History

The journal *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* was founded in 1969 by Arno Predonzan, with the aim of publishing original research articles in all fields of mathematics.

Rendiconti dell'Istituto di Matematica dell'Università di Trieste has been the first Italian mathematical journal to be published also on-line. The access to the electronic version of the journal is free. All published articles are available on-line.

In 2008 the *Dipartimento di Matematica e Informatica*, the owner of the journal, decided to renew it. The name of the journal however remained unchanged, but the subtitle *An International Journal of Mathematics* was added. The journal can be obtained by subscription, or by reciprocity with other similar journals. Currently more than 100 exchange agreements with mathematics departments and institutes around the world have been entered in.

The articles published by *Rendiconti dell'Istituto di Matematica dell'Università di Trieste* are reviewed/indexed by MathSciNet, Zentralblatt Math, Scopus, OpenStarTs.

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Foreword

Number 52 of our journal Rendiconti dell'Istituto di Matematica dell'Università di Trieste is divided in two issues. The first issue contains fifteen articles dedicated to Professor Julián López-Gómez on the occasion of his 60th birthday. The second issue is divided into two sections. The first one is a collection of ten articles dedicated to Professor Bruno Zimmermann on the occasion of his 70th birthday. This section has been edited with the collaboration of our colleague Mattia Mecchia as guest editor, whose valuable help we acknowledge with great pleasure. In the second section we publish four more papers which were normally submitted to the journal and did not enter in the previous two special sections.

Alessandro Fonda
Emilia Mezzetti
Pierpaolo Omari

Preface

This part of Volume 52 of RIMUT is dedicated to Professor Julián López-Gómez on the occasion of his sixtieth birthday: it contains fifteen invited papers in the field of Nonlinear Differential Equations, authored by some distinguished mathematicians who have collaborated with him in various ways during the last three decades.

The Editor of RIMUT who has taken care of this issue, Pierpaolo Omari, wishes to thank all Authors for their valuable contributions, as well as all Experts who collaborated in the reviewing process of the papers published herein for their highly professional work. Last, but not least, a warm thank is due as well to prof. Marcela Molina-Meyer, the wife of Julián, who kindly provided a lot of information about his biography.

Julián López-Gómez was born in Sacedón, Guadalajara province, on September 11th, 1959, in "La Alcarria", where, as Julián likes to point out, the bees produce the best honey of Spain! With wonderful landscapes and cliffs over the Tagus river, that ends in Lisbon, Sacedón had 1890 inhabitants when he was born. But Sacedón did not have any High School to complete Secondary Studies in 1969. Thus, since the salary of his father could not cover the fees of the High School internship in Guadalajara, Julián began his Secondary Studies on an independent basis at the age of 10 supervised by Don Timoteo, a senior professor with almost no knowledge of "Modern Mathematics", who nevertheless really loved Mathematics and trained him to calculate very fluidly in any basis. By his lack of knowledge of "Modern Mathematics", the marks of Julián during the annual exams in the public High School of Guadalajara were extremely poor the first three years: Julián could not understand neither the usual abstract symbols to design operations in algebraic structures, nor the standard set operations, nor what a Venn diagram was. As Julián says: "Bourbakist mathematics had not been designed for Don Timoteo and myself!"

When Julián reached 12 years, the salary of his father was almost doubled and he could finally enter at the age of 13 the internship of the Diocesan College at Guadalajara, the unique one existing there, to complete his Secondary Studies. Although he needed some complementary lectures of Latin and Mathematics at the beginning, he finished his studies very brilliantly getting the best marks of the Diocesan College since its foundation. At the Secondary School, he also enjoyed a top-level mathematical educator, Doña Concha, a young mathematician, who seemed to have a secret pact with the real numbers and the sequences of rational numbers. Her construction fascinated so much Julián that he begun to spend most of his time solving all the exercises proposed by her. Nonetheless, very fortunately for him, at the High School Julián also got a complete humanistic formation in Art, History, as well as in Spanish, French and Classical Literature: disciplines that he has constantly loved and cultivated, besides Mathematics, of course.

During 1975-76, Julián completed his University Orientation Course in Santa Ana School, where he was so lucky as to enjoy another excellent educator in Mathematics, Doña Teresa. As a consequence, he got the highest marks of his province in his Entrance Exam to Complutense University of Madrid, where he completed his Degree on Pure Mathematics in the period 1976-1981. His most influential Professors during his graduation were M. de Guzmán, C. Fernández-Pérez and A. Somolinos. Professor Fernández-Pérez was the advisor of Julián in the elaboration of his Degree Thesis, defended in November 1981. Simultaneously, Julián attended the inspiring "Seminar of Mathematical Biology" organized by A. Somolinos and F. Montero, where he got fascinated by the power of Nonlinear Differential Equations in modeling many important biological phenomena. This definitively pushed him to specialize in Nonlinear Analysis and Partial Differential Equations. In June 1984, Julián defended his PhD Thesis, titled "Critical cases of Hopf bifurcation at multiples eigenvalues", under the supervision of A. Casal, although at that time Julián worked on a rather independent basis as his mathematical background was already excellent for his age. Julián feels extremely fortunate of having had some of the best possible mathematical educators in the Spanish Transition. Two years later, he got a permanent Lectureship at the Polytechnic University of Madrid and, after six months, in November 1986, he moved to another Lectureship at the Complutense University, where he later got his present position of Full Professor, after winning a National Habilitation Competition against 58 competitors. Julián was the first habilitated mathematician in Applied Mathematics of Spain!

In his brilliant career Julián has advised 14 Doctoral Thesis and has played an important role in the elaboration of the Doctoral Thesis of some other students. He delivered an impressing number of advanced courses or seminars all over the world: Europe, North and South America, Asia, Australia, and North Africa. Till now he has authored about 200 papers and 13 books. He also serves, or has served, as Editor of several international mathematical journals, in particular of our one since 2014.

All this confirms that Professor Julián López-Gómez is a leading expert in the field of Nonlinear Analysis, Bifurcation Theory, and Partial Differential Equations.

Congratulations and best wishes, Julián!

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Complicated dynamics in a model of charged particles

Oltiana Gjata and Fabio Zanolin

"Dedicated to Professor Julián López-Gómez for his 60th birthday"

Abstract. *We give an analytical proof of the presence of complex dynamics for a model of charged particles in a magnetic field. Our method is based on the theory of topological horseshoes and applied to a periodically perturbed Duffing equation. The existence of chaos is proved for sufficiently large, but explicitly computable, periods.*

Keywords: Hamiltonian systems, period map, saddle points, homoclinic solutions, chaotic dynamics, topological horseshoes.
MS Classification 2010: 34C25, 34B18, 34C35.

1. Introduction

Recent numerical studies in [5] have shown chaotic aspects in a model describing the motion of charged particles inside a tokamak magnetic field.

A tokamak is a device, invented in the 1950s by the Soviet physicists Sakharov and Tamm, which employs a powerful magnetic field to confine hot plasma in the shape of a torus and keep it away from the machine walls. At the current stage of scientific knowledge and engineering capabilities, tokamaks are still considered among the most promising devices for a possible future production of energy through controlled atomic fusion. From this point of view, the study of mathematical and physical models describing the motion of charged particles inside toroidal (or cylindrical) magnetic fields like those generated by the tokamak coils is of great significance for the possible applications to plasma physics. In the recent past, periods of great expectation on the possibility of obtaining a stable controlled nuclear fusion process using the tokamaks were followed by periods of disappointment for the failure of some critical experiments. This happened due to the discovery of several new and unexpected instability phenomena that have compromised the performance of the device, including dangerous fluctuations of the plasma going in contact with the walls of the reactor. The sensitive dependence on initial conditions is one of the typical instability phenomena appearing in connection with so-called "chaotic behavior". Although stable and random motions can coexist and thus the presence of some chaotic dynamics may be compatible with results about the boundedness of the solutions, nevertheless in many cases (typical examples come from celestial mechanics, see [9, Introduction]) small instability effects due to chaos phenomena may produce relevant long term consequences. From this point of view, investigating the possibility of chaos in differential equations models for tokamak magnetic confinement, is not only a topic with its own theoretical interest, but it may also suggest some possible issues to be taken into account by the scientists involved in the design of these devices.

In [5] the Authors have considered two different configurations leading to Hamiltonian chaos for charged particle motions in a toroidal magnetic field. In the (r, θ, ϕ) coordinates for the torus (cf. [5, Fig. 1]) the tokamak magnetic field has the following form

$$\mathbb{B} = \frac{B_0 R}{\xi}(\hat{\mathbf{e}}_\phi + f(r)\hat{\mathbf{e}}_\theta), \tag{1}$$

where $\xi = R + r\cos(\theta)$ and $\hat{\mathbf{e}}_\phi$, $\hat{\mathbf{e}}_\theta$ are the unit vectors associated respectively with the ϕ and θ directions. The toroidal component along $\hat{\mathbf{e}}_\phi$ depends upon the external magnetic field generated by the coils around the device. The constant B_0, according to [5] is the typical magnetic intensity at the center of the torus. If the plasma is present, a generated current inside the tokamak leads to the creation of a poloidal component for the magnetic field, expressed by the term $f(r)\hat{\mathbf{e}}_\theta$ [1].

In a recent paper [7], we have examined the first configuration considered by the Authors in [5], namely the case in which the poloidal component is negligible. This situation is useful for the study of the motion on an hypothetic single charged particle inside the tokamak with no plasma inside.

In the present article we focus our attention to the second case discussed in [5] in which the effect of the plasma is substantial. In order to simplify the model, in [5, Section C and IV] the Authors consider a cylindrical magnetic geometry, which is the limit, when R tends to infinity, of the toroidal system. In this approximation, the direction $\hat{\mathbf{e}}_\phi$ becomes a stationary vector, subsequently identified to the z-component. In this manner, instead of an empty toroidal solenoid, we are led now to consider a cylindrical plasma tube. An application of Newton law to a charged particle of mass m and charge q moving in this magnetic field (see Section 2 for the details), leads to an integrable system with an associated effective Hamiltonian of the form

$$H_{eff} = \frac{m\dot{r}^2}{2} + \frac{mA^2}{2r^2} + \frac{(qB_0)^2}{8m}r^2 + \frac{q^2}{2m}F^2(r), \quad r > 0, \tag{2}$$

[1] The terms "toroidal" and "poloidal" refer to directions relative to a torus of reference. The poloidal direction follows a small circular ring around the surface, while the toroidal direction follows a large circular ring around the torus (according to Wikipedia). The introduction of these terms comes from [6] for the study of the Earth's magnetic field.

where A is a positive constant and $F(r) = \int^r f(x)\, dx$ (see [5, Appendix B]). Clearly the choice of f and then F greatly influences the Hamiltonian and hence the corresponding dynamics of the particles.

Writing (2) in a dimensionless form and deriving the corresponding differential equation for the new variable $x := r > 0$, we find that the trajectories of the charged particles can be described by a second-order Duffing equation

$$\ddot{x} + g(x) = 0$$

with a singularity at the origin. In [5] the Authors propose a mechanism to produce chaotic dynamics by a perturbation of (2). More precisely, the constant A in (2) (indicated in [5] by C'' in the dimensionless version of H_{eff}) is now considered as a slowly time dependent variable. Numerical evidence of chaos for the stroboscopic (Poincaré) map is provided by the analysis of the Poincaré section. Inspired by this example, we try to analyze this problem with a different approach, by considering a time-periodic perturbation of the associated Duffing equation. Our perturbation can be produced either by a slow modification of the constant A as in [5] o, by modifying the magnetic intensity B_0 . In each case, we produce chaotic dynamics by assuming that a formerly presumed constant coefficient in (2) becomes a slowly varying stepwise periodic function. The choice of a stepwise function (following [11, 12]) has the advantage that the corresponding differential equation system becomes a switched system for which we can apply recent results from the theory of topological horseshoes and therefore we can give a rigorous analytical proof of the existence of chaos.

In our investigation and following [5], we assume for the function F (the primitive of the amplitude of the poloidal field), the expression

$$F(x) := ax^2 \exp\left(-\frac{x^2}{c^2}\right),$$

where $a, c > 0$ are suitable constants. With such a choice of the function F and tuning suitably the constants a and c (the Authors in [5] provide physically meaningful values for these constants), we can produce, for the planar system

$$\dot{x} = y, \qquad \dot{y} = -g(x),$$

a phase-portrait which consists of two local centers surrounded by periodic orbits of increasing period and bounded by two homoclinic trajectories departing from an intermediate saddle point, thus altogether shaping a typical eight figure. After a small perturbation of the magnetic field we obtain another eight shaped figure which partially overlaps with the previous one. Near the intersections of the homoclinic trajectories associated with the two portraits we can define some appropriate rectangular regions where we can prove the existence of chaos on m-symbols ($m \geq 2$), for the Poincaré map, using the "stretching along the paths" (SAP) technique [13, 17]. It is well known that for periodic planar systems obtained as a perturbation of an autonomous system with a homoclinic orbit at a saddle point, the Melnikov method (see [8]) is a powerful tool to verify the existence of chaotic dynamics. Relevant developments for periodically perturbed Duffing equations are given in [3, 15]. In the applications of the Melnikov method one has to prove the existence of simple zeros for suitable integrals depending on the explicit analytical expression of the homoclinic solution. Unfortunately, in our example, such analytical expression is not available and this motivates the use of a different approach.

The plan of the paper is the following. In Section 2 we briefly describe the mathematical model considered in [5] in order to give a physical justification about the Hamiltonian defined in (2). In Section 3 we choose a special form for $F(x)$ (as proposed in [5]) which produces a double well potential in H_{eff}. Next in the same section, we also discuss the corresponding phase-portrait for the associated Duffing equation and then, as a further step, we introduce the time-periodic perturbation on the differential equation and define six rectangular regions where we will focus our analysis for the SAP technique. Section 4 contains our main result about chaotic dynamics whose proof is finally given in the subsequent Section 5.

2. Mathematical model

We follow the calculations in [5, Appendix B], in order to introduce the mathematical model that we are going to study. In [5] the Authors introduce a cylindrical magnetic geometry, which is considered as the limit, when R tends to infinity, of the toroidal system. The approximation to new geometric configuration leads to a magnetic field rewritten as

$$\mathbb{B} = B_0 \hat{\mathbf{e}}_z + f(r)\hat{\mathbf{e}}_\theta.$$

This is derived in [5] from (1) as a limit for $R \to \infty$ and considering the z-direction identified with the axes along with $\hat{\mathbf{e}}_\phi$, which is considered now as a constant. In order to avoid misunderstanding, it is important to notice (cf. [5, Appendix B]) that the z-direction here is not the one considered originally in [5, Fig. 1]. Moreover, with respect to (1), now the function f already incorporates the effect of B_0.

In order to find the differential system describing the dynamics of the particle of mass m and charge q moving in this magnetic field, we use the fact that the force acting on the charged particle is given by $\mathbf{F} = q\vec{v} \wedge \mathbb{B}$ (where $\vec{v}$ is the velocity of the particle). Next we recall also the expressions of the velocity and the acceleration in cylindrical coordinates, namely

$$\vec{v} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta + \dot{z}\hat{\mathbf{e}}_z$$

and

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta + \ddot{z}\hat{\mathbf{e}}_z.$$

Then, an application of the Newton second law, yields to

$$\begin{cases} \ddot{r} - r\dot{\theta}^2 = \frac{q}{m}(B_0 r\dot{\theta} - f(r)\dot{z}) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} = -\frac{qB_0}{m}\dot{r} \\ \ddot{z} = \frac{q}{m}\dot{r}f(r) \end{cases} \tag{3}$$

Multiplying by r the second equation and then integrating the second and the third equations, we obtain

$$\begin{cases} \dot{\theta} = \frac{A}{r^2} - \frac{qB_0}{2m} \\ \dot{z} = \frac{q}{m}F(r) \end{cases} \tag{4}$$

where A is a constant and $F(r) = \int^r f(x)dx$. Substituting the two equations of (4) into the first equation of (3), we obtain the second-order ODE

$$\ddot{r} - \frac{A^2}{r^3} + \left(\frac{qB_0}{2m}\right)^2 r + \frac{q^2}{m^2} f(r)F(r) = 0. \tag{5}$$

Multiplying equation (5) by $\dot{r}$ and then integrating we finally obtain

$$\int \dot{r}\ddot{r}dt - \int_{r=r(t)} \frac{A^2}{r^3}dr + \left(\frac{qB_0}{2m}\right)^2 \int_{r=r(t)} rdr + \frac{q^2}{m^2}\int_{r=r(t)} F(r)F'(r)dr = \text{constant}.$$

Thus we end up with an effective Hamiltonian, which is precisely the one considered in (2), namely

$$H_{eff} := \frac{m\dot{r}^2}{2} + \frac{mA^2}{2r^2} + \frac{(qB_0)^2}{8m}r^2 + \frac{q^2}{2m}F^2(r).$$

3. Geometric configurations

Following [5] we consider now the effective Hamiltonian

$$H_{\text{eff}} := \frac{\dot{r}^2}{2} + \frac{A^2}{2r^2} + \frac{B_0^2}{8}r^2 + F^2(r) \tag{6}$$

for

$$F(r) := ar^2 \exp\left(-\frac{r^2}{c^2}\right), \tag{7}$$

where A, a, c are suitable positive constants and B_0 is the intensity (magnitude) of the magnetic field. Without loss of generality, we have considered in (6) a unitary mass m and a unitary charge q (cf. formula ($B7$) in [5]). According to (2), the term depending on $f(r)$ should be of the form $F^2(r)/2$, but clearly there is no mistake in replacing it with $F^2(r)$ (just rename the original function f or replace a with $a\sqrt{2}$ in (7)). As in [5] we assume that the constants in the function F are adjusted in order to generate a double well potential in the effective Hamiltonian. We split H_{eff} as

$$H_{\text{eff}} = E_c + V_0(r) + F^2(r),$$

where E_c, is the kinetic energy and V_0 is the potential in absence of the component of the magnetic field given by $f(r)$. To explain the details, the potential $V_0(r)$ tends to infinity for $r \to 0^+$ and $r \to +\infty$ and it has a unique point of minimum at $r_0 > 0$, where $r_0^2 := 2A/B_0$. In [5], the Authors propose to fix the parameters a and c for the function F in order to produce a maximum point near r_0, so that the new potential $V_0(r) + F^2(r)$ assumes a double-well shape as in Figure 1 below. This is obtained by choosing c^2 close to r_0^2 and $a > 0$ sufficiently large.

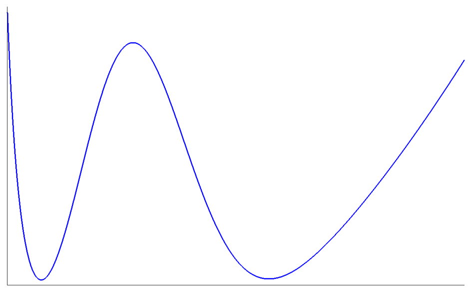

Figure 1: A possible profile of the modified potential $V_0(r) + F^2(r)$ for $r > 0$. The coefficients are tuned-up with a choice of $c^2 > r_0^2$.

The level lines of the effective Hamiltonian function in the right half-plane $\mathbb{R}_0^+ \times \mathbb{R}$ describe a phase-portrait with two centers separated by homoclinic orbits emanated from an intermediate saddle point. The typical portrait is like in Figure 2.

The level lines of H_{eff} are associated with the orbits of the second-order Duffing equation

$$\ddot{x} + g(x) = 0, \tag{8}$$

or, equivalently, the planar conservative system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(x), \end{cases} \tag{9}$$

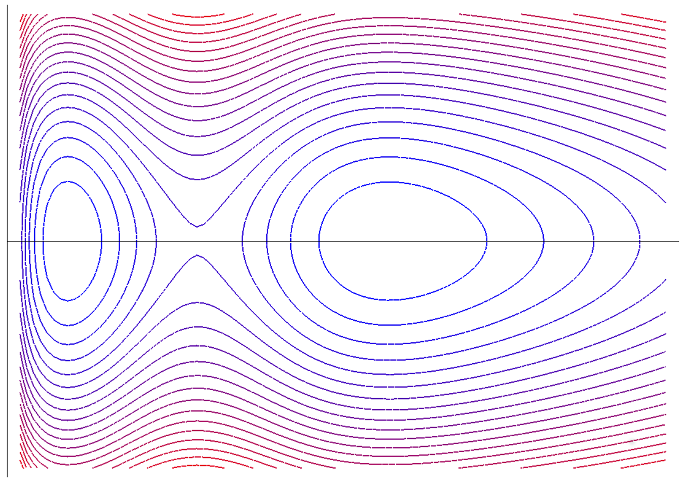

Figure 2: Some level lines associated with the Hamiltonian H_{eff} in the plane $(r, \dot{r})$ for $r > 0$.

for $x := r > 0$, $y = \dot{r}$ and

$$g(x) := \frac{d}{dx}\left(V_0(x) + F(x)^2\right) = -\frac{A^2}{x^3} + \frac{{B_0}^2}{4}x + 2F(x)f(x), \tag{10}$$

where we have set

$$f(x) := F'(x).$$

If we choose F in order to produce a potential as described in [5, Section IV] and in Figure 1, we find that the map g has precisely three simple zeros for $x > 0$ that we denote and order as

$$a < x_s < b.$$

In the phase-plane $\mathbb{R}_0^+ \times \mathbb{R}$, the points $(a, 0)$ and $(b, 0)$ are local centers, while $(x_s, 0)$ is a saddle point.

The level line of the Hamiltonian/energy function (from now on denoted simply by H) passing through $(x_s, 0)$ is given by

$$H(x, y) := \frac{y^2}{2} + V_0(x) + F^2(x) = c_s := V_0(x_s) + F^2(x_s).$$

Such level line is a double homoclinic loop, namely, it splits as

$$\mathcal{O}_l \cup \{(x_s, 0)\} \cup \mathcal{O}_r\,,$$

where $\mathcal{O}_l$ and $\mathcal{O}_r$ two homoclinic orbits at the saddle point $\{(x_s, 0)\}$. By convention, we suppose that $\mathcal{O}_l$ is contained in the strip $0 < x < x_s$ and surrounds $(a, 0)$, while $\mathcal{O}_r$ is contained in the half-plane strip $x > x_s$ and surrounds $(b, 0)$. We denote by $(\underline{a}, 0)$ and the $(\underline{b}, 0)$ the intersection points of $\mathcal{O}_l$ and, respectively, $\mathcal{O}_r$ with the x-axis. By definition, we have

$$0 < \underline{a} < a < x_s < b < \underline{b},$$

with $\underline{a}, x_s, \underline{b}$ the three solutions of $V_0(x) + F^2(x) = c_s$ (see Figure 1).
We also introduce the open regions

$$\mathcal{W}_l := \{(x, y) : 0 < x < x_s\,,\ H(x, y) < c_s\}$$

and

$$\mathcal{W}_r := \{(x, y) : x > x_s\,,\ H(x, y) < c_s\}.$$

By construction, we have

$$\partial\mathcal{W}_l = \mathcal{O}_l \cup \{(x_s, 0)\} \quad \text{and} \quad \partial\mathcal{W}_r = \mathcal{O}_r \cup \{(x_s, 0)\}$$

(see Figure 3).

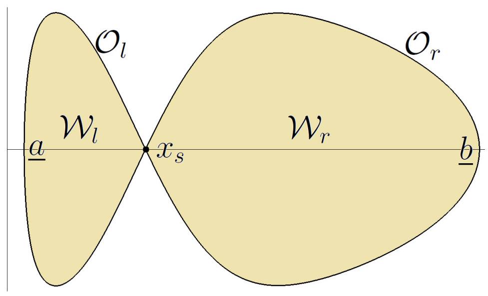

Figure 3: The saddle point $(x_s, 0)$ with the homoclinic orbits $\mathcal{O}_l, \mathcal{O}_l$ and the resulting regions $\mathcal{W}_l, \mathcal{W}_l$.

As a next step, we suppose that the modulus of the magnetic field B_0 is effected by a small change so that the three equilibrium points $(a, 0), (x_s, 0)$ and $(b, 0)$ are shifted along the x-axis. We suppose that the effect is small enough so that the new point $(x_s, 0)$ will belong to the region surrounded by $\mathcal{O}_l$ or the one surrounded by $\mathcal{O}_r$. More precisely, if we denote by $B_0^{(1)}$ and $B_0^{(2)}$ two different values of the magnetic field and associated the index $i = 1, 2$ to the corresponding equilibrium points and homoclinic orbits, we will assume that

$$H^{(1)}(x_s^{(2)}, 0) < H^{(1)}(x_s^{(1)}, 0) \quad \text{and} \quad H^{(2)}(x_s^{(1)}, 0) < H^{(2)}(x_s^{(2)}, 0). \tag{11}$$

We tacitly use the convention that the apex $i = 1, 2$ is associated to the points, orbits and regions of the phase-plane associated with the differential systems having Hamiltonians $H^{(1)}$ and $H^{(2)}$ for the magnetic fields $B_0^{(1)}$ and $B_0^{(2)}$. Under the assumption (11) the homoclinic loops associated with the two Hamiltonian systems, overlap as in Figure 4.

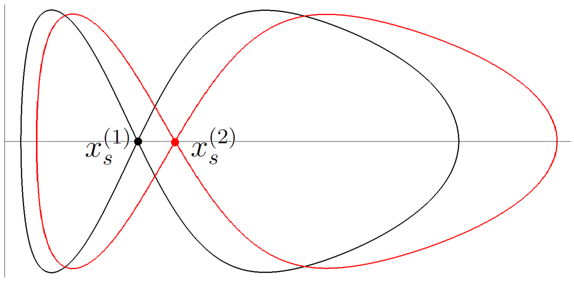

Figure 4: An example of the double homoclinic loops overlapping. The effect is obtained by moving the saddle point x_s. This occurs via a change of parameters in the equation. The aspect/ratio has been slightly modified in order to make the overlapping more evident.

Our plan is to construct some regions homeomorphic to rectangles which are obtained as intersections of suitable narrow bands around the homoclinics.

Let us consider the level line $H^{(1)}(x,y) = c^{(1)}$ with $c^{(1)} < H^{(1)}(x_s^{(1)}, 0)$ and $H^{(1)}(x_s^{(1)}, 0) - c^{(1)}$ small enough. This level line splits into two components, which are contained in the open regions $\mathcal{W}_l^{(1)}$ and $\mathcal{W}_r^{(1)}$, respectively. Now the equation $V_0(x) + F^2(x) = c^{(1)}$ has four solutions that we will denote $a_\pm^{(1)}$ and $b_\pm^{(1)}$, so that

$$\underline{a}^{(1)} < a_-^{(1)} < a^{(1)} < a_+^{(1)} < x_s^{(1)} < b_-^{(1)} < b^{(1)} < b_+^{(1)} < \underline{b}^{(1)}\,.$$

For the system associated with $B_0^{(2)}$, we can similarly determine some corresponding points with

$$\underline{a}^{(2)} < a_-^{(2)} < a^{(2)} < a_+^{(2)} < x_s^{(2)} < b_-^{(2)} < b^{(2)} < b_+^{(2)} < \underline{b}^{(2)}\,.$$

By suitably selecting the energy levels, it is always possible to enter in a setting such that the *crossing condition*

$$(CC) \qquad \begin{array}{rl} a_-^{(1)} < & \underline{a}^{(2)} < a_-^{(2)} < a_+^{(1)} \\ & b_-^{(1)} < a_+^{(2)} \\ b_-^{(2)} < & b_+^{(1)} < \underline{b}^{(1)} < b_+^{(2)} \end{array}$$

holds.

Let us consider now the ∞-shaped regions

$$\mathcal{A}^i := \{(x,y) : x > 0\,,\ c^{(i)} \leq H^{(i)}(x,y) \leq c_s^{(i)}\}, \quad \text{for } \ i = 1,2,$$

which are bounded by homoclinics $\mathcal{O}_l^{(i)}$ and $\mathcal{O}_r^{(i)}$.

As previously observed, the level line $H^{(i)}(x,y) = c^{(i)}$ has two components which are closed orbits contained in the regions $\mathcal{W}_l^{(i)}$ and $\mathcal{W}_r^{(i)}$, respectively. We set, for $i = 1,2$,

$$\Gamma_l^{(i)} := \{(x,y) : 0 < x < x_s^{(i)}\,,\ H^{(i)}(x,y) = c^{(i)}\} \subset \mathcal{W}_l^{(i)},$$

$$\Gamma_r^{(i)} := \{(x,y) : x > x_s^{(i)}\,,\ H^{(i)}(x,y) = c^{(i)}\} \subset \mathcal{W}_r^{(i)}$$

and denote by $\tau_l^{(i)}$ and $\tau_r^{(i)}$ the fundamental periods of the orbits $\Gamma_l^{(i)}$ and $\Gamma_r^{(i)}$, respectively.

The sets $\mathcal{A}^1$ and $\mathcal{A}^2$ intersects into six rectangular regions that we denote by $\mathfrak{a}_\pm, \mathfrak{b}_\pm, \mathfrak{c}_\pm$, respectively, labelling from left to right and using the sign $+$ or $-$ according to the fact that the region is contained in the upper or lower half-plane (see Figure 5).

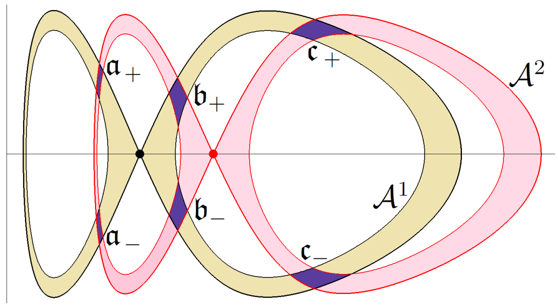

Figure 5: An example of intersection of $\mathcal{A}^1$ with $\mathcal{A}^2$ producing the six rectangular regions $\mathfrak{a}_\pm, \mathfrak{b}_\pm, \mathfrak{c}_\pm$.

Each one of the six regions introduced above can be "orientated" in two different manners. By an orientation of a topological rectangle $\mathcal{R}$, we mean the selection of two opposite sides whose union is denoted by $\mathcal{R}^-$. The two components of $\mathcal{R}^-$ are conventionally called the left and the right side (the order according to which we select to associate the terms "right" or "left" with the two sides of $\mathcal{R}^-$ is not relevant). The pair $(\mathcal{R}, \mathcal{R}^-)$ is called an *oriented rectangle*.

Now, let $\mathcal{R}$ be any of the $\mathfrak{a}_\pm, \mathfrak{b}_\pm, \mathfrak{c}_\pm$. We observe that we can give a natural orientation to the region $\mathcal{R}$ in two different manners, by choosing as $\mathcal{R}^-$ the two intersection of $\mathcal{R}$ with $H^{(1)} = c^{(1)}$ and with $H^{(1)} = c_s^{(1)}$ or the two intersection of $\mathcal{R}$ with $H^{(2)} = c^{(2)}$ and with $H^{(2)} = c_s^{(2)}$. The corresponding oriented rectangle $(\mathcal{R}, \mathcal{R}^-)$ will be denoted as $\breve{\mathcal{R}}$ in the former case and as $\widehat{\mathcal{R}}$ in the latter one. For example and with reference to Figure 5, the oriented rectangle $\widehat{\mathfrak{b}}_-$ is the region $\mathfrak{b}_-$ (center-below) in which we have selected as a couple of opposite sides forming $\mathfrak{b}_-^-$ the intersections of $\mathfrak{b}_-$ with the level lines $H^{(2)} = c^{(2)}$ and $H^{(2)} = c_s^{(2)}$. Analogously, the oriented rectangle $\breve{\mathfrak{c}}_+$ is the region $\mathfrak{c}_+$ (upper-right) in which we have selected as a couple of opposite sides forming $\mathfrak{c}_+^-$ the intersections of $\mathfrak{c}_+$ with the level lines $H^{(1)} = c^{(1)}$ and $H^{(1)} = c_s^{(1)}$.

At this point we are ready to introduce a dynamical aspect, by supposing that we switch periodically between the two systems associated with the Hamiltonians $H^{(1)}$ and $H^{(2)}$. More in detail, we consider the non-autonomous second-order scalar equation

$$\ddot{x} + g(t, x) = 0 \tag{12}$$

and also the associated first order system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g(t, x) \end{cases} \tag{13}$$

in the right-half plane $x > 0$, where $g : \mathbb{R} \times \mathbb{R}_0^+ \to \mathbb{R}$ is T-periodic in the t-variable and such that

$$g(t, x) := \begin{cases} g_1(x), & \text{for } 0 \le t < T_1 \\ g_2(x), & \text{for } T_1 \le t < T_1 + T_2 = T, \end{cases} \tag{14}$$

where

$$g_i(x) := \frac{\partial H^{(i)}}{\partial x}(x, y), \quad \text{for } i = 1, 2.$$

Equation (13) is a switched system (see [2] and the references therein) and its associated Poincaré map Φ can be decomposed as

$$\Phi = \Phi_2 \circ \Phi_1$$

where Φ_i is the Poincaré map on the time-interval $[0, T_i]$ associated with the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g_i(x) \end{cases} \tag{15}$$

for $i = 1, 2$.

Notice that, by the particular nature of the switched system (13), we can equivalently study the Poincaré map

$$\Phi = \Phi_1 \circ \Phi_2.$$

Indeed, in this latter case, we consider just a shift in time of the solutions.

4. Main result

After this preliminary discussion, we are now in position to state our main result which reads as follows.

THEOREM 4.1. *For any integer $m \geq 2$, there are T_1^* and $T_2^* > 0$ such that for each $T_1 > T_1^*$ and $T_2 > T_2^*$, the Poincaré map Φ induces chaotic dynamics on m symbols in each of the sets $\mathfrak{a}_\pm$, $\mathfrak{b}_\pm$ and $\mathfrak{c}_\pm$. Moreover, the result is robust in the sense that it is stable for small perturbations of system* (13).

