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Veronesean almost binomial almost complete intersections

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Abstract. *The second Veronese ideal I_n contains a natural complete intersection J_n of the same height, generated by the principal 2-minors of a symmetric $(n \times n)$-matrix. We determine subintersections of the primary decomposition of J_n where one intersectand is omitted. If I_n is omitted, the result is a direct link in the sense of complete intersection liaison. These subintersections also yield interesting insights into binomial ideals and multigraded algebra. For example, if n is even, I_n is a Gorenstein ideal and the intersection of the remaining primary components of J_n equals $J_n + \langle f \rangle$ for an explicit polynomial f constructed from the fibers of the Veronese grading map.*

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1. Introduction

Ideals generated by minors of matrices are a mainstay of commutative algebra. Here we are concerned with ideals generated by 2-minors of symmetric matrices. Ideals generated by arbitrary minors of symmetric matrices have been studied by Kutz [18] who proved, in the context of invariant theory, that the quotient rings are Cohen–Macaulay. Results of Goto show that the quotient ring is normal with divisor class group $\mathbb{Z}_2$ and Gorenstein if the format of the symmetric matrix has the same parity as the size of the minors [11, 12]. Conca extended these results to more general sets of minors of symmetric matrices [4] and determined Gröbner bases and multiplicity [5].

Here we are concerned only with the binomial ideal I_n generated by the 2-minors of a symmetric $(n \times n)$-matrix. This ideal cuts out the second Veronese variety and was studied classically, for example by Gröbner [15]. It contains a complete intersection J_n, generated by the principal 2-minors (Definition 2.2). Both ideals are of height $\binom{n}{2}$. Coming from liaison theory one may ask for the ideal $K_n = J_n : I_n$ on the other side of the complete intersection link via J_n. In this paper we determine K_n.

EXAMPLE 1.1: Consider the ideal $J_3 = \langle ad-b^2, af-c^2, df-e^2\rangle \subseteq \mathbb{Q}[a,b,c,d,e,f]$ generated by the principal 2-minors of the symmetric matrix $\left(\begin{smallmatrix} a & b & c \\ b & d & e \\ c & e & f \end{smallmatrix}\right)$. The ideal J_3 is a complete intersection because it has an initial ideal with this property. Using, for example, MACAULAY2 [14], one finds the prime decomposition $J_3 = I_3 \cap K_3$ where

$$I_3 = J_3 + \langle ae - bc, cd - be, ce - bf\rangle$$

is the second Veronese ideal, generated by all 2-minors, and

$$K_3 = J_3 + \langle ae + bc, cd + be, ce + bf\rangle$$

is the image of I_3 under the automorphism of $\mathbb{Q}[a,\dots,f]$ that maps b, c, and e to their negatives and the remaining indeterminates to themselves. As predicted by Theorem 2.11, the generator $ae+bc$ is the sum of monomials whose exponents are the lattice points of the fiber

$$\left\{ u \in \mathbb{N}^6 : \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{pmatrix} \cdot u = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$$

of the $\mathbb{Z}$-linear map V_3 that defines the fine grading of $\mathbb{Q}[a,\dots,f]/I_3$. We call this the *generating function* of the fiber. For $n \geq 4$ the extra generators are not binomials anymore and K_n is an intersection of ideals obtained from I_n by twisting automorphisms (Definition 2.7). In Example 2.12, for $n = 4$, we find $K_4 = J_4 + \langle p\rangle$ for one quartic polynomial p with eight terms.

Results on Gorenstein biliaison of ideals of minors of symmetric matrices have been obtained by Gorla [9, 10] but here we study direct complete intersection links. Our methods rely on the combinatorics of binomial ideals and since K_n is not binomial and we do not know of a natural binomial complete intersection contained in K_n, we cannot explore the linkage class more with the present method. Instead we are motivated by general questions about binomial ideals and their intersections. For example, [17, Problem 17.1] asks, when the intersection of binomial ideals is binomial. From the primary (in fact, prime) decomposition of J_n we remove I_n and intersect the remaining binomial prime ideals. The result is not binomial. If n is even, $K_n = J_n + \langle p\rangle$ for one additional polynomial p. In the terminology of [1], K_n is thus an *almost complete intersection*. It is also *almost binomial*, as it is principal modulo its binomial part—the binomial ideal spanned over $\Bbbk$ by all binomials in the ideal [16, Definition 2.1]. If n is odd, then there are n additional polynomials (Theorem 2.11). While these numbers can be predicted from general liaison theory, our explicit formulas reveal interesting structures at the boundary of binomiality and are thus a first step towards [17, Problem 17.1]

We determine K_n with methods from combinatorial commutative algebra, multigradings in particular (see [19, Chapters 7 and 8]). The principal observation that drives the proofs in Section 2 is that the Veronese-graded Hilbert function of the quotient $\Bbbk[\mathbf{x}]/J_n$ becomes eventually constant (Remark 2.14). The eventual value of the Hilbert function bounds the number of terms that a graded polynomial can have. The extra generators of K_n are the lowest degree polynomials that realize the bound. We envision that this structure could be explored independently and brought to unification with the theory of toral modules from [7]. Our results also have possible extensions to higher Veronese ideals as we outline in Section 3.

