_{\lambda \gamma, w_{0}}(\cdot)\right|_{0}$ is bounded on [0, $\infty$ ) (the paper [12] deals with the case $\gamma \equiv 1$ but the extension to the case of general $\gamma$ is straightforward, using a comparison theorem similar to Lemma 4.3, which is, as noted above, based on [12, Theorem 2.5]).

The argument in the proof of part $(a)$ of [14, Theorem 4.1] now shows that $w_{\lambda \gamma, w_{0}}$ must converge, in $W_{0,+}^{1, p}(\Omega)$, to an equilibrium solution of equation (6), with $\mu=\lambda$ and $\rho=\gamma$. But by assumption, $\lambda<\mu_{0}(\gamma)$, so Lemma 2.1 shows that the only equilibrium available is the trivial solution.
(b) This can be proved by following the proof of [12, Theorem 3.5].

## 5. Global stability or instability of the equilibria of (1)

For any $\lambda>0$ the time-dependent problem (13) has the trivial equilibrium solution $u=0$, and also, by Theorem 3.1, for any $\lambda \in\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$ there is at least one non-trivial, positive equilibrium. We will now consider the stability, and instability, of these equilibria.

Theorem 5.1. Suppose that $0 \neq v_{0} \in C_{+}^{0}(\bar{\Omega})$.
(a) $0<\lambda<\lambda_{\min }(g) \Longrightarrow T_{\lambda g, v_{0}}=\infty \quad$ and $\quad \lim _{t \rightarrow \infty}\left\|v_{\lambda g, v_{0}}(t)\right\|_{1, p}=0$.
(b) If $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$ and $e_{\lambda} \in \mathcal{E}^{+}$then:
(i) $\alpha<1$ and $v_{0}<\alpha e_{\lambda} \Longrightarrow T_{\lambda g, v_{0}}=\infty$ and $\lim _{t \rightarrow \infty}\left\|v_{\lambda g, v_{0}}(t)\right\|_{1, p}=0$;
(ii) $\beta>1$ and $v_{0}>\beta e_{\lambda} \Longrightarrow T_{\lambda g, v_{0}}<\infty$.
(c) $\lambda_{\max }(g)<\lambda \Longrightarrow T_{\lambda g, v_{0}}<\infty$.

Proof. Parts (a) and (c). The proofs of these parts of the theorem are simple modifications of the proofs of parts $(a)$ and $(c)$ of [14, Theorem 4.1]. We note that, for each $x \in \Omega$, the function $g(x, \cdot)$ is decreasing in [14], whereas it is increasing here, so the roles of $g_{0}$ and $g_{\infty}$, and $\mu_{0}\left(g_{0}\right)$ and $\mu_{0}\left(g_{\infty}\right)$, need to be interchanged in the comparison arguments used here, compared to those used in [14].

Part (b)-(i). We define $\widetilde{g}^{\alpha-}: \bar{\Omega} \times[0, \infty) \rightarrow(0, \infty)$ by

$$
\widetilde{g}^{\alpha-}(x, \xi):= \begin{cases}g\left(x, \alpha e_{\lambda}(\xi)\right), & \xi>\alpha e_{\lambda}(x),  \tag{16}\\ g(x, \xi), & \xi \leqslant \alpha e_{\lambda}(x)\end{cases}
$$

(and $\widetilde{g}_{\infty}^{\alpha-}$ will denote the limit of $\widetilde{g}^{\alpha-}$ as $\xi \rightarrow \infty$, as in (3)). Since $e_{\lambda}$ satisfies (9) we see, by scaling $e_{\lambda}$, that the function $w=\alpha e_{\lambda}$ satisfies the equation

$$
\begin{equation*}
-\Delta_{p}(w)=\lambda g\left(e_{\lambda}\right) \phi_{p}(w) \tag{17}
\end{equation*}
$$

that is, $\alpha e_{\lambda}$ is an equilibrium solution of (15), with $\gamma=g\left(e_{\lambda}\right)$. Also, by (2) and (16), $\widetilde{g}^{\alpha-} \leqslant \widetilde{g}_{\infty}^{\alpha-} \leqslant g\left(e_{\lambda}\right)$ on $\bar{\Omega} \times[0, \infty)$, and by assumption, $v_{0}<\alpha e_{\lambda}$, so by Lemma 4.3

$$
\begin{equation*}
v_{\lambda \tilde{g}^{\alpha-}, v_{0}}(t) \leqslant \alpha e_{\lambda}, \quad \text { on }[0, \infty) \tag{18}
\end{equation*}
$$

It follows immediately from (18) that $T_{\lambda \tilde{g}^{\alpha-}, v_{0}}=\infty$ (by Theorem 4.1), and $v_{\lambda g, v_{0}}=v_{\lambda \widetilde{g}^{\alpha-}, v_{0}}$ (by (16) and uniqueness of solutions).

Next, by (2) and (16), $\widetilde{g}_{\infty}^{\alpha-}<g\left(e_{\lambda}\right)$ on $\Omega$, so by (17) and Lemma 2.1,

$$
\lambda=\mu_{0}\left(g\left(e_{\lambda}\right)\right)<\mu_{0}\left(\widetilde{g}_{\infty}^{\alpha-}\right)=\lambda_{\min }\left(\widetilde{g}^{\alpha-}\right)
$$

Thus, part (a) of the theorem applies to the solution $v_{\lambda \tilde{g}^{\alpha-}, v_{0}}$, and since we have just shown that $v_{\lambda g, v_{0}}=v_{\lambda \widetilde{g}^{\alpha-}, v_{0}}$, this proves part $(b)-(i)$ of the theorem.

Part (b)-(ii). We now define $\widetilde{g}^{\beta+}: \bar{\Omega} \times[0, \infty) \rightarrow(0, \infty)$ by

$$
\tilde{g}^{\beta+}(x, \xi):= \begin{cases}g(x, \xi), & \xi \geqslant \beta e_{\lambda}(x)  \tag{19}\\ g\left(x, \beta e_{\lambda}(\xi)\right), & \xi<\beta e_{\lambda}(x)\end{cases}
$$

In this case the function $w=\beta e_{\lambda}$ satisfies (17), and a similar argument to that in the proof of part $(b)-(i)$ now shows that

$$
\begin{equation*}
v_{\lambda \tilde{g}^{\beta+}, v_{0}}(t) \geqslant \beta e_{\lambda} \quad \text { on }\left[0, T_{\lambda \tilde{g}^{\beta+}, v_{0}}\right), \tag{20}
\end{equation*}
$$

and hence, $v_{\lambda g, v_{0}}=v_{\lambda \widetilde{g}^{\beta+}, v_{0}}$. Also, by (2) and (19), $\widetilde{g}_{0}^{\beta+}>g\left(e_{\lambda}\right)$ on $\Omega$, so by (17) and Lemma 2.1,

$$
\lambda=\mu_{0}\left(g\left(e_{\lambda}\right)\right)>\mu_{0}\left(\widetilde{g}_{0}^{\beta+}\right)=\lambda_{\max }\left(\widetilde{g}^{\beta+}\right) .
$$

Thus, part (c) of the theorem applies to the solution $v_{\lambda g, v_{0}}=v_{\lambda \widetilde{g}^{\beta+}, v_{0}}$, and so proves part $(b)-(i i)$ of the theorem. This completes the proof of Theorem 5.1.

Part (b) of Theorem 5.1 shows that if $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$ then every non-trivial, positive equilibrium $e_{\lambda} \in \mathcal{E}^{+}$is unstable, and the trivial solution is not globally asymptotically stable. It also gives an indication of the global asymptotic behaviour of the positive solutions of (13), viz. if $v_{0}$ is 'large' then $v_{\lambda g, v_{0}}$ blows up in finite time, and if $v_{0}$ is 'small' then $v_{\lambda g, v_{0}}(t) \rightarrow 0$ as $t \rightarrow$ $\infty$. However, this result does not deal with all initial conditions $v_{0} \in C_{+}^{0}(\bar{\Omega})$. Specifically, it does not deal with any initial condition $v_{0}$ which 'crosses' all the non-trivial, positive equilibria. More unfortunately, it does not prove the stability of the trivial solution, in the sense that there are initial conditions $v_{0}$ with arbitrarily small norm (either $\left|v_{0}\right|_{0}$ or $\left\|v_{0}\right\|_{1, p}$ ) which do not satisfy the hypothesis in part $(b)-(i)$ of the theorem (for arbitrarily small $\epsilon$ there exist $v_{0}$ with $\left|v_{0}\right|_{0}<\epsilon$, but with $v_{0}(x)>e_{\lambda}(x)$ for $x$ near the boundary $\left.\partial \Omega\right)$. The following theorem rectifies some of these omissions, and proves stability of the trivial solution when $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$.

Theorem 5.2. Suppose that $\lambda_{\min }(g)<\lambda<\lambda_{\max }(g)$. Then there exists $\epsilon>0$ such that

$$
\begin{equation*}
\left|v_{0}\right|_{0}<\epsilon \Longrightarrow T_{\lambda g, v_{0}}=\infty \quad \text { and } \quad \lim _{t \rightarrow \infty}\left\|v_{\lambda g, v_{0}}(t)\right\|_{1, p}=0 \tag{21}
\end{equation*}
$$

Proof. For $\delta>0$, define $g_{\delta} \in C^{0}(\bar{\Omega})$ by

$$
g_{\delta}(x):=g(x, \delta), \quad x \in \bar{\Omega} .
$$

It follows from the properties of $g$, and the principal eigenvalue function $\mu_{0}(\cdot)$ (see Lemma 2.1 and [6]), that

$$
\begin{aligned}
g_{\delta}>g_{0} \text { on } \Omega & \text { and } \lim _{\delta>0}\left|g_{\delta}-g_{0}\right|_{0}=0 \\
& \Longrightarrow \mu_{0}\left(g_{\delta}\right)<\mu_{0}\left(g_{0}\right) \text { and } \lim _{\delta \searrow 0} \mu_{0}\left(g_{\delta}\right)=\mu_{0}\left(g_{0}\right)
\end{aligned}
$$

(the final limiting result is not explicitly stated in [6], but it can readily be proved using the minimisation characterisation of $\mu_{0}(\rho)$ in (1.3) of [6]; the argument is similar to the proof of [6, Proposition 4.3]). Hence, since $\lambda<$ $\lambda_{\max }(g)=\mu_{0}\left(g_{0}\right)$, we may choose $\delta$ sufficiently small that $\lambda<\mu_{0}\left(g_{\delta}\right)$.

Now, defining the function $\mathbf{1} \in C_{+}^{0}(\bar{\Omega})$ by $\mathbf{1}(x):=1, x \in \bar{\Omega}$, it follows from Lemma 4.4 (a) that

$$
\begin{equation*}
T_{\lambda g_{\delta}, \mathbf{1}}(t)=\infty \quad \text { and }\left|w_{\lambda g_{\delta}, \mathbf{1}}(t)\right|_{0} \rightarrow 0 \tag{22}
\end{equation*}
$$

Since the mapping $t \rightarrow\left|w_{\lambda g_{\delta}, \mathbf{1}}(t)\right|_{0}$ is continuous on $[0, \infty)$, we may define

$$
\begin{aligned}
\kappa & :=\max \left\{\left|w_{\lambda g_{\delta}, \mathbf{1}}(t)\right|_{0}: t \geqslant 0\right\}, \quad \epsilon:=\delta / \kappa, \\
\tilde{w}_{\epsilon}(x, t) & :=\epsilon w_{\lambda g_{\delta}, \mathbf{1}}\left(x, \epsilon^{p-2} t\right), \quad(x, t) \in \Omega \times[0, \infty),
\end{aligned}
$$

and we see that

$$
\begin{aligned}
\frac{d \tilde{w}_{\epsilon}}{d t} & =\epsilon^{p-1} \frac{d w_{\lambda g_{\delta}, \mathbf{1}}}{d t}=\epsilon^{p-1}\left(\Delta_{p}\left(w_{\lambda g_{\delta}, \mathbf{1}}\right)+\lambda g_{\delta} \phi_{p}\left(w_{\lambda g_{\delta}, \mathbf{1}}\right)\right) \\
& =\Delta_{p}\left(\tilde{w}_{\epsilon}\right)+\lambda g_{\delta} \phi_{p}\left(\tilde{w}_{\epsilon}\right), \\
\tilde{w}_{\epsilon} & =\epsilon \mathbf{1}, \quad\left|\tilde{w}_{\epsilon}(t)\right|_{0} \leqslant \delta, \quad t \geqslant 0
\end{aligned}
$$

Furthermore, since $g(x, \xi) \leqslant g_{\delta}(x)$ on $\bar{\Omega} \times[0, \delta]$, a similar comparison argument to that used in the proof of Theorem 5.1 (b) (i) now shows that

$$
\left|v_{0}\right|_{0}<\epsilon \Longrightarrow 0 \leqslant v_{\lambda g, v_{0}}(t) \leqslant \tilde{w}_{\epsilon}(t) \leqslant \delta, \quad t \geqslant 0
$$

which, by (22), proves that (21) holds with the $|\cdot|_{0}$ norm. It follows from this, by the argument in the proof of [14, Theorem 4.1], that (21) holds with the $\|\cdot\|_{1, p}$ norm, which completes the proof of Theorem 5.2.

## 6. Uniqueness of non-trivial, positive equilibria

The question of the uniqueness of the non-trivial, positive equilibria when $\lambda \in\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$, under conditions similar to our basic condition (2),
is clearly of interest here, so we briefly describe some recent results concerning this question. This problem has received considerable attention, but is still a long way from being resolved. The main results have been obtained for the case where $\Omega$ is a ball, say the unit ball $B_{1} \subset \mathbb{R}^{N}$, and the function $g$ is radially symmetric, that is, $g$ has the form $g(r, \xi)$, where $r$ denotes the usual Euclidean norm $|x|$ in $\mathbb{R}^{N}$. For simplicity, we only discuss the case where $g$ has the form $g(\xi)$.

We first observe that in this case, given our hypotheses on $g$, [2, Lemma 2] shows that any non-trivial solution $u \in W_{0,+}^{1, p}(\Omega)$ of (9) must be radially symmetric, that is, $u=u(r)$, with $u(1)=0$. Thus, the question of the uniqueness of the non-trivial solutions of the $\operatorname{PDE}(9)$ on $B_{1}$ reduces to considering the uniqueness of the solutions of an ODE problem on the interval $[-1,1]$. Of course, if we have such uniqueness then Theorem 5.1 (b) applies to the full PDE problem on the ball $B_{1} \subset \mathbb{R}^{N}$.

We now briefly describe some of the known results for this case, which apply to our problem.
The case $\mathrm{N}=1$.
This case is considered in [10], under the following hypothesis.

- The nonlinearity $g(\xi) \xi^{p-1}$ is 'strictly $p$-convex', as defined in [10, Definition 3] (which implies that (2) holds, see [10, Remark 6]).
Theorems 1 and 2 in [10] show that if $\lambda \in\left(\lambda_{\min }(g), \lambda_{\max }(g)\right)$ then (9) has a unique solution $e_{\lambda} \in W_{0,+}^{1, p}(\Omega)$ (these theorems combined cover all combinations of $\left.0 \leqslant \lambda_{\min }(g)<\lambda_{\max }(g) \leqslant \infty\right)$.

The case $\mathrm{N}>1$.
This case is considered in $[1,11]$. The results as stated in these papers do not quite cover the problem considered here, but by a slight adaptation of the arguments in [1] a uniqueness result can be obtained under the following hypotheses (in our notation):

- the function $\xi \rightarrow \xi g^{\prime}(\xi) / g(\xi)$ is increasing on $(0, \infty)$;
- $g^{\prime}(\xi)>0$ on $(0, \infty)$ (which implies that (2) holds).


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# The semilinear wave equation with non-monotone nonlinearity: a review 

Francisco Caicedo, Alfonso Castro, Rodrigo Duque and Arturo Sanjuán

Dedicated to Professor Jean Mawhin on his 75th birthday


#### Abstract

We review recent results on the existence of weak $2 \pi$ periodic solutions in time and space for a class of semilinear wave equations with non-monotone nonlinearity. Similar results exist for Dirichlet-periodic boundary conditions but, for the sake of clarity, we exclude them in this presentation.


Keywords: semilinear wave equation, bifurcation, operator range.
MS Classification 2010: 34B15, 35J65.

## 1. Introduction

In the study of the range of semilinear operators $L+N$, finding weak solutions to the wave equation

$$
\begin{equation*}
\square u+g(u):=u_{t t}-u_{x x}+g(u)=f(x, t) \tag{1}
\end{equation*}
$$

subject to double-periodic conditions

$$
\begin{equation*}
u(x, t)=u(x, t+2 \pi)=u(x+2 \pi, t) \text { for all } x, t \in \mathbb{R} \tag{2}
\end{equation*}
$$

provides a rich source of open questions. Up to minor modifications, the results here reviewed extend to (1) subject to the Dirichlet-periodic condition

$$
\begin{equation*}
u(0, t)=u(\pi, t)=0, \quad u(x, t)=u(x, t+2 \pi) \quad \text { for all } x \in(0, \pi), t \in \mathbb{R} \tag{3}
\end{equation*}
$$

Professor Jean Mawhin is a pioneer in this field. His work points out the role of the interaction of the numerical range of $N$ (i. e., the range of $g^{\prime}$ ) with the spectrum of $L$ (i.e., the spectrum of $-\square$, subject to either (2) or (3)), in the solvability of these problems. Such spectra are given by $\sigma(-\square)=\left\{j^{2}-\right.$ $\left.k^{2} ; j, k=0,1, \ldots\right\}$ for condition (2) and by $\sigma_{d}(-\square)=\left\{j^{2}-k^{2} ; j=0,1, \ldots ; k=\right.$ $1,2, \ldots\}$ for condition (3).

For example, from [22] it follows that if $g$ is monotone and

$$
\begin{equation*}
\left[\liminf _{|u| \rightarrow \infty} \frac{g(u)}{u}, \limsup _{|u| \rightarrow \infty} \frac{g(u)}{u}\right] \cap \sigma(-\square)=\emptyset, \tag{4}
\end{equation*}
$$

then (1)-(2), as well as (1)-(3), has a solution. The same result may be obtained from related developments in $[4,7,8,25,27,29,30]$. Arguing as in Theorem 3 of [14] one sees that (4) may be extended to

$$
\begin{equation*}
\liminf _{|u| \rightarrow \infty} \frac{g(u)}{u} \in\left(\lambda_{k}, \lambda_{k+1}\right), \quad \text { and } \quad \limsup _{|u| \rightarrow \infty} \frac{g(u)}{u}<\nu\left(\liminf _{|u| \rightarrow \infty} \frac{g(u)}{u}\right) \tag{5}
\end{equation*}
$$

where $\nu(a)>a, a \notin \sigma(-\square)$ is the smallest value for which $\square u+a u_{+}-\nu(a) u_{-}=$ 0 subject to (2) has a weak solution. That is $(a, \nu(a))$ belongs to the Fucik spectrum of $\square$ subject to (2).

Similar results occur in systems and wave equations in several space variables, see $[2,4,5,24,32,33]$. All these works assume the range of $g^{\prime}$ not to include eigenvalues of infinite multiplicity in its interior. Note that only 0 is an eigenvalue of infinite multiplicity both for (1)-(2) and (1)-(3).

When the periodicity condition (2) is replaced by

$$
\begin{equation*}
u(x, t)=u(x, t+2 \pi)=u(x+L, t) \text { for all } x, t \in \mathbb{R} \tag{6}
\end{equation*}
$$

and $L$ is not a rational multiple of $\pi$ the spectrum $\sigma(\square)$ may have multiple eigenvalues of infinite multiplicity and may not be a discrete. Here again professor Mawhin is a pioneer in the field with his work in [23, 21]. For additional analysis of this case the reader is referred to [28]. Little is known on the solvability of (1)-(6) when $L$ is not a rational multiple of $\pi$. In [9] existence results for cases where $\sigma(\square)$ is discrete and all the eigenvalues have finite multiplicity are found including cases where the range of $g^{\prime}$ may include multiple eigenvalues of infinite multiplicty.

If in (1) we replace $\square$ by an elliptic operator, $N$ need not be monotone as compactness arguments based on the absence of eigenvalues of infinite multiplicity suffice.

From now on we let $\Omega:=(0,2 \pi) \times(0,2 \pi)$ and

$$
\begin{align*}
\alpha_{k, j}(x, t) & =\sin (k x) \cos (j t), \quad \beta_{k, j}(x, t)=\sin (k x) \sin (j t),  \tag{7}\\
\gamma_{k, j}(x, t) & =\cos (k x) \cos (j t), \quad \text { and } \quad \delta_{k, j}(x, t)=\cos (k x) \sin (j t) .
\end{align*}
$$

Let $K$ be the closed subspace of $L^{2}(\Omega)$ spanned by

$$
\left\{\alpha_{k, k}, \beta_{k, k}, \gamma_{k, k}, \delta_{k, k} ; k=0,1,2, \ldots\right\} .
$$

That is, $K$ is the null space of the wave operatorsubject to (2). If $v \in K$ then there are unique $2 \pi$-periodic null-average functions $v_{1}$ and $v_{2}$ and a unique number $\bar{v}$ such that $v(x, t)=\bar{v}+v_{1}(t+x)+v_{2}(t-x)$.

We let $H$ denote the Sobolev space of functions $u$ such that $u$ as well as its first order partial derivatives belong to $L^{2}(\Omega)$. The norm in $L^{2}(\Omega)$ is denoted by $\|\|$ and the norm in $H$ by $\| \|_{1}$. We let $Y=K^{\perp} \cap H$. We say that $u=y+v \in Y \oplus K$ is a weak solution of (1)-(2) if

$$
\begin{equation*}
\int_{\Omega}\left\{\left(y_{t} \hat{y}_{t}-y_{x} \hat{y}_{x}\right)-(g(u)-f)(\hat{y}+\hat{v})\right\} d x d t=0 \tag{8}
\end{equation*}
$$

for all $\hat{y}+\hat{v} \in Y \oplus K$.

## 2. Existence of forced vibrations

In $[20,35]$ it was established that$+N$ subject to (2) has dense range in $L^{2}(\Omega)$ when $g^{\prime}(u) \cap \sigma(-\square)=\emptyset$ for $u$ large. That is, for all $f$ in a dense subset of $L^{2}(\Omega)$, the equation (1)-(2) has a weak solution. Note that here it is not assumed $g$ to be monotone. More precisely, if there are constants $\alpha, \beta, c \in \mathbb{R}$, $\alpha \leq \beta$, such that $\sigma(\square) \cap[\alpha, \beta]=\emptyset$, that $g: \mathbb{R} \rightarrow \mathbb{R}$ is globally Lipschitz continuous, and

$$
\begin{equation*}
-c+\frac{\alpha}{2} s^{2} \leq \int_{0}^{s} g(t) d t \leq c+\frac{\beta}{2} s^{2} \quad \text { for all } s \in \mathbb{R} \tag{9}
\end{equation*}
$$

then (1)-(2) has a solution for each $f$ in a dense set of $L^{2}(\Omega)$. However, to date, it is not known if such a range (the set of all such $f$ 's) is all of $L^{2}(\Omega)$.

The arguments in $[20,35]$ do not provide a characterization of the $f$ 's for which (1)-(2) has a solution. Nevertheless, in [12, 15, 16], sufficient conditions for $f$ to be in the range of $u \mapsto \square(u)+g(u)$ are provided when

$$
\begin{equation*}
g(s)=\lambda s+h(s), \quad \text { with } \quad-\lambda \notin \sigma(\square) \text { and } \quad \lim _{|u| \rightarrow+\infty} h^{\prime}(u)=0 . \tag{10}
\end{equation*}
$$

It is readily verified that functions satisfying (10) satisfy (9).
In order to find sufficient conditions on $f$ for (1)-(2), or (1)-(3), to have a solution the concept of functions flat on characteristics was introduced in [16]

Definition 2.1. We say that $\phi$ is not flat on characteristics if given $\epsilon>0$ there exists $\delta>0$ such that $m(\{x \in[0, \pi] ;|\phi(x, r \pm x)-\rho|<\delta\})<\epsilon$ for all $r, \rho \in \mathbb{R}$, where $m$ stands for the one dimensional Lebesgue measure.

In [12, Theorem 5.1] the following was proven.
Theorem 2.2. Let $-\lambda \notin \sigma(\square)$ and $f(x, t)=c q(x, t) \in L^{p}(\Omega), p \geq 2$ and $\phi$ the solution to $\square(\phi)+\lambda \phi=q(x, t), \phi(x, t)=\phi(x+2 \pi, t)=\phi(x, t+2 \pi), x, t \in \mathbb{R}$. If $\phi$ is not flat on characteristics then there exist $c_{0}$ such that for $|c| \geq c_{0}$ the equation (1)-(2) has a weak solution $u \in L^{p}(\Omega)$.

Earlier versions of Theorem 2.2 are found in $[15,16]$ where the existence of bounded solutions is considered. The proofs in $[12,15,16]$ are based on first establishing the existence of approximate solutions and then establishing the convergence of such approximations using the compactness of $(\square+\lambda I)^{-1}$ on the range of $\square$ and convergence in $K$ in $L^{p}$ using that the projection on $K$ of such approximations are large due to the size of the parameter $c$.

## 3. Non-existence of continuous solutions

If $\left|g^{\prime}\right|$ is bounded away from 0 (hence $g$ is strictly monotone) and $f$ is smooth, in $[7,29]$ it is shown smoothness of $f$ implies smoothness of solutions to (1)-(2). As credited by P. Rabinowitz in [29], the ideas for showing such regularity go back to L. Nirenberg.

On the other hand, for non-monotone nonlinearities one cannot expect regularity of the solutions as shown by the following theorem and lemma.

Theorem 3.1. Assume that $h(s)=g(s)-\lambda s$ is a differentiable function with support in $[0, D]$ for some $D>0$, that $\lambda>0$, that $-\lambda \notin \sigma(\square)$ and that $h^{\prime}(D / 2)<-\lambda D / 2$. Then there is $c_{0}>0$ such that if $|c|>c_{0}$ the problem (1)-(2) has no continuous solution for $f(x, t)=c \sin (x+t)$.

For the proof of Theorem 3.1 the reader is referred to [10, Theorem 2.1].
In contrast with Theorem 3.1 we have the following existence result.
Lemma 3.2. Let

$$
g(t)= \begin{cases}\tau_{1} t+h(t) & \text { if } t \leq 0  \tag{11}\\ \tau_{2} t+h(t) & \text { if } t>0\end{cases}
$$

with $\tau_{1}, \tau_{2}>0$, and $h$ continuous such that

$$
\begin{equation*}
\lim _{|s| \rightarrow \infty} \frac{h(s)}{s}=0 \tag{12}
\end{equation*}
$$

If $f(x, t)=p(x+t)$ or $f(x, t)=p(x-t)$, with $p: \mathbb{R} \rightarrow \mathbb{R}, p \in L^{2}[0,2 \pi]$, and $p(\xi+2 \pi)=p(\xi)$ for all $\xi \in \mathbb{R}$, then the equation (1)-(2) has a solution.

Note that the above lemma allows for resonance $\left(-\tau_{1},-\tau_{2} \in \sigma(\square)\right)$ and jumping nonlinearities $\left(\tau_{1} \neq \tau_{2}\right)$. Its proof goes as follows. One lets

$$
\Gamma=\{\gamma: \mathbb{R} \rightarrow \mathbb{R} ; \gamma \text { is increasing, continuous and } \gamma(t) \leq g(t) \text { for all } t \in \mathbb{R}\} .
$$

and

$$
\begin{equation*}
g_{1}(t):=\sup _{\gamma \in \Gamma} \gamma(t) . \tag{13}
\end{equation*}
$$

The function $g_{1}$ is continuous, non-decreasing, for all $\alpha \in \mathbb{R}$ the set $g^{-1}(\alpha)$ is a closed interval, and if $g(a)<g(t)$ for all $t>a$ then $g(a)=g_{1}(a)$.

For each $\xi \in \mathbb{R}$ there exists $a_{\xi}, b_{\xi} \in \mathbb{R}$ such that $\left.g_{1}^{-1}(\{\xi)\}\right)=\left[a_{\xi}, b_{\xi}\right]$. Given $f(x, t)=p(x+t)$, we define $v(s):=b_{p(s)}$. Due to $\tau_{1}>0, \tau_{2}>0$, and (12), $v \in L^{2}(0,2 \pi)$. Also $g(v(\xi))=p(\xi)$. Thus $u(x, t)=v(x+t) \in K$ and is a weak solution to (1)-(2). These solutions may have jump discontinuities along characteristic lines where $g_{1}^{-1}$ is not single valued. Furthermore, such solutions need not be unique. For example, if $p(s)=\xi$ is constant in a segment $[c, d]$, and $a_{\xi}<b_{\xi}$ then defining, for any $y \in(c, d), v_{y}(\zeta)=a_{p(\zeta)}$ for $\zeta \in[c, y)$, $v_{y}(\zeta)=b_{p(\zeta)}$ for $\zeta \in(y, d]$, and $u_{y}(x, t)=v_{y}(x+t)$ we have a continuum of solutions to (1)-(2).

## 4. Bifurcation

Finally we consider, subject to the periodicity condition (2), the one parameter equation

$$
\begin{equation*}
u_{t t}-u_{x x}+g(x, t, u, \lambda)=0, x, t, u, \lambda \in R \tag{14}
\end{equation*}
$$

with $g(x, t, u)=g(x+2 \pi, t)=g(x, t+2 \pi)$. If $g(x, t, u, \lambda)=\lambda G(x, t, u)$, $G(x, t, u)=0$, and $G_{u}(x, t, 0)=1$ one sees that $\left(0, \lambda_{k}\right)$ is a point of bifurcation for every $\lambda_{k} \in \sigma(-\square)$. More precisely, there is a connected set of nonzero solutions to (14)-(2) containing $\left(0, \lambda_{k}\right)$ in its closure. This fact is proven imitating the arguments for the case in which $\square$ is replaced by a second elliptic operator when $\lambda_{k} \neq 0$, and a more detailed analysis for $\lambda_{k}=0$ as shown in [31].

Bifurcation from infinity. Recently, bifurcation from infinity was considered in [13] resulting in the following theorem.

TheOrem 4.1. Let $-\lambda_{0} \in \sigma(\square), h: \mathbb{R} \rightarrow \mathbb{R}$ a bounded continuous function. Suppose there exists $M>0, \gamma>1$, and $A>0$ such that

$$
\begin{equation*}
\left|h^{\prime}(s)\right| \leq|s|^{-\gamma} \text { for all }|s| \geq M, \text { and } \lim _{s \rightarrow \pm \infty} h(s)= \pm A . \tag{15}
\end{equation*}
$$

If $g(s)=\lambda s+h(s)$, then there is $\epsilon_{0}$ such that if $0<\lambda_{0}-\lambda<\epsilon_{0}$ the problem (1)-(2) has a nontrivial weak solution $u_{\lambda}=v_{\lambda}+y_{\lambda} \in(K \oplus Y) \cap L^{\infty}(\Omega)$. Furthermore, if $\lambda \rightarrow \lambda_{0}$, then $\left\|v_{\lambda}\right\|+\left\|y_{\lambda}\right\|_{1} \rightarrow \infty$.

For $\lambda_{0} \neq 0$ the proof of Theorem 4.1 relies on the properties of sets of the form $\{(x, t) ;|p(x, t)|<\epsilon\}$, for $p$ a trigonometric polynomial of a given degree, using the Nazarov-Turan lemma, see [19]. The case $\lambda_{0}=0$, relies on the fact that constant functions belongs to the kernel $K$. This case does not extend to the boundary condition (3) due to the absence of constant functions in the kernel of $\square$ subject to this boundary condition.

Imperfect bifurcation. In [11], see also [6], the equation (14) for

$$
\begin{equation*}
g(x, t, u, \lambda)=\lambda(u+\lambda H)^{2 k}+\lambda R(t, x, u+\lambda H) \tag{16}
\end{equation*}
$$

subject to (2) and assuming that

$$
\begin{equation*}
\lim _{v \rightarrow 0} \frac{R_{v}(t, x, v)}{v^{2 k-1}}=0, \quad \text { and } k \text { a positive integer } \tag{17}
\end{equation*}
$$

is considered. Sufficient conditions on $H \neq 0$ are provided for the existence of solutions that accumulate at $(0,0)$. Since $H \neq 0$ this is known as imperfect bifurcation.

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# Remarks on nonautonomous bifurcation theory 

Matteo Franca and Russell Johnson<br>Dedicated to 75th birthday of Professor Jean Mawhin

Abstract. We study some elementary bifurcation patterns when the bifurcation parameter is subjected to fast oscillations.

Keywords: Nonautonomous bifurcation, Atkinson problem, Hale-Fink averaging. MS Classification 2010: 37B55, 34D35, 34C45, 34F10.

Foreword Professor Russell Johnson suddenly passed away on the 22 of July 2017, just after this manuscript had been accepted in its final form. Matteo Franca had the luck of being one of his PhD students and to profit of his vast knowledge and thoughtful advices since then, and wishes to take the chance to commemorate the excellent mathematician and magister to which he is indebted in several different ways.

## 1. Introduction

Problems of fast oscillations are of intrinsic mathematical interest and are of great importance in various areas of applied mathematics. It is well-known that, when the parameters of a mechanical system are subject to rapid oscillations, then the stability characteristics of that system may change in a substantial way. The interesting phenomenon of stabilization of a planar pendulum via vertical oscillations is a case in point [4], but it is far from being the only example. Others include the elimination of a Van der Pol oscillation, and the large-scale alteration of a stability diagram in a catalytic reactor [2, 3]. As a somewhat different example, we mention the quadruple ion trap (Paul trap), which is used as a component of a mass spectrometer.

In many problems of fast oscillations, one has a single parameter to deal with, namely the frequency of the oscillation. But in other problems there are one or more additional parameters, and these other parameters may determine a bifurcation pattern. In this paper we pose the question of how the presence of rapidly oscillating parameter disturbance can alter a bifurcation
pattern. We will give partial answers to this question in the context of the simplest bifurcation patterns, namely the saddle-node, transcritical, pitchfork, and Andronov-Hopf scenarios. We will emphasize the case in which the fast oscillations are almost periodic in the sense of Bohr, or more generally of "strictly ergodic" type. Of course this is considerably more general than considering periodic fast oscillations. Thus for example we can deal with a situation in which a parameter is perturbed by two or more periodic terms whose periods are not commensurate, rather than by a single periodic term.

We are motivated in particular by the discussion in the paper [3], where the authors provide insights and methods which are valuable for the study of problems involving fast oscillations and stability. Incidentally, the authors of that paper seem rather dismissive of the scenario in which a parameter is slowly varied, and of that in which a parameter undergoes a stochastic disturbance; see also [2].

The starting point of this discussion is the problem

$$
\begin{equation*}
x^{\prime}=x^{2}-\varepsilon \quad(x \in \mathbb{R}, \varepsilon \in \mathbb{R}) \text {. } \tag{1}
\end{equation*}
$$

Clearly equation (1) admits equilibria in $x= \pm \sqrt{\varepsilon}$ when $\varepsilon>0$, where $x=-\sqrt{\varepsilon}$ is asymptotically stable and $x=+\sqrt{\varepsilon}$ is unstable. These merge as $\varepsilon$ decreases through zero, and for $\varepsilon<0$ all the solutions of (1) are unbounded in finite time.

Let us now modify (1) by adding in a rapidly oscillating term, to obtain

$$
\begin{equation*}
x^{\prime}=x^{2}-\varepsilon+\frac{\alpha}{\mu} f\left(\frac{t}{\mu}\right) \quad \quad \quad=\frac{d}{d t} \tag{2}
\end{equation*}
$$

where $\alpha$ is a positive number, $\mu$ is a small positive parameter, and $f$ is an almost periodic function with zero mean value. Note that the oscillations are not only rapid, but also indefinitely large as $\mu \rightarrow 0^{+}$.

To study the effect of the oscillatory term on the bifurcation pattern, we set $\tau=\frac{t}{\mu}$, so that (2) takes the form

$$
\frac{d x}{d \tau}=\mu\left(x^{2}-\varepsilon\right)+\alpha f(\tau)
$$

Following [2], we consider

$$
\begin{equation*}
\frac{d x}{d \tau}=\alpha f(\tau) \tag{3}
\end{equation*}
$$

Let us assume that $f(\tau)$ admits an almost periodic primitive $F(\tau)$; this is actually a highly nontrivial hypothesis, unless $f(\tau)$ is assumed to be periodic. There is no loss of generality in assuming that $F$ has mean value zero. Then the general solution of $(3)$ is of course

$$
\begin{equation*}
x=c+\alpha F(\tau)=h(\tau, c) \tag{4}
\end{equation*}
$$

where $c$ is an arbitrary constant. Still following [2], we make the change of variables

$$
x=h(\tau, y)=y+\alpha F(\tau),
$$

which transforms (2) into

$$
\begin{equation*}
\frac{d y}{d \tau}=\mu\left\{[y+\alpha F(\tau)]^{2}-\varepsilon\right\} \tag{5}
\end{equation*}
$$

At this point we apply the method of infinite-interval averaging see Fink [10] or Hale [15]. We will discuss the method in Section 2; here we merely state the result.

There is a neighborhood $W$ of $y=0$, which can be fixed independently of small $\mu>0$, and an invertible change of variables in $w \in W$

$$
y=w+\mu G(\tau, w)
$$

such that, in the new variable $w,(5)$ goes into

$$
\begin{equation*}
\frac{d w}{d \tau}=\mu\left\{w-\varepsilon+\alpha^{2} \overline{F^{2}}\right\}+o(\mu) \tag{6}
\end{equation*}
$$

Here the overbar indicates the mean value of the given function $F$. We note explicitly that $\overline{F^{2}}$ is the mean value of $F^{2}$; further the function $G$ can be written explicitly:

$$
G(\tau, w)=\mathrm{e}^{-\mu \tau} \int_{-\infty}^{\tau} \mathrm{e}^{\mu s}\left[2 \alpha F(s) w+\alpha^{2} F^{2}(s)\right] d s
$$

One can show that, if $\varepsilon>0$ is fixed, $\varepsilon>\alpha^{2} \overline{F^{2}}$, and $\mu$ is chosen small enough (in dependence of $\varepsilon$ ), then (6) admits two almost periodic solutions, one of which is asymptotically stable and the other of which is unstable. If $\varepsilon<\alpha^{2} \overline{F^{2}}$, then for small $\mu$, all solutions of (6) will leave some fixed neighborhood $W$ of $w=0$ in finite time.

At this point one must note that there has been a macroscopic change in the bifurcation pattern in (6) as compared to (2), namely the bifurcation point $\varepsilon=0$ in (2) is transferred to $\varepsilon=\alpha^{2} \overline{F^{2}}$ in (6). Moreover, the very term "bifurcation pattern" is in the first moment not very well-defined, because the effect of the $o(\mu)$ terms on solutions of (6) may be quite pronounced if $\mu$ is not "small enough" relative to $\varepsilon-\alpha^{2} \overline{F^{2}}$, which of course takes its most interesting values near zero.

One may not find the " $\frac{\alpha}{\mu}$ "-factor in front of $f$ in (2) to be natural. Let us omit it and carry out the above calculations with $f$ in place of $\frac{\alpha}{\mu} f$. We obtain the analogue of (6):

$$
\begin{equation*}
\frac{d w}{d \tau}=\mu\left\{w-\varepsilon+\mu^{2} \overline{F^{2}}\right\}+o\left(\mu^{2}\right) \tag{7}
\end{equation*}
$$

The bifurcation value of the averaged equation is $\varepsilon=\mu^{2} \overline{F^{2}}$, which tends to zero as $\mu \rightarrow 0^{+}$. However there is still an interesting issue as far as the $\mu$-dependent terms are concerned, which we formulate as follows: set $\mu=\mu(\varepsilon)$ and ask "what happens" to the saddle-node pattern. We will take up this question in Section 3: rather remarkably it admits a reasonably clean-cut answer.

The reader may object at this point that one should be able to understand (2) in detail by applying repeated averaging, to obtain after $r+1$ steps

$$
\frac{d x}{d \tau}=\mu\left\{y^{2}-\varepsilon+\mu^{2} \overline{F^{2}}+\ldots+\mu^{r} f_{r}(y)\right\}+o\left(\mu^{r+1}\right)
$$

To this it can be replied that, in the first place, repeated averaging is not always possible in the almost periodic, non-periodic world, in fact in many circumstances it is the "exceptional case". Second, if one envisions substituting a generic function such as $\mu=\varepsilon^{1 / s}$ for $\mu$ with $s>r$, then one will still have to deal systematically with the $o\left(\mu^{1+r}\right)$-term. So if $\mu$ tends to zero sufficiently slowly we will have to consider the " $o$ "-term on the saddle-node bifurcation pattern.

This paper is structured as follows. In Section 2 we present a slightly generalized version of the Fink-Hale infinite-interval averaging theory. We make use of the Bebutov hull construction and other ideas of nonautonomous dynamics, which, in our opinion, clarify certain aspects of this method. Then, in Section 3, we discuss the classical bifurcation patterns when the relevant parameter is subjected to fast zero-mean oscillations. We will rediscuss the saddle node pattern, together with the transcritical, pitchfork, and AndronovHopf scenarios. Aside from presenting information on the bifurcation problems when fast oscillations are present, we want to illustrate what is now a rather extensive tool kit and body of results concerning nonautonomous differential systems. We make use of the basic Bebutov construction, some facts involving ergodic measures, exponential dichotomies, etc. We will also refer to previous papers concerning the nonautonomous bifurcation theory, as seems appropriate ([20, 25, 30]).

## 2. Preliminaries

In this section, we first present some facts from the field of topological dynamics (see [7, 35]) which will be useful in our discussion of infinite-interval averaging. Then we will describe our version of that averaging procedure.

Let $P$ be a topological space. A real flow on $P$ is determined by a family $\left\{\phi_{t} \mid t \in \mathbb{R}\right\}$ of homeomorphisms of $P$ with the following properties:

- $\phi_{0}(p)=p$ for all $p \in P ;$
- $\phi_{t} \circ \phi_{s}=\phi_{t+s}$ for all $t, s \in \mathbb{R}$;
- $\phi: P \times \mathbb{R} \rightarrow P:(t, p) \rightarrow \phi_{t}(p)$ is continuous.

Suppose now that $P$ is a compact metric space, and let $\left\{\phi_{t}\right\}$ be a flow on $P$. A regular Borel probability measure $\xi$ on $P$ is said to be $\phi_{t}$-invariant if $\xi\left(\phi_{t}(B)\right)=$ $\xi(B)$ for each Borel set $B \subset P$ and for each $t \in \mathbb{R}$. An invariant measure is said to be $\left\{\phi_{t}\right\}$-ergodic if, in addition, the following indecomposibility condition holds: if $B \subset P$ is a Borel set, and if $\xi\left(B \triangle \phi_{t}(B)\right)=0$ for each $t \in \mathbb{R}$, then either $\xi(B)=0$ or $\xi(B)=1$. Here $\triangle$ is the usual symmetric difference of sets: $A \triangle B=(A \backslash B) \cup(B \backslash A)$.

A famous theorem of Krylov and Bogoliubov ([23, 28]) states that, if $P$ is a compact metric space, and $\left\{\phi_{t}\right\}$ is a flow on $P$, then there exists at least one $\left\{\phi_{t}\right\}$-ergodic measure $\xi$ on $P$. If an ergodic measure $\xi$ on $P$ is the only $\left\{\phi_{t}\right\}$ ergodic measure on $P$ the flow $\left(P,\left\{\phi_{t}\right\}\right)$ is said to be uniquely ergodic. One can then apply a basic theorem of Birkhoff to the triple $\left(P,\left\{\phi_{t}\right\}, \xi\right)$, together with a refinement of that theorem. We state these results together.

Theorem 2.1. Let $P$ be a compact metric space, let $\left\{\phi_{t}\right\}$ be a flow on $P$, and let $\xi$ be $a\left\{\phi_{t}\right\}$-ergodic measure on $P$. If $h \in L^{1}(P, \xi)$, then

$$
\begin{equation*}
\lim _{t \rightarrow \pm \infty} \frac{1}{t} \int_{0}^{t} h\left(\phi_{s}(p)\right) d s=\int_{P} h d \xi \tag{8}
\end{equation*}
$$

for $\xi$-a.a. $p \in P$. If $h: P \rightarrow P$ is a continuous function and $\xi$ is uniquely ergodic, then (8) holds for all $p \in P$, and the limit is uniform in $p \in P$. That is, given $\varepsilon>0$ there exists $T>0$ such that, if $|t| \geq T$, then

$$
\left|\frac{1}{t} \int_{0}^{t} h\left(\phi_{s}(p)\right) d s-\int_{P} h d \xi\right| \leq \varepsilon \quad(p \in P)
$$

The first part of Theorem 2.1 can be stated and proved in the more general context of measurable flows; see e.g. [11]. The second part of the theorem is specific to a continuous flow $\left\{\phi_{t}\right\}$ defined on a compact space $P$ [14].

Let $P$ be a nonempty compact metric space. A flow $\left(P,\left\{\phi_{t}\right\}\right)$ on $P$ is said to be minimal if, for each $p \in P$, the orbit $\left\{\phi_{t}(p) \mid t \in \mathbb{R}\right\}$ is dense in $P$. A flow $\left(P,\left\{\phi_{t}\right\}\right)$ is said to be strictly ergodic if it is minimal and admits a unique ergodic measure $\xi$.

Again let $P$ be a nonempty compact metric space. A flow $\left(P,\left\{\phi_{t}\right\}\right)$ on $P$ is said to be Bohr almost periodic, or simply almost periodic, if there is a metric $d$ on $P$, which is compatible with the topology on $P$, such that

$$
d\left(\phi_{t}\left(p_{1}\right), \phi_{t}\left(p_{2}\right)\right)=d\left(p_{1}, p_{2}\right)
$$

for all $p_{1}, p_{2} \in P$ and for all $t \in \mathbb{R}$. If $\left(P,\left\{\phi_{t}\right\}\right)$ is almost periodic, then for each $p \in P$ the orbit closure $\operatorname{cls}\left\{\phi_{t}(p) \mid t \in \mathbb{R}\right\} \subset P$ is strictly ergodic (in particular it is minimal), and in fact $P$ is the union of its minimal sets. If $\left(P,\left\{\phi_{t}\right\}\right)$ is an
almost periodic minimal set, then one can give $P$ the structure of a compact Abelian topological group with multiplication $*$ and dense subgroup $\mathbb{R}$ in such a way that $\phi_{t}(P)=p * t(p \in P, t \in \mathbb{R})$. Let us note finally that, although a minimal almost periodic flow is strictly ergodic, the converse is not true, a fact which is illustrated by the Furstenberg flows [14].

We discuss a class of concrete minimal, almost periodic flows, namely the Kronecker flows. Let $d \geq 2$ and let $\mathbb{T}^{d}=\mathbb{R}^{d} / \mathbb{Z}^{d}$ be the $d$-torus. Let $\alpha_{1}, \ldots, \alpha_{d}$ be $\mathbb{Q}$-independent real numbers, let $p_{1}, \ldots, p_{d}$ be 1-periodic angular coordinates on $\mathbb{T}^{d}$, and set

$$
\phi_{t}\left(p_{1}, \ldots, p_{d}\right)=\left(p_{1}+\alpha_{1} t, \ldots, p_{d}+\alpha_{d} t\right) \quad \bmod \mathbb{Z}^{d}
$$

Then the flow $\left(\mathbb{T}^{d},\left\{\phi_{t}\right\}\right)$ is minimal and almost periodic. The unique $\left\{\phi_{t}\right\}$ invariant measure is the normalized Haar measure on $\mathbb{T}^{d}$. More generally, if $\alpha_{1}, \ldots, \alpha_{d}$ satisfy exactly $k \in\{0,1,2, \ldots, d-1\}$ independent homogeneous $\mathbb{Q}$-linear relations, then $\left(\mathbb{T}^{d},\left\{\phi_{t}\right\}\right)$ laminates into a disjoint union of almost periodic minimal flows, each of which is flow-isomorphic to a ( $d-k$ )-dimensional Kronecker flow.

Next we give a brief discussion of the Bebutov construction, which actually consists of a family of mutually similar constructions. Consider a timedependent differential system

$$
\begin{equation*}
\frac{d x}{d t}=f(t, x) \quad t \in \mathbb{R}, x \in \mathbb{R}^{d} \tag{9}
\end{equation*}
$$

The general goal is to apply the methods of topological dynamics to study the solutions of (9). This can be done if $f$ satisfies certain conditions, as we now indicate. We do not give proofs of the various assertions we make below: these are readily available in the literature (e.g., [34]) and in any case are usually quite easy to check directly.

First suppose that, for each compact subset $K \subset \mathbb{R}^{d}$, the restriction of $f$ to $\mathbb{R} \times K$ is uniformly continuous. Then there exist:
(i) a compact metrizable space $P$ which carries a flow $\left\{\phi_{t}\right\}$;
(ii) a continuous function $f_{*}: P \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$;
(iii) a point $p_{*} \in P$ such that $f(t, x)=f_{*}\left(\phi_{t}\left(p_{*}\right), x\right)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^{d}$.

The flow $\left\{\phi_{t}\right\}$ is induced by the translation in $t$, and the points of $P$ are actually functions $p(t, x)=\lim _{n \rightarrow \infty} f\left(t+t_{n}, x\right)$ for appropriate sequences $\left\{t_{n}\right\} \subset \mathbb{R}$. Here the limit is taken in the compact-open topology on $\mathbb{R} \times \mathbb{R}^{d}$. One usually abuses notation at this point and writes $f$ instead of $f_{*}$ (but not $p$ for $p_{*}$ ). The upshot is that equation (9) has been embedded into the family of differential equations

$$
\begin{equation*}
\frac{d x}{d t}=f\left(\phi_{t}(p), x\right) \quad p \in P, x \in \mathbb{R}^{n} \tag{9p}
\end{equation*}
$$

where ( 9 ) coincides with $\left(9 p_{*}\right)$.
Next suppose that each equation (9p) admits a unique global solution $x\left(t ; x_{0}, p\right)$ for each initial value $x_{0} \in \mathbb{R}^{n}$. Then the family of homeomorphisms

$$
\psi_{t}: P \times \mathbb{R}^{d} \rightarrow P \times \mathbb{R}^{d}:\left(p, x_{0}\right) \rightarrow\left(\phi_{t}(p), x\left(t ; x_{0}, p\right)\right)
$$

defines a flow on $P \times \mathbb{R}^{d}$. One speaks of a skew-product flow because the first factor does not depend on $x_{0}$. One can now use various techniques of topological dynamics to study the solutions of the various equation (9p), and in particular the solutions of equation (9), alias ( $9 p_{*}$ ).

One can take account of eventual smoothness properties of $f$ in $x \in \mathbb{R}^{d}$, say of order $r \geq 1$, as follows. Let $l=\left(l_{1}, \ldots, l_{d}\right)$ be a multiindex of integers such that $0 \leq l_{1}, \ldots, l_{d} \leq l_{1}+\ldots+l_{d}=|l| \leq r$. One requires that $f$ together with all its partial derivatives $D_{x}^{l} f=D_{x_{1}}^{l_{1}} \ldots D_{x_{d}}^{l_{d}} f$ of order $|l| \leq r$ be uniformly continuous on sets of the form $\mathbb{R} \times K$ where $K \subset \mathbb{R}^{d}$ is compact. If this condition holds, then there exist:
(i) a compact metric space $P$ with a flow $\left\{\phi_{t}\right\}$;
(ii) a continuous function $f_{*}: P \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ such that $D_{x}^{l} f_{*}: P \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ exists and is continuous for each multiindex $l=\left(l_{1}, \ldots, l_{d}\right)$ with $|l| \leq r$;
(iii) a point $p_{*} \in P$ such that $f(t, x)=f_{*}\left(\phi_{t}\left(p_{*}\right), x\right)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^{d}$.

One describes point (ii) by saying that $f_{*}$ is of class $C^{r}$ in $x$, uniformly in $p \in P$. In most of what we do below, we will simply assume that each function $f_{*}$ which is encountered is $C^{\infty}$ in $x$ uniformly in $p \in P$.

Let us now turn to the theory of infinite-interval averaging, which we formulate in a context in which a Bebutov flow is present. The ideas discussed here are drawn from the presentation in Fink [10] and Hale [15]. First we recall a basic lemma [22], which extends somewhat Lemma 14.1 of Fink's book.
Proposition 2.2. Let $P$ and $X$ be compact metric spaces, and let $\left(P,\left\{\phi_{t}\right\}\right)$ be $a$ uniquely ergodic flow with unique invariant measure $\xi$. Let $f: P \times X \rightarrow \mathbb{R}^{d}$ be a continuous function, and let

$$
\bar{f}(x)=\int_{P} f(p, x) d \xi(p)
$$

be the $\xi$-mean value of $f$. If $\mu$ is a positive number, define

$$
F(p, x, \mu)=\int_{-\infty}^{0} e^{\mu s}\left\{f\left(\phi_{s}(p), x\right)-\bar{f}(x)\right\} d s
$$

Then there is a continuous positive function $\zeta:(0, \infty) \rightarrow(0, \infty)$ such that $\zeta(\mu) \rightarrow 0$ as $\mu \rightarrow 0^{+}$, and

$$
\begin{equation*}
|\mu F(p, x, \mu)| \leq \zeta(\mu) \quad(\mu>0, p \in P, x \in X) \tag{10}
\end{equation*}
$$

Let us remark that the function $\zeta$ need not tend to zero at a prearranged rate - to be explicit, one cannot write, say, $\zeta(\mu)=C \mu^{s}$ where $C$ is a constant and $s>0$. We give an illustrative example in Appendix B. Let us remark that, if $X \subset \mathbb{R}^{m}$ is the closure of a bounded open set, and if $f$ is $C^{r}$ on $X$ uniformly in $p \in P$, then one can determine a continuous positive function $\zeta=\zeta(\mu)$ such that $\zeta(\mu) \rightarrow 0$ as $\mu \rightarrow 0^{+}$, and such that

$$
\left|\mu D_{x}^{l} F(p, x, \mu)\right| \leq \zeta(\mu)
$$

for all $p \in P, x \in X$, and for each multiindex $l=\left(l_{1}, \ldots, l_{d}\right)$ with $|l| \leq r$.
One can generalize these statements slightly by allowing $f$ to depend continuously and smoothly on $\mu$, for $\mu$ in some open interval containing $\mu=0$. This can be seen by substituting $x \in \mathbb{R}^{d}$ by $(x, \eta) \in \mathbb{R}^{d+1}$, applying Proposition 2.2 , then letting $\eta=\mu$.

Continuing the discussion, let $p \in P, x \in X$, and set

$$
\begin{equation*}
F_{\mu}(t, x)=F\left(\phi_{t}(p), x, \mu\right)=\mathrm{e}^{-\mu t} \int_{-\infty}^{t} \mathrm{e}^{\mu s}\left\{f\left(\phi_{s}(p), x\right)-\bar{f}(x)\right\} d s \tag{11}
\end{equation*}
$$

Of course $F_{\mu}$ depends on $p$ as well. Clearly

$$
\frac{d F_{\mu}}{d t}=-\mu F_{\mu}+f\left(\phi_{t}(p), x\right)-\bar{f}(x)
$$

We can apply this observation to ODES with rapidly varying time dependence. Since smoothness issues are of no particular relevance at present, let us assume that $f: P \times \mathbb{R}^{d} \times \mathbb{R} \rightarrow \mathbb{R}^{d}:(p, x, \mu) \rightarrow f(p, x, \mu)$ is a function which is $C^{\infty}$ in $(x, \mu)$, uniformly in $p \in P$. Consider the family of differential equations

$$
\begin{equation*}
\frac{d x}{d t}=\mu f\left(\phi_{t}(p), x, \mu\right) \quad x \in \mathbb{R}^{d}, \mu>0 \tag{12p}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
\frac{d x}{d s}=f\left(\phi_{\frac{s}{\mu}}(p), x, \mu\right) \quad \frac{s}{\mu}=t \tag{13p}
\end{equation*}
$$

where the rapid oscillations are displayed explicitly. For each fixed $p \in P$, make the following change of variables

$$
\begin{equation*}
x=y+\mu F_{\mu}(t, y) \tag{14}
\end{equation*}
$$

where $F_{\mu}$ is defined in (11).
We state the averaging theorem of Fink and Hale in the context of the family (12p).