Our definition of chaotic dynamics is linked to the concept of chaos according to Block and Coppel [1, 4], with a special emphasis to the presence of periodic points. More precisely, we say that a continuous and one-to-one map ψ induces chaotic dynamics on m symbols in a set $\mathcal{R}$ if there exists m pairwise disjoint compact subsets $K_1, \dots, K_m$ of $\mathcal{R}$ such that for each two-sides sequence $(s_i)_{i\in\mathbb{Z}}$ of m symbols there exists a trajectory $x_{i+1} = \psi(x_i)$ of ψ such that $x_i \in K_{s_i}$ for each $i \in \mathbb{Z}$. Moreover, if the sequence of symbols $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence, then also the sequence of points $(x_i)_{i\in\mathbb{Z}}$ is k-periodic. As a consequence of this definition, we have also that there exists a compact invariant set $\Lambda \subset \mathcal{R}$ having the set of periodic points of ψ as dense subset such that $\psi|_\Lambda$ is topologically semiconjugate (by a continuous and surjective map h) to the full shift automorphism on m-symbols $\sigma : \Sigma_m \to \Sigma_m := \{1, \dots, m\}^{\mathbb{Z}}$. Moreover, for each k-periodic two-sided sequence $\mathtt{s} := (s_i)_{i\in\mathbb{Z}}$, the set $h^{-1}(\mathtt{s})$ contains a k-periodic point of ψ (see [13, 16, 17]).

The proof of Theorem 4.1 is based on a variant of the theory of *topological horseshoes* [10], as developed in [16, 17]. In the first part of the next section we recall the basic tools and definitions that we are going to use.

5. Technical estimates and proof of the main result

Let $\widehat{\mathcal{M}} := (\mathcal{M}, \mathcal{M}^-)$ and $\widehat{\mathcal{N}} := (\mathcal{N}, \mathcal{N}^-)$ be oriented rectangles and let ψ be a continuous map. Let also m be a positive integer. We say that the triplet $(\widehat{\mathcal{M}}, \widehat{\mathcal{N}}, \psi)$ has the SAP (stretching along the paths) property with crossing number m, if there exist $K_1, \dots, K_m$ pairwise disjoint compact subsets of $\mathcal{M}$ such that any path γ in $\mathcal{M}$ connecting the two components of $\mathcal{M}^-$ possesses m sub-paths $\gamma_1, \dots \gamma_m$ with γ_i in K_i such that $\psi \circ \gamma_i$ is a path in $\mathcal{N}$ connecting the two components of $\mathcal{N}^-$. When this situation occurs, we write

$$\psi : \widehat{\mathcal{M}} \Bumpeq\!\!\!\!\longrightarrow^{m} \widehat{\mathcal{N}}.$$

We avoid mentioning the apex m when $m = 1$.

The above property is compatible with composition of maps, indeed we have that:

$$\phi : \widehat{\mathcal{L}} \Bumpeq\!\!\!\!\longrightarrow^{k} \widehat{\mathcal{M}},\ \psi : \widehat{\mathcal{M}} \Bumpeq\!\!\!\!\longrightarrow^{m} \widehat{\mathcal{N}} \implies \psi \circ \phi : \widehat{\mathcal{L}} \Bumpeq\!\!\!\!\longrightarrow^{km} \widehat{\mathcal{N}}.$$

The SAP property will be applied to prove the existence of complex dynamics for the Poincaré map, using the following result.

Lemma 5.1. *Let $\widehat{\mathcal{R}} := (\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle and $\psi : \mathcal{R} \to \mathbb{R}^2$ be a continuous and one-to-one map. Suppose that*

$$\psi : \widehat{\mathcal{R}} \mathrel{-\!\!\!\Leftrightarrow\!\!\!\!\to}^m \widehat{\mathcal{R}},$$

for some $m \geq 2$. Then ψ induces chaotic dynamics on m symbols on the set $\mathcal{R}$.

See [13, 16, 17, 18] for the general theory.

Remark 5.2: A byproduct of Lemma 5.1 implies the existence of at least m fixed points for ψ in $\mathcal{R}$. More precisely, each of the pairwise disjoint compact sets $K_1 \ldots, K_m$, involved in the definition of $\psi : \widehat{\mathcal{R}} \mathrel{-\!\!\!\Leftrightarrow\!\!\!\!\to}^m \widehat{\mathcal{R}}$, contains at least one fixed point of ψ.

The hypothesis of injectivity for the map ψ is not mandatory and the theory can be developed for arbitrary continuous maps. However, assuming ψ one-to-one is useful in order to have a semiconjugation with the Bernoulli shift on two-sided sequences (see [13] for a general discussion on this aspect). Since we apply this technique to the Poincaré map associated with a locally Lipschitz continuous differential system, the hypothesis of injectivity will be always satisfied. $\lhd$

Now we are going to describe the crossing relationships involving the sets $\overset{\smile}{\mathfrak{a}}_{\pm}$, $\overset{\smile}{\mathfrak{b}}_{\pm}$, $\overset{\smile}{\mathfrak{c}}_{\pm}$ and the dual ones $\overset{\frown}{\mathfrak{a}}_{\pm}$, $\overset{\frown}{\mathfrak{b}}_{\pm}$, $\overset{\frown}{\mathfrak{c}}_{\pm}$ by the maps Φ_i.

Lemma 5.3. *Given any positive integer ℓ_1, it holds that*

$$\Phi_1 : \overset{\smile}{\mathfrak{a}}_{+} \mathrel{-\!\!\!\Leftrightarrow\!\!\!\!\to}^{\ell_1} \overset{\frown}{\mathfrak{a}}_{-},$$

provided that $T_1 > \ell_1 \tau_l^{(1)}$.

Proof. Let $\gamma : [0,1] \to \mathfrak{a}_+$ be a (continuous) map such that $\gamma(0) \in \Gamma_l^{(1)}$ and $\gamma(1) \in \mathcal{O}_l^{(1)}$. Equivalently, $H^{(1)}(\gamma(0)) = c^{(1)}$ and $H^{(1)}(\gamma(1)) = c_s^{(1)}$. We examine the evolution of the set $\bar{\gamma} := \gamma([0,1])$ along the Poincaré map Φ_1. Observe that Φ_1 is associated with the system

$$\dot{x} = y, \quad \dot{y} = -g_1(x) \tag{16}$$

on the time-interval $[0, T_1]$.

Along the proof, we denote by $\zeta(t, z_0) = (x(t, z_0), y(t, z_0))$ the solution of (16) satisfying the initial condition $\zeta(0) = z_0$. By definition, $\Phi_1(z_0) = \zeta(T_1, z_0)$, for any $z_0 \in \mathbb{R}_0^+ \times \mathbb{R}$.

The point $\gamma(1)$ belongs to the homoclinic trajectory and therefore it remains on $\mathcal{O}_l^{(1)}$ for all the forward time, moving in the upper phase-plane from left to right but never meeting the saddle point $x_s^{(1)}$. As a consequence, $x(t,\gamma(1)) < x_s^{(1)}$ and $y(t,\gamma(1)) > 0$ for all $t \in [0,T_1]$. On the other hand, the point $\gamma(0)$ belongs to the periodic orbit $\Gamma_l^{(1)}$ of period $\tau_l^{(1)}$ and therefore, if $T_1 > \tau_l^{(1)}$, it makes at least ℓ_1 complete turns (in the clockwise sense) around the center $(a^{(1)},0)$ in the interval $[0,T_1]$.

If we introduce a polar coordinate system (θ,ρ), starting from the half-line $\{(x,0) : x < a^{(1)}\}$ and counting positive rotations in the clockwise sense, we have that $0 < \theta(\gamma(s)) < \pi$ for all $s \in [0,1]$ and then we define the sets

$$K_j := \{z \in \mathfrak{a}_+ \,:\, (2j-1)\pi < \theta(\Phi^{(1)}(z)) < 2j\pi\}, \quad \text{for } j = 1,\dots,\ell_1.$$

By the previous observation about the movement of the points $\gamma(1)$ and $\gamma(0)$ under the influence of the dynamical system of (16), we know that $\theta(\Phi_1(\gamma(1))) < \pi$, while $\theta(\Phi_1(\gamma(0))) > 2j\ell_1\pi$.

A simple continuity argument on the map $[0,1] \ni s \mapsto \theta(\Phi_1(\gamma(s)))$, implies the existence of ℓ_1 pairwise disjoint intervals $[\alpha_j,\beta_j] \subset [0,1]$ such that $(2j-1)\pi \le \theta(\Phi_1(\gamma(s))) \le 2j\pi$ for all $s \in [\alpha_j,\beta_j]$ with $\theta(\Phi_1(\gamma(\alpha_j)) = 2j\pi$ and $\theta(\Phi_1(\gamma(\beta_j)) = (2j-1)\pi$.
By definition, the path $\Phi_1 \circ \gamma$ restricted to the interval $[\alpha_j,\beta_j]$ is contained in the half-annulus

$$\mathcal{A}^1 \cap \{(x,y) : 0 < x < x_s^{(1)}\,,\ y \le 0\}$$

and therefore, it crosses the rectangle $\mathfrak{a}_-$ intersecting both components of $\mathfrak{a}_-^-$. Using again an elementary continuity argument of the map $s \mapsto \Phi_1(\gamma(s))$, for each $j = 1,\dots,\ell_1$, we determine a sub-interval $[\alpha_j',\beta_j'] \subset [\alpha_j,\beta_j]$ such that, $\Phi_1(\gamma(s)) \in \mathfrak{a}_-$ for all $s \in [\alpha_j',\beta_j']$. Moreover, $\Phi_1(\gamma(\alpha_j'))$ and $\Phi_1(\gamma(\beta_j'))$ belong to different components of $\mathfrak{a}_-$. Note also that, by construction, $\gamma(s) \in K_j$ for all $s \in [\alpha_j',\beta_j']$. We have thus verified the SAP property for $(\breve{\mathfrak{a}}_+, \overset{\frown}{\mathfrak{a}}_-, \Phi_1)$ with crossing number ℓ_1, provided that $T_1 > \ell_1\tau_l^{(1)}$ and the proof is complete. $\square$

At this point, we can repeat the same argument of the proof of Lemma 5.3 and consider all the possible combinations between the oriented rectangles and the maps Φ_i. We can summarize these conclusions by the following lemmas where the times τ_i^* can be easily determined from the periods of the closed orbits $\Gamma_l^{(i)}$ and $\Gamma_s^{(i)}$.

Lemma 5.4. *There exist times τ_1^* and τ_2^*, such that, for any positive integers ℓ_1,ℓ_2 it holds that:*

$\Phi_1 : \breve{\mathfrak{a}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_1} \overset{\frown}{\mathfrak{a}}_\pm\,,\quad \breve{\mathfrak{b}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_1} \overset{\frown}{\mathfrak{b}}_\pm\,,\ \overset{\frown}{\mathfrak{c}}_\pm\,,\quad \breve{\mathfrak{c}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_1} \overset{\frown}{\mathfrak{b}}_\pm\,,\ \overset{\frown}{\mathfrak{c}}_\pm\,,$
provided that $T_1 > \ell_1\tau_1^*$.

$\Phi_2 : \overset{\frown}{\mathfrak{a}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_2} \breve{\mathfrak{a}}_\pm\,,\ \breve{\mathfrak{b}}_\pm\,,\quad \overset{\frown}{\mathfrak{b}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_2} \breve{\mathfrak{a}}_\pm\,,\ \breve{\mathfrak{b}}_\pm\,,\quad \overset{\frown}{\mathfrak{c}}_\pm \Leftarrow\!\!\!\!\!\ominus\!\!\!\!\!\Rightarrow^{\ell_2} \breve{\mathfrak{c}}_\pm\,,$
provided that $T_2 > \ell_2\tau_2^*$.

In the above lemma, when we write a condition such as $\smile\!\!\!\!\!\mathfrak{a}_{\pm} \Leftrightarrow\!\!\!\!\rightarrow^{\ell} \frown\!\!\!\!\!\mathfrak{a}_{\pm}$, we mean that all the four possibilities in the choice of $\pm$ for the domain and codomain are possible.

The content of Lemma 5.4 is explained by means of Figure 6 and Figure 7.

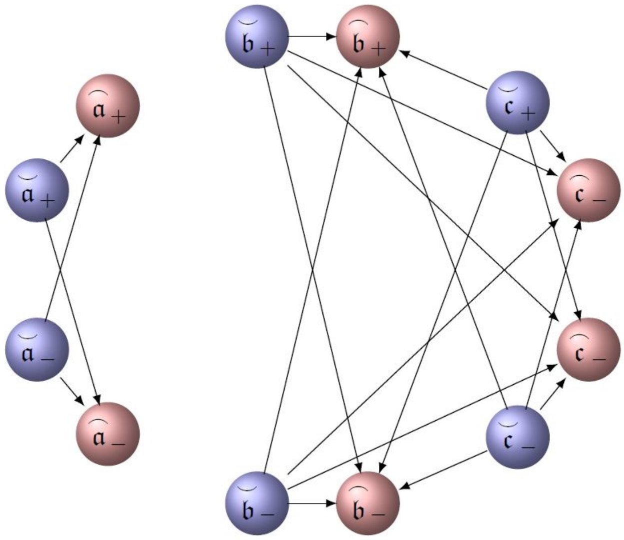

Figure 6: This graph represents all the possible connections by the partial Poincaré map Φ_1. The arrows correspond to the $\Leftrightarrow\!\!\!\!\rightarrow$ symbol. The integer ℓ_1 is not indicated but it can be arbitrarily chosen provided that $T_1 > \ell_1 \tau_1^*$.

Now, we are in position to conclude with the proof of our main result.

Proof of Theorem 4.1. Using Lemma 5.4 along with Lemma 5.1 we can guarantee that the Poincaré map $\Phi = \Phi_2 \circ \Phi_1$, as well as $\Phi = \Phi_1 \circ \Phi_2$ induces chaotic dynamics on any finite number of symbols, provided that T_1 and T_2 are large enough.

From the proof of Lemma 5.3 it is clear that the result is stable by small perturbations and the same holds for all the connections considered in Lemma 5.4.

In our case we have several possibilities of producing chaotic dynamics on $m \geq 2$ symbols on a rectangular region $\mathcal{R}$ chosen among the sets $\mathfrak{a}_{\pm}$, $\mathfrak{b}_{\pm}$ and $\mathfrak{c}_{\pm}$. In order to explain better how these possibilities arise, we fix out attention only on the Poincaré map $\Phi = \Phi_2 \circ \Phi_1$ (the other case is treated in a similar manner).

A first and more natural case is to take $\max\{\ell_1, \ell_2\} \geq 2$, so that

$$m = \ell_1 \times \ell_2 \geq 2$$

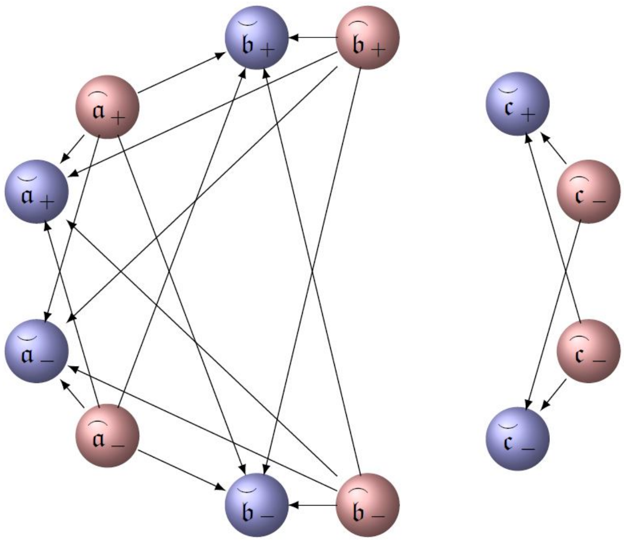

Figure 7: This graph represents all the possible connections by the partial Poincaré map Φ_2. The arrows correspond to the $\mathrel{\Diamond\!\!\!\!\longrightarrow}$ symbol. The integer ℓ_2 is not indicated but it can be arbitrarily chosen provided that $T_2 > \ell_2 \tau_2^*$.

and, considering the connections described in Lemma 5.4, we immediately see that Lemma 5.1 can be applied for $\widehat{\mathcal{R}}$ any of the sets $\overset{\smile}{\mathfrak{a}}_{\pm}$, $\overset{\smile}{\mathfrak{b}}_{\pm}$, $\overset{\smile}{\mathfrak{c}}_{\pm}$. However, a more careful analysis of the connection diagrams shows that in these sets the SAP property with crossing number greater or equal than two can be obtained also in the case when $\ell_1 = \ell_2 = 1$ (this may be more interesting from the point of view of the applications because we need a lesser restriction on the period). In fact, the following connections are available

$$
\begin{array}{ll}
\overset{\smile}{\mathfrak{a}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{a}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_+\,, & \overset{\smile}{\mathfrak{a}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{a}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_+ \\
\overset{\smile}{\mathfrak{a}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{a}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_-\,, & \overset{\smile}{\mathfrak{a}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{a}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_- \\
\overset{\smile}{\mathfrak{b}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{b}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{b}}_+\,, & \overset{\smile}{\mathfrak{b}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{b}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_+ \\
\overset{\smile}{\mathfrak{b}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{b}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{b}}_-\,, & \overset{\smile}{\mathfrak{b}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{b}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{b}}_- \\
\overset{\smile}{\mathfrak{c}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{c}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{c}}_+\,, & \overset{\smile}{\mathfrak{c}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{c}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{a}}_+ \\
\overset{\smile}{\mathfrak{c}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{c}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{c}}_-\,, & \overset{\smile}{\mathfrak{c}}_- \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\frown}{\mathfrak{c}}_+ \mathrel{\Diamond\!\!\!\!\longrightarrow} \overset{\smile}{\mathfrak{c}}_-
\end{array}
$$

and therefore, we find that

$$
\Phi : \overset{\smile}{\mathfrak{a}}_{\pm} \mathrel{\Diamond\!\!\!\!\longrightarrow}^2 \overset{\smile}{\mathfrak{a}}_{\pm}\,, \qquad \overset{\smile}{\mathfrak{b}}_{\pm} \mathrel{\Diamond\!\!\!\!\longrightarrow}^2 \overset{\smile}{\mathfrak{b}}_{\pm}\,, \qquad \overset{\smile}{\mathfrak{c}}_{\pm} \mathrel{\Diamond\!\!\!\!\longrightarrow}^2 \overset{\smile}{\mathfrak{c}}_{\pm}\,.
$$

In the last formula we use the convention that $[\,]_{\pm} \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow [\,]_{\pm}$ means that only the two possibilities $[\,]_{+} \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow [\,]_{+}$ and $[\,]_{-} \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow [\,]_{-}$ are available.

The situation becomes more complicated and interesting if we consider the iterates of the map Φ. For instance, for the map Φ^2, and taking $\widehat{\mathcal{R}} = \breve{\mathfrak{a}}_+$ as a starting set, new connections are available, such as

$$\breve{\mathfrak{a}}_+ \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow^2 \breve{\mathfrak{b}}_\pm \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow^2 \breve{\mathfrak{a}}_+ \quad\text{and}\quad \breve{\mathfrak{a}}_+ \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow^2 \breve{\mathfrak{a}}_\pm \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow^2 \breve{\mathfrak{a}}_+ \,.$$

Hence, counting all the possible connections for Φ^2, we obtain that

$$\Phi^2 : \breve{\mathfrak{a}}_+ \Bumpeq\!\!\!\!\!\!\!\!\longrightarrow^{16} \breve{\mathfrak{a}}_+ .$$

In fact, from $\breve{\mathfrak{a}}_+$ we come back again to $\breve{\mathfrak{a}}_+$ by Φ^2 passing through the four sets $\breve{\mathfrak{a}}_\pm$ and $\breve{\mathfrak{b}}_\pm$ and, each time we apply Φ we have two itineraries available. Similar combinations occur for the other oriented rectangles. $\square$

6. Final remarks

The existence of chaos in differential systems which are obtained as periodic perturbations of planar autonomous systems exhibiting homoclinic or heteroclinic trajectories is a well established fact (see [15, 8]). The methods of proof applied in those situations, such as the Melnikov method, usually permit to enter in the framework of Smale's horseshoe (cf. [19] and [14]) which guarantees the existence of a compact invariant set for the Poincaré map Φ, where Φ is *topologically conjugate* to the Bernoulli shift on a certain set of symbols. Our result provides a weaker form of chaos since only the semiconjugation is proved. On the other hand, in the concrete applications, some explicit knowledge of the homoclinic (or heteroclinic) solution, in terms of its analytic expression is often needed. A typical example is given by the classical periodically perturbed Duffing equation

$$\ddot{x} - x + x^3 = \varepsilon p(\omega t), \tag{17}$$

where the Melnikov function can be explicitly defined (see [8]) thanks to the knowledge of the analytic expression of the homoclinic solutions of

$$\dot{x} = y, \quad \dot{y} = x - x^3.$$

In the model studied in the present paper, two difficulties arise: first, we do not know an explicit form of the homoclinic solutions of system (9) and, secondly, the periodic perturbation leading to (12) from (8), which corresponds to a variation of the form $B_0 \mapsto B_0(t)$ in (10), appears to be more complicated than the perturbation considered in equation (17). Our approach, even if applied to the simplified situation of a stepwise function $B_0(t)$, allows to prove the presence of chaotic dynamics using only few geometric information on the geometry of the level curves of the associated energy functions. As already shown in [12] and in [11, Section 8], the choice of a stepwise coefficient has the advantage not only to simplify some technical estimates, but also to put in evidence the presence of interesting bifurcation phenomena for the solutions of the nonlinear equations which are involved.

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Population dynamics in hostile neighborhoods

Herbert Amann

Dedicated to Julián López-Gómez,
a mathematical friend for many years.

Abstract. *A new class of quasilinear reaction-diffusion equations is introduced for which the mass flow never reaches the boundary. It is proved that the initial value problem is well-posed in an appropriate weighted Sobolev space setting.*

Keywords: Degenerate quasilinear parabolic equations, reaction-diffusion systems, Sobolev space well-posedness.
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1. Introduction

It has been known since long that spacial interactions in population dynamics can be adequately modeled by systems of reaction-diffusion equations (see E.E. Holmes et al. [10], A. Okubo and S.A. Levin [17], or J.D. Murray [15], [16], for instance). In general, these systems possess a quasilinear structure and show an extremely rich qualitative behavior depending on the various structural assumptions which can meaningfully be imposed.

Reaction-diffusion equations are of great importance also in many other scientific areas as, for example, physics, chemistry, mechanical and chemical engineering, and the social sciences. Thus our mathematical results are not restricted to population dynamics. It is just a matter of convenience to describe the phenomenological background and motivation in terms of populations.

Throughout this paper, Ω is a bounded domain in $\mathbb{R}^m$ with a smooth boundary Γ lying locally on one side of Ω. (In population dynamics, $m = 1$, 2, or 3. But this is not relevant for what follows.) By ν we denote the inner (unit) normal vector field on Γ and use $\cdot$ or $(\cdot|\cdot)$ for the Euclidean inner product in $\mathbb{R}^m$.

We assume that Ω is occupied by n different species described by their densities $u_1, \dots, u_n$ and set $u := (u_1, \dots, u_n)$. The spacial and temporal change of u, that is, the (averaged) movement of the individual populations, is mathematically encoded in the form of conservation laws

$$\partial_t u_i + \operatorname{div} j_i(u) = f_i(u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \qquad 1 \le i \le n. \tag{1}$$

Here $j_i(u)$ is the (mass) flux vector, $f_i(u)$ the production rate of the i-th species, and (1) is a mass balance law (e.g., S.R. de Groot and P. Mazur [7]).

In order to get a significant model we have to impose constitutive assumptions on the n-tuple $j(u) = \bigl(j_1(u), \dots, j_n(u)\bigr)$ of flux vectors. In population dynamics it is customary to build on phenomenological laws which are basically variants and extensions of Fick's law, and we adhere in this paper to that practice. Thus we assume that

$$j_i(u) = -a_i(u) \operatorname{grad} u_i,$$

where the 'diffusion coefficient' $a_i(u) \in C^1(\Omega)$ may depend on the interaction of some, or all, species, hence on u. The fundamental assumption is then that

$$a_i(u)(x) > 0, \qquad x \in \Omega. \tag{2}$$

Besides of modeling the behavior of the populations in Ω, their conduct on the boundary Γ has also to be analyzed. We restrict ourselves to homogeneous boundary conditions. Then there are essentially two cases which are meaningful, namely the Dirichlet boundary condition

$$u_i = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+$$

or the no-flux condition

$$\nu \cdot j_i(u) = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}_+$$

for population i, or combinations thereof.

In the standard mathematical theory of reaction-diffusion equations it is assumed that $a_i(u)$ is uniformly positive on Ω. The focus of this paper is on the nonuniform case where $a_i(u)$ may tend to zero as we approach Γ.

In population dynamics nonuniformly positive one-population models have been introduced by W.S.C. Gurney and R.M. Nisbet [8] and M.E. Gurtin and R.C. MacCamy [9]. Arguing that the population desires to avoid overcrowding, they arrive at flux vectors of the form $j(u) = -u^k \operatorname{grad} u$ with $k \ge 1$. Thus their models are special instances of the porous media equation. Here the diffusion coefficient degenerates, in particular, near the Dirichlet boundary.

Ever since the appearance of the pioneering papers [8], [9], there have been numerous studies of the (weak) solvability of reaction-diffusion equations and systems exhibiting porous media type degenerations. We do not go into detail, since we propose a different approach.

In a series of papers, J. López-Gómez has studied (partly with coauthors) qualitative properties of one- and two-population models which he termed 'degenerate' (see [1, 6, 11, 12, 13] and the references therein). In those works the term 'degenerate', however, refers to the vanishing on some open subset of Ω of the 'logistic coefficient', which is part of $f(u)$.

For the following heuristic discussion, which describes the essence of our approach, we can assume that $n = 1$ and that a is independent of u.

If the Dirichlet boundary condition holds, then the population gets extinct if it reaches Γ. In the case of the no-flux condition, Γ is impenetrable, that is, the species can neither escape through the boundary nor can it get replenishment from the outside. In this sense we can say that the 'population lives in a hostile neighborhood'.

No slightly sensible species will move toward places where it is endangered to get killed, nor will it run head-on against an impenetrable wall. Instead, it will slow down drastically if it comes near such places. In mathematical terms this means that the flux in the normal direction has to decrease to zero near Γ. To achieve this, the diffusion coefficient a has to vanish sufficiently rapidly at Γ.

To describe more precisely what we have in mind, we study the motion of the population in a normal collar neighborhood of Γ. This means that we fix $0 < \varepsilon \leq 1$ such that, setting

$$S := \{\, q + y\nu(q) \;;\; 0 < y \leq \varepsilon,\ q \in \Gamma \,\},$$

the map

$$\varphi \colon \overline{S} \to [0, \varepsilon] \times \Gamma, \quad q + y\nu(q) \mapsto (y, q) \tag{3}$$

is a smooth diffeomorphism. Note that

$$y = \operatorname{dist}(x, \Gamma), \quad x = q + y\nu(q) \in S.$$

We extend ν to a smooth vector field on S, again denoted by ν, by setting

$$\nu(x) := \nu(q), \qquad x = q + y\nu(q) \in S. \tag{4}$$

Then the normal derivative

$$\partial_\nu u(x) := \nu(x) \cdot \operatorname{grad} u(x) \tag{5}$$

is well-defined for $x \in S$.

As usual, we denote by φ_* the push-forward and by φ^* the pull-back by φ of functions and tensors, in particular of vector fields. Then

$$\varphi_*(a \operatorname{grad}) = (\varphi_* a)\frac{\partial}{\partial y} \oplus (\varphi_* a) \operatorname{grad}_\Gamma \quad \text{on} \quad N := (0, \varepsilon] \times \Gamma, \tag{6}$$

where grad_Γ is the surface gradient on Γ with respect to the metric induced by the Euclidean metric on $\overline{\Omega}$. Note that, by (4) and (5),

$$\partial_y v = \partial_\nu u, \quad v = \varphi_* u. \tag{7}$$

Set $\Gamma_y := \varphi^{-1}\bigl(\{y\} \times \Gamma\bigr)$. Then $\mathrm{grad}_{\Gamma_y} = \mathrm{grad}_\Gamma$. Hence we obtain from (6) and (7) that

$$a \,\mathrm{grad}\, u = (a\partial_\nu u)\nu \oplus a\,\mathrm{grad}_\Gamma\, u \quad \text{on } S. \tag{8}$$

Thus, if we want to achieve that the flux $j(u) = -a\,\mathrm{grad}\, u$ decays in the normal direction, but not necessarily in directions parallel to the boundary, we have to replace (8) by

$$(a_1\partial_\nu u)\nu \oplus a\,\mathrm{grad}_\Gamma\, u,$$

where a_1 tends to zero as x approaches Γ. This we effectuate by replacing $j(u)$ by

$$j^s(u) := -\bigl((a\rho^{2s}\partial_\nu u)\nu \oplus a\,\mathrm{grad}_\Gamma\, u\bigr), \qquad u \in C^1(S), \tag{9}$$

for some $s \geq 1$, where $0 < \rho \leq 1$ on S and $\rho(x) = \mathrm{dist}(x, \Gamma)$ for x near Γ. Then, irrespective of the size of $\mathrm{grad}\, u$, $j^s(u)$ decays to zero as we approach Γ. The 'speed' of this decay increases if s gets bigger. Note, however, that the component orthogonal to ν is the same as in (8). This reflects the fact, known to everyone who has been hiking in high mountains—in the Swiss Alps, for example(!)—that one can move forward along a level line path in front of a steep slope with essentially the same speed as this can be done in the flat country. On the other hand, one slows down drastically—and eventually gives up—if one tries to go to the top along a line of steepest ascent.

In the next section we give a precise definition of the class of degenerate equations which we consider. Section 3 contains the definition of the appropriate weighted Sobolev spaces. In addition, we present the basic maximal regularity theorem for linear degenerate parabolic initial value problems.

The main result of this paper is Theorem 4.5 which is proved in Section 4. It guarantees the local well-posedness of quasilinear degenerate reaction-diffusion systems. In the last section we present some easy examples, discuss the differences between the present and the classical approach, and suggest possible directions of further research.

2. Degenerate reaction-diffusion operators

Let (M, g) be a Riemannian manifold. Then grad_g, resp. div_g, denotes the gradient, resp. divergence, operator on (M, g). The Riemannian metric on Γ, induced by the Euclidean metric on $\overline{\Omega}$, is written h. Then $\overline{N} = [0, \varepsilon] \times \Gamma$ is endowed with the metric $g_N := dy^2 + h$.

We fix $\chi \in C^\infty\big([0,\varepsilon],[0,1]\big)$ satisfying

$$\chi(y) = \begin{cases} 1, & 0 \le y \le \varepsilon/3, \\ 0, & 2\varepsilon/3 \le y \le \varepsilon. \end{cases}$$

Then

$$r(y) := \chi(y)y + 1 - \chi(y), \qquad 0 \le y \le \varepsilon.$$

We set

$$S(j) := \varphi^{-1}\big((0, j\varepsilon/3] \times \Gamma\big), \qquad j = 1, 2,$$

and $\rho := \varphi^* r = r \circ \varphi^{-1}$. Then $\rho \in C^\infty\big(S, (0,1]\big)$ and

$$\rho(x) = \begin{cases} \operatorname{dist}(x, \Gamma), & x \in S(1), \\ 1, & x \in S \backslash S(2). \end{cases} \tag{10}$$

Given a linear differential operator $\mathcal{B}$ on S, we denote by $\varphi_* \mathcal{B}$ its 'representation in the variables $(y,q) \in N$'. Thus $\varphi_*\mathcal{B}$, the push-forward of $\mathcal{B}$, is the linear operator on N defined by

$$(\varphi_*\mathcal{B})w := \varphi_*\big(\mathcal{B}(\varphi^* w)\big), \qquad w \in C^\infty(N).$$

First we consider a single linear operator, that is, $n = 1$ and

$$\mathcal{A}v := -\operatorname{div}(a \operatorname{grad} v)$$

with

$$a \in C^1(\Omega), \quad a(x) > 0 \text{ for } x \in \overline{\Omega}. \tag{11}$$

We set $\overline{a} := \varphi_* a \in C^1(N)$ and $\overline{\mathcal{A}} := \varphi_*\mathcal{A}$. Then we find

$$\overline{\mathcal{A}}w = -\operatorname{div}_{g_N}(\overline{a} \operatorname{grad}_{g_N} w) = -\partial_y(\overline{a}\partial_y w) - \operatorname{div}_h(\overline{a} \operatorname{grad}_h w)$$

for $w \in C^2(N)$. By pulling $\overline{\mathcal{A}}$ back to S we obtain the representation

$$\mathcal{A}u = -\partial_\nu(a\partial_\nu u) - \operatorname{div}_h(a \operatorname{grad}_h u), \qquad u \in C^2(S), \tag{12}$$

of $\mathcal{A}|S$.

We put

$$U := \Omega \backslash S(2) \tag{13}$$

and fix $s \in [1, \infty)$. Then we define a linear operator $\mathcal{A}_s$ on Ω by setting

$$\mathcal{A}_s v := -\operatorname{div}_s(a \operatorname{grad}_s v), \qquad v \in C^2(\Omega), \tag{14}$$

where

$$\operatorname{div}_s(a \operatorname{grad}_s v) := \begin{cases} \operatorname{div}(a \operatorname{grad} v), & v \in C^2(U), \\ \rho^s \partial_\nu (a\rho^s \partial_\nu v) + \operatorname{div}_h(a \operatorname{grad}_h v), & v \in C^2(S). \end{cases}$$

It follows from (10), (12), and (13) that $\mathcal{A}_s v$ is well-defined for $v \in C^2(\Omega)$. The map $\mathcal{A}_s$ is said to be a linear *s-degenerate reaction-diffusion* (or *divergence form*) *operator on* Ω.

REMARK 2.1: It has been shown in [5] that the right approach to study differential operators which are s-degenerate, is to endow S with the metric

$$g_s := \varphi^*(r^{-2s} dy^2 \oplus h).$$

Then

$$a \operatorname{grad}_{g_s} u = (a\rho^{2s}\partial_\nu u)\nu \oplus a \operatorname{grad}_h u, \qquad u \in C^1(S),$$

which equals $-j^s(u)$ of (9). Furthermore,

$$\mathcal{A}_s u = -\operatorname{div}_{g_s}(a \operatorname{grad}_{g_s} u), \qquad u \in C^2(S).$$

Thus $\mathcal{A}_s$ is a 'standard' linear reaction-diffusion operator if S is endowed with the metric g_s.