Denote by $c_n := \binom{n}{2}$ the entries of the second diagonal in Pascal's triangle. Throughout, let $[n] := \{1, \dots, n\}$ be the set of the first n integers. The second Veronese ideal lives in the polynomial ring $\Bbbk[\mathbb{N}^{c_{n+1}}]$ in c_{n+1} indeterminates over a field $\Bbbk$. For polynomial rings and quotients modulo binomial ideals we use monoid algebra notation (see, for instance, [17, Definition 2.15]). We make no a-priori assumptions on $\Bbbk$ regarding its characteristic or algebraic closure, although care is necessary in characteristic two. The variables of $\Bbbk[\mathbb{N}^{c_{n+1}}]$ are denoted x_{ij}, for $i, j \in [n]$ with the implicit convention that $x_{ij} = x_{ji}$. For brevity we avoid a comma between i and j. We usually think about upper triangular matrices, that is $i \leq j$. The Veronese ideal I_n is the toric ideal of the *Veronese multigrading* $\mathbb{N}V_n$, defined by the $(n \times c_{n+1})$-matrix V_n with entries

$$(V_n)_{i,jk} := \begin{cases} 2 & \text{if } i = j = k, \\ 1 & \text{if } i = j, \text{ or } i = k, \text{ but not both,} \\ 0 & \text{otherwise.} \end{cases}$$

That is, the columns of V_n are the non-negative integer vectors of length n and weight two. For $\mathbf{b} \in \mathbb{N}V_n$, the *fiber* is $V_n^{-1}[\mathbf{b}] = \{u \in \mathbb{N}^{c_{n+1}} : V_n u = \mathbf{b}\}$. Computing the V_n-degree of a monomial is easy: just count how often each row or column index appears in the monomial. For example, $\deg(x_{12}x_{nn}) = (1, 1, 0, \dots, 0, 2)$. We do not distinguish row and column vectors notationally, in particular we write columns as rows when convenient. Gröbner bases for a large class of toric ideals including I_n have been determined by Sturmfels [20, Theorem 14.2]. The *Veronese lattice* $L_n \subseteq \mathbb{Z}^{c_{n+1}}$ is the kernel of V_n. The rank of L_n is c_n since the rank of V_n is n and $c_{n+1} - n = c_n$. Lemma 2.1 gives a lattice basis. With $\{e_{ij}, i \leq j \in [n]\}$ a standard basis of $\mathbb{Z}^{c_{n+1}}$, we use the following notation

$$[ij|kl] := e_{ik} + e_{jl} - e_{il} - e_{jk} \in \mathbb{Z}^{c_{n+1}}.$$

Then $[ij|kl]$ is the exponent vector of the minor $x_{ik}x_{jl} - x_{il}x_{jk}$.

Example 1.2: The Veronese lattice $L_3 \subseteq \mathbb{Z}^6$ is of rank $3 = \binom{4}{2} - 3$ and minimally generated by the following elements

$$[13|13] = \begin{pmatrix} 1 & 0 & -2 \\ & 0 & 0 \\ & & 1 \end{pmatrix}, \ [13|23] = \begin{pmatrix} 0 & 1 & -1 \\ & 0 & -1 \\ & & 1 \end{pmatrix}, \ [23|23] = \begin{pmatrix} 0 & 0 & 0 \\ & 1 & -2 \\ & & 1 \end{pmatrix}.$$

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2. Decomposing and Recomposing

LEMMA 2.1. *The set*

$$\mathcal{B} = \{[in|jn] : i, j \in [n-1]\}$$

is a lattice basis of the Veronese lattice L_n.

Proof. Write the elements of $\mathcal{B}$ as the columns of a $(c_{n+1} \times c_n)$-matrix B. Deleting the rows corresponding to indices (i, n) for $i \in [n]$ yields the identity matrix I_{c_n}. Thus $\mathcal{B}$ spans a lattice of the correct rank and that lattice is saturated. Indeed, the Smith normal form of B must equal the identity matrix I_{c_n} concatenated with a zero matrix. Thus the quotient by the lattice spanned by $\mathcal{B}$ is free. □

The Veronese ideal contains a codimension c_n complete intersection J_n generated by the principal 2-minors.

DEFINITION 2.2. *The* principal minor ideal J_n *is generated by all principal* 2*-minors* $x_{ii}x_{jj} - x_{ij}^2$ *of a generic symmetric matrix. The* principal minor lattice L'_n *is the lattice generated by the corresponding exponent vectors* $[ij|ij]$, $i, j \in [n]$.

It can be seen that the principal minor lattice is minimally generated by $[ij|ij]$. It is an unsaturated lattice meaning that it cannot be written as the kernel of an integer matrix, or equivalently, that the quotient $\mathbb{Z}^{c_{n+1}}/L'_n$ has torsion. Since there are no non-trivial coefficients on the binomials in J_n, Proposition 2.5 below says that it is a lattice ideal with lattice L'_n. Its torsion subgroup is given in the following proposition.

Proposition 2.3. *The principal minor lattice is minimally generated by*

$$\mathcal{B}' = \{2[in|jn] : i \neq j \in [n-1]\} \cup \{[in|in] : i \in [n-1]\}.$$

Furthermore the group L_n/L'_n is (isomorphic to) $(\mathbb{Z}/2\mathbb{Z})^{c_{n-1}}$.

