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On upper and lower bounds for finite group-actions on bounded surfaces, handlebodies, closed handles and finite graphs[1]

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Abstract. *In the present paper, partly a survey, we discuss upper and lower bounds for finite group-actions on bounded surfaces, 3-dimensional handlebodies and closed handles, handlebodies in arbitrary dimensions and finite graphs (the common feature of these objects is that they have all free fundamental group).*

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1. Introduction

We consider finite group-actions of large order on various low-dimensional manifolds with free fundamental group, and also on higher-dimensional analogues and on finite graphs. All group-actions in the present paper will be faithful and smooth. The manifolds we consider are the following:

Section 1.1: 3-dimensional handlebodies and the closely related case of surfaces with nonempty boundary;

Section 1.2: closed 3-dimensional handles, i.e. connected sums $\sharp_g(S^1 \times S^2)$ of g copies of $S^1 \times S^2$, considering first arbitrary actions and then free actions (the case of free actions is in close analogy with the results in section 1.1);

Section 1.3: handlebodies in arbitrary dimensions;

Section 1.4: finite graphs, considering also finite group-actions on finite graphs embedded in spheres.

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1.1. Finite group-actions on 3-dimensional handlebodies and bounded surfaces

In analogy with the classical Hurwitz-bound $84(g-1)$ for the order of a finite, orientation-preserving group-action on a closed, orientable surface of genus $g \geq 2$, an uper bound for the order of a finite group of orientation-preserving diffeomorphisms of a a 3-dimension handlebody V_g of genus $g \geq 2$ is $12(g-1)$ ([18],[5, Theorem 7.2]). More generally, the following holds.

THEOREM 1.1 ([8]). *Let $m_{hb}(g)$ denote the maximum order of a finite, orientation-preserving group-action on a 3-dimensional handlebody of genus $g > 1$.*

i) There are the upper and lower bounds

$$4(g+1) \ \leq \ m_{hb}(g) \ \leq \ 12(g-1),$$

and both $4(g+1)$ and $12(g-1)$ occur for infinitely many genera g.

ii) If g is odd then $m_{hb}(g) = 8(g-1)$ or $m_{hb}(g) = 12(g-1)$, and both cases occur for infinitely many values of g.

iii) The possible values of $m_{hb}(g)$ are of the form $\frac{4n}{n-2}(g-1)$, for an integer $n \geq 3$, and infinitely many values of n occur resp. do not occur. Moreover if a value of n occurs then it occurs for infinitely many g.

For finite group-actions on bounded surfaces (compact with nonempty boundary, orientable or not), exactly the same results hold (and with the same proofs), using the setting in [6] (see [8, Section 3]). Note that, by taking the product with an interval (twisted if the surface is nonorientable), every finite group acting on a bounded surface of *algebraic genus* g (defined as the rank of the free fundamental group) admits also an orientation-preserving action on a handlebody of genus g.

THEOREM 1.2. *Let $m_{bs}(g)$ denote the maximum order of a finite, possibly orientation-reversing group-action on a bounded, orientable or nonorientable surface of algebraic genus $g > 1$.*

i) $m_{bs}(g) \ \leq \ m_{hb}(g)$.

ii) All statements of Theorem 1.1 remain true for $m_{bs}(g)$.

iii) There are values of g such that $m_{bs}(g)$ is strictly smaller than $m_{hb}(g)$.

Part iii) of Theorem 1.2 is proved in [1] by computational methods; the two smallest values of g such that $m_{bs}(g) < m_{hb}(g)$ are $g = 161$ and $g = 3761$.

1.2. Finite group-actions on closed handles

After the classical cases of 3-dimensional handlebodies and bounded surfaces, we consider actions of finite groups G on closed 3-dimensional analogues of handlebodies, the connected sums $H_g = \sharp_g(S^1 \times S^2)$ of g copies of $S^1 \times S^2$ (similar as $V_g = \sharp_g^{\partial}(S^1 \times D^2)$ is the boundary-connected sum of g copies of $S^1 \times D^2$; so H_g is the double of V_g along its boundary). We will call H_g a *closed handle* or just a *handle* of genus g

Since H_g admits S^1-actions (see [9]), it admits finite cyclic group-actions of arbitrarily large order acting trivially on the fundamental group. Let G_0 denote the normal subgroup of all elements of G acting trivially on the fundamental group (up to inner automorphisms); by [14, Proposition 2], G_0 is cyclic, the quotient H_g/G_0 is again a closed handle of the same genus g and the factor group G/G_0 acts faithfully on the fundamental group of the quotient $H_g/G_0 \cong H_g$. Hence one is led to consider actions of finite groups G on H_g which act faithfully on the fundamental group, i.e. induce an injection into the outer automorphism group $\mathrm{Out}\, F_g$ of the fundamental group of H_g, the free group F_g of rank g.