3. The Isomorphism Theorem

The natural framework for an efficient theory of strongly degenerate reaction-diffusion systems are weighted function spaces which we introduce now. We assume throughout that

- $1 < p < \infty$.

Suppose $s \geq 1$ and $k \in \mathbb{N}$. For $u \in C^k(S)$ set

$$v(y,q) := \varphi_* u(y,q) = u\big(q + y\nu(q)\big)$$

and

$$\|u\|_{W_p^k(S;s)} := \sum_{i=0}^{k} \Big(\int_0^\varepsilon \big\| \big(r(y)^s \partial_y\big)^i v(y,\cdot) \big\|_{W_p^{k-i}(\Gamma)}^p r(y)^{-s}\, dy \Big)^{1/p}.$$

Then the weighted Sobolev space $W_p^k(S;s)$ is the completion in $L_{1,\mathrm{loc}}(S)$ of the subspace of smooth compactly supported functions with respect to the norm $\|\cdot\|_{W_p^k(S;s)}$. The *weighted Sobolev space* $W_p^k(\Omega;s)$ consists of all u belonging to $L_{1,\mathrm{loc}}(\Omega)$ with

$$u \,|\, S \in W_p^k(\Omega;s), \quad u \,|\, U \in W_p^k(\Omega).$$

It is a Banach space with the norm

$$u \mapsto \|u\,|\,S\|_{W_p^k(S;s)} + \|u\,|\,U\|_{W_p^k(U)},$$

and $L_p(\Omega; s) := W_p^0(\Omega; s)$. Of course, $W_p^k(U)$ is the usual Sobolev space.

To define weighted spaces of bounded C^k functions we set

$$\|u\|_{BC^k(S;s)} := \sum_{i=0}^{k} \sup_{0<y<\varepsilon} \big\| \big(r(y)^s \partial_y\big)^i v(y,\cdot)\big\|_{C^k(\Gamma)}. \tag{15}$$

The weighted space $BC^k(S; s)$ is the linear subspace of all $u \in C^k(S)$ for which the norm (15) is finite. Then $BC^k(\Omega; s)$ is the linear space of all $u \in C^k(\Omega)$ with

$$u\,|\,S \in BC^k(S; s), \quad u\,|\,U \in BC^k(U).$$

It is a Banach space with the norm

$$u \mapsto \|u\,|\,S\|_{BC^k(S;s)} + \|u\,|\,U\|_{BC^k(U)}. \tag{16}$$

The topologies of the weighted spaces $W_p^k(\Omega; s)$ and $BC^k(\Omega; s)$ are independent of the particular choice of S (that is, of $\varepsilon > 0$) and the cut-off function χ.

Let $0 < T < \infty$ and set $J := [0, T]$. We introduce anisotropic weighted Sobolev spaces on $\Omega \times J$ by

$$W_p^{(2,1)}(\Omega \times J; s) := L_p\big(J, W_p^2(\Omega; s)\big) \cap W_p^1\big(J, L_p(\Omega; s)\big).$$

We denote by $(\cdot,\cdot)_{\theta,p}$ the real interpolation functor of exponent $\theta \in (0, 1)$. Then we institute a Besov space by

$$B_p^{2-2/p}(\Omega; s) := \big(L_p(\Omega; s), W_p^2(\Omega; s)\big)_{1-1/p,p}.$$

Lemma 3.1. *The weighted spaces possess the same embedding and interpolation properties as their non-weighted versions. In particular,*

$$BC^1(\Omega) \hookrightarrow BC^1(\Omega; s) \hookrightarrow BC(\Omega). \tag{17}$$

Proof. The first assertion is a consequence of Theorems 3.1 and 6.1 of [5]. The first embedding of (17) is obvious from $|\rho^s \partial_\nu u| \le |\partial_\nu u|$ and (16). It remains to observe that $BC(\Omega; s) = BC(\Omega)$. □

The following theorem settles the well-posedness problem for the linear initial value problem

$$\begin{aligned} \partial_t u + \mathcal{A}_s u &= f \quad \text{on } \Omega \times J, \\ \gamma_0 u &= u_0 \quad \text{on } \Omega \times \{0\}, \end{aligned}$$

where γ_0 is the trace operator at $t = 0$ and $\mathcal{A}_s$ is given by (14).

THEOREM 3.2. *Let $1 \leq s < \infty$ and $0 < T < \infty$. Assume that there exists $\underline{\alpha} > 0$ such that*

$$a \in BC^1(\Omega; s) \quad \textit{and} \quad a \geq \underline{\alpha}. \tag{18}$$

Then the map $(\partial_t + \mathcal{A}_s,\, \gamma_0)$ is a topological isomorphism from

$$W_p^{(2,1)}(\Omega \times J; s) \quad \textit{onto} \quad L_p(\Omega \times J; s) \times B_p^{2-2/p}(\Omega; s).$$

Proof. Note that

$$\operatorname{div}(a \operatorname{grad} v) = a\Delta v + \langle da, \operatorname{grad} v\rangle \quad \text{on } U, \tag{19}$$

where $\langle\cdot,\cdot\rangle$ stands for duality pairings. On S we get

$$\rho^s \partial_\nu (a \rho^s \partial_\nu v) = a(\rho^s \partial_\nu)^2 v + (\rho^s \partial_\nu a)\rho^s \partial_\nu v \tag{20}$$

and

$$\operatorname{div}_h (a \operatorname{grad}_h v) = a\Delta_h v + \langle da, \operatorname{grad}_h v\rangle, \tag{21}$$

where Δ_h is the Laplace–Beltrami operator on Γ.

Set $R(t) := t^s$ for $0 \leq t \leq 1$. Then we deduce from (18)–(21) and from Theorems 6.1 and 7.2 of [5] that $\mathcal{A}_s$ is a bc-regular R-degenerate uniformly strongly elliptic differential operator on Ω in the sense of [5, (1.6)]. (Observe that the regularity condition (1.9) in that paper is only sufficient and stronger than (18).) Hence the assertion follows from Theorem 1.3 of [5]. □

COROLLARY 3.3. *$\mathcal{A}_s$ has maximal $L_p(\Omega; s)$ regularity.*

Proof. [4] or [5]. □

4. Quasilinear degenerate systems

Now we turn to systems and consider quasilinear differential operators. Thus we assume:

$$\begin{aligned} &\text{(i)} \quad 1 \leq s < \infty.\\ &\text{(ii)} \quad X \text{ is a nonempty open subset of } \mathbb{R}^n.\\ &\text{(iii)} \quad a_i \in C^2\big(\overline{\Omega} \times X,\, (0,\infty)\big),\ 1 \leq i \leq n. \end{aligned} \tag{22}$$

Given $u \in C^1(\Omega, X)$,

$$a_i(u)(x) := a_i\big(x, u(x)\big), \qquad x \in \Omega. \tag{23}$$

Then

$$\mathcal{A}_{i,s}(u)v_i := -\operatorname{div}_s\big(a_i(u)\operatorname{grad}_s v_i\big), \qquad v_i \in C^2(\Omega),$$

and, setting $a := (a_1, \ldots, a_n)$,

$$\mathcal{A}_s(u)v := -\operatorname{div}_s\big(a(u)\operatorname{grad}_s v\big) := \big(\mathcal{A}_{1,s}(u)v_1, \ldots, \mathcal{A}_{n,s}(u)v_n\big)$$

for $v=(v_1,\dots,v_n)\in C^2(\Omega,\mathbb{R}^n)$. Note that $\mathcal{A}_s(u)$ is a diagonal operator whose diagonal elements are coupled by the u-dependence of their coefficients.

If $\mathfrak{F}(\Omega;s)$ stands for one of the spaces

$$W_p^k(\Omega;s),\ BC^k(\Omega;s),\ \text{or } B_p^{2-2/p}(\Omega;s),\ \text{then } \mathfrak{F}(\Omega,\mathbb{R}^n;s):=\mathfrak{F}(\Omega;s)^n.$$

Given subsets A and B of some topological spaces, $A\Subset B$ means that $\overline{A}$ is compact and contained in the interior of B.

We define

$$V:=\bigl\{\,v\in B_p^{2-2/p}(\Omega,\mathbb{R}^n;s)\ ;\ v(\Omega)\Subset X\,\bigr\}. \tag{24}$$

Lemma 4.1. *If $p>m+2$, then V is open in $B_p^{2-2/p}(\Omega,\mathbb{R}^n;s)$.*

Proof. It follows from Lemma 3.1 that

$$B_p^{2-2/p}(\Omega,\mathbb{R}^n;s)\hookrightarrow BC^1(\Omega,\mathbb{R}^n;s)\hookrightarrow BC(\Omega,\mathbb{R}^n). \tag{25}$$

Denote the embedding operator which maps the leftmost space into the rightmost one by ι. Let $u_0\in V$ so that $u_0(\Omega)\Subset X$.

Fix[1] $0<r<\operatorname{dist}\bigl(u_0(\Omega),\partial X\bigr)$ and set

$$K:=\bigl\{\,x\in X\ ;\ \operatorname{dist}\bigl(x,u_0(\Omega)\bigr)<r\,\bigr\}\Subset X \tag{26}$$

and

$$B(u_0,r):=\bigl\{\,u\in BC(\Omega,\mathbb{R}^n)\ ;\ \|u-u_0\|_\infty<r\,\bigr\}.$$

Then $u(\Omega)\subset K$ for $u\in B(u_0,r)$. Hence $B(u_0,r)$ is a neighborhood of u_0 in $BC(\Omega,K)$. Thus $\iota^{-1}\bigl(BC(\Omega,K)\bigr)$ is a neighborhood of u_0 in $B_p^{2-2/p}(\Omega,\mathbb{R}^n;s)$ and it is contained in V. This proves the claim. $\square$

For abbreviation,

$$E_0:=L_p(\Omega,\mathbb{R}^n;s),\quad E_1:=W_p^2(\Omega,\mathbb{R}^n;s),\quad E:=B_p^{2-2/p}(\Omega,\mathbb{R}^n;s).$$

As usual, $\mathcal{L}(E_1,E_0)$ is the Banach space of bounded linear operators from E_1 into E_0, and $C^{1\text{-}}$ means 'locally Lipschitz continuous'.

Lemma 4.2. *Suppose $p>m+2$. Then*

(i) *$\mathcal{A}_s(u_0)$ has maximal $L_p(\Omega,\mathbb{R}^n;s)$ regularity for $u_0\in V$.*

(ii) $\bigl(u\mapsto\mathcal{A}_s(u)\bigr)\in C^{1\text{-}}\bigl(V,\mathcal{L}(E_1,E_0)\bigr)$.

[1] $\operatorname{dist}\bigl(u_0(\Omega),\emptyset\bigr):=\infty$.

Proof. We denote by c constants ≥ 1 which may be different from occurrence to occurrence and write $BC_s^1 := BC^1(\Omega, \mathbb{R}^n; s)$ and $BC := BC(\Omega, \mathbb{R}^n)$.

Let $u_0 \in V$. Fix a bounded neighborhood V_K of u_0 in $\iota^{-1}\big(BC(\Omega, K)\big) \subset E$. This is possible by the preceding lemma.

Step 1. It is a consequence of (22-iii) and (26) that

$$1/c \leq a_i(u) \leq c, \qquad i = 1, \ldots, n, \quad u \in V_K. \tag{27}$$

Moreover, (22-iii) also implies that a_i and its Fréchet derivative ∂a_i are locally Lipschitz continuous. Hence

$$\begin{array}{l} a_i \text{ and } \partial a_i \text{ are bounded and uniformly} \\ \text{Lipschitz continuous on } \overline{\Omega} \times K \end{array} \tag{28}$$

(e.g., [2, Proposition 6.4]). From this and (25) we infer that

$$\|a_i(u) - a_i(v)\|_\infty \leq c\, \|u - v\|_E, \qquad 1 \leq i \leq n, \quad u, v \in V_K. \tag{29}$$

Step 2. Let i and j run from 1 to n and α from 1 to m. Then, using the summation convention writing $u = (u^1, \ldots, u^n)$,

$$\partial_\alpha\big(a_i(u)\big)(x) = (\partial_\alpha a_i)\big(x, u(x)\big) + (\partial_{m+j} a_i)\big(x, u(x)\big)\partial_\alpha u^j(x) \tag{30}$$

for $x \in \Omega$. By (25), V_K is bounded in BC_s^1. From this, (30), and (28) it follows

$$\sup_U \big|\partial_\alpha\big(a_i(u)\big)\big| \leq c, \qquad u \in V_K. \tag{31}$$

Similarly, by employing local coordinates on Γ,

$$\sup_S \big|\operatorname{grad}_\Gamma\big(a_i(u)\big)\big|_{T\Gamma} \leq c, \qquad u \in V_K, \tag{32}$$

where $|\cdot|_{T\Gamma}$ is the vector bundle norm on the tangent bundle $T\Gamma$ of Γ.

Let $x \in S$ and $\alpha = 1$ in (30). Then we get from $0 < \rho^s(x) \leq 1$, the boundedness of V_K in BC_s^1, and (29) that

$$\sup_S \big|\rho^s \partial_\nu\big(a_i(u)\big)\big| \leq c(1 + \sup_S |\rho^s \partial_\nu u|) \leq c, \qquad u \in V_K. \tag{33}$$

By collecting (27) and (31)–(33), we find (cf. (15) and (16))

$$a(u) \in BC_s^1, \quad \|a(u)\|_{BC_s^1} \leq c, \qquad u \in V_K. \tag{34}$$

Now (i) follows from (27) and Corollary 3.3.

Step 3. Let $u, v \in V$. Then

$$\begin{aligned}\partial_\alpha\bigl(a_i(u) - a_i(v)\bigr) = {} & \partial_\alpha(a_i)(u) - (\partial_\alpha a_i)(v) \\ & + \bigl((\partial_{m+j}a)(u) - (\partial_{m+j}a)(v)\bigr)\partial_\alpha u^j \\ & + (\partial_{m+j}a)(v)(\partial_\alpha u^j - \partial_\alpha v^j).\end{aligned}$$

Using (28) and (25), we obtain

$$\sup_U |(\partial_\alpha a)(u) - (\partial_\alpha a)(v)| \le c\,\|u - v\|_E, \qquad u, v \in V_K.$$

Similarly, employing also the boundedness of V_K in E and (34),

$$\sup_U \bigl|\bigl(\partial_{m+j}a)(u) - (\partial_{m+j}a)(v)\bigr)(\partial_\alpha u^j)\bigr| \le c\,\|u - v\|_E$$

and

$$\sup_U |(\partial_{m+j}a)(v)(\partial_\alpha u^j - \partial_\alpha v^j)| \le c\,\|u - v\|_E$$

for $u, v \in V_K$. Consequently,

$$\sup_U \bigl|\partial_\alpha\bigl(a(u) - a(v)\bigr)\bigr| \le c\,\|u - v\|_E, \qquad u, v \in V_K.$$

By analogous arguments we obtain, as in step (1),

$$\sup_S \bigl|\operatorname{grad}_\Gamma\bigl(a(u) - a(v)\bigr)\bigr|_{T\Gamma} \le c\,\|u - v\|_E$$

and

$$\sup_S \bigl|\rho^s\partial_\nu\bigl(a(u) - a(v)\bigr)\bigr| \le c\,\|u - v\|_E$$

for $u, v \in V_K$. In summary and recalling (29),

$$\|a(u) - a(v)\|_{BC^1_s} \le c\,\|u - v\|_E, \qquad u, v \in V_K.$$

This implies claim (ii). □

We also suppose

$$g \in C^1(\overline{\Omega} \times X, \mathbb{R}^{n\times n}) \tag{35}$$

and define $g(u)$ analogously to (23). Then, using obvious identifications,

$$f(u) := g(u)u, \qquad u \in C(\Omega, X). \tag{36}$$

Remark 4.3: This assumption on the production rate in (1) is motivated by models from population dynamics. It means that the reproduction (birth or death) rate is proportional to the size of the actually present crowd. Already the diagonal form

$$f_i(u) = g_i(u)u_i, \qquad 1 \le i \le n,$$

comprises the most frequently studied ecological models, namely the standard (two-population) models with competing (predator–prey or cooperative) species, for example. In those cases the g_i are affine functions of u.

Lemma 4.4. *Let $p > m+2$. Then*

$$\big(u \mapsto f(u)\big) \in C^{1\text{-}}\big(V, L_p(\Omega \times \mathbb{R}^n; s)\big).$$

Proof. Let $u_0 \in V$ and fix V_K as in the preceding proof. Then it is obvious from (25) and (35) that

$$\|g(u)\|_\infty \le c, \qquad u \in V_K,$$

and

$$\|g(u) - g(v)\|_\infty \le c\,\|u - v\|_E, \qquad u, v \in V_K.$$

Thus, since $E \hookrightarrow L_p(\Omega, \mathbb{R}^n; s) =: L_{p,s}$,

$$\|f(u)\|_{L_{p,s}} \le \|g(u)\|_\infty\,\|u\|_{L_{p,s}} \le c, \qquad u \in V_K,$$

and

$$\begin{aligned}\|f(u) - f(v)\|_{L_{p,s}} &\le \|g(u) - g(v)\|_\infty\,\|u\|_{L_{p,s}} + \|g(v)\|_\infty\,\|u - v\|_{L_{p,s}}\\ &\le c\,\|u - v\|_E\end{aligned}$$

for $u, v \in V_K$. □

Now we can prove the main result of this paper, a general well-posedness theorem for strong $L_p(\Omega, \mathbb{R}^n; s)$ solutions, by simply referring to known results. The reader may consult [2] or [3] for definitions and the facts on semiflows to which we appeal.

Theorem 4.5. *Let* (22), (35), *and* (36) *be satisfied and assume $p > m+2$. Define V by* (24). *Then the initial value problem for the s-degenerate quasilinear reaction-diffusion system*

$$\begin{aligned}\partial_t u - \operatorname{div}_s\big(a(u)\operatorname{grad}_s u\big) &= f(u) \quad \text{on } \Omega \times \mathbb{R}_+,\\ \gamma_0 u &= u_0 \quad \text{on } \Omega \times \{0\},\end{aligned} \tag{37}$$

has for each $u_0 \in V$ a unique maximal solution

$$u(\cdot, u_0) \in W_p^{(2,1)}\big(\Omega \times [0, t^+(u_0)), X; s\big).$$

The map $(t, u_0) \mapsto u(t, u_0)$ is a locally Lipschitz continuous semiflow on V. The exit time $t^+(u_0)$ is characterized by the following three (non mutually exclusive) alternatives:

$$\text{(i)}\quad t^+(u_0) = \infty.$$
$$\text{(ii)}\quad \liminf_{t \to t^+(u_0)} \operatorname{dist}\big(u(t, u_0)(\Omega), \partial X\big) = 0.$$
$$\text{(iii)}\quad \lim_{t \to t^+(u_0)} u(t, u_0) \textit{ does not exist in } B_p^{2-2/p}(\Omega, \mathbb{R}^n; s).$$

Proof. Due to Lemmas 4.1, 4.2, and 4.4, this follows from Theorem 5.1.1 and Corollary 5.1.2 in J. Prüss and G. Simonett [18]. □

REMARKS 4.6: (a) It is obvious from the above proofs that the regularity assumptions for a and g, concerning the variable $x \in \Omega$, are stronger than actually needed. We leave it to the interested reader to find out the optimal assumptions.

(b) Suppose that $a_i(u)$ is independent of u_j for $j \neq i$. Then the theorem remains true, with the obvious definitions of the weighted spaces, if we replace $\mathcal{A}_{i,s}(u)$ by $\mathcal{A}_{i,s_i}(u_i)$ with $1 \leq s_i < \infty$ for $1 \leq i \leq n$.

Proof. This follows by an inspection of the proof of Lemma 4.2. □

(c) For simplicity, we have assumed that $\Gamma = \partial\Omega$. It is clear that we can also consider the case where Γ is a proper open and closed subset of $\partial\Omega$ and regular boundary conditions are imposed on the remaining part.

(d) Similar results can be proved for strongly coupled systems, so-called cross-diffusion equations.

5. Examples and remarks

We close this paper by presenting some easy examples. In addition, we include some remarks on open problems and suggestions for further research. Throughout this section,

$$\bullet \quad 1 \leq s < \infty \quad \text{and} \quad p > m + 2.$$

EXAMPLE 5.1: (Two-population models) Let

$$a, b \in C^2(\overline{\Omega}, \mathbb{R}_+), \quad a_i, b_i \in C^1(\overline{\Omega}),\ i = 0, 1, 2, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}.$$

Consider the s-degenerate quasilinear system

$$\begin{aligned} \partial_t u - \operatorname{div}_s\big((a + u^\alpha v^\beta) \operatorname{grad}_s u\big) &= (a_0 + a_1 u + a_2 v)u, \\ \partial_t v - \operatorname{div}_s\big((b + u^\gamma v^\delta) \operatorname{grad}_s v\big) &= (b_0 + b_1 v + b_2 u)v \end{aligned} \tag{38}$$

on $\Omega \times \mathbb{R}_+$.

Suppose $\varepsilon > 0$ and

$$(u_0, v_0) \in W_p^2(\Omega; s), \qquad u_0, v_0 \geq \varepsilon.$$

Then there exist a maximal $t^+ = t^+(u_0, v_0) \in (0, \infty]$ and a unique solution

$$(u, v) \in W_p^{(2,1)}\big(\Omega \times [0, t^+), \mathbb{R}^2; s\big)$$

of (38) satisfying $u(t)(x) > 0$ and $v(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. Theorem 4.5 with $X = (0, \infty)^2$. □

Observe that the right side of (38) encompasses standard predator–prey as well as cooperation models, depending on the signs of the coefficient functions.

In the following examples we restrict ourselves to scalar equations.

Examples 5.2: (a) (Porous media equations) Let $\alpha \in \mathbb{R}\backslash\{0\}$ and assume that $g \in C^1(\overline{\Omega} \times \mathbb{R})$. Then

$$\partial_t u - \operatorname{div}_s(u^\alpha \operatorname{grad}_s u) = g(u)u \quad \text{in } \Omega \times \mathbb{R}_+$$

has for each $u_0 \in W_p^2(\Omega; s)$ with $u_0 \geq \varepsilon > 0$ a unique maximal solution

$$u \in W_p^{(2,1)}\big(\Omega \times (0, t^+); s\big)$$

satisfying $u(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. Theorem 4.5 with $X := (0, \infty)$. □

(b) (Diffusive logistic equations) Assume $\alpha, \lambda \in \mathbb{R}$ with $\alpha > 0$. Let $a \geq 0$. Set

$$\Delta_s := \operatorname{div}_s \operatorname{grad}_s .$$

If

$$u_0 \in W_p^2(\Omega; s), \qquad u_0 \geq \varepsilon > 0,$$

then there exist a maximal $t^+ \in (0, \infty]$ and a unique solution

$$u \in W_p^{(2,1)}\big(\Omega \times [0, t^+); s\big)$$

of

$$\partial_t u - \alpha \Delta_s u = (\lambda - au)u \quad \text{on } \Omega \times \mathbb{R}_+, \tag{39}$$

satisfying $u(t)(x) > 0$ for $x \in \Omega$ and $0 \leq t < t^+$.

Proof. This is essentially a subcase of Example 5.2. □

The most natural question which now arises is:

- How can we prove global existence?

An attempt to tackle this challenge, which is already hard in the case of standard boundary value problems, is even more demanding in the present setting. To point out where some of the difficulties originate from, we review in the following remarks some of the well-known techniques, which have successfully been applied to parabolic boundary value problems, and indicate why they do not straightforwardly apply to s-degenerate problems.

REMARK 5.3: (Maximum principle techniques) First we look at the diffusive logistic equation (39) and contrast it with the simple classical counterpart

$$\begin{aligned} \partial_t u - \alpha\Delta u &= (\lambda - u)u \quad &&\text{on } \overline{\Omega} \times \mathbb{R}_+, \\ u &= 0 &&\text{on } \Gamma \times \mathbb{R}_+. \end{aligned} \tag{40}$$

Suppose $\lambda > 0$. Then (40) has for each sufficiently smooth initial value u_0 satisfying $0 \leq u_0 \leq \lambda$ a unique global solution obeying the same bounds as u_0. This is a consequence of the maximum principle, since $(0, \lambda)$ is a pair of sub- and supersolutions. For the validity of this argument it is crucial that we deal with a boundary value problem.

In the s-degenerate case there is no boundary. Hence the preceding argument does not work, since there is no appropriate maximum principle.

REMARK 5.4: (Methods based on spectral properties) A further important technique, which is useful in the case of boundary value problems, rests on spectral properties of the linearization of associated stationary elliptic equations. The most prominent case is supposedly the 'principle of linearized stability' (and its generalizations to non-isolated equilibria, see [18, Chapter 5]). In addition, the better part of the qualitative studies on semi- and quasilinear parabolic boundary value problems, as well for a single equation as for systems, is based on spectral properties, in particular on the existence and the nature of eigenvalues.

In the case of boundary value problems, the associated linear elliptic operators have compact resolvents, due to the compactness of $\overline{\Omega}$. In our situation, Ω, more precisely, the Riemannian manifold $\Omega_s = (\Omega, g_s)$ introduced in Remark 2.1, is not compact. If $s = 1$, then it is a manifold with cylindrical ends in the sense of R. Melrose [14] and others[2]. For manifolds of this type—and more general ones—much is known about the (L_2) spectrum of the Laplace–Beltrami operator. In particular, the essential spectrum is not empty.

Nevertheless, we are in a simpler situation. In fact, the normal collar S can be represented as a half-cylinder over the compact manifold (Γ, h) and the

[2] I am grateful to Victor Nistor for pointing this out to me.

remaining 'interior part' U is flat (cf. [5, Section 5]). Thus there is hope to get sufficiently detailed information on the L_p spectrum of linear divergence form operators.

In particular, suppose $u_0 \in V \cap W_p^2(\Omega, \mathbb{R}^n; s)$ is a stationary point of (37). If it can be shown that the spectrum of $\mathcal{A}(u_0)$ is contained in an interval $[\alpha, \infty)$ with $\alpha > 0$, then, using the decay properties of the analytic semigroup generated by $-\mathcal{A}(u_0)$, it can be shown that u_0 is an asymptotically stable critical point of (37).

Remark 5.5: (The technique of a priori estimates) The general version of [18, Theorem 5.1.1] exploits the regularization properties of analytic semigroups. Suppose that alternative (ii) of Theorem 4.5 does not occur. Also assume that there can be established a uniform a priori bound in a Besov space $B_{p,s}^{\sigma-2/p} := B_p^{\sigma-2/p}(\Omega, \mathbb{R}^n; s)$ with $2/p < \sigma < 2$. Then we have global existence, *provided* the embedding $B_{p,s}^{2-2/p} \hookrightarrow B_{p,s}^{\sigma-2/p}$ is compact (see [18, Theorem 5.7.1]). Hence this technique is also not applicable to our equations.

However, we still have the possibility to use the interpolation-extrapolation techniques developed in [3] and [4] to switch to weak formulations. Then it might be possible to prove global existence by more classical techniques using a priori estimates with respect to suitable integral norms.

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Global structure of bifurcation curves related to inverse bifurcation problems

Tetsutaro Shibata

Dedicated to Professor Julián López-Gómez on his sixtieth birthday

Abstract. *We consider the nonlinear eigenvalue problem*

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0,$$
$$u(t) > 0\,, \quad t \in I := (0,1)\,, \quad u(0) = u(1) = 0,$$

which comes from the porous media type equation. Here, $D(u) = pu^{2n} + \sin u$ *(*$n \in \mathbb{N}$, $p > 0$*: given constants),* $g(u) = u$ *or* $g(u) = u + \sin u$. $\lambda > 0$ *is a bifurcation parameter which is a continuous function of* $\alpha = \|u_\lambda\|_\infty$ *of the solution* u_λ *corresponding to* λ*, and is expressed as* $\lambda = \lambda(\alpha)$*. Since our equation contains oscillatory term in diffusion term, it seems significant to study how this oscillatory term gives effect to the structure of bifurcation curves* $\lambda(\alpha)$*. We propose a question from a view point of inverse bifurcation problems and show that the simplest case* $D(u) = u^2 + \sin u$ *and* $g(u) = u$ *gives us the most impressible asymptotic formula for global behavior of* $\lambda(\alpha)$*.*

Keywords: precise structure of bifurcation curves, oscillatory nonlinear diffusion, inverse bifurcation problems.
MS Classification 2010: 34C23, 34F10.

1. Introduction

We study the following nonlinear eigenvalue problems

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0, \qquad t \in I := (0,1), \tag{1}$$
$$u(t) > 0, \qquad t \in I, \tag{2}$$
$$u(0) = u(1) = 0, \tag{3}$$

where $D(u) := pu^{2n} + \sin u$ ($n \in \mathbb{N}$, $p > 0$: given constants), $g(u) = u$ or $g(u) = u + \sin u$, and $\lambda > 0$ is a bifurcation parameter. We assume the following condition (A.1).

(A.1) $D(u) > 0$ for $u > 0$.

Under the condition (A.1), we know from [11] that for a given $\alpha > 0$, there is a unique solution pair (u_α, λ) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by $\alpha > 0$ as $\lambda(\alpha)$ and is continuous for $\alpha > 0$.

The purpose of this paper is to show how the oscillatory diffusion term $D(u)$ gives effect to the structure of bifurcation curves $\lambda(\alpha)$. To clarify our intention, let $n = p = 1$ in (1) for simplicity. Then (A.1) is satisfied and we have the equation

$$[\{u(t)^2 + \sin u(t)\}u'(t)]' + \lambda u(t) = 0, \qquad t \in I. \tag{4}$$

The other equations similar to (4) are

$$[u(t)^2 u(t)']' + \lambda(u(t) + \sin u(t)) = 0, \qquad t \in I, \tag{5}$$

$$[\{u(t)^2 + \sin u(t)\}u(t)']' + \lambda(u(t) + \sin u(t)) = 0, \qquad t \in I. \tag{6}$$

We propose the following question from a view point of inverse bifurcation problems

Question A. *Consider* (4)*, or* (5)*, or* (6) *with* (2)–(3)*. Then is it possible to distinguish* (4)*,* (5) *and* (6) *from the asymptotic behavior of* $\lambda(\alpha)$ *for* $\alpha \gg 1$ *or not?*

We explain the back ground of Question A more precisely. Bifurcation problems with $D(u) \equiv 1$ are one of the main interest in the study of differential equations, and many results have been established concerning the asymptotic behavior of bifurcation curves from mathematical point of view. We refer to [1, 2, 3, 4, 9, 10, 12, 13, 14] and the references therein. Besides, the bifurcation problems with nonlinear diffusion appear in the various fields. The case $D(u) = u^k$ ($k > 0$) appears as the porous media equation in material science and logistic type model equation in population dynamics. In the latter case, it implies that the diffusion rate $D(u)$ depends on both the population density u and a parameter $1/\lambda$. We refer to [11, 15, 19] and the references therein. Added to these, there are several papers studying the asymptotic behavior of oscillatory bifurcation curves. We refer to [6, 7, 8, 9, 16, 17, 18] and the references therein.

Recently, the following equation has been considered in [18].

$$[D(u(t))u(t)']' + \lambda g(u(t)) = 0, \qquad t \in I \tag{7}$$

with (2)–(3). Here, $D(u) = u^k$, $g(u) = u^{2m-k-1} + \sin u$, and $m \in \mathbb{N}$, k ($0 \le k < 2m-1$) are given constants. In particular, if we put $m = k = 2$, then we have the equation (5). In [18], the following result has been obtained.

THEOREM 1.1 ([18]). *Consider* (7) *with* (2)–(3). *Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = 4m\alpha^{2k+2-2m}\left\{A_{k,m}^2 - 2A_{k,m}\sqrt{\frac{\pi}{2m}}\,\alpha^{k+(1/2)-2m}\sin\left(\alpha - \frac{\pi}{4}\right)\right.$$
$$\left. + \ o(\alpha^{k+(1/2)-2m})\right\}, \tag{8}$$

where

$$A_{k,m} = \int_0^1 \frac{s^k}{\sqrt{1-s^{2m}}}ds.$$

If $m = k = 2$ in (8), then the asymptotic formula for $\lambda(\alpha)$ as $\alpha \to \infty$ for (5) is given by

$$\lambda(\alpha) = 8\alpha^2\left\{A_{2,2}^2 - A_{2,2}\sqrt{\pi}\alpha^{-3/2}\sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-3/2})\right\}. \tag{9}$$

Moreover, it was shown in [18] that if $m = k = 2$, then $\lambda(\alpha) = 4B_0^2\alpha^2(1+o(1))$ as $\alpha \to 0$, where B_0 is a positive constant. Therefore, the rough picture of $\lambda(\alpha)$ in (5) is depicted in Figure 1.

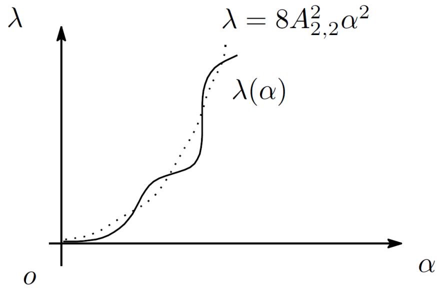

Figure 1: The graph of $\lambda(\alpha)$ for (5).

We understand from Theorem 1.1 that, by the effect of $\sin u$, $\lambda(\alpha)$ in Figure 1 oscillates and crosses the curve $\lambda = 8A_{2,2}\alpha^2$, which are the *original* bifurcation curve obtained from the equation (5) *without* $\sin u$, infinitely many times. Motivated by this, we would like to study how oscillatory nonlinear diffusion influences on the structure of $\lambda(\alpha)$. The question we have to ask here is whether the oscillatory term $\sin u$ in $D(u)$ gives the same influence on the asymptotic behavior of $\lambda(\alpha)$ as (8) or not.