Proof. It holds that $2[in|jn] = [in|in] + [jn|jn] - [ij|ij]$ and the map which includes the span of the elements $[ij|ij]$ into L'_n is unimodular. A presentation of the group can be read off the Smith normal form of the matrix whose columns are a lattice basis. Since $\mathcal{B}'$ is a basis of L'_n, the columns and rows can be arranged so that the diagonal matrix $\operatorname{diag}(2,\ldots,2,1,\ldots,1)$ with c_{n-1} entries 2 is the top $(c_n \times c_n)$-matrix of the Smith normal form. Any entry below a two is divisible by two and thus row operations can be used to zero out the the bottom part of the matrix. This yields the Smith normal form. □

Example 2.4: For $n = 3$, the basis $\mathcal{B}'$ is given in matrix notation as

$$\begin{pmatrix} 0 & 2 & -2 \\ & 0 & -2 \\ & & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -2 \\ & 0 & 0 \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ & 1 & -2 \\ & & 1 \end{pmatrix}.$$

The advantage of $\mathcal{B}'$ over the basis in Definition 2.2 is that the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$ is diagonal. This makes it easy to understand the quotient of the Veronese lattice modulo the principal minor lattice.

If $\operatorname{char}(\mathbb{k}) = 2$, then J_n is primary over I_n. In all other characteristics one can see that the Veronese ideal I_n is a minimal prime and in fact a primary component of J_n. These statements follow from [8] and are summarized in Proposition 2.10 below. Towards this observation, the next proposition says that J_n is a mesoprime ideal, that is, it equals the kernel of a monomial $\mathbb{k}$-algebra homorphism from $\mathbb{k}[\mathbb{N}^{c_{n+1}}]$ to a twisted group algebra [17, Definition 10.4]. The adjective twisted implements the general coefficients on the binomials in [17]. Here all coefficients are equal to $1_{\mathbb{k}}$. The ideal J_n is a lattice ideal as a kernel of a monomial homomorphism onto an ordinary group algebra.

Proposition 2.5. *J_n is a mesoprime binomial ideal and its associated lattice is L'_n.*

Proof. We show that $J_n = \langle x^{u^+} - x^{u^-} : u \in L'_n\rangle$, since the quotient by this ideal is the group algebra $\mathbb{k}[\mathbb{Z}^{c_{n+1}}/L'_n]$. By the correspondence between non-negative lattice walks and binomial ideals [6, Theorem 1.1] we prove that for any $u = u^+ - u^- \in L'_n$, the parts $u^+, u^- \in \mathbb{N}^{c_{n+1}}$ can be connected using *moves* $[ij|ij]$ without leaving $\mathbb{N}^{c_{n+1}}$.

The vectors u^+, u^- can be represented by upper triangular non-negative integer matrices. From Definition 2.2 it is obvious that all off-diagonal entries of $u^+ - u^-$ are divisible by two. Since

$$\langle x^{u^+} - x^{u^-} : u \in L'_n\rangle : x_{ij} = \langle x^{u^+} - x^{u^-} : u \in L'_n\rangle$$

for any variable x_{ij}, we can assume that u^+ and u^- have disjoint supports and thus individually have off-diagonal entries divisible by two. Consequently the moves $[ij|ij]$ allow to reduce all off-diagonal entries to zero, while increasing the diagonal entries. As visible from its basis, the lattice L'_n contains no nonzero diagonal matrices and thus u^+ and u^- have been connected to the same diagonal matrix. □

REMARK 2.6: From Proposition 2.3 it follows immediately that the group algebra $\Bbbk[\mathbb{Z}^{c_{n+1}}]/J_n\Bbbk[\mathbb{Z}^{c_{n+1}}]$ is isomorphic to $\Bbbk[\mathbb{Z}^n \oplus (\mathbb{Z}/2\mathbb{Z})^{c_{n-1}}]$. In particular $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ is finely graded by the monoid $\mathbb{N}V_n \oplus (\mathbb{Z}/2\mathbb{Z})^{c_{n-1}}$.

DEFINITION 2.7. *A* $\mathbb{Z}_2$-twisting *is a ring automorphism of a (Laurent) polynomial ring that maps the indeterminates either to themselves or to their negatives.*

A *fundamental parallelepiped* of the lattice L'_n is the quotient $\mathbb{Q}^{c_{n+1}}/L'_n$, embedded as a half-open polytope in $\mathbb{Q}^{c_{n+1}}$. The lattice points in it play an important role in the following developments. The most succinct way to encode them is using their generating function, a Laurent polynomial in the ring $\Bbbk[\mathbb{Z}^{c_{n+1}}]$. Its explicit form depends on the chosen coordinates. The next lemma is immediate from the definition of $\mathcal{B}'$.

LEMMA 2.8. *Let* $M = \{[in|jn] : i \neq j \in [n-1]\}$. *The generating function of the fundamental parallelepiped of* $\mathcal{B}'$ *is*

$$p_n = \prod_{m\in M} (x^m + 1) = \prod_{m\in M} (x^{m^+} + x^{m^-})$$

It is useful for the further development to pick the second representation of p_n in Lemma 2.8 as a representative of p_n in polynomial ring $\Bbbk[\mathbb{N}^{c_{n+1}}]$. Its image in the quotient $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ also has a natural representation. The terms of p_n can be identified with upper triangular integer matrices which arise as sums of positive and negative parts of elements of M. A positive part of $[in|jn] \in M$ has entries 1 at positions (i,j) and (n,n) while a negative part has two entries 1 in the last column, but not at (n,n). Modulo the moves $\mathcal{B}'$, any exponent matrix of a monomial of p_n can be reduced to have only entries 0 or 1 in its off-diagonal positions.