Proposition 2.3. If $\Delta>0$, let $B_{\Delta}$ be the closed ball of radius $\Delta$ in $\mathbb{R}^{d}$ centered at the origin. Let $P,\left\{\phi_{t}\right\}, f, F, F_{\mu}$ be as above. There exist positive numbers $\Delta, \Delta_{0}$ and $\mu_{0}$ with the following properties. First, for each $p \in P$ and $\mu \in$ $\left(0, \mu_{0}\right)$, the transformation (14) is of class $C^{\infty}$ on $B_{\Delta_{0}}$ and has a $C^{\infty}$ inverse on $B_{\Delta}$. Second, both the transformation (14) and its inverse are $C^{\infty}$ on $B_{\Delta_{0}} \times$ $\left(0, \mu_{0}\right)$ respectively $B_{\Delta} \times\left(0, \mu_{0}\right)$, uniformly in $p \in P$.

We now make the change of variables (14) in equation (12p), for each $p \in P$, and obtain

$$
\begin{equation*}
\left(I+\mu \frac{\partial F_{\mu}}{\partial y}\right) \frac{d y}{d t}=\mu\left\{\overline{f_{\mu}}(y)+\mu F_{\mu}(t, y)+f_{\mu}\left(\phi_{t}(p), x\right)-f_{\mu}\left(\phi_{t}(p), y\right)\right\} \tag{15p}
\end{equation*}
$$

for all $0<\mu \leq \mu_{0}, y \in B_{\Delta}$. Here $I$ is the identity matrix, and we have written $f_{\mu}(\cdot)=f(\cdot, \mu)$. Now

$$
f_{\mu}(p, x)-f_{\mu}(p, y)=\frac{\partial f_{\mu}}{\partial y}(p, y) \mu F_{\mu}+R
$$

where the remainder $R$ is of order $O\left(\left|\mu F_{\mu}\right|^{2}\right)$. Since $\mu F_{\mu}$ is of order $o(1)$ as $\mu \rightarrow 0^{+}$, it is natural to compare solutions of (12p) with those of the averaged equation

$$
\begin{equation*}
\frac{d y}{d t}=\mu \overline{f_{\mu}}(y) \tag{16}
\end{equation*}
$$

Generally speaking one does this on a case-by-case basis when dealing with problems on an infinite time interval.

It should be remarked that, in the case when $f$ has only finitely many $x$ derivatives, there is in general a loss of smoothness of one degree in passing from (12p) to (15p).

It may happen that the function $f$ in $(12 \mathrm{p})$ admits a primitive in the sense that there exists a continuous function $\tilde{F}: P \times \mathbb{R}^{d} \times \mathbb{R} \rightarrow \mathbb{R}^{d}$ such that

$$
\frac{d}{d t} \tilde{F}\left(\phi_{t}(p), x, \mu\right)=f\left(\phi_{t}(p), x, \mu\right)-\bar{f}(x, \mu)
$$

identically in $t, p, x, \mu$. In this case one replaces (14) by

$$
x=y+\mu \tilde{F}_{\mu}\left(\phi_{t}(p), y\right)
$$

where $\tilde{F}_{\mu}(\cdot)=\tilde{F}(\cdot, \mu)$. Then (15p) takes the form

$$
\left(I+\mu \frac{\partial \tilde{F}_{\mu}}{\partial y}\right) \frac{d y}{d t}=\mu\left\{\overline{f_{\mu}}(y)-\mu f_{1}\left(\phi_{t}(p), y, \mu\right)\right\}
$$

where the non autonomous term $\mu f_{1}(p, y, \mu)$ is of order $O(\mu)$ as $\mu \rightarrow 0^{+}$(and not just of order $o(1))$. It must be emphasized, however, that a primitive need
not exist if one has a non-periodic time dependence, i.e., if $\left(P,\left\{\phi_{t}\right\}\right)$ is not a periodic flow.

We consider one last issue, which regards the continuous/smooth convergence of the solutions of the rapidly oscillating equations (13p) as $\mu \rightarrow 0^{+}$. This issue has been treated in [9], and the results discussed there can be viewed as an amplification of classical theorems which are proved in the averaging theory on a finite interval [31].

We formulate a result which will be useful later. Consider the equations

$$
\begin{array}{lr}
\frac{d x}{d s}=f\left(\phi_{\frac{s}{\mu}}(p), x, \mu\right) & \mu>0  \tag{17}\\
\frac{d x}{d s}=\overline{f_{0}}(x) & \mu=0
\end{array}
$$

where $\overline{f_{0}}(x)=\int_{P} f(p, x, 0) d \xi(p)$. The first equation in (17) is equation (13p), and the second is (as we will see) the appropriate limiting equation. Assume that, for all $p \in P$, the solution $x\left(s ; x_{*}, p, \mu\right)$ of equation (13p) with initial condition $x_{*} \in \mathbb{R}^{2}$ is defined for all $s \in(-\infty, \infty)$, and that the same condition holds for each solution $x_{0}\left(s, x_{*}\right)$ of the equation $\frac{d x}{d s}=\overline{f_{0}}(x)$. Define $\Psi: P \times$ $\mathbb{R}^{2} \times[0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}^{2}$ as follows:

$$
\Psi\left(p, x_{*}, \mu, s\right)= \begin{cases}x\left(s ; x_{*}, p, \mu\right) & \mu>0 \\ x_{0}\left(s, x_{*}\right) & \mu=0\end{cases}
$$

Proposition 2.4. The map $\Psi$ is continuous. Furthermore for each multiindex $k=\left(k_{1}, k_{2}\right)$ with integer components $k_{1} \geq 0, k_{2} \geq 0$, the derivative $D_{x}^{k} \Psi$ exists for all $p \in P, x_{*} \in \mathbb{R}^{2}, \mu \in[0, \infty)$ and $s \in \mathbb{R}$, and is continuous. That is

$$
\begin{aligned}
\lim _{\mu \rightarrow 0^{+}} x\left(s ; x_{*}, p, \mu\right) & =x_{0}\left(s ; x_{*}\right) \\
\lim _{\mu \rightarrow 0^{+}} D_{x}^{k} x\left(s ; x_{*}, p, \mu\right) & =D_{x}^{k} x_{0}\left(s ; x_{*}\right)
\end{aligned}
$$

where the convergence is uniform in $\left(p, x_{*}, s\right) \in K \subset P \times \mathbb{R}^{2} \times \mathbb{R}$ whenever $K$ is compact.

A proof of this Proposition can be modelled on those of Propositions 2.5 and 2.6 in [9].

## 3. Analysis

In this section we analyze the elementary bifurcation patterns when the parameter is subjected to rapid oscillations. We will make use of the averaging method discussed in Section 2, of the methods used in $[2,3]$ for studying fast oscillation problems, and of various techniques from the field of Nonautonomous Dynamics.

### 3.1. The saddle-node pattern

The starting point is the differential equation

$$
\begin{equation*}
\frac{d x}{d t}=x^{2}-\varepsilon+f\left(\frac{t}{\mu}\right) \quad \quad \mu>0 \tag{18}
\end{equation*}
$$

where $\mu$ is a small positive parameter. It is convenient to carry out a Bebutov construction on $f=f(\tau)$ where $\tau=\frac{t}{\mu}$. We assume that there exist a strictly ergodic flow $\left(P,\left\{\phi_{\tau}\right\}\right)$, a continuous function $f_{*}: P \rightarrow \mathbb{R}$, and a point $p_{*} \in P$ such that $f(\tau)=f_{*}\left(\phi_{\tau}\left(p_{*}\right)\right)$. In this way, equation (18) can be embedded in the family of equations

$$
\begin{equation*}
\frac{d x}{d t}=x^{2}-\varepsilon+f_{*}\left(\phi_{\frac{t}{\mu}}(p)\right) \quad \mu>0, p \in P \tag{18p}
\end{equation*}
$$

If for example $f(\tau)$ is Bohr almost periodic, then $\left(P,\left\{\phi_{\tau}\right\}\right)$ is an almost periodic minimal flow. Let $\xi$ be the normalized Haar measure on $P$; then $\xi$ is the unique $\left\{\phi_{\tau}\right\}$-invariant measure on $P$. Assume that $\int_{P} f_{*} d \xi=0$.

Let us carry out an a priori analysis of (18), based on the discussion in [19, pp. 170-172]. For this, fix $\mu>0$ and make the change of variables

$$
x=-\frac{\psi^{\prime}}{\psi}, \quad,=\frac{d}{d t}
$$

which takes (18) to the form of a Schrödinger equation

$$
\begin{equation*}
-\frac{d^{2} \psi}{d t^{2}}=\left[f\left(\frac{t}{\mu}\right)-\varepsilon\right] \psi \tag{19}
\end{equation*}
$$

Clearly $\varepsilon$ plays the role of an eigenvalue parameter in equation (19). We will want to consider in addition the family

$$
\begin{equation*}
-\frac{d^{2} \psi}{d t^{2}}=\left[f_{*}\left(\phi_{\frac{t}{\mu}}(p)\right)-\varepsilon\right] \psi \tag{19p}
\end{equation*}
$$

By hypothesis, the flow $\left(P,\left\{\phi_{t}\right\}\right)$ is strictly ergodic, hence so is the flow of ( $P,\left\{\phi_{\frac{t}{\mu}}\right\}$ ) for each $\mu>0$. In this case, the following information is available; see, e.g., [18]. First, there is a critical value $\varepsilon=\varepsilon_{c}(\mu)$ such that, if $\varepsilon>\varepsilon_{c}(\mu)$, then the family of two-dimensional linear differential systems

$$
\frac{d}{d t}\binom{\psi}{\psi^{\prime}}=\left(\begin{array}{cc}
0 & 1  \tag{20p}\\
\varepsilon-f_{*}\left(\phi_{\frac{t}{\mu}}(p)\right) & 0
\end{array}\right)\binom{\psi}{\psi^{\prime}}
$$

admits an exponential dichotomy over $P$. Second, if $\varepsilon<\varepsilon_{c}(\mu)$, then all nonzero solutions of (20p) rotate infinitely often around the origin in the $\binom{\psi}{\psi^{\prime}}$-plane, both as $t \rightarrow+\infty$ and as $t \rightarrow-\infty$.

It can be further be shown that, if $\varepsilon>\varepsilon_{c}(\mu)$, and if $Q_{p}: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ denotes the dichotomy projection at $p \in P$, then $\operatorname{Ker} Q_{p} \subset \mathbb{R}^{2}$ and $\operatorname{Im} Q_{p} \subset \mathbb{R}^{2}$ (viewed as lines passing through the origin in $\mathbb{R}^{2}$ ) make angles $\theta^{ \pm}(p)$ with the vertical axis $\psi=0$ which are bounded away from zero $\bmod \pi$. Let us note parenthetically that the statements of the previous two paragraphs do not require strict ergodicity of $\left(P,\left\{\phi_{\tau}\right\}\right)$ but only minimality.

Returning to the $x$-variable via the transformation $x=-\frac{\psi^{\prime}}{\psi}$, we see that, if $\varepsilon>\varepsilon_{c}(\mu)$, then there are two compact subsets $M_{ \pm} \subset P \times \mathbb{R}$ which are invariant with respect to the local flow on $P \times \mathbb{R}^{2}$ induced by equations (18p). These sets determine an attractor-repeller pair, as follows from the fact that equations (20p) admit an exponential dichotomy for $\varepsilon>\varepsilon_{c}(\mu)$. In particular, if $\varepsilon>\varepsilon_{c}(\mu)$ and $p \in P$, then equation (18p) admits two globally defined bounded solutions, one of which attracts and the other of which repels nearby solutions.

On the other hand, if $\varepsilon<\varepsilon_{c}(\mu)$, then all solutions of (18p) are unbounded in finite time $(p \in P)$. So one has an analogue of the saddle-node bifurcation pattern as $\varepsilon$ increases through $\varepsilon_{c}(\mu)$, for each $\mu>0$. However the analogy is not complete for the following reason. At $\varepsilon=\varepsilon_{c}(\mu)$ the flow on $P \times \mathbb{R}$ induced by equations (18p) does admit a unique minimal subset $M_{c}$, which would seem to correspond to the zero solution of (1) when $\varepsilon=0$. However $M_{c}$ need not be homeomorphic to $P$. In fact if $\left(P,\left\{\phi_{t}\right\}\right)$ is a minimal almost periodic flow which is not periodic, it is possible to determine $f_{*}$ in such a way that $M_{c}$ with its flow is an almost automorphic but not almost periodic extension of $\left(P,\left\{\phi_{\frac{t}{\mu}}\right\}\right)$, [35]. Examples of this phenomenon may be constructed, beginning with equation (20p), by using a method of Millionščikov ([27]; also Vinograd [36]). Nowadays one can also find such examples using other techniques; see e.g. [5].

Let us remark at this point that a bifurcation analysis similar to that we have just given can be carried out for a more general equation of the form

$$
\frac{d x}{d t}=g(t, x, \varepsilon)
$$

where $\varepsilon$ is an appropriate bifurcation parameter, and $g$ is almost periodic in $t$ and concave as a function of $x$. See Núñez and Obaya [29].

Until now we have not taken account of the rapid oscillations present in equation (18). Their presence allows us to make several observations. The first one is
Proposition 3.1. As $\mu \rightarrow 0^{+}$, the critical value $\varepsilon_{c}(\mu)$ tends to zero.
Proof. The easiest way to prove this statement seems to be the following. Return to equations (20p). Since $\int_{P} f_{*} d \xi=0$, the averaged form of this equation is the constant system

$$
\frac{d}{d t}\binom{\psi}{\psi^{\prime}}=\left(\begin{array}{ll}
0 & 1  \tag{21}\\
\varepsilon & 0
\end{array}\right)\binom{\psi}{\psi^{\prime}}
$$

which has an exponential dichotomy whenever $\varepsilon>0$. Fix a positive number $\varepsilon$. There exists a positive number $\mu_{0}=\mu_{0}(\varepsilon)$ such that if $0<\mu<\mu_{0}$, then the family (20p) admits an exponential dichotomy over $P$. This follows from Proposition 2.4 and the well known Sacker-Sell perturbation theorem [32].

In view of the previous analysis, this implies that $\lim \sup _{\mu \rightarrow 0^{+}} \varepsilon_{c}(\mu) \leq 0$. On the other hand, if $\varepsilon<0$, we can analyze the $\xi$-rotation number $\alpha=\alpha(\mu)$ of the family (19p). See Appendix A for a discussion of the $\xi$-rotation number. By general results concerning the continuity of the rotation number, one has that $\lim _{\mu \rightarrow 0^{+}} \alpha(\mu)$ equals the rotation number of the constant system (21). This latter rotation number is strictly positive. So for small positive values of $\mu$, the rotation number of $\alpha(\mu)$ is strictly positive. This means that $\varepsilon_{c}(\mu) \geq 0$ for small positive values of $\mu$, hence $\lim \inf _{\mu \rightarrow 0^{+}} \varepsilon_{c}(\mu) \geq 0$. We conclude that indeed $\lim _{\mu \rightarrow 0^{+}} \varepsilon_{c}(\mu)=0$.

Let us now suppose that $\mu$ is a function of $\varepsilon: \mu=\mu(\varepsilon)$, which is continuous, positive when $\varepsilon \neq 0$, and such that $\lim _{\varepsilon \rightarrow 0} \mu(\varepsilon)=0$. This is a particularly interesting situation because the oscillations "become fast" near $\varepsilon=0$ in an $\varepsilon$-dependent way. We make a few remarks when $\mu$ depends on $\varepsilon$ in this way and $\varepsilon \rightarrow 0$. We adopt the point of view that (18) with $\mu=\mu(\varepsilon)$ is a bifurcation problem with parameter $\varepsilon$.

We assume that the function $f(\tau)$ admits a bounded primitive $F(\tau)$ : $F^{\prime}(\tau)=f(\tau)$, which can be chosen to have mean value zero. It is well-known that, when these conditions hold, there is a continuous function $F_{*}: P \rightarrow \mathbb{R}$ such that $\int_{P} F_{*} d \xi=0$ and

$$
F_{*}\left(\phi_{t}(p)\right)-F_{*}(p)=\int_{0}^{t} f_{*}\left(\phi_{\sigma}(p)\right) d \sigma \quad(t \in \mathbb{R}, p \in P)
$$

Following [2] we consider the equation

$$
\frac{d x}{d \tau}=\mu f(\tau) \quad \tau=t / \mu
$$

which has the general solution

$$
x=\mu F(\tau)+c=h(\tau, c)
$$

where $c$ is an arbitrary constant. Setting $x=h(\tau, y)$ in (18) leads to

$$
\frac{d y}{d \tau}=\mu\left\{(y+\mu F)^{2}-\varepsilon\right\}
$$

and an application of the averaging procedure discussed in Section 2 leads to

$$
\begin{equation*}
\frac{d y}{d t}=\mu\left\{y^{2}-\varepsilon+\mu^{2} \overline{F^{2}}+o(\mu) y+o\left(\mu^{2}\right)\right\} \tag{22}
\end{equation*}
$$

where $\overline{F^{2}}=\int_{P} F_{*}^{2} d \xi$. The averaged system is

$$
\begin{equation*}
\frac{d y}{d t}=\mu\left\{y^{2}-\varepsilon+\mu^{2} \overline{F^{2}}\right\} \tag{23}
\end{equation*}
$$

The following observations are in order. First, if $\mu(\varepsilon)=|\varepsilon|^{s}$ for $s>1 / 2$, then (23) admits a saddle-node bifurcation with critical value $\varepsilon=0$. On the other hand, if $\mu(\varepsilon)=|\varepsilon|^{1 / 2}$ and if $\overline{F^{2}}>1$, then no bifurcation occurs: (23) takes the form:

$$
\frac{d y}{d t}=\mu\left\{y^{2}-\varepsilon+|\varepsilon| \overline{F^{2}}\right\}
$$

and $-\varepsilon+|\varepsilon| \overline{F^{2}}>0$ when $\varepsilon \neq 0$. Of course the same conclusion holds when $0<s<1 / 2$. To take account of the terms $o(\mu) y+o\left(\mu^{2}\right)$, note that if $|y| \leq|\varepsilon|^{s}$, these are dominated by $|\varepsilon| \overline{F^{2}}-\varepsilon$, so the full equation (22) also admits no bifurcation in $\varepsilon=0$.

At this point one can envisage a curve $\mu=\mu(\varepsilon)$ whose graph intersects the critical curve in a transversal way in infinitely many points $\left(\mu_{n}, \varepsilon_{n}\right) \rightarrow(0,0)$, so that there will be infinitely many switches in the direction of the bifurcation. One may ask about the dynamics of equation (18p) at such a point $\left(\mu_{n}, \varepsilon_{n}\right)$. The answer follows from the earlier considerations: there is a unique minimal subset $M_{n} \subset P \times \mathbb{R}$, which is an almost automorphic extension of $P$.
REmark 3.1. One can analyze the following somewhat more general bifurcation problem in a similar way:

$$
\begin{equation*}
\frac{d x}{d t}=a x^{2}+2 b x+f-\varepsilon \tag{24}
\end{equation*}
$$

where $a, b$, and $f$ are functions of $t / \mu$, and $\mu>0$ is small. If $x=\frac{v}{u}$ and

$$
J \frac{d}{d t}\binom{u}{v}=\left[\left(\begin{array}{cc}
f & b  \tag{25}\\
b & a
\end{array}\right)-\varepsilon\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right)\right]\binom{u}{v}
$$

then $x$ satisfies (24). Here $J$ is the antisymmetric matrix $\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. Now (25) can be viewed as a spectral problem of Atkinson type $[1,8]$ with spectral parameter $-\varepsilon$. Such a problem has a theory which is, in general terms, analogous to that of the Schrödinger equation (19). In a bit more detail, assume that there exist a strictly ergodic flow $\left(P,\left\{\phi_{t}\right\}\right)$ with ergodic measure $\xi$, a point $p_{*} \in P$, and continuous functions $a_{*}, b_{*}, f_{*}: P \rightarrow \mathbb{R}$ such that $a(\tau)=a_{*}\left(\phi_{\tau}\left(p_{*}\right)\right), b(\tau)=b_{*}\left(\phi_{\tau}\left(p_{*}\right)\right), f(\tau)=f_{*}\left(\phi_{\tau}\left(p_{*}\right)\right)$. There is a family of equations parametrized by $p \in P$, which corresponds to (25). Now assume that $a_{*}$ is strictly positive on $P$. Then one can prove the existence of a critical curve $\varepsilon=\varepsilon_{c}(\mu)$, defined for $\mu>0$, such that if $\varepsilon=\varepsilon_{c}(\mu)$ then equations (25) admit an exponential dichotomy over $P$, and if $\varepsilon<\varepsilon_{c}(\mu)$ then the $\xi$-rotation number of the family (25) is positive. So if $\mu>0$, then a saddle-node type bifurcation occurs as $\varepsilon$ decreases through $\varepsilon_{c}(\mu)$.

### 3.2. The transcritical pattern

The starting point is the equation

$$
\begin{equation*}
\frac{d x}{d t}=x\left(\varepsilon+f\left(\frac{t}{\mu}\right)-x\right) . \tag{26}
\end{equation*}
$$

As before we write $\tau=\frac{t}{\mu}$, and assume that $f=f(\tau)$ has associated to it a Bebutov flow ( $P,\left\{\phi_{t}\right\}$ ) which is strictly ergodic, with unique ergodic measure $\xi$. For example, $f(\tau)$ might be a Bohr-almost periodic function. There exists a point $p_{*} \in P$ and a continuous function $f_{*}: P \rightarrow \mathbb{R}$ such that $f(\tau)=f_{*}\left(\phi_{t}\left(p_{*}\right)\right)$ for all $\tau \in \mathbb{R}$. We assume that $\int_{P} f_{*} d \xi=0$. Introduce the family of equations

$$
\begin{equation*}
\frac{d x}{d t}=x\left(\varepsilon+f_{*}\left(\phi_{t / \mu}(p)\right)-x\right) \tag{26p}
\end{equation*}
$$

We can carry out a priori analysis of the family (26p), in the following way. Let

$$
w=\frac{1}{x}
$$

so that (26p) takes the form

$$
\frac{d w}{d t}+\left[\varepsilon+f_{*}\left(\phi_{t / \mu}(p)\right)\right] w=1
$$

The substitution $w=\cot (\theta)=\frac{u}{v}$ leads to the family

$$
\frac{d}{d t}\binom{u}{v}=\left(\begin{array}{cc}
-a & 1  \tag{27p}\\
0 & a
\end{array}\right)\binom{u}{v}
$$

where $a=\frac{1}{2}\left[\varepsilon+f_{*}\right]$ and one views $\theta$ as the angular coordinate in the $(u, v)$ plane. We will restrict attention to the sector $0 \leq \theta \leq \pi$, that is the closed upper half $(u, v)$-plane.

For each fixed $\mu>0$ we can analyze the family (27p) along standard lines. Since $\int_{P} f_{*} d \xi=0$, one has that the dynamical spectrum of ( 27 p ) reduces to $\left\{-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right\}$ for all $\varepsilon \in \mathbb{R}$; this uses the unique ergodicity of $\left(P,\left\{\phi_{t}\right\}\right)$. This implies that the family (27p) admits an exponential dichotomy over $P$ whenever $\varepsilon \neq 0$ [21].

We can describe the dichotomy bundles as follows. One bundle $B_{0}$ is described by the relation $v=0$ for all $\varepsilon \neq 0$. That is $B_{0}$ coincides with the product space $P \times \ell$, where $\ell \subset \mathbb{R}^{2}$ is the horizontal line $\left\{\left.\binom{u}{0} \right\rvert\, u \in \mathbb{R}\right\}$. This bundle is unstable for $\varepsilon<0$ and stable for $\varepsilon>0$. The other bundle $B_{\varepsilon}$ can be parametrized by a continuous map $\Theta: P \rightarrow(0, \pi)$ in the sense that

$$
B_{\varepsilon}=\left\{\left(p,\binom{u}{v}\right) \in P \times \mathbb{R}^{2} \left\lvert\,\binom{ u}{v}=\left\|\binom{u}{v}\right\|\binom{\cos (\Theta(p))}{\sin (\Theta(p))}\right.\right\}
$$

Let $I$ respectively $I I$ denote the first respectively the second quadrant in the $(u, v)$ space. One can check that that, if $\epsilon<0$, then $B_{\epsilon}$ lies in $P \times I I$, while if $\epsilon>0$ then $B_{\epsilon}$ lies in $P \times I$. One can further check that $B_{\epsilon}$ is stable for $\varepsilon<0$ and unstable for $\varepsilon>0$.

All this means that, for each $p \in P$, the $x$-equation (26p) admits the solution $x(t) \equiv 0$ together with a solution $x(t)=X\left(\phi_{t / \mu}(p)\right)$, where $X: P \rightarrow \mathbb{R}$, $X(p)=\tan (\Theta(p))$ is continuous. It is clear that a clean analogue of the classical transcritical bifurcation pattern takes place at $\varepsilon=0$, for each $\mu>0$. In fact the solution $x \equiv 0$ is asymptotically stable for $\varepsilon<0$ and unstable for $\varepsilon>0$, while the solution $x(t)=X\left(\phi_{t / \mu}(p)\right)$ is unstable for $\varepsilon<0$ and asymptotically stable for $\varepsilon>0(p \in P)$. Thus $\varepsilon=0$ is a critical value of the parameter $\varepsilon$ for each $\mu>0$; we write $0=\varepsilon=\varepsilon_{c}(\mu)$ to reflect this fact.

We have encountered what appears to be a typical difference between the bifurcation behavior of systems with "additive noise" (namely $\varepsilon+f$ in the saddle-node pattern), and "multiplicative noise" (namely $(\varepsilon+f) x$ in the transcritical pattern). That is, the critical curve $\varepsilon_{c}(\mu)$ needs not be constant in the first case, but is constant in the second case.

It is of interest to study the solutions of (26) and of the family (26p) at the critical value $\varepsilon(\mu)=0$, for positive values of $\mu$. The simplest case is that in which $f(\tau)$ admits a bounded primitive: thus $\sup _{\tau \in \mathbb{R}}|F(\tau)|<\infty$ and $F^{\prime}(\tau)=f(\tau)$. Consider the problem

$$
\frac{d x}{d \tau}=\mu f(\tau) x \quad \tau=\frac{t}{\mu}
$$

which has the general solution $x=h(\tau, c)=c \mathrm{e}^{\mu F(\tau)}$ with arbitrary constant $c$. Following [3], set

$$
x=h(\tau, y)=y \mathrm{e}^{\mu F(\tau)},
$$

which transforms (26) into

$$
\begin{equation*}
\frac{d y}{d \tau}=\mu \mathrm{e}^{-\mu F(\tau)} \mathrm{e}^{2 \mu F(\tau)}\left(-y^{2}\right)=-\mu \mathrm{e}^{\mu F(\tau)} y^{2} \tag{28}
\end{equation*}
$$

It is clear that all positive solutions of (28) tend to zero as $\tau \rightarrow \infty$, and so by boundedness of $F$, all positive solutions of (26) tend to zero as $\tau \rightarrow \infty$.

If $f$ does not admit a bounded primitive, then the discussion of the positive solutions of (26) resp. (26p) requires a bit more effort. First of all, we claim that if $p \in P, x_{0}>0$, and $x(t)$ is the solution of $(26 \mathrm{p})$ such that $x(0)=x_{0}$, then $\lim \inf _{t \rightarrow \infty} x(t)=0$. We sketch a proof. Suppose for contradiction that there exist $p_{1} \in P, x^{1}>0$, and $\delta>0$ such that, if $x^{1}(t)$ is the solution of $\left(26 \mathrm{p}_{1}\right)$ with $x^{1}(0)=x^{1}$, then $x^{1}(t) \geq \delta$ for all $t>0$. It is clear that $x(t)$ is defined for all $t>0$ and is uniformly bounded. Consider the local flow $\left\{\psi_{t}\right\}$ on $P \times\{0 \leq x<\infty\}$ defined by $\psi_{t}\left(p, x_{0}\right)=\left(\phi_{t}(p), x(t)\right)$. Then the $\omega$-limit set $K$
of $\left(p_{1}, x^{1}\right)$ is a nonempty compact subset of $P \times\{0<x<\infty\}$, and the local flow $\left\{\psi_{t}\right\}$ extends to a global flow on $K$. Since $\left(P,\left\{\phi_{t}\right\}\right)$ is a minimal flow, we can conclude that, for each $p \in P$, there exists a solution $x_{p}(t)$ of $(26 \mathrm{p})$ with $x_{p}(0)>0$ such that $x_{p}(t)$ is uniformly bounded above and $x_{p}(t) \geq \delta$ for all $t \in \mathbb{R}$.

Next let $\tau=t / \mu$, and let $f_{*}: P \rightarrow \mathbb{R}$ be the "extension of $f$ to $P$ " introduced earlier. There exists $p_{*} \in P$ such that $F(\tau)=\int_{0}^{\tau} f_{*}\left(\phi_{\sigma}\left(p_{*}\right)\right) d \sigma$ is bounded above but unbounded below [33]. Write $x_{*}(\tau)=x_{p_{*}}(\mu \tau)$, so that

$$
x_{*}(\tau)=\mathrm{e}^{\mu F(\tau)}\left[x_{*}(0)-\int_{0}^{\tau} \mathrm{e}^{-\mu F(\sigma)} x_{*}^{2}(\sigma) d \sigma\right] .
$$

Since $x_{*}(\tau)>0$ for all $\tau>0$, it is clear that $\liminf _{\tau \rightarrow \infty} x_{*}(\tau)=0$, but this contradicts the hypothesis. The proof is complete.

It is not clear if "liminf" can be replaced by "lim" in the above result. We conjecture that it cannot, though we do not have as yet a suitable example. On the other hand, if $\mu>0$ is small, we can obtain more information by carrying out a Fink-Hale type averaging procedure, beginning with equation (26). Set

$$
\begin{aligned}
& \tau=t / \mu \\
& F_{\mu}(\tau)=\mathrm{e}^{-\mu \tau} \int_{-\infty}^{\tau} \mathrm{e}^{\mu \sigma} f(\sigma) d \sigma \\
& x=y+\mu F_{\mu}(\tau) y
\end{aligned}
$$

so that (26) takes the form

$$
\begin{equation*}
\left(1+\mu F_{\mu}\right) \frac{d y}{d \tau}=\mu\left\{(1+f) \mu F_{\mu} y-\left(1+\mu F_{y}\right)^{2} y^{2}\right\} \tag{29}
\end{equation*}
$$

From (29) we can draw the following conclusion. If $\mu>0$ and $x_{0}>0$, let $x_{\mu}\left(t, x_{0}\right)$ be the solution of (26) such that $x_{\mu}\left(0, x_{0}\right)=x_{0}$. Let $x_{*}>0$ be a positive number. Then there exists $\mu_{*}>0$ such that, if $0<\mu \leq \mu_{*}$ and if $x_{0} \geq x_{*}$, then there is a corresponding number $\delta_{*}=\delta_{*}\left(x_{*}, \mu_{*}\right)$ with the property that $\frac{d}{d t} x_{\mu}\left(t, x_{0}\right) \leq-\delta_{*}$ for all $t$ for which $x_{\mu}\left(t, x_{0}\right) \geq x_{*}$. See [20] for another approach to the nonautonomous transcritical bifurcation problems when fast oscillations are present.

### 3.3. The pitchfork bifurcation pattern

This bifurcation scenario can be studied essentially from the same point of view as the transcritical pattern. The starting point is the equation

$$
\begin{equation*}
\frac{d x}{d t}=x\left[\varepsilon+f\left(\frac{t}{\mu}\right)-x^{2}\right] \tag{30}
\end{equation*}
$$

together with the corresponding family

$$
\begin{equation*}
\frac{d x}{d t}=x\left[\varepsilon+f_{*}\left(\phi_{t / \mu}(p)\right)-x^{2}\right] . \tag{30p}
\end{equation*}
$$

Make the substitution $w=\frac{1}{2 x^{2}}$ to obtain

$$
\frac{d w}{d t}+2(\varepsilon+f) w=1
$$

then set $w=\cot (\theta)=\frac{u}{v}$ where

$$
\frac{d}{d t}\binom{u}{v}=\left(\begin{array}{cc}
-\varepsilon-f & 1  \tag{31}\\
0 & \varepsilon+f
\end{array}\right)\binom{u}{v}
$$

It is clear that we can study the rapidly oscillating bifurcation pattern using the arguments applied above. We summarize the conclusions that can be drawn. Set $\varepsilon=\varepsilon_{c}(\mu)=0$ for each $\mu>0$; this defines the critical curve. If $\varepsilon<\varepsilon_{c}(\mu)$, then all solutions of (30) tend to zero exponentially fast as $t \rightarrow \infty$. On the other hand, if $\varepsilon>\varepsilon_{c}(\mu)$, then the family

$$
\frac{d}{d t}\binom{u}{v}=\left(\begin{array}{cc}
-\left(\varepsilon+f_{*}\left(\phi_{t / \mu}(p)\right)\right) & 1  \tag{31p}\\
0 & \varepsilon+f_{*}\left(\phi_{t / \mu}(p)\right)
\end{array}\right)\binom{u}{v}
$$

admits an exponential dichotomy over $P$. The unstable bundle $B_{\varepsilon}$ can be parametrized by a continuous map $\Theta: P \rightarrow\left(0, \frac{\pi}{2}\right)$ in the sense that

$$
B_{\varepsilon}=\left\{\left(P,\binom{u}{v}\right) \in P \times \mathbb{R}^{2} \left\lvert\,\binom{ u}{v}=\left\|\binom{u}{v}\right\|\binom{\cos (\Theta(p))}{\sin (\Theta(p))}\right.\right\}
$$

Write $w(p)=\cot (\Theta(p))$, so that $w(p)>0$ for all $p \in P$. Then the functions

$$
X^{ \pm}(p)=\frac{ \pm 1}{\sqrt{2 w(p)}}
$$

give rise to solutions $x^{ \pm}(t)=X^{ \pm}\left(\phi_{t / \mu}(p)\right)$ of equation (30p) $(p \in P)$. These solutions are exponentially asymptotically stable. We conclude that there is a clean analogue of the classical pitchfork bifurcation pattern as $\varepsilon$ increases through zero. The behavior of solutions of the family (31p) when $\varepsilon=0$ can be studied as was done in the transcritical case; we omit the details.

### 3.4. The Andronov-Hopf pattern

We first consider a Van der Pol oscillator which exhibits an AH (AndronovHopf) bifurcation when a parameter $\varepsilon$ increases through zero. We will subject the parameter to rapid oscillations and analyze "what happens". Then we will make some remarks concerning equations with rapidly oscillating coefficients for which an AH-bifurcation takes place in the averaged equation. We will take account of the (few) general results known to us concerning the "nonautonomous Hopf bifurcation". For an introduction to the AH-bifurcation theory see [16].

Consider the equation

$$
\begin{equation*}
\left.\frac{d^{2} x}{d t^{2}}-(\varepsilon+f(t / \mu))-x^{2}\right) \frac{d x}{d t}+x=0 \tag{32}
\end{equation*}
$$

Compare this equation with (one version of) the Van der Pol oscillator:

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}-\left(\varepsilon-x^{2}\right) \frac{d x}{d t}+x=0 \tag{33}
\end{equation*}
$$

We see that indeed the bifurcation parameter $\varepsilon$ is subjected to fast oscillations. Let us note parenthetically that one sometimes refers to a different equation as that of Van der Pol, namely

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}-\varepsilon\left(1-x^{2}\right) \frac{d x}{d t}+x=0 \tag{34}
\end{equation*}
$$

this equation does not admit an AH-bifurcation in $\varepsilon=0$ because the origin $\left(x, x^{\prime}\right)=(0,0)$ in the phase plane is a center. Incidentally in [3] the parameter $\varepsilon$ in (34) is subjected to fast oscillations. We will only discuss the version (33) of the Van der Pol equation, or rather its perturbed form (32).

It is convenient to write equation (32) in phase coordinates $x_{1}=x, x_{2}=\frac{d x}{d t}$ :

$$
\begin{equation*}
\frac{d}{d t}\binom{x_{1}}{x_{2}}=\binom{x_{2}}{\left[\varepsilon+f\left(\frac{t}{\mu}\right)\right] x_{2}-x_{1}^{2} x_{2}-x_{1}} \tag{35}
\end{equation*}
$$

Let us write $\tau=t / \mu$ and apply the method of [3] to this equation. Consider the system

$$
\begin{equation*}
\frac{d}{d t}\binom{x_{1}}{x_{2}}=\binom{0}{\mu f(\tau) x_{2}} \tag{36}
\end{equation*}
$$

Assume that $f(\tau)$ admits a bounded primitive $F(\tau)$ which has mean value zero: $\bar{F}=\lim _{\tau \rightarrow \infty} \frac{1}{\tau} \int_{0}^{\tau} f(\sigma) d \sigma=0$. The general solution of (36) is

$$
\binom{x_{1}}{x_{2}}=\binom{c_{1}}{c_{2} \mathrm{e}^{\mu F(\tau)}}=h(\tau, c)
$$

where $c=\binom{c_{1}}{c_{2}}$. The substitution $x=h(\tau, y)$ takes (35) to the form

$$
\begin{equation*}
\frac{d}{d \tau}\binom{y_{1}}{y_{2}}=\mu\binom{y_{2} \mathrm{e}^{\mu F(\tau)}}{\left(\varepsilon-y_{1}^{2}\right) y_{2}-y_{1} \mathrm{e}^{-\mu F(\tau)}} \tag{37}
\end{equation*}
$$

Next write

$$
\begin{array}{cc}
a(\tau)=\mathrm{e}^{\mu F(\tau)}, & \bar{a}=\lim _{\tau \rightarrow \infty} \frac{1}{\tau} \int_{0}^{\tau} a(\sigma) d \sigma \\
b(\tau)=\mathrm{e}^{-\mu F(\tau)}, & \bar{b}=\lim _{\tau \rightarrow \infty} \frac{1}{\tau} \int_{0}^{\tau} b(\sigma) d \sigma
\end{array}
$$

Note that, if $f$ is not identically zero, then $\bar{a}>1$ and $\bar{b}>1$ by Jensen's inequality. We make the further assumption that $a(\tau)-\bar{a}$ and $b(\tau)-\bar{b}$ admit bounded primitives $A(\tau)$ and $B(\tau)$. Write $y=\binom{y_{1}}{y_{2}}$ and $z=\binom{z_{1}}{z_{2}}$, then introduce the averaging transformation

$$
y=z+\mu G(\tau, z) \quad G(\tau, z)=\binom{A(\tau) z_{2}}{-B(\tau) z_{1}}
$$

equation (37) takes the form

$$
\begin{align*}
& {\left[\mathbb{I}+\mu\left(\begin{array}{cc}
0 & A(\tau) \\
-B(\tau) & 0
\end{array}\right)\right] \frac{d z}{d \tau}=\mu\left\{\binom{\bar{a} z_{2}}{-\bar{b} z_{1}+\left(\varepsilon-z_{1}^{2}\right) z_{2}}\right.} \\
& \left.-\mu\binom{a B z_{1}}{b A z_{2}+2 A z_{1} z_{2}^{2}+\left(\varepsilon-z_{1}^{2}\right) B z_{1}}+o(\mu)\right\} \tag{38}
\end{align*}
$$

A brief analysis of equation (38) leads to the following conclusions. First of all, the averaged system

$$
\begin{equation*}
\frac{d z}{d \tau}=\mu\binom{\bar{a} z_{2}}{\left(\varepsilon-z_{1}^{2}\right) z_{2}-\bar{b} z_{1}} \tag{39}
\end{equation*}
$$

exhibits an AH-bifurcation in $\varepsilon=0$. However, the rate of rotation along the AH-limit cycle is (to zeroeth order in $\varepsilon$ ) increased by a factor of $(\bar{a} \bar{b})^{1 / 2}$. This factor tends to 1 as $\mu \rightarrow 0^{+}$. That being said, it is worth considering what happens if the term $f\left(\frac{t}{\mu}\right)$ in (32) is replaced by $\frac{\alpha}{\mu} f\left(\frac{t}{\mu}\right)$ as in [2]. In that case the rate of rotation along the limit cycle in (39) becomes $(\bar{a} \bar{b})^{1 / 2}$, where now $\bar{a}=\overline{\mathrm{e}^{\alpha F(\tau)}}$ and $\bar{b}=\overline{\mathrm{e}^{-\alpha F(\tau)}}$. This quantity is larger than 1 (if $f \not \equiv 0$ ) and is $\mu$-independent.

Second, the behavior of the solutions of the original equation (32) is of course not determined solely by those of equation (39), but is influenced also by the $\tau$-dependent terms in (38). These terms are of order $O(\mu)$ as $\mu \rightarrow 0^{+}$. So one cannot expect to construct an analogue of the AH-theory for fixed $\mu>0$, with bifurcation parameter $\varepsilon$ : if $\varepsilon$ is near zero, the $O(\mu)$-terms might wash away the structure needed to obtain any sort of "nonautonomous version of the limit cycle". On the other hand, if $\mu=O\left(\varepsilon^{s}\right)$ where $s>1$, then one suspects that some analogue of the AH-pattern will be present for $\varepsilon$ near zero. This is indeed the case, as we now explain in a more general context.

Consider a differential system

$$
\begin{equation*}
\frac{d x}{d t}=f\left(\frac{t}{\mu}, x, \varepsilon\right) \quad x \in\binom{x_{1}}{x_{2}} \in \mathbb{R}^{2} \tag{40}
\end{equation*}
$$

where $\varepsilon \in \mathbb{R}$ and $\mu>0$ are independent parameters, and $f(\cdot, 0, \cdot)=0$. We assume that a Bebutov construction can be carried out on $f$, with the following results.

There exist a strictly ergodic flow $\left(P,\left\{\phi_{\tau}\right\}\right)$ with unique ergodic measure $\xi$, a continuous function $f_{*}: P \times \mathbb{R}^{2} \times \mathbb{R} \rightarrow \mathbb{R}^{2}$, and a point $p_{*} \in P$ such that $f_{*}\left(\phi_{\tau}\left(p_{*}\right), x, \varepsilon\right)=f(\tau, x, \varepsilon)$ for all $\tau \in \mathbb{R}, x \in \mathbb{R}^{2}, \varepsilon \in \mathbb{R}$. We further assume that $f_{*}$ is a $C^{\infty}$ function of $(x, \varepsilon)$, uniformly in $p \in P$; that is, for each triple ( $\alpha_{1}, \alpha_{2}, \alpha_{3}$ ) of nonnegative integers the derivative

$$
\frac{\partial^{\alpha_{1}}}{\partial x_{1}^{\alpha_{1}}} \frac{\partial^{\alpha_{2}}}{\partial x_{2}^{\alpha_{2}}} \frac{\partial^{\alpha_{3}}}{\partial x_{3}^{\alpha_{3}}} f_{*}
$$

exists and is continuous on $P \times \mathbb{R}^{2} \times \mathbb{R}$.
Set $\tau=t / \mu$, then introduce the family of differential systems

$$
\begin{equation*}
\frac{d x}{d \tau}=\mu f_{*}\left(\phi_{\tau}(p), x, \varepsilon\right) \tag{40p}
\end{equation*}
$$

Write $\bar{f}(x, \varepsilon)=\int_{P} f_{*}(p, x, \varepsilon) d \xi(p)$. We assume that there exists a continuous function $F_{*}: P \times \mathbb{R}^{2} \times \mathbb{R} \rightarrow \mathbb{R}^{2}$ such that for each $\tau \in \mathbb{R}, p \in P, x \in \mathbb{R}^{2}$ one has

$$
F_{*}\left(\phi_{\tau}(p), x, \varepsilon\right)-F_{*}(p, x, \varepsilon)=\int_{0}^{\tau}\left[f_{*}\left(\phi_{\sigma}(p), x, \varepsilon\right)-\bar{f}(x, \varepsilon)\right] d \sigma
$$

Thus $F_{*}$ is a "bounded continuous primitive of $f_{*}$ ". Finally we assume that $F_{*}$ is $C^{\infty}$ in $(x, \varepsilon)$, uniformly in $p \in P$. When this conditions are fulfilled, the averaging transformation

$$
x=y+\mu F_{*}\left(\phi_{\tau}(p), y, \varepsilon\right)
$$

takes (40p) to the form

$$
\begin{equation*}
\frac{d y}{d \tau}=\mu\left\{\bar{f}(y, \varepsilon)+\mu g_{*}\left(\phi_{\tau}(p), y, \varepsilon, \mu\right)\right\} \tag{41p}
\end{equation*}
$$

for a function $g_{*}$ which is $C^{\infty}$ in $(x, \varepsilon, \mu)$, uniformly in $p \in P$. Here in the first moment $\mu$ must be restricted to a neighborhood of zero, but we assume that $g_{*}$ has been extended to all of $P \times \mathbb{R}^{2} \times \mathbb{R} \times \mathbb{R}$, so as to be $C^{\infty}$ in $(y, \varepsilon, \mu)$ uniformly in $p \in P$.

Next we observe that, except for the small coefficient $\mu$ in front of the parenthesis $\{\cdot\}$ in (41p), equations (41p) have the form of the family of equations studied in [12]. We digress to recall some facts stated there. Consider the equations

$$
\begin{equation*}
\frac{d x}{d t}=f(x, \varepsilon)+\mu g\left(\phi_{t}(p), x, \varepsilon, \mu\right) \tag{42p}
\end{equation*}
$$

where $f$ and $g$ are $C^{\infty}$ functions of all arguments, uniformly in $p \in P$. Suppose that $f(0, \varepsilon)=g(p, 0, \varepsilon, \mu)=0$ identically, and that

$$
\begin{equation*}
\frac{d x}{d t}=f(x, \varepsilon) \tag{43}
\end{equation*}
$$

exhibits a supercritical AH bifurcation in $\varepsilon=0$. More precisely it is required that $D_{x} f(0, \varepsilon)$ admits complex conjugate eigenvalues $\alpha(\varepsilon) \pm i \beta(\varepsilon)$ for $\varepsilon$ near zero, where $\alpha(0)=0, \alpha^{\prime}(0)>0$ and $\beta(0)>0$. It is further required that the second Lyapunov coefficient is positive [24], or equivalently that the vague attractor condition be valid in $\varepsilon=0$ [26].

One of the results of [12] can be stated as follows.
Theorem 3.2. Suppose that $\mu=O\left(\varepsilon^{s}\right)$ for some $s>1$. Then for sufficiently small $\varepsilon$, the family (42p) admits an integral manifold $M_{\varepsilon} \subset P \times \mathbb{R}^{2}$. If $\mu=0$ this integral manifold reduces to $P \times C_{\varepsilon}$ where $C_{\varepsilon} \subset \mathbb{R}^{2}$ is the support of the AH-periodic orbit of system (43). The flow on $M_{\varepsilon}$ determined by the solutions of the family (42p) is isomorphic to a flow in a circle extension of $P$.

For the precise meaning of the term "integral manifold" in this context see [12] and [37]. To say that $M_{\varepsilon}$ is a circle extension of $P$ amounts to saying that $M_{\varepsilon}$ is homemorphic to $P \times \mathbb{S}^{1}$ where $\mathbb{S}^{1}$ is the circle. The flow on $M_{\varepsilon}$ can be studied using results and ideas of [17], [6] and other papers. Actually the flow on $M_{\varepsilon}$ turns to be a so-called suspension flow, which allows a substantial analysis to be made. In particular the classical Furstenberg flows [14] play an important role in this "nonautonomous Andronov-Hopf" bifurcation scenario. See [12] for details.

Let us now return to the family (41p), which we rewrite so as to emphasize the presence of the rapid oscillations:

$$
\begin{equation*}
\frac{d y}{d \tau}=\bar{f}(y, \varepsilon)+\mu g_{*}\left(\phi_{t / \mu}(p), y, \varepsilon, \mu\right) \tag{43p}
\end{equation*}
$$

It turns out that the integral manifold result valid for equations (42p) can be proved for the family (43p) as well; see the forthcoming paper [13] for a discussion of this point. We conclude that, at least if $\mu=O\left(\varepsilon^{s}\right)$ and $s>1$, then a nonautonomous AH-bifurcation occurs in the family (41p) in the sense that the periodic orbit of $\frac{d y}{d y}=\bar{f}(y, \varepsilon)$ perturbs to a circle extension of $P$. It is not clear how to generalize the statement to the case $s=1$.

## A. Remarks on Atkinson problems

We state some basic facts concerning nonautonomous linear differential systems. Consider the equations

$$
\begin{equation*}
\frac{d x}{d t}=M(t) x \quad x \in \mathbb{R}^{d} \tag{44}
\end{equation*}
$$

where $M(\cdot)$ is a bounded uniformly continuous function defined on $\mathbb{R}$. We view $M(\cdot)$ as a point in the space $\mathcal{C}=C\left(\mathbb{R}, \mathcal{M}_{n}\right)$ of continuous maps from the reals into the set $\mathcal{M}_{n}$ of $n \times n$ real matrices. Give $\mathcal{C}$ the compact-open topology, and let $\phi_{t}(c)=c(t+\cdot)(t \in \mathbb{R}, c \in \mathcal{C})$. Then $\left\{\phi_{t} \mid t \in \mathbb{R}\right\}$ is the translation flow on $\mathcal{C}$. Define $P=\operatorname{cls}\left\{\phi_{t}(M) \mid t \in \mathbb{R}\right\}$ and set $p_{*}=M \in P, M_{*}(p)=p(0)$. Then $P$ is compact, $\left(P,\left\{\phi_{t}\right\}\right)$ is a flow, and (44) is the equation corresponding to $p_{*}$ of the family of linear differential systems

$$
\begin{equation*}
\frac{d x}{d t}=M_{*}\left(\phi_{t}(p)\right) x \tag{44p}
\end{equation*}
$$

Definition A.1. We say that the family of equations (44p) admits an exponential dichotomy over $P$ if there is a continuous projection-valued function $Q^{2}=Q: P \rightarrow \mathcal{M}_{n}$ such that

$$
\begin{array}{cll}
\left|\Phi_{p}(t) Q(p) \Phi_{p}(s)^{-1}\right| & \leq k e^{-\beta(t-s)} & \\
\left|\Phi_{p}(t)(I-Q(p)) \Phi_{p}(s)^{-1}\right| & \leq k e^{\beta(t-s)} & \\
t \leq s
\end{array}
$$

for positive constants $k, \beta$. Here $\Phi_{p}(t)$ is the $n \times n$ matrix solution of ( 44 p ) such that $\phi_{p}(0)$ is the $n \times n$ identity matrix; $\Phi_{p}(t)$ is the fundamental matrix solution of equation $(44 \mathrm{p})$.

If equations (44p) have an exponential dichotomy over $P$, then the stable and unstable bundles $\mathcal{B}_{+}$and $\mathcal{B}_{-}$are defined as follows:

$$
\begin{aligned}
& \mathcal{B}_{+}=\left\{(p, x) \in P \times \mathbb{R}^{d} \mid x \in \operatorname{Im} Q(p)\right\} \\
& \mathcal{B}_{-}=\left\{(p, x) \in P \times \mathbb{R}^{d} \mid x \in \operatorname{Ker} Q(p)\right\}
\end{aligned}
$$

These vector bundles over the base space $P$ are obviously invariant with respect to the linear skew-product flow $\left\{\psi_{t} \mid t \in \mathbb{R}\right\}$ on $P \times \mathbb{R}^{d}$ defined by $\psi_{t}(p, x)=$ $\left(\phi_{t}(x), \Phi_{p}(t) x\right)$.

Next set $d=2$. Let $\xi$ be a $\left\{\phi_{t}\right\}$-ergodic measure on $P$. We define the $\xi$-rotation number of the family (44p). Introduce polar coordinates $r, \theta$ in the $x$-plane. For each $p \in P$, equation (44p) induces a differential equation for $\theta$ which does not depend on $r$ :

$$
\begin{equation*}
\frac{d \theta}{d t}=g\left(\phi_{t}(p), \theta\right) \tag{45p}
\end{equation*}
$$

If $p \in P$ and $\theta \in \mathbb{R}$, then the solution $\theta(t)$ of (45p) is

$$
\theta(t)=\theta_{0}+\int_{0}^{t} g\left(\phi_{s}(p), \theta(s)\right) d s
$$

Observe that $\frac{\theta(t)}{t}=\frac{1}{t} \int_{0}^{t} g\left(\phi_{s}(p), \theta(s)\right) d s+o(1)$ as $|t| \rightarrow \infty$, so it is natural to compare the "average rotation" $\frac{\theta(t)}{t}$ with the time-averages of $g$, for various
values of $\left(p, \theta_{0}\right)$. There is no a priori guarantee that these time-averages exist. However, using the Birkhoff ergodic theorem (Theorem 2.1), the following result can be proved.

Proposition A.2. There is a Borel set $P_{0} \subset P$ of $\xi$-measure 1, such that if $p_{0} \in P$ and $\theta_{0} \in \mathbb{R}$, then $\lim _{|t| \rightarrow \infty} \frac{\theta(t)}{t}$ exists. The limit $\alpha=\alpha_{\xi}$ does not depend on the choice of $p_{0} \in P_{0}$ and $\theta_{0} \in \mathbb{R}$. If $\xi$ is the only $\left\{\phi_{t}\right\}$-ergodic measure on $P$, then the limit exists for all $\left(p_{0}, \theta_{0}\right) \in P \times \mathbb{R}$ and does not depend on $\left(p_{0}, \theta_{0}\right)$. In fact the limit is uniform in $\left(p_{0}, \theta_{0}\right) \in P \times \mathbb{R}$.

For obvious reasons, the number $\alpha_{\xi}$ is called the $\xi$-rotation number of the family (44p).

One can apply the concepts of exponential dichotomy and rotation number to the study of Atkinson-type spectral problems. We briefly discuss this matter; the notation used below is suggested by the application of Remark 3.1. Again let $P$ be a compact metric space with flow $\left\{\phi_{t}\right\}$. Let $a_{*}, b_{*}, f_{*}: P \rightarrow \mathbb{R}$ be continuous functions. Also let $\Gamma_{*}: P \rightarrow \mathcal{M}_{2}$ be a continuous function whose values are positive semi-definite matrices. Finally set $J=\left(\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right)$. Consider the family of differential systems:

$$
J \frac{d}{d t}\binom{u}{v}=\left[\left(\begin{array}{cc}
f_{*} & b_{*}  \tag{46p}\\
b_{*} & a_{*}
\end{array}\right)\left(\phi_{t}(p)\right)-\varepsilon \Gamma_{*}\left(\phi_{t}(p)\right)\right]\binom{u}{v}
$$

where $-\varepsilon$ is to be viewed as a spectral parameter.
Definition A.3. Let $\Phi_{p}(t)$ be the fundamental matrix solution of (46p) when $\varepsilon=0(p \in P)$. We say that the family (46p) satisfies the Atkinson condition if for each nonzero vector $\binom{u_{0}}{v_{0}} \in \mathbb{R}^{2}$ and for each $p \in P$ the following condition holds:

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left\|\Gamma_{*}\left(\phi_{t}(p)\right)\binom{u(t)}{v(t)}\right\|^{2} d t>0 \tag{45}
\end{equation*}
$$

Here of course $\binom{u(t)}{v(t)}$ is the solution of (46p) with initial value $\binom{u_{0}}{v_{0}}$.
The Atkinson condition (45) means that the positive semidefinite matrix $\Gamma_{*}$ "sees" each nonzero solution of ( 46 p ), for each $p \in P$. Note that, if $\Gamma_{*}=$ $\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$, then equations 46 p take the form $(25)($ with $\mu=1)$. It is easy to see that, if $\Gamma_{*}=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$ and if $a_{*}>0$ on $P$, then the Atkinson condition (45) holds for the family (46p).