Now we state our main results which give us the answer to this question. We begin with the typical case $n = 1$ and $g(u) = u$, since in this case, we are able to obtain up to the third term of $\lambda(\alpha)$. This fact seems to be a signifgicant progress in the study of asymptotic behavior of $\lambda(\alpha)$ as $\alpha \to \infty$.

THEOREM 1.2. *Assume (A.1). Let* $n = 1$. *Consider* (1)–(3) *with* $g(u) = u$.

(i) As $\alpha \to \infty$,

$$\lambda(\alpha) = 8pA_{2,2}^2\alpha^2 + \frac{8(p-1)}{p}A_{2,2}\sqrt{\pi}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right) + \frac{8}{p}\left\{2A_{2,2}\left(p - 2B\right)\cos\alpha\right\}\alpha^{-1} + O(\alpha^{-3/2}), \tag{10}$$

where

$$B := 2\int_0^{\pi/2} \frac{\sin^2\theta}{(1+\sin^2\theta)^{5/2}}\left(1 - \frac{1}{2}\sin^2\theta\right)d\theta. \tag{11}$$

In particular, let $p = 1$. *Then* (10) *is represented as*

$$\lambda(\alpha) = 8A_{2,2}^2\alpha^2 + (16A_{2,2}\left(1 - 2B\right)\cos\alpha)\alpha^{-1} + O(\alpha^{-3/2}). \tag{12}$$

(ii) As $\alpha \to 0$,

$$\lambda(\alpha) = 6\alpha(C_0^2 + 2pC_0C_1\alpha + O(\alpha^2)),$$

where

$$C_0 := \int_0^1 \frac{s}{\sqrt{1-s^3}}ds, \quad C_1 := \int_0^1 \frac{1}{\sqrt{1-s^3}}\left(s^2 - \frac{3s(1-s^4)}{8(1-s^3)}\right)ds.$$

Therefore, our first conclusion is that the the shape of the second term of (12) is completely different from that of (9), since we are able to show by direct calculation that $B \neq 1/2$. Namely, the answer to Question A is affirmative. The global structures of the bifurcation curves for (4) and (5) do not coincide each other.

For the case $n \geq 2$, we obtain up to the second term of $\lambda(\alpha)$.

THEOREM 1.3. *Assume (A.1). Let* $n \geq 2$. *Consider* (1)–(3) *with* $g(u) = u$. *Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = \frac{4(n+1)}{p}\left(p^2A_{2n,n+1}^2\alpha^{2n} + 2A_{2n,n+1}(p-1) \times \sqrt{\frac{\pi}{2(n+1)}}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1})\right).$$

In particular, let $p = 1$. *Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = 4(n+1)A_{2n,n+1}^2\alpha^{2n} + O(\alpha^{-1}). \tag{13}$$

REMARK 1.4: (i) Certainly, the future direction of this study will be to obtain the exact second term of (13), although it seems very difficult to get it technically.

(ii) Theorem 1.2-(ii) is only proved for the case $n = 1$ to show that the rough picture of $\lambda(\alpha)$ is almost the same as that of Fig. 1 if $p \neq 1$. Certainly, we easily obtain Theorem 1.2-(ii) for the case $n \geq 2$.

The following Theorem 1.5 gives us the negative answer to Question A.

THEOREM 1.5. *Consider* (6) *with* (2)–(3).

(i) The asymptotic formula (9) *holds as* $\alpha \to \infty$.

(ii) The following asymptotic formula holds as $\alpha \to 0$.

$$\lambda(\alpha) = 3\alpha \left(C_0^2 + 2C_0C_1\alpha + O(\alpha^2)\right).$$

We find from Theorems 1.2–1.5 that $\sin u$ in diffusion term has deep influences on the global behavior of $\lambda(\alpha)$.

We prove our results by using time-map method and stationary phase method.

2. Proof of Theorem 1.3-(i)

In this section, let $D(u) = pu^{2n} + \sin u$, $g(u) = u$ and $\alpha \gg 1$. We denote by C the various positive constants independent of $\alpha \gg 1$. We put

$$\Lambda := \left\{ \alpha > 0 \ \mid \ g(\alpha) > 0, \int_u^\alpha g(t)D(t)dt > 0 \ \text{ for all } \ u \in [0, \alpha) \right\}.$$

It follows from [11, (2.7)], that if $\alpha \in \Lambda$, then $\lambda(\alpha)$ is well defined. By (A.1), we have $D(t) > 0$, $g(t) > 0$ for $t > 0$. So $g(t)D(t) > 0$ for $t > 0$ holds. Hence, $\Lambda \equiv \mathbb{R}_+$. By this and the generalized time-map in [9, (2.5)] (cf. (15) below) and the time-map argument in [10, Theorem 2.1], we find that for any given $\alpha > 0$, there is a unique solution pair $(u_\alpha, \lambda) \in C^2(I) \bigcap C(\bar{I}) \times \mathbb{R}_+$ of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is a continuous function for $\alpha > 0$. It is well known that if $(u_\alpha, \lambda(\alpha)) \in C^2(I) \bigcap C(\bar{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$\begin{aligned}
&u_\alpha(t) = u_\alpha(1 - t), \quad 0 \leq t \leq 1, \\
&u_\alpha\left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_\alpha(t) = \alpha, \\
&u_\alpha'(t) > 0, \quad 0 < t < \frac{1}{2}.
\end{aligned}$$

We put

$$\begin{aligned}
G(u) &:= \int_0^u D(x)g(x)dx = \int_0^u (px^{2n} + \sin x)x dx \qquad (14) \\
&= \frac{p}{2n+2}u^{2n+2} - u\cos u + \sin u,
\end{aligned}$$

$$M_1 := \alpha\cos\alpha - \alpha s\cos(\alpha s), \;\; M_2 := \sin\alpha - \sin(\alpha s) \;\; (0 \le s \le 1).$$

For $0 \le s \le 1$ and $\alpha \gg 1$, we have

$$\frac{|M_1| + |M_2|}{\alpha^{2n+2}(1 - s^{2n+2})} \le C\alpha^{-2n}. \tag{15}$$

By this, Taylor expansion and putting $u = s\alpha$, we have from [9] that

$$\begin{aligned}
\sqrt{\frac{\lambda}{2}} &= \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha) - G(u)}}du \qquad (16)\\
&= \alpha\int_0^1 \frac{p\alpha^{2n}s^{2n} + \sin(\alpha s)}{\sqrt{\frac{p}{2n+2}\alpha^{2n+2}(1 - s^{2n+2}) - (M_1 - M_2)}}ds \\
&= \sqrt{\frac{2n+2}{p}}\alpha^{-n}\int_0^1 \frac{p\alpha^{2n}s^{2n} + \sin(\alpha s)}{\sqrt{1 - s^{2n+2}}\sqrt{1 - \frac{2n+2}{p\alpha^{2n+2}(1-s^{2n+2})}\{M_1 - M_2\}}}ds \\
&= \sqrt{\frac{2n+2}{p}}\alpha^{-n}\int_0^1 \frac{p\alpha^{2n}s^{2n} + \sin(\alpha s)}{\sqrt{1 - s^{2n+2}}} \\
&\qquad \times \left(1 + \frac{n+1}{p\alpha^{2n+2}(1 - s^{2n+2})}\{M_1 - M_2\}(1 + O(\alpha^{-2n}))\right)ds.
\end{aligned}$$

This implies that

$$\sqrt{\frac{\lambda}{2}} = \sqrt{\frac{2n+2}{p}}\alpha^{-n}\{J_1 + J_2 + J_3 + J_4 + J_5\}(1 + O(\alpha^{-2n})), \tag{17}$$

where

$$\begin{aligned}
J_1 &:= p\alpha^{2n}\int_0^1 \frac{s^{2n}}{\sqrt{1 - s^{2n+2}}}ds = pA_{2n,n}\alpha^{2n}, \\
J_2 &:= \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1 - s^{2n+2}}}ds, \qquad (18)\\
J_3 &:= \frac{n+1}{p\alpha}\int_0^1 \frac{s^{2n}}{(1 - s^{2n+2})^{3/2}}(\cos\alpha - s\cos(\alpha s))ds, \qquad (19)\\
J_4 &:= -\frac{n+1}{p\alpha^2}\int_0^1 \frac{s^{2n}}{(1 - s^{2n+2})^{3/2}}(\sin\alpha - \sin(\alpha s))ds, \\
J_5 &:= \frac{n+1}{p\alpha^{2n+2}}\int_0^1 \frac{\sin(\alpha s)}{(1 - s^{2n+2})^{3/2}}\{M_1 - M_2\}ds. \qquad (20)
\end{aligned}$$

To calculate $J_2 \sim J_5$, we use the following equality.

LEMMA 2.1 ([7, Lemma 2], [9, Lemma 2.25]). *Assume that the function $f(r) \in C^2[0,1]$, $w(r) = \cos(\pi r/2)$. Then as $\mu \to \infty$*

$$\int_0^1 f(r)e^{i\mu w(r)}dr = e^{i(\mu-(\pi/4))}\sqrt{\frac{2}{\mu\pi}}f(0) + O\left(\frac{1}{\mu}\right). \tag{21}$$

In particular, by taking the real and imaginary parts of (21)*, as $\mu \to \infty$,*

$$\begin{aligned}
\int_0^1 f(r)\cos(\mu w(r))dr &= \sqrt{\frac{2}{\mu\pi}}f(0)\cos\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right), \\
\int_0^1 f(r)\sin(\mu w(r))dr &= \sqrt{\frac{2}{\mu\pi}}f(0)\sin\left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right).
\end{aligned}$$

LEMMA 2.2. *As $\alpha \to \infty$,*

$$J_2 = \sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}).$$

Proof. Putting $s = \sin\theta$, $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$\begin{aligned}
J_2 &= \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1-s^2}\sqrt{1+s^2+\cdots+s^{2n}}}ds \qquad (22)\\
&= \int_0^{\pi/2} \frac{1}{\sqrt{1+\sin^2\theta+\cdots+\sin^{2n}\theta}}\sin(\alpha\sin\theta)d\theta \\
&= \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{\sqrt{1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)}}\sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx \\
&= \sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}).
\end{aligned}$$

Thus the proof is complete. □

LEMMA 2.3. *As $\alpha \to \infty$,*

$$J_3 = -\frac{1}{p}\sqrt{\frac{\pi}{2(n+1)\alpha}}\left\{\sin\left(\alpha - \frac{\pi}{4}\right) - \alpha^{-1}\cos\left(\alpha - \frac{\pi}{4}\right)\right\} + O(\alpha^{-1}). \tag{23}$$

Proof. We put $J_3 = (n+1)J_{31}/(p\alpha)$, $s = \sin\theta$ and $K(\theta) := \sin^{2n}\theta/(1+\sin^2\theta+$

$\cdots + \sin^{2n}\theta)^{3/2}$. Then by integration by parts,

$$\begin{aligned} J_{31} &= \int_0^1 \frac{s^{2n}}{(1-s^2)^{3/2}(1+s^2+\cdots+s^{2n})^{3/2}}(\cos\alpha - s\cos(\alpha s))ds \qquad (24)\\ &= \int_0^{\pi/2} \frac{1}{\cos^2\theta}K(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta \\ &= [\tan\theta K(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan\theta K'(\theta)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta \\ &\quad - \int_0^{\pi/2} \tan\theta K(\theta)\{-\cos\theta\cos(\alpha\sin\theta) + \alpha\sin\theta\cos\theta\sin(\alpha\sin\theta)\}d\theta \\ &:= J_{311} - J_{312} + J_{313}. \end{aligned}$$

By using l'Hôpital's rule, we have

$$\begin{aligned} &\lim_{\theta\to\pi/2} \frac{\cos\alpha - \sin\theta\cos(\alpha\sin\theta)}{\cos\theta} \\ &\quad = \lim_{\theta\to\pi/2} \frac{-\cos\theta\cos(\alpha\sin\theta) + \alpha\sin\theta\cos\theta\sin(\alpha\sin\theta)}{-\sin\theta} = 0. \end{aligned}$$

We see from this that $J_{311} = 0$. Moreover, by direct calculation, we see that $J_{312} = O(1)$. Now, putting $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$\begin{aligned} J_{313} &= \int_0^{\pi/2} \sin\theta K(\theta)\cos(\alpha\sin\theta)d\theta \qquad (25)\\ &\qquad - \alpha\int_0^{\pi/2} K(\theta)\sin^2\theta\sin(\alpha\sin\theta)d\theta \\ &= \frac{\pi}{2}\int_0^1 \frac{\cos^{2n+1}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}} \cos\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx \\ &\quad - \frac{\pi}{2}\alpha\int_0^1 \frac{\cos^{2n+2}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}} \sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx \\ &= (n+1)^{-3/2}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha-\frac{\pi}{4}\right) - (n+1)^{-3/2}\sqrt{\frac{\pi\alpha}{2}}\sin\left(\alpha-\frac{\pi}{4}\right) + O(1). \end{aligned}$$

This implies (23). Thus the proof is complete. □

LEMMA 2.4. *As* $\alpha \to \infty$,

$$J_4 = -\frac{1}{p}\sqrt{\frac{\pi}{2(n+1)\alpha}}\alpha^{-1}\cos\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-2}). \qquad (26)$$

Proof. We put $J_4 = -(n+1)J_{41}/(p\alpha^2)$ and $s = \sin\theta$. Then by the same argument as that to obtain J_{31} in (25), we obtain

$$\begin{aligned}
J_{41} &= \int_0^1 \frac{s^{2n}}{(1-s^{2n+2})^{3/2}}(\sin\alpha - \sin(\alpha s))ds \\
&= \int_0^{\pi/2} \frac{1}{\cos^2\theta} K(\theta)(\sin\alpha - \sin(\alpha\sin\theta))d\theta \\
&= [\tan\theta K(\theta)(\sin\alpha - \sin(\alpha\sin\theta))]_0^{\pi/2} \\
&\quad - \int_0^{\pi/2} \tan\theta K'(\theta)(\sin\alpha - \sin(\alpha\sin\theta))d\theta \\
&\quad + \alpha\int_0^{\pi/2} \sin\theta K(\theta)\cos(\alpha\sin\theta)d\theta \\
&= \frac{\pi}{2}\alpha\int_0^1 \frac{\cos^{2n+1}\left(\frac{\pi}{2}x\right)}{\left(1+\cos^2\left(\frac{\pi}{2}x\right)+\cdots+\cos^{2n}\left(\frac{\pi}{2}x\right)\right)^{3/2}}\cos\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx \\
&\qquad + O(1) \\
&= (n+1)^{-3/2}\sqrt{\frac{\pi\alpha}{2}}\cos\left(\alpha - \frac{\pi}{4}\right) + O(1).
\end{aligned}$$

By this, we obtain (26). Thus the proof is complete. $\square$

Proof of Theorem 1.3-(i). By (15) and (20), we see that $J_5 = O(\alpha^{-2n})$. By this, (17) and Lemmas 2.2–2.4, we obtain

$$\begin{aligned}
\sqrt{\frac{\lambda}{2}} = \sqrt{\frac{2n+2}{p}}\alpha^{-n}\Bigg[pA_{2n,n}\alpha^{2n} + \left(1-\frac{1}{p}\right) \\
\times\sqrt{\frac{\pi}{2(n+1)\alpha}}\sin\left(\alpha-\frac{\pi}{4}\right) + O(\alpha^{-1})\Bigg].
\end{aligned}$$

By this, we obtain

$$\begin{aligned}
\lambda = \frac{4(n+1)}{p}\alpha^{-2n}\Bigg[p^2A_{2n,n+1}^2\alpha^{4n} + 2\,(p-1)\,A_{2n,n+1} \\
\times\alpha^{2n-(1/2)}\sqrt{\frac{\pi}{2(n+1)}}\sin\left(\alpha-\frac{\pi}{4}\right) + O(\alpha^{2n-1})\Bigg].
\end{aligned} \tag{27}$$

This implies Theorem 1.3-(i). Thus the proof is complete. $\square$

3. Proof of Theorem 1.2-(i)

Let $n = 1$ in this section. Based on the calculation in the previous section and the argument of stationary phase method (cf. [9, Lemmas 2.24 and 2.25]), we calculate the third term of (10).

LEMMA 3.1. *Let $\alpha \gg 1$. Then*

$$\Phi(\alpha) := \int_0^1 e^{-i\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} e^{-i\pi/4} + \frac{i}{2\alpha} e^{-i\alpha} + O(\alpha^{-3/2}). \tag{28}$$

Proof. We put $t = \sqrt{\alpha} x$. Then by integral by parts, we obtain

$$\begin{aligned}\Phi(\alpha) &= \frac{1}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha}} e^{-it^2} dt \\ &= \frac{1}{\sqrt{\alpha}} \left\{ \int_0^\infty e^{-it^2} dt - \int_{\sqrt{\alpha}}^\infty e^{-it^2} dt \right\} \\ &=: \frac{1}{\sqrt{\alpha}} \left\{ \frac{\sqrt{\pi}}{2} e^{-i\pi/4} - \Phi_0(\alpha) \right\}.\end{aligned} \tag{29}$$

Then by integration by parts,

$$\begin{aligned}\Phi_0(\alpha) &= \int_{\sqrt{\alpha}}^\infty \frac{1}{-2it} (e^{-it^2})' dt \\ &= \left[\frac{1}{-2it} e^{-it^2} \right]_{\sqrt{\alpha}}^\infty + \frac{i}{2} \int_{\sqrt{\alpha}}^\infty \left(-\frac{1}{t} \right)' e^{-it^2} dt \\ &=: \frac{1}{2i\sqrt{\alpha}} e^{-i\alpha} + \frac{i}{2} \Phi_1(\alpha).\end{aligned} \tag{30}$$

Then

$$\begin{aligned}\Phi_1(\alpha) &= \int_{\sqrt{\alpha}}^\infty \frac{1}{t^2} e^{-it^2} dt = \int_{\sqrt{\alpha}}^\infty \frac{1}{t^2} \left(\frac{1}{-2it} \right) (e^{-it^2})' dt \\ &= \left[\frac{i}{2} \frac{1}{t^3} e^{-it^2} \right]_{\sqrt{\alpha}}^\infty + \frac{3i}{2} \int_{\sqrt{\alpha}}^\infty \frac{1}{t^4} e^{-it^2} dt = O(\alpha^{-3/2}).\end{aligned} \tag{31}$$

By (29)–(31), we obtain (28). Thus the proof is complete. □

LEMMA 3.2. *Let $\alpha \gg 1$, $g(x) := \cos(\pi x/2)$ and $k(x) \in C^3([0,1])$. Then as $\alpha \to \infty$,*

$$\begin{aligned}I &:= \int_0^1 k(x) e^{i\alpha g(x)} dx \\ &= \frac{4}{\pi} e^{i\alpha} \int_0^1 k\left(\frac{2}{\pi} \cos^{-1}(1 - t^2) \right) \frac{1}{\sqrt{2 - t^2}} e^{-i\alpha t^2} dt.\end{aligned} \tag{32}$$

Proof. We put $t = \sqrt{1 - \cos\left(\frac{\pi}{2} x\right)}$. Then by direct calculation, we obtain (32). □

LEMMA 3.3. *Let $f(x) \in C^3[0,1]$. Then as $\alpha \to \infty$,*

$$\begin{aligned} II &:= \int_0^1 f(x)e^{-i\alpha x^2}dx \\ &= f(0)\left\{\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}e^{-i\pi/4} + \frac{i}{2\alpha}e^{-i\alpha}\right\} + \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + O(\alpha^{-3/2}), \end{aligned} \tag{33}$$

where $h(x) := (f(x) - f(0))/x$.

Proof. We have

$$\begin{aligned} II &= \int_0^1 (f(0) + h(x)x)e^{-i\alpha x^2}dx \\ &= f(0)\int_0^1 e^{-i\alpha x^2}dx + III \\ &:= f(0)\int_0^1 e^{-i\alpha x^2}dx + \int_0^1 h(x)xe^{-i\alpha x^2}dx. \end{aligned} \tag{34}$$

By Lemma 3.1, we have

$$\begin{aligned} III &= \int_0^1 h(x)\left(\frac{1}{-2i\alpha}e^{-i\alpha x^2}\right)' dx \\ &= \left[h(x)\left(\frac{1}{-2i\alpha}e^{-i\alpha x^2}\right)\right]_0^1 + \frac{1}{2i\alpha}\int_0^1 h'(x)e^{-i\alpha x^2}dx \\ &= \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + \frac{1}{2i\alpha}\left(h'(0)\int_0^1 e^{-i\alpha x^2}dx + O(\alpha^{-1})\right) \\ &= \frac{i}{2\alpha}(h(1)e^{-i\alpha} - h(0)) + O(\alpha^{-3/2}). \end{aligned}$$

By this, Lemma 3.1 and (34), we obtain (33). Thus the proof is complete. □

LEMMA 3.4. *Consider J_2 defined in (18). Then as $\alpha \to \infty$,*

$$J_2 = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + \frac{1}{\alpha} + O(\alpha^{-3/2}). \tag{35}$$

Proof. Since $n = 1$, by (22), we have

$$J_2 = \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt{1+\cos^2\left(\frac{\pi}{2}x\right)}}\sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right)dx := \frac{\pi}{2}J_{21}. \tag{36}$$

We put $g(x) = \cos\left(\frac{\pi}{2}x\right)$ and $k(x) = 1/\sqrt{1+\cos^2\left(\frac{\pi}{2}x\right)}$. Then by using Lemma 3.2, we have

$$J_{21} = \mathrm{Im}\left(\frac{4}{\pi}e^{i\alpha}\int_0^1 \frac{1}{\sqrt{2-2t^2+t^4}}\frac{1}{\sqrt{2-t^2}}e^{-i\alpha t^2}dt\right). \tag{37}$$

We put $m(t) := 4 - 6t^2 + 4t^4 - t^6$. Then we have

$$K(t) := \frac{1}{\sqrt{2-2t^2+t^4}}\frac{1}{\sqrt{2-t^2}} = \frac{1}{\sqrt{m(t)}}, \tag{38}$$

$$h(t) := \frac{K(t)-K(0)}{t} = \frac{6t-4t^3+t^5}{2\sqrt{m(t)}(\sqrt{m(t)}+2)}. \tag{39}$$

Clearly, $h(t) \in C^3[0,1]$. By (38) and (39), we have

$$m(0) = 4,\ m(1) = 1,\ K(0) = \frac{1}{2},\ h(0) = 0,\ h(1) = \frac{1}{2}.$$

By this and Lemma 3.2, we obtain

$$\begin{aligned} J_{21} &= \frac{4}{\pi}e^{i\alpha}\left[\frac{\sqrt{\pi}}{4\sqrt{\alpha}}e^{-i\pi/4} + \frac{i}{2\alpha}e^{-i\alpha} + O(\alpha^{-3/2})\right], \\ \mathrm{Im}J_{21} &= \frac{1}{\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + \frac{2}{\alpha\pi} + O(\alpha^{-3/2}). \end{aligned}$$

By this and (36), we obtain (35). Thus the proof is complete. □

Lemma 3.5. *Consider J_3 defined in* (19). *Let B be the constant defined in* (11). *Then as $\alpha \to \infty$,*

$$J_3 = -\frac{1}{2p}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) - \frac{2B}{p\alpha}\cos\alpha + O(\alpha^{-3/2}).$$

Proof. We use the same notation as those in Lemma 2.3. We have $J_3 = \frac{2}{p\alpha}J_{31}$ and $J_{31} = -J_{312} + J_{313}$. Since $n = 1$, we have $K(\theta) = \sin^2\theta/(1+\sin^2\theta)^{3/2}$. We have

$$K'(\theta) = \frac{2\sin\theta\cos\theta}{(1+\sin^2)^{5/2}}\left(1 - \frac{1}{2}\sin^2\theta\right).$$

By this and Lemma 2.1, we obtain

$$\begin{aligned} J_{312} &= \int_0^1 \frac{2\sin^2\theta}{(1+\sin^2\theta)^{5/2}}\left(1-\frac{1}{2}\sin^2\theta\right)(\cos\alpha - \sin\theta\cos(\alpha\sin\theta))d\theta \\ &= B\cos\alpha + O(\alpha^{-1/2}). \end{aligned}$$

Next, by (24) and (25), we have

$$J_{313} = 2^{-3/2}\sqrt{\frac{\pi}{2\alpha}}\cos\left(\alpha - \frac{\pi}{4}\right)$$
$$- \frac{\pi}{2}\alpha \int_0^{\pi/2} \frac{\cos^4\left(\frac{\pi\theta}{2}\right)}{\left(1+\cos^2\left(\frac{\pi\theta}{2}\right)\right)^{3/2}} \sin\left(\alpha\cos\left(\frac{\pi\theta}{2}\right)\right) d\theta.$$

For $g(\theta) = \cos(\pi\theta/2)$, let

$$N := \int_0^1 \frac{\cos^4\left(\frac{\pi\theta}{2}\right)}{\left(1+\cos^2\left(\frac{\pi\theta}{2}\right)\right)^{3/2}} e^{-i\alpha g(\theta)} d\theta.$$

Then by the same calculation as that in (37), we have

$$N = \frac{4}{\pi}e^{i\alpha}\int_0^1 \frac{1}{\sqrt{m(t)}}\frac{(1-t^2)^4}{2-2t^2+t^4}e^{-i\alpha t^2}dt.$$

We put

$$K(t) := \frac{1}{\sqrt{m(t)}}M(t), \;\; M(t) := \frac{(1-t^2)^4}{2-2t^2+t^4}, \;\; H(t) := \frac{K(t)-K(0)}{t}. \tag{40}$$

Let $X := (1-t^2)^2$. Then by direct calculation, we have

$$H(t) = M(t)h(t) + \frac{(2X+1)(-2t+t^3)}{4(1+X)},$$

where $h(t)$ is a function defined in (39). By this and (40), we have

$$K(1) = 0, \; K(0) = \frac{1}{4}, \; X(1) = 0, \; X(0) = 1, \; M(1) = 0, \; M(0) = \frac{1}{2},$$
$$h(1) = \frac{1}{2}, \; h(0) = 0, \; H(1) = -\frac{1}{4}, \; H(0) = 0.$$

By this and Lemmas 3.2 and 3.3, we obtain

$$N = \frac{1}{2\sqrt{\alpha\pi}}e^{i(\alpha-\pi/4)} + O(\alpha^{-3/2}),$$
$$\mathrm{Im}N = \frac{1}{2\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-3/2}),$$
$$J_{313} = -\frac{\pi\alpha}{2}\left(\frac{1}{2\sqrt{\alpha\pi}}\sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-3/2})\right).$$

Then

$$J_3 = \frac{2}{p\alpha}(-J_{312} + J_{313}) = -\frac{1}{2p}\sqrt{\frac{\pi}{\alpha}} \sin\left(\alpha - \frac{\pi}{4}\right)$$
$$- \frac{2B}{p\alpha}\cos\alpha + O(\alpha^{-3/2}).$$

Thus the proof is complete. □

Proof of Theorem 1.2-(i). We know from Lemma 2.4 and the first line of the proof of Theorem 1.3-(i) that $J_4 = O(\alpha^{-3/2})$ and $J_5 = O(\alpha^{-2})$. Then by (17) and Lemmas 3.4 and 3.5, we obtain

$$\sqrt{\frac{\lambda}{2}} = \frac{2}{\sqrt{p}}\alpha^{-1}\left[pA_{2,2}\alpha^2 + \frac{p-1}{2p}\sqrt{\frac{\pi}{\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right)\right.$$
$$\left. + \frac{1}{p\alpha}(p - 2B\cos\alpha) + O(\alpha^{-3/2})\right].$$

By this and direct calculation, we obtain (10). Thus the proof of Theorem 1.2-(i) is complete. □

4. Proof of Theorem 1.5-(i)

In this section, let $D(u) = u^2 + \sin u$ and $g(u) = u + \sin u$. It follows from [11, (2.7)], we also find, as in Section 2, that for any given $\alpha > 0$, there is a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Moreover, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is a continuous function for $\alpha > 0$. Let $u \geq 0$. We put

$$\begin{aligned} G(u) &:= \int_0^u g(y)D(y)dy \\ &= \frac{1}{4}u^4 - u\cos u + \sin u + (2u\sin u - (u^2-2)\cos u - 2) \\ &\quad + \frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right) \\ &:= \frac{1}{4}u^4 + G_1(u). \end{aligned}$$

For $0 \le s \le 1$ and $\alpha \gg 1$, we have

$$\begin{aligned} G(\alpha) - G(\alpha s) &= \frac{1}{4}\alpha^4(1-s^4) + G_1(\alpha) - G_1(\alpha s) \\ &= \frac{1}{4}\alpha^4(1-s^4) - (\alpha\cos\alpha - \alpha s\cos(\alpha s)) + (\sin\alpha - \sin(\alpha s)) \\ &\quad + 2(\alpha\sin\alpha - \alpha s\sin(\alpha s)) - (\alpha^2\cos\alpha - \alpha^2 s^2\cos(\alpha s)) \\ &\quad + 2(\cos\alpha - \cos(\alpha s)) + \frac{1}{2}\alpha(1-s) - \frac{1}{4}(\sin 2\alpha - \sin 2\alpha s)) \\ &:= \frac{1}{4}\alpha^4(1-s^4) - I_1 + I_2 + I_3 - I_4 + I_5 + I_6 - I_7. \end{aligned} \tag{41}$$

It is easy to see that for $0 \le s \le 1$,

$$\left|\frac{I_4}{\alpha^4(1-s^4)}\right| \le C\alpha^{-1}, \tag{42}$$

$$\left|\frac{I_1}{\alpha^4(1-s^4)}\right|, \left|\frac{I_3}{\alpha^4(1-s^4)}\right| \le C\alpha^{-2}, \tag{43}$$

$$\left|\frac{I_2}{\alpha^4(1-s^4)}\right|, \left|\frac{I_5}{\alpha^4(1-s^4)}\right|, \left|\frac{I_6}{\alpha^4(1-s^4)}\right|, \left|\frac{I_7}{\alpha^4(1-s^4)}\right| \le C\alpha^{-3}. \tag{44}$$

By putting $u = \alpha s$, (42)–(44) and Taylor expansion, we have from [11, (2.5)] that

$$\begin{aligned} \sqrt{\frac{\lambda(\alpha)}{2}} &= \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha) - G(u)}} du \\ &= \int_0^\alpha \frac{u^2 + \sin u}{\sqrt{\frac{1}{4}(\alpha^4 - u^4) + G_1(\alpha) - G_1(u)}} du \\ &= \alpha \int_0^1 \frac{\alpha^2 s^2 + \sin\alpha s}{\sqrt{\alpha^4(1-s^4)/4 + G_1(\alpha) - G_1(\alpha s)}} ds \\ &= 2\alpha^{-1} \int_0^1 \frac{\alpha^2 s^2 + \sin\alpha s}{\sqrt{1-s^4}\sqrt{1 + \frac{4}{\alpha^4(1-s^4)}(G_1(\alpha) - G_1(\alpha s))}} ds \\ &= 2\alpha^{-1} \int_0^1 \frac{1}{\sqrt{1-s^4}}(\alpha^2 s^2 + \sin\alpha s) \\ &\qquad \times \left\{1 - \frac{2}{\alpha^4(1-s^4)}(G_1(\alpha) - G_1(\alpha s))(1 + O(\alpha^{-1}))\right\} ds. \end{aligned} \tag{45}$$

Now we show that the leading and second terms of the right hand side of

(45) are

$$L_1 := 2\alpha \int_0^1 \frac{s^2}{\sqrt{1-s^4}} ds = 2A_{2,2}\alpha, \tag{46}$$
$$L_4 := 4\alpha^{-5} \int_0^1 \frac{s^2}{(1-s^4)^{3/2}} I_4 ds.$$

Indeed, by (42)–(44), we obtain

$$2\alpha^{-1} \int_0^1 \frac{|\alpha^2 s^2 + \sin \alpha s|}{\sqrt{1-s^4}} \cdot \frac{1}{\alpha^4(1-s^4)} |I_1 + I_3 + I_5 + I_6 + I_7| ds = O(\alpha^{-1}).$$

Furthermore, by Lemma 2.2,

$$L_2 := 2\alpha^{-1} \int_0^1 \frac{\sin(\alpha s)}{\sqrt{1-s^4}} ds = (1 + o(1))\sqrt{\pi}\alpha^{-3/2} \sin\left(\alpha - \frac{\pi}{4}\right).$$

We calculate L_4 by Lemma 2.1.

LEMMA 4.1. *As* $\alpha \to \infty$,

$$L_4 = -\sqrt{\pi}\alpha^{-1/2} \sin\left(\alpha - \frac{\pi}{4}\right) + O(\alpha^{-1}). \tag{47}$$

Proof. We put $s = \sin\theta$. Then

$$\begin{aligned} L_4 &= 4\alpha^{-1} \int_0^1 \frac{s^2(\cos\alpha - s^2\cos(\alpha s))}{(1-s^2)^{3/2}(1+s^2)^{3/2}} ds \\ &= 4\alpha^{-1} \int_0^{\pi/2} \frac{1}{\cos^2\theta} Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)) d\theta, \end{aligned}$$

where $Y(\theta) = \sin^2\theta/(1+\sin^2\theta)^{3/2}$. By Integration by parts, we have

$$\begin{aligned} L_4 &= 4\alpha^{-1} \left[\tan\theta Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))\right]_0^{\pi/2} \\ &\quad - 4\alpha^{-1} \int_0^1 \tan\theta \left\{Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))\right\}' d\theta \\ &= 4\alpha^{-1}(L_{41} - L_{42}). \end{aligned} \tag{48}$$

By using l'Hôpital's rule, we have

$$\begin{aligned} &\lim_{\theta\to\pi/2} \frac{\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)}{\cos\theta} \\ &\quad = \lim_{\theta\to\pi/2} \frac{-2\sin\theta\cos\theta\cos(\alpha\sin\theta) + \alpha\sin^2\theta\cos\theta\sin(\alpha\sin\theta)}{-\sin\theta} = 0. \end{aligned}$$

By this, we see that $L_{41} = 0$. It is easy to see that

$$\int_0^1 \tan\theta \left\{ Y(\theta)'(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta)) \right\} d\theta = O(1).$$

By this, putting $\theta = \frac{\pi}{2}(1-x)$ and using Lemma 2.1, we have

$$\begin{aligned}
L_{42} &= \int_0^1 \tan\theta Y(\theta)(\cos\alpha - \sin^2\theta\cos(\alpha\sin\theta))' d\theta + O(1) \\
&= \alpha \int_0^{\pi/2} \frac{\sin^5\theta}{(1+\sin^2\theta)^{3/2}} \sin(\alpha\sin\theta) d\theta + O(1) \\
&= \frac{\pi}{2}\alpha \int_0^1 \frac{\cos^5\left(\frac{\pi}{2}x\right)}{(1+\cos^2\left(\frac{\pi}{2}x\right))^{3/2}} \sin\left(\alpha\cos\left(\frac{\pi}{2}x\right)\right) dx + O(1) \\
&= \frac{\sqrt{\pi\alpha}}{4}\sin\left(\alpha - \frac{\pi}{4}\right) + O(1).
\end{aligned}$$

By this and (48), we obtain (47). Thus the proof is complete. □

Proof of Theorem 1.5-(i). By (45), (46), Lemma 4.1, we obtain

$$\sqrt{\frac{\lambda}{2}} = 2A_{2,2}\alpha - \sqrt{\pi}\alpha^{-1/2}\sin\left(\alpha - \frac{\pi}{4}\right) + o(\alpha^{-1/2}).$$

This implies (9). Thus the proof of Theorem 1.5-(i) is complete. □

5. Proofs of Theorems 1.2-(ii) and 1.5-(ii)

In this section, let $0 < \alpha \ll 1$.