REMARK 2.9: A simple count yields that p_n has V_n-degree $(n-2,\ldots,n-2,2c_{n-1})$. In the natural representation of monomials of p_n as integer matrices with entries 0/1 off the diagonal, there is a lower bound for the value of the (n,n) entry. To achieve the lowest value, one would fill the last column with entries 1 using negative parts of elements of M, and then use positive parts (which increase (n,n)). For example, if n is even, there is one term of p_n whose last column arises from the negative parts of $[1n|2n],[3n|4n],\ldots,[(n-3)n|(n-2)n]$ and then positive parts of the remaining elements of M. If n is odd, then there is one term of p_n, whose n-th column is $(1,\dots,1,\sigma_{n-1})$ for some value σ_{n-1}. In fact, since $|M| = c_{n-1}$, the lowest possible value of the (n,n) entry is $\sigma_{n-1} = c_{n-1} - \lfloor \frac{n-1}{2} \rfloor$.

The primary decomposition of J_n is given by [8, Theorem 2.1 and Corollary 2.2].

Proposition 2.10. *If* $\mathrm{char}(\Bbbk) = 2$*, the* J_n *is primary over* I_n*. In all other characteristics, there exist* $\mathbb{Z}_2$*-twistings* ϕ_i *for* $i = 1,\dots,2^{c_{n-1}}$ *such that the complete intersection* J_n *has prime decomposition*

$$J_n = \bigcap_i \phi_i(I_n). \tag{1}$$

Theorem 2.11. *If* n *is odd, intersecting all but one of the components in* (1) *yields*

$$\bigcap_{i\neq l} \phi_i(I_n) = J_n + \langle \phi_l(p^+_{n,i}) : i \in [n]\rangle,$$

where $p^+_{n,i} \in \Bbbk[\mathbb{N}^{c_{n+1}}]$ *are homogeneous polynomials of degree* $\frac{(n-1)^2}{2}$ *that are given as generating functions of the fibers* $V_n^{-1}[(n-2,\dots,n-2)+e_i]$*. If* n *is even, then the same holds for a single polynomial* p^+_n *of degree* $\frac{n(n-2)}{2}$*, given as the generating function of* $V_n^{-1}[(n-2,\dots,n-2)]$*.*

The proof of Theorem 2.11 occupies the remainder of the section after the following example.

Example 2.12: The complete intersection J_4 is a lattice ideal for the lattice L'_4. In the basis $\mathcal{B}'$, it is generated by the six elements

$$\{2[i4|j4] : i < j \in [3]\} \cup \{[i4|i4] : i \in [3]\}.$$

Three of the six elements correspond to principal minors

$$x_{11}x_{44} - x_{14}^2, x_{22}x_{44} - x_{24}^2, x_{33}x_{44} - x_{34}^2.$$

The other elements give the binomials

$$x_{12}^2x_{44}^2 - x_{14}^2x_{24}^2, x_{13}^2x_{44}^2 - x_{14}^2x_{34}^2, x_{23}^2x_{44}^2 - x_{24}^2x_{34}^2.$$

These six binomials do not generate J_4, but J_4 equals the saturation with respect to the product of the variables [19, Lemma 7.6]. The $2^3 = 8$ minimal prime components of J_n are obtained by all possible twist combinations of the monomials $\pm x_{14}x_{24}$, $\pm x_{14}x_{34}$, $\pm x_{24}x_{34}$. Consider the mysterious polynomial

$$p_4 = (x_{12}x_{44} + x_{14}x_{24})(x_{13}x_{44} + x_{14}x_{34})(x_{23}x_{44} + x_{24}x_{34}),$$

which is the generating function of the fundamental parallelepiped of L_4' in the basis $\mathcal{B}'$ and of V_4-degree $(2,2,2,6)$. *In the Laurent polynomial ring* $\Bbbk[\mathbb{Z}^{10}]$, the desired ideal $J_4 : I_4$ equals $J_4 + \langle p_4 \rangle$. To do the computation in the polynomial ring, we need to saturate with respect to $\prod_{ij} x_{ij}$. If n is even, this saturation generates one polynomial, if n is odd, it generates n polynomials. Here, where $n = 4$, the ideal $J_4 : I_4$ is generated by J_4 and the single polynomial

$$\begin{aligned} p_4^+ = {} & x_{11}x_{22}x_{33}x_{44} + x_{11}x_{23}x_{24}x_{34} + x_{13}x_{14}x_{22}x_{34} + x_{12}x_{14}x_{24}x_{33} \\ & + x_{13}x_{14}x_{23}x_{24} + x_{12}x_{14}x_{23}x_{34} + x_{12}x_{13}x_{24}x_{34} + x_{12}x_{13}x_{23}x_{44}. \end{aligned}$$

Modulo the binomials in J_4, the polynomial p_4^+ equals p_4/x_{44}^2 (Lemma 2.18).

As a first step towards the proof of Theorem 2.11 we compute the monoid Q under which $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ is finely graded, meaning that its Q-graded Hilbert function takes values only zero or one. That is, we make Remark 2.6 explicit.