THEOREM 1.3 ([14]). *Let $m_{ch}(g)$ denote the maximum order of a finite, orientation-preserving group-action on a closed handle H_g of genus $g > 1$ which induces a faithful action on the fundamental group.*

i) For $g \geq 15$, there is the quadratic upper bound $m_{ch}(g) \leq 24g(g-1)$.

ii) For all g, there are the quadratic lower bounds $2g^2 \leq m_{ch}(g)$ if g is even, and $(g+1)^2 \leq m_{ch}(g)$ if g is odd.

We don't know the exact value of $m_{ch}(g)$ at present but believe that for large g it coincides with the lower bounds $2g^2$ resp. $(g+1)^2$ of the second part of Theorem 1.3; for small values of g there are group-actions of larger orders, e.g. $m_{ch}(2) = 12$, $m_{ch}(3) = 48$ and $m_{ch}(4) = 192$.

Next we consider the case of *free* actions of finite groups on closed handles H_g which is in strong analogy with the cases of arbitrary (i.e., not necessarily free) actions on handlebodies and bounded surfaces (where free means that every nontrivial element has empty fixed point set).

THEOREM 1.4. *Let $m_{chf}(g)$ denote the maximum order of a free, orientation-preserving finite group-action on a closed handle H_g of genus $g > 1$.*

i) For all $g > 1$,

$$2(g+1) \leq m_{chf}(g) \leq 6(g-1)$$

and both $2(g+1)$ and $6(g-1)$ occur for infinitely many genera g.

ii) If g is odd then $m_{chf}(g) = 4(g-1)$ or $m_{chf}(g) = 6(g-1)$, and both cases occur for infinitely many g.

iii) The possible values of $m_{chf}(g)$ are of the form $\frac{2n}{n-2}(g-1)$, for an integer $n \geq 3$, and infinitely many values of n occur resp. do not occur.

We note that exactly the same results hold for finite orientation-preserving group-actions on bounded, orientable surfaces of algebraic genus g.

The proof of Theorem 1.4 combines methods of the handlebody case (Theorem 1.1) with those for closed handles (Theorem 1.3); since it is shorter and less technical, as an illustration of the methods we give the proof in section 2.

1.3. Finite group-actions on handlebodies in arbitrary dimensions

A closed handle H_g is the boundary of a 4-dimensional handlebody, in particular the upper bounds of Theorem 1.3 i) hold also for finite group-actions on 4-dimensional handlebodies. More generally, an orientable handlebody V_g^d of dimension d and genus g is defined as a regular neighbourhood of a finite graph, with free fundamental group of rank g, embedded in the sphere S^d; alternatively, it is obtained from the closed disk D^d of dimension d by attaching along its boundary g copies of a handle $D^{d-1} \times [0,1]$ in an orientable way, or as the boundary-connected sum $\sharp_g^{\partial}(S^1 \times D^{d-1})$ of g copies of $S^1 \times D^{d-1}$.

After Thurston and Perelman, finite group-actions in dimension 3 are geometric; this is no longer true in higher dimensions, so in order to generalize Theorem 1.3 one has to consider some kind of standard actions also in higher dimensions. A natural way to proceed is to uniformize handlebodies V_g^d by Schottky groups (free groups of Möbius transformations of D^d acting by isometries on its interior, the Poincaré-model of hyperbolic space $\mathbb{H}^d$); this realizes the interior of a handlebody V_g^d as a hyperbolic manifold, and we will consider finite groups of isometries of such hyperbolic (Schottky type) handlebodies.

By [15], every finite subgroup of the outer automorphism group $\mathrm{Out}\, F_g$ of a free group $F_g \cong \pi_1(V_g^d)$ can be realized by the action of a group of isometries of a hyperbolic handlebody V_g^d (in the sense of the Nielsen realization problem), for a sufficiently large dimension d.

THEOREM 1.5 ([7]). *Let G be a finite group of isometries of a hyperbolic handlebody V_g^d, of dimension $d \geq 3$ and genus $g > 1$, which acts faithfully on the fundamental group.*

i) The order of G is bounded by a polynomial of degree $d/2$ in g if d is even, and of degree $(d+1)/2$ if d is odd.

ii) *The degree $d/2$ is best possible in even dimensions whereas in odd dimensions the optimal degree is either* $(d-1)/2$ *or* $(d+1)/2$.