Proof of Theorem 1.2-(ii). Let $n = 1$, namely, $D(u) = pu^2 + \sin u$. By (14), Taylor expansion and direct calculation, for $0 \le s \le 1$, we have

$$G(\alpha) - G(\alpha s) = \frac{1}{3}\alpha^3(1-s^3) + \frac{p}{4}\alpha^4(1-s^4) + O(\alpha^5)(1-s^5).$$

By this, putting $\theta = \alpha s$ and (16), we obtain

$$\sqrt{\frac{\lambda}{2}} = \alpha \int_0^1 \frac{\alpha s + p\alpha^2 s^2 + O(\alpha^3)}{\sqrt{\frac{1}{3}\alpha^3(1-s^3) + \frac{p}{4}\alpha^4(1-s^4) + O(\alpha^5)(1-s^5)}} ds \tag{49}$$

$$= \sqrt{3\alpha} \int_0^1 \frac{s + p\alpha s^2 + O(\alpha^2)}{\sqrt{(1-s^3) + \frac{3p}{4}\alpha(1-s^4) + O(\alpha^2)(1-s^5)}} ds$$

$$= \sqrt{3\alpha} \int_0^1 \frac{1}{\sqrt{1-s^3}} (s + p\alpha s^2 + O(\alpha^2)) \left(1 - \frac{3p(1-s^4)}{8(1-s^3)}\alpha + O(\alpha^2)\right) ds$$

$$= \sqrt{3\alpha} \left\{ \int_0^1 \frac{s}{\sqrt{1-s^3}} ds + p\alpha \int_0^1 \frac{1}{\sqrt{1-s^3}} \left(s^2 - \frac{3s(1-s^4)}{8(1-s^3)}\right) ds + O(\alpha^2) \right\}.$$

By this, we obtain Theorem 1.2-(ii). Thus the proof is complete. □

Proof of Theorem 1.5-(ii). By (41), Taylor expansion and direct calculation, for $0 \le s \le 1$, we have

$$G(\alpha) - G(\alpha s) = \frac{2}{3}\alpha^3(1-s^3) + \frac{1}{2}\alpha^4(1-s^4) + O(\alpha^5)(1-s^5).$$

By this, putting $\theta = \alpha s$ and (49), we obtain

$$\sqrt{\frac{\lambda}{2}} = \alpha \int_0^1 \frac{\alpha^2 s^2 + \sin(\alpha s)}{\sqrt{\frac{2}{3}\alpha^3(1-s^3) + \frac{1}{2}\alpha^4(1-s^4) + O(\alpha^5)(1-s^5)}} ds$$

$$= \sqrt{\frac{3}{2\alpha}} \int_0^1 \frac{\alpha s + \alpha^2 s^2 + O(\alpha^3)s^3}{\sqrt{1-s^3}\sqrt{1 + \frac{3}{4}\alpha\frac{1-s^4}{1-s^3} + O(\alpha^2)\frac{1-s^5}{1-s^3}}} ds$$

$$= \sqrt{\frac{3\alpha}{2}} \int_0^1 \frac{1}{\sqrt{1-s^3}} (s + \alpha s^2 + O(\alpha^2)) \left(1 - \frac{3}{8}\alpha\frac{1-s^4}{1-s^3} + O(\alpha^2)\right) ds$$

$$= \sqrt{\frac{3\alpha}{2}} \left(\int_0^1 \frac{s}{\sqrt{1-s^3}} ds + \alpha \left\{ \int_0^1 \frac{s^2}{\sqrt{1-s^3}} ds - \frac{3}{8} \int_0^1 \frac{s(1-s^4)}{(1-s^3)^{3/2}} ds \right\} + O(\alpha^2) \right).$$

By this, we obtain Theorem 1.5-(ii). Thus the proof is complete. □

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Asymptotic properties of a free boundary problem for a reaction-diffusion equation with multi-stable nonlinearity

Yoshio Yamada

"Dedicated to Professor Julian López-Gómez on the occasion of his 60th birthday"

Abstract. *This paper deals with a free boundary problem for a reaction-diffusion equation with moving boundary, whose dynamics is governed by the Stefan condition. We will mainly discuss the problem for the case of multi-stable nonlinearity, which is a function with a multiple number of positive stable equilibria. The first result is concerned with the classification of solutions in accordance with large-time behaviors. As a consequence, one can observe a multiple number of spreading phenomena corresponding for each positive stable equilibrium. Here it is seen that there exists a certain group of spreading solutions whose element accompanies a propagating terrace. We will derive sharp asymptotic estimates of free boundary and profile of every spreading solution including spreading one with propagating terrace.*

Keywords: free boundary problem, reaction-diffusion equation, asymptotic profile, spreading.
MS Classification 2010: 35R35, 34B18, 35B40, 35B51, 35K57, 92D25.

1. Introduction

This paper is concerned with the following free boundary problem for reaction-diffusion equations:

$$\text{(FBP)} \qquad \begin{cases} u_t = du_{xx} + f(u), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases}$$

where d, μ and h_0 are positive constants and $x = h(t)$ is a free boundary. Nonlinearity f is a function of class $C^1[0,\infty)$ satisfying

$$f(0) = f(u^*) = 0 \quad \text{with some } u^* > 0 \text{ and } f(u) < 0 \text{ for } u > u^* \tag{1}$$

and u_0 is a nonnegative function of class $C^2[0, h_0]$ such that

$$u_0'(0) = u_0(h_0) = 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{2}$$

Since Du and Lin published a pioneer work [4] on (FBP) in 2010, a lot of authors have studied (FBP) and related free boundary problems. Among them, we should refer to the paper of Du and Lou [5], who obtained very important results on large-time behaviors of solution $(u(t,\cdot), h(t))$ of (FBP) for typical types of nonlinearity f such as monostable, bistable and combustion types. Moreover, we should also note the work of Du, Matsuzawa and Zhou [9], who derived sharp asymptotic estimates of $(u(t,\cdot), h(t))$ as $t \to \infty$ in the case $\lim_{t\to\infty} h(t) = \infty$.

The main purpose of the present paper is to study (FBP) when f is a multi-stable function, that is, f has a multiple number of positive stable equilibria. For the sake of simplicity, we assume that $f \in C^1[0,\infty)$ satisfies the following conditions:

(PB) $f(u) = 0$ has solutions $u = 0, u_1^*, u_2^*, u_3^*$ $(0 < u_1^* < u_2^* < u_3^*)$,
$f'(0) > 0,\ f'(u_1^*) < 0,\ f'(u_2^*) \geq 0,\ f'(u_3^*) < 0,\ \displaystyle\int_{u_1^*}^{u_3^*} f(u)du > 0,$
and $f(u) \neq 0$ for $u \neq 0, u_1^*, u_2^*, u_3^*$.

When f satisfies (PB), we say that f is a function of *positive bistable type*. For such nonlinearity, we will show that solutions of (FBP) exhibit interesting large-time behaviors which are different from those discussed in previous works (see, e.g., Du and Lou [5] for monostable type and bistable type). Our first aim is to investigate what kind of asymptotic behaviors can be found for (FBP) with positive bistable nonlinearity. We will classify all solutions of (FBP) into the following four types:

(i) $\lim_{t\to\infty} h(t) < \infty$ and $\lim_{t\to\infty} u(t,x) = 0$ for $x \geq 0$ (vanishing),

(ii) $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = u_1^*$ for $x \geq 0$ (small spreading),

(iii) $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = u_3^*$ for $x \geq 0$ (big spreading),

(iv) $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = v_{dec}(x)$ for $x \geq 0$, where v_{dec} is a uniquely determined decreasing function such that $\lim_{x\to\infty} v_{dec}(x) = u_1^*$ (transition).

For numerical simulations of these typical types of solutions, see Figure 1. Here, if we consider (FBP) for $u_0 = \sigma u_0^*$ with parameter $\sigma \geq 0$ and any fixed nonnegative function u_0^* satisfying (2), we can prove the existence of two threshold numbers σ_1^* and σ_2^* ($\sigma_1^* < \sigma_2^*$) with the following properties:

The solution of (FBP) satisfies vanishing (i) for all $\sigma \in [0, \sigma_1^*]$, small spreading (ii) for all $\sigma \in (\sigma_1^*, \sigma_2^*)$, big spreading (iii) for all $\sigma \in (\sigma_2^*, \infty)$ and transition (iv) for exactly $\sigma = \sigma_2^*$. It should be noted that, for any stable equilibrium of f, one can observe the corresponding spreading phenomenon for (FBP).

Our second aim is to study asymptotic speed of $h(t)$ and asymptotic profile of $u(t,x)$ as $t \to \infty$ when $(u(t,x), h(t))$ exhibits spreading property (ii) or (iii) (or (iv)). It is shown by Du and Lou [5] that the study of asymptotic estimates of $u(t,x)$ and $h(t)$ is closely related with the following problem

$$\text{(SWP)} \qquad \begin{cases} dq'' - cq' + f(q) = 0, \quad q(z) > 0 \quad \text{for} \;\; z \in (0, \infty), \\ q(0) = 0, \;\; \mu q'(0) = c, \quad \lim\limits_{z \to \infty} q(z) = u^*, \end{cases}$$

with $u^* = u_1^*$ or $u^* = u_3^*$. When $(q(z), c) = (q^*(z), c^*)$ satisfies (SWP), $q^*(z)$ is called a *semi-wave with speed* c^*. Let (u, h) be a solution of (FBP) with $\lim_{t\to\infty} u(t,x) = u^*$ ($u^* = u_1^*$ or u_3^*) and let (SWP) possess a solution pair (q^*, c^*). Then it will be proved that (q^*, c^*) gives sharp estimates in the following sense:

$$\begin{aligned} &\lim_{t\to\infty} \{h(t) - c^* t\} = H^* \quad \text{with some} \;\; H^* \in \boldsymbol{R}, \\ &\lim_{t\to\infty} \sup_{0 \leq x \leq h(t)} |u(t,x) - q^*(h(t) - x)| = 0. \end{aligned}$$

The same estimates have been obtained by Du, Matsuzawa and Zhou [9] in the case that f is monostable or bistable type of nonlinearity.

The analysis of (SWP) can be carried out by using the phase plane analysis (see, e.g. [5]). It can be shown that (SWP) with $u^* = u_1^*$ always has a unique solution pair, whereas (SWP) with $u^* = u_3^*$ does not have a solution under a certain circumstance. Numerical simulations in this situation suggest that a spreading solution accompanies a propagating terrace (see Figure 2). In order to estimate such a terrace, we will use a travelling wave for the following problem:

$$\text{(TWP)} \qquad \begin{cases} dQ'' - cQ' + f(Q) = 0, \quad Q(z) > 0 \quad \text{for} \;\; z \in (-\infty, \infty), \\ \lim\limits_{z \to -\infty} Q(z) = u_1^*, \quad Q(0) = (u_1^* + u_3^*)/2, \quad \lim\limits_{z \to \infty} Q(z) = u_3^*. \end{cases}$$

We will prove that a semi-wave of (SWP) with $u^* = u_1^*$ together with a travelling wave of (TWP) gives a good approximation of any spreading solution of (FBP) with $\lim_{t\to\infty} u(t,x) = u_3^*$ in the case that there exists no solution pair of (SWP) with $u^* = u_3^*$.

The contents of the present paper are as follows. In Section 2 we will prepare some basic results of solutions for (FBP) with general nonlinearity f. Section 3 is devoted to the analysis of (FBP) for positive bistable nonlinearity. We will give a classification theorem and sharp estimates of spreading solutions when the corresponding semi-wave problem has a unique solution pair. In Section 4 we will estimate any spreading solution with propagating terrace by using solutions of (SWP) and (TWP). Finally, in Section 5, we will state two related topics. The first one is concerned with a free boundary problem in a radial symmetric environment of $\boldsymbol{R}^N$ and the second is the study of (FBP) with Neumann condition at $x = 0$ replaced by zero Dirichlet condition.

2. Basic results for (FBP)

In this section, we will collect some basic results on (FBP) with general nonlinearity f. The first result is the existence and uniqueness of a global solution to (FBP) (see Du-Lin [4, Theorems 2.1, 2.3 and Lemma 2.2] and Du-Lou [5, Theorem 2.4 and Lemma 2.8]).

THEOREM 2.1. *Let f and u_0 satisfy* (1) *and* (2)*, respectively. Then* (FBP) *admits a unique solution (u,h) in the following class*

$$(u,h) \in \left\{C^{(1+\alpha)/2,1+\alpha}(\overline{\Omega}) \cap C^{1+\alpha/2,2+\alpha}(\Omega)\right\} \times C^{1+\alpha/2}[0,\infty)$$

for any $\alpha \in (0,1)$ with $\Omega = \{(t,x) \in \boldsymbol{R}^2 |\ t > 0,\ 0 < x < h(t)\}$. Moreover, (u,h) possesses the following properties:

(i) *It holds that*

$$\begin{aligned} 0 < u(t,x) \le C_1 \quad &\text{for}\ \ t > 0 \ \ \text{and}\ \ 0 < x < h(t), \\ 0 < h'(t) \le C_2 \quad &\text{for}\ \ t > 0, \end{aligned}$$

where C_1 and C_2 are positive constants depending only on $\|u_0\|_{C[0,h_0]}$ and $\|u_0\|_{C^1[0,h_0]}$, respectively.

(ii) *$u_x(t,x) < 0$ for all $t \in (0,\infty)$ and $x \in [h_0,h(t)]$.*

The second result is the comparison theorem which is a very important tool in the analysis of dynamic behavior of solutions of (FBP) (see [4, Lemma 3.5]).

THEOREM 2.2. *For $T > 0$, let $(u^*,h^*) \in \{C^{0,1}(\overline{\Omega_T^*}) \cap C^{1,2}(\Omega_T^*)\} \times C^1[0,T]$ with $\Omega_T^* = \{(t,x) \in \boldsymbol{R}^2 |\ 0 < t < T,\ 0 < x < h^*(t)\}$ satisfy*

$$\begin{cases} u_t^* \ge d u_{xx}^* + f(u^*) & \text{for}\ \ (t,x) \in \Omega_T^*, \\ u_x^*(0,t) \le 0, \quad u^*(t,h^*(t)) = 0 & \text{for}\ \ t \in [0,T], \\ (h^*)'(t) \ge -\mu u_x^*(t,h^*(t)) & \text{for}\ \ t \in [0,T]. \end{cases} \tag{3}$$

Let $(u_*, h_*) \in \{C^{0,1}(\overline{\Omega_{*,T}}) \cap C^{1,2}(\Omega_{*,T})\} \times C^1[0,T]$ *satisfy* (3) *with inequality signs replaced by inverse inequality signs, where* $\Omega_{*,T} = \{(t,x) \in \boldsymbol{R}^2 |\ 0 < t < T,\ 0 < x < h_*(t)\}$. *If*

$$h^*(0) \geq h_*(0) \quad \text{and} \quad u^*(0,x) \geq u_*(0,x) \quad \text{for } \ 0 \leq x \leq h_*(0),$$

then

$$h^*(t) \geq h_*(t) \quad \text{for } \ t \in [0,T] \quad \text{and} \quad u^*(t,x) \geq u_*(t,x) \quad \text{for } \ (t,x) \in \Omega_{*,T}.$$

Remark 2.3: If (u^*, h^*) satisfies (3), $h^*(0) \geq h_0$ and

$$u^*(0,x) \geq u_0(x) \quad \text{for } \ 0 \leq x \leq h_0,$$

then (u^*, h^*) is called a *super-solution* of (FBP). Similarly, a *sub-solution* of (FBP) is defined with obvious modification.

We now introduce the notion of vanishing and spreading of solutions of (FBP).

Definition 2.4. *Let* (u,h) *be a solution of* (FBP). *Then* (u,h) *is called a vanishing solution if* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$ *and it is called a spreading solution if*

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} > 0.$$

As an application of the comparison theorem (Theorem 2.2), we will give one of sufficient conditions for the spreading.

Theorem 2.5. *For positive number* ℓ, *let* φ *be a solution of*

$$\begin{cases} d\varphi'' + f(\varphi) = 0, \quad \varphi > 0 \quad \text{in} \ \ (0,\ell), \\ \varphi'(0) = \varphi(\ell) = 0. \end{cases} \tag{4}$$

Suppose that (u_0, h_0) *satisfies* $h_0 \geq \ell$ *and* $u_0(x) \geq \varphi(x)$ *for* $x \in [0,\ell]$. *Then the solution* (u,h) *of* (FBP) *satisfies*

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} u(t,x) \geq v^*(x) \quad \text{for all } \ x \geq 0,$$

where v^* *is a minimal solution of*

$$\text{(SP)} \qquad \begin{cases} dv'' + f(v) = 0, \quad v > 0 \quad \text{in} \ \ (0,\infty), \\ v'(0) = 0 \end{cases}$$

satisfying $v^*(x) \geq \varphi(x)$ *for all* $x \in (0,\ell)$.

This theorem can be proved by repeating the arguments used in the proofs of Theorem 2.11 and Corollary 2.12 in [17].

The following result gives a necessary and sufficient condition for the vanishing of solutions.

THEOREM 2.6. *Assume $f'(0) \neq 0$. Then a solution (u, h) of* (FBP) *is vanishing if and only if $\lim\limits_{t\to\infty} h(t) < \infty$. In particular, if $f'(0) > 0$, then a vanishing solution satisfies*

$$\lim_{t\to\infty} h(t) \le \ell^* := \frac{\pi}{2}\sqrt{\frac{d}{f'(0)}}.$$

Proof. Let (u, h) be a solution of (FBP) such that $\lim_{t\to\infty} h(t) < \infty$. Then it is possible to prove the vanishing of the solutions, i.e, $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$ essentially in the same way as the proof of Theorem 2.10 of [17].

As to the necessity part, we will first discuss the case $f'(0) > 0$. When (u, h) is a vanishing solution, assume $\lim_{t\to\infty} h(t) > \ell^* = (\pi/2)\sqrt{d/f'(0)}$ to derive a contradiction. Then there exists a large number $T > 0$ such that $h(T) > \ell^*$. Here it should be noted that, for every $\ell > \ell^*$ there exists a unique solution $\varphi(x; \ell)$ of (4) and that $\lim\limits_{\ell\to\ell^*} \|\varphi(\cdot; \ell)\|_{C[0,\ell]} = 0$. Therefore, we can find a suitable $\ell \in (\ell^*, h(T))$ such that $u(T, x) \ge \varphi(x; \ell)$ for $x \in [0, \ell]$. Therefore, it follows from Theorem 2.5 that

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \liminf_{tto\infty} u(t, x) \ge v^*(x) > 0 \quad \text{for} \quad x > 0,$$

where v^* is a suitable positive solution of (SP). This is a contradiction to the vanishing of (u, h); so that h must satisfy $\lim_{t\to\infty} h(t) \le \ell^*$.

We next consider the case $f'(0) < 0$. Note that there exist positive constants η and δ such that

$$f(u) \le -\delta u \quad \text{for all} \quad u \in [0, \eta].$$

We define $(u^*(t, x), h^*(t))$ by

$$h^*(t) = H\left(1 - \frac{1}{2}e^{-\delta t}\right) \quad \text{and} \quad u^*(t, x) = \rho e^{-\delta t}\cos\left(\frac{\pi x}{2h^*(t)}\right),$$

where H and ρ are positive constants to be determined later. We will show that (u^*, h^*) satisfies (3). If ρ satisfies $\rho \le \eta$, then

$$\begin{aligned} u^*_t - du^*_{xx} - f(u^*) &= -\delta u^* + \rho e^{-\delta t} \cdot \frac{\pi x (h^*)'(t)}{2h^*(t)^2} \cdot \sin\left(\frac{\pi x}{2h^*(t)}\right) \\ &\qquad + \frac{\pi^2 d}{4h^*(t)^2} u^* - f(u^*) \\ &\ge -\delta u^* + \frac{\pi^2 d}{4h^*(t)^2} u^* + \delta u^* = \frac{\pi^2}{4h^*(t)^2} u^* > 0. \end{aligned}$$

Moreover, if H satisfies $H^2\delta \geq 2\mu\rho\pi$, then

$$\begin{aligned}(h^*)'(t) + \mu u^*_x(t, h^*(t)) &= \frac{H\delta}{2}e^{-\delta t} - \frac{\pi\mu\rho}{2h^*(t)}e^{-\delta t}\\ &\geq \frac{H^2\delta - 2\pi\mu\rho}{2H}e^{-\delta t} > 0.\end{aligned}$$

It is easy to see $u^*_x(t, 0) = 0$ and $u^*(t, h^*(t)) = 0$. Since (u, h) is a vanishing solution, we can take a sufficiently large $T > 0$ such that $\|u(T)\|_{C[0,h(T)]} \leq \eta$. Furthermore, choose sufficiently large H satisfying $h(T) \leq h^*(0) = H/2$ and $u(T, x) \leq \rho\cos(x/H)$ for $0 \leq x \leq h(T)$. Then Theorem 2.2 allows us to conclude

$$h(t + T) \leq h^*(t) \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u(t + T, x) \leq u^*(t, x)$$

for $t \geq 0$ and $0 \leq x \leq h(t + T)$. The above estimates implies $\lim_{t\to\infty} h(t) \leq \lim_{t\to\infty} h^*(t) = H$: so that the free boundary remains bounded. □

Theorem 2.7. *Assume $f'(0) \neq 0$ and let (u, h) be a solution of* (FBP) *satisfying $\lim_{t\to\infty} h(t) = \infty$. Then it holds taht for any $R > 0$*

$$\lim_{t\to\infty} u(t, x) = v^*(x) \quad \textit{uniformly in} \quad x \in [0, R], \tag{5}$$

where v^* is a non-increasing solution of (SP).

Proof. We consider an even extension of $u(t, x)$ for $x \in [-h(t), h(t)]$ and apply the general convergence theorem due to Du and Lou [5, Theorem 1.1] (see also [6]). It can be seen from $\lim_{t\to\infty} h(t) = \infty$ that $u(t, x)$ satisfies (5) for a nonnegative function v^*, which is a solution of

$$dv^*_{xx} + f(v^*) = 0 \;\text{ in }\; I := [0, \infty) \;\text{ and }\; v^*_x(0) = 0.$$

Suppose $v^*(x_0) = 0$ for some $x_o \in I$. Then $v^*_x(x_0) = 0$; so that the uniqueness of solutions for the initial value problem for second-order ordinary differential equations leads to $v^* \equiv 0$ in I. Then it follows that (u, h) must be a vanishing solution. Therefore, Theorem 2.6 implies $\lim_{t\to\infty} h(t) < \infty$, which is a contradiction to the assumption. Thus v must satisfy $v^*(x) > 0$ for all $x \in I$; so that v^* is a solution of (SP). The non-increasing property is an easy consequence of (ii) of Theorem 2.1. □

Let $\mathcal{S}$ be the set of non-increasing solutions of (SP). In order to determine the complete structure of $\mathcal{S}$, we will take two types of typical examples of f:

(M) *Monostable type*: $f \in C^1[0, \infty)$ and there exists a positive number u^* such that $f(0) = f(u^*) = 0$ with $f'(0) > 0, f'(u^*) < 0$, $f(u) > 0$ for $u \in (0, u^*)$ and $f(u) < 0$ for $u > u^*$.

(B) *Bistable type*: $f \in C^1[0,\infty)$ and there exist two positive numbers u^* and θ with $0 < \theta < u^*$ such that $f(0) = f(\theta) = f(u^*) = 0$, $f'(0) < 0, f'(u^*) < 0$, $f(u) > 0$ for $u \in (\theta, u^*)$, $f(u) < 0$ for $u \in (0,\theta) \cup (u^*, \infty)$ and $\displaystyle\int_0^{u^*} f(u)du > 0$.

When f is a monostable type of function, the phase plane analysis of (SP) enables us to prove $\mathcal{S} = \{u^*\}$. Then we can obtain the following result as in [4].

Theorem 2.8. *Let f satisfy* (M) *and let (u,h) be the solution of* (FBP). *Then (u,h) satisfies one of the following properties:*

(i) *Vanishing;* $\displaystyle\lim_{t\to\infty} h(t) \le (\pi/2)\sqrt{d/f'(0)}$ *and* $\displaystyle\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Spreading;* $\displaystyle\lim_{t\to\infty} h(t) = \infty$ *and* $\displaystyle\lim_{t\to\infty} u(t,x) = u^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

When f is a bistable type of function, one can see from the phase plane analysis that $\mathcal{S} = \{u^*, \theta, \hat{v}\}$. Here $\hat{v}$ is a monotone decreasing solution of (SP) satisfying $\hat{v}(0) = \hat{u}$ and $\lim_{x\to\infty} \hat{v}(x) = 0$, where $\hat{u} \in (\theta, u^*)$ is a unique number satisfying $\int_0^{\hat{u}} f(u)du = 0$. Furthermore, we can exclude the possibility of $\lim_{t\to\infty} u(t,x) = \theta$ by usig the zero number arguments (for details, see the proof of Theorem 3.1) . More precisely, it is possible to prove the following (see [5]):

Theorem 2.9. *Let f satisfy* (B) *and let (u,h) be the solution of* (FBP). *Then (u,h) satisfies one of the following properties:*

(i) *Vanishing;* $\displaystyle\lim_{t\to\infty} h(t) < \infty$ *and* $\displaystyle\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Spreading;* $\displaystyle\lim_{t\to\infty} h(t) = \infty$ *and* $\displaystyle\lim_{t\to\infty} u(t,x) = u^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iii) *Transition;* $\displaystyle\lim_{t\to\infty} h(t) = \infty$ *and* $\displaystyle\lim_{t\to\infty} u(t,x) = \hat{v}(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

3. Large-time behaviors of solutions for positive bistable nonlinearity

3.1. Positive bistable nonlinearity and classification of large-time behaviors

We will take multi-stable nonlinearity f in (FBP), that is, f has a multiple number of positive stable equilibrium points. A typical example is given by

$$f(u) = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}, \quad \text{with} \quad q, r > 0, \tag{6}$$

which is a combination of a logistic term $ru(1-u/q)$ and a predation term called Holling type III, $-u^2/(1+u^2)$. For ecological background of such f and its analysis, see the paper of Ludwig, Aronson and Weinberger [22]. It is known that, if q and r satisfy suitable conditions, then above f has two positive stable equilibria and satisfies (PB) given in Section 1.

In what follows, we always assume that f satisfies (PB). Note that $f(u)$ is a monostable type for $0 \le u \le u_1^*$ and is a bistable type for $u_1^* \le u \le u_3^*$. Our first result is the following classification result of solutions of (FBP) based on their large-time behaviors (see [19, Theorem 3.1]).

THEOREM 3.1. *Let (u,h) be the solution of* (FBP). *Then it satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \le (\pi/2)\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u_1^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = u_3^*$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iv) *Transition;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_{dec}(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$, *where* v_{dec} *is a solution of* (SP) *satisfying*

$$(v_{dec})'(x) < 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad \lim_{x\to\infty} v_{dec}(x) = u_1^*.$$

Proof. Let $\mathcal{S}$ be the set of non-increasing solutions of (SP). Using the phase plane analysis one can show

$$\mathcal{S} = \{u_1^*, u_2^*, u_3^*, v_{dec}\}.$$

By virtue of Theorems 2.6 and 2.7, it is sufficient to exclude the possibility $\lim_{t\to\infty} u(t,x) = u_2^*$ in order to complete the proof.

Assuming

$$\lim_{t\to\infty} u(t,\cdot) = u_2^* \quad \text{uniformly in} \quad [0,R] \tag{7}$$

for any $R > 0$, we will derive a contradiction. Let v be a periodic solution of (SP) satisfying $v(0) = \max_{x\geq 0} v(x) > u_2^*$. The phase plane analysis yields

$$u_1^* < \min_{x\geq 0} v(x) < u_2^* < \max_{x\geq 0} v(x) < u_3^*.$$

Set $w(t,x) = u(t,x) - v(x)$. Then

$$w_t = dw_{xx} + c(t,x)w,$$

where $c(t,x) = \displaystyle\int_0^1 f'(\theta u(t,x) + (1-\theta)v(x))d\theta$ is a bounded and continuous function. For a continuous function $\varphi(x)$ defined in a closed interval I, we denote by $\mathcal{Z}_I(\varphi)$ the number of zero-points of φ in I. Setting $I(t) = [0, h(t)]$ we consider $\mathcal{Z}_{I(t)}(w(t))$. Note that $w(t,h(t)) = -v(h(t)) < 0$ and $w(t,0) = u(t,0) - v(0) < 0$ for $t \geq T$ with sufficiently large $T > 0$. Then it follows from the zero number result of Angenent [1, Theorems C and D] that $t \to \mathcal{Z}_{I(t)}(w(t))$ is finite and non-increasing for $t \geq T$. However,

$$\mathcal{Z}_{I(t_2)}(w(t_2)) > \mathcal{Z}_{I(t_1)}(w(t_1)) \quad \text{for} \quad t_2 > t_1 \geq T$$

if $t_2 - t_1$ is large because $\lim_{t\to\infty} h(t) = \infty$, u satisfies (7) and $v(x)$ is periodic with respect to x. This result contradicts to the non-increasing property of $\mathcal{Z}_{I(t)}(w(t))$; so that (7) never happens. □

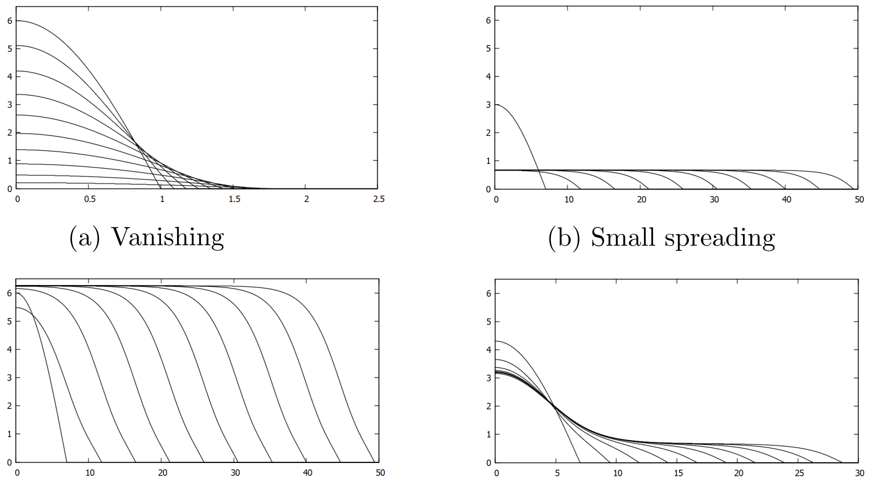
Remark 3.2: We consider (FBP) with initial condition replaced by

$$h(0) = h_0, \quad u(0,x) = \sigma u_0^*(x), \quad 0 \leq x \leq h_0,$$

where $\sigma \geq 0$ is a parameter and u_0^* is a nonnegative function satisfying (2). Denote by $(u_\sigma(t,x), h_\sigma(t))$ the solution of the above problem. By virtue of Theorems 3.7 and 3.8 in [19] there exist two threshold numbers σ_1^* and σ_2^* ($\sigma_1^* < \sigma_2^*$) with the following properties:

- For $\sigma \in [0, \sigma_1^*]$, (u_σ, h_σ) is a vanishing solution.
- For $\sigma \in (\sigma_1^*, \sigma_2^*)$, (u_σ, h_σ) is a small spreading solution.
- For $\sigma = \sigma_2^*$, (u_σ, h_σ) is a transition solution.
- For $\sigma > \sigma_2^*$, (u_σ, h_σ) is a big spreading solution.

As a result, the transition is a special solution which occurs as a borderline behavior between the small spreading and the big spreading.



(a) Vanishing (b) Small spreading

(c) Big spreading (d) Transition

Figure 1: Four types of large-time behaviors of $u(t, x)$ for (FBP) are shown as (a) vanishing, (b) small spreading, (c) big spreading and (d) transition. The right-end point of each curve represents $h(t)$ and moves forward as t goes on.

Numerical simulations for (FBP) are shown in Figure 1 for $d = 1, \mu = 0.1$ and f given by (6) with $q = 40/3$ and $r = 0.3$. As to small spreading, big spreading and transition of solutions, these simulations suggest that $u(t, x)$ proceeds like a " travelling wave " near the spreading front $x = h(t)$ for large t. We will investigate asymptotic behaviors of $(u(t, x), h(t))$ as $t \to \infty$ in the subsequent subsections.