LEMMA 2.13. *Fix* $\mathbf{b} \in \operatorname{cone}(V_n)$ *for some* n*. The equivalence classes of lattice points in the fiber* $V_n^{-1}[\mathbf{b}]$*, modulo the moves* $\mathcal{B}'$*, are in bijection with set of symmetric 0/1 matrices* $u \in \{0,1\}^{n\times n}$ *of the following form*

- $u_{ii} = 0$*, for all* $i \in [n]$
- $u_{in} = 0$*, for all* $i \in [n]$
- $\mathbf{b} - V_n u \in \mathbb{N}^n$.

Proof. Each equivalence class of upper triangular matrices has a representative whose off-diagonal entries are all either zero or one. The bijection maps such an equivalence class to the c_{n-1} entries that are off-diagonal and off the last column. To prove that this is a bijection it suffices to construct the inverse map. To this end, let u satisfy the properties in the statement. In each row $i = 1, \ldots, n$, there are two values unspecified: the diagonal entry and the entry in the last column. Given $\mathbf{b}_i$, using the representative modulo $\mathcal{B}'$ whose last column entries are either 0 or 1 fixes the diagonal entry by linearity. Therefore the map is a bijection. □

REMARK 2.14: If $\mathbf{b}_i \geq (n-2)$ for all $i \in [n]$, then any 0/1 upper triangular $(n-2)$-matrix is a possible choice for the off-diagonal off-last column entries of u in Lemma 2.13. An upper triangular $(n-2)$-matrix has c_{n-1} entries. Thus all those fibers have equivalence classes modulo $\mathcal{B}'$ that are in bijection with $\{0,1\}^{c_{n-1}}$. In particular, each of those fibers, has the same number of equivalence classes.

REMARK 2.15: Remark 2.14 implies that in the V_n-grading, $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ is *toral* as in [7, Definition 4.3]: its V_n-graded Hilbert function is globally bounded by $2^{c_{n-1}}$.

If n is odd, then $(n-2,\dots,n-2)\notin \mathbb{N}V_n$. Therefore the minimal (with respect to addition in the semigroup cone(V_n)) fibers that satisfy Remark 2.14 are $(n-1,n-2,\dots,n-2),\dots,(n-2,\dots,n-2,n-1)$. If n is even, there is only one minimal fiber.

For the proof of Theorem 2.11 it is convenient to work in the quotient ring $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$. Since $I_n \supseteq J_n$ and I_n is finely graded by $\mathbb{N}V_n$, each equivalence class is contained in a single fiber $V_n^{-1}[\mathbf{b}]$ and each fiber breaks into equivalence classes. The following definition sums the monomials corresponding to these classes for specific fibers.

Definition 2.16. *The* minimal saturated fibers *are the minimal fibers that satisfy Remark 2.14. The generating function of the equivalence class in a minimal saturated fiber is denoted by $p^+_{n,i}$. That is*

$$p^+_{n,i} = \sum_{\mathbf{a}\in V_n^{-1}[\mathbf{b}_i]/L'_n} \mathbf{x}^{\mathbf{a}} \in \Bbbk[\mathbb{N}^{c_{n+1}}]/J_n.$$

where $\mathbf{b}_i := (n-2,\dots,n-2)+e_i$ if n is odd and $\mathbf{b}_i = (n-2,\dots,n-2)$ if n is even.

If n is even, Definition 2.16 postulates only one polynomial which is simply denoted p^+_n when convenient. Sometimes, however, it can be convenient to keep the indices.

Remark 2.17: The construction of a generating function of equivalence classes of elements of the fiber in Definition 2.16 can be carried out for any fiber of V_n. For the fiber $V_n^{-1}[(n-2,\dots,n-2,2c_{n-1})]$ we get the polynomial p_n from Lemma 2.8.

The quantity $\sigma_{n-1} = c_{n-1} - \lfloor \frac{n-1}{2} \rfloor$ (that is $c_{n-1} - \frac{n-2}{2} = \frac{(n-2)^2}{2}$ for even n, and $c_{n-1} - \frac{n-1}{2} = \frac{(n-1)(n-3)}{2}$ for odd n) appeared in Remark 2.9 and shows up again in the next lemma: it almost gives the saturation exponent when passing from the Laurent polynomial ring to the polynomial ring.

Lemma 2.18. *As elements of $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$, if n is even then, $x_{nn}^{\sigma_{n-1}} p^+_{n,i} = p_n$, and if n is odd, then, $x_{nn}^{\sigma_{n-1}+1} p^+_{n,i} = x_{in} p_n$.*

Proof. If n is even, the product $x_{nn}^{\sigma_{n-1}} p^+_{n,i}$ has V_n-degree $(n-2,\dots,n-2,2c_{n-1})$. If n is odd, the degree of $x_{nn}^{\sigma_{n-1}+1} p^+_{n,i}$ equals $(n-2,\dots,n-2,2c_{n-1}+1)+e_i$. Now these products equal p_n if n is even and $x_{in}p_n$ if n is odd by Remarks 2.14 and 2.17. □

Lemma 2.19. *If n is odd, then for any triple of distinct indices $i,j,k\in[n]$, in $\Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ we have $x_{ij}p^+_{n,k} = x_{jk}p^+_{n,i}$.*

Proof. Since in a group algebra all monomials are invertible, Proposition 2.5 implies in particular that the variables are nonzerodivisors on $\mathbb{k}[\mathbb{N}^{c_{n+1}}]/J_n$. The multidegree of $p^+_{n,k}$ satisfies the conditions of Remark 2.14, thus there are bijections between the monomials of $x_{ij}p^+_{n,k}$ and $x_{jk}p^+_{n,i}$. Since all relations in J_n are equalities of monomials, multiplication with a variable does not touch coefficients. □

The following lemma captures an essential feature of our situation. Since the V_n-graded Hilbert function of $\mathbb{k}[\mathbb{N}^{c_{n+1}}]$ is globally bounded, there is a notion of *longest homogeneous polynomial* as one that uses all monomials in a given V_n-degree. For any multidegree $\mathbf{b}$ that satisfied $\mathbf{b}_i \geq (n-2)$, by Remark 2.14, if a longest polynomial of multidegree $\mathbf{b}$ is multiplied by a term, then it remains a longest polynomial.