So in odd dimensions the optimal degree remains open at present; note that, for $d = 3$, the bound $(d+1)/2 = 2$ is not best possible since it gives a quadratic bound instead of the actual linear bound $12(g-1)$, so maybe for all odd dimensions the optimal degree is $(d-1)/2$.

1.4. Finite group-actions on finite graphs

Let G be a finite group of automorphisms of a finite graph $\tilde{\Gamma}$ of rank $g > 1$ (defined as the rank of its free fundamental group), allowing closed and multiple edges. Note that, without changing the rank of a graph, we can delete all *free edges*, i.e. nonclosed edges with a vertex of valence 1 (an isolated vertex). By possibly subdividing edges, we can also assume that G acts *without inversions* (of edges), i.e. no element acts on an edge as a reflection in its midpoint. We say that a finite graph is *hyperbolic* if it has rank $g > 1$ and no free edges. In the following, all finite group-actions on graphs will be faithful and without inversions.

By [15], each finite subgroup G of the outer automorphism group $\mathrm{Out}\, F_g$ of a free group F_g can be induced by an action of G on a finite graph of rank g (this is again a version of the Nielsen realization problem). Conversely, if G acts on a hyperbolic graph then G induces an injection into $\mathrm{Out}\, F_g$ ([16, Lemma 1]). By [13], for $g \geq 3$ the largest possible order of a finite subgroup of $\mathrm{Out}\, F_g$ is $2^g g!$; in particular, there is no linear or polynomial bound in g for the order of G. In strong analogy with Theorems 1.1 and 1.4, the following holds (proved in section 3):

Theorem 1.6. i) *The maximal order of a finite group G acting with trivial edge stabilizers and without inversions on a finite hyperbolic graph of rank g is equal to* $6(g-1)$ *or* $4(g-1)$*, and both cases occur for infinitely many values of g.*

ii) *Let G be a finite group acting without inversions on a finite hyperbolic graph. If c denotes the order of an edge stabilizer of the action of G then* $|G| \leq 6c(g-1)$.

iii) *Equality* $|G| = 6c(g-1)$ *is obtained only for* $c = 1$*, 2, 4, 8 and 16. There are infinitely many values of c such that the second largest possibility* $|G| = 4c(g-1)$ *is obtained.*

Finally, Theorem 1.5 has the following application to finite group-actions on finite graphs embedded in spheres.

THEOREM 1.7 ([19]). *Let G be a finite subgroup of the orthogonal group $\mathrm{O}(d+1)$ acting on a pair (S^d, Γ), for a finite hyperbolic graph Γ of genus $g > 1$ embedded in S^d. Then the order of G is bounded above by a polynomial of degree $d/2$ in g if d is even and of degree $(d+1)/2$ if d is odd. The degree $d/2$ is best possible in even dimensions whereas in odd dimensions the optimal degree is either $(d-1)/2$ or $(d+1)/2$.*

For the case of the 3-sphere, by Theorem 1.1 an upper bound for the order of G is $12(g-1)$. The finitely many (hyperbolic) graphs Γ in S^3 for which this upper bound is attained are classified in [12], and the possible genera are $g =$ 2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241 and 601. By Theorem 1.3, an upper bound for the case of S^4 is $24g(g-1)$, for $g \geq 15$.

2. Finite group-actions on closed handles and the proof of Theorem 1.4

We briefly recall some concepts from [14]. Let G be a finite group acting on a handle H_g. By the equivariant sphere theorem, there is an equivariant decomposition of H_g into 0-handles S^3 connected by 1-handles $S^2 \times [-1, 1]$, and an associated finite graph $\tilde{\Gamma}$ with an action of g. In the language of [14], this induces on the quotient orbifold H_g/G the structure of a *closed handle-orbifold*, i.e. H_g/G decomposes into 0-handle orbifolds S^3/G_v connected by 1-handle orbifolds $(S^2/G_e) \times [-1, 1]$ (in the case of a free action, H_g/G is a 3-manifold and one may just use the classical decomposition into prime manifolds). This defines a finite graph of finite groups $(\Gamma, \mathcal{G})$ associated to the G-action, with underlying graph $\Gamma = \tilde{\Gamma}/G$; by subdividing edges, we assume here that G acts without inversions of edges on Γ. The vertices of Γ correspond to the 0-handle orbifolds, the edges to the 1-handle orbifolds. The vertex groups G_v of $(\Gamma, \mathcal{G})$ are the stabilizers in G of the 0-handles S^3 of H_g and isomorphic to finite subgroups of the orthogonal group $\mathrm{SO}(4)$, the edge groups G_e are stabilizers of 1-handles of H_g and isomorphic to finite subgroups of $\mathrm{SO}(3)$. The fundamental group $\pi_1(\Gamma, \mathcal{G})$ of the graph of groups $(\Gamma, \mathcal{G})$ is defined as the iterated free product with amalgamation and HNN-extension of the vertex groups along the edge groups (starting with a maximal tree), and is isomorphic to the orbifold fundamental group of the quotient orbifold H_g/G. There is a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$, and H_g is the orbifold covering of H_g/G associated to the kernel of ϕ (isomorphic to the free group F_g). Conversely, if $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$ is a surjection with torsionfree kernel onto a finite group G then its kernel is the fundamental group of a graph of groups with trivial vertex and edge groups (a free group) defining a handlebody which is a regular orbifold covering of the handle-orbifold associated to the graph of groups $(\Gamma, \mathcal{G})$.