As before, let $\xi$ be a $\left\{\phi_{t}\right\}$-ergodic measure on $P$. The $\xi$-rotation number of the family ( 46 p ) is a function $\alpha=\alpha(\varepsilon)$ of $\varepsilon$. For a more general version of
the following result (for linear systems of $2 d$-dimensional Hamiltonian ODEs) see [8].
Theorem A.4. Suppose that the topological support of $\xi$ equals $P$. Suppose that the family of differential systems (46p) satisfies the Atkinson condition (45). Then the function $\varepsilon \rightarrow \alpha(\varepsilon): \mathbb{R} \rightarrow \mathbb{R}$ is continuous and monotone increasing. The family (46p) admits an exponential dichotomy over $P$ at $\varepsilon=\varepsilon_{0}$ if and only if $\varepsilon_{0}$ is an element of an open interval $I \subset \mathbb{R}$ such that $\alpha(\varepsilon)$ is constant on $I$.

We can apply Theorem A. 4 to the situation discussed in Remark 3.1 by setting $\Gamma_{*}=\left(\begin{array}{ll}1 & 0 \\ 0 & 0\end{array}\right)$ and by assuming that $a_{*}>0$ on $P$. It turns out that, in this case, $\alpha(\varepsilon)=0$ for all sufficiently large $\varepsilon$. Since $\alpha(\varepsilon)$ is continuous and monotone nonincreasing, it is natural to define $\varepsilon_{c}=\min \{\varepsilon \in \mathbb{R} \mid \alpha(\varepsilon)=0\}$. Then Theorem A. 4 states that, if $\varepsilon>\varepsilon_{c}$, then the family (46p) admits an exponential dichotomy over $P$. If $\varepsilon>\varepsilon_{c}$ and $p \in P$, let $Q_{p}$ be the dichotomy projection of $(46 \mathrm{p})$. It turns out that the strict positivity of $a_{*}$ implies that neither the image nor the kernel of $Q_{p}$ can contain a vertical vector $\binom{0}{v} \in$ $\mathbb{R}^{2}$. So, reasoning as in Section 3, we can conclude that, if $\varepsilon>\varepsilon_{c}$, then the equation (24) admits two bounded solutions, one of which is attracting and the other is repelling.

## B. Fink averaging: an example

In Section 2 we stated that the function $\zeta(\mu)$ of equation (10) cannot, in general, be chosen to be of order $O\left(\mu^{s}\right)$ for any $s>0$. We will give an example to illustrate this point. We will construct a quasi-periodic function $f(t)$ which has mean value zero, and which has the following properties:

- $\mu \int_{-\infty}^{0} \mathrm{e}^{\mu t} f(t) d t$ is not $O\left(\mu^{s}\right)$ as $\mu \rightarrow 0^{+}$if $s>0$;
- $f^{\prime}(t)$ is a quasi-periodic function.

This means that, in the quasi-periodic averaging theory outlined in Section 2, the time-dependent quantity $\mu F_{\mu}(t, y)$ of (14) and (15p) cannot be made to be $O\left(\mu^{s}\right)$ for any $s>0$. In this sense the $o(1)$-estimate on $\mu F_{\mu}$ cannot be improved.

To begin the construction, let $P=\mathbb{R}^{2} / \mathbb{Z}^{2}$ be the standard 2-torus with angular coordinates $\theta_{1}, \theta_{2} \bmod 1$. Introduce the Kronecker flow

$$
\phi_{t}\left(\theta_{1}, \theta_{2}\right)=\left(\theta_{1}+t, \theta_{2}+\omega t\right) \quad \bmod 1
$$

where $\omega \in(0,1)$ is an irrational number with the following property: there are sequences $\left\{m_{k} \mid k \geq 1\right\}$ and $\left\{n_{k} \mid k \geq 1\right\}$ of integers such that $n_{k} \rightarrow \infty$ and

$$
\begin{equation*}
\left|m_{k}+n_{k} \omega\right| \leq \mathrm{e}^{-n_{k}} \quad(k=1,2, \ldots) . \tag{46}
\end{equation*}
$$

If $(m, n) \in \mathbb{Z}^{2}$ write

$$
\varepsilon_{m n}\left(\theta_{1}, \theta_{2}\right)=\mathrm{e}^{2 \pi i m \theta_{1}} \mathrm{e}^{2 \pi i n \theta_{2}},
$$

and set

$$
\begin{equation*}
f_{*}\left(\theta_{1}, \theta_{2}\right)=\sum_{(m, n) \neq(0,0)} f_{m, n} \varepsilon_{m n}\left(\theta_{1}, \theta_{2}\right) \tag{47}
\end{equation*}
$$

where the coefficients $f_{m, n}$ satisfy $f_{-m,-n}=\overline{f_{m, n}}$ for all integer values of $m, n$. Here and below the overbar indicates the complex conjugate. The coefficients $f_{m, n}$ will be chosen so that the right-hand side of (47) converges uniformly on $P$. Let $\xi$ be the normalized Haar measure on $P$, and note that $\int_{P} f_{*} d \xi=0$.

Next choose positive real numbers $\beta_{k}$ such that $\sum_{k=1}^{\infty} \beta_{k}^{-1}<\infty$. Set

$$
\mu_{k}=\beta_{k}^{-\beta_{k}} \quad(k=1,2, \ldots),
$$

then choose integers $m_{k}, n_{k}$ such that

$$
\left|m_{k}+n_{k} \omega\right| \leq \mu_{k}^{\beta_{k}}=\beta_{k}^{-\beta_{k}^{2}} \quad(k=1,2, \ldots)
$$

This can be done by using (46) to choose $n_{k}$ such that $n_{k}>\beta_{k}^{2} \ln \left(\beta_{k}\right)$. Set

$$
\begin{array}{cc}
f_{m_{k}, n_{k}}=\beta_{k}^{-1}=f_{-m_{k},-n_{k}} & (k=1,2, \ldots) \\
f_{m, n}=0 & \text { other values of }(m, n) \in \mathbb{Z}^{2} .
\end{array}
$$

Finally define $f(t)=f_{*}(t, \omega t)$ so that $f(t)$ is obtained by evaluating $f_{*}$ along the orbit through $p_{*}=(0,0) \in P$.

Observe that, for each $\mu>0$,

$$
\int_{-\infty}^{0} \mathrm{e}^{\mu s} f(s) d s=\sum_{k=1}^{\infty} f_{m_{k}, n_{k}}\left[\frac{1}{\mu+2 \pi i\left(m_{k}+n_{k} \omega\right)}+\frac{1}{\mu-2 \pi i\left(m_{k}+n_{k} \omega\right)}\right]
$$

since $f_{m_{k}, n_{k}}$ is real. For each $\ell=1,2, \ldots$ choose $\mu=\mu_{\ell}$ and note that

$$
\mu_{\ell} \int_{-\infty}^{0} \mathrm{e}^{\mu_{\ell} s} f(s) d s=\sum_{k=1}^{\infty} f_{m_{k}, n_{k}} \frac{2\left|\mu_{\ell}\right|^{2}}{\left|\mu_{\ell}\right|^{2}+\left|2 \pi\left(m_{k}+n_{k} \omega\right)\right|^{2}} .
$$

If $k=\ell$, then the corresponding term of the series is real, positive, and greater than

$$
f_{m_{\ell}, n_{\ell}}=\beta_{\ell}^{-1}=\mu_{\ell}^{s_{\ell}}
$$

where $s_{\ell}=\frac{1}{\beta_{\ell}} \rightarrow 0$ as $\ell \rightarrow \infty$. We used the fact that $m_{\ell}+n_{\ell} \omega$ is small compared to $\mu_{\ell}$ for $\ell=1,2, \ldots$. If $k \neq \ell$, then the corresponding term of the series is positive. So we can conclude that $\mu \int_{-\infty}^{0} \mathrm{e}^{\mu s} f(s) d s$ cannot be $O\left(\mu^{s}\right)$ as $\mu \rightarrow 0^{+}$, for any number $s>0$.

Concerning the derivative of $f(t)$, we note that

$$
f^{\prime}(t)=-4 \pi \sum_{k=1}^{\infty}\left(m_{k}+n_{k} \omega\right) f_{m_{k}, n_{k}} \sin \left[2 \pi\left(m_{k}+n_{k} \omega\right) t\right]
$$

Since $\left|m_{k}+n_{k} \omega\right|<\mathrm{e}^{-\left(\left|m_{k}\right|+\left|n_{k}\right|\right) / 2}$, we see that $f^{\prime}(t)$ extends to an analytic function, call it $f_{*}^{\prime}\left(\theta_{1}, \theta_{2}\right)$, on the torus $P$. So $f^{\prime}(t)$ is certainly a quasi-periodic function.

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# Positive radial solutions for systems with mean curvature operator in Minkowski space 

Daniela Gurban and Petru Jebelean<br>Dedicated to Jean Mawhin for his 75th anniversary

Abstract. We are concerned with a Dirichlet system, involving the mean curvature operator in Minkowski space

$$
\mathcal{M}(\mathrm{w})=\operatorname{div}\left(\frac{\nabla \mathrm{w}}{\sqrt{1-|\nabla \mathrm{w}|^{2}}}\right)
$$


#### Abstract

in a ball in $\mathbb{R}^{N}$. Using topological degree arguments, critical point theory and lower and upper solutions method, we obtain non-existence, existence and multiplicity of radial, positive solutions. The examples we provide involve Lane-Emden type nonlinearities in both sublinear and superlinear cases.


Keywords: Minkowski curvature operator, system, positive solution, nonexistence/existence, multiplicity, Leray-Schauder degree, critical point, lower and upper solutions.
MS Classification 2010: 35J66, 34B15, 34B18.

## 1. Introduction

In this paper we study the existence and multiplicity of positive solutions for radial systems of type

$$
\begin{cases}\mathcal{M}(\mathrm{u})+g_{1}(|x|, \mathrm{u}, \mathrm{v})=0 & \text { in } \mathcal{B}(R),  \tag{1}\\ \mathcal{M}(\mathrm{v})+g_{2}(|x|, \mathrm{u}, \mathrm{v})=0 & \text { in } \mathcal{B}(R), \\ \left.\mathrm{u}\right|_{\partial \mathcal{B}(R)}=0=\left.\mathrm{v}\right|_{\partial \mathcal{B}(R)} & \end{cases}
$$

where $\mathcal{M}$ stands for the mean curvature operator in Minkowski space

$$
\mathcal{M}(\mathrm{w})=\operatorname{div}\left(\frac{\nabla \mathrm{w}}{\sqrt{1-|\nabla \mathrm{w}|^{2}}}\right)
$$

$\mathcal{B}(R)=\left\{x \in \mathbb{R}^{N}:|x|<R\right\}, N \geq 2$ is an integer and the functions $g_{1}, g_{2}:$ $[0, R] \times[0, \infty)^{2} \rightarrow[0, \infty)$ are continuous.

In recent years, a particular attention was paid to Dirichlet problems (for a single equation) involving the operator $\mathcal{M}$, either in a general bounded domain $[3,4,11,12,13,24]$ or in a ball $[6,5,10,25]$. These problems are originated in differential geometry and are related to maximal or constant mean curvature hypersurfaces (spacelike submanifolds of codimension one in the flat Minkowski space $\mathbb{L}^{N+1}=\left\{(x, t): x \in \mathbb{R}^{N}, t \in \mathbb{R}\right\}$ endowed with the Lorentzian metric $\left.\sum_{j=1}^{N}\left(d x_{j}\right)^{2}-(d t)^{2}\right)$, having the property that their mean extrinsic curvature is respectively zero or constant $[1,8,20,28]$. On the other hand, it is known that systems with classical Laplacian (or other more general elliptic operators) bring in discussion new and specific phenomena, which does not occur in the study of a single equation. From the wide literature, for a basic outlook on the subject we restrict ourselves to mention here the papers [7, 14, 16, 17, 18, 29] and the references therein. It is worth to point out that, among various nonlinearities, an important role is played by those of Lane-Emden type, having either the form $k_{1} \mathrm{u}^{p}+k_{2} \mathrm{v}^{q}$ (see, e.g. [15, 26, 30]) or $k_{3} \mathrm{u}^{\alpha} \mathrm{v}^{\beta}$ (see, e.g. [16, 19, 22]). In view of the above, it appears as a natural direction the study of systems involving the mean curvature operator $\mathcal{M}$.

In the recent paper [21], among others, the authors deal with gradient systems of type

$$
\begin{cases}\mathcal{M}(\mathrm{u})+\lambda F_{\mathrm{u}}(x, \mathrm{u}, \mathrm{v})=0, & \text { in } \Omega  \tag{2}\\ \mathcal{M}(\mathrm{v})+\lambda F_{\mathrm{v}}(x, \mathrm{u}, \mathrm{v})=0, & \text { in } \Omega \\ \mathrm{u}|\partial \Omega=0=\mathrm{v}| \partial \Omega & \end{cases}
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^{N}$ and $\lambda>0$ is a real parameter. They obtain existence and multiplicity (at least two) of nontrivial non-negative solutions for large values of the parameter, when the nonlinearities $F_{\mathrm{u}}$ and $F_{\mathrm{v}}$ have a superlinear behavior near origin. On the other hand, in paper [5], for the problem

$$
\mathcal{M}(\mathrm{u})+\lambda \mu(|x|) \mathrm{u}^{\alpha}=0 \text { in } \mathcal{B}(R),\left.\quad \mathrm{u}\right|_{\partial \mathcal{B}(R)}=0 \quad(\alpha>1)
$$

with $\mu>0$ on $(0, R]$, it was shown a sharper result: there exists $\Lambda>0$ such that it has zero, at least one or at least two positive solutions according to $\lambda \in(0, \Lambda)$, $\lambda=\Lambda$ or $\lambda>\Lambda$. It is the main goal of this paper to improve the result from [21] in the case when $F$ has the particular form $F(x, \mathrm{u}, \mathrm{v})=\mu(|x|) \mathrm{u}^{p+1} \mathrm{v}^{q+1}$, with the positive exponents $p, q$ satisfying $\max \{p, q\}>1$ (this guaranties a superlinear behavior of both $F_{\mathrm{u}}$ and $F_{\mathrm{v}}$ near origin, with respect to ( $\mathrm{u}, \mathrm{v}$ ) ) and $\Omega=\mathcal{B}(R)$. By adapting the strategy from [5], we prove (Theorem 5.1, Corollary 5.2) that the result from [5] for a single equation remains valid for the system (2) with the above choice of $F$ and $\Omega$. Notice, in this case $g_{i}$ in (1) have the form

$$
g_{1}(|x|, \mathrm{u}, \mathrm{v})=\lambda \mu(|x|)(p+1) \mathrm{u}^{p} \mathrm{v}^{q+1}, \quad g_{2}(|x|, \mathrm{u}, \mathrm{v})=\lambda \mu(|x|)(q+1) \mathrm{u}^{p+1} \mathrm{v}^{q}
$$

which, in particular, include Hénon-Lane-Emden nonlinearities for $\mu(|x|)=|x|^{\sigma}$ $(\sigma>0)$. We also deal with the case when $g_{1}$ (resp. $g_{2}$ ) has a sublinear growth near origin with respect to $u$ (resp. v). In this respect, we obtain (Theorem 3.1, Corollary 3.3) the existence of a solution with either one or both components positive. This enables us to consider Lane-Emden non-potential nonlinearities having the form $k_{1} \mathrm{u}^{p}+k_{2} \mathrm{v}^{q}$. Here we have in view extensions of some results obtained in [6] for a single equation to systems of type (1).

As usual, setting $r=|x|$ and $\mathrm{u}(x)=u(r), \mathrm{v}(x)=v(r)$, the Dirichlet problem (1) reduces to the mixed boundary value problem:

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+r^{N-1} g_{1}(r, u, v)=0,}  \tag{3}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+r^{N-1} g_{2}(r, u, v)=0,} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

where

$$
\varphi(y)=\frac{y}{\sqrt{1-y^{2}}} \quad(y \in \mathbb{R},|y|<1)
$$

By a solution of (3) we mean a couple of functions $(u, v) \in C^{1}[0, R] \times C^{1}[0, R]$ with $\left\|u^{\prime}\right\|_{\infty}<1,\left\|v^{\prime}\right\|_{\infty}<1$ and $r \mapsto r^{N-1} \varphi\left(u^{\prime}(r)\right), r \mapsto r^{N-1} \varphi\left(v^{\prime}(r)\right)$ of class $C^{1}$ on $[0, R]$, which satisfies problem (3). Here and below, we denote by $\|\cdot\|_{\infty}$ the usual sup-norm on $C:=C[0, R]$. We say that $u \in C$ is positive if $u>0$ on $[0, R)$. By a positive solution of (3) we understand a solution $(u, v)$ with both $u$ and $v$ positive.

The paper is organized as follows. In Section 2 we present some preliminary results concerning the reformulation of system (3) as a fixed point problem as well as a variational problem - in the case when it has a potential structure. Two lemmas about the positivity of the components of the solution are also provided. Section 3 is devoted to the case when $g_{1}$ and $g_{2}$ have a sublinear behavior near origin. The lower and upper solution method and some degree estimations in the superlinear case are presented in Section 4. The main non-existence, existence and multiplicity result for an one-parameter system is stated and proved in Section 5.

## 2. Preliminaries

Throughout this paper, the space $C^{1}:=C^{1}[0, R]$ will be considered with the norm $\|u\|_{1}=\|u\|_{\infty}+\left\|u^{\prime}\right\|_{\infty}$. We shall use the product space $C^{1} \times C^{1}$ endowed with the norm $\|(u, v)\|=\max \left\{\|u\|_{\infty},\|v\|_{\infty}\right\}+\max \left\{\left\|u^{\prime}\right\|_{\infty},\left\|v^{\prime}\right\|_{\infty}\right\}$ and its closed subspace

$$
\mathcal{C}_{M}^{1}=\left\{(u, v) \in C^{1} \times C^{1}: u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)\right\}
$$

we shall denote $B_{\rho}:=\left\{(u, v) \in \mathcal{C}_{M}^{1}:\|(u, v)\|<\rho\right\}$. For given $f_{1}, f_{2}:[0, R] \times$ $\mathbb{R}^{2} \rightarrow \mathbb{R}$ continuous functions, let us consider the problem

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+r^{N-1} f_{1}(r, u, v)=0}  \tag{4}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+r^{N-1} f_{2}(r, u, v)=0} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

and the linear operators

$$
\begin{gathered}
S: C \rightarrow C, S u(r)=\frac{1}{r^{N-1}} \int_{0}^{r} t^{N-1} u(t) d t \quad(r \in[0, R]), S u(0)=0 \\
K: C \rightarrow C^{1}, K u(r)=\int_{r}^{R} u(t) d t \quad(r \in[0, R])
\end{gathered}
$$

It is easy to see that $K$ is bounded and $S$ is compact. Hence, the nonlinear operator $K \circ \varphi^{-1} \circ S: C \rightarrow C^{1}$ is compact. Denoting by $N_{f_{i}}$ the Nemytskii operator associated to $f_{i}(i=1,2)$, i.e.,

$$
N_{f_{i}}: C \times C \rightarrow C, \quad N_{f_{i}}(u, v)=f_{i}(\cdot, u(\cdot), v(\cdot)) \quad(u, v \in C)
$$

we have that $N_{f_{i}}$ is continuous and takes bounded sets into bounded sets. Below, we denote by $d_{L S}$ the Leray-Schauder degree. We have the following fixed point reformulation of problem (3).

Proposition 2.1. A couple of functions $(u, v) \in \mathcal{C}_{M}^{1}$ is a solution of (4) if and only if it is a fixed point of the compact nonlinear operator

$$
\mathcal{N}_{f}: \mathcal{C}_{M}^{1} \rightarrow \mathcal{C}_{M}^{1}, \quad \mathcal{N}_{f}=\left(K \circ \varphi^{-1} \circ S \circ N_{f_{1}}, K \circ \varphi^{-1} \circ S \circ N_{f_{2}}\right)
$$

In addition, any fixed point $(u, v)$ of $\mathcal{N}_{f}$ satisfies

$$
\begin{equation*}
\left\|u^{\prime}\right\|_{\infty}<1, \quad\left\|v^{\prime}\right\|_{\infty}<1, \quad\|u\|_{\infty}<R, \quad\|v\|_{\infty}<R \tag{5}
\end{equation*}
$$

and

$$
d_{L S}\left[I-\mathcal{N}_{f}, B_{\rho}, 0\right]=1 \text { for all } \rho \geq R+1
$$

In particular, problem (4) has at least one solution in $B_{\rho}$ for all $\rho \geq R+1$.
Proof. The inequalities in (5) follow from the fact that the range of $\varphi^{-1}$ is $(-1,1)$. We consider the compact homotopy

$$
\mathcal{H}:[0,1] \times \mathcal{C}_{M}^{1} \rightarrow \mathcal{C}_{M}^{1}, \quad \mathcal{H}(\tau, \cdot)=\tau \mathcal{N}_{f}(\cdot)
$$

Using

$$
\mathcal{H}\left([0,1] \times \mathcal{C}_{M}^{1}\right) \subset B_{R+1}
$$

together with the invariance property of Leray-Schauder degree, we have

$$
d_{L S}\left[I-\mathcal{N}_{f}, B_{\rho}, 0\right]=d_{L S}\left[I, B_{\rho}, 0\right]=1 \text { for all } \rho \geq R+1
$$

When system (4) has the form

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}=r^{N-1} F_{u}(r, u, v)}  \tag{6}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}=r^{N-1} F_{v}(r, u, v)} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

with $F=F(r, u, v):[0, R] \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ continuous, such that $F_{u}$ and $F_{v}$ exist and are continuous on $[0, R] \times \mathbb{R}^{2}$, then a variational approach is available. For this, let

$$
K_{0}=\left\{u \in W^{1, \infty}[0, R]:\left\|u^{\prime}\right\|_{\infty} \leq 1, u(R)=0\right\}
$$

We know (see $[2,6]$ ) that $K$ is a compact subset of $C$. So, we have that $K_{0} \times K_{0} \subset C \times C$ is compact and convex. By means of $\psi: C \rightarrow(-\infty,+\infty]$ defined by

$$
\psi(u)=\left\{\begin{array}{lll}
\int_{0}^{R} r^{N-1}\left[1-\sqrt{1-u^{\prime 2}}\right] d r & \text { for } & u \in K_{0} \\
+\infty & \text { for } & u \in C \backslash K_{0}
\end{array}\right.
$$

we introduce $\Psi: C \times C \rightarrow(-\infty,+\infty]$ by

$$
\Psi(u, v)=\psi(u)+\psi(v), \text { for all }(u, v) \in C \times C
$$

Using the arguments in [2] we deduce that $\Psi$ is proper, convex and lower semicontinuous. Also, the mapping

$$
(u, v) \mapsto \mathcal{F}(u, v):=\int_{0}^{R} r^{N-1} F(r, u, v),(u, v \in C)
$$

is of class $C^{1}$ on $C \times C$ and its Fréchet derivative is given by

$$
\left\langle\mathcal{F}^{\prime}(u, v),\left(w_{1}, w_{2}\right)\right\rangle=\int_{\Omega} r^{N-1}\left[F_{u}(r, u, v) w_{1}+F_{v}(r, u, v) w_{2}\right],\left(u, v, w_{1}, w_{2} \in C\right)
$$

The energy functional associated to (6) will be $\mathcal{I}:=\Psi+\mathcal{F}$. This has the structure required by Szulkin's critical point theory [27]. Accordingly, $(u, v) \in$ $K_{0} \times K_{0}$ is a critical point of $\mathcal{I}$ if it is a solution of the variational inequality

$$
\begin{equation*}
\Psi\left(w_{1}, w_{2}\right)-\Psi(u, v)+\left\langle\mathcal{F}^{\prime}(u, v),\left(w_{1}-u, w_{2}-v\right)\right\rangle \geq 0, \quad \forall w_{1}, w_{2} \in C \tag{7}
\end{equation*}
$$

Proposition 2.2. If $(u, v) \in C \times C$ is a critical point of $\mathcal{I}$, then it is a solution of system (6). Moreover, system (6) has a solution which is a minimum point of $\mathcal{I}$ on $C \times C$.

Proof. Let $(u, v)$ be a critical point of $\mathcal{I}$. By taking in (7) $w_{2}=v$, one gets

$$
\psi\left(w_{1}\right)-\psi(u)+\int_{0}^{R} r^{N-1} F_{u}(r, u, v)\left(w_{1}-u\right) \geq 0, \quad \text { for all } w_{1} \in C
$$

i.e., $u \in K_{0}$ is a critical point of $\psi(\cdot)+\mathcal{F}(\cdot, v)$, which by virtue of $[6$, Proposition 4] satisfies

$$
\left\{\begin{array}{l}
\left(r^{N-1} \varphi\left(u^{\prime}\right)\right)^{\prime}=r^{N-1} F_{u}(r, u, v) \\
u^{\prime}(0)=0=u(R)
\end{array}\right.
$$

Similarly, one obtains that $v$ verifies

$$
\left\{\begin{array}{l}
\left(r^{N-1} \varphi\left(v^{\prime}\right)\right)^{\prime}=r^{N-1} F_{v}(r, u, v) \\
v^{\prime}(0)=0=v(R)
\end{array}\right.
$$

The rest of the proof follows exactly as in [6, Proposition 4].

Next, let $g_{1}, g_{2}:[0, R] \times[0, \infty)^{2} \rightarrow[0, \infty)$ be continuous. We are interested about positive solutions for the system (3). With this aim, we consider the modified problem

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+r^{N-1} g_{1}\left(r, u_{+}, v_{+}\right)=0}  \tag{8}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+r^{N-1} g_{2}\left(r, u_{+}, v_{+}\right)=0} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

where, as usual we have denoted $\xi_{+}:=\max \{0, \xi\}$.
Let $J_{1}, J_{2} \subset \mathbb{R}$. In the terminology of [9, 23], a function $f=f(r, s, t)$ : $[0, R] \times J_{1} \times J_{2} \rightarrow \mathbb{R}$ is said to be quasi-monotone nondecreasing with respect to $t$ (resp. s) if for fixed $r, s$ (resp. $r, t$ ) one has

$$
f\left(r, s, t_{1}\right) \leq f\left(r, s, t_{2}\right) \text { as } t_{1} \leq t_{2} \quad\left(\text { resp. } f\left(r, s_{1}, t\right) \leq f\left(r, s_{2}, t\right) \text { as } s_{1} \leq s_{2}\right)
$$

Lemma 2.3. Assume that $(u, v)$ is a nontrivial solution of problem (8) and $\left(H_{g}^{1}\right) g_{1}(r, \xi, 0)>0<g_{2}(r, 0, \xi), \forall \xi>0, \forall r \in(0, R]$,
then $u \geq 0 \leq v$ and either $u$ or $v$ is positive and strictly decreasing.
If in addition to hypothesis $\left(H_{g}^{1}\right)$ one has that $g_{1}(r, s, t)$ (resp. $g_{2}(r, s, t)$ ) is quasi-monotone nondecreasing with respect to $t$ (resp. s) and it holds

$$
\left(H_{g}^{2}\right) g_{1}(r, 0, \xi)>0<g_{2}(r, \xi, 0), \forall \xi>0, \forall r \in(0, R]
$$

then $(u, v)$ is a positive solution with both $u$ and $v$ strictly decreasing.

Proof. From

$$
\begin{equation*}
r^{N-1} \varphi\left(u^{\prime}\right)=-\int_{0}^{r} \tau^{N-1} g_{1}\left(\tau, u_{+}, v_{+}\right) d \tau \tag{9}
\end{equation*}
$$

it follows $u^{\prime} \leq 0$, which means that $u$ is decreasing. Similarly, one obtains that $v$ is decreasing. Then $u(R)=0$ implies $u \geq 0$ and analogously, $v$ is $\geq 0$. If we assume $u \equiv 0$, on account of

$$
r^{N-1} \varphi\left(v^{\prime}\right)=-\int_{0}^{r} \tau^{N-1} g_{2}(\tau, 0, v) d \tau
$$

and $v(0)>0$, from $\left(H_{g}^{1}\right)$ one obtains $v^{\prime}<0$; thus $v$ is strictly decreasing and $v>0$ on $[0, R)$. Similarly, if $v \equiv 0$ one has that $u$ is positive and strictly decreasing.

To prove the second part, suppose that $u$ is positive and let us show that $v$ is positive, too. If $v(0)=0$, from the second equation we get $g_{2}(r, u(r), 0)=0$ for all $r \in[0, R]$, contradicting $\left(H_{g}^{2}\right)$. So, we have $v(0)>0$. Then, using that $g_{2}(r, s, t)$ is quasi-monotone nondecreasing with respect to $s$, it follows

$$
r^{N-1} \varphi\left(v^{\prime}\right)=-\int_{0}^{r} \tau^{N-1} g_{2}(\tau, u, v) d \tau \leq-\int_{0}^{r} \tau^{N-1} g_{2}(\tau, 0, v) d \tau<0
$$

Hence, $v^{\prime}<0$ and $v$ is strictly decreasing.
Lemma 2.4. Assume that
$\left(H_{g}^{3}\right)(i) g_{1}(r, s, t)>0<g_{2}(r, s, t), \forall s, t>0, \forall r \in(0, R] ;$
(ii) $g_{1}(r, \xi, 0)=g_{2}(r, 0, \xi)=0, \forall \xi>0, \forall r \in(0, R]$.

If $(u, v)$ is a nontrivial solution of problem (8), then $(u, v)$ is a positive solution with both $u$ and $v$ strictly decreasing.
Proof. From the second equation we have

$$
\begin{equation*}
r^{N-1} \varphi\left(v^{\prime}\right)=-\int_{0}^{r} \tau^{N-1} g_{2}\left(\tau, u_{+}, v_{+}\right) d \tau \tag{10}
\end{equation*}
$$

which gives $v^{\prime} \leq 0$, meaning that $v$ is decreasing. Similarly, one obtains that $u$ is decreasing. From $u(R)=0$ we have $u \geq 0$ and analogously $v \geq 0$. Assuming that $u \equiv 0$, from $v \not \equiv 0$, equality (10), (ii) in $\left(H_{g}^{3}\right)$ and $v(R)=0$ we get $v \equiv 0$, which is a contradiction. It follows that $u \not \equiv 0$. A similar argument shows that $v \not \equiv 0$. Then, from (10), hypothesis $(i)$ in $\left(H_{g}^{3}\right)$ and $u(0)>0<v(0)$ we get $v^{\prime}<0$, thus $v$ is strictly decreasing and $v>0$ on $[0, R)$. Similarly, $u$ is positive and strictly decreasing.

Remark 2.5. Under the assumptions of Lemmas 2.3 and 2.4 any nontrivial solution of problem (8) actually solves the system (3).

## 3. Sublinear nonlinearities near origin

In this section we deal with positive solutions of problem (3) when $g_{1}$ (resp. $g_{2}$ ) has a sublinear growth near origin with respect to $u$ (resp. $v$ ).

Theorem 3.1. Assume that $g_{1}, g_{2}:[0, R] \times[0, \infty)^{2} \rightarrow[0, \infty)$ are continuous and satisfy hypothesis $\left(H_{g}^{1}\right)$ in Lemma 2.3. If $g_{1}(r, s, t)$ (resp. $g_{2}(r, s, t)$ ) is quasi-monotone nondecreasing with respect to $t$ (resp.s) and

$$
\begin{align*}
& \lim _{s \rightarrow 0_{+}} \frac{g_{1}(r, s, 0)}{s}=+\infty \text { uniformly with } r \in[0, R]  \tag{11}\\
& \lim _{t \rightarrow 0_{+}} \frac{g_{2}(r, 0, t)}{t}=+\infty \text { uniformly with } r \in[0, R] \tag{12}
\end{align*}
$$

then problem (3) has a solution ( $u, v$ ) with $u \geq 0 \leq v$ and either $u$ or $v$ positive and strictly decreasing. If in addition, $\left(H_{g}^{2}\right)$ in Lemma 2.3 holds true, then problem (3) has a positive solution ( $u, v$ ) with both $u$ and $v$ strictly decreasing.

Proof. We make use of some ideas from [6]. First, we show that there exists $\rho \in(0, R+1)$ such that problem

$$
\left\{\begin{array}{l}
\left(r^{N-1} \varphi\left(u^{\prime}\right)\right)^{\prime}+r^{N-1}\left[g_{1}\left(r, u_{+}, v_{+}\right)+\tau\right]=0  \tag{13}\\
\left(r^{N-1} \varphi\left(v^{\prime}\right)\right)^{\prime}+r^{N-1}\left[g_{2}\left(r, u_{+}, v_{+}\right)+\tau\right]=0 \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

has at most the trivial solution in $\bar{B}_{\rho}$, for all $\tau \in[0,1]$. By contradiction, assume that there exist $\left\{\tau_{k}\right\} \subset[0,1],\left\{\left(u_{k}, v_{k}\right)\right\} \subset \mathcal{C}_{M}^{1} \backslash\{(0,0)\},\left\|\left(u_{k}, v_{k}\right)\right\| \rightarrow 0$, such that ( $u_{k}, v_{k}$ ) is a nontrivial solution of (13) with $\tau=\tau_{k}$, for all $k \in \mathbb{N}$. From Lemma 2.3 we have that either $u_{k}$ or $v_{k}$ is positive and strictly decreasing. We may assume that e.g., $u_{k}$ is positive for all $k \in \mathbb{N}$. Choose $m>0$, with

$$
\begin{equation*}
\frac{m(R / 3)^{N}}{N(2 R / 3)^{N-1}}>\frac{3}{R} \tag{14}
\end{equation*}
$$

Then, using (11) (similar reasoning with (12) when all $v_{k}$ are positive) we can find $k_{0} \in \mathbb{N}$ such that

$$
\begin{equation*}
g_{1}\left(r, u_{k}(r), 0\right) \geq m \varphi\left(u_{k}(r)\right) \text { for all } r \in[0, R] \text { and } k \geq k_{0} . \tag{15}
\end{equation*}
$$

Moreover, integrating over $[0, r]$ the first equation in (13) with $\tau=\tau_{k}, u=u_{k}$, $v=v_{k}$, using that $g_{1}\left(r, \xi_{+}, \eta_{+}\right)$is quasi-monotone nondecreasing with respect to $\eta$ and taking into account (15) one has

$$
-\varphi\left(u_{k}^{\prime}\right) \geq m S\left[\varphi\left(u_{k}\right)\right]
$$

Next, following exactly the estimations in the proof of [6, Proposition 1] we obtain

$$
\frac{\varphi\left(3 u_{k}(R / 3) / R\right)}{\varphi\left(u_{k}(R / 3)\right)} \geq \frac{m(R / 3)^{N}}{N(2 R / 3)^{N-1}}
$$

for $k$ sufficiently large. By passing with $k \rightarrow \infty$, and taking into account that $u_{k}(R / 3) \rightarrow 0$ we get a contradiction with (14).

Note that (13) has no solution in $\bar{B}_{\rho}$, for any $\tau \in(0,1]$.
Next, we consider the compact homotopy $\mathcal{H}:[0,1] \times \bar{B}_{\rho} \rightarrow \mathcal{C}_{M}^{1}$,

$$
\mathcal{H}(\tau,(u, v))=\mathcal{N}_{g+\tau}(u, v)
$$

where by $\mathcal{N}_{g+\tau}$ we have denoted the fixed point operator associated to (13). Notice, the Leray-Schauder condition on the boundary

$$
(u, v) \neq \mathcal{H}(\tau,(u, v)), \text { for all }(\tau,(u, v)) \in[0,1] \times \partial B_{\rho}
$$

is fulfilled. Then, from the invariance under homotopy of the Leray-Schauder degree we have

$$
d_{L S}\left[I-\mathcal{H}(0, \cdot), B_{\rho}, 0\right]=d_{L S}\left[I-\mathcal{H}(1, \cdot), B_{\rho}, 0\right] .
$$

So, assuming that $d_{L S}\left[I-\mathcal{H}(1, \cdot), B_{\rho}, 0\right] \neq 0$, we infer that there exists $(u, v) \in$ $B_{\rho}$ with $\mathcal{H}(1,(u, v))=(u, v)$, a contradiction. Consequently,

$$
d_{L S}\left[I-\mathcal{H}(1, \cdot), B_{\rho}, 0\right]=0
$$

Using Proposition 2.1 together with the excision property of Leray-Schauder degree one obtains

$$
d_{L S}\left[I-\mathcal{N}_{g}, B_{R+1} \backslash \bar{B}_{\rho}, 0\right]=1
$$

where $\mathcal{N}_{g}$ is the fixed point operator associated to problem (8). Therefore, there exists a solution $(u, v) \in B_{R+1} \backslash \bar{B}_{\rho}$ of (8). The conclusion follows by Lemma 2.3 and Remark 2.5.

Remark 3.2. From [6, Theorem 1] it is known that, if $g:[0, R] \times[0, \infty) \rightarrow$ $[0, \infty)$ is continuous, $g(r, s)>0$, for all $(r, s) \in(0, R] \times(0, \infty)$ and

$$
\lim _{s \rightarrow 0_{+}} \frac{g(r, s)}{s}=+\infty \text { uniformly with } r \in[0, R]
$$

then the mixed boundary value problem

$$
\left\{\begin{array}{l}
\left(r^{N-1} \varphi\left(u^{\prime}\right)\right)^{\prime}+r^{N-1} g(r, u)=0 \\
u^{\prime}(0)=0=u(R)
\end{array}\right.
$$

has a positive solution. It is easily seen that this result also follows from Theorem 3.1 by taking $g_{1}(r, \xi, \eta)=g(r, \xi)=g_{2}(r, \eta, \xi)$.
Corollary 3.3. Let $g_{1}, g_{2}:[0, R] \times[0, \infty)^{2} \rightarrow[0, \infty)$ be continuous and satisfy hypothesis $\left(H_{g}^{1}\right)$ in Lemma 2.3. If $g_{1}(r, s, t)$ (resp. $g_{2}(r, s, t)$ ) is quasi-monotone nondecreasing with respect to $t$ (resp. s) and (11), (12) hold true, then the system (1) has a solution ( $\mathrm{u}, \mathrm{v}$ ) with $\mathrm{u} \geq 0 \leq \mathrm{v}$ and either u or v positive and radially strictly decreasing. If in addition, $\left(H_{g}^{2}\right)$ in Lemma 2.3 is satisfied, then problem (1) has a positive solution ( $\mathrm{u}, \mathrm{v}$ ) with both u and v strictly decreasing.
Example 3.4. Let $p_{1}, q_{2} \in(0,1)$ and $q_{1} \geq 0 \leq p_{2}$.
(i) The system

$$
\begin{cases}\mathcal{M}(\mathrm{u})+\mathrm{u}^{p_{1}}+\mathrm{uv}^{q_{1}}=0, & \text { in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v})+\mathrm{vu}^{p_{2}}+\mathrm{v}^{q_{2}}=0, & \text { in } \mathcal{B}(R), \\ \left.\mathrm{u}\right|_{\partial \mathcal{B}(R)}=0=\left.\mathrm{v}\right|_{\partial \mathcal{B}(R)} & \end{cases}
$$

has a solution ( $\mathrm{u}, \mathrm{v}$ ) with $\mathrm{u} \geq 0 \leq \mathrm{v}$ and either u or v positive and radially strictly decreasing.
(ii) The system

$$
\left\{\begin{array}{l}
\mathcal{M}(\mathrm{u})+\mathrm{u}^{p_{1}}+\mathrm{v}^{q_{1}}=0, \quad \text { in } \mathcal{B}(R), \\
\mathcal{M}(\mathrm{v})+\mathrm{u}^{p_{2}}+\mathrm{v}^{q_{2}}=0, \quad \text { in } \mathcal{B}(R), \\
\left.\mathrm{u}\right|_{\partial \mathcal{B}(R)}=0=\left.\mathrm{v}\right|_{\partial \mathcal{B}(R)}
\end{array}\right.
$$

has a solution ( $\mathrm{u}, \mathrm{v}$ ) with $\mathrm{u}>0<\mathrm{v}$ on $\mathcal{B}(R)$ and both u and v radially strictly decreasing.

## 4. Lower and upper solutions; degree estimations

A lower solution of (4) is a couple of functions $\left(\alpha_{u}, \alpha_{v}\right) \in C^{1} \times C^{1}$, such that $\left\|\alpha_{u}^{\prime}\right\|_{\infty}<1,\left\|\alpha_{v}^{\prime}\right\|_{\infty}<1$, the mappings $r \mapsto r^{N-1} \varphi\left(\alpha_{u}^{\prime}(r)\right), r \mapsto r^{N-1} \varphi\left(\alpha_{v}^{\prime}(r)\right)$ are of class $C^{1}$ on $[0, R]$ and satisfies

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(\alpha_{u}^{\prime}\right)\right]^{\prime}+r^{N-1} f_{1}\left(r, \alpha_{u}, \alpha_{v}\right) \geq 0} \\
{\left[r^{N-1} \varphi\left(\alpha_{v}^{\prime}\right)\right]^{\prime}+r^{N-1} f_{2}\left(r, \alpha_{u}, \alpha_{v}\right) \geq 0} \\
\alpha_{u}(R) \leq 0, \quad \alpha_{v}(R) \leq 0
\end{array}\right.
$$

An upper solution $\left(\beta_{u}, \beta_{v}\right) \in C^{1} \times C^{1}$ is defined by reversing the above inequalities.

Proposition 4.1. If (4) has a lower solution $\left(\alpha_{u}, \alpha_{v}\right)$ and an upper solution $\left(\beta_{u}, \beta_{v}\right)$ such that $\alpha_{u}(r) \leq \beta_{u}(r), \alpha_{v}(r) \leq \beta_{v}(r)$ for all $r \in[0, R]$ and $f_{1}(r, s, t)$ (resp. $f_{2}(r, s, t)$ ) is quasi-monotone nondecreasing with respect to $t$ (resp. s), then (4) has a solution $(u, v)$ such that $\alpha_{u}(r) \leq u(r) \leq \beta_{u}(r)$ and $\alpha_{v}(r) \leq$ $v(r) \leq \beta_{v}(r)$ for all $r \in[0, R]$.

Proof. Define the modified functions

$$
\begin{aligned}
& \Gamma_{1}(r, u, v)=f_{1}\left(r, \gamma_{1}(r, u), \gamma_{2}(r, v)\right)-u+\gamma_{1}(r, u) \\
& \Gamma_{2}(r, u, v)=f_{2}\left(r, \gamma_{1}(r, u), \gamma_{2}(r, v)\right)-v+\gamma_{2}(r, v)
\end{aligned}
$$

where $\gamma_{i}$ are given by

$$
\gamma_{1}(r, \xi)=\max \left\{\alpha_{u}(r), \min \left\{\xi, \beta_{u}(r)\right\}\right\}, \gamma_{2}(r, \xi)=\max \left\{\alpha_{v}(r), \min \left\{\xi, \beta_{v}(r)\right\}\right\}
$$

Then $\Gamma_{1}, \Gamma_{2}:[0, R] \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ are continuous and we consider the modified problem

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+r^{N-1} \Gamma_{1}(r, u, v)=0,}  \tag{16}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+r^{N-1} \Gamma_{2}(r, u, v)=0,} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

From Proposition 2.1 it follows that problem (16) has at least one solution. We show now that if $(u, v)$ is a solution of (16) then $\alpha_{u}(r) \leq u(r) \leq \beta_{u}(r)$ and $\alpha_{v}(r) \leq v(r) \leq \beta_{v}(r)$ for all $r \in[0, R]$. We only prove that $\alpha_{u} \leq u$ on $[0, R]$, the remainder can be obtain analogously.

By contradiction, suppose that exists $r_{0} \in[0, R]$ such that

$$
\begin{equation*}
\max _{[0, R]}\left(\alpha_{u}-u\right)=\alpha_{u}\left(r_{0}\right)-u\left(r_{0}\right)>0 \tag{17}
\end{equation*}
$$

If $r_{0} \in(0, R)$, then $\alpha_{u}^{\prime}\left(r_{0}\right)=u^{\prime}\left(r_{0}\right)$ and there exists a sequence $\left\{r_{k}\right\} \subset\left(0, r_{0}\right)$ converging to $r_{0}$ such that $\alpha_{u}^{\prime}\left(r_{k}\right)-u^{\prime}\left(r_{k}\right) \geq 0$. Since $\varphi$ is increasing, this implies

$$
r_{k}^{N-1} \varphi\left(\alpha_{u}^{\prime}\left(r_{k}\right)\right)-r_{0}^{N-1} \varphi\left(\alpha_{u}^{\prime}\left(r_{0}\right)\right) \geq r_{k}^{N-1} \varphi\left(u^{\prime}\left(r_{k}\right)\right)-r_{0}^{N-1} \varphi\left(u^{\prime}\left(r_{0}\right)\right)
$$

which yields

$$
\left[r^{N-1} \varphi\left(\alpha_{u}^{\prime}(r)\right)\right]_{r=r_{0}}^{\prime} \leq\left[r^{N-1} \varphi\left(u^{\prime}(r)\right)\right]_{r=r_{0}}^{\prime}
$$

Hence, because $\left(\alpha_{u}, \alpha_{v}\right)$ is a lower solution of (3) and $f_{1}$ is quasi-monotone nondecreasing with respect to $v$, we obtain

$$
\begin{aligned}
{\left[r^{N-1} \varphi\left(\alpha_{u}^{\prime}(r)\right)\right]_{r=r_{0}}^{\prime} } & \leq\left[r^{N-1} \varphi\left(u^{\prime}(r)\right)\right]_{r=r_{0}}^{\prime} \\
& =r_{0}^{N-1}\left[-f_{1}\left(r_{0}, \alpha_{u}\left(r_{0}\right), \gamma_{2}\left(r_{0}, v\left(r_{0}\right)\right)\right)+u\left(r_{0}\right)-\alpha_{u}\left(r_{0}\right)\right] \\
& <r_{0}^{N-1}\left[-f_{1}\left(r_{0}, \alpha_{u}\left(r_{0}\right), \gamma_{2}\left(r_{0}, v\left(r_{0}\right)\right)\right)\right] \\
& \leq r_{0}^{N-1}\left[-f_{1}\left(r_{0}, \alpha_{u}\left(r_{0}\right), \alpha_{v}\left(r_{0}\right)\right)\right] \\
& \leq\left[r^{N-1} \varphi\left(\alpha_{u}^{\prime}(r)\right)\right]_{r=r_{0}}^{\prime}
\end{aligned}
$$

a contradiction. If $r_{0}=R$ then $\alpha_{u}(R)-u(R)>0$, contradiction with $\alpha_{u}(R) \leq$ 0 . Finally, if $r_{0}=0$, then there exists $r_{1} \in(0, R]$ such that $\alpha_{u}(r)-u(r)>0$ for all $r \in\left[0, r_{1}\right]$ and $\alpha_{u}^{\prime}\left(r_{1}\right)-u^{\prime}\left(r_{1}\right) \leq 0$. It follows that

$$
r_{1}^{N-1} \varphi\left(\alpha_{u}^{\prime}\left(r_{1}\right)\right) \leq r_{1}^{N-1} \varphi\left(u^{\prime}\left(r_{1}\right)\right) .
$$

Integrating the first equation in problem (16) from 0 to $r_{1}$ and using the fact that $\left(\alpha_{u}, \alpha_{v}\right)$ is a lower solution of (4) and $f_{1}$ is quasi-monotone nondecreasing with respect to $v$ we get

$$
\begin{aligned}
r_{1}^{N-1} \varphi\left(u^{\prime}\left(r_{1}\right)\right) & =\int_{0}^{r_{1}} r^{N-1}\left[-f_{1}\left(r, \alpha_{u}(r), \gamma_{2}(r, v(r))\right)+u(r)-\alpha_{u}(r)\right] d r \\
& <\int_{0}^{r_{1}} r^{N-1}\left[-f_{1}\left(r, \alpha_{u}(r), \gamma_{2}(r, v(r))\right)\right] d r \\
& \leq \int_{0}^{r_{1}} r^{N-1}\left[-f_{1}\left(r, \alpha_{u}(r), \alpha_{v}(r)\right)\right] d r \\
& \leq \int_{0}^{r_{1}}\left[r^{N-1} \varphi\left(\alpha_{u}^{\prime}(r)\right)\right]^{\prime} d r \\
& =r_{1}^{N-1} \varphi\left(\alpha_{u}^{\prime}\left(r_{1}\right)\right),
\end{aligned}
$$

a contradiction. Consequently, $\alpha_{u}(r) \leq u(r)$ for all $r \in[0, R]$.
Lemma 4.2. Assume that (4) has a lower solution $\left(\alpha_{u}, \alpha_{v}\right)$ and an upper solution $\left(\beta_{u}, \beta_{v}\right)$ such that $\alpha_{u}(r) \leq \beta_{u}(r), \alpha_{v}(r) \leq \beta_{v}(r)$ for all $r \in[0, R]$ and $f_{1}(r, s, t)$ (resp. $f_{2}(r, s, t)$ ) is quasi-monotone nondecreasing with respect to $t$ (resp. s). Let

$$
\mathcal{A}_{\alpha, \beta}:=\left\{(u, v) \in \mathcal{C}_{M}^{1}: \alpha_{u} \leq u \leq \beta_{u}, \alpha_{v} \leq v \leq \beta_{v}\right\} .
$$

Assume also that (4) has an unique solution $\left(u_{0}, v_{0}\right)$ in $\mathcal{A}_{\alpha, \beta}$ and there exists $\rho_{0}>0$ such that $\bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{0}\right) \subset \mathcal{A}_{\alpha, \beta}$. Then

$$
d_{L S}\left[I-\mathcal{N}_{f}, B\left(\left(u_{0}, v_{0}\right), \rho\right), 0\right]=1, \quad \text { for all } 0<\rho \leq \rho_{0}
$$

where $\mathcal{N}_{f}$ stands for the fixed point operator associated to (4).
Proof. Let $\mathcal{N}_{\Gamma}$ be the fixed point operator associated to (16). From Proposition 2.1 and the proof of Proposition 4.1 it follows that any fixed point $(u, v)$ of $\mathcal{N}_{\Gamma}$ is contained in $\mathcal{A}_{\alpha, \beta}$ and it is fixed point of $\mathcal{N}_{f}$. Using again Proposition 2.1 together with the excision property of the Leray-Schauder degree one has that

$$
d_{L S}\left[I-\mathcal{N}_{\Gamma}, B\left(\left(u_{0}, v_{0}\right), \rho\right), 0\right]=1 \text { for all } 0<\rho \leq \rho_{0}
$$

The conclusion follows from the fact that $\mathcal{N}_{\Gamma}=\mathcal{N}_{f}$ on $\mathcal{A}_{\alpha, \beta} \supset \bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{0}\right)$.

Lemma 4.3. Assume that $g_{1}, g_{2}:[0, R] \times[0, \infty)^{2} \rightarrow[0, \infty)$ are continuous and satisfy hypothesis $\left(H_{g}^{3}\right)$ in Lemma 2.4. If there is some $M>0$ such that either

$$
\begin{equation*}
\lim _{s \rightarrow 0_{+}} \frac{g_{1}(r, s, t)}{s}=0 \text { uniformly with } r \in[0, R], t \in[0, M] \tag{18}
\end{equation*}
$$

or

$$
\begin{equation*}
\lim _{t \rightarrow 0_{+}} \frac{g_{2}(r, s, t)}{t}=0 \text { uniformly with } r \in[0, R], s \in[0, M] \tag{19}
\end{equation*}
$$

then there exists $\rho_{0}>0$ such that

$$
d_{L S}\left[I-\mathcal{N}_{g}, B_{\rho}, 0\right]=1 \text { for all } 0<\rho \leq \rho_{0}
$$

where $\mathcal{N}_{g}$ is the fixed point operator associated to problem (8).
Proof. Let $0<\varepsilon<N / R^{2}$. Assume (18) (similar reasoning when (19) holds true). Then there exists $s_{\varepsilon}>0$ such that for all $s \in\left(0, s_{\varepsilon}\right)$,

$$
\begin{equation*}
g_{1}(r, s, t) \leq \varepsilon \varphi(s) \text { for all } r \in[0, R], t \in[0, M] \tag{20}
\end{equation*}
$$

Consider the compact homotopy

$$
\mathcal{H}:[0,1] \times \mathcal{C}_{M}^{1} \rightarrow \mathcal{C}_{M}^{1}, \quad \mathcal{H}(\tau, u, v)=\tau \mathcal{N}_{g}(u, v) .
$$

We show that there exists $\rho_{0}>0$ such that

$$
(u, v) \neq \mathcal{H}(\tau, u, v), \text { for all }(\tau, u, v) \in[0,1] \times\left(\bar{B}_{\rho_{0}} \backslash\{(0,0)\}\right)
$$

By contradiction, assume that

$$
\left(u_{k}, v_{k}\right)=\tau_{k} \mathcal{N}_{g}\left(u_{k}, v_{k}\right),
$$

with $\tau_{k} \in[0,1],\left(u_{k}, v_{k}\right) \in \mathcal{C}_{M}^{1} \backslash\{(0,0)\}$ for all $k \in \mathbb{N}$ and $\left\|\left(u_{k}, v_{k}\right)\right\| \rightarrow 0$. From Lemma 2.4 we have that both $u_{k}$ and $v_{k}$ are strictly positive on $[0, R)$. We may assume (passing if necessary to a subsequence) that $\left\|u_{k}\right\|_{\infty} \leq s_{\varepsilon}$ and $\left\|v_{k}\right\|_{\infty} \leq M$ for all $k \in \mathbb{N}$. Using (20) it follows that

$$
g_{1}\left(r, u_{k}(r), v_{k}(r)\right) \leq \varepsilon \varphi\left(\left\|u_{k}\right\|_{\infty}\right) \text { for all } r \in[0, R], k \in \mathbb{N} .
$$

For any $k \in \mathbb{N}$ we obtain

$$
\begin{aligned}
\left\|u_{k}\right\|_{\infty} & \leq \int_{0}^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_{0}^{t} r^{N-1} g_{1}\left(r, u_{k}(r), v_{k}(r)\right) d r\right) d t \\
& \leq \int_{0}^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_{0}^{t} r^{N-1} \varepsilon \varphi\left(\left\|u_{k}\right\|_{\infty}\right) d r\right) d t \\
& \leq R \varphi^{-1}\left(\varepsilon \frac{R}{N} \varphi\left(\left\|u_{k}\right\|_{\infty}\right)\right)
\end{aligned}
$$

It follows that

$$
\frac{\varphi\left(\left\|u_{k}\right\|_{\infty} / R\right)}{\varphi\left(\left\|u_{k}\right\|_{\infty}\right)} \leq \frac{\varepsilon R}{N}, \quad \forall k \in \mathbb{N}
$$

By passing with $k \rightarrow \infty$, we get $1 / R \leq \varepsilon R / N$, contradicting the choice of $\varepsilon$.
Then, from the invariance under homotopy of Leray-Schauder degree we have that for any $\rho \in\left(0, \rho_{0}\right]$,

$$
d_{L S}\left[I-\mathcal{N}_{g}, B_{\rho}, 0\right]=d_{L S}\left[I, B_{\rho}, 0\right]=1
$$

which completes the proof.

## 5. Non-existence, existence and multiplicity

In this section we study the one-parameter gradient system

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+\lambda r^{N-1} \mu(r)(p+1) u^{p} v^{q+1}=0,}  \tag{21}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+\lambda r^{N-1} \mu(r)(q+1) u^{p+1} v^{q}=0,} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0),
\end{array}\right.
$$

under the hypothesis:
( $H$ ) The positive exponents $p, q$ satisfy $\max \{p, q\}>1$ and the function $\mu$ : $[0, R] \rightarrow[0, \infty)$ is continuous and $\mu(r)>0$ for all $r \in(0, R]$.

Theorem 5.1. Assume ( $H$ ). Then there exists $\Lambda>0$ such that the system (21) has zero, at least one or at least two positive solutions according to $\lambda \in(0, \Lambda)$, $\lambda=\Lambda$ or $\lambda>\Lambda$.