LEMMA 2.20. *The V_n-graded Hilbert functions of the $\mathbb{k}[\mathbb{N}^{c_{n+1}}]/J_n$-modules, $\langle p_n\rangle$ and $\langle p^+_{n,i}\rangle$, $i = 1,\dots,n$ take only zero and one as their values.*

Proof. We only prove the statement for $\langle p_n\rangle$ since the same argument applies also to $\langle p^+_{n,i}\rangle$. The claim is equivalent to the statement that any $f \in \langle p_n\rangle$ is a term (that is, a monomial times a scalar) times p_n. Let $f = gp_n$ with a V_n-homogeneous g. Let $t_1,\dots,t_s$ be the terms of g. Since p_n is the sum of all monomials of degree $\deg(p_n)$, and multiplication by a term does not produce any cancellation, the number of terms of t_ip_n equals that of p_n. By Remark 2.14, the monomials in degree $\deg(t_ip_n)$ are in bijection with the monomials in degree $\deg(p_n)$, and therefore all t_ip_n are scalar multiples of the generating function of the fiber for $\deg(t_ip_n)$ and this generating function is equal to mp_n for any monomial m of multidegree $\deg(gp_n) - \deg(p_n)$. □

LEMMA 2.21. *For any $i \in [n]$, $\langle p^+_{n,i}\rangle : \left(\prod_{ij} x_{ij}\right)^\infty = \langle p^+_{n,k} : k \in [n]\rangle$.*

Proof. If n is odd, the containment of $p^+_{n,k}$ in the left hand side follows immediately from Lemma 2.19. If n is even, it is trivial. For the other containment, let f be a V_n-homogeneous polynomial that satisfies $mf \in \langle p^+_{n,i}\rangle$ for some monomial m. We want $f \in \langle p^+_{n,k} : k \in [n]\rangle$. By Lemma 2.20, $mf = tp^+_{n,i}$ for some term t. Since mf has the same number of terms as f and also the same number of terms as $tp^+_{n,i}$, this number must be $2^{c_{n-1}}$. By Remark 2.14, the only V_n-homogeneous polynomials with $2^{c_{n-1}}$ terms are monomial multiples of the $p^+_{n,k}$ for $k \in [n]$. □

PROPOSITION 2.22. $(J_n + \langle p_n\rangle) : \left(\prod_{ij} x_{ij}\right)^\infty = J_n + \langle p^+_{n,j}, j \in [n]\rangle$.

Proof. Throughout we work in the quotient ring $S := \Bbbk[\mathbb{N}^{c_{n+1}}]/J_n$ and want to show

$$\langle p_n\rangle : \left(\prod_{ij} x_{ij}\right)^{\infty} = \langle p^+_{n,j}, j \in [n]\rangle.$$

Lemma 2.18 gives the inclusion $\supseteq$, since it shows that, modulo J_n, a monomial multiple of $p^+_{n,i}$ is equal to either p_n or $x_{in}p_n$ and thus lies in $\langle p_n\rangle$. For the other containment let

$$f \in \langle p_n\rangle : \left(\prod_{ij} x_{ij}\right)^{\infty},$$

that is $mf \in \langle p_n\rangle$ for some monomial m in S. This implies $mf = gp_n$ for some polynomial $g \in S$. By Lemma 2.18, $x_{in}mf = g'p^+_{n,i}$ for some $g' \in S$. So, $x_{in}mf \in \langle p^+_{n,i}\rangle$ and thus $f \in \langle p^+_{n,i}\rangle : x_{in}m$. Lemma 2.21 shows that $f \in \langle p_{n,k} : k \in [n]\rangle$. □

Having identified the minimal saturated fibers, the longest polynomials, and computed the saturation with respect to the variables x_{ij}, we are now ready to prove Theorem 2.11.

Proof of Theorem 2.11. After a potential renumbering, assume ϕ_1 is the identity. It suffices to prove the theorem for the omission of the Veronese ideal $i = 1$ from the intersection. The remaining cases follow by application of ϕ_l to the ambient ring.