3.2. Large-time behaviors of solutions and semi-wave problem

We will study large-time behaviors of solutions of (FBP) which possess properties (ii) and (iii) of Theorem 3.1. In the case $\lim_{t\to\infty} h(t) = \infty$, we infer from the preceding numerical simulations that such a spreading solution converges to a pair $(u(t, x), h(t))$ of the following form as $t \to \infty$:

$$h(t) = ct + H \quad (H: \ \text{constant}), \quad u(t, x) = q(h(t) - x), \quad 0 \le x \le h(t), \quad (8)$$

where c is a positive constant and $q = q(z)$ is a positive function defined for $z \ge 0$. Substitution of (8) into the first equation of (FBP) yields

$$dq'' - cq' + f(q) = 0, \quad q(z) > 0 \ \text{ for } \ z > 0. \quad (9)$$

At $x = h(t)$ in (FBP), we get

$$q(0) = 0 \quad \text{and} \quad \mu q'(0) = c. \tag{10}$$

Moreover, since $\lim_{t\to\infty} u(t,x) = u_i^*$ $(i = 1, 3)$ uniformly in $x \in [0, R]$ for any $R > 0$, q must satisfy

$$\lim_{z\to\infty} q(z) = u_i^* \quad (i = 1, 3). \tag{11}$$

Summarizing (9), (10) and (11) we arrive at (SWP) given in Section 1. This problem was first introduced by Du and Lou [5] and it is called a *semi-wave problem*. They have shown the existence of a unique solution pair $(q, c) = (q^*, c^*)$ when f is a monostable type or a bistable type.

Let f satisfy (PB) and consider a small spreading solution or a big spreading solution of (FBP). When we discuss a small (resp. big) spreading solution, the corresponding semi-wave problem (SWP) with $u^* = u_1^*$ (resp. $u^* = u_3^*$) is denoted by (SWP-1) (resp. (SWP-3)). The solvability of these problems has been established by Kawai and Yamada [19, Theorem 4.1].

Proposition 3.3. (i) *For every $\mu > 0$,* (SWP-1) *has a unique solution pair $(q, c) = (q_S, c_S)$.*

(ii) *Case A: For every $\mu > 0$,* (SWP-3) *has a unique solution pair $(q, c) = (q_B, c_B)$.*
Case B: There exists a positive number μ^* such that (SWP-3) *has a unique solution pair $(q, c) = (q_B, c_B)$ for $\mu \in (0, \mu^*)$, whereas* (SWP-3) *has no solution for $\mu \in [\mu^*, \infty)$.*

(iii) *$q_S'(z) > 0$, $q_B'(z) > 0$ for $z \geq 0$ and $c_B > c_S$ when (q_B, c_B) exists.*

The semi-wave of (SWP) is useful for the study of asymptotic behaviors of spreading solutions as $t \to \infty$. The following result gives a rough estimate of the spreading speed of $h(t)$ (see [19, Theorem 4.2]).

Proposition 3.4. *Let c_S, c_B and μ^* be positive constants given in Proposition 3.3.*

(i) *If (u, h) is a small spreading solution of* (FBP)*, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

(ii) *Let (u, h) be a big spreading solution of* (FBP)*. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \begin{cases} c_B & \text{if (SWP-3) has a solution pair } (q_B, c_B), \\ c_S & \text{if (SWP-3) has no solution pair.} \end{cases}$$

(iii) *If (u, h) is a transition solution of* (FBP)*, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S.$$

3.3. Sharp asymptotic estimates of spreading solutions

We will show that the unique solution pair of (SWP) with $u^* = u_1^*$ or u_3^* gives a good approximation of any spreading solution of (FBP) for large t whenever the corresponding semi-wave exists.

We begin with the analysis of a small spreading solution $(u(t,x), h(t))$ of (FBP). The following result gives a rough estimate of (u,h) with use of (q_S, u_S).

LEMMA 3.5. *Let (u,h) be a small spreading solution of* (FBP). *Then there exist positive constants δ, M_1, T_1 and H_1 such that*

$$\begin{aligned} &h(t) \le c_S t + H_1, \\ &u(t,x) \le (1 + M_1 e^{-\delta t}) q_S(c_S t + H_1 - x), \end{aligned}$$

for all $t \ge T_1$ and $0 \le x \le h(t)$.

Proof. Define (u^*, h^*) by

$$\begin{cases} h^*(t) = c_S(t-T) + \rho(e^{-\delta T} - e^{-\delta t}) + H, & t \ge T, \\ u^*(t,x) = (1 + M e^{-\delta t}) q_S(h^*(t) - x), & t \ge T, \quad 0 \le x \le h^*(t), \end{cases}$$

where δ is a positive constant satisfying

$$f'(u) \le -\delta \quad \text{for} \quad u \in [u_1^* - \eta, u_1^* + \eta]$$

with some $\eta > 0$ and ρ, M, T and H are constants to be determined later. We will show that (u^*, h^*) is a super-solution of (FBP) for $t \ge T$; that is,

$$u_t^* \ge d u_{xx}^* + f(u^*), \qquad t \ge T, \quad 0 \le x \le h^*(t), \tag{12}$$

$$u_x^*(t,0) \le 0, \qquad u^*(t, h^*(t)) = 0, \qquad t \ge T, \tag{13}$$

$$(h^*)'(t) \ge -\mu u_x^*(t, h^*(t)), \qquad t \ge T, \tag{14}$$

$$h^*(T) \ge h(T), \qquad u^*(T,x) \ge u(T,x), \qquad 0 \le x \le h(T). \tag{15}$$

Clearly, (13) holds and, moreover,(14) is satisfied if $\rho\delta \ge M c_S$. If we follow the arguments in the work of Du, Matsuzawa and Zhou [9, Lemma 3.2], we can prove (12) provided that ρ is sufficiently large. Finally, taking sufficiently large T such that $u(T,x) \le u_1^* + \varepsilon$ for $0 \le x \le h(T)$ with sufficiently small $\varepsilon > 0$ and choosing sufficiently large M and H such that

$$h^*(T) = H \ge h(T) \quad \text{and} \quad u^*(T,x) = (1 + M e^{-\delta T}) q_S(H - x) \ge u(T,x),$$

for $0 \le x \le h(T)$, one can verify (15).

The application of Theorem 2.2 leads to

$$h(t) \le h^*(t) \quad \text{and} \quad u(t,x) \le u^*(t,x) \tag{16}$$

for $t \ge T$ and $0 \le x \le h(t)$. Since $q_S(z)$ is strictly increasing in $z \ge 0$, it is easy to derive the assertion from (16). $\square$

Similarly, one can also show the following rough estimate from below.

LEMMA 3.6. *Let (u, h) be a small spreading solution of* (FBP). *Then there exist positive constants δ, M_2, T_2 and $H_2 \in \boldsymbol{R}$ such that*

$$h(t) \geq c_S t + H_2,$$
$$u(t,x) \geq (1 - M_1 e^{-\delta t}) q_S(c_S t + H_2 - x),$$

for all $t \geq T_2$ and $0 \leq x \leq c_S t + H_2$.

For the proof of this lemma, see, e.g. [9, Lemma 3.3].

We can get sharper estimates than Lemmas 3.5 and 3.6 if we repeat the arguments in [9] (see also [13, Proposition 1.3]).

THEOREM 3.7. *Let (u, h) be a small spreading solution of* (FBP) *and let (q_S, c_S) be the solution pair of* (SWP-1). *Then there exists $H_S \in \boldsymbol{R}$ such that*

$$\lim_{t\to\infty} (h(t) - c_S t) = H_S \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_S$$

and

$$\lim_{t\to\infty} \sup_{0\leq x\leq h(t)} |u(t,x) - q_S(h(t) - x)| = 0.$$

Theorem 3.7 shows that (q_S, c_S) plays a very important role in the estimate of any small spreading solution (u, h) of (FBP): c_S gives an asymptotic constant speed of the free boundary $x = h(t)$ and a simple function $q(z)$ is enough to approximate $u(t,x)$ in the form of $q(h(t) - x)$ over the whole interval $[0, h(t)]$ for large t. An analogous result is also valid for any big spreading solution when (SWP-3) has a unique solution pair (q_B, c_B).

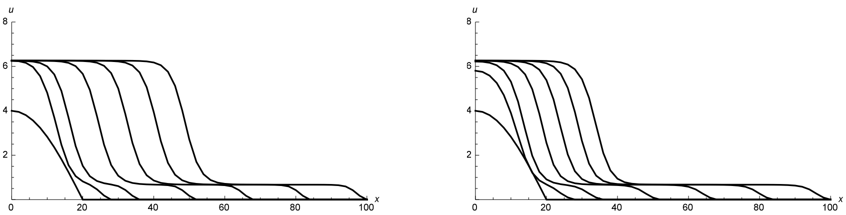
THEOREM 3.8. *Let (u, h) be a big spreading solution of* (FBP) *and assume that* (SWP-3) *admits a unique solution pair (q_B, c_B). Then there exists $H_B \in \boldsymbol{R}$ such that*

$$\lim_{t\to\infty} (h(t) - c_B t) = H_B \quad \textit{and} \quad \lim_{t\to\infty} h'(t) = c_B$$

and

$$\lim_{t\to\infty} \sup_{0\leq x\leq h(t)} |u(t,x) - q_B(h(t) - x)| = 0.$$

This theorem gives a sharp estimate of any big spreading solution $(u(t,x), h(t))$ over the whole interval $[0, h(t)]$ when the corresponding semi-wave exists. We will discuss its asymptotic estimate for the remaining case in the next section.



$\mu = 3$ $\mu = 10$

Figure 2: Numerical simulations of (FBP) for $d = 1$ and $f(u) = u(0.5 - 0.055u) - u^2/(1 + u^2)$ with $u_1^* \approx 0.672$ and $u_3^* = 6.258$

4. Sharp asymptotic estimates of solutions with propagating terrace

We will derive asymptotic estimates of a big spreading solution (u, h) of (FBP) under the following condition

(A) Semi-wave problem (SWP-3) has no solution pair.

By Proposition 3.4 such a big spreading solution satisfies

$$\lim_{t\to\infty} \frac{h(t)}{t} = c_S,$$

where c_S is the speed of semi-wave q_S for (SWP-1). Thus (q_S, c_S) will be helpful to approximate $(u(t, x), h(t))$ around the spreading front $x = h(t)$. On the other hand,

$$\lim_{t\to\infty} u(t, x) = u_3^* \quad \text{uniformly in} \quad x \in [0, R]$$

for any $R > 0$. Taking account of these facts we guess that there must be a function like a "travelling wave", which connects u_1^* with u_3^*. Numerical simulations of such big spreading solutions are given in Figure 2 when f satisfies (A). These simulations suggest the following dynamics:

A big spreading solution proceeds like a small spreading solution around the spreading front $x = h(t)$ and a propagating function (connecting u_1^* and u_3^*) subsequently appears with slower speed in the intermediate range.

As a candidate of such a connecting function, we will take a travelling wave for (TWP) (see Section 1). It is known that (TWP) has a unique solution $(Q^*(z), c^*)$. Moreover, it follows from the result of [19, Remark 4.1] that condition (A) assures

$$c^* < c_S. \tag{17}$$

Hereafter we will study a big spreading solution (u, h) by assuming (17). We will briefly explain the arguments developed by Kaneko, Matsuzawa and

Yamada [13] to obtain sharp asymptotic estimates for $(u(t,x), h(t))$ with use of both (q_S, c_S) and (Q^*, c^*).

As the first step, define

$$u^*(t,x) = Q^*(c^*t + H - \rho e^{-\delta t} - x) + Me^{-\delta t}, \tag{18}$$

where $\delta > 0$ is a constant satisfying

$$f'(u) \leq -\delta \quad \text{for} \quad u \in [u_1^* - \eta, u_1^* + \eta] \cup [u_3^* - \eta, u_3^* + \eta]$$

with some $\eta > 0$. Then one can choose sufficiently small $M > 0$ and large positive ρ, H and T such that

$$\begin{array}{ll} u_t^* \geq du_{xx}^* + f(u^*) & \text{for } \ t \geq T, \quad 0 \leq x \leq h(t), \\ u_x^*(t,0) < 0, \qquad u^*(t,h(t)) > 0 & \text{for } \ t \geq T, \\ u^*(T,x) \geq u(T,x) & \text{for } \ 0 \leq x \leq h(T). \end{array}$$

The comparison principle for parabolic equations yields

$$u(t,x) \leq u^*(t,x)$$

for $t \geq T$ and $0 \leq x \leq h(t)$. Since $Q^*(z)$ is strictly increasing in z, the above estimate together with (18) allows us to show the following result (see [13, Lemma 3.5]).

Lemma 4.1. *Let (u,h) a big spreading solution of* (FBP). *Then there exist positive constants δ, M_1, H_1 and T_1 such that*

$$u(t,x) \leq Q^*(c^*t + H_1 - x) + M_1 e^{-\delta t}$$

for all $t \geq T_1$ and $0 \leq x \leq h(t)$.

The second step is to derive the following rough estimate for (u,h) from below.

Lemma 4.2. *Let (u,h) be a big spreading solution of* (FBP). *Then there exists constants $T_2 > 0$ and $H_2 \in \boldsymbol{R}$ such that*

$$h(t) \geq c_S t + H_2, \quad u(t,x) \geq q_s(c_S t + H_2 - x)$$

for all $t \geq T_2$ and $0 \leq x \leq c_S t + H_2$.

Proof. We define

$$h_*(t) = c_S t + H, \quad u_*(t,x) = q_*(c_S t + H - x),$$

where H is a number to be determined later. It is easy to verify

$$\begin{array}{ll} (u_*)_t = d(u_*)_{xx} + f(u_*), & t > 0, \quad 0 \le x \le h_*(t), \\ u_*(t, h_*(t)) = 0, & t > 0, \\ (h_*)'(t) = -\mu(u_*)_x(t, h_*(t)), & t > 0. \end{array}$$

We can choose a sufficiently large $T > 0$ such that

$$u_*(t,0) < u_1^* < u(t+T,0) \quad \text{for} \quad t \ge 0;$$

then

$$h_*(0) = H < h(T), \quad u_*(0,x) = q_S(H-x) \le u(T,x) \quad \text{for} \quad 0 \le x \le H$$

with small $H > 0$. Therefore, the comparison principle allows us to derive

$$h_*(t) \le h(t+T), \quad u_*(t,x) \le u(t+T,x)$$

for $t \ge 0$ and $0 \le x \le h_*(t)$; so that the assertion follows from the above inequalities. □

In Lemmas 4.1 and 4.2, note (17) and $\lim_{z \to -\infty} Q^*(z) = u_1^*$. Therefore, if c satisfies $c^* < c < c_S$, then $u(t, ct) \to u_1^*$ as $t \to \infty$. More precisely, it is possible to show the following result from Lemmas 4.1 and 4.2:

Proposition 4.3. *Let* (u,h) *be any big spreading solution of* (FBP). *Then*

$$\lim_{t\to\infty} \sup_{c_1 t \le x \le c_2 t} |u(t,x) - u_1^*| = 0$$

for any c_1 *and* c_2 *satisfying* $c^* < c_1 < c_2 < c_S$.

Roughly speaking, Proposition 4.3 implies that $u(t,x)$ stays at almost constant u_1^* when x lies in an intermediate range $[c_1 t, c_2 t]$ of $(0, h(t))$ with $c^* < c_1 < c_2 < c_S$. Taking account of this fact we will be able to obtain a similar result to Lemma 3.5. As the third step, one can repeat the proof of Lemma 3.5 with some modification and prove the following lemma (see also [13, Lemma 3.9]).

Lemma 4.4. *Let* (u,h) *be a big spreading solution of* (FBP). *Then for any* $c \in (c^*, c_S)$ *there exist positive constants* δ, M_3, T_3 *and* $H_3 \in \boldsymbol{R}$ *such that*

$$\begin{array}{l} h((t) \le c_S t + H_3, \\ u(t,x) \le (1 + M_3 e^{-\delta t}) q_S(c_S t + H_3 - x), \end{array}$$

for all $t \ge T_3$ *and* $ct \le x \le h(t)$.

Lemmas 4.2 and 4.4 yield rough estimates of any big spreading solution $u(t,x)$ over $[ct,h(t)]$ for any $c\in(c^*,c_S)$ if t is sufficiently large. Therefore, the arguments developed by Du, Matsuzawa and Zhou [9] are valid and allow us to get the following sharp estimate (for details, see the proofs of (1.11) and (1.12) in [13]).

Theorem 4.5. *Let* (u,h) *be any big spreading solution of* (FBP). *Then there exists* $H_s\in\boldsymbol{R}$ *such that*

$$\lim_{t\to\infty}(h(t)-c_St)=H_S,\quad \lim_{t\to\infty}h'(t)=c_S$$

and, for any $c\in(c^*,c_S)$,

$$\lim_{t\to\infty}\sup_{ct\le x\le h(t)}|u(t,x)-q_S(h(t)-x)|=0.$$

The final step is to estimate $u(t,x)$ from below when x lies in $[0,ct]$ for any $c\in(c^*,c_S)$.

Lemma 4.6. *Let* (u,h) *be any big spreading solution of* (FBP). *Then, for any* $c\in(c^*,c_S)$, *there exist positive constants* δ,M_4,T_4 *and* $H_4\in\boldsymbol{R}$ *such that*

$$u(t,x)\ge Q^*(c^*t+H_4-x)-M_4e^{-\delta t}$$

for all $t\ge T_4$ *and* $0\le x\le ct$.

For the proof of this lemma, see [13, Lemma 3.8].

Since we have established Lemmas 4.1 and 4.6, we are ready to approximate $u(t,x)$ over $[0,ct]$ for any $c\in(c^*,c_S)$ by using travelling wave (Q^*,c^*) of (TWP). Indeed, we have the following theorem whose proof can be found in [13, Section 5].

Theorem 4.7. *Let* (u,h) *be any big spreading solution of* (FBP). *Then there exists* $H^*\in\boldsymbol{R}$ *such that for any* $c\in(c^*,c_S)$

$$\lim_{t\to\infty}\sup_{0\le x\le ct}|u(t,x)-Q^*(c^*t+H^*-x)|=0.$$

Owing to Theorems 4.5 and 4.7, we have obtained sharp asymptotic estimates of any big spreading solution under assumption (A). In this situation, the semi-wave of (SW-1) gives a good approximation of $u(t,x)$ near the spreading front $x=h(t)$, whereas $u(t,x)$ is sharply estimated by the travelling wave of (TWP) over the other range in $[0,h(t)]$. In particular, we can say that for large t a big spreading solution proceed at almost constant speed c_S and it is accompanied by a propagating terrace with slower speed c^*.

REMARK 4.8: It is also possible to derive sharp estimates for a transition solution. Indeed, a transition solution (u, h) satisfies the same assertion as Theorem 4.5 for any $c \in (0, c_S)$ and, furthermore,

$$\lim_{t\to\infty} \sup_{0\le x\le ct} |u(t,x) - v_{dec}(x)| = 0$$

for any $c \in (0, c_S)$ (see [13, Theorem C]).

5. Concluding remarks

5.1. Free boundary problem in $\boldsymbol{R}^N$

In this subsection we will consider a free boundary problem for a reaction-diffusion equation in $\boldsymbol{R}^N$. We focus on the problem in a radially symmetric environment. So it is formulated in the following form for a pair of unknown function $u = u(t, r)$ with $r = |x|$ ($x \in \boldsymbol{R}^N$) and $h = h(t)$:

$$\begin{cases} u_t = d\Delta u + f(u), & t > 0, \quad 0 < r < h(t), \\ u_r(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \le r \le h_0, \end{cases} \tag{19}$$

where d, μ and h_0 are positive constants, $\Delta u = u_{rr} + (N-1)u_r/r$ and u_0 is a nonnegative function satisfying (2). When f satisfies (1), it is shown by Du and Guo [2] that (19) admits a unique global solution which possesses similar properties to those in Theorem 2.1. Moreover, basic properties on the comparison principle and large-time behaviors of solutions hold true as in the one-dimensional case (see [2], [7] and [12]). In particular, if f satisfies (M) (resp. (B)), it is also possible to show the same classification result as Theorem 2.8 (resp. Theorem 2.9) established for $N = 1$. For the study of free boundary problems for general domain, see, for instance, [3], [7] and [8].

We will investigate (19) for positive bistable nonlinearity. In addition to (PB), we put the following condition on f:

(PB-1) $f(u)/(u - \overline{u})$ is non-increasing for $u \in (\overline{u}, u_3^*)$, where $\overline{u} \in (u_2^*, u_3^*)$ is a unique number determined by $\displaystyle\int_{u_2^*}^{\overline{u}} f(s)ds = 0$.

Then it is possible to prove the following classification theorem which corresponds to Theorem 3.1 (see [14, Theorem A]).

THEOREM 5.1. *Let f satisfy* (PB) *and* (PB-1). *Then the solution* (u, h) *of* (19) *satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \le \sqrt{d\lambda_1/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$, *where* λ_1 *is the principal eigenvalue of*

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \ \Omega := \{x \in \boldsymbol{R}^N \mid \ |x| < 1\}, \\ \varphi = 0 & \text{on } \ \partial\Omega := \{x \in \boldsymbol{R}^N \mid \ |x| = 1\}. \end{cases}$$

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = u_1^*$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$.

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = u_3^*$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$.

(iv) *Transition;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,r) = V_{dec}(r)$ *uniformly in* $r \in [0,R]$ *for any* $R > 0$, *where* V_{dec} *is a decreasing solution of*

$$\begin{cases} dV_{rr} + (N-1)V_r/r + f(V) = 0, \quad V(r) > 0 \qquad \text{for } \ r > 0, \\ V_r(0) = 0, \end{cases} \tag{20}$$

and it satisfies $\lim_{r\to\infty} V_{dec}(r) = u_1^*$.

Note that (20) corresponds to stationary problem (SP) for $N = 1$. In the proof of Theorem 5.1, it is important to study the set of non-increasing solutions of (20). We need to take a different approach from the phase plane analysis which is efficient for $N = 1$.

As to large-time behaviors of spreading solutions $(u(t,r), h(t))$ of (19), semi-waves for (SWP) are still available in the analysis. Indeed, rough estimates of the free boundary are given by the following result (see [14, Theorem C]).

Proposition 5.2. *Assume that* f *satisfies* (PB) *and* (PB-1). *Let* (u,h) *be the solution of* (19). *Then the same conclusions as Proposition 3.4 hold true.*

This proposition implies that there is no difference between $N = 1$ and $N \ge 2$ in order to give rough estimates of $h(t) \to \infty$ as $t \to \infty$.

The dependence on the space dimension N appears in sharp estimates of $h(t)$ of spreading solutions. They have been obtained by Du, Matsuzawa and Zhou [10] in the case that f satisfies (M) or (B). When f is positive bistable nonlinearity satisfying (PB) and (PB-1), we can prove similar results for small spreading solutions or big spreading solutions. Let (u,h) be a small spreading solution or a big spreading solution of (19) and let the corresponding semi-wave problem (SWP) possess a solution pair $(q^*(z), c^*)$ for $u^* = u_1^*$ or $u^* = u_3^*$. Then it is possible to show the following estimate ([15]):

There exists a constant $R^* \in \boldsymbol{R}$ such that

$$\lim_{t\to\infty} \{h(t) - c^* t + (N-1)c_* \log t\} = R^*,$$
$$\lim_{t\to\infty} \sup_{0\le r\le h(t)} |u(t,r) - q^*(h(t)-r)| = 0,$$

where $c_* = 1/(\zeta c^*)$,

$$\zeta = 1 + \frac{c^*}{\mu^2 \int_0^\infty \{(q^*)'(z)\}^2 e^{-c^* z} dz}$$

(see also [10, Theorem 4.1]). For a big spreading solution (u,h) whose corresponding semi-wave problem (SWP-3) has no solution pair, we can also give sharp estimates with use of semi-wave (q_S, c_S) of (SWP-1) and travelling wave (Q^*, c^*) of (TWP):
There exist $R_S, R_B \in \boldsymbol{R}$ such that

$$\lim_{t\to\infty} \{h(t) - c_S t + (N-1)c_{S*} \log t\} = R_S$$

with $c_{S*} = 1/(\zeta c_S)$,

$$\zeta = 1 + \frac{c_S}{\mu^2 \int_0^\infty \{(q_S)'(z)\}^2 e^{-c_S z} dz}$$

and, for sufficiently large $L > 0$,

$$\lim_{t\to\infty} \sup_{c_S t - L\log t\le r\le h(t)} |u(t,r) - q_S(h(t)-r)| = 0,$$
$$\lim_{t\to\infty} \sup_{0\le r\le c_S t - L\log t} \sup \left| u(t,r) - Q^*\left(c^* t - \frac{N-1}{c^*}\log t + R_B - r\right)\right| = 0.$$

For details of these results, see [15].

5.2. Free boundary problem with Dirichlet boundary condition

In this subsection we will consider (FBP) with zero Neumann condition at $x = 0$ replaced by zero Dirichlet condition. The problem is written as follows:

$$\begin{cases} u_t = du_{xx} + f(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{21}$$

where f is a function satisfying (1) and u_0 is a nonnegative function of class $C^2[0, h_0]$ such that

$$u_0(0) = u_0(h_0) = 0 \quad \text{and} \quad u_0 \not\equiv 0. \tag{22}$$

Basic results such as the existence and uniqueness of global solutions (Theorem 2.1) and the comparison principle (Theorem 2.2) are valid with obvious modification (see Kaneko and Yamada [17]). The notion of vanishing and spreading of solutions to (21) is the same as Definition 2.4. It should be noted that the following theorem holds true in place of Theorem 2.6 (see [16] and [17]).

THEOREM 5.3. *Assume $f'(0) \neq 0$. Then a solution (u, h) of (21) is vanishing if and only if $\lim_{t\to\infty} h(t) < \infty$. Moreover, if $f'(0) > 0$, a spreading solution satisfies*

$$\lim_{t\to\infty} h(t) \leq \pi\sqrt{d/f'(0)}.$$

For the case $\lim_{t\to\infty} h(t) = \infty$, it is also possible to prove the following theorem similarly to Theorem 2.7 (see [11, Proposition 4.7]).

THEOREM 5.4. *Assume $f'(0) \neq 0$ and let (u, h) be the solution of (21) satisfying $\lim_{t\to\infty} h(t) = \infty$. Then it holds that for any $R > 0$*

$$\lim_{t\to\infty} u(t, x) = v^*(x) \quad \textit{uniformly in} \quad x \in [0, R],$$

where v^* is a bounded solution of

$$\begin{cases} dv'' + f(v) = 0, \quad v(x) > 0 \quad \textit{for} \quad x \in (0, \infty), \\ v(0) = 0. \end{cases} \tag{23}$$

We now consider positive bistable nonlinearity f, which satisfies (PB). Let $\mathcal{S}$ be the set of bounded solutions of stationary problem (23). The phase plane analysis is available to get

$$\mathcal{S} = \{v_1, v_3\},$$

where v_i is an increasing solution of (23) satisfying

$$\lim_{x\to\infty} v_i(x) = u_i^*$$

for each $i = 1, 3$. Therefore, Theorems 5.3 and 5.4 enable us to show the following classification theorem (see Endo, Kaneko and Yamada [11, Theorem 4.1]).

THEOREM 5.5. *Under assumption (PB), the solution (u, h) of (21) satisfies one of the following properties:*

(i) *Vanishing;* $\lim_{t\to\infty} h(t) \leq \pi\sqrt{d/f'(0)}$ *and* $\lim_{t\to\infty} \|u(t)\|_{C[0,h(t)]} = 0$.

(ii) *Small spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_1(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

(iii) *Big spreading;* $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t,x) = v_3(x)$ *uniformly in* $x \in [0,R]$ *for any* $R > 0$.

REMARK 5.6: Differently from the classification result for the Neumann boundary condition (Theorem 3.1), a transition solution does not appear as a borderline one in Theorem 5.5. But small spreading solutions can be divided into two subgroups;

(a) small spreading solutions with $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} < u_2^*$,

(b) small spreading solutions with $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} \geq u_2^*$

(see [11, Remark 5]). We have a conjecture that a small spreading solution in the latter subgroup exhibits a borderline behavior between small spreading solutions in the former subgroup and big spreading solutions. For the related problem, see the works of Liu and Lou [20, 21]. They discussed the existence of a transition solution with a moving peak as a borderline behavior for f satisfying (B).

In the study of large-time behaviors of solutions with $\lim_{t\to\infty} h(t) = \infty$, semi-waves of (SWP) also play a crucial role. Indeed, we can obtain the following result (see [11, Theorems 5.3 and 5.5]).

THEOREM 5.7. *Under assumption* (PB)*, let* (u,h) *be a small spreading solution of* (21) *satisfying* $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} < u_2^*$ *and let* (q_S, c_S) *be the solution pair of* (SWP-1)*. Then there exists* $h_S \in \boldsymbol{R}$ *such that*

$$\lim_{t\to\infty} \{h(t) - c_S t\} = h_S \qquad \text{and} \qquad \lim_{t\to\infty} h'(t) = c_S$$

and

$$\lim_{t\to\infty} \sup_{h(t)/2 \leq x \leq h(t)} |u(t,x) - q_S(h(t)-x)| = 0.$$

Moreover, for any $c \in (0, c_S)$*,*

$$\lim_{t\to\infty} \sup_{0 \leq x \leq ct} |u(t,x) - v_1(x)| = 0.$$

REMARK 5.8: Let (u,h) be a small spreading solution of (21) sch that u satisfies $\liminf_{t\to\infty} \|u(t)\|_{C[0,h(t)]} \geq u_2^*$. Then $u(t,x)$ has a moving peak at $x = x_t^*$ such that $u(t,x_t^*) \geq x_2^* - \delta$ with some $\delta > 0$ for sufficiently large t. On the other hand, it satisfies $\lim_{t\to\infty} u(t,x) = v_1(x) < u_1^*$ for each $x \in [0,\infty)$. Therefore, $u(t,x)$ cannot be estimated by only q_S and v_1. Approximation of such a spreading solution is an interesting open problem.

When (u, h) is a big spreading solution of (21) and (SWP-3) has a solution pair (q_B, c_B), it is seen from [18] that similar results to Theorem 5.7 hold true (see also [11, Theorems 5.4 and 5.5]).

When (SWP-3) has no solution pair, a big spreading solution will be approximated with use of semi-wave $(q_S.c_S)$ of (SWP-1), travelling wave (Q^*, c^*) of (TWP) and stationary solution v_3 of (23). We will discuss this problem elsewhere.

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Non-standard bifurcation approach to nonlinear elliptic problems

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Dedicated to Julián on his 60th birthday, who, together with Prof. Manuel Delgado, contributed to the development of the elliptic PDEs in Seville and Belém

Abstract. *Bifurcation is a very useful method to prove the existence of positive solutions for nonlinear elliptic equations. The existence of an unbounded continuum of positive solutions emanating from zero or from infinity can be deduced in many problems. In this paper, we show the applicability of this method in some problems where the classical bifurcation results can not be directly applied.*

Keywords: Bifurcation, positive solutions, family of supersolutions.
MS Classification 2010: 35J60, 35B32, 35A16.

1. Introduction

Consider a nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda u + b(x)g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded and regular domain, $g : \mathbb{R} \mapsto [0, \infty)$ is a continuous map, $b \in C(\overline{\Omega})$ and λ is a real parameter.

The bifurcation method is one of the most well-known tools in order to study (nonnegative and nontrivial) solutions of (1). In fact, the bifurcation method provides the existence of an unbounded continuum $\mathcal{C}_0 \subset \mathbb{R} \times C_0^1(\overline{\Omega})$ of solutions of (1) emanating from the trivial solution at $\lambda = \lambda_1$, where λ_1 stands for the principal eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions, under the condition

$$\lim_{s \to 0^+} \frac{g(s)}{s} = 0 \tag{H_0}$$

see for instance [25] and [19].

In a similar way, if g verifies

$$\lim_{s\to+\infty}\frac{g(s)}{s}=0 \tag{H_∞}$$

then an unbounded continuum $\mathcal{C}_\infty$ of solutions of (1) emanates from infinity at $\lambda=\lambda_1$, [26]. In both cases, the results are similar if the limits are finite and not necessarily zero, see [4]. We point out that when (H_0) and (H_∞) are both satisfied, $\mathcal{C}_0$ and $\mathcal{C}_\infty$ do not have necessarily to coincide, see for instance [6].

We assume now that g verifies only (H_0) and not (H_∞). Then, the global behaviour of the continuum $\mathcal{C}_0$ depends strongly on g and the sign of b. Let us summarize the main results in this case. For that, we need to introduce some notation. Define the sets

$$\begin{aligned}
B_+ &:= \{x\in\Omega : b(x)>0\},\\
B_- &:= \{x\in\Omega : b(x)<0\},\\
B_0 &= \operatorname{int}(\Omega\setminus(\overline{B_+}\cup\overline{B_-})),
\end{aligned}$$

for which we will assume for simplicity that they are regular sets and that B_0 is also connected.

Given a subdomain $D\subset\Omega$, we denote by λ_1^D the principal eigenvalue of the Laplacian under homogeneous Dirichlet boundary conditions. Moreover, given $(\lambda,u)\in\mathcal{C}_0$ we define $\mathrm{Proj}_{\mathbb{R}}(\lambda,u)=\lambda$.

Finally, assume that there exists $p>1$ such that

$$\lim_{s\to+\infty}\frac{g(s)}{s^p}=g_0>0. \tag{S_∞}$$

Hence, when g verifies only (H_0) and not (H_∞), the main results can be summarized as follows:

1. If $b(x)\le b_1<0$ for all $x\in\overline{\Omega}$ for some $b_1\in\mathbb{R}$, then $\mathrm{Proj}_{\mathbb{R}}(\mathcal{C}_0)=(\lambda_1,+\infty)$ and as consequence there exists at least a positive solution for $\lambda>\lambda_1$.
2. If $b\le 0$, $b\ne 0$ in Ω and $B_0\neq\emptyset$, then $\mathrm{Proj}_{\mathbb{R}}(\mathcal{C}_0)=(\lambda_1,\lambda_1^{B_0})$. In this case, a bifurcation to infinity appears at $\lambda=\lambda_1^{B_0}$. Moreover, there exists at least a positive solution for $\lambda\in(\lambda_1,\lambda_1^{B_0})$.
3. Assume that b changes sign, (S_∞) and that $p<p^*$, for some $p^*\le(N+2)/(N-2)$. Then, $(-\infty,\lambda_1)\subset\mathrm{Proj}_{\mathbb{R}}(\mathcal{C}_0)\subset(-\infty,\overline{\lambda})$, for some $\overline{\lambda}<\infty$. In this case, there exists at least a positive solution for $\lambda<\lambda_1$.