Proof. We assume that $p>1, q>0$ and we divide the proof in two steps.

1. Existence of $\Lambda$; the cases $\lambda \in(0, \Lambda)$ and $\lambda=\Lambda$. First, notice that, by Lemma 2.4 and Remark 2.5, $(u, v)$ is a positive solution of problem (21) if and only if $(u, v)$ is a nontrivial solution of

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+\lambda r^{N-1} \mu(r)(p+1) u_{+}^{p} v_{+}^{q+1}=0}  \tag{22}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+\lambda r^{N-1} \mu(r)(q+1) u_{+}^{p+1} v_{+}^{q}=0} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

and in this case, $u, v$ are strictly decreasing. We set

$$
\mathcal{S}:=\{\lambda>0:(21) \text { has a positive solution }\} .
$$

Let $\lambda>0$ and $(u, v)$ be a positive solution of (21). Integrating the first equation in (21) on $[0, r]$, one obtains

$$
-r^{N-1} \varphi\left(u^{\prime}(r)\right)=\lambda(p+1) \int_{0}^{r} t^{N-1} \mu(t) u^{p}(t) v^{q+1}(t) d t \text { for all } r \in[0, R]
$$

Since $u, v$ are strictly decreasing on $[0, R]$, we deduce

$$
\begin{aligned}
-r^{N-1} u^{\prime}(r) & \leq-r^{N-1} \varphi\left(u^{\prime}(r)\right) \\
& \leq \lambda(p+1) \mu_{M} u^{p}(0) v^{q+1}(0) r^{N} / N
\end{aligned}
$$

where $\mu_{M}:=\max _{[0, R]} \mu$. This gives

$$
\begin{equation*}
u(0) \leq \lambda(p+1) \mu_{M} u^{p}(0) v^{q+1}(0) R^{2} /(2 N) \tag{23}
\end{equation*}
$$

From $0<u(0), v(0)<R$ and $p>1$ we obtain

$$
\begin{equation*}
\lambda>2 N /\left[(p+1) \mu_{M} R^{p+q+2}\right] \tag{24}
\end{equation*}
$$

The energy functional $\mathcal{I}_{\lambda}: C \times C \rightarrow(-\infty,+\infty]$ associated to problem (22) is
$\mathcal{I}_{\lambda}(u, v)=\frac{2 R^{N}}{N}-\int_{0}^{R} r^{N-1}\left[\sqrt{1-u^{2}}+\sqrt{1-v^{\prime 2}}\right] d r-\lambda \int_{0}^{R} r^{N-1} \mu(r) u_{+}^{p+1} v_{+}^{q+1} d r$
for $(u, v) \in K_{0} \times K_{0}$ and $\mathcal{I}_{\lambda} \equiv+\infty$ on $C \times C \backslash K_{0} \times K_{0}$. Computing the value of $\mathcal{I}_{\lambda}$ at $u_{0}(r)=v_{0}(r)=R-r$ we obtain that $\mathcal{I}_{\lambda}\left(u_{0}, v_{0}\right)<0$, for $\lambda>0$ large enough. Hence, for such $\lambda$, the functional $\mathcal{I}_{\lambda}$ has a negative minimum and, as $\mathcal{I}_{\lambda}(0,0)=0$, from Proposition 2.2 we have that problem (22) has a nontrivial solution. In particular, $\mathcal{S} \neq \emptyset$. We denote

$$
\Lambda=\Lambda(R):=\inf \mathcal{S}(<+\infty)
$$

and we show that $\Lambda \in \mathcal{S}$. For this, let $\left\{\lambda_{k}\right\} \subset \mathcal{S}$ be a sequence converging to $\Lambda$ and $\left(u_{k}, v_{k}\right) \in \mathcal{C}_{M}^{1}$ with $u_{k}>0<v_{k}$ on $[0, R)$ such that

$$
\begin{aligned}
u_{k} & =K \circ \varphi^{-1} \circ S \circ\left[\lambda_{k}(p+1) \mu u_{k}^{p} v_{k}^{q+1}\right], \\
v_{k} & =K \circ \varphi^{-1} \circ S \circ\left[\lambda_{k}(q+1) \mu u_{k}^{p+1} v_{k}^{q}\right] .
\end{aligned}
$$

From (5) and Arzela-Ascoli theorem we obtain that there exists $(u, v) \in C \times C$ such that, passing eventually to a subsequence, $\left\{\left(u_{k}, v_{k}\right)\right\}$ converges to $(u, v)$ in $C \times C$. Hence, $u \geq 0 \leq v$ and

$$
\begin{aligned}
& u=K \circ \varphi^{-1} \circ S \circ\left[\Lambda(p+1) \mu u^{p} v^{q+1}\right] \\
& v=K \circ \varphi^{-1} \circ S \circ\left[\Lambda(q+1) \mu u^{p+1} v^{q}\right] .
\end{aligned}
$$

Using (23) we infer that there is a constant $c>0$ such that $u_{k}(0)>c$ for all $k$ sufficiently large. This leads to $u(0) \geq c$, hence by Lemma 2.4 we get $u>0<v$ on $[0, R)$. Consequently, $\Lambda \in \mathcal{S}$. Also, from (24) it is clear that

$$
\Lambda>2 N /\left[(p+1) \mu_{M} R^{p+q+2}\right] .
$$

2. The case $\lambda>\Lambda$. First, we show that $(\Lambda, \infty) \subset \mathcal{S}$. With this aim, let $\lambda_{0} \in(\Lambda, \infty)$ be arbitrarily chosen and $\left(u_{\Lambda}, v_{\Lambda}\right)$ be a positive solution for (21) with $\lambda=\Lambda$. Then, $\left(u_{\Lambda}, v_{\Lambda}\right)$ is a lower solution of (22) with $\lambda=\lambda_{0}$. In order to construct an upper solution for (22), we first observe that if $H_{1}>0<H_{2}$, the mixed boundary value problem

$$
\left\{\begin{array}{l}
{\left[r^{N-1} \varphi\left(u^{\prime}\right)\right]^{\prime}+r^{N-1} H_{1}=0}  \tag{25}\\
{\left[r^{N-1} \varphi\left(v^{\prime}\right)\right]^{\prime}+r^{N-1} H_{2}=0} \\
u^{\prime}(0)=u(R)=0=v(R)=v^{\prime}(0)
\end{array}\right.
$$

has as the unique (positive) solution the couple

$$
\begin{array}{ll}
u_{H_{1}}(r)=\frac{N}{H_{1}}\left[\sqrt{1+\frac{H_{1}^{2}}{N^{2}} R^{2}}-\sqrt{1+\frac{H_{1}^{2}}{N^{2}} r^{2}}\right], \quad r \in[0, R], \\
v_{H_{2}}(r)=\frac{N}{H_{2}}\left[\sqrt{1+\frac{H_{2}^{2}}{N^{2}} R^{2}}-\sqrt{1+\frac{H_{2}^{2}}{N^{2}} r^{2}}\right], \quad r \in[0, R] .
\end{array}
$$

Below, $\tilde{R}$ will be $>R$. For fixed $\tilde{\lambda}>\lambda_{0}$, let $\left(u_{H_{1}}, v_{H_{2}}\right)$ be the solution of (25) corresponding to

$$
\begin{aligned}
H_{1} & =\tilde{\lambda}(p+1) \mu_{M} \tilde{R}^{p+q+1} \\
H_{2} & =\tilde{\lambda}(q+1) \mu_{M} \tilde{R}^{p+q+1}
\end{aligned}
$$

Using that $R<\tilde{R}$, together with

$$
\begin{array}{ll}
\lambda_{0}(p+1) \mu(r) u_{H_{1}}^{p} v_{H_{2}}^{q+1}(r) \leq \tilde{\lambda}(p+1) \mu_{M} \tilde{R}^{p+q+1}, & r \in[0, \tilde{R}], \\
\lambda_{0}(q+1) \mu(r) u_{H_{1}}^{p+1} v_{H_{2}}^{q}(r) \leq \tilde{\lambda}(q+1) \mu_{M} \tilde{R}^{p+q+1}, & r \in[0, \tilde{R}],
\end{array}
$$

it follows that $\left(u_{H_{1}}, v_{H_{2}}\right)$ is an upper solution for (22) with $\lambda=\lambda_{0}$. From the fact that

$$
u_{H_{1}}(R)=N\left[\sqrt{\frac{\tilde{R}^{-2(p+q+1)}}{\left(\tilde{\lambda}(p+1) \mu_{M}\right)^{2}}+\frac{\tilde{R}^{2}}{N^{2}}}-\sqrt{\frac{\tilde{R}^{-2(p+q+1)}}{\left(\tilde{\lambda}(p+1) \mu_{M}\right)^{2}}+\frac{R^{2}}{N^{2}}}\right]
$$

there exists $\tilde{R}$ sufficiently large, such that $u_{H_{1}}(R)>u_{\Lambda}(0)$ and similarly, we may assume that $v_{H_{2}}(R)>v_{\Lambda}(0)$. Taking into account that $u_{H_{1}}, v_{H_{2}}, u_{\Lambda}, v_{\Lambda}$
are strictly decreasing it follows that $u_{\Lambda}<u_{H_{1}}$ and $v_{\Lambda}<v_{H_{2}}$ on $[0, R]$. From Proposition 4.1 we obtain that $\lambda_{0} \in \mathcal{S}$.

Next, we show that for $\lambda_{0} \in(\Lambda, \infty)$ problem (22) with $\lambda=\lambda_{0}$ has a second positive solution. For this, let $\left(u_{\Lambda}, v_{\Lambda}\right)$ be the lower solution and ( $u_{H_{1}}, v_{H_{2}}$ ) be the upper solution constructed as above. We fix $\left(u_{0}, v_{0}\right)$ a positive solution of (21) with $\lambda=\lambda_{0}$ such that $\left(u_{0}, v_{0}\right) \in \mathcal{A}:=\mathcal{A}_{\left(u_{\Lambda}, v_{\Lambda}\right),\left(u_{H_{1}}, v_{H_{2}}\right)}$ (see Lemma 4.2).

Firstly, we claim that there exists $\varepsilon>0$ such that $\bar{B}\left(\left(u_{0}, v_{0}\right), \varepsilon\right) \subset \mathcal{A}$. Note that, for all $r \in[0, R]$ we have

$$
\begin{aligned}
u_{H_{1}}(r) & =\int_{r}^{\tilde{R}} \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_{0}^{t} s^{N-1}\left[\tilde{\lambda}(p+1) \mu_{M} \tilde{R}^{p+q+1}\right] d s\right) d t \\
& >\int_{r}^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_{0}^{t} s^{N-1}\left[\tilde{\lambda}(p+1) \mu_{M} \tilde{R}^{p+q+1}\right] d s\right) d t \\
& \geq \int_{r}^{R} \varphi^{-1}\left(\frac{1}{t^{N-1}} \int_{0}^{t} s^{N-1}\left[\lambda_{0}(p+1) \mu(s) u_{0}^{p}(s) v_{0}^{q+1}(s)\right] d s\right) d t \\
& =u_{0}(r) .
\end{aligned}
$$

Analogously we obtain that $v_{H_{2}}(r)>v_{0}(r)$. Thus, there exists $\varepsilon_{1}>0$ such that if $(u, v) \in \mathcal{C}_{M}^{1}$ then

$$
\begin{equation*}
\left\|u-u_{0}\right\|_{\infty} \leq \varepsilon_{1} \Rightarrow u \leq u_{H_{1}} \text { and }\left\|v-v_{0}\right\|_{\infty} \leq \varepsilon_{1} \Rightarrow v \leq v_{H_{2}} \tag{26}
\end{equation*}
$$

Using similar arguments we have $u_{\Lambda}(r)<u_{0}(r)$ and $v_{\Lambda}(r)<v_{0}(r)$ on [0, R/2]. So, we can find $\varepsilon_{1}^{\prime}>0$ such that if $(u, v) \in \mathcal{C}_{M}^{1}$ then

$$
\begin{equation*}
\left\|u-u_{0}\right\|_{\infty} \leq \varepsilon_{1}^{\prime} \Rightarrow u_{\Lambda} \leq u \text { and }\left\|v-v_{0}\right\|_{\infty} \leq \varepsilon_{1}^{\prime} \Rightarrow v_{\Lambda} \leq v \text { on }[0, R / 2] \tag{27}
\end{equation*}
$$

On the other hand, for $r \in[R / 2, R]$ one obtains $u_{0}^{\prime}(r)<u_{\Lambda}^{\prime}(r)$ and $v_{0}^{\prime}(r)<$ $v_{\Lambda}^{\prime}(r)$. Thus, there is some $\varepsilon_{1}^{\prime \prime} \in\left(0, \varepsilon_{1}^{\prime}\right)$ such that if $(u, v) \in \mathcal{C}_{M}^{1}$, then

$$
\left\|u^{\prime}-u_{0}^{\prime}\right\|_{\infty} \leq \varepsilon_{1}^{\prime \prime} \Rightarrow u_{\Lambda}^{\prime}>u^{\prime} \text { and }\left\|v^{\prime}-v_{0}^{\prime}\right\|_{\infty} \leq \varepsilon_{1}^{\prime \prime} \Rightarrow v_{\Lambda}^{\prime}>v^{\prime} \text { on }[R / 2, R] .
$$

From $u_{\Lambda}(R)=0=u(R)$ we deduce that $u>u_{\Lambda}$ (and, similarly $v>v_{\Lambda}$ ) on $[R / 2, R)$. This means that

$$
\begin{equation*}
\left\|u^{\prime}-u_{0}^{\prime}\right\|_{\infty} \leq \varepsilon_{1}^{\prime \prime} \Rightarrow u_{\Lambda} \leq u \text { and }\left\|v^{\prime}-v_{0}^{\prime}\right\|_{\infty} \leq \varepsilon_{1}^{\prime \prime} \Rightarrow v_{\Lambda} \leq v \text { on }[R / 2, R] \tag{28}
\end{equation*}
$$

The claim follows from (26), (27) and (28), by taking $0<\varepsilon<\min \left\{\varepsilon_{1}, \varepsilon_{1}^{\prime \prime}\right\}$.
Next, if (22) has a second solution contained in $\mathcal{A}$, then it is nontrivial and the proof is complete. If not, by Lemma 4.2 we infer that

$$
d_{L S}\left[I-\mathcal{N}_{\lambda_{0}}, B\left(\left(u_{0}, v_{0}\right), \rho\right), 0\right]=1 \text { for all } 0<\rho \leq \varepsilon
$$

where $\mathcal{N}_{\lambda_{0}}$ stands for the fixed point operator associated to (22) with $\lambda=\lambda_{0}$. Also, from Proposition 2.1 we have

$$
d_{L S}\left[I-\mathcal{N}_{\lambda_{0}}, B_{\rho}, 0\right]=1 \text { for all } \rho \geq R+1,
$$

and from Lemma 4.3 we get

$$
d_{L S}\left[I-\mathcal{N}_{\lambda_{0}}, B_{\rho}, 0\right]=1 \text { for all } \rho>0 \text { sufficiently small. }
$$

Let $\rho_{1}, \rho_{2}>0$ be sufficiently small and $\rho_{3} \geq R+1$ such that $\bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{1}\right) \cap$ $\bar{B}_{\rho_{2}}=\emptyset$ and $\bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{1}\right) \cup \bar{B}_{\rho_{2}} \subset B_{\rho_{3}}$. From the additivity-excision property of Leray-Schauder degree it follows that

$$
d_{L S}\left[I-\mathcal{N}_{\lambda_{0}}, B_{\rho_{3}} \backslash\left[\bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{1}\right) \cup \bar{B}_{\rho_{2}}\right], 0\right]=-1 .
$$

Therefore, $\mathcal{N}_{\lambda_{0}}$ has a fixed point $(u, v) \in B_{\rho_{3}} \backslash\left[\bar{B}\left(\left(u_{0}, v_{0}\right), \rho_{1}\right) \cup \bar{B}_{\rho_{2}}\right]$. We obtain that (22) has a second positive solution.

Corollary 5.2. Assume $(H)$. Then there exists $\Lambda>0$ such that the problem

$$
\begin{cases}\mathcal{M}(\mathrm{u})+\lambda \mu(|x|)(p+1) \mathrm{u}^{p} \mathrm{v}^{q+1}=0 & \text { in } \mathcal{B}(R), \\ \mathcal{M}(\mathrm{v})+\lambda \mu(|x|)(q+1) \mathrm{u}^{p+1} \mathrm{v}^{q}=0 & \text { in } \mathcal{B}(R), \\ \left.\mathrm{u}\right|_{\partial \mathcal{B}(R)}=0=\left.\mathrm{v}\right|_{\partial \mathcal{B}(R)} & \end{cases}
$$

has zero, at least one or at least two positive solutions according to $\lambda \in(0, \Lambda)$, $\lambda=\Lambda$ or $\lambda>\Lambda$.

Remark 5.3. Analyzing the proof of Theorem 5.1, the reader will emphasize that the potentiality of the system (21) is only involved in showing that the set $\mathcal{S}$ is nonempty. This means that a topological proof of this fact could allow to consider non-potential systems which are superlinear near origin.

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# On a family of Kepler problems with linear dissipation 

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## Dedicated to Jean Mawhin


#### Abstract

We consider the dissipative Kepler problem for a family of dissipations that is linear in the velocity. Under mild assumptions on the drag coefficient, we show that its forward dynamics is qualitatively similar to the one obtained in [15] and [16] for a constant drag coefficient. In particular, we extend to this more general framework the existence of a continuous vector-valued first integral I obtained as the limit along the trajectories of the Runge-Lenz vector. We also establish the existence of asymptotically circular orbits, so improving the result about the range of I contained in [16].


Keywords: Keywords: Kepler equation, drag linear in the velocity, first integral. MS Classification 2010: MSClassification 70F16, 70F40, 35A24.

## 1. Introduction

In our previous papers [15] and [16] we studied the global dynamics of a Kepler problem with linear drag

$$
\begin{equation*}
\ddot{x}+\epsilon \dot{x}=-\frac{x}{|x|^{3}}, \quad x \in \mathbb{R}^{2} \backslash\{0\}, \epsilon>0 . \tag{1}
\end{equation*}
$$

The main conclusion was the existence of a vector-valued first integral $I=$ $\left(I_{1}, I_{2}\right), I_{i}=I_{i}(x, \dot{x})$. This integral was obtained in a rather indirect way and we do not know if it has an explicit formula. In contrast it has a very intuitive dynamical description. The vector $I(x, \dot{x})$ can be interpreted as the eccentricity vector of an ellipse $\mathcal{E}$ such that the solution $x(t)$ tends to the origin along a spiral modelled after $\mathcal{E}$ (see Figure 1). Also, we proved that the existence of $I$ implies that such spiral is described with angular velocity which increases exponentially with time.

The aim of this work is to extend this type of results to the family of


Figure 1: In red, an orbit $x(t)$ of (1) with $\epsilon=0.01$ plotted for $t \in[0,35]$. In blue, the approximate shape of $\mathcal{E}$, obtained by plotting the final segment of the curve $y(t)=e^{2 \epsilon t} x(t) \rightarrow \mathcal{E}$ (see [16] for more details).
dissipative Kepler problems

$$
\begin{equation*}
\ddot{x}+\mathcal{D}(|x|) \dot{x}=-\frac{x}{|x|^{3}}, \quad x \in \mathbb{R}^{2} \backslash\{0\}, \tag{2}
\end{equation*}
$$

where $\mathcal{D}:\left[0,+\infty\left[\rightarrow \mathbb{R}^{+}\right.\right.$is a locally Lipschitz continuous function which satisfies

$$
\begin{equation*}
\mathcal{D}(r) \geq A_{1} \quad \text { for any } r \geq 0 \tag{3}
\end{equation*}
$$

for a suitable positive real number $A_{1}$.
It is a curious fact that the spiralling faster and faster towards the Sun of a celestial body was already described by Euler in a letter written in 1749 and published in Philosophical Transactions [8]. There Euler postulated the existence of small resistance forces around the planets and he described the consequent gradual approach of the Earth to the Sun as follows: "...The effect of this Resistance will gradually bring the Planets nearer and nearer to the Sun; and as their Orbits thereby become less, their periodical Times will also be diminished."

More than one century later Poincaré went back to the study of the effect of a resistive medium on the motion of a planet in his course "Leçons sur les hypothéses cosmogoniques" [20]. In these lectures he discussed several hypotheses on the formation of the solar system. In Chapter VI, devoted to an hypothesis due to T.J.J. See, Poincaré considered the class of dissipative Kepler problems

$$
\begin{equation*}
\ddot{x}+R \frac{\dot{x}}{|\dot{x}|}=-\frac{x}{|x|^{3}}, \quad x \in \mathbb{R}^{2} \backslash\{0\}, \tag{4}
\end{equation*}
$$

where $R=h|x|^{-\beta}|\dot{x}|^{\alpha}$ and $\alpha$ and $\beta$ are positive constants. After some computations with astronomical coordinates Poincaré found out that the semi major
axis of an orbit of elliptic type is, essentiellement, decreasing with time and observed that this fact implies an increase of the orbital velocity of the planet.

Moreover, from his computations he concluded that if the exponents $\alpha$ and $\beta$ are sufficiently large then the value of the orbital eccentricity decreases after each complete revolution. Poincaré also presented a qualitative argument ${ }^{1}$ to justify the decrease of the eccentricity of an orbit in presence of a general resistive force.

Both these arguments suggest that dissipation has a circularizing effect on orbits, that is, that their eccentricity will eventually approach zero. In this connection, we note that our results for the linear drag ( $\alpha=1, \beta=0$ ) imply that for an open set of initial conditions the eccentricity of the corresponding orbit will converge to a positive constant, and so we cannot expect a circularization effect for many orbits of (1). This fact has been observed previously in [12] (for more information on the notion of circularization see [9] and [1]).

When $\beta=0$ the family (4) was already considered by Jacobi in his book on mechanics [13] but he only discussed some formal aspects. Another member of the family (4) that has been considered in the recent literature is the so called Poynting-Plummer-Danby drag (see $[1,6,7]$ and the references therein), corresponding to $\alpha=1$ and $\beta=2$. In this case it is possible to obtain in closed form the equation of the orbits. We point out that for this family of resistive forces the qualitative behaviour of the solutions differs sharply from the qualitative behaviour we obtained in [15] and [16] for the solutions of (1) and that, in this paper, we show also to hold for the solutions of (2). In fact, for the Poynting-Plummer-Danby drag, many non rectilinear solutions, corresponding to an open set of initial conditions, collide in finite time and with finite velocity at the singularity, winding around the origin just a finite number of times before collision. This is nicely described in the unpublished master thesis of Mauricio Misquero Castro ${ }^{2}$.

The Runge-Lenz vector, denoted by $R$, is a well-known first integral of the conservative Kepler problem. If its norm is less than one, then $R$ corresponds to a family of elliptic orbits whose eccentricity is $|R|$. In the presence of friction this vector is no longer a constant of motion but it is still useful and it has been employed in the literature on dissipative problems (see [11, 14, 17]). We will show that for linear dissipations the Runge-Lenz vector has a limit $I=$ $\lim _{t \rightarrow+\infty} R(t)$ that becomes a first integral such that $|I| \leq 1$. This approach to construct integrals is inspired by the ideas on asymptotic integrals developed by Moser in [18] for the study of the Störmer problem (see also [19]).

We notice that in our setting the circularization of an orbit is equivalent to $I=0$. Orbits satisfying this condition were called asymptotically circular

[^1]in our previous paper [16], but at that time we were unable to decide whether they existed or not. In this work we show that they actually exist, although they are not typical. In this aspect the results of this paper improve those in [16] even for the case $\mathcal{D}=\epsilon$. This approach to construct integrals is inspired by the ideas on asymptotic integrals developed by Moser in [18] for the study of the Störmer problem (see also [19]).

This paper can be seen as a contribution to the construction of a qualitative theory of the Kepler problem with dissipation. Many interesting problems in this topic are still to be addressed. For example, to determine the region of parameters $(\alpha, \beta)$ producing circularization on an open set of initial conditions seems a challenging question. Also, the study of more realistic drags involved in satellite dynamics appears to be relevant (see [2]).

The rest of the paper is organized as follows.
In the second section we study the forward dynamics of (2), showing that the singularity is a global attractor. Our proof makes use of an extension to singular systems of the LaSalle invariance principle, which may have an independent interest. In Section 3 we extend to (2) the results given in [15] about the asymptotic values of the energy of solutions. This is done by adapting the approach based on the Levi-Civita transformation for the dissipative setting already considered in [15] for the linear drag. We recall that the Levi-Civita regularization in a dissipative setting was introduced by [3] for the numerical study of the global dynamics of a restricted three body problem with drag. In the fourth section we construct the asymptotic first integral for (2) and we show that it is continuous and invariant under planar rotations. In the fifth section we prove that its range is the unit disk. This is achieved by establishing the existence of asymptotically circular orbits of (2). It is interesting to note that for this aim we employ the Brouwer degree to show that there is a continuation from the circular solutions of the conservative case. Finally, in the Appendix we sketch the proofs of some results about rectilinear motions.

## 2. Dynamics in forward time: attraction towards the singularity

In this section we study the behaviour of the solutions of (2) when $t \rightarrow \omega$, where $\omega$ is the right endpoint of their maximal interval of definition. We show that the singularity $x=0$ is a global attractor of (2). First we prove that non rectilinear solutions are defined up to $\omega=+\infty$ and are bounded. Then we state and apply a version LaSalle's invariance principle which is well suited for singular equations. An analogous result is given in [4]. Finally, we show that all the rectilinear motions collide in finite time with $x=0$.

We point out that in [15] the property that the origin is a global attractor was obtained by applying the results in [5], where such result is proved for
a family of resistive forces of the form $F(x, \dot{x})=-\frac{k(|\dot{x}|)}{|\dot{x}|} \dot{x}$ which includes the linear drag. Here, rather than trying to adapt to our setting such results we have preferred to provide a direct proof of the global attractiveness of the singularity.

The dissipative Kepler problem described by equation (2) with is equivalent to the system

$$
\left\{\begin{array}{l}
\dot{x}=v  \tag{5}\\
\dot{v}+\mathcal{D}(|x|) v=-\frac{x}{|x|^{3}}
\end{array}\right.
$$

in the phase space $\Omega=\left(\mathbb{R}^{2} \backslash\{0\}\right) \times \mathbb{R}^{2}$.
Throughout the paper we will always assume that $\mathcal{D}:\left[0,+\infty\left[\rightarrow \mathbb{R}^{+}\right.\right.$is a locally Lipschitz continuous function which satisfies (3).

In what follows, for any fixed $\left(x_{0}, v_{0}\right) \in \Omega$, we will sometimes employ the notation $x\left(t ; x_{0}, v_{0}\right)$ for the solution of (2) such that $x(0)=x_{0}, v(0)=v_{0}$. If we consider the functions of the real variables $(x, v)$ given respectively by

$$
\begin{equation*}
E(x, v)=\frac{1}{2}|v|^{2}-\frac{1}{|x|}, \quad(\text { energy }) \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
C(x, v)=x \wedge v, \quad(\text { angular momentum }) \tag{7}
\end{equation*}
$$

then along the solutions of (2) it is

$$
\begin{equation*}
\dot{E}(t):=\frac{d E}{d t}(x(t), \dot{x}(t))=-\mathcal{D}(|x(t)|)|\dot{x}(t)|^{2} \tag{8}
\end{equation*}
$$

and

$$
\dot{C}(t):=\frac{d C}{d t}(x(t), \dot{x}(t))=-\mathcal{D}(|x(t)|) C(t)
$$

from which it follows

$$
\begin{equation*}
C(t)=C(0) \mathrm{e}^{-\int_{0}^{t} \mathcal{D}(|x(\tau)|) d \tau}, \quad C(0):=x(0) \wedge \dot{x}(0) \tag{9}
\end{equation*}
$$

We rewrite now equation (2) using polar coordinates. If we consider the change of variables $x=r \mathrm{e}^{i \theta}$, the new coordinates satisfy the following differential system:

$$
\left\{\begin{array}{l}
\ddot{r}-r \dot{\theta}^{2}+\mathcal{D}(r) \dot{r}=-\frac{1}{r^{2}}  \tag{10}\\
\frac{d}{d t}\left(r^{2} \dot{\theta}\right)=-\mathcal{D}(r) r^{2} \dot{\theta}
\end{array}\right.
$$

Recalling that $|x \wedge \dot{x}|= \pm r^{2} \dot{\theta}$, by (9) we get that the radial component of the solutions of (2) satisfies the integro-differential equation

$$
\begin{equation*}
\ddot{r}-\alpha^{2} \frac{\mathrm{e}^{-2 \int_{0}^{t} \mathcal{D}(r(s)) d s}}{r^{3}}+\mathcal{D}(r) \dot{r}=-\frac{1}{r^{2}} \tag{11}
\end{equation*}
$$

where $\alpha=|C(0)|$.
We are now in a position to state our result about the non rectilinear motions of (2).

Proposition 2.1. Let $x(t)=r(t) e^{i \theta(t)}$ be a maximal solution of (2) with $\alpha \neq 0$, and let $[0, \omega[$ be its domain in forward time. Then $\omega=+\infty$, and $r(t)=|x(t)|$ is bounded on $[0,+\infty[$.

Proof. We prove first that $r(t)$ is bounded on $[0, \omega[$ and then we show that $\omega=+\infty$.

To get the boundedness of the solutions we argue as follows. Either $r(t) \leq$ $\alpha^{2}$ when $t$ is sufficiently close to $\omega$ and we have nothing to prove, or there exists a sequence $t_{n} \rightarrow \omega$ such that $r\left(t_{n}\right)>\alpha^{2}$. If this is the case, there are two possible occurrences:
i) $r(t)>\alpha^{2}$ in $[\tau, \omega[$ for some $\tau \in[0, \omega[$;
ii) there exists a sequence of intervals $I_{n}=\left[a_{n}, b_{n}\right] \subseteq[0, \omega[$ such that $r(t)>$ $\alpha^{2}$ if and only if $\left.t \in\right] a_{n}, b_{n}[$.

If i) holds, by (11) it follows that if $t \in[\tau, \omega[$

$$
\begin{equation*}
\frac{d}{d t}\left(e^{\int_{\tau}^{t} \mathcal{D}(r(s)) d s} \dot{r}\right)=e^{\int_{\mathcal{\tau}}^{t} \mathcal{D}(r(s)) d s}(\ddot{r}+\mathcal{D}(r) \dot{r}) \leq 0 \tag{12}
\end{equation*}
$$

and by integrating this inequality we obtain

$$
\dot{r}(t) \leq e^{-\int_{\tau}^{t} \mathcal{D}(r(s)) d s} \dot{r}(\tau) \leq e^{-A_{1}(t-\tau)}|\dot{r}(\tau)|, \quad t \in[\tau, \omega[,
$$

which implies

$$
r(t) \leq r(\tau)+\frac{|\dot{r}(\tau)|}{A_{1}}\left(1-e^{-A_{1}(t-\tau)}\right)<r(\tau)+\frac{|\dot{r}(\tau)|}{A_{1}}, \quad t \in[\tau, \omega[.
$$

The proof of boundedness of $r(t)$ on $[0, \omega[$ in case i) is concluded.
In case ii) we note that, since for any $n$ we have $r\left(a_{n}\right)=\left|x\left(a_{n}\right)\right|=\alpha^{2}$, then from $E\left(a_{1}\right) \geq E\left(a_{n}\right)$, it follows that $\left|\dot{r}\left(a_{n}\right)\right| \leq\left|\dot{x}\left(a_{n}\right)\right| \leq\left|\dot{x}\left(a_{1}\right)\right|$ for any $n$. Then, taking into account that on $I_{n}(12)$ holds, in a similar manner as above we get $\dot{r}(t) \leq e^{-A_{1}\left(t-a_{n}\right)}\left|\dot{r}\left(a_{n}\right)\right|, t \in I_{n}$, and then

$$
r(t) \leq \alpha^{2}+\frac{\left|\dot{x}\left(a_{1}\right)\right|}{A_{1}}, \quad t \in I_{n} .
$$

Since the constant that bounds the solution is the same for all the intervals $I_{n}$ and since $r(t) \leq \alpha^{2}$ on the set $\left[0,+\omega\left[\backslash \cup_{n} I_{n}\right.\right.$, the proof of the boundedness of $r(t)$ in case ii) is finished.

We conclude that $r(t)$ is bounded on $[0, \omega[$.
To prove that $\omega=+\infty$ assume by contradiction that $\omega<+\infty$. The standard theory for initial value problems implies that one of the following cases hold:
(i) there exists a sequence $t_{n} \uparrow \omega$ such that $x\left(t_{n}\right) \rightarrow 0$;
(ii) $|x(t)| \geq \delta$ if $t \in\left[0, \omega\left[\right.\right.$ for some $\delta>0$ and $\lim _{t \uparrow \omega}|\dot{x}(t)|=+\infty$.

If condition (i) were valid then $E\left(t_{n}\right) \rightarrow+\infty$ as $n \rightarrow+\infty$ and this is incompatible with (8). In fact,

$$
E\left(t_{n}\right) \geq \frac{\alpha^{2}}{2 r^{2}\left(t_{n}\right)} e^{-2 \int_{0}^{t_{n}} \mathcal{D}(r(s)) d s}-\frac{1}{r\left(t_{n}\right)} \geq \frac{1}{2 r^{2}\left(t_{n}\right)}\left[\alpha^{2} e^{-2 M t_{n}}-r\left(t_{n}\right)\right]
$$

where $M:=\sup _{t \in[0, \omega[ } \mathcal{D}(r(t))$ is finite since $\mathcal{D}(r)$ is continuous on $[0,+\infty[$ and $r(t)$ is bounded in $[0, \omega[$. Assume now that (ii) holds. From $|x(t)| \geq \delta$ for any $t \in[0, \omega[$ we get

$$
\frac{1}{2}|\dot{x}(t)|^{2}-\frac{1}{\delta} \leq E(t) \leq E(0)
$$

Since this inequality gives a bound for $|\dot{x}(t)|$ on $[0, \omega[$ we get a contradiction with the limit in (ii).

To prove that all the non rectilinear solutions of equation (2) tend to the singularity as $t \rightarrow+\infty$ we need the following general auxiliary result, which is an extension to singular systems of the LaSalle invariance principle.

Proposition 2.2. Let $\Omega \subset \mathbb{R}^{d}$ be an open set and assume that the existence and uniqueness of solution holds for the system $\dot{x}=f(x)$ with $f: \Omega \rightarrow \mathbb{R}^{d}$ continuous. Let $\phi_{t}(x)$ denote the value at time $t$ of the solution of $\dot{x}=f(x)$ which starts from $x$ at $t=0$ and let $I_{x} \subset \mathbb{R}$ be its maximal interval of definition. Assume there exists a continuous function $V: \Omega \rightarrow \mathbb{R}$ such that

$$
\begin{equation*}
V\left(\phi_{t}(x)\right)<V(x), \quad t \in I_{x}, t>0, x \in \Omega \tag{13}
\end{equation*}
$$

If $x_{*} \in \Omega$ is such that $\left[0,+\infty\left[\subset I_{x_{*}}\right.\right.$, then

$$
L_{\omega}\left(x_{*}\right) \cap \Omega=\emptyset,
$$

where $L_{\omega}\left(x_{*}\right)$ denotes the $\omega$-limit set of $x_{*}$.
Note that in the above statement the limit set is defined as

$$
L_{\omega}\left(x_{*}\right)=\cap_{t \geq 0} \overline{\left\{\phi_{\tau}\left(x_{*}\right): \tau \geq t\right\}}
$$

where the closure is taken in $\mathbb{R}^{d}$.

Remark 2.1. The following variant of the Proposition will be useful later. We can assume that the condition (13) only holds for points $x$ lying on a closed subset $F$ of $\Omega$. If the set $F$ is invariant under the flow then the conclusion on the limit set will be valid for the orbits lying on $F$.
Proof. By contradiction, assume that there exists $t_{n} \rightarrow+\infty$ such that $\phi_{t_{n}}\left(x_{*}\right) \rightarrow$ $\xi \in \Omega$. By the continuous dependence of the solutions of $\dot{x}=f(x)$ on the initial value, given $\sigma>0$ such that $\sigma \in I_{\xi}$ we know that, for large $n, \sigma \in I_{\phi_{t_{n}}\left(x_{*}\right)}$ and $\phi_{\sigma+t_{n}}\left(x_{*}\right) \rightarrow \phi_{\sigma}(\xi)$. For each $n$ there exists $\mu(n)>n$ such that $t_{\mu(n)}>t_{n}+\sigma$. Then,

$$
V\left(\phi_{t_{\mu(n)}}\left(x_{*}\right)\right)<V\left(\phi_{t_{n}+\sigma}\left(x_{*}\right)\right) .
$$

Letting $n \rightarrow+\infty$ we get

$$
V(\xi) \leq V\left(\phi_{\sigma}(\xi)\right)
$$

and this is a contradiction
As a corollary we get:
Proposition 2.3. Let $x(t)=r(t) e^{i \theta(t)}$ be a non rectilinear solution of (2). Then

$$
\lim _{t \rightarrow+\infty} x(t)=0 .
$$

Proof. Assume by contradiction that there exists a sequence $t_{n} \rightarrow+\infty$ such that $\left|x\left(t_{n}\right)\right| \geq \delta>0$ for a suitable $\delta$. From the energy inequality

$$
E\left(t_{n}\right)=\frac{1}{2}\left|\dot{x}\left(t_{n}\right)\right|^{2}-\frac{1}{\left|x\left(t_{n}\right)\right|} \leq E(0)
$$

we deduce that $\left|\dot{x}\left(t_{n}\right)\right|^{2} \leq E(0)+\frac{1}{\delta}$. As by Proposition $2.1 x\left(t_{n}\right)$ is bounded, it must be $L_{\omega}(x(0), \dot{x}(0)) \cap \Omega \neq \emptyset$. Since $\ddot{x}(t) \neq 0$ when $\dot{x}(t)=0$, we deduce that the zeros of $\dot{x}(t)$ are isolated. Then the formula $\dot{E}=-\mathcal{D}(|x|)|\dot{x}|^{2}$ implies that the energy function $E$ is strictly decreasing on the solutions of (2) and (13) holds. Now we can apply the previous proposition with $V=E$ and get a contradiction.

As to the solutions with zero angular momentum, the so called rectilinear motions, they satisfy the equation

$$
\begin{equation*}
\ddot{r}+\mathcal{D}(r) \dot{r}=-\frac{1}{r^{2}}, \tag{14}
\end{equation*}
$$

obtained from (11) by setting $\alpha=0$.
For this class of solutions we state the following result. Its proof is analogous to the one of Proposition 3.1 in [15], and we only sketch it in the Appendix.
Proposition 2.4. All solutions of (14) are collision solutions, that is $\omega$ is finite and

$$
\lim _{t \rightarrow \omega^{-}} r(t)=0, \quad \lim _{t \rightarrow \omega^{-}} \dot{r}(t)=-\infty
$$

## 3. The Levi-Civita transformation and the asymptotic behaviour of the energy

In this section we study the behaviour of the energy of the solutions of (2) as they approach the singularity.

The starting point is to adapt to equation (2) the Levi-Civita regularization that was introduced in a dissipative setting in [15] to deal with the linear drag. We recall that, after the natural identification of $x=\left(x_{1}, x_{2}\right)$ with the complex number $x_{1}+i x_{2}$, the Levi-Civita regularization is defined by the change of variables

$$
\begin{equation*}
x=w^{2}, d s=\frac{d t}{|x|} \tag{15}
\end{equation*}
$$

Using this regularization, equation (2) is transformed into the system of ODEs in the new time $s$

$$
\begin{equation*}
w^{\prime}=v, \quad v^{\prime}=\frac{E w}{2}-\mathcal{D}\left(|w|^{2}\right)|w|^{2} v, \quad E^{\prime}=-2 \mathcal{D}\left(|w|^{2}\right)\left(E|w|^{2}+1\right) \tag{16}
\end{equation*}
$$

This system has to be considered on the invariant manifold

$$
\begin{equation*}
\mathcal{M}=\left\{(w, v, E) \in \mathbb{C}^{2} \times \mathbb{R}: E|w|^{2}+1-2|v|^{2}=0\right\} \tag{17}
\end{equation*}
$$

which contains all the physically meaningful solutions.
A solution of (2) starting from $\left(x_{0}, v_{0}\right) \in \Omega$ is transformed in a solution of (16) starting from $\left(w_{0}, \hat{v}_{0}, E_{0}\right) \in \mathcal{M}$, where $w_{0}$ is a square root of $x_{0}, \hat{v}_{0}=\frac{\left|x_{0}\right| v_{0}}{2 w_{0}}$ and $E_{0}=\frac{1}{2}\left|v_{0}\right|^{2}-\frac{1}{\left|x_{0}\right|}$. Vice-versa, a solution of (16) starting on $\mathcal{M}$ and such that $w(0) \neq 0$ corresponds to the solution $x(t):=w^{2}(S(t))$ of (2), where $S(t)$ is the inverse function of $T(s):=\int_{0}^{s}|w(\sigma)|^{2} d \sigma$.

Notice that if the points $\left(x_{0}, v_{0}\right)$ belong to a compact subset $K$ of $\Omega$, then the triplets $\left(w_{0}, \hat{v}_{0}, E_{0}\right)$ lie on a compact subset $\mathcal{K}$ of $\mathcal{M}$.

Lemma 3.1. Let $\left(w_{0}, v_{0}, E_{0}\right)$ be a point of $\mathcal{M}$ and let $(w(s), v(s), E(s))$ denote the solution of (16) passing through this point at $s=0$. Then this solution is well defined on $[0,+\infty[$ and

$$
\lim _{s \rightarrow+\infty} E(s)=-\infty
$$

Proof. Let $[0, \sigma[$ be the maximal interval to the right of the solution. By a contradiction argument we assume that $\sigma<+\infty$. The third equation of (16) and the invariance of $\mathcal{M}$ imply that

$$
E^{\prime}(s)=-4 \mathcal{D}\left(|w(s)|^{2}\right)|v(s)|^{2} \leq 0
$$

In particular $E(s) \leq E(0)$ for each $s \in[0, \sigma[$. Again, the invariance of $\mathcal{M}$ leads to the differential inequality

$$
\frac{d}{d s}|w(s)| \leq\left|w^{\prime}(s)\right|=\sqrt{\frac{1+E(s)|w(s)|^{2}}{2}} \leq \frac{1+|E(0)|^{\frac{1}{2}}|w(s)|}{\sqrt{2}}
$$

It follows that $|w(s)|$ remains bounded in $[0, \sigma[$. This fact implies that the function $\mathcal{D}\left(|w(s)|^{2}\right)$ is bounded on $[0, \sigma[$ and by the last equation of (16) we conclude that the same is true for $|E(s)|$. The definition of $\mathcal{M}$ implies now that $|v(s)|$ is bounded on $[0, \sigma[$. It follows that the solution $(w(s), v(s), E(s))$ cannot blow up at $s=\sigma$ and this gives a contradiction with $\sigma<+\infty$. We conclude that the solution is well defined on $[0,+\infty[$.

Since on this interval we have $E^{\prime}(s) \leq 0$, then $E_{\infty}=\lim _{s \rightarrow+\infty} E(s)$ exists and belongs to $[-\infty, E(0)]$. We prove now that $E_{\infty}=-\infty$. Let us assume by contradiction that $E_{\infty} \in \mathbb{R}$ and distinguish two cases:
(i) $E_{\infty} \geq 0$;
(ii) $E_{\infty}<0$.

If (i) holds, we know that $E(s) \geq E_{\infty} \geq 0$ if $s \geq 0$. After integrating the third equation of (16), we have

$$
E(s)=E(0)-2 \int_{0}^{s} \mathcal{D}\left(|w(\xi)|^{2}\right)\left(E(\xi)|w(\xi)|^{2}+1\right) d \xi \leq E(0)-2 A_{1} s \rightarrow-\infty
$$

as $s \rightarrow-\infty$, and we get a contradiction.
Assume now that (ii) holds. We note that system (16), defined on $\Omega=$ $\mathbb{C}^{2} \times \mathbb{R}$, is in the conditions of the remark after Proposition 2.2 with $F=\mathcal{M}$ and $V=E$. Once we are on $\mathcal{M}$ we know from the discussions of the case (i) that it is not restrictive to assume that $E(s)<0$ if $s \geq 0$, and we claim that the zeros of $v(s)$ on $\left[0,+\infty\left[\right.\right.$ are isolated. Indeed, $v(s)=0$ implies $|w(s)|^{2}=\frac{1}{|E(s)|}>0$ and then $v^{\prime}(s)=\frac{E(s) w(s)}{2} \neq 0$. Thus

$$
E(s)-E(0)=-4 \int_{0}^{s} \mathcal{D}\left(|w(\xi)|^{2}\right)|v(\xi)|^{2} d \xi<0
$$

when $s>0$ and then condition (13) holds on $\mathcal{M}$ with $V=E$. As a consequence, the $\omega$-limit set of our solution is empty. From the identity

$$
E(s)|w(s)|^{2}+1=2|v(s)|^{2} \geq 0
$$

we deduce that

$$
\limsup _{s \rightarrow+\infty}|w(s)|^{2} \leq \frac{1}{\left|E_{\infty}\right|}
$$

Also,

$$
\limsup _{s \rightarrow+\infty}|v(s)|^{2} \leq \frac{1}{2}
$$

Then the forward orbit $\{(w(s), v(s), E(s)): s \geq 0\}$ is bounded and the $\omega$-limit set is a non empty compact set of $\mathcal{M}$. This is the searched contradiction.

As an immediate consequence of this lemma we have the following:
Proposition 3.2. If $x(t)$ is a solution of (2) with non zero angular momentum, then

$$
\lim _{t \rightarrow+\infty} E(t)=-\infty
$$

Proof. By choosing a branch of the square root, a non rectilinear solution $x(t)$ of (2) is transformed by (15) in a solution of (16) on $\mathcal{M}$ such that $w(s)=$ $\sqrt{x(T(s))}$, where $T(s)$ is the inverse function of $s=S(t)=\int_{0}^{t} \frac{1}{|x(\tau)|} d \tau$. By Proposition 2.1 we conclude that $s \rightarrow+\infty$ when $t \rightarrow+\infty$, and the claim follows from Lemma 3.1.

As to the energy of the rectilinear solutions $x=r(t)$ of (2) we have the following result. Its proof is analogous to the one of the corresponding results given in [15] for the linear drag (see Proposition 3.1 and Proposition 4.2 therein) and therefore it is just outlined in the Appendix. Here we stress that the LeviCivita regularization is used to get the second part of the statement.

Proposition 3.3. Collisions occur with finite energy. Energy at collision may have any arbitrarily prescribed real value.

## 4. Existence and properties of the Runge-Lenz-type first integral

As proved in the previous sections, a solution of (2) (and hence of (5)) such that $x(0)=x_{0}$ and $\dot{x}(0)=v_{0}$ is defined for $t \in\left[0, \omega\left[\right.\right.$ where $\omega=\omega\left(x_{0}, v_{0}\right)$ is finite in the case of a rectilinear motion, whereas $\omega=+\infty$ for a non rectilinear motion.

We recall that, if we consider the energy $E(x, v)$, the angular momentum $C(x, v)$ and the vector

$$
\begin{equation*}
R(x, v)=v \wedge(x \wedge v)-\frac{x}{|x|}, \quad(\text { Runge }- \text { Lenz vector }) \tag{18}
\end{equation*}
$$

then the two following functional relationships hold among them as functions of the real variables $(x, v)$ (see also [10], 3-9):

$$
\begin{equation*}
|x|+<R, x>=|C|^{2}, \text { for any } x \in \mathbb{R}^{2} \backslash\{0\}, \tag{19}
\end{equation*}
$$

where $\langle v, w\rangle$ denotes the inner product between the vectors $v$ and $w$, and

$$
\begin{equation*}
|R|^{2}-1=2|C|^{2} E \tag{20}
\end{equation*}
$$

In the conservative case, $E, C$ and $R$ are first integrals of the Kepler problem. In particular, if $0<|R|<1$, the vector $R$ is the eccentricity vector corresponding to the Keplerian ellipse defined by (19), the unit vector $\frac{R}{|R|}$ is the direction of its major axis and $e=|R|$ is its eccentricity.

To end our preparatory work, we state the following lemma, needed to prove the continuity on $\Omega$ of the Runge-Lenz-type first integral $I$ we define below in Theorem 4.2.
Lemma 4.1. Let $K$ be a compact subset of $\Omega$. Then, there exist numbers $m_{K}>$ 0 and $\mu_{K}>0$ such that

$$
\begin{equation*}
\left|x\left(t ; x_{0}, v_{0}\right)\right| \leq m_{K} \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\dot{x}\left(t ; x_{0}, v_{0}\right)\right|\left|x\left(t ; x_{0}, v_{0}\right)\right|^{\frac{1}{2}} \leq \mu_{K} \tag{22}
\end{equation*}
$$

for any $\left(x_{0}, v_{0}\right) \in K$ and $t \in[0, \omega[$.
Proof. To prove the first estimate, we proceed as in the last part of the proof of Lemma 2.2 in [16], to which the reader should refer for the details. As pointed out in the previous section, the Levi-Civita regularization transforms the solutions of (2) starting in $\left(x_{0}, v_{0}\right) \in K$ into solutions of system (16) starting in a compact set $\mathcal{K} \subset \mathcal{M}$. By Lemma 3.1 these solutions are defined on $[0,+\infty[$ and are such that their energy $E(s)$ becomes, eventually, negative, say less than $-\frac{1}{2} \cdot{ }^{3}$ If we consider the $w$ component of a solution of (16), the invariance of $\mathcal{M}$ gives the bound $|w(s)|^{2}<2$ for sufficiently large $s$. Then, by a standard compactness argument, solutions of (16) starting in $\mathcal{K}$ are such that the previous bound on $|w(s)|$ holds for $s$ greater than a suitable $s^{*}$ uniformly in $\mathcal{K}$. For such solutions the existence of a uniform bound for $|w(s)|$ on $[0,+\infty[$ easily follows. Going back to the original variables one gets (21) for $|x|=|w|^{2}$ when $\left(x_{0}, v_{0}\right) \in K$.

To prove the second estimate we observe that since the energy is decreasing, $E(t) \leq E(0)$ for any $t \in[0, \omega[$, and we get the following bound on the velocity:

$$
\begin{equation*}
|\dot{x}(t)| \leq \sqrt{2\left(E(0)+\frac{1}{|x(t)|}\right)}, \quad t \in[0, \omega[. \tag{23}
\end{equation*}
$$

Multiplying (23) by $|x(t)|^{\frac{1}{2}}$ we obtain (22) with $\mu_{K}:=\sqrt{2\left(E_{K} m_{K}+1\right)}$ and $E_{K}:=\max _{K}\left|E\left(x_{0}, v_{0}\right)\right|$.

[^2]We are now in a position to state the main result of this section. This result provides a continuous vector first integral $I=\left(I_{1}, I_{2}\right)$ which is invariant under the group of planar rotations and whose components are two functionally independent scalar first integrals (see Remark 1 in [16]).

As in the case of the linear drag, $I$ can be interpreted as an asymptotic eccentricity vector and its norm as an asymptotic eccentricity. In particular, solutions with $|I|<1$ tend to the origin along a spiral determined asymptotically by $I$.

Theorem 4.2. There exists a continuous vector field

$$
I: \Omega \rightarrow \mathbb{R}^{2}, \quad I=I(x, v)
$$

satisfying
(i) $I(\sigma x, \sigma v)=\sigma I(x, v)$, for each $(x, v) \in \Omega$ and each rotation $\sigma \in S O(2)$.
(ii) The range of $I$ is the closed unit disk, that is

$$
\begin{equation*}
I(\Omega)=\mathbb{D} \tag{24}
\end{equation*}
$$

where $\mathbb{D}=\left\{y \in \mathbb{R}^{2}:|y| \leq 1\right\}$.
(iii) Each solution $(x(t), v(t))$ of (5), defined on a maximal right interval of the form $[0, \omega[$, satisfies

$$
\begin{equation*}
I(x(t), v(t))=\lim _{\tau \rightarrow \omega} R(x(\tau), v(\tau)) \tag{25}
\end{equation*}
$$

Proof. Below we will prove the continuity of $I$ and properties (i) and (iii). The proof of $(i i)$ is postponed to the next section, to properly highlight the fact that it relies on the existence of asymptotically circular orbits.

Throughout the proof, $K$ will be a fixed compact set contained in $\Omega$, and $\left(x_{0}, v_{0}\right)$ will be a point of $K$. Let $(x(t), v(t))$ be the solution of system (5) such that $(x(0), v(0))=\left(x_{0}, v_{0}\right)$. We denote by $R(t)=R(x(t), v(t))$ and denote by $\dot{R}$ its derivative with respect to time.

Recall that we have $\dot{C}=-\mathcal{D}(|x(t)|) C$ and that

$$
\begin{equation*}
\frac{d}{d t}\left(\frac{x}{|x|}\right)=C \wedge\left(\frac{x}{|x|^{3}}\right) \tag{26}
\end{equation*}
$$

for any smooth function $x=x(t)$. By differentiating the equality defining $R(t)$ and then integrating the result from 0 to $t$ we get

$$
\begin{equation*}
R(t)=R(0)-2 \int_{0}^{t} \mathcal{D}(|x(\tau)|) \dot{x}(\tau) \wedge C(\tau) d \tau \tag{27}
\end{equation*}
$$

If $x_{0} \wedge v_{0}=0$, then for the corresponding rectilinear motion we have $R(t)=$ $R(0)=R\left(x_{0}, v_{0}\right)=-\frac{x_{0}}{\left|x_{0}\right|}$ for any $t \in[0, \omega[$ so that, trivially, we define $I\left(x_{0}, v_{0}\right):=\lim _{t \rightarrow \omega} R(t)=R\left(x_{0}, v_{0}\right)$. Let us consider the case $x_{0} \wedge v_{0} \neq 0$. We claim that the estimate below holds:

$$
\begin{equation*}
|(\mathcal{D}(|x(t)|)-\mathcal{D}(0)) \dot{x}(t) \wedge C(t)| \leq M e^{-A_{1} t} \text { if } t \geq 0 \tag{28}
\end{equation*}
$$

where the constant $M$ is uniform with respect to $K$. To prove (28) let $m_{K}$ and $\mu_{K}$ be the numbers provided by Lemma 4.1. Since $\mathcal{D}$ is locally Lipschitz continuous on $\left[0,+\infty\left[\right.\right.$, we can find a Lipschitz constant $L_{K}$ on the compact interval $\left[0, m_{K}\right]$. In particular

$$
|\mathcal{D}(r)-\mathcal{D}(0)| \leq L_{K} r \quad \text { if } \quad 0 \leq r \leq m_{K}
$$

Thus, for any $t \geq 0$ we have

$$
\begin{align*}
|(\mathcal{D}(|x(t)|)-\mathcal{D}(0)) \dot{x}(t) \wedge C(t)| \leq L_{K}|x(t)| \mid & \dot{x}(t)||C(t)| \\
& \leq L_{K} m_{K}^{\frac{1}{2}} \mu_{K}\left|x_{0}\right|\left|v_{0}\right| e^{-A_{1} t} \tag{29}
\end{align*}
$$

where we have used (3) and

$$
\begin{equation*}
|C(t)| \leq\left|x_{0}\right|\left|v_{0}\right| e^{-A_{1} t}, \quad t \geq 0 \tag{30}
\end{equation*}
$$

Once (28) has been proved, we rewrite the Runge-Lenz vector in the form

$$
\begin{equation*}
R(t)=R(0)+2 \mathcal{D}(0) x_{0} \wedge C(0)-2 \mathcal{D}(0) x(t) \wedge C(t)-2 I_{1}(t)+2 \mathcal{D}(0) I_{2}(t) \tag{31}
\end{equation*}
$$

with

$$
I_{1}(t)=\int_{0}^{t}(\mathcal{D}(|x(\tau)|)-\mathcal{D}(0)) \dot{x}(\tau) \wedge C(\tau) d \tau
$$

and

$$
I_{2}(t)=\int_{0}^{t} x(\tau) \wedge \dot{C}(\tau) d \tau
$$

Formula (31) is obtained by adding and subtracting $\mathcal{D}(0)$ in the scalar factor of the integral in (27) and then applying an integration by parts. From (30) we deduce that if $t \geq 0$

$$
\begin{gather*}
|x(t) \wedge C(t)| \leq m_{K}\left|x_{0}\right|\left|v_{0}\right| e^{-A_{1} t}  \tag{32}\\
|x(t) \wedge \dot{C}(t)| \leq m_{K} D_{K}\left|x_{0}\right|\left|v_{0}\right| e^{-A_{1} t} \tag{33}
\end{gather*}
$$

where $D_{K}=\max _{\left[0, m_{K}\right]} \mathcal{D}(r)$. Together with (28) these inequalities imply that $I=\lim _{t \rightarrow+\infty} R(t)$ exists.