Consider the extensions $J_n\Bbbk[\mathbb{Z}^{c_{n+1}}]$ and $I_n\Bbbk[\mathbb{Z}^{c_{n+1}}]$ to the Laurent polynomial ring. By the general Theorem 2.23

$$\bigcap_{i\neq 1} \phi_i(I_n\Bbbk[\mathbb{Z}^{c_{n+1}}]) = J_n\Bbbk[\mathbb{Z}^{c_{n+1}}] + \langle p_n\rangle.$$

Pulling back to the polynomial ring, we have

$$\bigcap_{i\neq 1} \phi_i(I_n) = (J_n + \langle p_n\rangle) : (\prod_{x_{ij}} x_{ij})^{\infty}.$$

Contingent on Theorem 2.23, the result now follows from Proposition 2.22. □

We have reduced the proof of Theorem 2.11 to a general result on intersection in the Laurent polynomial ring. It is a variation of [8, Theorem 2.1]. According to [8, Section 2], any binomial ideal in the Laurent polynomial ring $\Bbbk[\mathbb{Z}^n]$ is defined by its lattice $L \subseteq \mathbb{Z}^n$ of exponents and a partial character $\rho : L \to \Bbbk^*$. Such an ideal is denoted $I(\rho)$ where the lattice L is part of the definition of ρ. Let now L be a saturated lattice, $\rho : L \to \Bbbk^*$ a partial character, and $1 : L \to \Bbbk^*$ the trivial character that maps all of L to $1 \in \Bbbk$. The ideal $I(\rho)$

can be constructed by appropriately twisting the ideal $I(1)$. Specifically, if $\Bbbk$ is algebraically closed, there exists an automorphism ϕ_ρ of $\Bbbk[\mathbb{Z}^n]$ that maps each variable to a scalar multiple of the same variable and such that $\phi_\rho(I(1)) = I(\rho)$. Suitable coefficients $a_1, \ldots, a_n$ of the variables that define such an automorphism can be computed by solving the equations $a^{-m_i} = \rho(m_i)$ for any lattice basis $m_1, \ldots, m_r$ of L. These equations are solvable over an algebraically closed field and the resulting automorphisms generalize the $\mathbb{Z}_2$-twistings from Definition 2.7.

THEOREM 2.23. *Let* $\Bbbk$ *be a field such that* $\operatorname{char}(\Bbbk)$ *is either zero or does not divide the order of the torsion part of* $\mathbb{Z}^n/L$ *and* $I(\rho) \subseteq \Bbbk[\mathbb{Z}^n]$ *be the binomial ideal for some partial character* $\rho : L \to \Bbbk^*$. *Let* $I(\rho) = I(\rho'_1) \cap \ldots \cap I(\rho'_k)$ *be a primary decomposition of* $I(\rho)$ *over the algebraic closure* $\overline{\Bbbk}$ *of* $\Bbbk$. *Omitting one component* $I(\rho'_{i^*})$ *yields*

$$\bigcap_{i \neq i^*} I(\rho'_i) = I(\rho) + \phi_{\rho'_{i^*}}(p_L)$$

where p_L *is the generating function of a fundamental parallelepiped of the lattice* L.

Proof. A linear change of coordinates in $\mathbb{Z}^n$ corresponds to a multiplicative change of coordinates in $\Bbbk[\mathbb{Z}^n]$. Since the inclusion of $L \subseteq \mathbb{Z}^n$ can be diagonalized using the Smith normal form, one can reduce to the case that $I(\rho)$ is generated by binomials $x_i^{q_i} - c_i$ for some coefficients $c_i \in \Bbbk$. This case follows by multiplication of the results in the univariate case. In the univariate case, the factors of $x^n - c$ are the n-th roots $\zeta_1, \ldots, \zeta_n$ of c. Then ρ is defined by $n \mapsto c$ and ρ'_i by $1 \mapsto \zeta_i$. One has

$$\prod_{i \neq i^*} (x - \zeta_i) = \phi_{i^*}((x^n - 1)/(x - 1)),$$

where ϕ_{i^*} is the automorphism of $\Bbbk[\mathbb{Z}]$ defined by $x \mapsto \zeta_{i^*}^{-1} x$. □

The assumption on $\operatorname{char}(\Bbbk)$ in Theorem 2.23 can be relaxed at the cost of a case distinction similar to that in [8, Theorem 2.1].

The explicit form of p_L depends on a choice of lattice basis. Because the notions lattice basis ideal and lattice ideal are not the same in the polynomial ring (they are in the Laurent polynomial ring), one needs to pull back using colon ideals to get a result in the polynomial ring. Even if in the Laurent polynomial ring the subintersection in Theorem 2.23 is principal modulo $I(\rho)$, it need not be principal in the polynomial ring (as visible in Theorem 2.11). It would be very nice to find more effective methods for binomial subintersections in the polynomial ring, but at the moment the following remark is all we have.

REMARK 2.24: Under the field assumptions in Theorem 2.23, let $I \subseteq \Bbbk[\mathbb{N}^n]$ be a lattice ideal in a polynomial ring with indeterminates $x_1, \ldots, x_n$. There exists a partial character $\rho : L \to \Bbbk^*$ such that $I = I(\rho) \cap \Bbbk[\mathbb{N}^n]$. The intersection of all but one minimal primary components of I is

$$(I(\rho) + \phi_\rho(p_L)) \cap \Bbbk[\mathbb{N}^n] = (I + \phi_\rho(p)m) : \left(\prod_{i=1}^{n} x_i\right)^\infty .$$

where p_L is the generating function of a fundamental parallelepiped of L, and m is any monomial such that $\phi_\rho(p_L)m \in \Bbbk[\mathbb{N}^n]$.