There is a large literature on the above problem. Let us focus on those papers that mainly use the bifurcation technique to get the results. Thanks to the a priori bounds and the non-existence of positive solutions for $\lambda \le \lambda_1$, the case $b(x) \le b_1 < 0$ is the simplest one. For the case $b < 0$ and $B_0 \neq \emptyset$ we refer to [21] as a general reference, see also [3, 14, 15, 16, 23] and the references therein. For the case b changing sign, see for instance [2, 8, 22].

A similar study could be done if g verifies (H_∞) and not (H_0). However, in this case, the behaviour of $\mathcal{C}_\infty$ is less known in general. Let us focus on the particular case $g(u) = u^q$, $0 < q < 1$. Hence, we have the following results:

1. If $b \le b_1 < 0$ for all $x \in \Omega$ for some $b_1 \in \mathbb{R}$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_\infty) = (\lambda_1, +\infty)$.
2. If $b(x) \ge b_0 > 0$ for all $x \in \overline{\Omega}$ for some $b_0 \in \mathbb{R}$, then $\mathrm{Proj}_{\mathbf{R}}(\mathcal{C}_\infty) = (-\infty, \lambda_1)$.

See for instance [7, 12, 13, 24, 26].

In this paper, our main goal is to study the set of nonnegative and nontrivial solutions of (1) when conditions (H_0) and (H_∞) are not fullfilled. For that, we are going to study the following specific equation

$$\begin{cases} -\Delta u = \lambda u + b(x)(u^q + u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where

$$0 < q < 1 < p$$

although most of the results obtained here are also true for more general set of functions.

Problem (2) can be included in a more general problem

$$\begin{cases} -\Delta u = \lambda u + a(x)u^q + b(x)u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

for a and b verifying several structural assumptions. Problem (3) has been analyzed in [1] when $b(x) = \gamma \ge 0$ under homogeneous Neumann boundary conditions. In [10] the author studied (3) when b changes sign and some further conditions on a and b. The author proved the existence of two nonnegative and nontrivial solutions when $\lambda < \lambda^*$ for some $\lambda^* \in \mathbb{R}$. First, the sub-supersolution method is used to prove the existence of a solution, which is a local minimum of the associated functional. Finally, using mainly the mountain pass theorem the existence of the second solution is shown. The case $\lambda = 0$ and $a(x) = \gamma c(x)$, regarding now γ as a real parameter, has been studied for many authors from the pioneering work [5], see for instance [9, 11, 18] and references therein.

We will study (2) for different conditions on b using bifurcation methods. In the first results, we deal with the case b changing sign and b negative, respectively. In both cases, we can not apply directly the bifurcation method, but we can consider a truncated problem where the bifurcation method can be applied and then use a compactness method. Our main results can be stated as follows (see Figure 1):

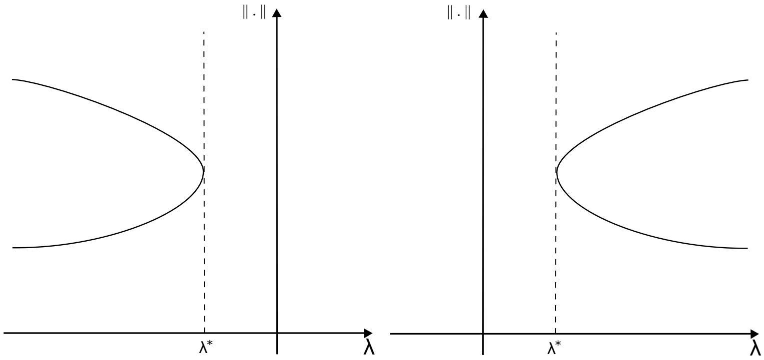

Figure 1: Minimal bifurcation diagrams of (2) in the cases b changing sign and b negative, respectively.

THEOREM 1.1. *Assume that* $0 < q < 1 < p$.

1. *Assume that* b *changes sign and that for* x *close* ∂B_+,
$$b^+(x) \approx [dist(x, \partial B_+)]^\gamma, \qquad \gamma \geq 0,$$
and
$$1 < p < \min\{(N+2)/(N-2), (N+1+\gamma)/(N-1)\}. \tag{4}$$
Then, there exists $\lambda^* \in \mathbb{R}$ *such that for (2) possesses at least two nonnegative and nontrivial solutions for* $\lambda < \lambda^*$.

2. *Assume that* $b(x) \leq b_1 < 0$ *for all* $x \in \overline{\Omega}$ *and for some* $b_1 \in \mathbb{R}$. *Then, there exists* $\lambda_* \in \mathbb{R}$ *such that for (2) possesses at least two nonnegative and nontrivial solutions for* $\lambda > \lambda_*$.

Surprisingly, in the case $b \leq 0$ and $B_0 \neq \emptyset$, we obtain the existence of two continua bifurcating from the trivial solution and from infinity at the same point $\lambda = \lambda_1^{B_0}$. The main result is (see Figure 2):

THEOREM 1.2. *Assume that* $b \leq 0$, $b \neq 0$ *in* Ω *and* $B_0 \neq \emptyset$. *If* $\lambda \leq \lambda_1$, *(2) does not possess nonnegative and nontrivial solutions. Moreover:*

1. *From the trivial solution emanates at* $\lambda = \lambda_1^{B_0}$ *an unbounded continuum* $\mathcal{C}_0 \subset \mathbb{R} \times L^\infty(\Omega)$ *of nonnegative and nontrivial solutions of (2). Moreover,* $\lambda_1^{B_0}$ *is the unique bifurcation point from the trivial solution.*

2. $\lambda = \lambda_1^{B_0}$ *is a bifurcation point from infinity of nonnegative and nontrivial solutions, and it is the only one. Moreover, there exists an unbounded*

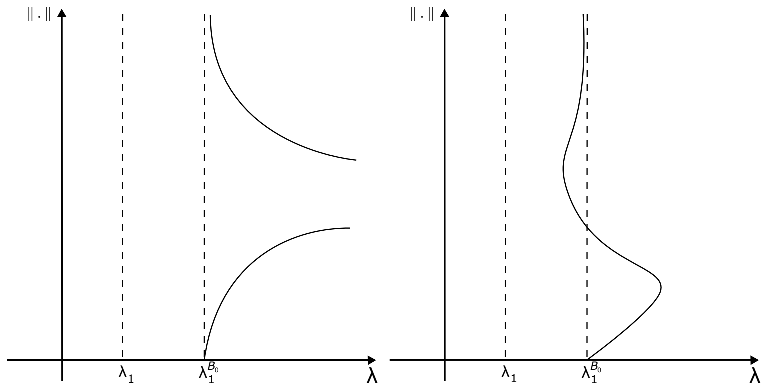

Figure 2: Minimal bifurcation diagrams of (2) when $b \leq 0$ and $B_0 \neq \emptyset$. In the first case both continua $\mathcal{C}_0$ and $\mathcal{C}_\infty$ are different; in the second one both coincide.

continuum $\mathcal{C}_\infty$ of nonnegative and nontrivial solutions of (2) such that

$$\mathcal{D}_\infty = \left\{(\lambda, u) : u \neq 0, \left(\lambda, \frac{u}{\|u\|_\infty^2}\right) \in \mathcal{C}_\infty\right\} \cup \{(\lambda_1^{B_0}, 0)\}$$

is connected and unbounded.

In this case, we are not able to ascertain the global behaviour of these continua, mainly to the lack of the strong maximum principle in (2).

An outline of this work is as follows: Section 2 contains some properties of the principal eigenvalue of an elliptic problem. Section 3 is devoted to show the relative position between a family of supersolutions and a continuum of solutions of a nonlinear elliptic problem. In Section 3 we study in detail the truncated problems using the bifurcation method. In Sections 4 and 5 the main results are proved.

2. Eigenvalue problems

In this section we recall some useful properties of elliptic eigenvalue problems. Given a subdomain $D \subset \Omega$ we consider

$$\begin{cases} -\Delta u + m(x)u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{5}$$

where $m \in L^\infty(\Omega)$. The following result is well-known (see [20], where a detailed study of (5) and more general eigenvalue problems can be found)

LEMMA 2.1. *There exists a principal eigenvalue of (5), denoted by* $\lambda_1^D(-\Delta+m)$. *It is simple and isolated, and it is the only one whose eigenfunction associated can be chosen to be positive in* D. *If we denote by* φ_1 *a positive eigenfunction associated to* $\lambda_1^D(-\Delta+m)$, *then* $\varphi_1 \in C^{1,\alpha}(\overline{D})$, $\alpha \in (0,1)$ *and* $\partial\varphi_1/\partial n < 0$ *on* ∂D *where* n *is the outward unit vector normal to* ∂D.

Moreover, the following properties hold:

1. *Asume that* m *changes sing. Then* $t \mapsto \lambda_1^D(-\Delta+tm)$ *is continuous, concave and*
$$\lim_{t\to\pm\infty} \lambda_1^D(-\Delta+tm) = -\infty.$$
2. *Assume that* $m(x) \le m_0 < 0$ *for all* $x \in D$. *Then,* $t \mapsto \lambda_1^D(-\Delta+tm)$ *is continuous, decreasing and*
$$\lim_{t\to\pm\infty} \lambda_1^D(-\Delta+tm) = \mp\infty.$$

When $D = \Omega$, we omit the superscript and we denote $\lambda_1(-\Delta+m) = \lambda_1^\Omega(-\Delta+m)$. Moreover, when $m \equiv 0$ we simply write λ_1^D instead of $\lambda_1^D(-\Delta)$.

3. Relative position between a subcontinuum of solutions and a continuous family of supersolution

The main goal of this section is to generalize some results of [15]. Consider the general elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is a continuous function and locally Lipschitz in the second variable.

We define the positive cone in $C^1(\overline{\Omega})$

$$\mathcal{Q} := \{u \in C^1(\overline{\Omega}) : u(x) \ge 0 \text{ for all } x \in \overline{\Omega}\},$$

whose interior and exterior are

$$\text{int}(\mathcal{Q}) = \{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ for all } x \in \overline{\Omega}\},$$

and

$$\text{ext}(\mathcal{Q}) = C^1(\overline{\Omega}) \setminus \mathcal{Q}.$$

We have the following result

LEMMA 3.1. *Let $\overline{u} \in C^1(\overline{\Omega})$ be a supersolution of (6) with $\overline{u} > 0$ on $\partial\Omega$ and $u \in C_0^1(\overline{\Omega})$ a solution of (6). Then,*

$$\overline{u} - u \notin \partial\mathcal{Q}.$$

Proof. By contradiction assume that $\overline{u} - u \in \partial\mathcal{Q} = \overline{\mathcal{Q}} \setminus \mathrm{int}(\mathcal{Q})$. Then,

$$w(x) = \overline{u}(x) - u(x)$$

verifies that $w(x) \geq 0$ for all $x \in \Omega$ and $w(x_0) = 0$ for some $x_0 \in \Omega$. Observe that,

$$\begin{cases} -\Delta w + Mw \geq f(x,\overline{u}) + M\overline{u} - (f(x,u) + Mu) \geq 0 & \text{in } \Omega, \\ w > 0 & \text{on } \partial\Omega, \end{cases}$$

for some $M > 0$ large enough. The strong maximum principle asserts that $w(x) > 0$ for all $x \in \overline{\Omega}$. This is a contradiction. □

The main result of this section reads as follows:

THEOREM 3.2. *Let $\mathcal{C}$ be a subcontinuum of solutions $\mathcal{C} \subset I \times C_0^1(\overline{\Omega})$ of (6), where $I \subset \mathbb{R}$ is a real interval. Let $U : I \mapsto C^1(\overline{\Omega})$ a continuous family of supersolutions of (6) with $U(\lambda) > 0$ on $\partial\Omega$. If for some $(\lambda_0, u_0) \in \mathcal{C}$, $u_0 \leq U(\lambda_0)$, then $u < U(\lambda)$ for all $(\lambda, u) \in \mathcal{C}$.*

Proof. Consider the continuous map $T : I \times C^1(\overline{\Omega}) \mapsto C^1(\overline{\Omega})$ given by

$$T(\lambda, u) := U(\lambda) - u. \tag{7}$$

Since T is continuous, then $T(\mathcal{C})$ is connected. By Lemma 3.1 we conclude that $T(\mathcal{C}) \cap \partial\mathcal{Q} = \emptyset$. Then, either $T(\mathcal{C})$ is completely inside $\mathrm{int}(\mathcal{Q})$ or completely outside. Since $T(\lambda_0, u_0) \in \mathrm{int}(\mathcal{Q})$, we deduce that $T(\mathcal{C}) \subset \mathrm{int}(\mathcal{Q})$. □

In fact, from the proof of Theorem 3.2, we obtain:

COROLLARY 3.3. *Let $\mathcal{C} \subset I \times C_0^1(\overline{\Omega})$ a subcontinuum of solutions of (6) and T the map defined in (7). Then, either*

1. *$T(\mathcal{C}) \subset$ int($\mathcal{Q}$), and therefore, $u < U(\lambda)$, or*

2. *$T(\mathcal{C}) \subset$ ext($\mathcal{Q}$).*

4. Study of the truncated problems

For $\delta > 0$ we define

$$f_\delta(s) := \begin{cases} \delta^{q-1}s & \text{if } s \in [0, \delta], \\ s^q & \text{if } s > \delta. \end{cases}$$

Let us consider now the truncated problem

$$\begin{cases} -\Delta u = \lambda u + b(x)(f_\delta(u) + u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8}$$

We point out that the nonlinear term is locally Lipschitz continuous in the second variable, and then by the strong maximum principle, any nonnegative and nontrivial solution of (8) is positive in all Ω.

4.1. b changes sign

First, we prove a non-existence result.

LEMMA 4.1. *Consider (λ, u) a positive solution of (8). Then*

$$\lambda \leq \overline{\lambda} \qquad \textit{for some } \overline{\lambda} < \infty.$$

Moreover, if $B_0 \neq \emptyset$, then

$$\lambda \leq \lambda_1^{B_0}.$$

Proof. Take a ball $B \subset B_+$ such that $b(x) \geq b_0 > 0$ for $x \in B$. Let φ_1^B be a positive eigenfunction associated to λ_1^B and consider

$$\varphi = \begin{cases} \varphi_1^B & \text{in } B, \\ 0 & \text{in } \Omega \setminus \overline{B}. \end{cases}$$

Since $\varphi \in H_0^1(\Omega)$, then on multiplying (8) by φ and using that $\partial\varphi_1^B/\partial n < 0$ on ∂B, we deduce that

$$0 \geq \int_B \left(\lambda - \lambda_1^B + b_0 \frac{f_\delta(u) + u^p}{u}\right) u\varphi_1^B,$$

which is a contradiction for λ large, for instance, for $\lambda \geq \lambda_1^B$.

Assume now that $B_0 \neq \emptyset$. Let $\varphi_1^{B_0}$ be a positive eigenfunction associated to $\lambda_1^{B_0}$ and consider

$$\varphi = \begin{cases} \varphi_1^{B_0} & \text{in } B_0, \\ 0 & \text{in } \Omega \setminus \overline{B_0}. \end{cases}$$

Now, we can follow the previous argument and conclude that

$$\lambda \leq \lambda_1^{B_0}.$$

□

In the following theorem we show a priori bounds for the solutions of (8). In the first part, we obtain a priori bounds with respect to the parameter λ, and then, for a fix λ, with respect to δ. These results will be crucial in order to pass to the limit as $\delta \to 0$. For its proof, we will closely follow [2].

THEOREM 4.2. *Assume that for x close ∂B_+,*

$$b^+(x) \approx [dist(x, \partial B_+)]^\gamma, \qquad \gamma \geq 0,$$

and

$$1 < p < \min\{(N+2)/(N-2), (N+1+\gamma)/(N-1)\}. \tag{9}$$

1. *Then, for every bounded interval $\Lambda \subset \mathbb{R}$ there exists a positive constant M such that*
$$\|u\|_\infty \leq M$$
for any positive solution (λ, u) of (8), with $\lambda \in \Lambda$.

2. *Fix $\lambda \in \mathbb{R}$ and consider a sequence $\delta_n \to 0$. Denote by u_n a positive solution of (8). Then, there exists a positive constant $C > 0$ such that*
$$\|u_n\|_\infty \leq C.$$

Proof. 1. This paragraph follows by Theorem 4.3 in [2].

2. In this case we can follow again the proof of Theorem 4.3 in [2], using a Gidas-Spruck argument [17] taking into account that
$$f_\delta(u) \leq u^q.$$

□

We are ready to show the main result in this case (see Figure 3):

THEOREM 4.3. *Assume that b changes sign, $0 < q < 1 < p$ and p verifying (9). Then, there exists an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from $u \equiv 0$ at*

$$\lambda = \lambda_1(\delta) := \lambda_1(-\Delta - b(x)\delta^{q-1}).$$

For any $\delta < 1$.

1. *There exists $\lambda_1^+(\delta) < \lambda_1(\delta)$ such that (8) does not possess positive solution (λ, u) for $\lambda \leq \lambda_1^+(\delta)$ with $\|u\|_\infty \leq \delta$.*

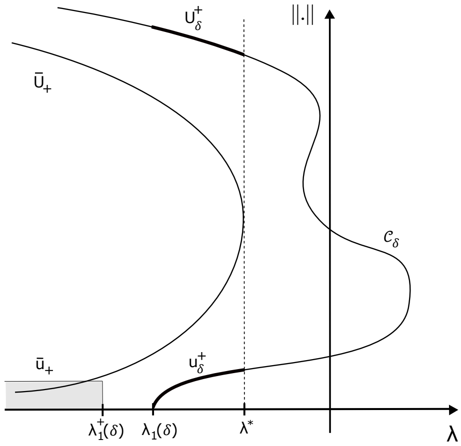

Figure 3: Minimal bifurcation diagram of (8) when b changes sign.

2. *There exist a real value $\lambda^* \in \mathbb{R}$ and two continuous families of supersolutions $\overline{u}_+, \overline{U}_+ : (-\infty, \lambda^*) \mapsto C^1(\overline{\Omega})$, all independent of δ, with $\overline{u}_+(\lambda) > 0, \overline{U}_+(\lambda) > 0$ on $\partial\Omega$. Moreover, $\overline{u}_+(\lambda) < \overline{U}_+(\lambda)$ for $\lambda < \lambda^*$ and $\overline{u}_+(\lambda^*) = \overline{U}_+(\lambda^*)$. Furthermore,*
$$\overline{u}_+(\lambda) \to 0 \quad \textit{and} \quad \overline{U}_+(\lambda) \to +\infty \quad \textit{in } L^\infty(\Omega) \textit{ as } \lambda \to -\infty.$$

3. *For any $\lambda \in (\lambda_1(\delta), \lambda^*)$ there exist at least two solutions u_δ^+ and U_δ^+ of (8) with $(\lambda, u_\delta^+), (\lambda, U_\delta^+) \in \mathcal{C}_\delta$ such that*
$$\overline{u}_+(\lambda) - u_\delta^+ \in \textit{int}(\mathcal{Q}) \quad \textit{and} \quad \overline{U}_+(\lambda) - U_\delta^+ \in \textit{ext}(\mathcal{Q}).$$

Proof. Since
$$\lim_{s \to 0^+} \frac{f_\delta(s)}{s} = \delta^{q-1},$$
it follows the existence of an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from the trivial solution at $\lambda = \lambda_1(-\Delta - b(x)\delta^{q-1}) = \lambda_1(\delta)$.

Thanks to Lemma 4.1 and the first paragraph of Theorem 4.2, we conclude the existence of $\lambda^+ \in \mathbb{R}$ such that
$$(-\infty, \lambda^+) \subset \mathrm{Proj}_{\mathbb{R}}(\mathcal{C}_\delta) \subset (-\infty, \overline{\lambda}). \tag{10}$$

For any $0 < \delta < 1$ consider a positive solution u of (8) such that $\|u\|_\infty \leq \delta < 1$. Observe that $u^p \leq u$ because $p > 1$. Then,
$$-\Delta u = \lambda u + b(x)(f_\delta(u) + u^p) = \lambda u + b(x)(\delta^{q-1} u + u^p) \leq u(\lambda + b_M(\delta^{q-1} + 1)),$$

where $b_M = \max_{x\in\overline{\Omega}} b(x)$, and hence

$$\lambda \geq \lambda_1(-\Delta - b_M(\delta^{q-1}+1)) = \lambda_1 - b_M(\delta^{q-1}+1).$$

It suffices to take

$$\lambda_1^+(\delta) := \lambda_1 - b_M(\delta^{q-1}+1).$$

We now build the families of supersolutions. Notice that $K > 0$ is a supersolution of (8) if

$$0 \geq \lambda K + b(x)(f_\delta(K) + K^p).$$

Observe that

$$b(x)(f_\delta(K)+K^p) \leq b_M(\delta^{q-1}K\chi_{\{K\leq\delta\}} + K^q\chi_{\{K>\delta\}} + K^p) \leq b_M(\delta^q + K^q + K^p).$$

Using now that $\delta < 1$, we have that K is supersolution of (8) if

$$b_M(K^{-1} + K^{q-1} + K^{p-1}) \leq -\lambda.$$

The function

$$h(x) := b_M(x^{-1} + x^{q-1} + x^{p-1})$$

attains a minimum at $x_{min} > 0$, $h(x_{min}) = h_0 > 0$ and $h'(x) < 0$ if $x < x_{min}$ while that $h'(x) > 0$ if $x > x_{min}$. Then, taking $\lambda^* = -h_0$ for any $\lambda < \lambda^*$ there exist two positive constants K_i, $i = 1, 2$, such that $h(K_i) = -\lambda$, with $K_1 < K_2$ and $K_1(\lambda) \to 0$ and $K_2(\lambda) \to +\infty$ as $\lambda \to -\infty$. Then, it suffices to take

$$\overline{u}_+(\lambda) = K_1(\lambda), \qquad \overline{U}_+(\lambda) = K_2(\lambda).$$

Now, we apply Theorem 3.2 with $I = (-\infty, \lambda*]$. By (10), the nonexistence of positive solutions with $\|u\|_\infty \leq \delta$ for $\lambda \leq \lambda_1^+(\delta)$, that $\mathcal{C}_\delta$ bifurcates at $\lambda = \lambda_1(\delta)$ and $\overline{u}_+(\lambda_1(\delta)) > 0$, it follows the existence of a positive solution u_δ^+ of (8) for any $\lambda \in (\lambda_1(\delta), \lambda^*]$ with $(\lambda, u_\delta^+) \in \mathcal{C}_\delta$ such that

$$\overline{u}_+(\lambda) - u_\delta^+ \in \operatorname{int}(\mathcal{Q}).$$

Moreover, we can conclude the existence of a positive solution of (8) for some $\lambda > \lambda^*$.

Now, we claim that there exists a subcontinuum $\mathcal{D}_\delta \subset \mathcal{C}_\delta$ such that

$$\overline{U}_+(\lambda) - U_\delta^+ \in \operatorname{ext}(\mathcal{Q}) \qquad (\lambda, U_\delta^+) \in \mathcal{D}_\delta, \quad \lambda \in (-\infty, \lambda^*]. \tag{11}$$

It is already known the existence of positive solutions $(\lambda, u) \in \mathcal{C}_\delta$ of (8) for all $\lambda \in (-\infty, \lambda^*]$. Moreover, it is not posible that $\overline{u}_+(\lambda) - u \in \operatorname{int}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{C}_\delta$. Hence, there exists $(\lambda_0, u_0) \in \mathcal{C}_\delta$ such that $\overline{u}_+(\lambda_0) - u_0 \in \operatorname{ext}(\mathcal{Q})$. Thus, from Corollary 3.3, there exists a subcontinuum $\mathcal{D}_\delta$ such that $\overline{u}_+(\lambda) - u \in \operatorname{ext}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$. Again, by Corollary 3.3, this subcontinuum has two possibilities, either

1. $\overline{U}_+(\lambda) - u \in \mathrm{ext}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, or
2. $\overline{U}_+(\lambda) - u \in \mathrm{int}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$.

We show that the second possibility is not possible, proving the claim (11). Indeed, if $\overline{U}_+(\lambda) - u \in \mathrm{int}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, since $\overline{u}_+(\lambda) - u \in \mathrm{ext}(\mathcal{Q})$ for all $(\lambda, u) \in \mathcal{D}_\delta$, then we have for $\lambda = \lambda^*$ that

$$\overline{U}_+(\lambda^*) - u_{\lambda^*} \in \mathrm{int}(\mathcal{Q}), \qquad \overline{u}_+(\lambda^*) - u_{\lambda^*} \in \mathrm{ext}(\mathcal{Q}),$$

which is impossible because $\overline{U}_+(\lambda^*) = \overline{u}_+(\lambda^*)$. This completes the proof. □

4.2. $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$

First, we show a necessary condition on λ for the existence of positive solution of (8).

LEMMA 4.4. *Assume $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$ and consider (λ, u) a positive solution of (8). Then,*

$$\lambda \geq \lambda_1.$$

Proof. In this case, we have that $-\Delta u \leq \lambda u$ in Ω, whence we deduce the result. □

With respect to the a priori bounds, we have:

LEMMA 4.5. *Assume $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$ and consider (λ, u) a positive solution of (8). Then, there exists $C(\lambda) > 0$ such that*

$$\|u\|_\infty \leq \max\{\delta, C(\lambda)\}. \tag{12}$$

Proof. Let $x_M \in \Omega$ be such that $u_M = u(x_M) = \max_{x \in \overline{\Omega}} u(x)$. Assume that $u_M > \delta$. Then,

$$\lambda u_M + b(x_M)(u_M^q + u_M^p) \geq 0,$$

and hence

$$-b_L(u_M^{q-1} + u_M^{p-1}) \leq \lambda,$$

where $b_L = \min_{x \in \overline{\Omega}} b(x)$. This finishes the result. □

Our main result is the following (see Figure 4):

THEOREM 4.6. *Assume that $b(x) \leq b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$. Then, there exists an unbounded continuum $\mathcal{C}_\delta$ in $\mathbb{R} \times C_0^1(\overline{\Omega})$ of positive solutions of (8) emanating from $u \equiv 0$ at*

$$\lambda = \lambda_1(\delta) := \lambda_1(-\Delta - b(x)\delta^{q-1}).$$

Moreover, there exists δ_0 such that for $0 < \delta < \delta_0$, we have:

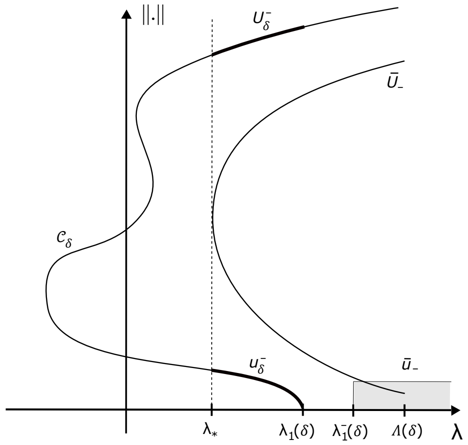

Figure 4: Minimal bifurcation diagram of (8) when b is negative.

1. *The existence of* $\lambda_1^-(\delta) > \lambda_1(\delta)$ *such that (8) does not possess positive solution* (λ, u) *for* $\lambda \geq \lambda_1^-(\delta)$ *with* $\|u\|_\infty \leq \delta$.

2. *There exist a real value* $\lambda_* \in \mathbb{R}$, *independent of* δ, *and two continuous families of supersolutions* $\overline{u}_-, \overline{U}_- : [\lambda_*, \Lambda(\delta)] \mapsto C^1(\overline{\Omega})$ *with* $\overline{u}_-, \overline{U}_- > 0$ *on* $\partial\Omega$, *where* $\Lambda(\delta) = \delta^{q-1} + \delta^{p-1}$. *Such families satisfy*

   $\overline{u}_-(\lambda) < \overline{U}_-(\lambda)$ *for* $\lambda \in [\lambda_*, \Lambda(\delta)]$ *and* $\overline{u}_-(\lambda_*) = \overline{U}_-(\lambda_*), \overline{u}_-(\Lambda(\delta)) = \delta$.

   Furthermore, $\overline{U}_-(\Lambda(\delta)) \to +\infty$ *and* $\overline{u}_-(\Lambda(\delta)) \to 0$ *in* $L^\infty(\Omega)$ *as* $\delta \to 0$.

3. *For* $\lambda \in (\lambda_*, \lambda_1(\delta))$ *there exist at least two solutions* u_δ^- *and* U_δ^- *of (8) such that*

$$\overline{u}_-(\lambda) - u_\delta^- \in int(\mathcal{Q}) \quad and \quad \overline{U}_\delta^- - U_\delta^- \in ext(\mathcal{Q}).$$

Proof. The proof is rather similar to the one of Theorem 4.3. We point out only the main differences.

Assume that $\|u\|_\infty \leq \delta$, then

$$-\Delta u = \lambda u + b(x)(\delta^{q-1}u + u^p) \geq \lambda u + b(x)(\delta^{q-1} + 1)u.$$

Therefore

$$\lambda \leq \lambda_1^-(\delta) := \lambda_1(-\Delta - b(x)(\delta^{q-1} + 1)).$$

Taking $K > 0$, we have that K is a supersolution of (8) provided that

$$h_\delta(K) := (\delta^{q-1}\chi_{\{K\leq\delta\}} + K^{q-1}\chi_{\{K>\delta\}} + K^{p-1}) \geq \frac{\lambda}{-b_M}.$$

Observe that $h_\delta(K)$ can be rewritten as

$$h_\delta(K) = \begin{cases} \delta^{q-1} + K^{p-1} & \text{if } K \le \delta, \\ K^{q-1} + K^{p-1} & \text{if } K > \delta. \end{cases}$$

A detailed study of $h_\delta(K)$ leads to the result. Indeed, since the infimum of the map $x \mapsto h(x) := x^{q-1} + x^{p-1}$ is attained in $x_{min} = ((p-1)/(1-q))^{1/(p-q)}$ and its value is $h(x_{min}) = h_0 > 0$, then, for δ small, x_{min} is also the minimum of $h_\delta(K)$. Then, for δ small, we have that the function h_δ has the following properties:

1. $x \in [0, \delta] \mapsto h_\delta(x) \in [\delta^{q-1}, \delta^{q-1} + \delta^{p-1}]$ is increasing.
2. $x \in [\delta, x_{min}] \mapsto h_\delta(x) \in [h_0, \delta^{q-1} + \delta^{p-1}]$ is decreasing.
3. $x \in [x_{min}, +\infty) \mapsto h_\delta(x) \in [h_0, +\infty)$ is increasing.

Hence, taking $\Lambda(\delta) = \delta^{q-1} + \delta^{p-1}$, for

$$\frac{\lambda}{-b_M} \in [h_0, \Lambda(\delta)],$$

there exist $K_1(\lambda) < K_2(\lambda)$ such that $h_\delta(K_i(\lambda)) = \frac{\lambda}{-b_M}$ with $\delta < K_1(\lambda) < K_2(\lambda)$. In fact, observe that in this region, $h_\delta(x) = x^{q-1} + x^{p-1}$, and therefore $K_i(\lambda)$ does not depend on δ. Moreover,

$$K_1(\lambda) \to \delta \quad \text{as } -\lambda/b_M \to \Lambda(\delta).$$

□

5. Proof Theorem 1.1

1. Let us fix $\lambda < \lambda^*$. By Lemma 2.1, $\lambda_1(\delta) \to -\infty$ as $\delta \to 0$. Hence, there exists δ_0 such that for $\delta \le \delta_0$ we have that $\lambda_1(\delta) < \lambda$. Then, $\lambda \in (\lambda_1(\delta), \lambda^*))$ and by Theorem 4.3 there exist two positive solutions, $u_\delta^+ < U_\delta^+$ of (8) for $\delta \le \delta_0$.

   On the other hand, thanks to the a priori bound given by the second paragraph of Theorem 4.2, we get that $\|U_\delta^+\|_\infty \le M$ for a constant M that does not depend on δ. Observe that

   $$f_\delta(U_\delta^+) \le (U_\delta^+)^q,$$

   and then $\{U_\delta^+\}$ is bounded in $W^{2,r}(\Omega)$ for any $r > 1$. Hence, we can pass to the limit and conclude that $U_\delta^+ \to U_0^+$ in $C^1(\overline{\Omega})$ as $\delta \to 0$, with U_0^+

a nonnegative solution of (2). Moreover, since $\overline{U}_+(\lambda) - U_\delta^+ \in \mathrm{ext}(\mathcal{Q})$ for all $\delta \le \delta_0$, it follows the existence of $x_0 \in \Omega$ such that

$$U_0^+(x_0) \ge \overline{U}_+(\lambda)(x_0) > 0. \tag{13}$$

Hence, U_0^+ is a nonnegative and nontrivial solution of (2).

On the other hand, since $u_\delta^+ < \overline{u}_+(\lambda)$ we can conclude that $u_\delta^+ \to u_0^+ \ge 0$ in $C^1(\overline{\Omega})$ as $\delta \to 0$. We will prove that $u_0^+ \ne 0$. Assume by contradiction that $u_\delta^+ \to 0$ in $C^1(\overline{\Omega})$. Take a ball $B \subset B_+$ such that $b(x) \ge b_0 > 0$ in $\overline{B}$. Since λ is fixed, let us take M large enough such that

$$\lambda_1^B - \lambda \le b_0 M.$$

For this M, let us take δ small such that $u_\delta^q \ge M u_\delta$ and

$$\lambda_1^B - \lambda \le b_0 \min\{\delta^{q-1}, M\}.$$

On multiplying (8) by φ_1^B and integrating in B, we obtain

$$-\int_B \Delta u_\delta^+ \varphi_1^B = \lambda \int_B u_\delta^+ \varphi_1^B + \int_B b(x)(f_\delta(u_\delta^+) + (u_\delta^+)^p)\varphi_1^B.$$

Then,

$$\begin{aligned}\lambda_1^B \int_B u_\delta^+ \varphi_1^B + \int_{\partial B} \partial\varphi_1^B/\partial n u_\delta^+ &> \lambda \int_B u_\delta^+ \varphi_1^B \\ &\quad + b_0 \int_B \big(\delta^{q-1} u_\delta^+ \chi_{\{u_\delta^+ \le \delta\}} + M u_\delta^+ \chi_{\{u_\delta^+ > \delta\}}\big)\varphi_1^B.\end{aligned}$$

Using that $\partial\varphi_1^B/\partial n < 0$ on $\partial\Omega$, we conclude that

$$\lambda_1^B > \lambda + b_0 \min\{\delta^{q-1}, M\},$$

a contradiction. Hence, u_0^+ is a nontrivial and nonnegative solution of (2). Moreover, since

$$u_0^+ \le \overline{u}_+(\lambda) < \overline{U}_+(\lambda),$$

and (13), it follows that $u_0^+ \ne U_0^+$. Thus, there exist at least two positive solutions of (2).