We will assume in the following that the graph of groups $(\Gamma, \mathcal{G})$ has no *trivial* *edges*, i.e. edges with two different vertices such that the edge group coincides with one of the two vertex groups (by contracting such edges).

We denote by

$$\chi(\Gamma, \mathcal{G}) = \sum \frac{1}{|G_v|} - \sum \frac{1}{|G_e|}$$

the Euler characteristic of the graph of groups $(\Gamma, \mathcal{G})$ (the sum is taken over all vertex groups G_v resp. edge groups G_e of $(\Gamma, \mathcal{G})$); then

$$g - 1 = -\chi(\Gamma, \mathcal{G})\ |G|$$

(see [10, 11, 17] for the general theory of graphs of groups, groups acting on trees and groups acting on finite graphs).

Remark 2.1: The approach to finite group-actions on 3-dimensional handlebodies (Theorem 1.1) is analogous, using the equivariant Dehn lemma/loop theorem instead of the equivariant sphere theorem. The 0-handles are disks D^3 connected by 1-handles $D^2 \times [-1, 1]$, the vertex groups of the graph of groups $(\Gamma, \mathcal{G})$ are finite subgroups of SO(3) and the edge groups finite subgroups of SO(2) (i.e., cyclic groups). In the case of maximal order $12(g-1)$, $\pi_1(\Gamma, \mathcal{G})$ is one of the following four products with amalgamation ([5, 15]):

$$\mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{S}_3, \quad \mathbb{D}_3 *_{\mathbb{Z}_3} \mathbb{A}_4, \quad \mathbb{D}_4 *_{\mathbb{Z}_4} \mathbb{S}_4, \quad \mathbb{D}_5 *_{\mathbb{Z}_5} \mathbb{A}_5$$

where $\mathbb{D}_n$ denotes the dihedral group of order $2n$, $\mathbb{A}_4$ and $\mathbb{A}_5$ the alternating groups of orders 12 and 60, and $\mathbb{S}_4$ the symmetric group of order 24.

Remark 2.2: For the case of finite group-actions on bounded surfaces (Theorem 1.2), one decomposes the action along properly embedded arcs; the 0-handles are disks D^2 connected by 1-handles $D^1 \times [-1, 1]$, the vertex groups of $(\Gamma, \mathcal{G})$ are finite subgroups of O(2) (cyclic or dihedral) and the edge groups subgroups of $O(1) \cong \mathbb{Z}_2$ (i.e., of order two generated by a reflection of D^1, or trivial). In the case of maximal order $12(g-1)$, $\pi_1(\Gamma, \mathcal{G})$ is the free product with amalgamation $\mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{D}_3$ (the first of the four groups in part i).

Proof of Theorem 1.4. i) Suppose that G acts freely on a closed handle H_g, $g > 1$. Since there are no orientation-preserving free actions of a finite group on S^2, the edge groups of the associated graph of groups $(\Gamma, \mathcal{G})$ are all trivial. It is easy to see then that the minimum positive value for $-\chi(\Gamma, \mathcal{G})$ is realized exactly by the graph of groups $(\Gamma, \mathcal{G})$ with exactly one edge and vertex groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, with $\pi_1(\Gamma, \mathcal{G}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ and $-\chi(\Gamma, \mathcal{G}) = 1 - 1/2 - 1/3 = 1/6$ (we will say that $(\Gamma, \mathcal{G})$ is of type (2,3) in the following), and hence $|G| = 6(g-1)$ is the largest possible order.