At this point it is convenient to make explicit the functional dependence of $I$ on the initial condition $\left(x_{0}, v_{0}\right) \in \Omega$ and write it as

$$
\begin{align*}
I\left(x_{0}, v_{0}\right)=R\left(x_{0}, v_{0}\right)+2 \mathcal{D}(0) x_{0} \wedge C\left(x_{0}, v_{0}\right)- & 2 I_{1}\left(+\infty ; x_{0}, v_{0}\right)+ \\
& +2 \mathcal{D}(0) I_{2}\left(+\infty ; x_{0}, v_{0}\right) \tag{34}
\end{align*}
$$

where we set $I_{1}\left(+\infty ; x_{0}, v_{0}\right)=I_{2}\left(+\infty ; x_{0}, v_{0}\right)=0$ if $C\left(x_{0}, v_{0}\right)=x_{0} \wedge v_{0}=0$.
To prove the continuity of this function at each point we consider first the case $\left(x_{0}, v_{0}\right) \in \Omega$ with $x_{0} \wedge v_{0} \neq 0$. We can select a small closed ball centered at $\left(x_{0}, v_{0}\right)$ such that the angular momentum does not vanish on it. This will be our set $K$. Then, estimates (28) and (33), together with the results on continuous dependence of solutions with respect to initial conditions, allow to get the continuity of $I_{1}(+\infty ; \cdot, \cdot)$ and $I_{2}(+\infty ; \cdot, \cdot)$ by applying standard results on functions defined by parametric Lebesgue integrals. In the case $x_{0} \wedge v_{0}=0$ it must be noticed that if $\left(x_{0 n}, v_{0 n}\right)$ is a sequence converging to $\left(x_{0}, v_{0}\right)$ with $x_{0 n} \wedge v_{0 n} \neq 0$, then the corresponding solution satisfies

$$
C_{n}(t):=x_{n}(t) \wedge \dot{x}_{n}(t)=e^{-\int_{0}^{t} \mathcal{D}\left(\left|x_{n}(\tau)\right|\right) d \tau} x_{0 n} \wedge v_{0 n} \rightarrow 0
$$

as $n \rightarrow+\infty$ for each $t \geq 0$. Similarly, $\lim _{n \rightarrow+\infty} \dot{C}_{n}(t)=0$ for each $t \geq 0$. From the estimates

$$
|(\mathcal{D}(|x(t)|)-\mathcal{D}(0)) \dot{x}(t) \wedge C(t)| \leq L_{K} m_{K}^{1 / 2} \mu_{K}|C(t)|,|x(t) \wedge \dot{C}(t)| \leq m_{K}|\dot{C}(t)|
$$

we deduce that $I_{i}\left(+\infty ; x_{0 n}, v_{0 n}\right) \rightarrow 0$. Note that the estimates (29) and (33) imply that the convergence is dominated.

Then $I\left(x_{0 n}, v_{0 n}\right) \rightarrow I\left(x_{0}, v_{0}\right)=-\frac{x_{0}}{\left|x_{0}\right|}$ as $n \rightarrow \infty$. Since the same property trivially holds for sequences $\left(x_{0 n}, v_{0 n}\right)$ converging to $\left(x_{0}, v_{0}\right)$ and such that $x_{0 n} \wedge v_{0 n}=0$, the continuity of $I$ on $\Omega$ is proved.

Properties (i) and (iii) follow immediately from the definition of $I$.

## 5. Existence of asymptotically circular orbits

In this section we complete the proof of Theorem 4.2 by showing that the range of $I$ is the closed unit disk. This property will be a consequence of the continuity of $I$, of its invariance under rotations, and of the existence of asymptotically circular orbits of (2), that is orbits for which $I=0$. We will show below how to obtain these orbits using the Brouwer degree to continue the circular ones of the conservative Kepler problem.

We start by considering the set

$$
\mathcal{C}_{+}=\left\{(\xi, \eta) \in \Omega: \eta=|\xi|^{-\frac{3}{2}} J \xi\right\}
$$

with $J=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$.
Then $R(\xi, \eta)=0, E(\xi, \eta)=-\frac{1}{2|\xi|}<0$ and $C(\xi, \eta) \neq 0$. We fix a point $(\xi, \eta) \in \mathcal{C}_{+}$and define the function $F: \Omega \rightarrow \mathbb{R}^{4}$

$$
F(x, v):=\binom{R(x, v)}{x-\xi} .
$$

Lemma 5.1. The point $(\xi, \eta)$ is a nondegenerate zero of $F$. Actually

$$
\operatorname{det} F^{\prime}(\xi, \eta)=2|\xi|>0
$$

Proof. Clearly $F(\xi, \eta)=0$. We have

$$
F^{\prime}(\xi, \eta)=\left(\begin{array}{cc}
\partial_{x} R(\xi, \eta) & \partial_{v} R(\xi, \eta) \\
I d & 0
\end{array}\right)
$$

where $I d$ denote the identity matrix of order two, so that

$$
\operatorname{det} F^{\prime}(\xi, \eta)=\operatorname{det}\left[\partial_{v} R(\xi, \eta)\right]
$$

Since

$$
R(x, v)=\binom{x_{1} v_{2}^{2}-x_{2} v_{1} v_{2}}{x_{2} v_{1}^{2}-x_{1} v_{1} v_{2}}-\frac{x}{|x|},
$$

it follows that

$$
\operatorname{det}\left[\partial_{v} R(\xi, \eta)\right]=|\xi|^{-3}\left|\begin{array}{cc}
-\xi_{2} \xi_{1} & 2 \xi_{1}^{2}+\xi_{2}^{2} \\
-2 \xi_{2}^{2}-\xi_{1}^{2} & \xi_{1} \xi_{2}
\end{array}\right|=2|\xi|
$$

and our proof is concluded.
Let us fix a small open ball $B \subset \mathbb{R}^{4}$ centred at $(\xi, \eta)$ satisfying the following properties:

- $(\xi, \eta)$ is the only zero of $F$ in $\bar{B}$;
- there exists $\delta>0$ such that $E\left(x_{0}, v_{0}\right) \leq-\delta<0$ if $\left(x_{0}, v_{0}\right) \in \bar{B}$;
- $C\left(x_{0}, v_{0}\right) \neq 0$ if $\left(x_{0}, v_{0}\right) \in \bar{B}$.

In particular the Brouwer degree of $F$ in $\bar{B}$ is well defined and

$$
\begin{equation*}
\operatorname{deg}(F, B, 0)=1 \tag{35}
\end{equation*}
$$

For each $\epsilon>0$ the change of variables $x(t)=\epsilon^{\frac{2}{3}} y\left(\frac{t}{\epsilon}\right)$ transforms equation (2) into

$$
\begin{equation*}
\ddot{y}+\epsilon \mathcal{D}\left(\epsilon^{\frac{2}{3}}|y|\right) \dot{y}=-\frac{y}{|y|^{3}} . \tag{36}
\end{equation*}
$$

The Runge-Lenz vector has the invariance property

$$
R(x, v)=R\left(\epsilon^{\frac{2}{3}} x, \epsilon^{-\frac{1}{3}} v\right)
$$

and so

$$
R(x(t), \dot{x}(t))=R(y(t / \epsilon), \dot{y}(t / \epsilon)) \text { for any } t \in[0,+\infty[.
$$

Letting $t \rightarrow+\infty$ we obtain the identity $I_{1}\left(x_{0}, v_{0}\right)=I_{\epsilon}\left(\epsilon^{-\frac{2}{3}} x_{0}, \epsilon^{\frac{1}{3}} v_{0}\right)$ or, equivalently,

$$
\begin{equation*}
I_{\epsilon}\left(x_{0}, v_{0}\right)=I_{1}\left(\epsilon^{\frac{2}{3}} x_{0}, \epsilon^{-\frac{1}{3}} v_{0}\right) \tag{37}
\end{equation*}
$$

where $I_{\epsilon}\left(x_{0}, v_{0}\right):=\lim _{t \rightarrow+\infty} R\left(y\left(t ; x_{0}, v_{0}, \epsilon\right), \dot{y}\left(t ; x_{0}, v_{0}, \epsilon\right)\right)$ and $y\left(t ; x_{0}, v_{0}, \epsilon\right)$ is the solution of the Cauchy problem for (36). The identity (37) shows that it is sufficient to prove the existence of an asymptotically circular motion for (36) for some $\epsilon>0$.
Lemma 5.2. The function $\tilde{I}:[0,1] \times \bar{B} \rightarrow \mathbb{R}^{2}$, given by $\tilde{I}\left(\epsilon, x_{0}, v_{0}\right):=I_{\epsilon}\left(x_{0}, v_{0}\right)$ is continuous.
Proof. The continuity of $\tilde{I}$ on $] 0,1] \times \bar{B}$ is a consequence of (37) and of the continuity of $I_{1}$ established in Theorem 4.2. The continuity at $\epsilon=0$ is a consequence of the expansion

$$
\begin{equation*}
I_{\epsilon}\left(x_{0}, v_{0}\right)=R\left(x_{0}, v_{0}\right)+O\left(\epsilon^{\frac{2}{3}}\right), \quad \text { uniformly in }\left(x_{0}, v_{0}\right) \in \bar{B} \tag{38}
\end{equation*}
$$

To prove (38) we simplify the notation by setting $y=y\left(t ; x_{0}, v_{0}, \epsilon\right), \dot{y}=$ $\dot{y}\left(t ; x_{0}, v_{0}, \epsilon\right), C=y \wedge \dot{y}$ and observe that

$$
\begin{equation*}
|y| \leq \frac{1}{\delta}, \quad|y|^{\frac{1}{2}}|\dot{y}| \leq \sqrt{2} \tag{39}
\end{equation*}
$$

These estimates are a consequence of the inequality $\frac{1}{2}|\dot{y}|^{2}-\frac{1}{|y|} \leq-\delta$. Also,

$$
\begin{equation*}
|C| \leq\left|x_{0}\right|\left|v_{0}\right| e^{-\epsilon A_{1} t} \quad \text { and } \quad|\dot{C}| \leq \epsilon M_{\delta}\left|x_{0}\right|\left|v_{0}\right| e^{-\epsilon A_{1} t} \tag{40}
\end{equation*}
$$

where $M_{\delta}:=\max _{r \in\left[0, \frac{1}{\delta}\right]} \mathcal{D}(r)$. From the proof of Theorem 4.2 we see that $I_{\epsilon}$ can be expressed in the form

$$
\begin{align*}
I_{\epsilon}\left(x_{0}, v_{0}\right)=R\left(x_{0}, v_{0}\right)+2 \epsilon \mathcal{D}(0) x_{0} \wedge C\left(x_{0}, v_{0}\right) & -2 I_{1, \epsilon}\left(+\infty ; x_{0}, v_{0}\right)+ \\
& +2 \epsilon \mathcal{D}(0) I_{2, \epsilon}\left(+\infty ; x_{0}, v_{0}\right) . \tag{41}
\end{align*}
$$

If we denote by $L_{\delta}$ the Lipschitz constant of $\mathcal{D}$ on $\left[0, \frac{1}{\delta}\right]$, by using (39) and the first inequality of (40) we get

$$
\begin{align*}
\left|I_{1, \epsilon}\right|=\epsilon \mid \int_{0}^{+\infty} & \left.\left(\mathcal{D}\left(\epsilon^{\frac{2}{3}}|y|\right)-\mathcal{D}(0)\right) \dot{y} \wedge C d t\left|\leq \epsilon L_{\delta} \epsilon^{\frac{2}{3}} \int_{0}^{+\infty}\right| y||\dot{y}|| C \right\rvert\, d t \\
& \leq \epsilon^{\frac{2}{3}} L_{\delta} \sqrt{\frac{2}{\delta}}\left|x_{0}\right|\left|v_{0}\right| \epsilon \int_{0}^{+\infty} e^{-\epsilon A_{1} t} d t=\epsilon^{\frac{2}{3}} \frac{L_{\delta}}{A_{1}} \sqrt{\frac{2}{\delta}}\left|x_{0}\right|\left|v_{0}\right| \tag{42}
\end{align*}
$$

Now, from the first inequality of (39) and the second inequality of (40) we get

$$
\left|I_{2, \epsilon}\right|=\left|\int_{0}^{+\infty} y \wedge \dot{C} d t\right| \leq \frac{1}{\delta} M_{\delta}\left|x_{0}\right|\left|v_{0}\right| \epsilon \int_{0}^{+\infty} e^{-A_{1} \epsilon t} d t=\frac{M_{\delta}}{\delta A_{1}}\left|x_{0}\right|\left|v_{0}\right|
$$

that together with (42) gives (38).
We are now in a position to prove that for the drag in (2) there exist orbits whose eccentricity tends asymptotically to zero.

Proposition 5.3. There exists $\left(x_{0}, v_{0}\right) \in \Omega$ such that the corresponding solution of (2) is asymptotically circular, that is $\left(x_{0}, v_{0}\right)$ satisfies $I\left(x_{0}, v_{0}\right)=0$.

Proof. Consider the family of functions $F_{\epsilon}:[0,1] \times \Omega \rightarrow \mathbb{R}^{4}$, where

$$
F_{\epsilon}(x, v):=\binom{I_{\epsilon}(x, v)}{x-\xi} .
$$

By Lemma 5.2 the family $F_{\epsilon}$ is continuous in $[0,1] \times \bar{B}$ and, moreover, by (38) we have that $F_{0}=F$. Then, since $\operatorname{deg}(F, B, 0)=1$, the homotopy invariance of the degree guarantees that for sufficiently small $\epsilon$ there exists a zero, necessarily of the form $(\xi, v(\epsilon))$, of $F_{\epsilon}$ in $B$. Hence, $I_{\epsilon}(\xi, v(\epsilon))=0$ and by (37) we conclude that the point $\left(x_{0}, v_{0}\right):=\left(\epsilon^{\frac{2}{3}} \xi, \epsilon^{-\frac{1}{3}} v(\epsilon)\right) \in \Omega$ is the initial condition of an asymptotically circular orbit of (2).

Finally, we prove our claim about the range of $I$.

### 5.1. Proof of (ii) of Theorem 4.2.

If $x_{0} \wedge v_{0} \neq 0$ then $C(t) \neq 0$ for any $t \in[0,+\infty[$ and, by Proposition 3.2, $E(t) \rightarrow-\infty$ when $t \rightarrow+\infty$. As a consequence, by (20) we get $|R(t)|^{2}-1<0$ if $t$ is large enough, and $|I| \leq 1$ follows taking the limit in $t$. In the case $x_{0} \wedge v_{0}=0$, we have $I\left(x_{0}, v_{0}\right)=-\frac{x_{0}}{\left|x_{0}\right|}$ so that $\left|I\left(x_{0}, v_{0}\right)\right|=1$. Since by Proposition 5.3 the first integral $I$ takes the value 0 , by its continuity and by its invariance under planar rotations we get $I(\Omega)=\mathbb{D}$.

The geometrical and dynamical consequences of the existence of $I$ are analogous to the ones described in [16] for the linear drag. Namely, if $x(t)=$ $x\left(t ; x_{0}, v_{0}\right)=r(t) e^{i \theta(t)}$ is a non rectilinear motion of (2), then the trajectory

$$
y(t)=e^{2 \int_{0}^{t} \mathcal{D}(|x(s)|) d s} x(t)
$$

tends asymptotically to the curve

$$
|y|+<y, I\left(x_{0}, v_{0}\right)>=\left|C\left(x_{0}, v_{0}\right)\right|^{2} .
$$

When $\left|I\left(x_{0}, v_{0}\right)\right|<1$ this is an ellipse whose eccentricity vector is $I\left(x_{0}, v_{0}\right)$, and in such a case the following holds: $|x(t)|=r(t)$ tends to zero exponentially with time, whereas the modulus of the angular velocity $|\dot{\theta}(t)|$ increases exponentially with time. The proofs of these facts follow taking into account that $A_{1} \leq$ $\mathcal{D}\left(\left|x\left(t ; x_{0}, v_{0}\right)\right|\right) \leq M=\max _{t \geq 0} \mathcal{D}\left(\left|x\left(t ; x_{0}, v_{0}\right)\right|\right)$ and following the steps in [16] to obtain the exponential estimates on the growth of $|x|$ and $|\dot{\theta}|$.

## 6. Appendix

### 6.1. Proof of Proposition 2.4

We start by regularizing the first order system equivalent to equation (14) by the time rescaling $\tau=\tau(t)=\int_{0}^{t} \frac{d \sigma}{r^{2}(\sigma)}$. We obtain the system

$$
\left\{\begin{array}{l}
r^{\prime}=r^{2} u  \tag{43}\\
u^{\prime}=-\mathcal{D}(r) r^{2} u-1
\end{array}\right.
$$

where the derivatives are taken with respect to the time $\tau$. Now we proceed in a manner that is analogous to the one employed in the proof of Proposition 3.2 of [15]. We start by noticing that $r=0$ is an orbit of (43) and that this system does not have any equilibria. Also, the set $Q=\{(r, u): r>0, u<0\}$ is a positively invariant set for (43) on which the $r$ component of the solutions of (43) is decreasing. A key ingredient of the proof is the existence of the first integral of (43) given by

$$
\begin{equation*}
H:=u+\Delta(r)+\tau \tag{44}
\end{equation*}
$$

where $\Delta(r):=\int_{0}^{r} \mathcal{D}(\sigma) d \sigma$ satisfies the estimate $\Delta(r) \geq A_{1} r$. Using the first integral and the estimate, one proves that all solutions with $r(0)>0$ eventually enter the set $Q$. In fact, by a contradiction argument, one sees that the negation of this property implies the existence of a bounded orbit having as its $\omega$-limit an equilibrium of (43) in the first quadrant. Then, in an analogous manner, it is easily shown that all solutions are defined for $\tau \in[0,+\infty[$. Since if $(r(0), u(0))=$ $\left(r_{0}, u_{0}\right) \in Q$, then $r(\tau) \in\left[0, r_{0}\right]$ for any $\tau \in[0,+\infty[$, and since from (44) we have

$$
u(\tau)+\Delta(r(\tau))=u_{0}+\Delta\left(r_{0}\right)-\tau
$$

we conclude that

$$
\begin{equation*}
\frac{u(\tau)}{\tau} \rightarrow-1 \text { as } \tau \rightarrow+\infty \tag{45}
\end{equation*}
$$

As a consequence, we get that $u(\tau) \rightarrow-\infty$ as $\tau \rightarrow+\infty$ and, integrating the first equation of (43), we get also that

$$
\begin{equation*}
\tau^{2} r(\tau)=\frac{\tau^{2}}{\frac{1}{r_{0}}+\int_{0}^{\tau}|u(\sigma)| d \sigma} \rightarrow 2 \text { as } \tau \rightarrow+\infty \tag{46}
\end{equation*}
$$

We conclude that $r(\tau) \rightarrow 0$ as $\tau \rightarrow+\infty$. To end our proof we have to show that the maximal interval $[0, \omega[$ of a solution $r(t)$ of (14) is bounded. If $t=T(\tau)$ is the inverse function of $\tau=\tau(t)$, then $r(\tau):=r(T(\tau))$ is the first component of a solution of (43), and we have

$$
\omega=\int_{0}^{+\infty} T^{\prime}(\tau) d \tau=\int_{0}^{+\infty} r^{2}(\tau) d \tau \in \mathbb{R}
$$

since by (46) $r^{2}(\tau)$ behaves like $\frac{4}{\tau^{4}}$ for large $\tau$.

### 6.2. Proof of Proposition 3.3

This proof follows the steps of the one given for the linear drag in Proposition 3.2 of [15]. Let $r(t)$ be a maximal solution of (14) defined on $[0, \omega[, \omega \in \mathbb{R}$. Its energy, expressed in the time $\tau$, is given by $E(\tau):=E(r(\tau), u(\tau))$, where $(r(\tau), u(\tau))=(r(T(\tau)), \dot{r}(T(\tau)))$ is a solution of (43) defined on $[0,+\infty[$. Then,

$$
E^{\prime}(\tau)=-\mathcal{D}(r(\tau)) u^{2}(\tau) r^{2}(\tau), \quad \tau \in[0,+\infty[
$$

By (45) and (46) we get that fixed any positive $\eta$

$$
\left|E^{\prime}(\tau)\right| \leq \max _{[0, M]} \mathcal{D}(r) \frac{4+\eta}{\tau^{2}}
$$

for sufficiently large $\tau$, where $M=\max _{[0,+\infty[ } r(\tau) \in \mathbb{R}$ exists since by Proposition 2.4 all solutions of (14) tend to zero. We conclude that $E^{\prime} \in L^{1}[0,+\infty[$ and hence $E(\omega)=E(0)+\int_{0}^{+\infty} E^{\prime}(\tau) d \tau \in \mathbb{R}$.

The proof of the fact that the energy may take any arbitrarily prescribed value $E_{1} \in \mathbb{R}$ at collision is completely analogous to the proof of Proposition 4.2 in [15], and we give it for the reader's sake. Let $E_{1}$ be a prescribed value of the energy. Let $(w(s), v(s), E(s)) \in \mathcal{M}$ be the solution of (16) such that $(w(0), v(0), E(0))=\left(0,-1 / \sqrt{2}, E_{1}\right) \in \mathcal{M}$. Let $s=S(t)$ be the local inverse of $T(s)=t_{1}-\int_{s}^{0} w^{2}(\sigma) d \sigma$ in a suitable left neighbourhood of $s=0$, where $t_{1}$ is arbitrarily fixed in $\mathbb{R}$. Then, the function $r(t):=w(S(t))^{2}$, defined in a left neighbourhood of $t_{1}$, will solve

$$
\ddot{r}=-\mathcal{D}(r) \dot{r}+\frac{1}{r^{2}} \mathcal{J}-\frac{1}{r^{2}},
$$

where $\mathcal{J}(E, w, v):=E|w|^{2}-2|v|^{2}+1$. Since $\mathcal{M}$ is invariant, we get that $\mathcal{J}=0$ along the solutions of (16) and we conclude that $r(t)$ is a solution (14) which collides with the singularity at time $t_{1}$ having energy $E_{1}$ at collision.

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# Principal eigenvalues of weighted periodic-parabolic problems 

Inmaculada Antón and Julián López-Gómez

Dedicated to J. L. Mawhin at the occasion of his 75th birthday


#### Abstract

Based on a recent characterization of the strong maximum principle, [3], this paper gives some periodic-parabolic counterparts of some of the results of Chapters 8 and 9 of J. López-Gómez [22]. Among them count some pivotal monotonicity properties of the principal eigenvalue $\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]$, as well as its concavity with respect to the periodic potential $V$ through a point-wise periodic-parabolic Donsker-Varadhan min-max characterization. Finally, based on these findings, this paper sharpens, substantially, some classical results of A. Beltramo and P. Hess [4], K. J. Brown and S. S. Lin [6], and P. Hess [14] on the existence and uniqueness of principal eigenvalues for weighted boundary value problems.


Keywords: periodic-parabolic problems, maximum principle, principal eigenvalue, global properties.
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## 1. Introduction

For any given $T>0$, this paper studies the existence of principal eigenvalues, $\lambda$, for the $T$-periodic-parabolic weighted boundary value problem

$$
\begin{cases}\partial_{t} \varphi+\mathfrak{L} \varphi=\lambda W(x, t) \varphi & \text { in } \Omega \times[0, T],  \tag{1}\\ \mathfrak{B} \varphi=0 & \text { on } \partial \Omega \times[0, T],\end{cases}
$$

under the following general assumptions:
(A1) $\Omega$ is a bounded subdomain (open and connected set) of $\mathbb{R}^{N}, N \geq 1$, of class $\mathcal{C}^{2+\theta}$ for some $0<\theta \leq 1$, whose boundary, $\partial \Omega$, consists of two disjoint open and closed subsets, $\Gamma_{0}$ and $\Gamma_{1}$, such that $\partial \Omega:=\Gamma_{0} \cup \Gamma_{1}$ (as they are disjoint, $\Gamma_{0}$ and $\Gamma_{1}$ must be of class $\mathcal{C}^{2+\theta}$ ).
(A2) $\mathfrak{L}$ is a non-autonomous differential operator of the form

$$
\mathfrak{L}=\mathfrak{L}(x, t):=-\sum_{i, j=1}^{N} a_{i j}(x, t) \frac{\partial^{2}}{\partial x_{i} \partial x_{j}}+\sum_{j=1}^{N} b_{j}(x, t) \frac{\partial}{\partial x_{j}}+c(x, t)
$$

with $a_{i j}=a_{i j}, b_{j}, c \in F$ for all $i, j \in\{1, \ldots, N\}$, where

$$
\begin{equation*}
F:=\left\{u \in \mathcal{C}^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R} ; \mathbb{R}): u(\cdot, T+t)=u(\cdot, t) \text { for all } t \in \mathbb{R}\right\} \tag{2}
\end{equation*}
$$

Similarly, $W \in F$. So, $\mathfrak{L}-\lambda W$ has exactly the same type as $\mathfrak{L}$, because $c-\lambda W \in F$. Moreover, the operator $\mathfrak{L}$ is assumed to be uniformly elliptic in $\bar{Q}_{T}$, where $Q_{T}$ stands for the (open) parabolic cylinder

$$
Q_{T}:=\Omega \times(0, T) .
$$

In other words, there exists $\mu>0$ such that

$$
\sum_{i, j=1}^{N} a_{i j}(x, t) \xi_{i} \xi_{j} \geq \mu|\xi|^{2} \quad \text { for all }(x, t, \xi) \in \bar{Q}_{T} \times \mathbb{R}^{N}
$$

where $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^{N}$.
(A3) $\mathfrak{B}: \mathcal{C}\left(\Gamma_{0}\right) \oplus \mathcal{C}^{1}\left(\Omega \cup \Gamma_{1}\right) \rightarrow C(\partial \Omega)$ stands for the boundary operator

$$
\mathfrak{B} \xi:= \begin{cases}\xi & \text { on } \Gamma_{0} \\ \frac{\partial \xi}{\partial \nu}+\beta(x) \xi & \text { on } \Gamma_{1}\end{cases}
$$

for each $\xi \in \mathcal{C}\left(\Gamma_{0}\right) \oplus \mathcal{C}^{1}\left(\Omega \cup \Gamma_{1}\right)$, where $\beta \in \mathcal{C}^{1+\theta}\left(\Gamma_{1}\right)$ and

$$
\nu=\left(\nu_{1}, \ldots, \nu_{N}\right) \in \mathcal{C}^{1+\theta}\left(\partial \Omega ; \mathbb{R}^{N}\right)
$$

is an outward pointing nowhere tangent vector field. Occasionally, we will emphasize the dependence of $\mathfrak{B}$ on $\beta$ by setting $\mathfrak{B}=\mathfrak{B}[\beta]$. Naturally, we simply set $\mathfrak{D}=\mathfrak{B}$ if $\Gamma_{1}=\emptyset$ (Dirichlet b.c.), or $\mathfrak{N}=\mathfrak{B}$ if $\Gamma_{0}=\emptyset$ and $\beta=0$ (Neumann b.c.).

Thus, the functions $c(x, t)$ and $\beta(x)$ can change sign, in strong contrast with the classical setting of A. Beltramo and P. Hess [4], substantially refined by P. Hess [14, Ch. II], where $c, \beta \geq 0$ and either $\Gamma_{0}$, or $\Gamma_{1}$, is empty. Note that $\mathfrak{B}$ is the Dirichlet boundary operator on $\Gamma_{0}$, and the Neumann, or a first order regular oblique derivative boundary operator, on $\Gamma_{1}$. Naturally, either $\Gamma_{0}$, or $\Gamma_{1}$, can be empty.

Subsequently, besides the space $F$ introduced in (2), we also consider the Banach space of Hölder continuous $T$-periodic functions

$$
E:=\left\{u \in \mathcal{C}^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times \mathbb{R} ; \mathbb{R}): u(\cdot, T+t)=u(\cdot, t) \text { for all } t \in \mathbb{R}\right\}
$$

and the periodic-parabolic operator

$$
\mathcal{P}:=\partial_{t}+\mathfrak{L}(x, t) .
$$

By a principal eigenvalue of the eigenvalue problem (1) it is meant a value of $\lambda \in \mathbb{R}$ for which (1) possesses a positive eigenfunction, $\varphi \in E$. The main goal of this paper is analyzing the existence and multiplicity of eigenvalues of (1) by adapting to the periodic-parabolic context the methodology of J. López-Gómez [18], later refined in [19] and [22, Ch. 9], in order to sharpen the classical results of P. Hess and T. Kato [15]. Naturally, the principal eigenvalues of the weighted problem (1) are given by the zeroes of the principal eigenvalue

$$
\begin{equation*}
\Sigma(\lambda)=\sigma\left[\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right], \quad \lambda \in \mathbb{R}, \tag{3}
\end{equation*}
$$

of the tern $\left(\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right)$, whose existence and uniqueness, under the general setting of this paper, goes back to $[2,3]$.

Throughout this paper, a function $h \in E$ is said to be a supersolution of the tern $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$ if

$$
\begin{cases}\mathcal{P} h \geq 0 & \text { in } Q_{T}, \\ \mathfrak{B} h \geq 0 & \text { on } \partial Q_{T}=\partial \Omega \times[0, T]\end{cases}
$$

If, in addition, some of these inequalities is strict, $¥$, then $h$ is said to be a strict supersolution of $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$. A significant portion of the mathematical analysis carried out in this paper is based on the next result, going back to Theorem 1.2 of [3] in its greatest generality. Based on the abstract theory of D. Daners and P. Koch-Medina [10], it extends to a periodic-parabolic context the corresponding elliptic counterparts of J. López-Gómez \& M. Molina-Meyer [23] and H. Amann \& J. López-Gómez [2]. A special version, for $\beta \geq 0$, had been recently given by R. Peng and X. Q. Zhao [25].

Theorem 1.1. Suppose (A1), (A2) and (A3). Then, the following conditions are equivalent:
(a) $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]>0$.
(b) $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$ possesses a non-negative strict supersolution $h \in E$.
(c) The resolvent operator of $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$ is strongly positive, i.e., any strict supersolution $u \in E$ of $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$ satisfies $u \gg 0$, in the sense that $u(x, t)>0$ for all $t \in[0, T]$ and $x \in \Omega \cup \Gamma_{1}$, and

$$
\partial_{\nu} u(x, t)<0 \quad \text { for all } t \in[0, T] \text { and } x \in u^{-1}(0) \cap \Gamma_{0} .
$$

In other words, $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$ satisfies the strong maximum principle.
Based on Theorem 1.1 one can easily derive all monotonicity properties of $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$ given in Section 2, as well as infer the point-wise min-max characterizations of the principal eigenvalue of Donsker-Varadhan type given in Section 3. In Section 4, based on these min-max characterizations, we will
adopt the methodology of J. López-Gómez [19, 21, 22], in order to establish the concavity of $\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]$ with respect to the periodic potential $V \in F$. The most pioneering results in this vain go back to T. Kato [16]. Our proof is based on a technical device of H. Berestycki, L. Nirenberg and S. R. S. Varadhan [5] based on the Donsker-Varadhan characterization of the principal eigenvalue, [13]. Later, in Section 5, the concavity with respect to $V$ will provide us with the concavity of $\Sigma(\lambda)$ with respect to the parameter $\lambda \in \mathbb{R}$ and the real analyticity of $\Sigma(\lambda)$, which is derived from a classical result of F. Rellich [26] sharpened by T. Kato [17]. From all these results one can easily derive some important global properties of $\Sigma(\lambda)$ that provide us with some substantial improvements of those collected by P. Hess in Chapter II of [14], where it was imposed, in addition, that $c \geq 0$ and $\beta \geq 0$, and that either $\Gamma_{0}$, or $\Gamma_{1}$, is empty. Actually, in Sections 6 and 7 we characterize the existence, uniqueness, multiplicity and simplicity of the principal eigenvalues of (1) in all possible cases. Crucially, in this paper we are not requiring ( $\mathcal{P}, \mathfrak{B}, Q_{T}$ ) to satisfy the strong maximum principle. So, our analysis is much sharper and versatile than the classical one of P. Hess [14, Ch. II]. As a result, the problem (1) can admit two principal eigenvalues with the same sign, which is a situation not previously considered, even in the elliptic counterpart of (1), by the classical theory of A. Manes \& A. M. Micheletti [24] and P. Hess \& T. Kato [15].

## 2. Some basic properties of the principal eigenvalue

This section collects some useful properties of $\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]$ that are direct consequences from Theorem 1.1. The next one establishes its monotonicity with respect to the potential $V$.

Proposition 2.1. Let $V_{1}, V_{2} \in F$ such that $V_{1} \lesseqgtr V_{2}$. Then,

$$
\sigma\left[\mathcal{P}+V_{1}, \mathfrak{B}, Q_{T}\right]<\sigma\left[\mathcal{P}+V_{2}, \mathfrak{B}, Q_{T}\right] .
$$

Proof. Let $\varphi_{1} \in E, \varphi_{1} \gg 0$, be an eigenfunction associated to the principal eigenvalue $\sigma_{1}:=\sigma\left[\mathcal{P}+V_{1} ; \mathfrak{B}, Q_{T}\right]$. Then,

$$
\left(\mathcal{P}+V_{2}-\sigma_{1}\right) \varphi_{1}=\left(V_{2}-V_{1}\right) \varphi_{1} \not \geqslant 0 \quad \text { in } Q_{T} .
$$

Thus, $\varphi_{1}$ provides us with a positive strict supersolution of the tern $\left(\mathcal{P}+V_{2}-\right.$ $\left.\sigma_{1}, \mathfrak{B}, Q_{T}\right)$. Therefore, by Theorem 1.1,

$$
\begin{aligned}
0<\sigma\left[\mathcal{P}+V_{2}-\sigma_{1}, \mathfrak{B}, Q_{T}\right] & =\sigma\left[\mathcal{P}+V_{2}, \mathfrak{B}, Q_{T}\right]-\sigma_{1} \\
& =\sigma\left[\mathcal{P}+V_{2}, \mathfrak{B}, Q_{T}\right]-\sigma\left[\mathcal{P}+V_{1}, \mathfrak{B}, Q_{T}\right]
\end{aligned}
$$

which ends the proof.

The next two consequences of Proposition 2.1 provide us with the continuous dependence of the principal eigenvalue with respect to $V$.

Corollary 2.2. Let $V_{n} \in F, n \geq 1$, be a sequence of potentials such that

$$
\lim _{n \rightarrow \infty} V_{n}=V \quad \text { in } \quad \mathcal{C}\left(\bar{Q}_{T}\right)
$$

Then,

$$
\lim _{n \rightarrow \infty} \sigma\left[\mathcal{P}+V_{n}, \mathfrak{B}, Q_{T}\right]=\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right] .
$$

Proof. For every $\varepsilon>0$ there exists a natural number $n_{0}=n_{0}(\varepsilon)>1$ such that

$$
V-\varepsilon \leq V_{n} \leq V+\varepsilon \quad \text { in } \quad \bar{Q}_{T} \quad \text { for all } n \geq n_{0}
$$

Thus, thanks to Proposition 2.1, for every $n \geq n_{0}$,

$$
\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]-\varepsilon \leq \sigma\left[\mathcal{P}+V_{n}, \mathfrak{B}, Q_{T}\right] \leq \sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]+\varepsilon,
$$

which ends the proof.
Naturally, as a byproduct, Corollary 2.2 yields
Corollary 2.3. For every $W \in F$, the map $\Sigma: \mathbb{R} \rightarrow \mathbb{R}$ defined by (3) is continuous.

Next, we will adapt Propositions 3.1, 3.2 and 3.5 of C. Cano-Casanova and J. López-Gómez [7] to the periodic-parabolic setting of this paper. Essentially, they establish the monotonicities of the principal eigenvalue with respect to $\beta$ and $\Omega$, as well as the dominance of $\sigma\left[\mathcal{P}, \mathfrak{D}, Q_{T}\right]$.
Proposition 2.4. Suppose $\Gamma_{1} \neq \emptyset$ and $\beta_{1}, \beta_{2} \in \mathcal{C}^{1+\theta}\left(\Gamma_{1}\right)$ satisfy $\beta_{1} \lesseqgtr \beta_{2}$. Then,

$$
\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right]<\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{2}\right], Q_{T}\right]
$$

Proof. Let $\varphi_{1} \in E, \varphi_{1} \gg 0$, be a principal eigenfunction associated to the principal eigenvalue $\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right]$. Then,

$$
\left(\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right]\right) \varphi_{1}=0 \quad \text { in } \quad Q_{T}
$$

$\varphi_{1}=0$ on $\Gamma_{0}$, and

$$
\mathfrak{B}\left[\beta_{2}\right] \varphi_{1}=\mathfrak{B}\left[\beta_{1}\right] \varphi_{1}+\left(\beta_{2}-\beta_{1}\right) \varphi_{1}=\left(\beta_{2}-\beta_{1}\right) \varphi_{1} \geqslant 0 \quad \text { on } \quad \Gamma_{1}
$$

because $\beta_{2} \nsucceq \beta_{1}$ and $\varphi_{1}(x, t)>0$ for all $t \in[0, T]$ and $x \in \Omega \cup \Gamma_{1}$. Thus, $\varphi_{1}$ provides us with a strict positive supersolution of

$$
\left(\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right], \mathfrak{B}\left[\beta_{2}\right], Q_{T}\right)
$$

Therefore, owing to Theorem 1.1,

$$
0<\sigma\left[\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right], \mathfrak{B}\left[\beta_{2}\right], Q_{T}\right]=\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{2}\right], Q_{T}\right]-\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}\right], Q_{T}\right] .
$$

The proof is complete.

Proposition 2.5. $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]<\sigma\left[\mathcal{P}, \mathfrak{D}, Q_{T}\right]$ if $\Gamma_{1} \neq \emptyset$.
Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$. Then, according to Theorem 1.1,

$$
\varphi(x, t)>0 \quad \text { for all } x \in \Omega \cup \Gamma_{1} \text { and } t \in[0, T] .
$$

Thus, $\mathfrak{D} \varphi(x, t)=\varphi(x, t)>0$ for all $x \in \Gamma_{1}$ and $t \in[0, T]$. Hence,

$$
\mathfrak{D} \varphi=\varphi \ngtr 0 \quad \text { on } \partial \Omega \times[0, T] .
$$

So, $\varphi$ provides us with a positive strict supersolution of

$$
\left(\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right], \mathfrak{D}, Q_{T}\right)
$$

and therefore, by Theorem 1.1,

$$
0<\sigma\left[\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right], \mathfrak{D}, Q_{T}\right]=\sigma\left[\mathcal{P}, \mathfrak{D}, Q_{T}\right]-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]
$$

which ends the proof.
Suppose $\Gamma_{1} \neq \emptyset$. Then, for every proper subdomain of $\Omega, \Omega_{0}$, of class $\mathcal{C}^{2+\theta}$ with

$$
\begin{equation*}
\operatorname{dist}\left(\Gamma_{1}, \partial \Omega_{0} \cap \Omega\right)>0 \tag{4}
\end{equation*}
$$

we denote by $\mathfrak{B}\left[\Omega_{0}\right]$ the boundary operator defined by

$$
\mathfrak{B}\left[\Omega_{0}\right] \xi:= \begin{cases}\xi & \text { on } \quad \partial \Omega_{0} \cap \Omega  \tag{5}\\ \mathfrak{B} \xi & \text { on } \quad \partial \Omega_{0} \cap \partial \Omega\end{cases}
$$

for each $\xi \in \mathcal{C}\left(\Gamma_{0}\right) \oplus \mathcal{C}^{1}\left(\Omega \cup \Gamma_{1}\right)$. In particular, $\mathfrak{B}\left[\Omega_{0}\right]=\mathfrak{D}$ if $\bar{\Omega}_{0} \subset \Omega$, because, in such case, $\partial \Omega_{0} \subset \Omega$. When $\Gamma_{1}=\emptyset$, by definition, $\mathfrak{B}=\mathfrak{D}$ and we simply set $\mathfrak{B}\left[\Omega_{0}\right]:=\mathfrak{D}$. The next result establishes the monotonicity of the principal eigenvalue with respect to $\Omega$.
Proposition 2.6. Let $\Omega_{0}$ be a proper subdomain of $\Omega$ of class $\mathcal{C}^{2+\theta}$ satisfying (4) if $\Gamma_{1} \neq \emptyset$. Then,

$$
\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]<\sigma\left[\mathcal{P}, \mathfrak{B}\left[\Omega_{0}\right], \Omega_{0} \times(0, T)\right]
$$

where $\mathfrak{B}\left[\Omega_{0}\right]$ is the boundary operator defined by (5).

Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$. By definition,

$$
\left(\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]\right) \varphi=0 \quad \text { in } \Omega_{0} \times(0, T),
$$

because $\Omega_{0} \subset \Omega$. Moreover, by construction,

$$
\begin{cases}\varphi>0 & \text { on }\left(\partial \Omega_{0} \cap \Omega\right) \times[0, T] \\ \varphi=0 & \text { on }\left(\partial \Omega_{0} \cap \Gamma_{0}\right) \times[0, T] \\ \partial_{\nu} \varphi+\beta \varphi=0 & \text { on }\left(\partial \Omega_{0} \cap \Gamma_{1}\right) \times[0, T]\end{cases}
$$

Note that $\partial \Omega_{0} \cap \Omega \neq \emptyset$, because $\Omega_{0} \nsubseteq \Omega$. Thus, $\left.\varphi\right|_{\Omega_{0}}$ provides us with a positive strict supersolution of the tern

$$
\left(\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right], \mathfrak{B}\left[\Omega_{0}\right], \Omega_{0} \times(0, T)\right)
$$

Therefore, thanks again to Theorem 1.1,

$$
\begin{aligned}
0 & <\sigma\left[\mathcal{P}-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right], \mathfrak{B}\left[\Omega_{0}\right], \Omega_{0} \times(0, T)\right] \\
& =\sigma\left[\mathcal{P}, \mathfrak{B}\left[\Omega_{0}\right], \Omega_{0} \times(0, T)\right]-\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]
\end{aligned}
$$

which ends the proof.

As an immediate consequence of Propositions 2.4 and 2.6, the next result holds.

Corollary 2.7. Suppose $\Gamma_{1} \neq \emptyset$. Then, for every subdomain of class $\mathcal{C}^{2+\theta}$ of $\Omega, \Omega_{0}$, satisfying (4) if $\Gamma_{1} \neq \emptyset$, and any $\beta_{1}, \beta_{2} \in \mathcal{C}^{1+\theta}\left(\Gamma_{1}\right)$ with $\beta_{1} \lesseqgtr \beta_{2}$,

$$
\begin{equation*}
\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{1}, \Omega\right], Q_{T}\right]<\sigma\left[\mathcal{P}, \mathfrak{B}\left[\beta_{2}, \Omega_{0}\right], \Omega_{0} \times(0, T)\right] \tag{6}
\end{equation*}
$$

The same conclusion holds if $\beta_{1} \leq \beta_{2}$ and $\Omega_{0} \subsetneq \Omega$.
We conclude this section with an extremely useful consequence of the uniqueness of the principal eigenvalue. It should be compared with [14, Lem. 15.3].

Proposition 2.8. Let $V \in F$ be independent of $x \in \Omega$, i.e., $V(x, t)=V(t)$ for all $(x, t) \in Q_{T}$. Then,

$$
\begin{equation*}
\sigma\left[\mathcal{P}+V(t), \mathfrak{B}, Q_{T}\right]=\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]+\frac{1}{T} \int_{0}^{T} V(t) d t \tag{7}
\end{equation*}
$$

Proof. Let $\varphi \gg 0$ be a principal eigenfunction associated to $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$. The proof consists in searching for a real function $h \in \mathcal{C}^{1}(\mathbb{R})$ such that

$$
\psi(x, t):=e^{h(t)} \varphi(x, t), \quad(x, t) \in \bar{Q}_{T}
$$

provides us with a principal eigenfunction of $\left(\mathcal{P}+V(t), \mathfrak{B}, Q_{T}\right)$. Since

$$
(\mathcal{P}+V(t)) \psi(x, t)=\left(\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]+h^{\prime}(t)+V(t)\right) \psi(x, t)
$$

it becomes apparent that making the choice

$$
h(t)=\frac{t}{T} \int_{0}^{T} V-\int_{0}^{t} V, \quad t \in[0, T]
$$

we have that $h(0)=h(T)=0$ and

$$
h^{\prime}(t)+V(t)=\frac{1}{T} \int_{0}^{T} V
$$

for all $t \in[0, T]$. Thus,

$$
(\mathcal{P}+V(t)) \psi(x, t)=\left(\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]+\frac{1}{T} \int_{0}^{T} V\right) \psi(x, t)
$$

Therefore, by the uniqueness of the principal eigenvalue, (7) holds.

As a byproduct of (7), for every $V \in F$ independent on $x \in \Omega$, we have that

$$
\Sigma(\lambda):=\sigma\left[\mathcal{P}+\lambda V(t), \mathfrak{B}, Q_{T}\right]=\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]+\lambda \bar{V}
$$

for all $\lambda \in \mathbb{R}$, where, as usual, we are denoting by $\bar{V}$ the average

$$
\bar{V}:=\frac{1}{T} \int_{0}^{T} V(t) d t .
$$

Thus, the graph of $\Sigma(\lambda)$ is a straight line with slope $\bar{V}$. Note that $\bar{V}$ can have any sign if $V$ changes sign, which cannot occur in the elliptic counterpart of the theory.

We conclude this section with the next fundamental result.
THEOREM 2.9. $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$ is an algebraically simple eigenvalue of $\left(\mathcal{P}, \mathfrak{B}, Q_{T}\right)$.
Proof. Through this proof, we set $\sigma:=\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$. By the construction of $\sigma$ in [3], $\sigma$ is geometrically simple. To show that it is algebraically simple we should see that, for any given associated eigenfunction, $\varphi \gg 0$, the boundary value problem

$$
\begin{cases}(\mathcal{P}-\sigma) u=\varphi & \text { in } Q_{T}, \\ \mathfrak{B} u=0 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

cannot admit a solution in $E$. On the contrary, suppose that it admits a solution, $u \in E$. Then, for all $\omega>0$, we have that

$$
\begin{cases}(\mathcal{P}+\omega) u=(\sigma+\omega) u+\varphi & \text { in } Q_{T}, \\ \mathfrak{B} u=0 & \text { on } \partial \Omega \times[0, T] .\end{cases}
$$

Thus, according to Theorem 1.1, for sufficiently large $\omega>0$, we have that

$$
\begin{equation*}
u=(\sigma+\omega)(\mathcal{P}+\omega)^{-1} u+(\mathcal{P}+\omega)^{-1} \varphi \tag{8}
\end{equation*}
$$

On the other hand, since $\mathcal{P} \varphi=\sigma \varphi$, it becomes apparent that

$$
(\mathcal{P}+\omega)^{-1} \varphi=\frac{1}{\sigma+\omega} \varphi \quad \text { and } \quad \operatorname{spr}(\mathcal{P}+\omega)^{-1}=\frac{1}{\sigma+\omega}
$$

Thus, dividing by $\sigma+\omega$ the identity (8) yields

$$
\left(\operatorname{spr}(\mathcal{P}+\omega)^{-1}-(\mathcal{P}+\omega)^{-1}\right) u=\frac{\varphi}{(\omega+\sigma)^{2}} \gg 0
$$

In particular,

$$
\varphi \in R\left[\operatorname{spr}(\mathcal{P}+\omega)^{-1}-(\mathcal{P}+\omega)^{-1}\right]
$$

which contradicts Theorem 6.1(f) of [22].

## 3. Donsker-Varadhan min-max characterizations

This section gives two point-wise min-max characterizations of the principal eigenvalue $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$. These results adapt to a periodic-parabolic context the celebrated formula of M. D. Donsker and S. R. S. Varadhan [13]. The first one can be stated as follows.

Theorem 3.1. Let $C$ denote the set

$$
C:=\left\{\psi \in E: \psi(x, t)>0 \text { for all }(x, t) \in Q_{T} \text { and } \mathfrak{B} \psi \geq 0 \text { on } \partial \Omega \times[0, T]\right\}
$$

Then,

$$
\begin{equation*}
\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]=\sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}=\max _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi} . \tag{9}
\end{equation*}
$$

Proof. Set $\sigma_{1}:=\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]$ and pick $\lambda<\sigma_{1}$. Then,

$$
\sigma\left[\mathcal{P}-\lambda, \mathfrak{B}, Q_{T}\right]=\sigma_{1}-\lambda>0
$$

and hence, by Theorem 1.1, $\left(\mathcal{P}-\lambda, \mathfrak{B}, Q_{T}\right)$ satisfies Theorem 1.1(c). Thus, the problem

$$
\begin{cases}(\mathcal{P}-\lambda) \psi=1 & \text { in } Q_{T}, \\ \mathfrak{B} \psi=0 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

admits a unique solution in $E, \psi_{1}$, and $\psi_{1} \gg 0$. In particular, $\psi_{1} \in C$ and hence, $C \neq \emptyset$. Moreover, since $\psi_{1}(x, t)>0$ for all $(x, t) \in Q_{T}$, it follows that

$$
\lambda<\frac{\mathcal{P} \psi_{1}}{\psi_{1}} \text { in } Q_{T}
$$

Thus,

$$
\begin{equation*}
\lambda \leq \inf _{Q_{T}} \frac{\mathcal{P} \psi_{1}}{\psi_{1}} \leq \sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi} \tag{10}
\end{equation*}
$$

As this estimate holds for each $\lambda<\sigma_{1}$, it becomes apparent that

$$
\sigma_{1} \leq \sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

To prove the equality, we can argue by contradiction. Suppose

$$
\sigma_{1}<\sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

Then, there are $\epsilon>0$ and $\psi \in C$ such that

$$
\sigma_{1}+\epsilon<\frac{\mathcal{P} \psi(x, t)}{\psi(x, t)} \quad \text { for all }(x, t) \in Q_{T}
$$

As this entails

$$
\begin{cases}\left(\mathcal{P}-\sigma_{1}-\epsilon\right) \psi>0 & \text { in } Q_{T} \\ \mathfrak{B} \psi \geq 0 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

the function $\psi$ provides us with a supersolution of $\left(\mathcal{P}-\sigma_{1}-\epsilon, \mathfrak{B}, Q_{T}\right)$. Thus, by Theorem 1.1,

$$
0<\sigma\left[\mathcal{P}-\sigma_{1}-\epsilon, \mathfrak{B}, Q_{T}\right]=-\epsilon<0
$$

which is impossible. Therefore,

$$
\sigma_{1}=\sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

which provides us with the first identity of (9).
Finally, let $\varphi_{1} \in E, \varphi_{1} \gg 0$, be a principal eigenfunction associated to $\sigma_{1}$. Then, by definition,

$$
\begin{cases}\mathcal{P} \varphi_{1}=\sigma_{1} \varphi_{1} & \text { in } Q_{T}, \\ \mathfrak{B} \varphi_{1}=0 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

and $\varphi_{1} \in C$. Thus,

$$
\sigma_{1}=\inf _{Q_{T}} \frac{\mathcal{P} \varphi_{1}}{\varphi_{1}} .
$$

Consequently, we also have that

$$
\sigma_{1}=\max _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

The proof is completed.
The next results allows us shortening $C$ in the statement of Theorem 3.1.
THEOREM 3.2. Let $C_{+}$be the subset of $C$ defined by
$C_{+}:=\left\{\psi \in E: \psi(x, t)>0\right.$ for all $(x, t) \in \bar{Q}_{T}$ and $\mathfrak{B} \psi \geq 0$ on $\left.\partial \Omega \times[0, T]\right\}$.
Then,

$$
\begin{equation*}
\sigma_{1}:=\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]=\sup _{\psi \in C_{+}} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi} \tag{11}
\end{equation*}
$$

Proof. Let $\lambda<\sigma_{1}$ be. Then, arguing as in Theorem 3.1, it follows from Theorem 1.1 that $\left(\mathcal{P}-\lambda, \mathfrak{B}, Q_{T}\right)$ satisfies Theorem 1.1(c). Now, consider the auxiliary problem

$$
\begin{cases}(\mathcal{P}-\lambda) \psi=1 & \text { in } Q_{T}  \tag{12}\\ \mathfrak{B} \psi=1 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

and a function $h \in E$ such that

$$
\mathfrak{B} h=1 \quad \text { on } \quad \partial \Omega \times[0, T] .
$$

Then, the change of variable

$$
\psi=h+w
$$

transforms (12) into

$$
\begin{cases}(\mathcal{P}-\lambda) w=1-(\mathcal{P}-\lambda) h & \text { in } Q_{T} \\ \mathfrak{B} w=0 & \text { on } \partial \Omega \times[0, T]\end{cases}
$$

Then, owing to Theorem 1.1(c), the function

$$
\psi:=h+(\mathcal{P}-\lambda)^{-1}[1-(\mathcal{P}-\lambda) h]
$$

provides us with the unique solution of (12) in $E$. By Theorem 1.1(c), $\psi \gg 0$. In particular, $\psi(x, t)>0$ for all $x \in \Omega \cup \Gamma_{1}$ and $t \in[0, T]$. Moreover, since $\mathfrak{B} h=1$ on $\partial \Omega \times[0, T]$, we also have that $h=\psi=1$ on $\Gamma_{0}$ and hence, $\psi(x, t)>0$ for all $x \in \partial \Omega$ and $t \in[0, T]$. So, $\psi \in C_{+}$. As, due to (12), we also have that

$$
\lambda<\frac{\mathcal{P} \psi_{1}(x, t)}{\psi_{1}(x, t)} \quad \text { for all } \quad(x, t) \in Q_{T}
$$

it becomes apparent that

$$
\begin{equation*}
\lambda \leq \inf _{Q_{T}} \frac{\mathcal{P} \psi_{1}}{\psi_{1}} \leq \sup _{\psi \in C_{+}} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi} . \tag{13}
\end{equation*}
$$

Therefore, since this inequality holds for every $\lambda<\sigma_{1}$, we find that

$$
\sigma_{1} \leq \sup _{\psi \in C_{+}} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

Finally, since $C_{+} \subset C$,

$$
\sigma_{1} \leq \sup _{\psi \in C_{+}} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi} \leq \sup _{\psi \in C} \inf _{Q_{T}} \frac{\mathcal{P} \psi}{\psi}
$$

Consequently, (11) follows from Theorem 3.1.

## 4. Concavity with respect to the potential

This section establishes the concavity of the map

$$
\begin{array}{lll}
F & \longrightarrow & \mathbb{R} \\
V & \mapsto & \sigma[V]:=\sigma\left[\mathcal{P}+V, \mathfrak{B}, Q_{T}\right]
\end{array}
$$

with respect to potential $V$. This sharpens some classical results of T. Kato [16] and Lemma 5.2 of P . Hess [14], assuming positivity of $c(x, t)$ and $\beta(x)$. Although D. Daners and P. Koch removed these restrictions on Section 14 of [10] under slightly less general boundary conditions than our's, in this paper we are providing an elementary proof of this feature avoiding the use of abstract functional analytic methods. Our proof reveals in a rather direct way the role played by the ellipticity of the differential operator $\mathfrak{L}$ in the underlying theorem, which can be stated as follows.