3. Extensions

The broadest possible generalization of the results in Section 2 may start from an arbitrary toric ideal $I \subseteq \Bbbk[\mathbb{N}^n]$, corresponding to a grading matrix $V \in \mathbb{N}^{d\times n}$, and a subideal $J \subseteq I$, for example a lattice basis ideal. One can then ask when the quotient $\Bbbk[\mathbb{N}^n]/J$ is toral in the grading V. The techniques in Section 2 depend heavily on this property and the very controllable stabilization of the Hilbert function. One can get the feeling that this happens if $J \subseteq I$ is a lattice ideal for some lattice that is of finite index in the saturated lattice $\ker_{\mathbb{Z}}(V)$. However, such a J cannot always be found: by a result of Cattani, Curran, and Dickenstein, there exist toric ideals that do not contain a binomial complete intersection of the same dimension [3].

A more direct generalization of the results of Section 2 was suggested to us by Aldo Conca. The d-th Veronese grading $V_{d,n}$ has as its columns all vectors of length n and weight d. The corresponding toric ideal is the d-th Veronese ideal $I_{d,n} \subseteq S = \Bbbk[\mathbb{N}^N]$ and it contains a natural complete intersection $J_{d,n}$ defined as follows. The set of columns of $V_{d,n}$ includes the multiples of the unit vectors $D := \{de_i, i = 1, \dots, n\}$. For any column $v \notin D$, let $f_v = x_v^d - \prod_i x_{de_i}^{v_i}$. Then $J = \langle f_v : v \notin D\rangle \subseteq I_{d,n}$ is a complete intersection with $\operatorname{codim}(J_{d,n}) = \operatorname{codim}(I_{d,n})$. It is natural to conjecture that a statement similar to Proposition 2.5 is true. In this case, however, the group L/L' (cf. Proposition 2.3) has higher torsion. This implies that the binomial primary decomposition of J exists only if $\Bbbk$ has corresponding roots of unity. By results of Goto and Watanabe [13, Chapter 3] on the canonical module (cf. [2, Exercise 3.6.21]) the ring S/I is Gorenstein if and only if $d|n$, so that $J : I$ is equal to $J + (p)$ for some polynomial p exactly in this situation.

In Section 2, the notation can be kept in check because there is a nice representation of monomials as upper triangular matrices (Proposition 2.5, Lemma 2.13, etc.). To manage the generalization, it will be an important task to find a similarly nice representation. It is entirely possible that something akin to the string notation of [20, Section 14] does the job. Additionally, experimentation with Macaulay2—which has informed the authors of this paper—will be hard. For example, for $d = 3, n = 4$, the group L/L' from Proposition 2.3 is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^{13}$ which means that a prime decomposition of $J_{3,4}$ has 1594323 components. Computing subintersections of it is out of reach. It may be possible to compute a colon ideal like $(J_{3,4} : I_{3,4})$ directly, but off-the-shelf methods failed for us.

References

[1] R. Berger, *Differentialmoduln eindimensionaler lokaler Ringe*, Mathematische Zeitschrift **81** (1963), no. 4, 326–354.

[2] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge University Press, 1998.

[3] E. Cattani, R. Curran, and A. Dickenstein, *Complete intersections in toric ideals*, Proceedings of the American Mathematical Society **135** (2007), no. 2, 329–335.

[4] A. Conca, *Divisor class group and canonical class of determinantal rings defined by ideals of minors of a symmetric matrix*, Archiv der Mathematik **63** (1994), no. 3, 216–224.

[5] A. Conca, *Gröbner bases of ideals of minors of a symmetric matrix*, Journal of Algebra **166** (1994), no. 2, 406–421.

[6] P. Diaconis, D. Eisenbud, and B. Sturmfels, *Lattice walks and primary decomposition*, Mathematical Essays in Honor of Gian-Carlo Rota (B. Sagan and R. Stanley, eds.), Progress in Mathematics, vol. 161, Birkhauser, Boston, 1998, pp. 173–193.

[7] A. Dickenstein, L. Matusevich, and E. Miller, *Combinatorics of binomial primary decomposition*, Mathematische Zeitschrift **264** (2010), no. 4, 745–763.

[8] D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Mathematical Journal **84** (1996), no. 1, 1–45.

[9] E. Gorla, *The G-biliaison class of symmetric determinantal schemes*, Journal of Algebra **310** (2007), no. 2, 880–902.

[10] E. Gorla, *Symmetric ladders and G-biliaison*, Liaison, Schottky Problem and Invariant Theory, Springer, 2010, pp. 49–62.

[11] S. Goto, *The divisor class group of a certain Krull domain*, Journal of Mathematics of Kyoto University **17** (1977), 47–50.

[12] S. Goto, *On the Gorensteinness of determinantal loci*, Journal of Mathematics of Kyoto University **19** (1979), no. 2, 371–374.

[13] S. Goto and K. Watanabe, *On graded rings I*, Journal of the Mathematical Society of Japan **30** (1978), no. 2, 179–213.

[14] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[15] W. Gröbner, *Über Veronesesche Varitäten und deren Projektionen*, Archiv der Mathematik **16** (1965), no. 1, 257–264.

[16] A. Jensen, T. Kahle, and L. Katthän, *Finding binomials in polynomial ideals*, Research in the Mathematical Sciences **4** (2017), no. 16, 10p.

[17] T. Kahle and E. Miller, *Decompositions of commutative monoid congruences and binomial ideals*, Algebra and Number Theory **8** (2014), no. 6, 1297–1364.

[18] R. E. Kutz, *Cohen–Macaulay rings and ideal theory in rings of invariants of algebraic groups*, Transactions of the American Mathematical Society **194** (1974), 115–129.

[19] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*, GTM, vol. 227, Springer, Berlin, 2005.

[20] B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.

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