Let M be the 3-manifold which is the connected sum of two lens spaces with fundamental groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, with $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Let ϕ be a surjection from $\mathbb{Z}_2 * \mathbb{Z}_3$ to a cyclic or dihedral group G of order 6. The regular covering of M associated to the kernel F_2 of ϕ (torsionfree, hence free) is a closed handle of genus 2 on which G acts as the group of covering transformations, and this realizes the largest possible order $|G| = 6(g-1)$. By factorizing $\mathbb{Z}_2 * \mathbb{Z}_3$ by characteristic subgroups of F_2 of arbitrary large finite indices, one obtains examples for the maximal order $|G| = 6(g-1)$ for arbitrarily large values of g (see also the remark after the proof for explicit examples realizing the maximum order).

Concerning the lower bound $2(g+1)$, we consider a graph of groups $(\Gamma, \mathcal{G})$ with exactly one edge, of type $(2, n)$, with $\pi_1(\Gamma, \mathcal{G}) \cong \mathbb{Z}_2 * \mathbb{Z}_n$ and $-\chi(\Gamma, \mathcal{G}) = 1 - 1/2 - 1/n = (n-2)/2n$. Let M be the connected sum of two lens spaces, with $\pi_1(M) \cong \mathbb{Z}_2 * \mathbb{Z}_n$, and let $\phi : \mathbb{Z}_2 * \mathbb{Z}_n \to \mathbb{D}_n$ be a surjection onto the dihedral group of order $2n$. The covering of M associated to the kernel of ϕ is a closed handle H_g of genus g with a free G-action; also, $g - 1 = (n-2)|G|/2n = n - 2$, hence $n = g + 1$ and $|G| = 2(g+1)$. (Note that, if n is even, there is also a surjection of $\mathbb{Z}_2 * \mathbb{Z}_n$ onto the cyclic group $\mathbb{Z}_n$ which gives an order $|G| = n = 2g < 2(g+1)$.)

It remains to show that $|G| = 2(g+1)$ is the largest possible order for infinitely many values of g. It is easy to see that the graphs of groups $(\Gamma, \mathcal{G})$ with trivial edge groups and with a possible surjection of $\pi_1(\Gamma, \mathcal{G})$ onto a group of order $|G| > 2(g+1)$ have exactly one edge, of type $(2, n)$, (3,3), (3,4) or (3,5).

We exclude first the case of a graph of groups $(\Gamma, \mathcal{G})$ of type $(2, n)$. A finite quotient G of $\pi_1(\Gamma, \mathcal{G})$ with torsionfree kernel has order xn, for some positive integer x, hence $|G| = xn = 2n(g-1)/(n-2)$ and $x(n-2) = 2(g-1)$. Suppose that $g - 1$ is a prime number. As seen above, the cases $x = 1$ and $x = 2$ give orders $2g$ and $2(g+1)$, so we can assume that $x > 2$ and hence $n = 3$ or $n = 4$.

Let $n = 3$, so G has order $6(g-1)$. Suppose in addition that $g > 7$; then 6 and $g - 1$ are coprime and, by a result of Schur-Zassenhaus, G is a semidirect product of $\mathbb{Z}_{g-1}$ and a group $\bar{G}$ of order 6. We can also assume that 3 does not divide $g - 2$ (or, equivalently, that $g - 1$ is one of the infinitely many primes congruent to 2 mod 3, by a result of Dirichlet). Then an element of order 3 in $\bar{G}$ acts trivially on $\mathbb{Z}_{g-1}$ by conjugation, the element of order 2 acts trivially or dihedrally, and this implies easily that G cannot be generated by two elements of orders 2 and 3.

Now let $n = 4$; then G has order $4(g-1)$ and is a semidirect product of $\mathbb{Z}_{g-1}$ and $\mathbb{Z}_4$ (since $g - 1$ is prime). Suppose that $g - 1$ is one of the infinitely many primes congruent to 11 mod 12, so 4 does not divide $g-2$ (and, as before, 3 does not divide $g - 2$). Then the element of order two in $\mathbb{Z}_4$ acts trivially on $\mathbb{Z}_{g-1}$, is the unique element of order two in G and the square of every element of order 4, so clearly G cannot be generated by two elements of orders 2 and 4.

It remains to exclude the types (3,3), (3,4) and (3,5). If $(\Gamma, \mathcal{G})$ is of type

(3,3) then $|G| = 3(g-1)$. As before, since 3 does not divide $g-2$, there is a unique subgroup of order 3 in G and G is not generated by two elements of order 3. In the cases (3,4) and (3,5) one has $|G| = 12(g-1)/5$ and $|G| = 15(g-1)/7$; since $g = 8$ is already excluded, also these two cases are not possible.