Theorem 4.1. For every $V_{1}, V_{2} \in F$ and $\varrho \in[0,1]$, the following inequality holds

$$
\begin{equation*}
\sigma\left[\varrho V_{1}+(1-\varrho) V_{2}\right] \geq \varrho \sigma\left[V_{1}\right]+(1-\varrho) \sigma\left[V_{2}\right] . \tag{14}
\end{equation*}
$$

Proof. Throughout this proof, we will set

$$
\xi:=\left(\xi_{1}, \ldots, \xi_{N}\right), \psi:=\left(\psi_{1}, \ldots, \psi_{N}\right) \in \mathbb{R}^{N} .
$$

Since $\mathfrak{L}$ is strongly uniformly elliptic in $\bar{Q}_{T}$ with $a_{i j}=a_{j i}$, setting

$$
A(x, t):=\left(a_{i j}(x, t)\right)_{1 \leq i, j \leq N},
$$

it is apparent that, for every $(x, t) \in \bar{\Omega} \times[0, T]$, the bilinear form

$$
\mathfrak{a}(\xi, \psi):=\sum_{i, j=1}^{N} a_{i j}(x, t) \xi_{i} \psi_{j}=\langle A(x, t) \xi, \psi\rangle, \quad \xi, \psi \in \mathbb{R}^{N}
$$

defines a scalar product in $\mathbb{R}^{N}$. Thus, setting

$$
|\xi|_{\mathfrak{a}}:=\sqrt{\mathfrak{a}(\xi, \xi)}, \quad \xi \in \mathbb{R}^{N}
$$

we find from the Cauchy-Schwarz inequality that

$$
\begin{align*}
2 \mathfrak{a}(\xi, \psi) & =2 \sum_{i, j=1}^{N} a_{i j}(x, t) \xi_{i} \psi_{j} \leq 2|\xi|_{\mathfrak{a}}|\psi|_{\mathfrak{a}} \leq|\xi|_{\mathfrak{a}}^{2}+|\psi|_{\mathfrak{a}}^{2} \\
& =\sum_{i, j=1}^{N} a_{i j}(x, t) \xi_{i} \xi_{j}+\sum_{i, j=1}^{N} a_{i j}(x, t) \psi_{i} \psi_{j} \tag{15}
\end{align*}
$$

for all $\xi, \psi \in \mathbb{R}^{N}$ and $(x, t) \in \bar{\Omega} \times[0, T]$. From this inequality it is easily seen that the map $\mathcal{Q}: E \rightarrow F$ defined by

$$
\mathcal{Q}(u)=-\sum_{i, j=1}^{N} a_{i j}(x, t) \frac{\partial u}{\partial x_{i}} \frac{\partial u}{\partial x_{j}}=-\mathfrak{a}(\nabla u, \nabla u), \quad u \in E,
$$

is concave. Indeed, by (15), the following chain of inequalities holds for every $u_{1}, u_{2} \in E$ and $\varrho \in[0,1]:$

$$
\begin{aligned}
\mathcal{Q}\left(\varrho u_{1}+(1-\varrho) u_{2}\right) & =-\mathfrak{a}\left(\nabla\left(\varrho u_{1}+(1-\varrho) u_{2}\right), \nabla\left(\varrho u_{1}+(1-\varrho) u_{2}\right)\right) \\
& =\varrho^{2} \mathcal{Q}\left(u_{1}\right)+(1-\varrho)^{2} \mathcal{Q}\left(u_{2}\right)-2 \varrho(1-\varrho) \mathfrak{a}\left(\nabla u_{1}, \nabla u_{2}\right) \\
& \geq \varrho^{2} \mathcal{Q}\left(u_{1}\right)+(1-\varrho)^{2} \mathcal{Q}\left(u_{2}\right)+\varrho(1-\varrho)\left(\mathcal{Q}\left(u_{1}\right)+\mathcal{Q}\left(u_{2}\right)\right) \\
& =\varrho \mathcal{Q}\left(u_{1}\right)+(1-\varrho) \mathcal{Q}\left(u_{2}\right)
\end{aligned}
$$

Therefore, the map $G: E \rightarrow F$ defined by

$$
G(u):=(\mathcal{P}-c) u+c+\mathcal{Q}(u), \quad u \in E,
$$

is concave, because $\mathcal{Q}(u)$ is concave and $u \mapsto(\mathcal{P}-c) u$ is linear and, hence, concave. Our interest in $G$ comes from the fact that, for every $\psi \in C_{+}$,

$$
\begin{equation*}
\frac{\mathcal{P} \psi}{\psi}=G(\log \psi) \tag{16}
\end{equation*}
$$

which can be established through a direct, elementary, calculation, whose details are omitted here.

Subsequently, we considerer $V_{1}, V_{2} \in F, \varrho \in[0,1]$ and $\psi_{1}, \psi_{2} \in C_{+}$arbitrary. Since $\psi \in C_{+}$implies $\psi, 1 / \psi \in \mathcal{C}\left(\bar{Q}_{T}\right)$ and $\nabla \psi \in \mathcal{C}\left(\bar{Q}_{T}, \mathbb{R}^{N}\right)$, we have that $\psi_{1}^{\varrho}, \psi_{2}^{1-\varrho} \in C_{+}$. Thus, the concavity of $G(u)$ yields

$$
\begin{aligned}
& \frac{\left[\mathcal{P}+\varrho V_{1}+(1-\varrho) V_{2}\right]\left(\psi_{1}^{\varrho} \psi_{2}^{1-\varrho}\right)}{\psi_{1}^{\varrho} \psi_{2}^{1-\varrho}} \\
& \quad=\varrho V_{1}+(1-\varrho) V_{2}+\frac{\mathcal{P}\left(\psi_{1}^{\varrho} \psi_{2}^{1-\varrho}\right)}{\psi_{1}^{\varrho} \psi_{2}^{1-\varrho}} \\
& \quad=\varrho V_{1}+(1-\varrho) V_{2}+G\left(\log \left[\psi_{1}^{\varrho} \psi_{2}^{1-\varrho}\right]\right) \\
& \quad=\varrho V_{1}+(1-\varrho) V_{2}+G\left(\varrho \log \psi_{1}+(1-\varrho) \log \psi_{2}\right) \\
& \quad \geq \varrho V_{1}+(1-\varrho) V_{2}+\varrho G\left(\log \psi_{1}\right)+(1-\varrho) G\left(\log \psi_{2}\right) \\
& \quad=\varrho \frac{\left(\mathcal{P}+V_{1}\right) \psi_{1}}{\psi_{1}}+(1-\varrho) \frac{\left(\mathcal{P}+V_{2}\right) \psi_{2}}{\psi_{2}} \\
& \quad \geq \varrho \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{1}\right) \psi_{1}}{\psi_{1}}+(1-\varrho) \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{2}\right) \psi_{2}}{\psi_{2}} .
\end{aligned}
$$

Consequently, since the previous inequality holds for every $\psi_{1}, \psi_{2} \in C_{+}$, we find that

$$
\sup _{\psi \in C_{+}} \inf _{Q_{T}} \frac{\left[\mathcal{P}+\varrho V_{1}+(1-\varrho) V_{2}\right] \psi}{\psi} \geq \varrho \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{1}\right) \psi_{1}}{\psi_{1}}+(1-\varrho) \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{2}\right) \psi_{2}}{\psi_{2}} .
$$

Therefore, by Theorem 3.2,

$$
\begin{aligned}
\sigma\left[\varrho V_{1}+(1-\varrho) V_{2}\right] & \geq \varrho \sup _{\psi_{1} \in C_{+}} \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{1}\right) \psi_{1}}{\psi_{1}}+(1-\varrho) \sup _{\psi_{2} \in C_{+}} \inf _{Q_{T}} \frac{\left(\mathcal{P}+V_{2}\right) \psi_{2}}{\psi_{2}} \\
& =\varrho \sigma\left[V_{1}\right]+(1-\varrho) \sigma\left[V_{2}\right]
\end{aligned}
$$

which ends the proof.

## 5. Analyticity of $\Sigma(\lambda):=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right]$

The main result of this section establishes the analyticity of the principal eigenvalue $\Sigma(\lambda)$ (see (3)) with respect to $\lambda$. It extends Lemma 15.1 of P. Hess [14], under the assumption that $c(x, t)$ and $\beta(x)$ are non-negative, to our more general setting. Unfortunately, the proof of [14, Lem. 15.1] contains a gap, as there was not detailed how to infer the analyticity from M. G. Crandall and P. H. Rabinowitz [8]. For it, one might adapt the proof of [20, Lem. 2.1.1]. The main result of this section reads as follows.

Theorem 5.1. For every $V \in F$, the map

$$
\begin{equation*}
\Sigma(\lambda):=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right], \quad \lambda \in \mathbb{R} \tag{17}
\end{equation*}
$$

is real analytic and concave in the sense that $\Sigma^{\prime \prime}(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Furthermore, either $\Sigma^{\prime \prime} \equiv 0$ in $\mathbb{R}$, or there exists a discrete subset $Z \subset \mathbb{R}$ such that $\Sigma^{\prime \prime}(\lambda)<0$ for all $\lambda \in \mathbb{R} \backslash Z$.

Proof. Set

$$
\mathcal{T}(\lambda):=\mathcal{P}+\lambda V, \quad \lambda \in \mathbb{R}
$$

and regard $\mathcal{T}(\lambda), \lambda \in \mathbb{R}$, as a family of closed operators with domain $E$ and values in $F$. Then, for every $\lambda_{0} \in \mathbb{R}$, we can express

$$
\mathcal{T}(\lambda) u=\mathcal{T} u+\left(\lambda-\lambda_{0}\right) \mathcal{T}^{(1)} u, \quad u \in E
$$

where

$$
\mathcal{T}:=\mathcal{P}+\lambda_{0} V, \quad \mathcal{T}^{(1)}:=V,
$$

and there exists a constant $C>0$ such that

$$
\begin{equation*}
\left\|\mathcal{T}^{(1)} u\right\|_{F}=\|V u\|_{F} \leq C\|u\|_{E}+\|\mathcal{T} u\|_{F} \tag{18}
\end{equation*}
$$

where

$$
\|v\|_{F}:=\|v\|_{\infty}+\sup _{\substack{x, y \in \Omega, x \neq y, y \\ t \in[0, T]}} \frac{|v(x, t)-v(y, t)|}{|x-y|^{\theta}}+\sup _{\substack{t, s \in[0, T], t \neq s, x \in \bar{\Omega}}} \frac{|v(x, t)-v(x, s)|}{|t-s|^{\frac{\theta}{2}}}
$$

for all $v \in F$, and

$$
\begin{aligned}
\|u\|_{E}:=\|u\|_{\mathcal{C}^{2,1}\left(\bar{Q}_{T}\right)} & +\sum_{|\alpha| \leq 2} \sup _{\substack{x, y \in \Omega, x \neq y, t \in[0, T]}} \frac{\left|D_{x}^{\alpha} v(x, t)-D_{x}^{\alpha} v(y, t)\right|}{|x-y|^{\theta}} \\
& +\sum_{|\beta| \leq 1} \sup _{\substack{t, s \in[0, T], t \neq s, x \in \Omega}} \frac{\left|D_{t}^{\beta} v(x, t)-D_{t}^{\beta} v(x, s)\right|}{|t-s|^{\frac{\theta}{2}}}
\end{aligned}
$$

for all $u \in E$. Note that, by definition,

$$
\begin{equation*}
\|u\|_{F} \leq\|u\|_{E} \quad \text { for all } u \in E \tag{19}
\end{equation*}
$$

To prove (18), we can argue as follows. By definition of the norm, for every $u \in E$,

$$
\begin{aligned}
\|V u\|_{F}=\|V u\|_{\infty} & +\sup _{\substack{x, y \in \Omega, x \neq y, t \in[0, T]}} \frac{|V(x, t) u(x, t)-V(y, t) u(y, t)|}{|x-y|^{\theta}} \\
& +\sup _{\substack{t, s \in[0, T], t \neq s, x \in \Omega}} \frac{|V(x, t) u(x, t)-V(x, s) u(x, s)|}{|t-s|^{\frac{\theta}{2}}} .
\end{aligned}
$$

Obviously, the first term can be estimated as follows

$$
\|V u\|_{\infty} \leq\|V\|_{\infty}\|u\|_{\infty} \leq\|V\|_{F}\|u\|_{F}
$$

To estimate the second term, let $x, y \in \Omega$ be with $x \neq y$ and pick $t \in[0, T]$. Then,

$$
\begin{aligned}
& \frac{|V(x, t) u(x, t)-V(y, t) u(y, t)|}{|x-y|^{\theta}} \leq \frac{|V(x, t) u(x, t)-V(x, t) u(y, t)|}{|x-y|^{\theta}} \\
& \quad+\frac{|V(x, t) u(y, t)-V(y, t) u(y, t)|}{|x-y|^{\theta}} \\
& \leq\|V\|_{\infty} \frac{|u(x, t)-u(y, t)|}{|x-y|^{\theta}}+\frac{|V(x, t)-V(y, t)|}{|x-y|^{\theta}}\|u\|_{\infty} \\
& \leq\|V\|_{\infty}\|u\|_{F}+\|V\|_{F}\|u\|_{\infty} \leq 2\|V\|_{F}\|u\|_{F}
\end{aligned}
$$

and hence,

$$
\sup _{\substack{x, y \in \Omega, x \neq y \\ t \in[0, T]}} \frac{|V(x, t) u(x, t)-V(y, t) u(y, t)|}{|x-y|^{\theta}} \leq 2\|V\|_{F}\|u\|_{F} .
$$

Similarly,

$$
\frac{|V(x, t) u(x, t)-V(x, s) u(x, s)|}{|t-s|^{\frac{\theta}{2}}} \leq 2\|V\|_{F}\|u\|_{F} .
$$

Hence, taking sups yields

$$
\sup _{\substack{t, s \in[0, T], t \neq s, x \in \Omega}} \frac{|V(x, t) u(x, t)-V(x, s) u(x, s)|}{|t-s|^{\frac{\theta}{2}}} \leq 2\|V\|_{F}\|u\|_{F} .
$$

Thus, setting $C:=5\|V\|_{F}$ and using (19), we find that, for every $u \in E$,

$$
\left\|\mathcal{T}^{(1)} u\right\|_{F}=\|V u\|_{F} \leq 5\|V\|_{F}\|u\|_{F} \leq C\|u\|_{F}+\|\mathcal{T} u\|_{F} \leq C\|u\|_{E}+\|\mathcal{T} u\|_{F}
$$

and so, (18) holds. Consequently, according to Theorem 2.6 of Section VII.2.2 of T. Kato [17], which extends a previous result of F. Rellich [26] for self-adjoint families, $\mathcal{T}(\lambda)$ is a real holomorphic family of type (A). Thus, by Remark 2.9 of Section VII.2.3 of T. Kato [17], it follows from Theorem 2.9 that $\Sigma(\lambda)$ is real analytic in $\lambda$, as well as the map

$$
\begin{array}{llr}
\mathbb{R} & \rightarrow F \\
\lambda & \mapsto \varphi(\lambda)
\end{array}
$$

where $\varphi(\lambda) \gg 0$ is the unique eigenfunction of $\Sigma(\lambda)$ such that $\int_{Q_{T}} \varphi^{2}(\lambda)=1$.

Now, we will show that

$$
\begin{equation*}
\Sigma^{\prime \prime}(\lambda) \leq 0 \quad \text { for all } \quad \lambda \in \mathbb{R} \tag{20}
\end{equation*}
$$

Although this is a rather standard fact on concave functions from elementary calculus, by the sake of completeness we will give complete details here. According to Theorem 4.1, for every $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ and $\varrho \in(0,1]$,

$$
\begin{aligned}
\Sigma\left(\varrho \lambda_{1}+(1-\varrho) \lambda_{2}\right) & =\sigma\left[\mathcal{P}+\varrho \lambda_{1} V+(1-\varrho) \lambda_{2} V, \mathfrak{B}, Q_{T}\right] \\
& \geq \varrho \sigma\left[\mathcal{P}+\lambda_{1} V, \mathfrak{B}, Q_{T}\right]+(1-\varrho) \sigma\left[\mathcal{P}+\lambda_{2} V, \mathfrak{B}, Q_{T}\right] \\
& =\varrho \Sigma\left(\lambda_{1}\right)+(1-\varrho) \Sigma\left(\lambda_{2}\right) .
\end{aligned}
$$

Thus,

$$
\Sigma\left(\lambda_{2}+\varrho\left(\lambda_{1}-\lambda_{2}\right)\right) \geq \Sigma\left(\lambda_{2}\right)+\varrho\left(\Sigma\left(\lambda_{1}\right)-\Sigma\left(\lambda_{2}\right)\right)
$$

and hence,

$$
\frac{\Sigma\left(\lambda_{2}+\varrho\left(\lambda_{1}-\lambda_{2}\right)\right)-\Sigma\left(\lambda_{2}\right)}{\varrho} \geq \Sigma\left(\lambda_{1}\right)-\Sigma\left(\lambda_{2}\right)
$$

Therefore, for every $\varrho \in(0,1]$ and $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ with $\lambda_{1}>\lambda_{2}$,

$$
\begin{equation*}
\frac{\Sigma\left(\lambda_{2}+\varrho\left(\lambda_{1}-\lambda_{2}\right)\right)-\Sigma\left(\lambda_{2}\right)}{\varrho\left(\lambda_{1}-\lambda_{2}\right)} \geq \frac{\Sigma\left(\lambda_{1}\right)-\Sigma\left(\lambda_{2}\right)}{\lambda_{1}-\lambda_{2}} . \tag{21}
\end{equation*}
$$

Consequently, letting $\varrho \downarrow 0$ yields

$$
\lim _{\varrho \rightarrow 0} \frac{\Sigma\left(\lambda_{2}+\varrho\left(\lambda_{1}-\lambda_{2}\right)\right)-\Sigma\left(\lambda_{2}\right)}{\varrho\left(\lambda_{1}-\lambda_{2}\right)} \geq \frac{\Sigma\left(\lambda_{1}\right)-\Sigma\left(\lambda_{2}\right)}{\lambda_{1}-\lambda_{2}}
$$

for every $\lambda_{1}>\lambda_{2}$. In other words,

$$
\Sigma^{\prime}\left(\lambda_{2}\right) \geq \frac{\Sigma\left(\lambda_{1}\right)-\Sigma\left(\lambda_{2}\right)}{\lambda_{1}-\lambda_{2}} \quad \text { if } \lambda_{1}>\lambda_{2}
$$

So, by the mean value theorem, we find that, for every $\lambda_{1}, \lambda_{2} \in \mathbb{R}$ with $\lambda_{1}>\lambda_{2}$, there exists $\lambda \in\left(\lambda_{2}, \lambda_{1}\right)$ such that

$$
\begin{equation*}
\Sigma^{\prime}\left(\lambda_{2}\right) \geq \Sigma^{\prime}(\lambda) \tag{22}
\end{equation*}
$$

So, $\Sigma^{\prime \prime}(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Indeed, if there would exist $\lambda_{2} \in \mathbb{R}$ such that $\Sigma^{\prime \prime}\left(\lambda_{2}\right)>0$, then $\Sigma^{\prime}$ should be increasing in a neighborhood of $\lambda_{2}$, which contradicts (22). Finally, since $\Sigma$ is real analytic, also $\Sigma^{\prime \prime}$ is real analytic and therefore, either $\Sigma^{\prime \prime}=0$, or the set of zeroes of $\Sigma^{\prime \prime}$ must be discrete, possibly empty. The proof is complete.

Naturally, combining Proposition 2.1 with Theorem 5.1 the next result holds.

Proposition 5.2. For any given $V \in F$, the map

$$
\Sigma(\lambda):=\Sigma_{V}(\lambda)=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right], \quad \lambda \in \mathbb{R}
$$

satisfies the following properties:
(a) $V \ngtr 0$ implies $\Sigma^{\prime}(\lambda)>0$ for all $\lambda \in \mathbb{R}$.
(b) $V \lesseqgtr 0$ implies $\Sigma^{\prime}(\lambda)<0$ for all $\lambda \in \mathbb{R}$.

Proof. Suppose that $V \ngtr 0$ on $Q_{T}$. Then, by Proposition 2.1 and Theorem 5.1, we find that $\Sigma^{\prime}(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. Moreover, by analyticity, either $\Sigma^{\prime} \equiv 0$, or $\Sigma^{\prime}$ vanishes, at most, on a discrete set. Since $V \ngtr 0, \Sigma(\lambda)$ cannot be constant. Thus, it satisfies the second option. Let us suppose that $\Sigma^{\prime}\left(\lambda_{0}\right)=0$ for some $\lambda_{0} \in \mathbb{R}$. Then, by Theorem 5.1,

$$
0 \leq \Sigma^{\prime}(\lambda)=\Sigma^{\prime}(\lambda)-\Sigma^{\prime}\left(\lambda_{0}\right)=\int_{\lambda_{0}}^{\lambda} \Sigma^{\prime \prime} \leq 0 \quad \text { for all } \lambda \geq \lambda_{0}
$$

So, $\Sigma^{\prime}=0$ in $\left[\lambda_{0}, \infty\right)$ which is impossible. Therefore, $\Sigma^{\prime}(\lambda)>0$ for all $\lambda \in \mathbb{R}$, which ends the proof of Part (a).

Now, suppose that $V \lesseqgtr 0$ in $Q_{T}$. Then,

$$
\begin{equation*}
\Sigma_{V}(\lambda)=\Sigma_{-V}(-\lambda) \text { for all } \lambda \in \mathbb{R} \tag{23}
\end{equation*}
$$

and hence, since $-V \ngtr 0$, Part (a) yields

$$
\Sigma_{V}^{\prime}(\lambda)=-\Sigma_{-V}^{\prime}(-\lambda)<0
$$

for all $\lambda \in \mathbb{R}$, which ends the proof of Part (b).
6. Global behavior of $\Sigma(\lambda):=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right]$

The next result provides us with a simple periodic-parabolic counterpart of [22, Th. 9.1]. Note that both results differ substantially.

Theorem 6.1. Given $V \in F$, consider the map $\Sigma(\lambda)$ defined in (17). Then:
(a) If there exists $x_{+} \in \Omega$ such that $V\left(x_{+}, t\right)>0$ for all $t \in[0, T]$, or, alternatively,

$$
\begin{equation*}
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t>0 \tag{24}
\end{equation*}
$$

then,

$$
\begin{equation*}
\lim _{\lambda \downarrow-\infty} \Sigma(\lambda)=-\infty \tag{25}
\end{equation*}
$$

(b) If there exists $x_{-} \in \Omega$ such that $V\left(x_{-}, t\right)<0$ for all $t \in[0, T]$, or, alternatively,

$$
\begin{equation*}
\int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t<0 \tag{26}
\end{equation*}
$$

then,

$$
\begin{equation*}
\lim _{\lambda \uparrow \infty} \Sigma(\lambda)=-\infty \tag{27}
\end{equation*}
$$

(c) If there exist $x_{+}, x_{-} \in \Omega$ such that $V\left(x_{+}, t\right)>0$ and $V\left(x_{-}, t\right)<0$ for all $t \in[0, T]$, then (25) and (27) are satisfied and hence, for some $\lambda_{0} \in \mathbb{R}$,

$$
\begin{equation*}
\Sigma\left(\lambda_{0}\right)=\max _{\lambda \in \mathbb{R}} \Sigma(\lambda) \tag{28}
\end{equation*}
$$

Moreover, $\Sigma^{\prime}\left(\lambda_{0}\right)=0, \Sigma^{\prime}(\lambda)>0$ if $\lambda<\lambda_{0}$, and $\Sigma^{\prime}(\lambda)<0$ if $\lambda>\lambda_{0}$. So, $\lambda_{0}$ is unique.

Proof. Suppose that there exists $x_{+} \in \Omega$ such that $V\left(x_{+}, t\right)>0$ for all $t \in$ $[0, T]$. Then, by continuity, there exists $R>0$ such that

$$
B_{+}:=B_{R}\left(x_{+}\right) \Subset \Omega \quad \text { and } \quad \min _{\bar{B}_{+} \times[0, T]} V=\omega>0
$$

Thus, according to Proposition 2.6,

$$
\Sigma(\lambda)=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right]<\sigma\left[\mathcal{P}+\lambda V, \mathfrak{D}, B_{+} \times(0, T)\right]
$$

and hence, by Proposition 2.1, we find that

$$
\Sigma(\lambda)<\sigma\left[\mathcal{P}, \mathfrak{D}, B_{+} \times(0, T)\right]+\lambda \omega \quad \text { for all } \lambda<0
$$

Letting $\lambda \downarrow-\infty$ in this inequality yields (25).
Now, suppose (24). Then, thanks to Propositions 2.1 and 2.8 , it becomes apparent that, for every $\lambda<0$,

$$
\begin{aligned}
\Sigma(\lambda) & =\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right] \leq \sigma\left[\mathcal{P}+\lambda \min _{x \in \bar{\Omega}} V(x, t), \mathfrak{B}, Q_{T}\right] \\
& =\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]+\frac{\lambda}{T} \int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t .
\end{aligned}
$$

Therefore, by (24), letting $\lambda \downarrow-\infty$ in this inequality also provides us with (25). This completes the proof of Part (a). Part (b) follows easily from (23), by applying Part (a) to the potential $-V$.

Finally, suppose that there exist $x_{+}, x_{-} \in \Omega$ such that

$$
V\left(x_{+}, t\right)>0 \quad \text { and } \quad V\left(x_{+}, t\right)<0 \quad \text { for all } t \in[0, T] .
$$

Then, by Parts (a) and (b), (25) and (27) hold. Thus, there exists $\lambda_{0} \in \mathbb{R}$ satisfying (28). Obviously, $\Sigma^{\prime}\left(\lambda_{0}\right)=0$. Suppose that $\Sigma^{\prime}\left(\lambda_{-}\right) \leq 0$ for some $\lambda_{-}<\lambda_{0}$. Then,

$$
0 \leq-\Sigma^{\prime}\left(\lambda_{-}\right)=\Sigma^{\prime}\left(\lambda_{0}\right)-\Sigma^{\prime}\left(\lambda_{-}\right)=\int_{\lambda_{-}}^{\lambda_{0}} \Sigma^{\prime \prime} \leq 0
$$

and hence,

$$
\Sigma^{\prime}\left(\lambda_{-}\right)=-\int_{\lambda_{-}}^{\lambda_{0}} \Sigma^{\prime \prime}=0
$$

So, $\Sigma^{\prime \prime}=0$ on $\left[\lambda_{-}, \lambda_{0}\right]$, which implies $\Sigma^{\prime \prime}=0$ in $\mathbb{R}$, by analyticity. Consequently, there are two constants, $a, b \in \mathbb{R}$, such that,

$$
\Sigma(\lambda)=a \lambda+b \quad \text { for all } \lambda \in \mathbb{R}
$$

By (25) and (27), this is impossible. Therefore, $\Sigma^{\prime}(\lambda)>0$ for all $\lambda<\lambda_{0}$ Similarly, $\Sigma^{\prime}(\lambda)<0$ for all $\lambda>\lambda_{0}$. This ends the proof.

As illustrated by Figure 1, the two sufficient conditions for (25) established by Theorem 6.1(a) are supplementary, even when $V \ngtr 0$.


Figure 1: Two admissible nodal configurations of $V$.
In Figure 1, the dark regions represent the set of $(x, t) \in Q_{T}$ where $V(x, t)>$ 0 , while the white regions are the portions of $Q_{T}$ where $V(x, t)=0$. In Case (A), $V(x, t)>0$ for all $t \in[0, T]$ as soon as $x \in \Omega$ is chosen appropriately, but

$$
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t=0
$$

Contrarily, in Case (B), there cannot exist a point $x \in \Omega$ for which $V(x, t)>0$ for all $t \in[0, T]$, though

$$
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t \geq \int_{t_{1}}^{t_{2}} \min _{x \in \bar{\Omega}} V(x, t) d t>0
$$

provided $V(x, t)>0$ for all $(x, t) \in \bar{\Omega} \times\left(t_{1}, t_{2}\right)$. Similarly, the two sufficient conditions for (27) established by Theorem 6.1(b) are supplementary, even in case $V \lesseqgtr 0$.

Note that, since

$$
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t \leq \int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t
$$

conditions (24) and (26) cannot hold simultaneously. Moreover, if there exists $x_{+} \in \Omega$ for which $V\left(x_{+}, t\right)>0$ for all $t \in[0, T]$, then

$$
\int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t \geq \int_{0}^{T} V\left(x_{+}, t\right) d t>0
$$

and hence, (26) fails. Similarly, if there exists $x_{-} \in \Omega$ such that $V\left(x_{-}, t\right)<0$ for all $t \in[0, T]$, then

$$
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t \leq \int_{0}^{T} V\left(x_{-}, t\right) d t<0
$$

and so, (24) fails.
Note that, under the assumptions of Theorem 6.1(a),

$$
\begin{equation*}
\int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t>0 \tag{29}
\end{equation*}
$$

Similarly, any of the assumptions of Theorem 6.1(b) implies

$$
\begin{equation*}
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t<0 \tag{30}
\end{equation*}
$$

Therefore, the next result provides us with a substantial extension of Theorem 6.1. The first assertions of Parts (a) and (b) generalize [14, Lem. 15.4], going back to A. Beltramo and P. Hess [4], where it was assumed that $c \geq 0$ and $\beta \geq 0$, and Proposition 3.2 of D. Daners [9], where no assumption on the sign of $c(x, t)$ was imposed, but only for Dirichlet boundary conditions.

Theorem 6.2. Given $V \in F$, consider the map $\Sigma(\lambda)$ defined in (17). Then:
(a) Condition (29) implies $\lim _{\lambda \downarrow-\infty} \Sigma(\lambda)=-\infty$, and

$$
\begin{equation*}
\int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t<0 \tag{31}
\end{equation*}
$$

implies $\lim _{\lambda \downarrow-\infty} \Sigma(\lambda)=\infty$.
(b) Condition (30) implies $\lim _{\lambda \uparrow \infty} \Sigma(\lambda)=-\infty$, and

$$
\begin{equation*}
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t>0 \tag{32}
\end{equation*}
$$

implies $\lim _{\lambda \uparrow \infty} \Sigma(\lambda)=\infty$.
(c) If

$$
\int_{0}^{T} \min _{x \in \bar{\Omega}} V(x, t) d t<0<\int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t
$$

then $\Sigma\left(\lambda_{0}\right)=\max _{\lambda \in \mathbb{R}} \Sigma(\lambda)$ holds for some $\lambda_{0} \in \mathbb{R}$. Moreover, $\Sigma^{\prime}\left(\lambda_{0}\right)=$ $0, \Sigma^{\prime}(\lambda)>0$ if $\lambda<\lambda_{0}$, and $\Sigma^{\prime}(\lambda)<0$ if $\lambda>\lambda_{0}$. Thus, $\lambda_{0}$ is unique.

Proof. Since Part (b) follows easily from Part (a) and, arguing as in Theorem 6.1, Part (c) is an easy consequence of Parts (a) and (b), it suffices to prove Part (a). Suppose (29). Then, arguing as in A. Beltramo and P. Hess [4], there exists a $T$-periodic function $\kappa \in \mathcal{C}^{2}(\mathbb{R} ; \Omega)$ such that

$$
\int_{0}^{T} V(\kappa(t), t) d t>0
$$

Essentially, $\kappa(t)$ follows the points where $V(\cdot, t)$ takes the maximum, even if they lie on the boundary! Let $\psi: \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}^{N} \times \mathbb{R}$ be the $\mathcal{C}^{2}$-diffeomorphism defined by

$$
(y, t)=\psi(x, t):=(x-\kappa(t), t)
$$

Then, the original boundary value problem

$$
\begin{cases}\mathcal{P} \varphi+\lambda V \varphi=\Sigma(\lambda) \varphi & \text { in } \Omega \times \mathbb{R}  \tag{33}\\ \mathfrak{B} \varphi=0 & \text { on } \partial \Omega \times \mathbb{R}\end{cases}
$$

where $\varphi \in E, \varphi \gg 0$, is transformed into

$$
\begin{cases}\mathcal{P}_{\psi} \varphi_{\psi}+\lambda V_{\psi} \varphi_{\psi}=\Sigma(\lambda) \varphi_{\psi} & \text { in } \psi(\Omega \times \mathbb{R}),  \tag{34}\\ \mathfrak{B}_{\psi} \varphi_{\psi}=0 & \text { on the lateral boundary of } \psi(\Omega \times \mathbb{R}),\end{cases}
$$

where $\mathcal{P}_{\psi}$ is a certain periodic-parabolic operator of the same type as $\mathcal{P}$ (see the proof of $\left[14\right.$, Lem. 15.4]), $\mathfrak{B}_{\psi}$ is a boundary operator of the same type as $\mathfrak{B}$ whose explicit expression is not important here, and

$$
V_{\psi}=\left.V \circ \psi^{-1}\right|_{\psi(\bar{\Omega} \times \mathbb{R})}, \quad \varphi_{\psi}=\left.\varphi \circ \psi^{-1}\right|_{\psi(\bar{\Omega} \times \mathbb{R})}
$$

By construction,

$$
p:=\int_{0}^{T} V_{\psi}(0, t) d t=\int_{0}^{T} V(\kappa(t), t) d t>0
$$

Moreover, since $V_{\psi}$ is uniformly continuous, there exists $\varepsilon>0$ such that $\bar{B}_{\varepsilon} \times$ $\mathbb{R} \subset \psi(\Omega \times \mathbb{R})$ and

$$
V_{\psi}(y, t) \geq c(t)=V_{\psi}(0, t)-\frac{p}{2 T} \quad \text { for all }(y, t) \in \bar{B}_{\varepsilon} \times \mathbb{R}
$$

where $B_{\varepsilon}$ stands for the ball of radius $\varepsilon$ centered at 0 .
According to (34), the restriction $h:=\left.\varphi_{\psi}\right|_{\bar{B}_{\varepsilon} \times \mathbb{R}}$ provides us with a positive strict supersolution of

$$
\left(\mathcal{P}_{\psi}+\lambda V_{\psi}-\Sigma(\lambda), \mathfrak{D}, B_{\varepsilon} \times(0, T)\right)
$$

Thus, thanks to Theorem 1.1,

$$
\sigma\left[\mathcal{P}_{\psi}+\lambda V_{\psi}-\Sigma(\lambda), \mathfrak{D}, B_{\varepsilon} \times(0, T)\right]>0
$$

Equivalently,

$$
\Sigma(\lambda)<\sigma\left[\mathcal{P}_{\psi}+\lambda V_{\psi}, \mathfrak{D}, B_{\varepsilon} \times(0, T)\right]
$$

Since $V_{\psi} \geq c$, we have that $\lambda V_{\psi} \leq \lambda c$ for all $\lambda<0$. Hence, by Propositions 2.1 and 2.8 , it becomes apparent that

$$
\Sigma(\lambda)<\sigma\left[\mathcal{P}_{\psi}+\lambda c(t), \mathfrak{D}, B_{\varepsilon} \times(0, T)\right]=\sigma\left[\mathcal{P}_{\psi}, \mathfrak{D}, B_{\varepsilon} \times(0, T)\right]+\frac{\lambda}{T} \int_{0}^{T} c(t) d t
$$

On the other hand, by the definition of $c(t)$ and $p$, we have that

$$
\int_{0}^{T} c(t) d t=\int_{0}^{T} V_{\psi}(0, t) d t-\frac{p}{2}=p-\frac{p}{2}=\frac{p}{2}
$$

Therefore,

$$
\Sigma(\lambda)<\sigma\left[\mathcal{P}_{\psi}, \mathfrak{D}, B_{\varepsilon} \times(0, T)\right]+\frac{p \lambda}{2 T} \quad \text { for all } \lambda<0
$$

Since $p>0$, letting $\lambda \rightarrow-\infty$ shows that $\Sigma(\lambda) \rightarrow-\infty$. This ends the proof of the first claim.

Finally, suppose (31). Then, for every $\lambda<0$, we have that

$$
\lambda V(x, t) \geq \lambda \max _{x \in \bar{\Omega}} V(x, t)
$$

and hence, by Propositions 2.1 and 2.8,

$$
\Sigma(\lambda) \geq \sigma[\mathcal{P}, \mathfrak{B}, \Omega \times(0, T)]+\frac{\lambda}{T} \int_{0}^{T} \max _{x \in \bar{\Omega}} V(x, t) d t
$$

Thanks to (31), letting $\lambda \downarrow-\infty$ in the previous estimate yields $\Sigma(\lambda) \rightarrow \infty$ and concludes the proof.

Although the construction in the first part of the proof follows mutatis mutandis the proof of Lemma 15.4 of P. Hess [14], the second half seems new. Anyway, thanks to Theorem 1.1, it is considerably shorter than the extremely intricate comparison argument of the proof of [14, Lem. 15.4].

## 7. Principal eigenvalues of the weighted boundary value problem

This section studies the weighted boundary value problem

$$
\begin{cases}\mathcal{P} \varphi=\lambda W(x, t) \varphi & \text { in } Q_{T},  \tag{35}\\ \mathfrak{B} \varphi=0 & \text { on } \partial \Omega \times[0, T],\end{cases}
$$

where $W \in F$ and $\lambda \in \mathbb{R}$. Denoting $V:=-W$ and setting

$$
\Sigma(\lambda):=\sigma\left[\mathcal{P}+\lambda V, \mathfrak{B}, Q_{T}\right]=\sigma\left[\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right], \quad \lambda \in \mathbb{R},
$$

it is apparent that $\lambda^{*} \in \mathbb{R}$ is a principal eigenvalue of (35) if $\Sigma\left(\lambda^{*}\right)=0$.
The next theorem characterizes the existence of the principal eigenvalue of (35) when $W \ngtr 0$, i.e., $V=-W \lesseqgtr 0$.

TheOrem 7.1. Suppose $W \ngtr 0$, which implies $\int_{0}^{T} \max _{x \in \bar{\Omega}} W(x, t) d t>0$. Then, (35) possesses a principal eigenvalue if and only if

$$
\begin{equation*}
\Sigma(-\infty):=\lim _{\lambda \downarrow-\infty} \Sigma(\lambda)>0 . \tag{36}
\end{equation*}
$$

Moreover, it is unique if it exists and if we denote it by $\lambda^{*}$, then, $\lambda^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8], i.e.,

$$
\begin{equation*}
W \varphi^{*} \notin R\left[\mathcal{P}-\lambda^{*} W\right] \tag{37}
\end{equation*}
$$

for all principal eigenfunction $\varphi^{*} \gg 0$ of (35) associated to $\lambda^{*}$.
Proof. Since $V=-W \lesseqgtr 0$, according to Proposition 5.2, $\Sigma^{\prime}(\lambda)<0$ for all $\lambda \in \mathbb{R}$. Thus, the limit (36) is well defined. It might be finite, or infinity. Indeed, if

$$
\begin{equation*}
\min _{\bar{Q}_{T}} W>0, \tag{38}
\end{equation*}
$$

then, for every $\lambda<0$, we have that

$$
\Sigma(\lambda)=\sigma\left[\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right] \geq \sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]-\lambda \min _{\bar{Q}_{T}} W
$$

and hence, letting $\lambda \downarrow-\infty$ yields $\Sigma(-\infty)=\infty$. Now, instead of (38), assume that there exists an open set $\Omega_{0} \Subset \Omega$ such that

$$
W=0 \quad \text { on } \quad \Omega_{0} \times[0, T] .
$$

Then,

$$
\Sigma(\lambda)=\sigma\left[\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right] \leq \sigma\left[\mathcal{P}, \mathfrak{D}, \Omega_{0} \times(0, T)\right]
$$

for all $\lambda \in \mathbb{R}$ and hence,

$$
\Sigma(-\infty) \leq \sigma\left[\mathcal{P}, \mathfrak{D}, \Omega_{0} \times(0, T)\right]
$$

On the other hand, by Theorem 6.1(b),

$$
\begin{equation*}
\lim _{\lambda \uparrow \infty} \Sigma(\lambda)=-\infty \tag{39}
\end{equation*}
$$

Suppose $\Sigma(-\infty)>0$. Then, $\Sigma\left(\lambda_{1}\right)>0$ for some $\lambda_{1} \in \mathbb{R}$ and hence, by (39), there exists a unique $\lambda^{*} \in \mathbb{R}$ such that $\Sigma\left(\lambda^{*}\right)=0$. Conversely, if there exists $\lambda^{*} \in \mathbb{R}$ such that $\Sigma\left(\lambda^{*}\right)=0$, then, $\Sigma(\lambda)>0$ for all $\lambda<\lambda^{*}$ and therefore, $\Sigma(-\infty)>0$.

It remains to prove (37). Let $\varphi(\lambda)$ denote the principal eigenfunction associated to $\Sigma(\lambda)$ normalized so that $\int_{Q_{T}} \varphi^{2}(\lambda)=1$. By Theorem 5.1, $\Sigma(\lambda)$ and $\varphi(\lambda)$ are real analytic in $\lambda$. Thus, differentiating with respect to $\lambda$ the identity

$$
(\mathcal{P}-\lambda W) \varphi(\lambda)=\Sigma(\lambda) \varphi(\lambda), \quad \lambda \in \mathbb{R}
$$

we find that

$$
(\mathcal{P}-\lambda W) \varphi^{\prime}(\lambda)-W \varphi(\lambda)=\Sigma^{\prime}(\lambda) \varphi(\lambda)+\Sigma(\lambda) \varphi^{\prime}(\lambda), \quad \lambda \in \mathbb{R}
$$

Thus, since $\Sigma\left(\lambda^{*}\right)=0$, particularizing at $\lambda=\lambda^{*}$ yields

$$
\begin{equation*}
\left(\mathcal{P}-\lambda^{*} W\right) \varphi^{\prime}\left(\lambda^{*}\right)=W \varphi\left(\lambda^{*}\right)+\Sigma^{\prime}\left(\lambda^{*}\right) \varphi\left(\lambda^{*}\right) \tag{40}
\end{equation*}
$$

Set $\varphi^{*}:=\varphi\left(\lambda^{*}\right)$. To prove (37) we can argue by contradiction. Suppose that

$$
W \varphi^{*} \in R\left[\mathcal{P}-\lambda^{*} W\right] .
$$

Then, (40) implies

$$
\Sigma^{\prime}\left(\lambda^{*}\right) \varphi^{*} \in R\left[\mathcal{P}-\lambda^{*} W\right]
$$

and, since $\Sigma^{\prime}\left(\lambda^{*}\right)<0$, it becomes apparent that

$$
N\left[\mathcal{P}-\lambda^{*} W\right]=\operatorname{span}\left[\varphi^{*}\right] \quad \text { and } \quad \varphi^{*} \in R\left[\mathcal{P}-\lambda^{*} W\right]
$$

As, for every $\omega>0$, we have that

$$
\left(\mathcal{P}-\lambda^{*} W+\omega\right) \varphi^{*}=\omega \varphi^{*}
$$

and, owing to Theorem $1.1,\left(\mathcal{P}-\lambda^{*} W+\omega\right)^{-1}$ is strongly order preserving, because

$$
\sigma\left[\mathcal{P}-\lambda^{*} W+\omega, \mathfrak{B}, Q_{T}\right]=\omega>0
$$

by the Krein-Rutman theorem (see [22, Th. 6.3]), it becomes apparent that

$$
\frac{1}{\omega}=\operatorname{spect}\left(\mathcal{P}-\lambda^{*} W+\omega\right)^{-1}
$$

On the other hand, since $\varphi^{*} \in R\left[\mathcal{P}-\lambda^{*} W\right]$, there exists $u \in E$ such that

$$
\left(\mathcal{P}-\lambda^{*} W+\omega\right) u=\omega u+\varphi^{*} .
$$

Equivalently,

$$
\frac{1}{\omega} u-\left(\mathcal{P}-\lambda^{*} W+\omega\right)^{-1} u=\frac{1}{\omega} \varphi^{*}>0
$$

which contradicts Theorem $6.3(\mathrm{f})(\mathrm{b})$ of $[22]$ and ends the proof.
Remark 7.2. Based on a very recent technical device of D. Daners and C. Thornett [12], one can characterize the non-negative potentials $W$ for which $\Sigma(-\infty)<\infty$. This analysis will appear in [11].

Remark 7.3. Under the assumptions of Theorem 7.1, when $\Sigma(-\infty)>0$ we have that

$$
\begin{cases}\lambda^{*}>0 & \text { if } \Sigma(0)>0 \\ \lambda^{*}=0 & \text { if } \Sigma(0)=0 \\ \lambda^{*}<0 & \text { if } \Sigma(0)<0\end{cases}
$$

as it has been illustrated in Figure 2.


Figure 2: The graph of $\Sigma(\lambda)$ when $W \geqslant 0$ and $\Sigma(-\infty)>0$.
Essentially, the proof of (37) is based on the fact that $\Sigma^{\prime}\left(\lambda^{*}\right) \neq 0$. Thus, the last assertion of Theorem 7.1 holds true as soon as

$$
\Sigma\left(\lambda^{*}\right)=0 \quad \text { and } \quad \Sigma^{\prime}\left(\lambda^{*}\right) \neq 0
$$

Consequently, the proof of Theorem 7.1 can be easily adapted to get the next result, whose proof is omitted here.

THEOREM 7.4. Suppose $W \leq 0$, which implies $\int_{0}^{T} \min _{x \in \bar{\Omega}} W(x, t) d t<0$. Then, (35) possesses a principal eigenvalue if and only if

$$
\Sigma(\infty):=\lim _{\lambda \uparrow \infty} \Sigma(\lambda)>0
$$

Moreover, it is unique if it exists and if we denote it by $\lambda^{*}$, then, $\lambda^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].

According to Proposition 5.2, when $W \lesseqgtr 0$ we have that $\Sigma^{\prime}(\lambda)>0$ for all $\lambda \in \mathbb{R}$. Figure 3 shows the graph of $\Sigma(\lambda)$ in this case. Since $\Sigma^{\prime}(\lambda)>0$ for all $\lambda \in \mathbb{R}$, we have that $\lambda^{*}<0$ if $\Sigma(0)>0, \lambda^{*}=0$ if $\Sigma(0)=0$, and $\lambda^{*}>0$ if $\Sigma(0)<0$.


Figure 3: The graph of $\Sigma(\lambda)$ when $W \lesseqgtr 0$ and $\Sigma(\infty)>0$.
According to Theorems 7.1 and 7.4 , if $W \neq 0$ has constat sign, then, the problem (35) has a principal eigenvalue, if and only if,

$$
\sigma\left[\mathcal{P}-\lambda W, \mathfrak{B}, Q_{T}\right]>0 \quad \text { for some } \quad \lambda \in \mathbb{R}
$$

In the general case when $W$ changes sign, as a byproduct of Theorem 6.1(c), the next result holds.

Theorem 7.5. Suppose

$$
\begin{equation*}
\int_{0}^{T} \min _{x \in \bar{\Omega}} W(x, t) d t<0<\int_{0}^{T} \max _{x \in \bar{\Omega}} W(x, t) d t \tag{41}
\end{equation*}
$$

Then, by Theorem 6.1(c),

$$
\lim _{\lambda \downarrow-\infty} \Sigma(\lambda)=\lim _{\lambda \uparrow \infty} \Sigma(\lambda)=-\infty
$$

Moreover, there exists a unique $\lambda_{0} \in \mathbb{R}$ such that

$$
\Sigma\left(\lambda_{0}\right)=\max _{\lambda \in \mathbb{R}} \Sigma(\lambda)
$$

Furthermore, $\Sigma^{\prime}\left(\lambda_{0}\right)=0, \Sigma^{\prime}(\lambda)>0$ if $\lambda<\lambda_{0}$, and $\Sigma^{\prime}(\lambda)<0$ if $\lambda>\lambda_{0}$. Therefore, (35) possesses a principal eigenvalue, if and only if, $\Sigma\left(\lambda_{0}\right) \geq 0$. Moreover, $\lambda_{0}$ provides us with unique principal eigenvalue of (35) if $\Sigma\left(\lambda_{0}\right)=$ 0 , while (35) possesses two principal eigenvalues, $\lambda_{-}^{*}<\lambda_{+}^{*}$, if $\Sigma\left(\lambda_{0}\right)>0$. Actually, in this case,

$$
\lambda_{-}^{*}<\lambda_{0}<\lambda_{+}^{*}
$$

and $\lambda_{-}^{*}$ and $\lambda_{+}^{*}$ are simple eigenvalues of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].

Since $\Sigma^{\prime}\left(\lambda_{0}\right)=0$, zero cannot be a simple eigenvalue of $\left(\mathcal{P}-\lambda_{0} W, W\right)$ if $\Sigma\left(\lambda_{0}\right)=0$. When $\Sigma\left(\lambda_{0}\right)>0$, then:

$$
\begin{array}{lll}
\lambda_{-}^{*}<0<\lambda_{+}^{*} & \text { if } & \Sigma(0)>0 \\
0=\lambda_{-}^{*}<\lambda_{+}^{*} & \text { if } & \Sigma(0)=0 \quad \text { and } \quad \Sigma^{\prime}(0)>0 \\
\lambda_{-}^{*}<\lambda_{+}^{*}=0 & \text { if } & \Sigma(0)=0 \quad \text { and } \\
0<\Sigma_{-}^{*}<\lambda_{+}^{*}(0)<0 \\
\text { if } & \Sigma(0)<0 \quad \text { and } & \Sigma^{\prime}(0)>0 \\
\lambda_{-}^{*}<\lambda_{+}^{*}<0 & \text { if } & \Sigma(0)<0 \quad \text { and }
\end{array} \Sigma^{\prime}(0)<0 .
$$

In particular, (35) admits two eigenvalues with contrary sign if, and only if, $\sigma\left[\mathcal{P}, \mathfrak{B}, Q_{T}\right]>0$. Figure 4 shows the graph of $\Sigma(\lambda)$ when $\Sigma(0) \neq 0$.

$\Sigma(0)<0$

$\Sigma(0)>0$

$\Sigma(0)<0$

Figure 4: The graph of $\Sigma(\lambda)$ when $W$ changes sign and $\Sigma\left(\lambda_{0}\right)>0$.
Naturally, from this abstract theory the following generalized version of a classical result of K. J. Brown and S. S. Lin [6] holds.

Corollary 7.6. Suppose $\Sigma(0)=0$ and $W \in F$ satisfies (41). Then:
(a) The problem (35) possesses a negative principal eigenvalue, $\lambda_{-}^{*}<0$, if, and only if, $\Sigma^{\prime}(0)<0$. Moreover, in such case, $\lambda_{-}^{*}$ is the unique non-zero eigenvalue of (35) and $\Sigma^{\prime}\left(\lambda_{-}^{*}\right)>0$. Therefore, $\lambda_{-}^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].
(b) The problem (35) possesses a positive principal eigenvalue, $\lambda_{+}^{*}>0$, if, and only if, $\Sigma^{\prime}(0)>0$. Moreover, in such case, $\lambda_{+}^{*}$ is the unique non-zero eigenvalue of (35) and $\Sigma^{\prime}\left(\lambda_{+}^{*}\right)<0$. Therefore, $\lambda_{+}^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].
When, in addition, $\Sigma^{\prime}(0)=0$, then $\lambda=0$ is the unique principal eigenvalue of (35), as illustrated in the third picture of Figure 5.

Figure 5 sketches each of the possible cases considered by Corollary 7.6.


Figure 5: The graph of $\Sigma(\lambda)$ when $W$ changes sign and $\Sigma(0)=0$.
In the classical elliptic context of K. J. Brown and S. S. Lin [6] and the periodic-parabolic counterpart of P . Hess [14], it is imposed that $\Gamma_{0}=\emptyset, \beta=0$ on $\Gamma_{1}=\partial \Omega$, and $c=0$ in $Q_{T}$. In other words, $\mathfrak{B}$ is the Neumann operator on $\partial \Omega$ and $c=0$. Thus, since $\mathcal{P} 1=0$ in $Q_{T}$ and $\mathcal{B} 1=0$ on $\partial \Omega$, it is apparent that $\lambda=0$ provides us with an eigenvalue of the problem (35), and that $\varphi=1$ is a principal eigenfunction associated to $\lambda=0$. Thus, $\Sigma(0)=0$ and

$$
(\mathcal{P}-\lambda W) \varphi(\lambda)=\Sigma(\lambda) \varphi(\lambda), \quad \lambda \in \mathbb{R}
$$

where $\varphi(0)=1$ and $\varphi(\lambda)$ is real analytic. Hence, differentiating with respect to $\lambda$ and particularizing at $\lambda=0$, it becomes apparent that

$$
\mathcal{P} \varphi^{\prime}(0)-W=\Sigma^{\prime}(0)
$$

Therefore, integrating in $Q_{T}$ yields

$$
\begin{equation*}
\Sigma^{\prime}(0)=-\frac{1}{\left|Q_{T}\right|} \int_{Q_{T}} W(x, t) d x d t \tag{42}
\end{equation*}
$$

because

$$
\begin{equation*}
\int_{Q_{T}} \mathcal{P} \varphi^{\prime}(0)=\int_{Q_{T}} \partial_{t} \varphi^{\prime}(0)+\int_{Q_{T}} \mathfrak{L} \varphi^{\prime}(0)=0 . \tag{43}
\end{equation*}
$$

Indeed, since $\varphi^{\prime}(0) \in F$, for every $x \in \bar{\Omega}$, we have that

$$
\int_{0}^{T} \partial_{t} \varphi^{\prime}(0)=\varphi^{\prime}(0)(x, T)-\varphi^{\prime}(0)(x, 0)=0 .
$$

Moreover, for every $t \in[0, T]$, integrating by parts in $\Omega$ it becomes apparent that

$$
\int_{\Omega} \mathfrak{L} \psi^{\prime}(0) d x=\int_{\Omega} \varphi^{\prime}(0) \mathfrak{L}^{*} 1 d x=0 .
$$

Therefore, (43), and hence (42), holds. Consequently, Corollary 7.6 can be reformulated in terms of the sign of the total mass $\int_{Q_{T}} W$, providing us with the following periodic-parabolic counterpart of the main theorem of K. J. Brown and S. S. Lin [6].

Corollary 7.7. Suppose $\Gamma_{0}=\emptyset, \beta=0$ on $\Gamma_{1}=\partial \Omega, c=0$ in $Q_{T}$, and $W \in F$ satisfies (41). Then:
(a) The problem (35) possesses a negative principal eigenvalue, $\lambda_{-}^{*}<0$, if, and only if, $\int_{Q_{T}} W>0$. Moreover, in such case, $\lambda_{-}^{*}$ is the unique nonzero eigenvalue of (35) and $\Sigma^{\prime}\left(\lambda_{-}^{*}\right)>0$. Therefore, $\lambda_{-}^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].
(b) The problem (35) possesses a positive principal eigenvalue, $\lambda_{+}^{*}>0$, if, and only if, $\int_{Q_{T}} W<0$. Moreover, in such case, $\lambda_{+}^{*}$ is the unique nonzero eigenvalue of (35) and $\Sigma^{\prime}\left(\lambda_{+}^{*}\right)<0$. Therefore, $\lambda_{+}^{*}$ is a simple eigenvalue of $(\mathcal{P}-\lambda W, W)$ as discussed by Crandall and Rabinowitz [8].
If $\int_{Q_{T}} W=0$, then $\lambda=0$ is the unique principal eigenvalue of (35).

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# Dirichlet problems without asymptotic conditions on the nonlinear term 

Gabriele Bonanno<br>Dedicated with immense esteem to Jean Mawhin on occasion of his 75th birthday


#### Abstract

This paper is devoted, with my great esteem, to Jean Mawhin. Jean Mawhin, who is for me a great teacher and a very good friend, is a fundamental reference for the research in nonlinear differential problems dealt both with topological and variational methods. Here, owing to this occasion in honor of Jean Mawhin, Dirichlet problems depending on a parameter are investigated, ensuring the existence of non-zero solutions without requiring asymptotic conditions neither at zero nor at infinity on the nonlinear term which, in addition, is not forced by subcritical or critical growth. The approach is based on a combination of variational and topological tools that in turn are developed by starting from a fundamental estimate.