We have shown that $m_{chf}(g) = 2(g+1)$ for infinitely many genera g, and this concludes the proof of part i) of Theorem 1.4.

ii) For an odd integer g, we consider the semidirect product $G = (\mathbb{Z}_{(g-1)/2} \rtimes \mathbb{Z}_4) \rtimes \mathbb{Z}_2$, of order $4(g-1)$. Denoting by x a generator of $\mathbb{Z}_{(g-1)/2}$, by y a generator of $\mathbb{Z}_4$ and by t a generator of $\mathbb{Z}_2$, the actions of the semidirect product are given by $yxy^{-1} = x^{-1}$, $txt^{-1} = x^{-1}$ and $tyt^{-1} = xy$. There is a surjection with torsionfree kernel $\phi : \mathbb{Z}_2 * \mathbb{Z}_4 \to G$ which maps a generator of $\mathbb{Z}_2$ to t and a generator of $\mathbb{Z}_4$ to y. As before, ϕ defines a free action of G on a closed handle H_g of genus g, so $m_{chf}(g) \geq 4(g-1)$; this leaves the possibilities $m_{chf}(g) = 4(g-1)$ and $m_{chf}(g) = 6(g-1)$.

Suppose that $g = 2p+1$, for a prime $p > 12$. We show that there is no surjection ϕ of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto a group G of order $6(g-1) = 12p$, and hence $m_{chf}(g) = 4(g-1)$. By the Sylow theorems, such a group G has a normal subgroup $\mathbb{Z}_p$, and the factor group is the alternating group $\mathbb{A}_4$ (since this is the only group of order 12 generated by two elements of orders 2 and 3). Again by the theorem of Schur-Zassenhaus, G is a semidirect product $\mathbb{Z}_p \rtimes \mathbb{A}_4$. If the action of $\mathbb{A}_4$ on $\mathbb{Z}_p$ is trivial then clearly such a surjection ϕ does not exist. Suppose that the action of $\mathbb{A}_4$ on $\mathbb{Z}_p$ is nontrivial; since the automorphism group of $\mathbb{Z}_p$ is cyclic, the action of $\mathbb{A}_4$ factors through a nontrivial action of the factor group $\mathbb{Z}_3$ of $\mathbb{A}_4$, and the subgroup $\mathbb{D}_2$ of $\mathbb{A}_4$ acts trivially. By the Sylow theorems, up to conjugation we can assume that a surjection ϕ maps the factor $\mathbb{Z}_3$ of $\mathbb{Z}_2 * \mathbb{Z}_3$ to $\mathbb{A}_4$. Since any involution in $\mathbb{A}_4$ acts trivially on $\mathbb{Z}_p$, every element of order 2 in G is in $\mathbb{A}_4$, and hence ϕ is not surjective.

Finally, any surjection $\phi : \mathbb{Z}_2 * \mathbb{Z}_3 \to \mathbb{A}_4$ defines a free action of $\mathbb{A}_4$ on a closed handle of genus 3. Factorizing by characteristic subgroups of arbitrary large indices of the kernel F_3 of ϕ, one obtains $m_{chf}(g) = 6(g-1)$ for infinitely many odd values of g.

iii) Suppose that $n-2$ is prime, and let $g = n-1$ and $|G| = 2n$. Then it follows easily as above that $m_{chf}(g) = 2n(g-1)/(n-2)$, hence infinitely many values of n occur. We will show that also infinitely many values of n do not occur.

Let n be congruent to 2 mod 8, $n \neq 2$, and suppose that there exists g such that $m_{chf}(g) = 2n(g-1)/(n-2)$. Then 8 divides $(n-2)m_{chf}(g) = 2n(g-1)$ and also $4(g-1)$, so $g-1$ is even and g is odd. By ii), $m_{chf}(g) = 2n(g-1)/(n-2) \geq 4(g-1)$, and this gives the contradiction $n \leq 4$.