Keywords: Nonlinear eigenvalue problems; critical point; sub-super solutions. MS Classification 2010: 35J60; 34B15.

## 1. Introduction

Nonlinear eigenvalue problems have been widely investigated over years (see, for instance, $[1,2,3,10,12,20,21,26,27,30,33,37]$ and the references therein) and even today they are a major topic of nonlinear analysis (see, for instance, $[8,9,23,24,31,34])$. In this paper, the following Dirichlet problem depending on a positive parameter $\lambda$ is investigated

$$
\begin{cases}-\Delta u=\lambda f(u) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded domain in $R^{n}, n \geq 3$, and $f: R \rightarrow R$ is a continuous function. Precisely, by requiring only an algebraic condition on the nonlinear term, which expresses a suitable growth of $f$ in an arbitrary real interval $[d, s]$, the existence of at least one non-zero solution for $\left(D_{\lambda}\right)$ is obtained for each $\lambda$
belonging to a precise real interval (see Corollary 3.2). Our results are true also for $n=1$ and, as an example, here, a special case is presented.

Theorem 1.1. Let $f:[0,+\infty[\rightarrow[0,+\infty[$ be a continuous function. Assume that there are two positive constants $d, s$, with $d<s$, such that

$$
\begin{equation*}
\frac{\max _{t \in[0, s]} f(t)}{s}<\frac{\int_{0}^{d} f(t) d t}{d^{2}} \tag{1.1}
\end{equation*}
$$

$$
\begin{align*}
& \text { Then, for each } \lambda \in 8 \frac{d^{2}}{\int_{0}^{d} f(t) d t}, 8 \frac{s}{\max _{t \in[0, s]} f(t)}[\text {, the problem } \\
&\left\{\begin{array}{l}
-u^{\prime \prime}=\lambda f(u) \\
u(0)=u(1)=0
\end{array}\right. \\
&\text { in }] 0,1[
\end{align*}
$$

admits at least one positive classical solution $u_{\lambda} \in C^{2}([0,1])$ such that $\left\|u_{\lambda}\right\|_{\infty} \leq$ $s$.

In Theorem 1.1, no asymptotic condition at zero and at infinity on $f$ is requested. The unique assumption is essentially a suitable growth on $f$ in an arbitrary interval $[d, s]$, that is, condition (1.1). Clearly, if $f$ is sublinear at zero, that is

$$
\begin{equation*}
\lim _{t \rightarrow 0^{+}} \frac{f(t)}{t}=+\infty \tag{1.2}
\end{equation*}
$$

condition (1.1) in Theorem 1.1 is satisfied and the interval of parameters becomes

$$
] 0,8 \sup _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)}[
$$

Of course, condition (1.2) is in turn more general than the classical

$$
\begin{equation*}
f(0)>0 \tag{1.3}
\end{equation*}
$$

On the contrary, condition (1.1) can be satisfied also in the cases for which $f$ is superlinear, or linear, at zero, that is, Dirichlet problems ( $T_{\lambda}$ ) (and, more generally, $\left(D_{\lambda}\right)$ ) may admit positive solutions even if condition (1.2) is not verified.

The existence of non-zero solutions for nonlinear Dirichlet problems $\left(D_{\lambda}\right)$ has been widely studied in several papers by topological methods (see for instance the paper of Amann [1]) as well as, by variational methods (see for
instance the paper of Crandall-Rabinowitz [20]). In these latest papers, one of the key assumptions in order to obtain solutions for ordinary and elliptic case respectively, is condition (1.3). Moreover, also nonlinear problems with specific equations having a nonlinear term satisfying (1.2) and for which $f(0)=0$ have been studied. In this direction, we recall the paper of Boccardo-EscobadoPeral [10], where the existence of one non-zero solution, without requiring the restriction of a subcritical growth on the nonlinear term, is established, as well as the paper of Ambrosetti-Brezis-Cerami [3], where the existence of two positive solutions, under a growth at most critical, has been obtained again for a combined effect of concave and convex nonlinearities. It is worth noticing that, in all previous cited papers, the existence of the best parameter $\lambda^{*}$, for which the problem $\left(D_{\lambda}\right)$ admits positive solutions for each $\lambda \leq \lambda^{*}$, has been proved. However, such a parameter $\lambda^{*}$ has not been numerically determined, but only lower or upper bound estimations have been obtained. Indeed, on estimates from above, that is upper bounds of $\lambda^{*}$, there is a very wide literature (see, for instance, $[3,19,20,22,34]$ and the references therein), while, at the best of our knowledge, only few papers are devoted to estimate from below the best value $\lambda^{*}$. Precisely, a lower bound of $\lambda^{*}$ has been established in [34] for the specific nonlinear term $f(u)=u^{q}+u^{p}, 0<q<1<p$, and only for $n=1$. In [7], in the case $n=2$, and in [11] when $f(0) \neq 0$. In this paper, as a consequence of our main result a lower bound of the best parameter $\lambda^{*}$ is obtained. For instance, in the ordinary case, from Theorem 1.1 the following estimate is established

$$
\lambda^{*} \geq 8 \sup _{s>0} \frac{s}{\max _{[0, s]} f}
$$

Summarizing, in this paper two novel aspects, which are different among them, are pointed out. On one hand, the existence of non-zero solutions to $\left(D_{\lambda}\right)$ without requiring the sublinearity at zero of the nonlinear term (see Corollary 3.2 and Example 3.10) and, on the other hand, when the nonlinear term is sublinear at zero, a precise lower bound of the best parameter for which $\left(D_{\lambda}\right)$ admits positive solutions is given (see Corollary 3.3, Remark 3.12 and Example 3.11).

The paper is organized as follows. The main result, Theorem 3.1, is presented in Section 3 and it establishes the existence of positive solutions for elliptic Dirichlet problems without requiring any condition at zero and at infinity. As a consequence, Corollary 3.2 and Corollary 3.3 are obtained. The first one is the parametric version of Theorem 3.1 and the second one is a special case when the nonlinear term is sublinear at zero. It is also pointed out that such results are true for the ordinary case (see Corollary 3.6). It is worth noticing that Corollary 3.2 can be applied to problems where the nonlinear term may be not sublinear at zero for which the classical results as [1] and [20] cannot be applied (see Remark 3.8 and Example 3.10) and Corollary 3.3 establishes a lower bound of the best parameter $\lambda^{*}$ (see Example 3.11 and Remark 3.12). Previously, in

Section 2, the result given in [11], that is Theorem 2.1, is recalled. Here, a variational proof, different from the topological proof established in [11], based on the fixed point theorem obtained by Arino-Gautier-Penot [5], is proposed. We point out that a fundamental tool for such proofs, both variational and topological, is a fruitful estimate due to Talenti in [36] (see the beginning of Section 2).

## 2. Preliminaries and introductory results

Fix a bounded domain $\Omega \subseteq R^{n}, n \geq 3$, with a $C^{1,1}-$ boundary $\partial \Omega$ and $v \in L^{\infty}(\Omega)$. Moreover, consider the problem

$$
\begin{cases}-\Delta u=v(x) & \text { in } \Omega  \tag{P}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

It is well known that $(P)$ admits a unique strong solution $u \in W_{0}^{1,2}(\Omega) \cap$ $W^{2, p}(\Omega)$, for all $p \geq 1$ (see, for instance, [25, Theorem 9.15]); in particular, $u \in L^{\infty}(\Omega)$ (see, for instance, [25, Theorem 7.10]). Moreover, by [36, Theorem 2 and Remark 1] one has

$$
\begin{equation*}
\|u\|_{\infty} \leq B\|v\|_{\infty} \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
B=\frac{1}{2 n \pi}\left(\Gamma\left(1+\frac{n}{2}\right)|\Omega|\right)^{\frac{2}{n}} . \tag{2.2}
\end{equation*}
$$

Now, we point out the following result.
Theorem 2.1. Let $f: R \rightarrow R$ be a continuous function. Assume that there is $r>0$ such that

$$
\begin{equation*}
\max _{t \in[-B r, B r]}|f(t)| \leq r, \tag{2.3}
\end{equation*}
$$

where $B$ is given by (2.2).
Then, the problem

$$
\begin{cases}-\Delta u=f(u) & \text { in } \Omega  \tag{D}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

admits at least one strong solution $u_{0} \in W_{0}^{1,2}(\Omega) \cap W^{2, p}(\Omega)$, for all $p \geq 1$, such that $\left\|u_{0}\right\|_{\infty} \leq B r$.

Proof. Let $f_{r}: R \rightarrow R$ be the continuous function defined as follows

$$
f_{r}(t)= \begin{cases}f(t) & \text { if }|t| \leq B r \\ f(B r) & \text { if } t>B r \\ f(-B r) & \text { if } t<-B r\end{cases}
$$

Moreover, put $F_{r}(t)=\int_{0}^{t} f_{r}(\tau) d \tau$ for all $t \in R$. Clearly, one has $f_{r}(t) \leq$ $\max _{t \in[-B r, B r]}|f(t)|$ for all $t \in R$, for which from (2.3) we get

$$
\begin{equation*}
f_{r}(t) \leq r \tag{2.4}
\end{equation*}
$$

for all $t \in R$. Now, take $X=W_{0}^{1,2}(\Omega)$ endowed with the norm

$$
\|u\|=\left(\int_{\Omega}|\nabla u(x)|^{2} d x\right)^{\frac{1}{2}}
$$

and put

$$
\Phi(u)=\frac{1}{2}\|u\|^{2} \quad \Psi_{r}(u)=\int_{\Omega} F_{r}(u(x)) d x \quad I_{r}(u)=\Phi(u)-\Psi_{r}(u)
$$

for all $u \in X$. Standard computations show that $I_{r}$ is continuously Fréchet differentiable and weakly lower semi-continuous. Moreover, from (2.4) it follows that $I_{r}$ is coercive. Therefore, the direct method of the calculus of variations (see, for instance, [29, Theorem 1.1]) ensures the existence of a global minimizer $u_{0}$. It follows that $I_{r}^{\prime}\left(u_{0}\right)=0$ and $u_{0}$ is a weak solution of the problem

$$
\begin{cases}-\Delta u=f_{r}(u) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Owing to (2.1) one has $\left\|u_{0}\right\|_{\infty} \leq B\left\|f_{r}\left(u_{0}\right)\right\|_{\infty}$. So, from (2.4) one has $\left\|u_{0}\right\|_{\infty} \leq B \sup _{t \in R}\left|f_{r}(t)\right| \leq B r$, that is

$$
\left\|u_{0}\right\|_{\infty} \leq B r
$$

Therefore, one has $f\left(u_{0}(x)\right)=f_{r}\left(u_{0}(x)\right)$ for all $x \in \Omega$ for which $u_{0}$ is also a weak solution of $(D)$ and the conclusion is achieved.

As a consequence of Theorem 2.1 the following result is obtained.

Corollary 2.2. Let $f: R \rightarrow R$ be a nonnegative continuous function such that $f(0)>0$. Put

$$
\bar{\lambda}=\frac{1}{B} \sup _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)},
$$

where $B$ is given by (2.2).
Then for each $\lambda \in] 0, \bar{\lambda}\left[\right.$, problem $\left(D_{\lambda}\right)$ admits at least one positive strong solution $u_{\lambda} \in W_{0}^{1,2}(\Omega) \cap W^{2, p}(\Omega)$, for all $p \geq 1$.

Proof. Let $f^{*}: R \rightarrow R$ be the nonnegative continuous function defined as follows

$$
f^{*}(t)= \begin{cases}f(t) & \text { if } t \geq 0 \\ f(0) & \text { if } t<0\end{cases}
$$

and fix $\lambda \in] 0, \bar{\lambda}\left[\right.$. So, there is $s>0$ such that $\lambda<\frac{1}{B} \frac{s}{\max _{t \in[0, s]} f^{*}(t)}$. Clearly, by setting $r=\frac{s}{B}$, one has $\max _{t \in[-B r, B r]}\left|\lambda f^{*}(t)\right|<r$. Hence, Theorem 2.1 ensures the existence of one weak solution $u_{\lambda}$ for the problem

$$
\begin{cases}-\Delta u=\lambda f^{*}(u) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

which is non-zero since $f^{*}(0) \neq 0$ and, then it is positive owing to the strong maximum principle. It follows that $u_{\lambda}$ is also a weak solution of $\left(D_{\lambda}\right)$ and the conclusion is achieved.

Remark 2.3. If in Corollary 2.2, we assume in addition that $\lim _{t \rightarrow+\infty} \frac{f(t)}{t}=+\infty$ then the conclusion also for $\lambda=\bar{\lambda}$ holds true and, moreover, one has

$$
\left.\left.\left\|u_{\lambda}\right\|_{\infty} \leq \bar{s} \quad \forall \lambda \in\right] 0, \bar{\lambda}\right],
$$

where $\bar{s}>0$ is such that $\bar{\lambda}=\frac{1}{B} \frac{\bar{s}}{\max _{[0, \bar{s}]} f}$.
Indeed, one has $\lim _{s \rightarrow 0^{+}} \frac{s}{\max _{[0, s]} f}=\lim _{s \rightarrow+\infty} \frac{s}{\max _{[0, s]} f}=0$ for which the function $\frac{s}{\max _{[0, s]} f}$ admits a point of global maximum $\bar{s}$ in $] 0,+\infty[$ and $\bar{\lambda}=$ $\frac{1}{B} \max _{s \in] 0,+\infty[ } \frac{s}{\max _{[0, s]} f}$, so that the same proof of Corollary 2.2 ensures the conclusion.

Remark 2.4. Clearly, the existence of a non-trivial solution to problem $\left(D_{\lambda}\right)$ in Corollary 2.2 is deduced from the assumption $f(0)>0$. Moreover, such a condition, without requiring that $f$ be nonnegative everywhere and by standard computations (see, for instance, [16, Lemma 2.3]), ensures that the obtained solution is nonnegative in $\Omega$.

Remark 2.5. Theorem 2.1 and Corollary 2.2 also for the ordinary case, that is $n=1$, are true. Indeed, fixed $v \in L^{\infty}(] a, b[)$, the problem

$$
\left\{\begin{array}{l}
\left.-u^{\prime \prime}=v(x) \quad \text { in }\right] a, b[ \\
u(a)=u(b)=0
\end{array}\right.
$$

admits a unique solution $u \in W^{2, \infty}(] a, b[)$ such that

$$
\|u\|_{\infty} \leq \frac{(b-a)^{2}}{8}\|v\|_{\infty}
$$

(see, for instance, [6, Lemma 1(1) and Lemma 2(3)]). As an example, we report below a version of Corollary 2.2 for $n=1$.

Corollary 2.6. Let $f: R \rightarrow R$ be a nonnegative continuous function such that $f(0)>0$. Put

$$
\bar{\lambda}=8 \sup _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)}
$$

Then for each $\lambda \in] 0, \bar{\lambda}\left[\right.$, problem $\left(T_{\lambda}\right)$ admits at least one positive classical solution $u_{\lambda}$.

Remark 2.7. We recall that for a precise class of nonnegative continuous functions $f: R \rightarrow R$ satisfying, in particular, the following conditions:

1. $f(0)>0$;
2. $\lim _{t \rightarrow+\infty} \frac{f(t)}{t}=+\infty$,

Crandall and Rabinowitz in [20] established the existence of $\lambda^{*}>0$ such that, for each $\lambda \in] 0, \lambda^{*}\left[\right.$, the problem $\left(D_{\lambda}\right)$ admits at least two positive weak solutions. Moreover, they also proved that such value $\lambda^{*}$ is the best value for which the problem admits solutions. However, no lower bound of $\lambda^{*}$ is given there. We observe that Corollary 2.2 allows us to establish a lower bound of $\lambda^{*}$. Precisely, one has

$$
\lambda^{*} \geq \bar{\lambda}=\frac{1}{B} \max _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)}
$$

(see also Remark 2.3). We recall that, in order to obtain the second solution in the elliptic case, the classical $A R$ - condition, stronger than condition 2 ., is requested (see [4, 35]).

The same remark, also for the ordinary case, can be pointed out. In fact, Amann in [1] established the same type of result for a two-point boundary value problem, by obtaining a positive value $\lambda^{*}$ for which the ordinary problem admits two positive solutions for $\lambda<\lambda^{*}$, one solution for $\lambda=\lambda^{*}$, and no solution for $\lambda>\lambda^{*}$. As an example, from the result of Amann [1], we obtain that there is $\lambda^{*}>0$ such that the problem

$$
\left\{\begin{array}{l}
\left.-u^{\prime \prime}=\lambda e^{u} \quad \text { in }\right] 0,1[ \\
u(0)=u(1)=0
\end{array}\right.
$$

admits positive classical solutions if and only if $\left.\lambda \in] 0, \lambda^{*}\right]$. So, owing to Corollary 2.6 we obtain a lower bound of $\lambda^{*}$, that is,

$$
\lambda^{*} \geq \frac{8}{e}
$$

Taking also [22, Theorem 3.2, page 367] into account, it follows that

$$
\lambda^{*} \in\left[\frac{8}{e}, \frac{\pi^{2}}{e}\right]
$$

Remark 2.8. We recall that Theorem 2.1 has been established in [11] (see also [13]) by topological methods (see [11, Theorem 1]). We observe that in order to obtain a non-zero solution by such a result we must assume $f(0) \neq 0$ (see Corollary 2.2). So, we point out here that the proof of Theorem 2.1 is variational and it gives us an additional information, that is, the solution is a global minimizer of the associated functional $I_{r}$. Such information allows us to obtain a positive solution under an assumption which is more general than $f(0) \neq 0$, as it is shown in Section 3.

Remark 2.9. The proof of Theorem 2.1 presented here is based on the direct method of the calculus of variations, which is a fundamental tool of variational methods. The proof obtained in [11] instead is based on the fixed point theorem for sequentially weakly continuous maps proved by Arino-Gautier-Penot in [5], which is a standard tool in topological methods. Both the proofs are based on an estimate given by Talenti established in [36], which is, hence, fundamental for our purposes. We wish to recall that such a result has been applied in order to obtain solutions to nonlinear differential problems for the first time in [28] (see also $[17,18]$ ), where also set-valued techniques have been used.

## 3. Main results

In this Section, we present our main result, Theorem 3.1, and its consequences and applications. To this end, put $R(x)=\sup \{\delta: B(x, \delta) \subseteq \Omega\}$ for all $x \in \Omega$, and $R=\sup _{x \in \Omega} R(x)$, for which there exists $x_{0} \in \Omega$ such that $B\left(x_{0}, R\right) \subseteq \Omega$. We have the following result.

Theorem 3.1. Let $f: R \rightarrow R$ be a nonnegative continuous function. Assume that
(a) there is $r>0$ such that

$$
\max _{t \in[0, B r]} f(t) \leq r
$$

where $B$ is given by (2.2);
(b) there is $d>0$, with $d<B r$, such that

$$
\int_{0}^{d} f(t) d t>\frac{2\left(2^{n}-1\right)}{R^{2}} d^{2}
$$

Then, problem $(D)$ admits at least one strong positive solution $u_{0} \in W_{0}^{1,2}(\Omega) \cap$ $W^{2, p}(\Omega), p \geq 1$, such that $\left\|u_{0}\right\|_{\infty} \leq B r$.

Proof. Without loss of generality, we can assume $f(t)=f(0)$ for all $t<0$. From the proof of Theorem 2.1 we obtain that the solution $u_{0}$ of $(D)$ is a global minimizer for the functional $I_{r}$. Now, put

$$
u_{d}(x):= \begin{cases}0 & \text { if } x \in \Omega \backslash B\left(x_{0}, R\right) \\ \frac{2 d}{R}\left(R-\left|x-x_{0}\right|\right) & \text { if } x \in B\left(x_{0}, R\right) \backslash B\left(x_{0}, R / 2\right) \\ d & \text { if } x \in B\left(x_{0}, R / 2\right) .\end{cases}
$$

Clearly, one has that $u_{d} \in X$ and $\left\|u_{d}\right\|_{\infty}=d<B r$ for which $F_{r}(d)>$ $\frac{2\left(2^{n}-1\right)}{R^{2}} d^{2}$. It follows $\frac{\Psi_{r}\left(u_{d}\right)}{\Phi\left(u_{d}\right)} \geq \frac{R^{2}}{2\left(2^{n}-1\right)} \frac{F_{r}(d)}{d^{2}}>1$. Therefore, one has $I_{r}\left(u_{d}\right)<I_{r}(0)$ and hence we obtain $I_{r}\left(u_{0}\right) \leq I_{r}\left(u_{d}\right)<0$, for which $u_{0} \neq 0$ and from the maximum principle the conclusion follows.

As a consequence of Theorem 3.1 we obtain the following result.

Corollary 3.2. Let $f: R \rightarrow R$ be a nonnegative continuous function. Put $F(t)=\int_{0}^{t} f(\xi) d \xi$ for all $t \in R$ and assume that there are two positive constants $s, d$, with $d<s$, such that

$$
\begin{equation*}
\frac{\max _{t \in[0, s]} f(t)}{s}<\left(\frac{R^{2}}{2 B\left(2^{n}-1\right)}\right) \frac{F(d)}{d^{2}} . \tag{3.1}
\end{equation*}
$$

Then for each $\lambda \in] \frac{2\left(2^{n}-1\right)}{R^{2}} \frac{d^{2}}{F(d)}, \frac{1}{B} \frac{s}{\max _{t \in[0, s]} f(t)}\left[\right.$, problem $\left(D_{\lambda}\right)$ admits at least one positive strong solution $u_{\lambda} \in W_{0}^{1,2}(\Omega) \cap W^{2, p}(\Omega)$, $p \geq 1$, such that $\left\|u_{\lambda}\right\|_{\infty} \leq s$.

Proof. Fix $\lambda$ as in the conclusion. Therefore, one has

$$
B \frac{\max _{t \in[0, s]} f(t)}{s}<\frac{1}{\lambda}<\left(\frac{R^{2}}{2\left(2^{n}-1\right)}\right) \frac{\int_{0}^{d} f(\xi) d \xi}{d^{2}}
$$

So, setting $r=\frac{s}{B}$ it follows $\frac{\max _{t \in[0, B r]} \lambda f(t)}{r}<1$ and $\frac{R^{2}}{2\left(2^{n}-1\right)} \frac{\int_{0}^{d} \lambda f(\xi) d \xi}{d^{2}}>$ 1, for which Theorem 3.1 ensures the conclusion.

Finally, as a special case of Corollary 3.2, we point out the following result.
Corollary 3.3. Let $f: R \rightarrow R$ be a nonnegative continuous function such that

$$
\lim _{t \rightarrow 0^{+}} \frac{f(t)}{t}=+\infty
$$

Put

$$
\bar{\lambda}=\frac{1}{B} \sup _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)},
$$

where $B$ is given by (2.2).
Then for each $\lambda \in] 0, \bar{\lambda}\left[\right.$, problem $\left(D_{\lambda}\right)$ admits at least one positive strong solution $u_{\lambda} \in W_{0}^{1,2}(\Omega) \cap W^{2, p}(\Omega), p \geq 1$.

Proof. Fix $\lambda<\bar{\lambda}$. Therefore, there is $s>0$ such that $\lambda<\frac{1}{B} \frac{s}{\max _{t \in[0, s]} f(t)}$. From $\lim _{t \rightarrow 0^{+}} \frac{R^{2}}{2\left(2^{n}-1\right)} \frac{F(t)}{t^{2}}=+\infty$ one has that there is $\left.d \in\right] 0, s$ [ such that $\frac{R^{2}}{2\left(2^{n}-1\right)} \frac{F(d)}{d^{2}}>\frac{1}{\lambda}$ for which $\frac{2\left(2^{n}-1\right)}{R^{2}} \frac{d^{2}}{F(d)}<\lambda<\frac{1}{B} \frac{s}{\max _{t \in[0, s]} f(t)}$. Hence, Corollary 3.2 ensures the conclusion.

Remark 3.4. Condition (b) in Theorem 3.1 is imposed in order to obtain that the solution is non-trivial. We recall that in literature this type of condition has been already considered (see, for instance, [32, Theorem 3.7, $\left.\left(h_{18}\right)\right]$ and $[15$, Theorem 3.1, (3.1)]). Moreover, in order to obtain nonnegative solutions to problem $(D)$, without requiring that f be nonnegative everywhere, it is enough to assume in Theorem 3.1 only $f(0) \geq 0$ (see Remark 2.4).

Remark 3.5. Theorem 3.1 and Corollaries 3.2 and 3.3 hold also for $n=1$ (see Remark 2.5). So, in particular, we obtain Theorem 1.1 presented in the Introduction and the corollary below.

Corollary 3.6. Let $f: R \rightarrow R$ be a nonnegative continuous function such that

$$
\lim _{t \rightarrow 0^{+}} \frac{f(t)}{t}=+\infty
$$

Put

$$
\bar{\lambda}=8 \sup _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)} .
$$

Then for each $\lambda \in] 0, \bar{\lambda}\left[\right.$, problem $\left(T_{\lambda}\right)$ admits at least one positive solution $u_{\lambda} \in C^{2}([0,1])$.

Remark 3.7. If in Corollary 3.3, or in Corollary 3.6, we assume in addition that $\lim _{t \rightarrow+\infty} \frac{f(t)}{t}=+\infty$ then the conclusion also for $\lambda=\bar{\lambda}$ holds true and, moreover, one has

$$
\left.\left.\left\|u_{\lambda}\right\|_{\infty} \leq \bar{s} \quad \forall \lambda \in\right] 0, \bar{\lambda}\right],
$$

where $\bar{s}>0$ is such that $\bar{\lambda}=\frac{1}{B} \frac{\bar{s}}{\max _{[0, \bar{s}]} f}$ (see Remark 2.3).
Remark 3.8. Corollary 3.2 ensures the existence of positive solutions to $\left(D_{\lambda}\right)$ without any condition at zero or at infinity on the nonlinear term. We note that, in literature, a condition at zero as (1.3) (or, in some cases, as (1.2)) is requested (see $[1,2,10,20,26,37]$ ). Therefore, such a result can be applied to problems where the nonlinear term is not sublinear at zero, as Example 3.10 below shows. Clearly, results in $[1,2,10,20,26,37]$ cannot be applied to the problem in Example 3.10.

Remark 3.9. When the nonlinear term, in particular, is sublinear at zero, Corollary 3.3 ensures the existence of positive solutions to $\left(D_{\lambda}\right)$ for each positive $\lambda \leq \bar{\lambda}$. In literature, there are several results in this direction again for specific equations (see for instance $[3,10]$ ) establishing the existence of the best $\lambda^{*}$ for which the problem $\left(D_{\lambda}\right)$ admits solutions. However, no estimate on $\lambda^{*}$ is given in [3] and [10]. In [34] a lower bound of $\lambda^{*}$ is guaranteed, but only for the ordinary case (see [34, Corollary 1]. Our result ensures a lower bound of $\lambda^{*}$,
that is, $\lambda^{*} \geq \bar{\lambda}$, which can be used also for elliptic case differently to result obtained in [34] which can be applied only to ordinary case (see Remark 3.12 and Example 3.11).

Example 3.10. Let $f: R \rightarrow R$ be the function defined as follows

$$
f(t)= \begin{cases}t \sqrt{|t|} & \text { if } t<1 \\ \sqrt{t} & \text { if } 1 \leq t \leq 10 \\ h(t) & \text { if } t>10\end{cases}
$$

where $h:[10,+\infty[\rightarrow R$ is a completely arbitrary function. Owing to Corollary 3.2 , the problem

$$
\left\{\begin{array}{l}
\left.-u^{\prime \prime}=25 f(u) \quad \text { in }\right] 0,1[, \\
u(0)=u(1)=0
\end{array}\right.
$$

admits at least one positive classical solution $u_{0}$ such that $\left\|u_{0}\right\|_{\infty} \leq 10$. It is enough to choose $d=1, s=10$ by verifying that one has $8 \frac{1}{\int_{0}^{1} t \sqrt{t} d t}<25<8 \frac{10}{\sqrt{10}}$. We explicitly observe that in this case, the nonlinearity $f$ is not sublinear at zero and its behavior at infinity is completely arbitrary.

Example 3.11. Consider the problem

$$
\begin{cases}-\Delta u=\mu u^{q}+u^{p} & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $0<q<1<p$ and $\mu$ is a positive parameter, and put

$$
\begin{equation*}
\bar{\mu}=\left(\frac{1}{B}\right)^{\frac{p-q}{p-1}} \frac{(p-1)(1-q)^{\frac{1-q}{p-1}}}{(p-q)^{\frac{p-q}{p-1}}} \tag{3.2}
\end{equation*}
$$

Owing to Corollary 3.3 the problem $\left(D_{\mu}\right)$ admits at least one positive solution for each $\mu \leq \bar{\mu}$. So that $\bar{\mu}$ is a lower bound of the best parameter $\Lambda$ guaranteed by [3] (see also [34]) for which $\left(D_{\mu}\right)$ admits two solutions. Indeed, applying Corollary 3.3 to

$$
\begin{cases}-\Delta u=\lambda\left(\mu u^{q}+u^{p}\right) & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

the existence of solutions is obtained for each $\lambda \leq \bar{\lambda}$, where
$\bar{\lambda}=\frac{1}{B} \max _{s>0} \frac{s}{\max _{t \in[0, s]} f(t)}=\frac{1}{B} \max _{s>0} \frac{s}{\mu s^{q}+s^{p}}=\frac{1}{B} \frac{1}{\mu^{\frac{p-1}{p-q}}\left[\left(\frac{1-q}{p-1}\right)^{\frac{q-1}{p-q}}+\left(\frac{1-q}{p-1}\right)^{\frac{p-1}{p-q}}\right]}$,
for which $\bar{\lambda} \geq 1$ being $\mu \leq \bar{\mu}$.
As an example, by picking $\Omega=\left\{x \in R^{3}:|x|<1\right\}$ and $q=\frac{1}{2}, p=\frac{3}{2}$ we obtain $\bar{\mu}=9$.

Remark 3.12. Problem $\left(D_{\mu}\right)$ has been introduced in [3] (see also [10]) establishing the existence of $\Lambda>0$ for which it admits solutions if and only if $\mu \leq \Lambda$ (also a growth at most critical is assumed in order to obtain a second solution for $\mu<\Lambda$ ). No estimate on such parameter is provided. As a consequence of Corollary 3.3 we obtain $\bar{\mu}$ as a lower bound of $\Lambda$, that is

$$
\Lambda \geq \bar{\mu}
$$

(see (3.2) in Example 3.11). In [34], only for the ordinary case, a lower bound of $\Lambda$ is given. Our estimate instead can be applied also to the elliptic case (see Example 3.11).

Remark 3.13. To observe that the proof of our main result is actually a combination of variational and topological tools may be interesting. Indeed, the assumption (a) of Theorem 3.1 is equivalent to assume that $-\Delta^{-1} r$ (that is, the unique solution of $-\Delta u=r$ in $\Omega, u_{\partial \Omega}=0$ ) is an upper solution of $(D)$. We also observe that a totally variational proof in order to obtain solutions for $\left(D_{\lambda}\right)$ has been obtained in [15] by applying the local minimum theorem established in [14].

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# A periodic problem for first order differential equations with locally coercive nonlinearities 

Elisa Sovrano and Fabio Zanolin<br>Dedicated to Professor Jean Mawhin on the occasion of his 75th birthday


#### Abstract

In this paper we study the periodic boundary value problem associated with a first order $O D E$ of the form $x^{\prime}+g(t, x)=s$ where $s$ is a real parameter and $g$ is a continuous function, T-periodic in the variable $t$. We prove an Ambrosetti-Prodi type result in which the classical uniformity condition on $g(t, x)$ at infinity is considerably relaxed. The Carathéodory case is also discussed.


Keywords: Periodic solutions, Multiplicity results, Local coercivity, Coincidence degree. MS Classification 2010: 34B15, 34C25.

## 1. Introduction

This paper is concerned with the study of the periodic boundary value problem associated with the first order scalar ODE

$$
\begin{equation*}
x^{\prime}+g(t, x)=s \tag{s}
\end{equation*}
$$

where $s$ is a real parameter and $g$ is a continuous function, $T$-periodic in the variable $t$.

Interest in this kind of parameter-dependent equations can be found in connection to the celebrated Ambrosetti-Prodi problem that was first investigated in the setting of the Dirichlet problem for elliptic PDEs (see [1, 2, 5]). The study of the Ambrosetti-Prodi problem for second order ODEs with periodic boundary conditions is a broad and active research area in which many investigators have been involved (see, for instance, $[8,23,26]$ for some significant contributions in this field). In this latter context, the analysis is focused on the existence, nonexistence and multiplicity of (periodic) solutions for parameter dependent equations of the form

$$
x^{\prime \prime}+F\left(t, x, x^{\prime}\right)=s
$$

For the generalized Liénard equation

$$
\begin{equation*}
x^{\prime \prime}+f(x) x^{\prime}+g(t, x)=s, \tag{s}
\end{equation*}
$$

a relevant contribution in this direction is represented by the work of Fabry, Mawhin and Nkashama [8]. We recall here their result.

Theorem 1.1. Suppose that $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, $g$ is $T$-periodic in $t$ and satisfies hypothesis

$$
\begin{equation*}
\lim _{|x| \rightarrow+\infty} g(t, x)=+\infty \text {, uniformly in } t . \tag{H}
\end{equation*}
$$

Then, there exists a number $s_{0}$ such that
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{L C}_{s}\right)$ has no $T$-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{L C}_{s}\right)$ has at least one T-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{L C}_{s}\right)$ has at least two T-periodic solutions.
The above theorem has motivated a rich area of research, including investigations on problems with singularities [9] and on nonlinear operators of $p$ Laplacian type [20].

The Ambrosetti-Prodi problem for first order periodic ODEs was studied by McKean and Scovel in [22] and by Vidossich in [29]. A version of Theorem 1.1 for equation $\left(\mathscr{E}_{s}\right)$ was carried out by Mawhin in $[16,17]$ and it can be stated as follows.

Theorem 1.2. Suppose that $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and T-periodic in $t$. Assume ( $H$ ). Then, there exists a number $s_{0}$ such that
$1^{\circ}$ for $s<s_{0}$, equation ( $\mathscr{E}_{s}$ ) has no T-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least one T-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least two T-periodic solutions.
Notice that the results obtained for equation $\left(\mathscr{E}_{s}\right)$ can be stated also for

$$
x^{\prime}=q(t, x) \pm \theta,
$$

where $\theta$ a real parameter. More precisely, we can reduce the above equation to $\left(\mathscr{E}_{s}\right)$, mainly in two ways. One is due to the obvious position $g(t, x)=-q(t, x)$ and $s= \pm \theta$. The other one follows from the change of variable $t \mapsto-t$, so that $g(t, x)=q(-t, x)$ and $s=\mp \theta$ (see also [16, Remark 1]).

As described in [18], a possible application of Theorem 1.2 is to the Riccati differential equation

$$
x^{\prime}+\gamma_{2}(t) x^{2}+\gamma_{1}(t) x+\gamma_{3}(t)=0 .
$$

In this case, the coercivity condition $(H)$ is satisfied if

$$
\gamma_{2}(t) \geq \kappa>0, \quad \text { for all } t .
$$

The motivation to study this topic is well presented in [18], by means of several interesting references describing the state of the art up to the middle of the Eighties.

Remark 1.3. The works $[16,17,18]$ of Mawhin, for equation $\left(\mathscr{E}_{s}\right)$, have stimulated a great deal of researches in this area. Even if, at first glance, the search of periodic solutions for equation $\left(\mathscr{E}_{s}\right)$ could appear "elementary", it has been and, especially, it is still a source of interesting and, sometimes, challenging problems. Among the problems leading directly or indirectly to first order equations, we recall the study on the number of limit cycles for planar polynomial autonomous systems, which is connected to Hilbert sixteenth problem, and questions arising from single species population dynamics connected to periodic Kolmogorov equations (see the detailed presentations, as well as the comprehensive list of references, in [7, 25] that cover a great part of the literature concerning these equations up to the early 2000s).

In [28] we have proposed a possible variant of Theorem 1.1 for equation $\left(\mathscr{L C}_{s}\right)$ in which the coercivity condition $(H)$ is replaced by a local one, thus avoiding the uniformity in the variable $t$. Taking into account this generalization, the natural question which arises in the context of first order equations is whether the same outcome holds in the setting of Theorem 1.2. A clue that this conjecture is true can be found in the study of the Kolmogorov equation $x^{\prime}=x h(t, x)$ and in the particular case of the Verhulst (logistic) equation, namely for $h(t, x)=r(t)-\gamma_{2}(t) x$. Indeed, from [3,27, 31, 32], a classical result for equation

$$
x^{\prime}+\gamma_{2}(t) x^{2}-r(t) x=0,
$$

with $r, \gamma_{2}: \mathbb{R} \rightarrow \mathbb{R}$ continuous and $T$-periodic functions, is the existence of exactly two $T$-periodic solutions, the trivial one and another one positive, provided that

$$
\int_{0}^{T} r(t) d t>0 \quad \text { and } \quad \gamma_{2}(t) \geq 0 \forall t, \quad \gamma_{2} \not \equiv 0
$$

In the present paper we propose an extension of Theorem 1.2, in the spirit of [28]. In particular, we replace condition $(H)$ by an average-type assumption at infinity of Gaetano Villari's type, which reads as follows.
( $G V$ ) Given $K_{1}>0$ and $K_{2}>0$, for each $\sigma$ there exists $d_{\sigma}>0$ such that $\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t>\sigma$ for each $x \in C_{T}$ with $|x(t)| \geq d_{\sigma}$ for all $t \in[0, T]$ and such that $|x|_{\max } \leq K_{1}|x|_{\text {min }}+K_{2}$.

We remark also that an immediate consequence of condition $(H)$ is that the function $g$ is bounded from below. In our case, such lower bound is no more guaranteed and therefore we impose the following one-sided growth assumption:
$\left(G_{0}\right) \exists a_{0}, b_{0} \in L^{1}\left([0, T], \mathbb{R}^{+}\right): g(t, x) \geq-a_{0}(t)|x|-b_{0}(t), \forall x \in \mathbb{R}, t \in[0, T]$.
In this setting, we are in position to present our main result.
Theorem 1.4. Suppose that $g: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and T-periodic in $t$. Assume $\left(G_{0}\right)$ and $(G V)$. Then, there exists a number $s_{0}$ such that
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has no $T$-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least one T-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least two $T$-periodic solutions.
A possible corollary of Theorem 1.4 is the following.
Corollary 1.5. Let $\gamma_{0}, \gamma_{1}, \gamma_{p}: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and $T$-periodic functions and let $p>1$. Suppose that $\gamma_{p}(t) \geq 0$ for all $t$ with $\gamma_{p} \not \equiv 0$. Then, for equation

$$
\begin{equation*}
x^{\prime}+\gamma_{p}(t)|x|^{p}+\gamma_{1}(t) x+\gamma_{0}(t)=s, \tag{E}
\end{equation*}
$$

the following result holds. There exists a number $s_{0}$ such that:
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{R}_{\mathscr{E}}\right)$ has no $T$-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{R}_{s}\right)$ has at least one $T$-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{R}_{s}\right)$ has at least two T-periodic solutions.
Looking again at the uniform condition $(H)$ and applying it to $\left(\mathscr{R}_{\mathscr{S}}\right)$, then we need to require that $\gamma_{p}(t)$ is positive and uniformly bounded away from zero. On the other hand, by Corollary 1.5, we observe that the coercivity condition in our setting is of local type. Notice also that $g(t, x)=\gamma_{p}(t)|x|^{p}+\gamma_{1}(t) x+\gamma_{0}(t)$ is not necessarily bounded from below but it satisfies the growth assumption $\left(G_{0}\right)$.

The scheme of the proof already developed in $[8,16,17]$ is reconsidered here to prove Theorem 1.4. In more detail, we combine topological degree arguments and the method of upper-lower solutions with some new tools adapted from [28]. We will also take advantage of some preliminary lemmas needed to treat the case of first order equations. We stress the fact that all our results will be formulated in the Carathéodory setting. In this manner we also improve some previous results in [24].

## 2. Preliminaries

In this section we deal with the periodic boundary value problem associated with the first order differential equation

$$
\begin{equation*}
x^{\prime}+\psi(t, x)=0, \tag{1}
\end{equation*}
$$

where we assume that $\psi:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function. As usual, by a $T$-periodic solution of (1) we mean a generalized solution $x:[0, T] \rightarrow \mathbb{R}$ of the equation (1) which satisfies the boundary condition

$$
x(0)=x(T)
$$

Equivalently, one could extend the map $\psi(\cdot, x)$ on $\mathbb{R}$ by $T$-periodicity and then consider $T$-periodic solutions $x: \mathbb{R} \rightarrow \mathbb{R}$ with $x$ absolutely continuous (AC). In the frame of Mawhin's coincidence degree theory we will find a priori bounds and will provide existence results for periodic solutions of equation (1).

The standard setting to enter in the framework of the coincidence degree is the following. Let

$$
X=C_{T}:=\{x \in C([0, T]): x(0)=x(T)\}
$$

endowed with the norm $\|x\|_{X}:=\|x\|_{\infty}$ and $Z=L^{1}([0, T])$ with the norm $\|x\|_{Z}:=\|x\|_{1}$. Let $L: X \supseteq \operatorname{dom} L \rightarrow Z$ be defined as $L x:=-x^{\prime}$, with

$$
\operatorname{dom} L=W_{T}^{1,1}:=\{x \in X: x \in A C\} .
$$

According to [14], a natural choice for the projections is given by

$$
Q x:=\frac{1}{T} \int_{0}^{T} x(t) d t, \quad \forall x \in Z, \quad P x=Q x, \quad \forall x \in X
$$

This way, we have $\operatorname{ker} L=\operatorname{Im} P=\mathbb{R}$ and $\operatorname{coker} L=\operatorname{Im} Q=\mathbb{R}$. Moreover, we take $J$ as the identity in $\mathbb{R}$. Notice that, for each $w \in Z$, the vector $v=K_{P}(I-Q) w$ is the (unique) solution of the linear boundary value problem

$$
\left\{\begin{array}{l}
-v^{\prime}(t)=w(t)-\frac{1}{T} \int_{0}^{T} w(t) d t \\
v(0)=v(T), \quad \int_{0}^{T} v(t) d t=0
\end{array}\right.
$$

Lastly, as nonlinear operator $N$, we take the associated Nemytskii operator, namely

$$
(N x)(t):=\psi(t, x(t)), \quad \forall x \in X
$$

By a standard argument, it is possible to verify that the operator $N$ is $L$ completely continuous and, moreover, the map $\tilde{x}(\cdot)$ is a $T$-periodic solution
of (1) if and only if $\tilde{x} \in \operatorname{dom} L$ with $L \tilde{x}=N \tilde{x}$. Analogously, solutions to the abstract equation $L x=\lambda N x$, with $0<\lambda \leq 1$, correspond to $T$-periodic solutions of

$$
\begin{equation*}
x^{\prime}+\lambda \psi(t, x)=0, \quad 0<\lambda \leq 1 . \tag{2}
\end{equation*}
$$

In the next two lemmas we provide some a priori bounds for the solutions of the parameter dependent equation (2) that will be useful for the application of Theorem 5.1 in the Appendix to the equation (1).

Lemma 2.1. Let $\psi:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying
$\left(H_{0}\right) \quad \exists a_{0}, b_{0} \in L^{1}\left([0, T], \mathbb{R}^{+}\right): \psi(t, x) \geq-a_{0}(t)|x|-b_{0}(t), \forall x \in \mathbb{R}$ and a.e. $t \in[0, T]$.

Then, there exist constants $C \geq 1$ and $K>0$ such that any $T$-periodic solution of (2) satisfies

$$
\begin{cases}x_{\max } \leq C^{-1} x_{\min }+C^{-1} K & \text { if } x_{\min }<-K,  \tag{3}\\ |x(t)| \leq K, \forall t & \text { if }-K \leq x_{\min }<0, \\ x_{\max } \leq C x_{\min }+K, & \text { if } x_{\min } \geq 0 .\end{cases}
$$

Moreover, in any case

$$
\begin{equation*}
|x|_{\max } \leq C|x|_{\min }+K \tag{4}
\end{equation*}
$$

with $C=1$ when $a_{0} \equiv 0$.
Proof. Without loss of generality, let us suppose that $x_{\min }<x_{\max }$ and let $t_{0}<t_{1}<t_{0}+T$ be such that $x\left(t_{0}\right)=x_{\min }$ and $x\left(t_{1}\right)=x_{\max }$. The theory of differential inequalities guarantees that, for all $t \in\left[t_{0}, t_{1}\right]$, we have that $x(t) \leq y(t)$, where $y$ is the solution of the initial value problem

$$
\begin{equation*}
y^{\prime}=a_{0}(t)|y|+b_{0}(t), \quad y\left(t_{0}\right)=x\left(t_{0}\right)=x_{\min } . \tag{5}
\end{equation*}
$$

Notice that the solution $y(t)$ of the equation in (5) is monotone non-decreasing and therefore $y(t) \geq y\left(t_{0}\right)$ for all $t \in\left[t_{0}, t_{1}\right]$.

First of all, let us suppose that $x_{\min }=y\left(t_{0}\right)<0$ and let $\left[t_{0}, \hat{t}[\right.$ be the maximal open interval contained in $\left[t_{0}, t_{1}[\right.$ such that $y(t)<0$. Accordingly,

$$
y^{\prime}(t)=-a_{0}(t) y(t)+b_{0}(t), \quad \text { for a.e. } t \in\left[t_{0}, \hat{t}\right] .
$$

An integration of the linear equation on $\left[t_{0}, t\right] \subseteq\left[t_{0}, \hat{t}\right]$ yields to

$$
\begin{aligned}
y(t) & =y\left(t_{0}\right) \exp (-\mathcal{A}(t))+\int_{t_{0}}^{t} b_{0}(\xi) \exp (\mathcal{A}(\xi)-\mathcal{A}(t)) d \xi \\
& \leq y\left(t_{0}\right) \exp (-\mathcal{A}(t))+\mathcal{B}(t)
\end{aligned}
$$

where we have set

$$
\mathcal{A}(t):=\int_{t_{0}}^{t} a_{0}(\xi) d \xi, \quad \mathcal{B}(t):=\int_{t_{0}}^{t} b_{0}(\xi) d \xi
$$

Using the fact that $y\left(t_{0}\right)<0$, it follows that

$$
\begin{aligned}
x(t) & \leq y(t) \leq \exp (-\mathcal{A}(t)) y\left(t_{0}\right)+\mathcal{B}(t) \\
& \leq \exp \left(-\int_{0}^{T} a_{0}(t) d t\right) x_{\min }+\int_{0}^{T} b_{0}(t) d t
\end{aligned}
$$

holds for all $t \in\left[t_{0}, \hat{t}\right]$. By setting

$$
K:=\exp \left(\int_{0}^{T} a_{0}(t) d t\right) \int_{0}^{T} b_{0}(t) d t
$$

we immediately obtain that $y(t)<0$ for all $t \in\left[t_{0}, \hat{t}\right]$ if $x_{\text {min }}<-K$ and therefore, by the maximality of $\hat{t}$ we conclude that $\hat{t}=t_{1}$. Hence,

$$
x_{\max }=x\left(t_{1}\right) \leq y\left(t_{1}\right) \leq \exp \left(-\int_{0}^{T} a_{0}(t) d t\right) x_{\min }+\int_{0}^{T} b_{0}(t) d t
$$

and this proves the first inequality in (3) for

$$
C:=\exp \left(\int_{0}^{T} a_{0}(t) d t\right)
$$

On the other hand, if $-K \leq x_{\min }<0$, either $x(t) \leq 0$ for all $t \in\left[t_{0}, t_{1}\right]$, or $x_{\text {max }}>0$ and there exists a first time $\hat{t} \in\left[t_{0}, t_{1}[\right.$ such that $x(\hat{t})=0$. By assumption, $-K \leq x_{\text {min }} \leq x(t) \leq 0$ for all $t \in\left[t_{0}, \hat{t}\right]$, while $x(t) \leq v(t)$ on $\left[\hat{t}, t_{1}\right]$, where $v$ is the solution of

$$
v^{\prime}=a_{0}(t) v+b_{0}(t), \quad v(\hat{t})=x(\hat{t})=0
$$

An integration of the linear equation on $\left[\hat{t}, t_{1}\right]$ yields to

$$
\begin{aligned}
x_{\max } & =x\left(t_{1}\right) \leq v\left(t_{1}\right)=\int_{\hat{t}}^{t_{1}} b_{0}(\xi) \exp (\mathcal{A}(t)-\mathcal{A}(\xi)) d \xi \\
& \leq \exp \left(\int_{0}^{T} a_{0}(t) d t\right) \int_{0}^{T} b_{0}(t) d t=K
\end{aligned}
$$

Hence, in any case, we can conclude that $-K \leq x_{\min } \leq x(t) \leq x_{\max } \leq K$, for all $t$ and the second inequality in (3) is verified.

At last, let us suppose that $x_{\text {min }}=y\left(t_{0}\right) \geq 0$, so that (5) takes the form

$$
y^{\prime}=a_{0}(t) y+b_{0}(t), \quad \text { for a.e. } t \in\left[t_{0}, t_{1}\right] .
$$

An integration of the linear equation yields to

$$
\begin{aligned}
y(t) & =y\left(t_{0}\right) \exp (\mathcal{A}(t))+\int_{t_{0}}^{t} b_{0}(\xi) \exp (\mathcal{A}(t)-\mathcal{A}(\xi)) d \xi \\
& \leq\left(y\left(t_{0}\right)+\mathcal{B}(t)\right) \exp (\mathcal{A}(t)) \leq\left(x_{\min }+\mathcal{B}(t)\right) \exp (\mathcal{A}(t))
\end{aligned}
$$

Therefore,

$$
x_{\max }=x\left(t_{1}\right) \leq y\left(t_{1}\right) \leq C x_{\min }+K
$$

and the third inequality in (3) is verified.
Finally, (4) follows straightforwardly from (3)
Remark 2.2. It is crucial to observe that the constants $C$ and $K$ in Lemma 2.1 depend only on $a_{0}$ and $b_{0}$ and do not depend on the function $\psi$ or the parameter $\lambda \in] 0,1]$.

For the main results of this section let us introduce the following definitions.
Definition 2.3. Let $\alpha \in W_{T}^{1,1}$. We say that $\alpha$ is a lower solution of (1) if

$$
\begin{equation*}
\alpha^{\prime}(t)+\psi(t, \alpha(t)) \leq 0, \quad \text { for a.e. } t \in[0, T] . \tag{6}
\end{equation*}
$$

If $\alpha$ is not a solution, we say that it is proper. In particular, if

$$
\begin{equation*}
\alpha^{\prime}(t)+\psi(t, \alpha(t))<0, \quad \text { for a.e. } t \in[0, T], \tag{7}
\end{equation*}
$$

we say that the lower solution $\alpha$ is strongly proper.
An upper solution of (1) is defined in the same manner, just by reversing the inequality in (6) (respectively in (7), when it is strongly proper). Given $u, v \in C_{T}$, we denote by $u \leq v$ if $u(t) \leq v(t)$ for all $t \in[0, T]$ and by $u \prec v$ if $u \leq v$ and $u \not \equiv v$.

In the next definition we recall Villari's conditions [30] which is presented here in a slightly modified form. For other generalizations in different contexts, we refer to $[4,11,21]$.

Definition 2.4. We say that $\psi(t, x)$ satisfies the Villari's condition at $-\infty$ (respectively, at $+\infty$ ) if, given $K_{1}>0$ and $K_{2}>0$, there exists a constant $d_{0}>0$ such that

$$
\exists \delta= \pm 1: \delta \int_{0}^{T} \psi(t, x(t)) d t>0
$$

for each $x \in C_{T}$ such that $x(t) \leq-d_{0}, \forall t \in[0, T]$ (respectively, $x(t) \geq d_{0}$, $\forall t \in[0, T])$ and $|x|_{\max } \leq K_{1}|x|_{\min }+K_{2}$.

Now we are in position to state the following.
Theorem 2.5. Let $\psi:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying $\left(H_{0}\right)$ and the Villari's condition at $-\infty$ with $\delta=1$. Suppose there exists $\alpha \in$ $W_{T}^{1,1}$ which is a strongly proper lower solution for equation (1). Then, (1) has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \prec \alpha$. Moreover, there exists $R_{0} \geq d_{0}$ such that any $T$-periodic solution of (1) with $x \leq \alpha$, satisfies $x(t)>-R_{0}$ for all $t \in[0, T]$.
Let us make a comment before proceeding with the proof of the theorem. In presence of a lower solution, one can expect to find a solution $\tilde{x} \geq \alpha$. Indeed, what we are going to do, is to treat $\alpha$ as an upper solution of the problem. Our notation is consistent with the one in [7, 25], nevertheless other authors overturn the terminology (cf. [24]). Actually, for Theorem 2.5 the terminology is not relevant and what matters is that $\alpha$ satisfies (7).

Proof. Following a standard approach, we define the truncated function

$$
\hat{\psi}(t, x):= \begin{cases}\psi(t, x) & \text { for } x \leq \alpha(t) \\ \psi(t, \alpha(t)) & \text { for } x \geq \alpha(t)\end{cases}
$$

and consider the parameter dependent equation

$$
\begin{equation*}
x^{\prime}+\lambda \hat{\psi}(t, x)=0, \quad 0<\lambda \leq 1 \tag{8}
\end{equation*}
$$

First of all, as a consequence of $\left(H_{0}\right)$, we remark that

$$
\hat{\psi}(t, x) \geq-a_{0}(t)|x|-b_{1}(t), \forall x \in \mathbb{R} \text { and a.e. } t \in[0, T]
$$

where $b_{1}(t)=b_{0}(t)+a_{0}(t)|\alpha(t)|$. Therefore $\hat{\psi}$ satisfies $\left(H_{0}\right)$, too. According to Lemma 2.1 (applied to $\hat{\psi}$ in place of $\psi$ ) any $T$-periodic solution $x$ of (8) satisfies

$$
|x|_{\max } \leq K_{1}|x|_{\min }+K_{2}
$$

for some suitable constants $K_{1} \geq 1$ and $K_{2}>0$ possibly depending in $\alpha$ but independent on $x$ and $\lambda$.