This concludes the proof of Theorem 1.4. □

Remark 2.3: We give an explicit construction realizing the maximum order

$|G| = 6(g-1)$ for infinitely many values of g. Let $g = p+1$, for a prime p such that 6 divides $p-1$. We shall define a surjection ϕ of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto a semidirect product $G = \mathbb{Z}_p \rtimes \mathbb{Z}_6$; this defines an action of G on a closed handle of genus $g = p+1$ of maximal possible order $|G| = 6(g-1)$, hence $m_{chf}(g) = 6(g-1)$. Writing $\mathbb{Z}_p$ additively and $\mathbb{Z}_6$ multiplicatively, suppose that a generator t of $\mathbb{Z}_6$ acts by conjugation on $\mathbb{Z}_p$ by an automorphism of order 6, in particular t^3 acts dihedrally on $\mathbb{Z}_p$; let α be the automorphism of order 3 of $\mathbb{Z}_p$ induced by t^2. Fixing a generator a of $\mathbb{Z}_p$, one has $\alpha(a) = a + b$ for some $b \neq 0$ in $\mathbb{Z}_p$; then $\alpha^3(a) = a$ implies $b + \alpha(b) + \alpha^2(b) = 0$. Considering the factors of $\mathbb{Z}_2 * \mathbb{Z}_3$, let ϕ map a generator of $\mathbb{Z}_2$ to $bt^3 = t^3(-b)$ and a generator of $\mathbb{Z}_3$ to bt^2. Since also b generates $\mathbb{Z}_p$, clearly ϕ is a surjection.

3. Finite group-actions on finite graphs and the proof of Theorem 1.6

Let G be a finite group acting without inversions on a finite, hyperbolic graph $\tilde{\Gamma}$ of rank g. Considering the quotient graph $\Gamma = \tilde{\Gamma}/G$, we associate to each vertex group and edge group of Γ the stabilizer in G of a preimage in $\tilde{\Gamma}$ (starting with a lift of a maximal tree in Γ to $\tilde{\Gamma}$); this defines a finite graph of finite groups $(\Gamma, \mathcal{G})$ and a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$, injective on vertex groups, with kernel F_g. Conversely, by the theory of groups acting on trees and graphs of groups (see [10, 11, 17]), such a surjection $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$ defines an action of G on a finite graph $\tilde{\Gamma}$ of rank $g = -\chi(\Gamma, \mathcal{G})\ |G| + 1$; the action of G on $\tilde{\Gamma}$ is faithful if and only if every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial (since a finite normal subgroup must be contained in all edge groups; see [7, Lemma 1]).

Proof of Theorem 1.6. i) Let G be a finite group which acts with trivial edge stabilizers and without inversions on a finite graph $\tilde{\Gamma}$, of rank $g > 1$. Then the associated graph of groups $(\Gamma, \mathcal{G})$ has trivial edge groups, and clearly $-\chi(\Gamma, \mathcal{G}) = 1/6$ is the smallest positive value which can be obtained for the Euler characteristic $\chi(\Gamma, \mathcal{G})$ (realized by the graph of groups with one edge and edge groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$). Hence $|G| \leq 6(g-1)$ and, as in the proof of Theorem 1.4, the upper bound $6(g-1)$ is obtained for infinitely many values of g.

For an integer $m > 1$, choose a surjection $\phi : \mathbb{Z}_2 * \mathbb{D}_2 \to G$ where G is the dihedral group $\mathbb{D}_{2m}$ or the group $\mathbb{Z}_2 \times \mathbb{D}_m$, of order $4m$. Then ϕ defines an action of G on a finite graph of rank $g = m+1$, hence $|G| = 4m = 4(g-1)$. On the other hand, if $g-1$ is a prime such that 3 does not divide $g-2$ then it follows as in the proof of Theorem 1.4 that there does not exist a surjection of $\mathbb{Z}_2 * \mathbb{Z}_3$ onto any group of order $6(g-1)$, so $4(g-1)$ is the maximal possible order for infinitely many g.

ii) As before, the action of G on $\tilde{\Gamma}$ is associated to a surjection with torison-free kernel $\phi : \pi_1(\Gamma, \mathcal{G}) \to G$, for a finite graph of finite groups $(\Gamma, \mathcal{G})$. We can assume that $(\Gamma, \mathcal{G})$ has no trivial edges, i.e. edges with two different vertices such that the edge group coincides with one of the vertex groups (by contracting such an edge). Since the action of G on $\tilde{\Gamma}$ is faithful, every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial. Let e be an edge of Γ with an edge group of order c; let $\chi = \chi(\Gamma, \mathcal{G})$ denote the Euler-characteristic of $(\Gamma, \mathcal{G})$ and n the order of G.

Suppose first that e is a closed edge (a loop). If e is the only edge of $(\Gamma, \mathcal{G})$ then

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} = \frac{1}{2c}, \quad g - 1 = -\chi n \geq \frac{n}{2c}, \quad n \leq 2c(g-1)$$

(since every finite normal subgroup of $\pi_1(\Gamma, \mathcal{G})$ is trivial, the edge group of e cannot coincide with the vertex group).

If e is closed and not the only edge then

$$-\chi \geq \frac{1}{c}, \quad g - 1 = -\chi n \geq \frac{n}{c}, \quad n \leq c(g-1).$$

Suppose that e is not closed. If e is the only edge of $(\Gamma, \mathcal{G})$ then both vertices of e are isolated and

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} - \frac{1}{3c} = \frac{1}{6c}, \quad g - 1 = -\chi\, n \ \geq \ \frac{n}{6c}, \quad n \leq 6c(g-1).$$

If e is not closed, not the only edge and has exactly one isolated vertex then

$$-\chi \geq \frac{1}{c} - \frac{1}{2c} = \frac{1}{2c}, \quad g - 1 = -\chi\, n \ \geq \ \frac{n}{2c}, \quad n \leq 2c(g-1).$$

Finally, if e is not closed, not the only edge and has no isolated vertex then

$$-\chi \geq \frac{1}{c}, \quad g - 1 = -\chi\, n \ \geq \ \frac{n}{c}, \quad n \leq c(g-1).$$

Concluding, in all cases we have $|G| \leq 6c(g-1)$, proving ii).

iii) By [3] and [4], there are only finitely many free products with amalgamation of two finite groups, without nontrivial finite normal subgroups, such that the amalgamated subgroup has indices 2 and 3 in the two factors (and the same holds also for indices 3 and 3). These *effective* (2,3)-*amalgams* are classified in [3], there are exactly seven such amalgams (described below), and the amalgamated subgroups have order 1, 2, 4, 8 or 16. It follows then from the proof of ii) that equality $|G| = 6c(g-1)$ can be obtained only for these values of c.

On the other hand, by [2] there are infinitely many effective (2,4)-amalgams, and hence $|G| = 4c(g-1)$ is obtained for infinitely many values of c.

This concludes the proof of Theorem 1.6. □

REMARK 3.1: The seven effective $(2,3)$-amalgams, with amalgamated subgroups of orders $c = 1, 2, 4, 8$ or 16, are the following:

$$\mathbb{Z}_2 * \mathbb{Z}_3, \quad \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{D}_3, \quad \mathbb{D}_2 *_{\mathbb{Z}_2} \mathbb{D}_3, \quad \mathbb{D}_4 *_{\mathbb{D}_2} \mathbb{D}_6, \quad \mathbb{D}_8 *_{\mathbb{D}_4} \mathbb{S}_4, \quad \tilde{\mathbb{D}}_8 *_{\mathbb{D}_4} \mathbb{S}_4,$$

$$\mathbb{K}_{32} *_{(\mathbb{D}_4 \times \mathbb{Z}_2)} (\mathbb{S}_4 \times \mathbb{Z}_2),$$

where $\tilde{\mathbb{D}}_8$ denotes the quasidihedral group of order 16 and K_{32} a group of order 32.

REMARK 3.2: Finally, we describe the two families which realize the largest possible orders for all g. As noted before, the largest order of a finite group G of automorphisms of a finite graph of rank $g > 2$ without free edges (or equivalently, of a finite subgroup G of $\operatorname{Out} F_g$) is $2^g g!$, and this is realized by the automorphism group $(\mathbb{Z}_2)^g \rtimes \mathbb{S}_g$ of a finite graph with one vertex and g closed edges (a bouquet of g circles or a multiple closed edge), subdividing edges to avoid inversions. Considering the quotient graph/graph of groups, this action is associated to a surjection

$$\phi : ((\mathbb{Z}_2)^g \rtimes \mathbb{S}_g) *_{((\mathbb{Z}_2)^{g-1} \rtimes \mathbb{S}_{g-1})} ((\mathbb{Z}_2)^g \rtimes \mathbb{S}_{g-1}) \to (\mathbb{Z}_2)^g \rtimes \mathbb{S}_g$$

For $g \geq 3$, this realizes the unique action of maximal possible order $2^g g!$ (the unique finite subgroup of $\operatorname{Out} F_g$ of maximal order, up to conjugation).

The second family of large orders is given by the automorphism groups $S_{g+1} \times \mathbb{Z}_2$ of the graphs with two vertices and $g+1$ connecting edges (a multiple nonclosed edge, subdividing edges again to avoid inversions), associated to the surjections

$$\phi : \mathbb{S}_{g+1} *_{\mathbb{S}_g} (\mathbb{S}_g \times \mathbb{Z}_2) \to \mathbb{S}_{g+1} \times \mathbb{Z}_2$$

These realize the largest possible order for $g = 2$, and again for $g = 3$.

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