Next, we choose a constant $d_{1} \geq d_{0}$ with $d_{1}>\|\alpha\|_{\infty}$ and we claim that $\max x>-d_{1}$. Indeed, if we suppose by contradiction that $x(t) \leq-d_{1}$ for all $t \in[0, T]$, then $x(t)<\alpha(t)$ for all $t \in[0, T]$ and so $x(t)$ is a $T$-periodic solution of (2). Hence, an integration on $[0, T]$ of (2) (divided by $\lambda>0$ ), yields to $\int_{0}^{T} \psi(t, x(t)) d t=0$, which clearly contradicts Villari's condition at $-\infty$ as $-d_{1} \leq-d_{0}$. Having proved that $x(t)>-d_{1}$ for some $t \in[0, T]$ and hence $\max x>-d_{1}$, we obtain that

$$
\min x>-R_{0}, \quad \text { for } \quad R_{0}:=K_{1} d_{1}+K_{2} .
$$

Now, we claim that there exists $\bar{t} \in[0, T]$ such that $x(\bar{t})<\alpha(\bar{t})$. If, by contradiction, $x(t) \geq \alpha(t)$ for all $t \in[0, T]$, then $x$ is a $T$-periodic solution of $x^{\prime}+\lambda \psi(t, \alpha(t))=0$, for $0<\lambda \leq 1$ and then an integration on $[0, T]$ of this equation (divided by $\lambda>0$ ), yields to $\int_{0}^{T} \psi(t, \alpha(t)) d t=0$. On the other hand, an integration of $(7)$ on $[0, T]$ gives $\int_{0}^{T} \psi(t, \alpha(t)) d t<0$, thus a contradiction. Having proved that $x(t)<\|\alpha\|_{\infty}$ for some $t \in[0, T]$ and hence $\min x<\|\alpha\|_{\infty}$, we can also deduce that

$$
\max x<K_{1}\|\alpha\|_{\infty}+K_{2} .
$$

Writing equation

$$
\begin{equation*}
-x^{\prime}=\hat{\psi}(t, x) \tag{9}
\end{equation*}
$$

as a coincidence equation of the form $L x=\hat{N} x$ in the space $C_{T}$, from the a priori bounds, we find that the coincidence degree $D_{L}(L-\hat{N}, \mathcal{O})$ is well defined for any open and bounded set $\mathcal{O} \subset C_{T}$ of the form

$$
\mathcal{O}:=\left\{x \in C_{T}:-R^{-}<x(t)<R^{+}, \forall t \in[0, T]\right\}
$$

where $R^{-} \geq R_{0}, R^{+} \geq K_{1}\|\alpha\|_{\infty}+K_{2}$.
As a last step, we consider the averaged scalar map

$$
\hat{\psi}^{\#}: \mathbb{R} \rightarrow \mathbb{R}, \quad \hat{\psi}^{\#}(\xi):=\frac{1}{T} \int_{0}^{T} \hat{\psi}(t, \xi) d t, \quad \forall \xi \in \mathbb{R}
$$

We have $-\left.J Q \hat{N}\right|_{\text {ker } L}=-\hat{\psi}^{\#}$ and $\hat{\psi}^{\#}\left(-R^{-}\right)>0>\hat{\psi}^{\#}\left(R^{+}\right)$.
In more detail, since $R^{-} \geq d_{1}$, the first inequality follows from Villari's condition, while $\int_{0}^{T} \psi(t, \alpha(t)) d t<0$ and the choice $R^{+} \geq\|\alpha\|_{\infty}$, imply the second inequality. An application of Theorem 5.1 guarantees that $D_{L}(L-\hat{N}, \mathcal{O})=1$ and hence equation (9) has a $T$-periodic solution $\tilde{x}$ with $-R^{-}<\tilde{x}(t)<R^{+}$, for all $t \in[0, T]$.

In order to conclude, we check that $\tilde{x} \prec \alpha$. This is a standard fact, however we give the details for the reader's convenience. From the previous part of the proof we already know that any $T$-periodic solution of (8) is below $\alpha$, at least for some $t$, thus the same must occur for $\tilde{x}$. Let $t_{*}$ be such that $\tilde{x}\left(t_{*}\right)<\alpha\left(t_{*}\right)$. Suppose, by contradiction, that there exists a $t^{*}$ such that $\tilde{x}\left(t^{*}\right)>\alpha\left(t^{*}\right)$. Let $\left[t_{1}, t_{2}\right]$ be such that $t_{1}<t^{*}<t_{2}$ with $v(t)>0$ for all $\left.t \in\right] t_{1}, t_{2}[$ and, moreover, $v\left(t_{1}\right)=v\left(t_{2}\right)=0$. On the interval $\left[t_{1}, t_{2}\right]$, we have that $\tilde{x}^{\prime}(t)+\psi(t, \alpha(t))=0$ and hence, recalling (7), we find that $v^{\prime}(t)>0$, for a.e. $t \in\left[t_{1}, t_{2}\right]$. An integration on $\left[t_{1}, t_{2}\right]$ gives immediately a contradiction. We have thus proved that $\tilde{x}(t) \leq \alpha(t)$ for all $t \in[0, T]$ and therefore $\tilde{x}$ is a $T$-periodic solution of (1) satisfying $\tilde{x} \leq \alpha$. Moreover, since $\alpha$ is proper, we conclude that $\tilde{x} \prec \alpha$.

Remark 2.6. Notice that, under additional hypothesis ensuring that the $T$ periodic solutions $x$ with $x \leq \alpha$ are such that $x \ll \alpha$, namely $x(t)<\alpha(t)$ for all $t$, we can also prove that:
there exist $R_{0} \geq d_{0}$ such that for each $R>R_{0}$, we have $D_{L}(L-N, \Omega)=1$ for $\Omega=\left\{x \in C_{T}:-R<x(t)<\alpha(t) \forall t \in[0, T]\right\}$.
A possible additional hypothesis guaranteeing $x \ll \alpha$ could be
(A) For all $t_{0} \in[0, T]$ and $u_{0} \in \mathbb{R}$ and $\varepsilon>0$, there exists $\delta>0$ such that $\left|t-t_{0}\right|<\delta,\left|u-u_{0}\right|<\delta \Rightarrow\left|\psi(t, u)-\psi\left(t, u_{0}\right)\right|<\varepsilon$.

Observe that $(A)$ is always satisfied when $\psi$ is continuous. Such kind of conditions are widely discussed in [6] for second order equations.

We propose now a dual version Theorem 2.5 whose proof can be obtained via minor changes.

Theorem 2.7. Let $\psi:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying $\left(H_{0}\right)$ and the Villari's condition at $+\infty$ with $\delta=1$. Suppose there exists $\alpha \in$ $W_{T}^{1,1}$ which is a strongly proper lower solution for equation (1). Then, (1) has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \succ \alpha$. Moreover, there exists $R_{0} \geq d_{0}$ such that any $T$-periodic solution of (1) with $x \geq \alpha$, satisfies $x(t)<R_{0}$ for all $t \in[0, T]$.

Proof. We define the truncated function

$$
\hat{\psi}(t, x):= \begin{cases}\psi(t, x) & \text { for } x \geq \alpha(t) \\ \psi(t, \alpha(t)) & \text { for } x \leq \alpha(t)\end{cases}
$$

and consider the parameter dependent equation (8). The proof now follows the same scheme as that of Theorem 2.5 till to the introduction of an open bounded set $\mathcal{O}:=\left\{x \in C_{T}:-S^{-}<x(t)<S^{+}, \forall t \in[0, T]\right\}$ where $S^{-}$and $S^{+}$are suitable constants obtained similarly as $R^{-}$and $R^{+}$. In this case, one can compute the coincidence degree and find that $D_{L}(L-\hat{N}, \mathcal{O})=-1$, thus ensuring the existence of a $T$-periodic solution $\tilde{x} \in \mathcal{O}$. Finally, by the same argument as above, we prove that $\tilde{x} \succ \alpha$.

It is a well-known fact (cf. [14]), that results like Theorem 2.5 or Theorem 2.7, obtained by using strict inequalities, can be relaxed by considering weak inequalities. Accordingly, from Theorem 2.5, the following result holds.
Corollary 2.8. Let $\psi:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function satisfying $\left(H_{0}\right)$ and such that, given $K_{1}>0$ and $K_{2}>0$, there exists $d_{0}>0$ for which $\int_{0}^{T} \psi(t, x(t)) d t \geq 0$ for each $x \in C_{T}$ with $x(t) \leq-d_{0}, \forall t \in[0, T]$ and $|x|_{\max } \leq$ $K_{1}|x|_{\text {min }}+K_{2}$. Suppose there exists a lower solution $\alpha \in W_{T}^{1,1}$ for equation (1). Then, (1) has at least a T-periodic solution $\tilde{x}$ such that $\tilde{x} \leq \alpha$.

Proof. We introduce the auxiliary functions

$$
\ell(x):=\max \left\{-1,-x-\|\alpha\|_{\infty}-1\right\}, \quad \psi^{\varepsilon}(t, x):=\psi(t, x)+\varepsilon \ell(x), \varepsilon>0
$$

and apply Theorem 2.5 to equation $x^{\prime}+\psi^{\varepsilon}(t, x)=0$. Moreover, one can easily check that the constant $R_{0}$ can be taken uniformly with respect to $\varepsilon$. The conclusion then follows via Ascoli-Arzelà theorem.

A corollary similar to the above one can be stated with respect to Theorem 2.7.

## 3. Existence and multiplicity theorems

Here we discuss the number of $T$-periodic solutions for the parameter dependent equation

$$
\begin{equation*}
x^{\prime}+g(t, x)=s \tag{s}
\end{equation*}
$$

Throughout this section we suppose that $g:[0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions.

Moreover, in the sequel, the following hypotheses will be considered:
$\left(G_{0}\right) \exists a_{0}, b_{0} \in L^{1}\left([0, T], \mathbb{R}^{+}\right): g(t, x) \geq-a_{0}(t)|x|-b_{0}(t), \forall x \in \mathbb{R}$ and a.e. $t \in[0, T] ;$
$\left(G_{1}\right) \exists x_{0}, g_{0} \in \mathbb{R}: g\left(t, x_{0}\right) \leq g_{0}$ for a.e. $t \in[0, T] ;$
$\left(G_{2}^{-}\right)$given $K_{1}>0$ and $K_{2}>0$, for each $\sigma$ there exists $d_{\sigma}>0$ such that $\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t>\sigma$ for each $x \in C_{T}$ such that $x(t) \leq-d_{\sigma}$ for all $t \in[0, T]$ and $|x|_{\max } \leq K_{1}|x|_{\min }+K_{2}$;
$\left(G_{2}^{+}\right)$given $K_{1}>0$ and $K_{2}>0$, for each $\sigma$ there exists $d_{\sigma}>0$ such that $\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t>\sigma$ for each $x \in C_{T}$ such that $x(t) \geq d_{\sigma}$ for all $t \in[0, T]$ and $|x|_{\max } \leq K_{1}|x|_{\text {min }}+K_{2}$.

Theorem 3.1. Assume $\left(G_{0}\right)$, $\left(G_{1}\right)$ and, either $\left(G_{2}^{-}\right)$or $\left(G_{2}^{+}\right)$. Then, there exists $s_{0} \in \mathbb{R} \cup\{-\infty\}$ such that for every $s>s_{0}$ equation $\left(\mathscr{E}_{s}\right)$ has at least one $T$-periodic solution.

Proof. For any given parameter $s \in \mathbb{R}$, we set

$$
\psi_{s}(t, x):=g(t, x)-s,
$$

so that equation $\left(\mathscr{E}_{s}\right)$ is of the form (1). Just to fix a case, let us suppose that $\left(G_{2}^{-}\right)$holds.

We start by choosing a parameter $s_{1}>g_{0}$. In this situation, the constant function $\alpha(t) \equiv x_{0}$ is a strongly proper lower solution. Indeed, we have

$$
\alpha^{\prime}(t)+g(t, \alpha(t))-s_{1}=g\left(t, x_{0}\right)-s_{1} \leq-\left(s_{1}-g_{0}\right)<0
$$

On the other hand, for $\sigma=s_{1}$, condition $\left(G_{2}^{-}\right)$implies the Villari's condition at $-\infty$ with $\delta=1$. Hence, an application of Theorem 2.5 guarantees the existence of at least one $T$-periodic solution $x$ of $\left(\mathscr{E}_{S_{1}}\right)$ with $x \prec x_{0}$.

Next, we claim that if, for some $\tilde{s}<s_{1}$ the equation has a $T$-periodic solution (that we will denote by $w$ ), then equation $\left(\mathscr{E}_{s}\right)$ has a $T$-periodic solution for each $s \in] \tilde{s}, s_{1}\left[\right.$. We write equation $\left(\mathscr{E}_{s}\right)$ as

$$
x^{\prime}+g(t, x)-\tilde{s}-(s-\tilde{s})=0
$$

so that $\alpha(t) \equiv w(t)$ is a strongly proper lower solution of $\left(\mathscr{E}_{s}\right)$. Indeed, we have

$$
\alpha^{\prime}(t)+g(t, \alpha(t))-s=w^{\prime}(t)+g(t, w(t))-s=-(s-\tilde{s})<0
$$

On the other hand, for $\sigma=s$, condition $\left(G_{2}^{-}\right)$implies the Villari's condition at $-\infty$ with $\delta=1$. An application of Theorem 2.5 guarantees the existence of at least one $T$-periodic solution $x$ of $\left(\mathscr{E}_{s}\right)$ with $x \prec w$ and the claim is proved.

Since we can take $s_{1}$ arbitrarily large, we conclude that the set of the parameters $s$ for which equation $\left(\mathscr{E}_{s}\right)$ has $T$-periodic solutions is an interval $\mathcal{J}$ with $\sup \mathcal{J}=+\infty$. Setting

$$
s_{0}:=\inf \left\{s \in \mathbb{R}:\left(\mathscr{E}_{s}\right) \text { has at least one } T \text {-periodic solution }\right\} \in \mathbb{R} \cup\{-\infty\}
$$

the thesis follows. The same argument applies if, instead of $\left(G_{2}^{-}\right)$, we assume $\left(G_{2}^{+}\right)$and apply Theorem 2.7.

Remark 3.2. Let us make some comments that arise from Theorem 3.1. The first one is about the critical parameter $s_{0}$. Without supplementary conditions, we cannot say, a priori, whether $s_{0}=-\infty$ or $s_{0} \in \mathbb{R}$ and, in this latter case, if the equation ( $\mathscr{E}_{s_{0}}$ ) has $T$-periodic solutions. Simple examples can be provided for each of these cases. However, from the proof, it is clear that $s_{0} \leq g_{0}$. As a second comment, we observe that the Villari's conditions $\left(G_{2}^{ \pm}\right)$guarantee the existence of upper solutions. In fact, suppose that $w$ is a $T$-periodic solution of $\left(\mathscr{E}_{s_{1}}\right)$ for some $s_{1}>g_{0}$. Then $\beta(t) \equiv w(t)$ is a strongly proper upper solution of $\left(\mathscr{E}_{s}\right)$ for any $s<s_{1}$. Indeed, we have $\beta^{\prime}(t)+g(t, \beta(t))-s=w^{\prime}(t)+g(t, w(t))-s=$ $s_{1}-s>0$. Hence, a posteriori along the proof, we have discovered that for $s \in] g_{0}, s_{1}[$, there are both a strongly proper upper solution $\beta$ and a strongly proper lower solution $\alpha$ with $\beta \prec \alpha$ or $\alpha \prec \beta$, according to the assumption $\left(G_{2}^{-}\right)$or $\left(G_{2}^{+}\right)$, respectively. Thus we enter in the setting of [25] where a detailed analysis is performed about continua of $T$-periodic solutions and their stability.

The previous result concerns the case in which the conditions $\left(G_{2}^{ \pm}\right)$are applied in a separately way. The next theorem considers the situations in which Villari's conditions hold at the same time.

Theorem 3.3. Assume $\left(G_{0}\right)$, $\left(G_{1}\right)$, ( $\left.G_{2}^{-}\right)$and $\left(G_{2}^{+}\right)$. Then there exists $s_{0} \in \mathbb{R}$ such that:
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has no T-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least one T-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has at least two T-periodic solutions.
Proof. Without loss of generality, we can suppose that the map $\sigma \mapsto d_{\sigma}$ is defined on $[0,+\infty)$ and is monotone non-decreasing.

We claim that there exists a constant $\nu_{0} \leq 0$ such that, if $s<\nu_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has no $T$-periodic solution.

Indeed, let $x$ be a $T$-periodic solution of $\left(\mathscr{E}_{s}\right)$ for any $s \leq 0$. The function $\psi_{s}(t, x)=g(t, x)-s$ satisfies condition $\left(H_{0}\right)$, uniformly for $s \leq 0$. Hence, according to Lemma 2.1 and Remark 2.2, there exist two constants $C \geq 1$ and $K>0$ such that (4) holds for each $T$-periodic solution of $\left(\mathscr{E}_{s}\right)$. Consider now condition $\left(G_{2}^{+}\right)$that we read now for $\sigma=0$ and $K_{1}=C, K_{2}=K$. It implies that if $x(t) \geq d_{0}$ for all $t \in[0, T]$, then $\int_{0}^{T} g(t, x(t)) d t>0$. On the other hand, $x^{\prime}+g(t, x)=s \leq 0$ and a contradiction follows. This implies that $x_{\min }<d_{0}$. In the same manner, using $\left(G_{2}^{-}\right)$for $\sigma=0$ and $K_{1}=C, K_{2}=K$, we can prove that $x_{\max }>-d_{0}$. In conclusion, we have proved that $|x|_{\min }<d_{0}$. Therefore, from (4) we find that

$$
\begin{equation*}
|x|_{\max }<R^{*}:=C d_{0}+K \tag{10}
\end{equation*}
$$

We stress the fact that (10) holds for any possible Tperiodic solution of ( $\mathscr{E}_{s}$ ) with $s \leq 0$. Now, let $\rho \in L^{1}\left([0, T], \mathbb{R}^{+}\right)$be such that

$$
|g(t, \xi)| \leq \rho(t), \quad \forall \xi \in\left[-R^{*}, R^{*}\right] \quad \text { and a.e. } t \in[0, T]
$$

Let us consider again $x^{\prime}+g(t, x)=s$ with $s \leq 0$. Integrating the equation on $[0, T]$, we have

$$
s T=\int_{0}^{T} g(t, x(t)) d t \geq-\|\rho\|_{1}
$$

We have thus proved that if there exists a $T$-periodic solution of $\left(\mathscr{E}_{s}\right)$ for $s \leq 0$, then, necessarily

$$
s \geq \nu_{0}:=-\frac{1}{T}\|\rho\|_{1}
$$

Hence, if $s<\nu_{0}$, equation $\left(\mathscr{E}_{s}\right)$ has no $T$-periodic solution. The claim is proved.

After this preliminary observation, we proceed now as in the proof of Theorem 3.1. We fix (arbitrarily) $s_{1}>g_{0}$ and using $\left(G_{2}^{-}\right)$, as well as $\left(G_{2}^{+}\right)$, we prove the existence of at least two $T$-periodic solutions $x^{(-)}$and $x^{(+)}$with $x^{(-)} \prec x_{0} \prec x^{(+)}$.

Next, we claim that if, for some $\tilde{s}<s_{1}$ the equation has a $T$-periodic solution (that we will denote by $w$ ), then equation $\left(\mathscr{E}_{s}\right)$ has at least two $T$ periodic solutions for each $s \in] \tilde{s}, s_{1}[$.

We write equation $\left(\mathscr{E}_{s}\right)$ as

$$
x^{\prime}+g(t, x)-\tilde{s}-(s-\tilde{s})=0
$$

so that $\alpha(t) \equiv w(t)$ is a strongly proper lower solution of $\left(\mathscr{E}_{s}\right)$ (as in Theorem 3.1). On the other hand, for $\sigma=s$, condition $\left(G_{2}^{-}\right)$implies the Villari's condition at $-\infty$ with $\delta=1$ and, similarly, $\left(G_{2}^{+}\right)$implies the Villari's condition at $+\infty$ with $\delta=1$ An application of Theorem 2.5 and Theorem 2.7 guarantees the existence of at least one $T$-periodic solution $u^{(-)}$of $\left(\mathscr{E}_{s}\right)$ with $u^{(-)} \prec w$ and the existence of at least one $T$-periodic solution $u^{(+)}$of $\left(\mathscr{E}_{s}\right)$ with $u^{(+)} \succ w$. Clearly, $u^{(-)} \not \equiv u^{(+)}$.

Since we can take $s_{1}$ arbitrarily large, we conclude that the set of the parameters $s$ for which equation $\left(\mathscr{E}_{s}\right)$ has $T$-periodic solutions is an interval $\mathcal{J}$ with $\sup \mathcal{J}=+\infty$. Setting

$$
s_{0}:=\inf \left\{s \in \mathbb{R}:\left(\mathscr{E}_{s}\right) \text { has at least one } T \text {-periodic solution }\right\} \in \mathbb{R} \cup\{-\infty\}
$$

we know that $s_{0}$ is finite, indeed, $\nu_{0} \leq s_{0} \leq g_{0}$. Moreover, by the above discussion, we also know, that for each $s>s_{0}$ equation ( $\mathscr{E}_{s}$ ) has at least two $T$-periodic solutions. By construction, we also know that for $s<s_{0}$, there is no $T$-periodic solution for $\left(\mathscr{E}_{s}\right)$.

To conclude the proof, we have to check that for $s=s_{0}$ there is at least one $T$-periodic solution. This will be achieved following an argument borrowed from [8]. Let $s_{2}<s_{0}<s_{1}$ be fixed and let $\theta_{n}$ be a decreasing sequence of parameters with $\theta_{n} \rightarrow s_{0}$ and $\left.\left.\theta_{n} \in\right] s_{0}, s_{1}\right]$ for all $n$. By the estimates developed previously, we know that, for each $n$ there exists at least one (actually two) $T$-periodic solution $w_{n}$ of equation $x^{\prime}+g(t, x)=\theta_{n}$ with $\left\|w_{n}\right\|_{\infty} \leq M$, where $M$ is a uniform a priori bound obtained as $R^{*}$ in (10). An application of the Ascoli-Arzelà theorem, passing to the limit as $n \rightarrow \infty$, provides the existence of at least one $T$-periodic solution of $\left(\mathscr{E}_{s}\right)$ for $s=s_{0}$. This completes the proof.

Remark 3.4. Notice that assuming the Villari's condition ( $G V$ ) is equivalent to require both $\left(G_{2}^{-}\right)$and $\left(G_{2}^{+}\right)$. As in [17, Remark 2], we also observe that all the results remain true if $s$ in $\left(\mathscr{E}_{s}\right)$ is replaced by $s \varphi(t)$ with $\varphi \in L^{\infty}(0, T)$ and positive (i.e. essinf $\varphi>0$ ).

## 4. Applications

In this section we show a few applications of the preceding theorems in order to treat some classical examples in literature. In particular, we focus our attention to consequences of Theorem 3.3.

As a first example, we consider the periodic problem associated with

$$
\begin{equation*}
x^{\prime}+\gamma(t) \phi(x)=s+p(t) . \tag{E}
\end{equation*}
$$

In this case, a multiplicity result reads as follow.
Corollary 4.1. Let $\phi: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and suppose that

$$
\lim _{|x| \rightarrow \infty} \phi(x)=+\infty
$$

Let $\gamma, p \in L^{\infty}(0, T)$ with $\gamma(t) \geq 0$ for a.e. $t \in[0, T]$ and $\int_{0}^{T} \gamma(t) d t>0$. Then, there exists $s_{0} \in \mathbb{R}$ such that:
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{W}_{\mathscr{E}}\right)$ has no $T$-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{W}_{\mathscr{E}}^{s}\right.$ ) has at least one $T$-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{W}_{\mathscr{E}}^{s}\right.$ ) has at least two $T$-periodic solutions.
Proof. We apply Theorem 3.3 for

$$
g(t, x):=\gamma(t) \phi(x)-p(t)
$$

Let us set $\phi_{0}:=\min _{\xi \in \mathbb{R}} \phi(\xi)$. For any $d>\max \left\{\phi_{0}, 0\right\}$, we introduce the following constants:

$$
\zeta^{-}(d):=\min \{\phi(x): x \leq-d\}, \quad \zeta^{+}(d):=\min \{\phi(x): x \geq d\}
$$

From $\left(H_{\phi}\right)$ it follows that $\zeta^{ \pm}(d) \rightarrow+\infty$ for $d \rightarrow+\infty$.
Let $x \in C_{T}$ be such that $|x(t)| \geq d>0$ for all $t \in[0, T]$. If $x(t) \leq-d, \forall t$, then

$$
\begin{aligned}
\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t & =\frac{1}{T} \int_{0}^{T} \gamma(t) \phi(x(t)) d t-\frac{1}{T} \int_{0}^{T} p(t) d t \\
& \geq \frac{\zeta^{-}(d)}{T} \int_{0}^{T} \gamma(t) d t-\frac{1}{T} \int_{0}^{T} p(t) d t
\end{aligned}
$$

In the other case, if $x(t) \geq d, \forall t$, then

$$
\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t \geq \frac{\zeta^{+}(d)}{T} \int_{0}^{T} \gamma(t) d t-\frac{1}{T} \int_{0}^{T} p(t) d t
$$

Hence, the Villari's condition $(G V)$ is satisfied by the properties of $\zeta^{ \pm}(d)$.
Hypothesis $\left(G_{0}\right)$ is satisfied by choosing as $b_{0}(t)$ the positive part of $p(t)-$ $\gamma(t) \phi_{0}$ and $a_{0} \equiv 0$. Also $\left(G_{1}\right)$ holds for $x_{0}=0$ and any constant $g_{0} \geq$ $\|\gamma\|_{\infty} \phi(0)+\|p\|_{\infty}$. Now, an application of Theorem 3.3 gives the result.

Corollary 4.1 extends [24, Corollary 3.1], where the periodic problem for equation ( $\mathscr{W}_{\mathscr{E}}^{s}$ ) was considered only for $\gamma \equiv 1$.

Our second example deals with a generalized Riccati equation of the form

$$
\begin{equation*}
x^{\prime}+\gamma_{p}(t)|x|^{p}+\gamma_{1}(t) x+\gamma_{0}(t)=s \tag{s}
\end{equation*}
$$

Also in this case a multiplicity result can be stated.
Corollary 4.2. Let $\gamma_{0} \in L^{\infty}(0, T)$ and $\gamma_{1}, \gamma_{p} \in L^{1}(0, T)$, with $\gamma_{p}(t) \geq 0$ for a.e. $t \in[0, T]$ and $\int_{0}^{T} \gamma_{p}(t) d t>0$. Then, there exists $s_{0} \in \mathbb{R}$ such that:
$1^{\circ}$ for $s<s_{0}$, equation $\left(\mathscr{R}_{\mathscr{E}}\right)$ has no $T$-periodic solutions;
$2^{\circ}$ for $s=s_{0}$, equation $\left(\mathscr{R}_{s}\right)$ has at least one $T$-periodic solution;
$3^{\circ}$ for $s>s_{0}$, equation $\left(\mathscr{R}_{\mathscr{E}_{s}}\right)$ has at least two T-periodic solutions.
Proof. We show, that all the hypotheses of Theorem 3.3 are fulfilled for

$$
g(t, x):=\gamma_{p}(t)|x|^{p}+\gamma_{1}(t) x+\gamma_{0}(t)
$$

Condition $\left(G_{0}\right)$ holds for $a_{0}:=\left|\gamma_{1}\right|$ and $b_{0}:=\left|\gamma_{0}\right|$. Concerning hypothesis $\left(G_{1}\right)$ we observe that it is satisfied with $x_{0}=0$ and $g_{0} \geq\left\|\gamma_{0}\right\|_{\infty}$. Finally, we verify the validity of the Villari's condition $(G V)$. Let us suppose that $K_{1} \geq 1$ and $K_{2}>0$ are fixed and $x \in C_{T}$ is such that $|x|_{\max } \leq K_{1}|x|_{\min }+K_{2}$.

$$
\begin{aligned}
\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t & =\frac{1}{T} \int_{0}^{T}\left(\gamma_{p}(t)|x(t)|^{p}+\gamma_{1}(t) x(t)+\gamma_{0}(t)\right) d t \\
& \geq|x|_{\min }^{p}\left\|\gamma_{p}\right\|_{1}-|x|_{\max }\left\|\gamma_{1}\right\|_{1}-\left\|\gamma_{0}\right\|_{1} \\
& \geq|x|_{\min }^{p}\left\|\gamma_{p}\right\|_{1}-|x|_{\min } K_{1}\left\|\gamma_{1}\right\|_{1}-K_{2}\left\|\gamma_{1}\right\|_{1}-\left\|\gamma_{0}\right\|_{1}
\end{aligned}
$$

Therefore,

$$
\frac{1}{T} \int_{0}^{T} g(t, x(t)) d t \rightarrow+\infty, \quad \text { as } \quad|x|_{\min } \rightarrow+\infty
$$

so that $(G V)$ is satisfied.
REmark 4.3. The nonlinear term $\gamma_{p}(t)|x|^{p}+\gamma_{1}(t) x+\gamma_{0}(t)$ in equation ( $\mathscr{R}_{\mathscr{E}}^{s}$ ) is convex in $x$ (and strictly on a set of positive measure). We can then apply a result of Mawhin in [17, Proposition 3] which guarantees that there are at most two $T$-periodic solutions for each $s \in \mathbb{R}$. As a consequence, in the situation of

Corollary 4.2, we conclude that for each $s>s_{0}$ equation $\left(\mathscr{R}_{\mathscr{E}}{ }_{s}\right)$ has exactly two $T$-periodic solutions $x^{(-)}<x^{(+)}$. Moreover, $x^{(+)}$is asymptotically stable and $x^{(-)}$is unstable (cf. [25]). Figure 1 shows an example for this case. The same conclusion holds also for Corollary 4.1 if we assume that $\phi$ is strictly convex.

(a) The four solutions in the interval $[-60,0]$ show evidence of the presence of an unstable periodic solution.

(b) The four solutions in the interval [ 0,120 ] show evidence of an asymptotically stable periodic solution.

Figure 1: A numerical simulation for equation $\left(\mathscr{R}_{\mathscr{G}}\right)$. The example is obtained for $\gamma_{2}(t)=\max \{0, \sin t-0.9\}, \gamma_{1}(t)=\cos t, \gamma_{0}(t)=0, p=1.1$ and $s=1$. We have considered the solutions corresponding to four initial points $x(0)=$ $-90,-50$ (magenta), 0 (black), 120 . Consistently with Remark 4.3 we give evidence of two $2 \pi$-periodic solutions.

## 5. Appendix: Mawhin's coincidence degree

For the reader's convenience, we briefly recall here a few basic facts from coincidence degree theory which are used in the present paper. We refer to [10, 15, 19] for the general theory.

Let $X, Z$ be real normed spaces and let $\Omega$ be an open bounded set in $X$. We consider a coincidence equation of the form

$$
\begin{equation*}
L x=N x, \quad x \in \operatorname{dom} L \cap \Omega, \tag{11}
\end{equation*}
$$

where $L: X \supseteq \operatorname{dom} L \rightarrow Z$ is a linear (non-invertible) Fredholm mapping of index zero and $N: X \rightarrow Z$ is a nonlinear operator. We also consider two linear and continuous projections $P: X \rightarrow \operatorname{ker} L$ and $Q: Z \rightarrow \operatorname{Im} L$, as well as, the (continuous) right inverse of $L$, denoted by $K_{P}: \operatorname{Im} L \rightarrow \operatorname{dom} L \cap X_{0}$, where $X_{0}:=\operatorname{ker} P \equiv X / \operatorname{ker} L$ is a complementary subspace of $\operatorname{ker} L$ in $X$. In this manner (11) is equivalent to the fixed point problem

$$
\begin{equation*}
x=\Phi(x):=P x+J Q N x+K_{P}(I-Q) N x, \quad x \in \Omega, \tag{12}
\end{equation*}
$$

where $J: \operatorname{coker} L=\operatorname{Im} Q \equiv Z / \operatorname{Im} L \rightarrow \operatorname{ker} L$ is a linear isomorphism. We further suppose that $N$ is a continuous operator which maps bounded sets to bounded sets and such that, for any bounded set $B$ in $X$, the set $K_{P}(I-Q) N(B)$ is relatively compact (i.e., $N$ is $L$-completely continuous [19]). As a consequence, the operator $\Phi$, defined in (12), is completely continuous, too.

If we suppose that

$$
L x \neq N x, \quad \forall x \in \operatorname{dom} L \cap \partial \Omega
$$

then also $I-\Phi$ never vanishes on $\partial \Omega$ and, therefore, we can define the coincidence degree

$$
D_{L}(L-N, \Omega):=\operatorname{deg}(I-\Phi, \Omega, 0)
$$

where "deg" denotes the Leray-Schauder degree. Notice that, usually one defines the coincidence degree with absolute value, namely $\left|D_{L}(L-N, \Omega)\right|=$ $|\operatorname{deg}(I-\Phi, \Omega, 0)|$ in order to make the degree independent from the choice of the projections $P, Q$, the isomorphism $J$ and the orientations of $\operatorname{ker} L$ and $\operatorname{coker} L$ (see [19]). In our applications no sign ambiguity will arise because we fix the natural orientations on $\operatorname{ker} L$ and $\operatorname{coker} L$, which are identified by $\mathbb{R}$ and we choose $P, Q$ and $J$ in an obvious way.

If we denote by " $\operatorname{deg}_{B}$ " the (finite dimensional) Brouwer degree, then, according to Mawhin's continuation theorem (see [12, 13]), the following result holds.

Theorem 5.1. Let $L$ and $N$ be as above and let $\Omega \subseteq X$ be an open and bounded set. Suppose that $L x \neq \lambda N x, \forall x \in \operatorname{dom} L \cap \partial \Omega, \forall \lambda \in] 0,1]$ and $Q N(x) \neq 0$, $\forall x \in \partial \Omega \cap \operatorname{ker} L$. Then,

$$
D_{L}(L-N, \Omega)=\operatorname{deg}_{B}\left(-\left.J Q N\right|_{\operatorname{ker} L}, \Omega \cap \operatorname{ker} L, 0\right)
$$

As a consequence, if $\operatorname{deg}_{B}\left(-\left.J Q N\right|_{\operatorname{ker} L}, \Omega \cap \operatorname{ker} L, 0\right) \neq 0$, then (11) has at leat one solution.

We also point out that the classical properties of the Leray-Schauder degree, such as additivity/excision, homotopic invariance, hold also in the coincidence degree framework.

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# On sheaves of differential operators 

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#### Abstract

Given a $\mathscr{C}^{\infty}$ manifold $X$, denote by $\mathscr{C}_{X}^{m}$ the sheaf of mtimes differentiable real-valued functions and by $\mathscr{D}_{X}^{m, r}$ the sheaf of differential operators of order $\leq m$ with coefficient functions of class $\mathscr{C}^{r}$. We prove that the natural morphism $\mathscr{D}_{X}^{m-r, r} \longrightarrow \mathscr{H}$ om $_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)$ is an isomorphism.


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## 1. Introduction

Sheaves were invented by Jean Leray [6] as a special mathematical tool which provides a unified approach for establishing connections between local and global properties of topological spaces (in particular geometric objects). It is a powerful method for studying many problems in contemporary algebra, geometry, topology, and analysis (see [5] for more details and references therein). There are many natural examples of sheaves [5].

Leray defined cohomology groups for continuous maps, and related them to the cohomology of the source space by means of the spectral sequence that was introduced for this purpose. Henri Cartan reformulated sheaf theory and, together with Jean-Pierre Serre, gave striking applications to the theory of analytic spaces in their seminal work [2]. Subsequently Serre, and Grothendieck extended these methods to algebraic geometry. Indeed, the latter's use of schemes led to a complete reconceptualization of the subject and the development of new and powerful methods. Finally Sato introduced $D$-modules, creating micro-local analysis (see [9] and any references therein). For this reason it seems natural to apply this theory to differential operators.

In this paper, we investigate the relationship between the sheaf of linear differential operators that satisfies a certain condition to be given in Section 2 and the sheaf of $\mathbb{R}$-linear morphisms of certain sheaves.

The paper is organized as follows. In Section 2, we recall some basic definitions and state the main theorem. Finally, we prove in Section 3 the main theorem by cases.

## 2. Basic Facts and Main Theorem

Let $X$ be an $n$-dimensional $\mathscr{C}^{\infty}$-manifold and $m$ a nonnegative integer. We denote by $\mathscr{C}_{X}^{m}$ the sheaf of real-valued functions of class $\mathscr{C}^{m}$ on $X$. Furthermore, for $0 \leq r \leq \infty$, we denote by $\mathscr{D}_{X}^{m, r}$ the sheaf of differential operators of order $\leq m$ with coefficients of class $\mathscr{C}^{r}$. Note that, for any nonnegative integer $r$, the sheaf $\mathscr{D}_{X}^{0, r}$ coincide with the sheaf $\mathscr{C}_{X}^{r}$, i.e.

$$
\mathscr{D}_{X}^{0, r}=\mathscr{C}_{X}^{r}
$$

As is usually the case in the literature, we recall that $\mathbb{R}_{X}$ denotes the constant sheaf on the $\mathscr{C}^{\infty}$ manifold $X$, and $\mathscr{C}_{X}^{\infty}$ denotes the sheaf of $C^{\infty}$ realvalued functions on $X$.

Moreover, we also recall that, for any local coordinate system $\left(x_{i}\right)_{1 \leq i \leq n}$ of $X$, a section $P$ of the sheaf $\mathscr{D}_{X}^{m, r}$ on $U$, is given by (see [3, p. 13])

$$
\begin{equation*}
P=\sum_{|\alpha| \leq m} a_{\alpha}(x) \partial_{x}^{\alpha} \tag{1}
\end{equation*}
$$

where $a_{\alpha}$ are real-valued functions of class $\mathscr{C}^{r}$.
In (1), $\alpha$ stands for the multi-index $\alpha:=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, where, for every $1 \leq i \leq n, \alpha_{i} \in\{0,1,2, \ldots\}$, and

$$
\partial_{x}^{\alpha}:=\partial_{1}^{\alpha_{1}} \cdots \partial_{n}^{\alpha_{n}}
$$

We also set by classical conventions:

$$
|\alpha|:=\sum \alpha_{i} \text { and } \alpha!:=\alpha_{1}!\cdots \alpha_{n}!
$$

The number $|\alpha|$ is called the order or degree of $\alpha$.
For $x_{0} \in X$, one defines the sheaf $\mathscr{M}_{X, x_{0}}^{m}$ as the subsheaf of $\mathscr{C}_{X}^{m}$ of functions vanishing up to order $m$ at $x_{0}$. Note that $\mathscr{M}_{X, x_{0}}^{m}(U)=\mathscr{C}_{X}^{m}(U)$ for $x_{0} \notin U$. More precisely, the module $\mathscr{M}_{X, x_{0}}^{m}(U)$ consists of $\mathscr{C}^{m}$-functions $\varphi: U \longrightarrow \mathbb{R}$ such that, for all $|\alpha| \leq m$,

$$
\left(\partial_{I_{U}}^{\alpha} \varphi\right)\left(x_{0}\right)=0
$$

Let us denote by

$$
\mathscr{H} \operatorname{om}_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)
$$

the sheaf of $\mathbb{R}$-linear morphisms from the sheaf of real-valued $\mathscr{C}^{m}$-functions to the sheaf of real-valued $\mathscr{C}_{X}^{r}$-functions on $X$.

For any nonnegative integers $m$ and $r$ such that $m \geq r$, we consider the natural morphism

$$
\begin{align*}
\theta: \mathscr{D}_{X}^{m-r, r} & \longrightarrow \mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)  \tag{2}\\
P & \longmapsto \theta(P)
\end{align*}
$$

defined, for any section $\varphi$ of $\mathscr{C}_{X}^{m}$, by $\theta(P) \varphi:=P(\varphi)$.
On the other hand, we set

$$
\begin{equation*}
\mathscr{D}_{X}^{m-r, r}=0, \quad \text { if } m-r<0 . \tag{3}
\end{equation*}
$$

Our main result is as follows.
THEOREM 2.1. For any nonnegative integers $m$ and $r$, the natural morphism

$$
\begin{aligned}
\theta: \mathscr{D}_{X}^{m-r, r} & \longrightarrow \mathscr{H}_{0} m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right) \\
P & \longmapsto \theta(P): \varphi \longmapsto \quad \theta(P) \varphi:=P(\varphi),
\end{aligned}
$$

is an isomorphism.
Theorem 2.1 is associated, in a natural way, with Peetre's theorem $([7,8])$. Peetre proves the following:

Theorem 2.2 (Peetre [7, 8]). Let $X$ be a smooth manifold. Let $\mathscr{D}_{X}$ and $\mathscr{C}_{X}^{\infty}$ denote the sheaves of differential operators of finite order and of $\mathscr{C}_{X}^{\infty}$ real-valued functions on $X$, respectively. Then we have

$$
\begin{equation*}
\mathscr{D}_{X} \cong \mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{\infty}, \mathscr{C}_{X}^{\infty}\right) \tag{4}
\end{equation*}
$$

Note that the Peetre's Theorem appeared first in 1959 (see [7] for more details). The proof was incomplete and this was pointed out by M. Carleson [8]. In that proof, Peetre considered the family of functions $\left\{a_{\alpha}\right\}$ given in (1) to be finite at each of the local chart. This gap, in the proof, was later rectified by the same author in the article [8] published a year later, in 1960. The new proof given in [8] is quite different from the original, and the modified technique led to a more general representation formula for linear maps $P$ of $\mathscr{D}_{X}$ into suitable subspaces of $\mathscr{D}_{X}, P$ being assumed to shrink supports, so as to correspond with a sheaf homomorphism.

## 3. Proof of Theorem 2.1

To prove Theorem 2.1, we need some intermediary results which are summarized into lemmas below.

First, let us recall the following classical result (see, for instance [4, Lemma 1.1.1, p. 5]).

Lemma 3.1. Let $\left\{U_{i}\right\}_{i \in I}$ be a finite open covering of the unit sphere $\mathbb{S}^{n-1}$. Then, there exists a family of nonnegative real-valued functions of class $\mathscr{C}^{\infty}$ $\sigma_{i}: \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ such that
(i) $\operatorname{supp} \sigma_{i} \subseteq U_{i}$, for all $i$,
(ii) $0 \leq \sigma_{i}(x) \leq 1$, for all $x \in S^{n-1}, i \in I$,
(iii) $\sum_{i \in I} \sigma_{i}(x)=1$, for all $x \in S^{n-1}$.

In keeping with the notations of Lemma 3.1, we let, for every $i \in I, \psi_{i}$ : $\mathbb{R}^{n} \backslash\{0\} \longrightarrow \mathbb{R}$ be the map given by

$$
\begin{equation*}
\psi_{i}(x)=\sigma_{i}\left(\frac{x}{\|x\|}\right) . \tag{5}
\end{equation*}
$$

Clearly, $\psi_{i}$ is $\mathscr{C}^{\infty}$ on $\mathbb{R}^{n} \backslash\{0\}$. Next, let $m \in \mathbb{N}$ and $\eta: \mathbb{R}^{n} \longrightarrow \mathbb{R}$ be a $\mathscr{C}^{m}$ real-valued function such that $\left(\partial^{\alpha} \eta\right)(0)=0$, for all $|\alpha| \leq m$. For every $i \in I$ and every multi-index $\alpha$, set

$$
\left(\partial^{\alpha}\left(\psi_{i} \cdot \eta\right)\right)(x)= \begin{cases}\sum_{\{\beta: \beta \leq \alpha\}}\binom{\alpha}{\beta}\left(\partial^{\beta} \psi_{i}\right)(x)\left(\partial^{\alpha-\beta} \eta\right)(x), & \text { if } x \neq 0  \tag{6}\\ 0, & \text { if } x=0\end{cases}
$$

It is clear that

$$
\begin{equation*}
\eta=\sum_{i \in I} \psi_{i} \cdot \eta . \tag{7}
\end{equation*}
$$

Therefore, we have the following.
Lemma 3.2. Let $U$ be an open neighborhood of 0 in $\mathbb{R}^{n}$. For $m \geq 0$ and $\eta \in \mathscr{M}_{\mathbb{R}^{n}, 0}^{m}(U)$, every function $\psi_{i} \eta \in \mathscr{C}_{\mathbb{R}^{n}}^{m}(U \backslash\{0\})$ extends as a function of $\mathscr{M}_{\mathbb{R}^{n}, 0}^{m}(U)$.

Proof. Consider the map

$$
\lambda: \mathbb{R}^{n} \backslash\{0\} \longrightarrow \mathbb{S}^{n-1}, \quad \lambda(x)=x /\|x\| .
$$

Then $\psi_{i}=\sigma_{i} \circ \lambda$. One checks that for any $\beta \in \mathbb{N}^{n}$, there exists a constant $C>0$ such that

$$
\left\|\partial^{\beta} \lambda(x)\right\| \leq C \cdot\|x\|^{-|\beta|}
$$

and a similar result holds for $\psi_{i}$ :

$$
\left\|\partial^{\beta} \psi_{i}(x)\right\| \leq C \cdot\|x\|^{-|\beta|} .
$$

On the other hand, since $\eta \in \mathscr{M}_{\mathbb{R}^{n}, 0}^{m}(U)$, for $|\alpha| \leq m$, one has, by Taylor's formula,

$$
\partial^{\beta} \eta(x)=\|x\|^{m-|\beta|} \varepsilon(x), \text { with } \varepsilon(x) \rightarrow 0 \text { when } x \rightarrow 0 .
$$

Therefore

$$
\left\|\partial^{\beta} \psi_{i} \cdot \partial^{\alpha-\beta} \eta\right\| \leq C \cdot\|x\|^{-|\beta|} \cdot\|x\|^{m-|\alpha|+|\beta|} \varepsilon(x)
$$

that is,

$$
\left\|\partial^{\beta} \psi_{i} \cdot \partial^{\alpha-\beta} \eta\right\| \leq C\|x\|^{m-|\alpha|} \varepsilon(x)
$$

Since, by the formula $(6), \partial^{\alpha}(\psi \cdot \eta)$ is a linear combination of $\partial^{\beta} \psi_{i} \cdot \partial^{\alpha-\beta} \eta$, the result follows.

Furthermore, we have the following.
Lemma 3.3. For any open neighborhood $U$ of 0 in $\mathbb{R}^{n}$ and nonnegative integer $m$, if $u \in \mathscr{H}$ om $_{\mathbb{R}_{\mathbb{R}^{n}}}\left(\mathscr{C}_{\mathbb{R}^{n}}^{m}, \mathscr{C}_{\mathbb{R}^{n}}^{0}\right)(U)$, then

$$
u\left(\mathscr{M}_{\mathbb{R}^{n}, 0}^{m}\right) \subseteq \mathscr{M}_{\mathbb{R}^{n}, 0}^{0}
$$

Proof. First, let us consider the unit sphere $\mathbb{S}^{n-1}$, and denote by $N$ and $S$ the north and south poles of $\mathbb{S}^{n-1}$.

Next, consider the following open covering of $\mathbb{S}^{n-1}:\left\{U_{1}, U_{2}\right\}$, where $U_{1}$ contains $N$ and does not intersect some open neighborhood $V_{1}$ of $S$, and, similarly, $U_{2}$ contains $S$ and does not intersect some open neighborhood $V_{2}$ of $N$. By Lemma 3.1, we let $\left\{\sigma_{1}, \sigma_{2}\right\}$ be a partition of unity subordinate to the covering $\left\{U_{1}, U_{2}\right\}$, and let $\psi_{1}, \psi_{2}$ be functions derived from the $\sigma_{i}$ as in (5). We denote by $\mathbb{R}^{+} V_{i}$ the open cone generated by $V_{i}, i=1,2$. It is obvious that $\psi_{i}$ vanishes on $\mathbb{R}^{+} V_{i}$, and so does $\left(\left.\psi_{i}\right|_{U}\right) \sigma \equiv \psi_{i} \cdot \sigma$, for any $\sigma \in \mathscr{M}_{\mathbb{R}^{n}, 0}^{m}(U)$. As $u: \mathscr{C}_{\left.\mathbb{R}^{n}\right|_{U}}^{m} \longrightarrow \mathscr{C}_{\left.\mathbb{R}^{n}\right|_{U}}^{0}$ is a sheaf morphism, it follows that

$$
u\left(\psi_{i} \sigma\right)_{\left.\right|_{\mathbb{R}+V_{i}}}=0
$$

thus, since $u\left(\psi_{i} \cdot \sigma\right)$ is continuous,

$$
u\left(\psi_{i} \cdot \sigma\right)_{\left.\right|_{\overline{\mathbb{R}}+V_{i}}}=0
$$

from which we deduce that $u\left(\psi_{i} \cdot \sigma\right)(0)=0$, for every $i=1,2$. Thus,

$$
u(\sigma)(0)=u\left(\psi_{1} \sigma\right)(0)+u\left(\psi_{2} \sigma\right)(0)=0
$$

and hence,

$$
u(\sigma) \in \mathscr{M}_{\mathbb{R}^{n}, 0}^{0}(U)
$$

which completes the proof.
We are now set for the proof of a particular case of Theorem 2.1: the isomorphism

$$
\mathscr{D}_{X}^{m-r, r} \cong \mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)
$$

where the integers $m, r$ are such that $0 \leq r \leq m$.

Definition 3.4. Let $(U, \phi) \equiv\left(U,\left(x_{1}, \cdots, x_{n}\right)\right)$ be a local chart in an $n$-dimensional $\mathscr{C}^{\infty}$-manifold $X$ and $\mathscr{P}^{m}$ be the ring of polynomials in $\left(x_{i}\right)_{1 \leq i \leq n}$ of degree $\leq m$. We define by $\mathscr{P}_{\phi(U)}^{m}$ the constant sheaf on $\phi(U)$, whose stalk is $\mathscr{P}^{m}$.

In keeping with the notations of Definition 3.4 above, we have the following.
Lemma 3.5. Let $(U, \phi)$ be a local chart of $X$, and $u \in \mathscr{H}$ om $_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{0}\right)(U)$. If $u\left(\phi^{*}\left(\mathscr{P}_{\phi(U)}^{m}\right)\right)=0$, where $\phi^{*}\left(\mathscr{P}_{\phi(U)}^{m}\right)$ is the inverse image of $\mathscr{P}_{\phi(U)}^{m}$, then $u=0$.

Proof. One may assume that $X$ is open in $\mathbb{R}^{n}$. Let $\varphi \in \mathscr{C}_{X}^{m}(V)$, where $V$ is a sub-open of $X$ containing $x_{0}$. Then we have

$$
\varphi=q+\psi
$$

where $q \in \mathscr{P}_{X}^{m-1}(V)$ and $\psi \in \mathscr{M}_{\mathbb{R}^{n}, x_{0}}^{m}(V)$. Then, by virtue of the hypothesis and Lemma 3.3, we have

$$
u(\varphi) \in \mathscr{M}_{\mathbb{R}^{n}, x_{0}}^{0}(V)
$$

therefore

$$
u(\varphi)\left(x_{0}\right)=0
$$

But since this holds for all $x_{0} \in V$, sub-open $V$ of $X$, and $\varphi \in \mathscr{C}_{X}^{m}(V)$, we deduce that $u=0$.

We are going to consider two cases to prove the Theorem 2.1.

### 3.1. Case $0 \leq r \leq m$

Lemma 3.6. Let $X$ be an $n$-dimensional $\mathscr{C}^{\infty}$-manifold and $\mathscr{D}_{X}^{m, r}$ the sheaf of differential operators of order $\leq m$ and whose coefficients are of class $\mathscr{C}^{r}$. Then, the natural morphism

$$
\begin{align*}
\theta: \mathscr{D}_{X}^{m-r, r} & \longrightarrow \mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right) \\
P & \longmapsto \quad \theta(P): f \longmapsto \quad \theta(P) f:=P(f), \tag{8}
\end{align*}
$$

is an isomorphism.
Proof. The morphism (8) is clearly injective. Indeed, let $P$ be a section of $\mathscr{D}_{X}^{m-r, r}$ such that $\theta(P)(f)=0$ for all polynomials $f$ (in a local chart), then $P=0$. Let us now show that it is surjective.

To this end, let $u \in \mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)(U)$, where $U$ is an open subset of $X$. We will show that $u$ is in fact a differential operator of order $\leq m-r$
and whose coefficient functions are of class $\mathscr{C}^{r}$. For this purpose, consider the differential operator

$$
P=\sum_{|\beta| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}
$$

with the coefficients $a_{\beta}$ being of class $C^{r}$ and defined by induction on $|\beta|$ in the following way. Let $\mathbb{I}: U \longrightarrow \mathbb{R}$ be the constant function defined by $\mathbb{I}(x)=1$, for any $x \in U$; and we set

$$
a_{0}(x)=u(\mathbb{I}) \equiv a_{0}
$$

For any multi-index $\alpha$, suppose that we have defined $a_{\beta}$ for all $|\beta|<|\alpha| \leq m-r$; define $a_{\alpha}$ by setting

$$
\begin{equation*}
a_{\alpha}(x)=\left(u-\sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right)\left(x^{\alpha}\right), \tag{9}
\end{equation*}
$$

where $x^{\alpha}=x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$. Clearly, $a_{\alpha} \in \mathscr{C}_{X}^{r}(U)$. Denote by $\wedge_{\alpha}$ the set of all multi-indices $\alpha^{\prime}$ such that $\left|\alpha^{\prime}\right|=|\alpha| \leq m-r$. By easy calculations, one shows that

$$
\partial_{x_{1}}^{\alpha_{1}^{\prime}} \cdots \partial_{x_{n}}^{\alpha_{n}^{\prime}}\left(x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}\right)= \begin{cases}\alpha_{1}!\alpha_{2}!\cdots \alpha_{n}! & \text { if } \alpha_{i}=\alpha_{i}^{\prime}, i=1, \ldots, n \\ 0 & \text { otherwise }\end{cases}
$$

Without loss of generality, suppose that $\alpha^{\prime} \neq \alpha$ in $\wedge_{\alpha}$, and $\alpha_{1}^{\prime}=\alpha_{1}$. Then, for some $2 \leq j \leq n, \alpha_{j}^{\prime}>\alpha_{j}$, we have

$$
\partial_{x}^{\alpha^{\prime}}\left(x^{\alpha}\right)=\partial_{x_{1}}^{\alpha_{1}^{\prime}} \cdots \partial_{x_{n}}^{\alpha_{n}^{\prime}}\left(x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}\right)=0
$$

It follows that

$$
\left(\sum_{\alpha^{\prime} \in \wedge_{\alpha}} a_{\alpha^{\prime}}(x) \partial_{x}^{\alpha^{\prime}}\right)\left(x^{\alpha}\right)=0
$$

On the other hand, since for any $\beta$ such that $|\beta|>|\alpha|$, we have $\partial_{x}^{\beta}\left(x^{\alpha}\right)=0$, it follows, using (9), that

$$
\begin{aligned}
P\left(x^{\alpha}\right) & =\left(\sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right)\left(x^{\alpha}\right)+a_{\alpha}(x) \partial_{x}^{\alpha}\left(x^{\alpha}\right) \\
& =\left(\sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right)\left(x^{\alpha}\right)+\alpha!\left(u-\sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right)\left(x^{\alpha}\right)
\end{aligned}
$$

with $\alpha!:=\alpha_{1}!\alpha_{2}!\cdots \alpha_{n}!$. Thus, we deduce that, for every $x^{\alpha}$, with $|\alpha| \leq m-r$,

$$
\left(u-\frac{1}{\alpha!}\left[P-(1-\alpha!) \sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right]\right)\left(x^{\alpha}\right)=0
$$

which implies that

$$
\left(u-\frac{1}{\alpha!}\left[P-(1-\alpha!) \sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right]\right)\left(\mathcal{P}_{\phi(U)}^{m-r}\right)=0
$$

Hence, by Lemma 3.5,

$$
u=\frac{1}{\alpha!}\left(P-(1-\alpha!) \sum_{|\beta|<|\alpha| \leq m-r} a_{\beta}(x) \partial_{x}^{\beta}\right)
$$

and the proof is complete.
In particular we deduce, from Lemma 3.6, that $\mathscr{H} o m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{m}\right) \cong \mathscr{C}_{X}^{m}$.

### 3.2. Case $m<r$

Lemma 3.7. For any nonnegative integers $m$ and $r$ such that $m<r$,

$$
\mathscr{H} \circ m_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right)=0
$$

Proof. Since $\mathscr{C}_{X}^{r} \subseteq \mathscr{C}_{X}^{m}$, then

$$
\mathscr{H} \operatorname{om}_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{r}\right) \subseteq \mathscr{H} \operatorname{om}_{\mathbb{R}_{X}}\left(\mathscr{C}_{X}^{m}, \mathscr{C}_{X}^{m}\right) \cong \mathscr{C}_{X}^{m}
$$

Therefore, we are reduced to prove that given $m<r$, if $u \in \mathscr{C}^{m}(U)$ and also $u \cdot f \in \mathscr{C}^{r}(U)$ for any $f \in \mathscr{C}^{m}(U)$, then $u=0$. Indeed, assume that $u$ is not identically 0 and let $x_{0}$ with $u\left(x_{0}\right) \neq 0$. Let $v=u^{-1}$. Then $f=v \cdot u \cdot f$ would be of class $\mathscr{C}^{r}$ in a neighborhood of $x_{0}$.

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[^1]:    $1 "$ en gros et sans calcul"
    ${ }^{2}$ Castro, M. Misquero, El problema de Kepler disipativo, Master thesis, Universidad de Granada, (2016)

[^2]:    ${ }^{3}$ The discussion in [16] on the inequality (62) appearing in the proof of Lemma 2.2 was incomplete. This inequality is valid and follows from Lemma 3.1.