2. Assume that $b(x) \le b_1 < 0$ for all $x \in \overline{\Omega}$ for some $b_1 \in \mathbb{R}$. Let us fix $\lambda > \lambda_*$. Let us take $\delta_0 > 0$ small such that

$$\lambda < \min\{\lambda_1(\delta), \Lambda(\delta)\} \quad \text{for any } \delta \le \delta_0.$$

Observe that this is possible thanks to the expression of $\Lambda(\delta)$ and Lemma 2.1.

Then, by Theorem 4.6 there exist two positive solutions $u_\delta^- < U_\delta^-$ of (8). With a similar argument to the one used in the first paragraph, we can show that $U_\delta^- \to U_0^-$ in $C^1(\overline{\Omega})$ as $\delta \to 0$, where U_0^- is a nonnegative solution of (2) and $U_0^- \neq 0$ in Ω.

On the other hand, we have that $u_\delta^- \to u_0^- \geq 0$ in $C^1(\overline{\Omega})$ as $\delta \to 0$ and $u_0^- \neq 0$. Indeed, arguing by contradiction, assume that $u_\delta^- \to 0$ in $C^1(\overline{\Omega})$. Then, for $M > 0$ we have that for $0 < \delta$ close to zero that $(u_\delta^-)^q \geq M u_\delta^-$. Hence,

$$\begin{aligned} -\Delta u_\delta^- &\leq \lambda u_\delta^- + b_M \big(\delta^{q-1} u_\delta^- \chi_{\{u_\delta^- \leq \delta\}} + (u_\delta^-)^q \chi_{\{u_\delta^- > \delta\}}\big) \\ &= (\lambda + b_M \min\{\delta^{q-1}, M\}) u_\delta^-, \end{aligned}$$

whence

$$\lambda_1 \leq \lambda + b_M \min\{\delta^{q-1}, M\},$$

again a contradiction for M large and δ sufficiently close to zero.

6. The case with bifurcation

Finally, we deal with the case $b \leq 0$, $b \neq 0$ in Ω and $B_0 \neq \emptyset$. For that, we will prove directly that from the trivial solution and from infinity emanate unbounded continua of nonnegative and nontrivial solutions of (2).

We will use the Leray-Schauder degree of K_λ in $B_\rho := \{u \in C(\overline{\Omega}) : \|u\|_\infty < \rho\}$, with respect to zero, denoted by $\deg(K_\lambda, B_\rho)$. The isolated index of u of K_λ is denoted by $i(K_\lambda, u)$. Let us define the map

$$K_\lambda : C_0(\overline{\Omega}) \to C_0(\overline{\Omega}); \qquad K_\lambda(u) := u - T(\lambda, u)$$

where

$$T(\lambda, u) := (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p)),$$

$u^+ := \max\{u, 0\}$, $C_0(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : u = 0 \quad \text{on } \partial\Omega\}$ and $(-\Delta)^{-1}$ denotes the inverse of the laplacian-operator under homogeneous Dirichlet boundary conditions.

It is easy to show that u is nonnegative solution of (2) if and only if u is zero of the map K_λ. Moreover, by the standard regularization properties of T, T is a compact operator on $C_0(\overline{\Omega})$.

6.1. Bifurcation from zero

LEMMA 6.1. *If* $\lambda < \lambda_1^{B_0}$, *then* $i(K_\lambda, 0) = 1$.

Proof. Define the map $\mathcal{H}_1 : [0,1] \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ by

$$\mathcal{H}_1(t,u) = (-\Delta)^{-1}(t(\lambda u^+ + b(x)((u^+)^q + (u^+)^p))).$$

We show now that $\mathcal{H}_1$ is an admisible homotopy, for which it is sufficient to show that there exists $\gamma > 0$ such that

$$u \neq \mathcal{H}_1(t,u) \qquad \forall u \in \overline{B}_\gamma, u \neq 0 \text{ and } t \in [0,1].$$

Assume that there exist $u_n \in C_0(\overline{\Omega})\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$, such that

$$u_n = \mathcal{H}_1(t_n, u_n).$$

This is,

$$-\Delta u_n = t_n(\lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p)) \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

On multiplying the above equality by $u_n^- := \min\{u_n, 0\}$ and integrating in Ω, we infer that $u_n \geq 0$ in Ω.

Let us define

$$z_n = \frac{u_n}{\|u_n\|_2}.$$

Then, z_n verifies

$$-\Delta z_n = t_n(\lambda z_n + b(x)(\|u_n\|_2^{q-1} z_n^q + \|u_n\|_2^{p-1} z_n^p)) \text{ in } \Omega, \quad z_n = 0 \text{ on } \partial\Omega. \tag{14}$$

Since $b \leq 0$, on multiplying the above equality by z_n and integrating in Ω, we obtain that

$$\|z_n\|_{H_0^1} \leq C \qquad \text{for some } C > 0,$$

and hence, up a subsequence,

$$\begin{aligned} z_n &\rightharpoonup z \quad \text{in } H_0^1(\Omega), \\ z_n &\to z \quad \text{in } L^2(\Omega), \end{aligned}$$

for some $z \in H_0^1(\Omega)$, $z \geq 0$ and $\|z\|_2 = 1$.

Next, we show that

$$t_n \|u_n\|_2^{q-1} \to \infty. \tag{15}$$

Assume that for a subsequence $t_n\|u_n\|_2^{q-1} \to r^* \in [0, \infty)$. In such case, since $\|u_n\|_2 \to 0$ and $q < 1$ we obtain that $t_n \to 0$. Then, passing to the limit in (14), we obtain that

$$-\Delta z = r^* b(x) z^q \quad \text{in } \Omega, \qquad z = 0 \quad \text{on } \partial\Omega,$$

whence we deduce that $z = 0$, a contradiction.

We have that $z \equiv 0$ in $\Omega \setminus B_0$. Indeed, assume that $z(x) > 0$ in $D \subset \Omega \setminus B_0$. Take $\varphi \in C_c^\infty(D)$, then

$$-\int_D z_n \Delta\varphi = t_n \left(\lambda \int_D z_n\varphi + \|u_n\|_2^{q-1} \int_D b(x) z_n^q \varphi + \|u_n\|_2^{p-1} \int_D b(x)\varphi z_n^p\right).$$

Since $z_n \to z$ in $L^2(\Omega)$, we deduce that

$$\int_D b(x) z_n^q \varphi \to \int_D b(x) z^q \varphi < 0,$$

whence using (15)

$$-\int_D z_n \Delta\varphi \to -\infty,$$

a contradiction.

For any $\varphi \in H_0^1(B_0)$, prolongating this function by zero, and passing to the limit in (14), we get that

$$\int_{B_0} \nabla z \cdot \nabla\varphi = t^* \lambda \int_{B_0} z\varphi,$$

where $t_n \to t^* \in [0,1]$, and then

$$t^*\lambda = \lambda_1^{B_0}.$$

Hence, $\lambda \geq \lambda_1^{B_0}$, a contradiction.

Take $\epsilon \in (0, \delta]$, we have

$$\begin{aligned} i(K_\lambda, 0) &= \deg(K_\lambda, B_\epsilon) = \deg(I - \mathcal{H}_1(1,\cdot), B_\epsilon) \\ &= \deg(I - \mathcal{H}_1(0,\cdot), B_\epsilon) = \deg(I, B_\epsilon) = 1, \end{aligned}$$

where I denotes the identity map. The proof is complete. $\square$

LEMMA 6.2. *If $\lambda > \lambda_1^{B_0}$, then $i(K_\lambda, 0) = 0$.*

Proof. Let us take a positive and regular function $\varphi > 0$ in Ω. Let us define the map $\mathcal{H}_2 : [0,1] \times C_0(\overline{\Omega}) \to C_0(\overline{\Omega})$ by

$$\mathcal{H}_2(t,u) = (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p) + t\varphi).$$

We show now that $\mathcal{H}_2$ is an admisible homotopy, for which it is sufficient to prove that there exists $\gamma > 0$ such that

$$u \neq \mathcal{H}_2(t,u) \quad \forall u \in \overline{B}_\gamma, u \neq 0 \text{ and } t \in [0,1].$$

Assume that there exist $u_n \in C_0(\overline{\Omega})\backslash\{0\}$ with $\|u_n\|_\infty \to 0$ and $t_n \in [0,1]$, such that

$$u_n = \mathcal{H}_2(t_n, u_n).$$

This is,

$$-\Delta u_n = \lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p) + t_n\varphi \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Again, it can be shown that $u_n \geq 0$ in Ω. Let us define

$$z_n = \frac{u_n}{\|u_n\|_2}.$$

Hence, z_n verifies that

$$-\Delta z_n = \lambda z_n + b(x)(\|u_n\|_2^{q-1} z_n^q + \|u_n\|_2^{p-1} z_n^p) + \frac{t_n}{\|u_n\|_2}\varphi \quad \text{in } \Omega, \quad z_n = 0 \quad \text{on } \partial\Omega. \tag{16}$$

Now, on multiplying (16) by $\psi \in C_c^\infty(B_0)$, the formula of integration by parts gives

$$\frac{t_n}{\|u_n\|_2}\int_{B_0} \varphi\psi = -\lambda \int_{B_0} z_n\psi - \int_{B_0} z_n \Delta\psi.$$

Since $\|z_n\|_2 = 1$ it follows that

$$\frac{t_n}{\|u_n\|_2} \leq C \tag{17}$$

and then, for a subsequence, $t_n/\|u_n\|_2 \to t^* \geq 0$.

Since $\|z_n\|_2 = 1$ and $b \leq 0$, it follows from (16) and (17) that

$$\|z_n\|_{H_0^1} \leq C.$$

Arguing as in Lemma 6.1 we deduce that $z = 0$ in $\Omega \setminus B_0$. Moreover, passing to the limit in B_0 we conclude that

$$-\Delta z = \lambda z + t^*\varphi \quad \text{in } B_0, \quad z = 0 \quad \text{on } \partial B_0.$$

Since $t^* \geq 0$, we get that $\lambda \leq \lambda_1^{B_0}$ and a contradiction arises immediately.

Take $\epsilon \in (0, \gamma]$, we have that

$$\begin{aligned} i(K_\lambda, 0) &= \deg(K_\lambda, B_\epsilon) = \deg(I - \mathcal{H}_2(0,\cdot), B_\epsilon) \\ &= \deg(I - \mathcal{H}_2(1,\cdot), B_\epsilon) = 0. \end{aligned}$$

This last equality holds because we have proved that the equation

$$-\Delta u = \lambda u + b(x)(u^q + u^p) + \varphi$$

has not solution in $\overline{B}_\epsilon$.

□

6.2. Bifurcation from infinity

LEMMA 6.3. *Assume that* $\lambda < \lambda_1^{B_0}$. *Then, there exists* $R > 0$ *such that for any* $u \in C_0(\overline{\Omega})$ *with* $\|u\|_\infty \geq R$ *and for any* $t \in [0,1]$,

$$u \neq (-\Delta)^{-1}(t(\lambda u^+ + b(x)((u^+)^q + (u^+)^p))).$$

Proof. Assume by contradiction that there exist two sequences $\|u_n\|_\infty \to \infty$ and $t_n \in [0,1]$ such that

$$-\Delta u_n = t_n(\lambda u_n^+ + b(x)((u_n^+)^q + (u_n^+)^p)) \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega.$$

Using elliptic regularity results, it is not hard to show that $\|u_n\|_2 \to \infty$. Now, the proof follows exactly as in Lemma 6.1, arguing now with $t_n\|u_n\|_\infty^{p-1}$ instead of $t_n\|u_n\|_\infty^{q-1}$. □

LEMMA 6.4. *Assume that* $\lambda > \lambda_1^{B_0}$ *and let* $\varphi \in C_0^1(\overline{\Omega})$, $\varphi > 0$ *in* Ω. *Then, there exists* $R > 0$ *such that for any* $u \in C_0(\overline{\Omega})$ *with* $\|u\|_\infty \geq R$ *and for any* $t \in [0,1]$,

$$u \neq (-\Delta)^{-1}(\lambda u^+ + b(x)((u^+)^q + (u^+)^p) + t\varphi).$$

Proof. In this case, the proof is rather similar to the proof of Lemma 6.2. □

Proof of Theorem 1.2. From Lemmas 6.1 and Lemma 6.2, it follows the existence of a continuum $\mathcal{C}_0$ of nonnegative and nontrivial solution of (2) emanating from the trivial solution at $\lambda = \lambda_1^{B_0}$. Moreover, it can be shown that this is the unique point of bifurcation form zero, and hence we can conclude that $\mathcal{C}_0$ is unbounded.

For the existence of $\mathcal{C}_\infty$ we perform the change of variable $z = u/\|u\|_\infty^2$ ($u \neq 0$). See, for instance [26] and [6]. Now, thanks to Lemmas 6.3 and 6.4, the existence of $\mathcal{C}_\infty$ can be deduced. We omit the details. □

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Stability and periodicity of solutions to the Oldroyd-B model on exterior domains

Matthias Hieber and Thieu Huy Nguyen

Dedicated to our good friend Julian Lopez-Gomez on the occasion of his 60^{th}-Birthday

Abstract. *Consider the Oldroyd-B system on exterior domains with nonzero external forces f. It is shown that this system admits under smallness assumptions on f a bounded, global solution (u, τ), which is stable in the sense that any other global solution to this system starting in a sufficiently small neighborhood of $(u(0), \tau(0))$ is tending to (u, τ). In addition, if the outer force is T-periodic and small enough, the Oldroyd-B system admits a T-periodic solution. Note that no smallness condition on the coupling coefficient is assumed.*

Keywords: Oldroyd-B fluids, periodic solutions, exterior domains, asympotic stability.
MS Classification 2010: 76A10, 35B10, 76D03, 35Q3.

1. Introduction

In this note we consider stability and periodicity questions related to viscoelastic fluids of Oldroyd-B type with non vanishing external forces on exterior domains. This type of fluids are described by the following set of equations

$$\begin{cases} \mathrm{Re}(u_t+(u\cdot\nabla)u)-(1-\alpha)\Delta u+\nabla p &= \operatorname{div}\tau+f \quad \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u &= 0 \quad \text{in } \Omega\times(0,\infty),\\ \mathrm{We}(\tau_t+(u\cdot\nabla)\tau+g_a(\tau,\nabla u))+\tau &= 2\alpha D(u) \quad \text{in } \Omega\times(0,\infty),\\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty),\\ u(0) &= u_0 \quad \text{in } \Omega,\\ \tau(0) &= \tau_0 \quad \text{in } \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^3$ denotes a domain with smooth boundary $\partial\Omega$, u the velocity of the fluid, and the tensor τ represents the elastic part of the stress tensor. Furthermore, Re and We denote the Reynolds and Weissenberg number of the fluid, respectively. The term g_a is given by

$$g_a(\tau, \nabla u) := \tau W(u) - W(u)\tau - a(D(u)\tau + \tau D(u)) \tag{2}$$

for some $a \in [-1, 1]$ and $D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and $W(u) = \frac{1}{2}(\nabla u - (\nabla u)^T)$ denote the deformation and vorticity tensors, respectively. The constant $\alpha \in (0, 1)$ is the coupling coefficient between the two equations and represents in particular the strengthness of the coupling between the parabolic fluid type equation for u and the hyperbolic transport type equation for τ.

This set of equations has been introduced first by J.G. Oldroyd [24] and the analysis of this set of equations for viscoelastic fluids gained a lot of attention since then.

If $\Omega \subset \mathbb{R}^3$ is a *bounded domain* with smooth boundary, Guillopé and Saut [13] proved the existence and uniqueness and exponential stability of small solutions to (1) in the case of small coupling parameters α. They further proved the existence of periodic and stationary solutions to (1) by adapting Serrin's method to this situation. For extensions of this results to the L^p-setting we refer to the work of Fernandez-Cara et al [9]. Molinet and Talhouk [23] extended the result of Guillopé and Saut [13] to the case of non small coupling parameters $\alpha \in (0, 1)$. For results concerning the critical L^p-framework, we refer to the work of Zi, Fang, and Zhang [25].

For the case $\Omega = \mathbb{R}^3$, Lions and Masmoudi [21] proved the existence of global weak solutions provided $a = 0$. For further results in this direction we refer to the works [4] and [19]. Blow-up criteria for Oldroyd-B type fluids were developed by Kupfermann, Mangoubi and Titi [18] in the case where the Navier-Stokes equation is replaced by the stationary Stokes system and in the general case by Lei, Masmoudi and Zhou [20] as well as by Feng, Zhu and Zi [8]. For global regularity results in the two dimensional setting, we refer to the work of Constantin and Kriegl [5].

If $\Omega \subset \mathbb{R}^3$ is an *exterior domain*, existence and uniqueness of solutions to (1) for small data were proved by Hieber, Naito and Shibata in [14] for small coupling parameter α and by Fang, Hieber and Zi in [7] for any $\alpha \in (0, 1)$. For optimal decay rates for the case $\Omega = \mathbb{R}^3$, see [16].

For recent results on ill-posedness of these equations within the L^∞-setting we refer to the work of Elgindi and Masmoudi [6].

In this article we are interested in the global existence, stability and periodicity of solutions to the Oldroyd-B equations in *exterior domains* in the presence of external forces f of the form $f = \operatorname{div} F$ for certain F. One might think of applying the method developed in [11] to the given situation, however, it is unclear whether the Oldroyd semigroup constructed in [10] satisfies suitable decay estimates.

Note that the methods for obtaining results on stability, bifurcation and periodicity of solutions for viscoelastic fluids are quite different from the ones often used in the theory of second order parabolic equations, where comparison principles allow to develop a very rich and powerful theory. For beautiful results in this direction, we refer to the work of Julian Lopez-Gomez and mention here only his book [22] as well as the recent articles [2] and [1].

2. Existence of Bounded Solutions

We consider the Oldroyd-B equation with an external force f of the form $f = \operatorname{div} F$

$$\begin{cases} u_t+(u\cdot\nabla)u-(1-\alpha)\Delta u+\nabla p &= \operatorname{div}\tau+\operatorname{div}F \quad \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u &= 0 \quad \text{in } \Omega\times(0,\infty),\\ \tau_t+(u\cdot\nabla)\tau+g_a(\tau,\nabla u)+\tau &= 2\alpha D(u) \quad \text{in } \Omega\times(0,\infty),\\ u &= 0 \quad \text{on } \partial\Omega\times(0,\infty),\\ u(0) &= u_0 \quad \text{in } \Omega,\\ \tau(0) &= \tau_0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

where $\Omega\subset\mathbb{R}^3$ is an exterior domain with boundary of class C^3. Let $A := -\mathbb{P}\Delta$ be the Stokes operator in the solenoidal space $L^2_\sigma(\Omega)$ with domain $D(A) = H^2(\Omega)\cap H^1_0(\Omega)\cap L^2_\sigma(\Omega)$ and set $V := H^1_0(\Omega)\cap L^2_\sigma(\Omega)$. For fixed $T>0$ we put

$$\begin{aligned} E_1(T) &:= L^2(0,T;H^3(\Omega))\cap L^\infty(0,T;D(A)),\\ E_2(T) &:= L^2(0,T;V)\cap L^\infty(0,T;L^2_\sigma(\Omega)),\\ G_1(T) &:= L^\infty(0,T;H^2(\Omega)),\\ G_2(T) &:= L^\infty(0,T;H^1(\Omega)). \end{aligned}$$

Our first result concerns the local existence of a unique, strong solution to (3) under certain conditions on F.

Proposition 2.1 (Local Existence). *Let Ω be an exterior domain with C^3-boundary and let $u_0\in D(A)$ and $\tau_0\in H^2(\Omega)$. Then there exist $T_*>0$ and $M>0$ such that for $F\in G_1(T_*)$ and $F'\in G_2(T_*)$ with $\|F\|_{G_1(T_*)}+\|F'\|_{G_2(T_*)}<M$, equation (3) has a unique solutions (u,p,τ) on $(0,T_*)$ with*

$$\begin{aligned} u &\in E_1(T_*)\cap C([0,T_*],D(A)),\\ u' &\in E_2\cap C([0,T_*],D(A))),\\ p &\in L^2(0,T_*;H^2_{loc}(\Omega)) \text{ with } \nabla p\in L^2(0,T_*;H^1(\Omega)),\\ \tau &\in C([0,T_*];H^2(\Omega)) \text{ with } \tau'\in C([0,T_*];H^1(\Omega)). \end{aligned}$$

In order to prove Proposition 2.1 we make use of the following version of Banach's fixed point theorem, see [17].

LEMMA 2.2 ([17]). *Let X be either reflexive Banach space or have a separable pre-dual. Let K be a convex, closed and bounded subset of X and assume that X is continuously embedded into a Banach space Y. Let $\Phi : X \to X$ maps K into K and assume there is $0 < q < 1$ such that*

$$\|\Phi(x) - \Phi(y)\|_Y \leqslant q\|x - y\|_Y \quad \textit{for all } x, y \in K.$$

Then there exists a unique fixed point of Φ in K.

Proof of Proposition 2.1. The proof follows the strategy described in [7, Prop. 3.1], however with a forcing term of the form $f = \operatorname{div} F$. For the reader's convenience we give here a short outline of the proof. For real numbers $B_1, B_2 > 0$ we set

$$\begin{aligned} K(T) :=\{&(v,\theta) \in E_1(T) \times G_1(T) : v' \in E_2(T), \theta' \in G_2(T), v(0) = u_0, \theta(0) = \tau_0 \\ &\text{and } \|v\|^2_{E_1(T)} + \|v'\|^2_{E_2(T)} \leqslant B_1, \|\theta\|_{G_1(T)} \leqslant B_1, \|\theta'\|_{G_2(T)} \leqslant B_2\} \end{aligned}$$

Then, for $(v, \theta) \in K(T)$ we define the mapping

$$\Phi(v, \theta) := (u, \tau),$$

where (u, τ) is the unique solution of the linearized problem of (3), i.e.,

$$\left\{ \begin{array}{rll} u_t + (1-\alpha)Au & = -\mathbb{P}\mathrm{div}\,(v \otimes v) + \mathbb{P}\mathrm{div}\,\theta + \mathbb{P}\mathrm{div}\,F & \text{in } \Omega \times (0,\infty), \\ \tau_t + (u \cdot \nabla)\tau + \tau & = 2\alpha D(v) - g_a(\tau, \nabla v)) & \text{in } \Omega \times (0,\infty), \\ u & = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(0) & = u_0 & \text{in } \Omega, \\ \tau(0) & = \tau_0 & \text{in } \Omega. \end{array} \right. \tag{4}$$

Regularity results for the Stokes and the transport equation imply the existence of a constant $C > 0$ such that

$$\begin{aligned} &\|u\|^2_{L^2(H^3)\cap L^\infty(D(A))} + \|u'\|^2_{L^2(V)\cap L^\infty(L^\infty_\sigma)} \\ &\quad \leqslant C\big[\|u_0\|^2_{H^2} + \|v(0)\|^2_{H^2} + \|v\|^2_{L^2(H^3)} + \|v'\|_{L^\infty(H^1)}\|v\|_{L^2(H^3)} \\ &\qquad + \|\theta + F\|_{L^\infty(H^2)} + \|\theta' + F'\|_{L^\infty(H^1)}\big] \end{aligned}$$

and

$$\|\tau\|_{L^\infty(H^2)} + \|\tau'\|_{L^\infty(H^1)} \leqslant [2 + C\|v\|_{L^\infty(H^2)}]\Big(\|\tau_{H^2} + \frac{2\alpha}{C}\Big) \exp C\|v\|_{L^1(H^3)}.$$

Hence, choosing B_1, B_2 and T_1 appropriately, we see that Φ maps $K(T_1)$ into $K(T_1)$.

Next, similarly as in [7], for two solutions (u_i, τ_i) corresponding to given (v_i, θ_i) for $i = 1, 2$ we verify that

$$\begin{aligned}\|u_1 - u_2\|^2_{L^\infty(L^2)} + \|\tau_1 - \tau_2\|^2_{L^\infty(L^2)} + \int_0^T (\|\nabla u_1 - \nabla u_2\|^2_{L^2} + \|\tau_1 - \tau_2\|^2_{L^2})dt \\ \leqslant \frac{1}{4}\Big(\|v_1 - v_2\|^2_{L^\infty(L^2)} + \|\theta_1 - \theta_2\|^2_{L^\infty(L^2)} \\ + \int_0^T (\|\nabla v_1 - \nabla v_2\|^2_{L^2} + \|\theta_1 - \theta_2\|^2_{L^2})dt\Big)\end{aligned}$$

provided $T \leqslant T_* := \min\big\{T_1, \frac{\delta}{1+2B_1^2}, \frac{1}{B_1}, \frac{1-\alpha}{4C(1+2B_1)(1+2C\exp(2C))}\big\}$ with $\delta := \frac{1-\alpha}{4+8C\exp(2C)}$. Therefore, the mapping Φ is a contraction from

$$Y(T_*) := \big\{(v, \theta) \in L^\infty(0, T; L^2(\Omega))^2, \nabla v \in L^\infty(0, T; L^2(\Omega))\big\}$$

into itself and the assertion of Proposition 2.1 follows from Lemma 2.2. □

Our global existence result to (3) in the presence of outer forces f of the form $f = \operatorname{div} F$ reads as follows.

THEOREM 2.3 (Global Existence). *Let $F \in L^\infty(0, \infty; H^2(\Omega))$ such that $F' \in L^\infty(0, \infty; H^1(\Omega))$. Then there exists $\varepsilon_0 > 0$ such that if*

$$\|u_0\|_{D(A)} + \|\tau_0\|_{H^2} < \varepsilon_0 \text{ and}$$
$$\max\{\|F\|_{L^\infty(H^2)}, \|F'\|_{L^\infty(H^1)}\} < \min\{\varepsilon_0, 1 - \alpha\},$$

then equation (3) admits a unique, global strong solution (u, p, τ) on $(0, \infty)$ satisfying

$$\begin{aligned} u &\in C_b([0, \infty); D(A)) \text{ with } \nabla u \in L^2([0, \infty); H^2(\Omega)) \text{ and} \\ u' &\in L^2([0, \infty); H^1_0(\Omega) \cap L^2_\sigma(\Omega)), \\ \nabla p &\in L^2([0, \infty), H^1(\Omega)) \cap L^\infty([0, \infty), H^1(\Omega)), \\ \tau &\in C_b([0, \infty); H^2(\Omega)) \cap L^2([0, \infty); H^2(\Omega)) \text{ and } \tau' \in L^2([0, \infty); L^2(\Omega)).\end{aligned}$$

Proof. The proof of Theorem 2.3 follows essentially the lines of the proof of Theorem 1.1 in [7], but we need to take into account the contributins due to the external force $\operatorname{div} F$. For the convenience of the reader, we sketch the main ideas of the proof here. Let (u, τ) be the local solution of (3) constructed in Proposition 2.1. Our aim is to to derive a priori estimates for u, τ, u' and τ'. Since the norms of F are assumed to be small, our strategy is to absorb these terms into the left-hand sides of the equations thanks to energy-type estimates.

Since the equation $(3)_2$ for τ does not contain external forces, estimates (4.1) and (4.2) of [7] yield

$$\frac{d}{dt}\|\tau\|_{H^2}^2 + \|\tau\|_{H^2}^2 \leqslant C\alpha^2\|\nabla u\|_{H^2}^2 + \frac{C}{\alpha^2}\|\tau\|_{H^2}^4.$$

Applying the Helmholtz projection $\mathbb{P}$ to the second line of (3) gives

$$u_t + \mathbb{P}(u\cdot\nabla)u + (1-\alpha)Au = \mathbb{P}\mathrm{div}\,\tau + \mathbb{P}\mathrm{div}\,F. \tag{5}$$

Similarly as in [7] we obtain

$$\begin{aligned}\|\nabla u\|_{H^2}^2 \leqslant C\Big(&\|Au\|_{H^2}^2 + \|\nabla u\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|\nabla u_t\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|\nabla\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2\\ &+ \frac{1}{(1-\alpha)^2}\|\nabla\mathbb{P}\mathrm{div}\,F\|_{L^2}^2 + \frac{1}{(1-\alpha)^2}\|Au\|_{L^2}^4 + \frac{1}{(1-\alpha)^2}\|\nabla u\|_{L^2}^4\Big).\end{aligned}$$

Next, taking the inner product of (5) with u yields

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + (1-\alpha)\|\nabla u\|_{L^2}^2 = (\mathrm{div}\,\tau \mid u) + (\mathrm{div}\,F \mid u).$$

Similarly as in [7] we arrive at

$$\begin{aligned}\frac{d}{dt}\big(\|u\|_{L^2}^2 + \frac{1}{2}\|\tau\|_{L^2}^2\big) + (1-\alpha-\|F\|_{L^2})\|\nabla u\|_{L^2}^2 + \frac{1}{2\alpha}\|\tau\|_{L^2}^2&\\ \leqslant \frac{C}{(1-\alpha)\alpha^2}\|\tau\|_{H^2}^4 + \frac{1}{2\alpha}\|F\|_{L^2}&\end{aligned}$$

and obtain the differential inequality

$$\frac{d}{dt}U(t) + V(t) \leqslant CH(t)V(t),$$

where

$$\begin{aligned}U(t) :=&(1-\alpha)(\kappa_4 C_0+1)(\|\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2 + \|\operatorname{curl}\mathrm{div}\,\tau\|_{L^2}^2) + \frac{\kappa_6+1}{1-\alpha}\|u\|_{L^2}^2\\ &+ \frac{\kappa_6+1}{2\alpha(1-\alpha)}\|\tau\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \frac{1}{2}\|F\|_{L^2}\\ &+ \frac{(\kappa_1+1)(3-\alpha-\|F\|_{L^2}^2)}{1-\alpha}\|\nabla u\|_{L^2}^2\\ &+ \frac{\kappa_5+1}{1-\alpha}\|u_t\|_{L^2}^2 + \frac{\kappa_5+1}{2\alpha(1-\alpha)}\|\tau_t\|_{L^2}^2,\end{aligned}$$

$$
\begin{aligned}
V(t) :=& \frac{\kappa_1+1}{1-\alpha}\|u_t\|_{L^2}^2 + \|Au\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla u_t\|_{L^2}^2 \\
&+ \frac{\kappa_5+1}{\alpha(1-\alpha)}\|\tau_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\tau\|_{L^2}^2 + \|\mathbb{P}\mathrm{div}\,\tau\|_{L^2}^2 \\
&+ \|\operatorname{curl}\operatorname{div}\tau\|_{L^2}^2 + \|F\|_{L^2}^2, \\
H(t) :=& \|u_t\|_{L^2}^2 + \|Au\|_{L^2}^2 + \|\tau\|_{H^2}^2 + \|\tau_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4.
\end{aligned}
$$

Following (4.28) in [7], there is a constant $M_1 = M_1(\alpha) > 0$ such that

$$
H(t) \leqslant M_1\big(U(t) + U(t)^2 + U(t)^3\big), \quad t \geq 0. \tag{6}
$$

Arguing as in (4.28) in [7] we see that for $\delta_0 > 0$ with $\delta + \delta^2 + \delta^3 < \frac{1}{2CM_1}$ and for $\epsilon_0 >$ such that $C(\epsilon_0^4 + \epsilon_0^4) < \delta_0$ we have

$$
\sup_{0\leqslant t\leqslant T_*} U(t) + \frac{1}{2}\int_0^{T_*} V(s)ds \leqslant \delta_0.
$$

Hence,

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant T_*}\big(\|u(t)\|_{D(A)}^2 + \|u'(t)\|_{L^2}^2 + \|\tau(t)\|_{H^2}^2 + \|\tau'(t)\|_{L^2}^2\big) \\
&\quad + \frac{1}{2}\int_0^{T_*}\big(\|\nabla u(t)\|_{H^2}^2 + \|\nabla u'(t)\|_{L^2}^2 + \|\tau(t)\|_{H^2}^2 + \|\tau'(t)\|_{L^2}^2\big)dt \leqslant C,
\end{aligned}
$$

and the local solution (u, p, τ) can be extended to all $t > 0$. $\square$

3. Stability of the Oldroyd-B Equations with Small External Forces

In this section we consider the stability of bounded solutions to the system (3). Applying the Helmholtz projection to (3) we obtain

$$
\begin{cases}
u_t + (u\cdot\nabla)u + (1-\alpha)Au &= \mathbb{P}\mathrm{div}\,\tau + \mathbb{P}\mathrm{div}\,F, \\
\tau_t + (u\cdot\nabla)\tau + g_a(\tau,\nabla u) + \tau &= 2\alpha D(u), \\
u(0) &= u_0, \\
\tau(0) &= \tau_0,
\end{cases} \tag{7}
$$

In the following we will prove that the bounded global solution (u, τ) to (7) obtained in Theorem 2.3 is stable in the sense that any other global solution to (3) starting in a sufficiently small neighborhood of $(u(0), \tau(0))$ is tending to (u, τ). To this end, we introduce the spaces

$$
W_1 := H^3(\Omega) \cap H_0^1(\Omega) \cap L_\sigma^2(\Omega), \qquad W_2 := H^2(\Omega)
$$

and set $W := W_1 \times W_2$. Moreover, for $r > 0$ and $(x_1, x_2) \in W$ we set

$$\mathcal{B}(x_1, x_2, r) := \{(y_1, y_2) \in W : \|(y_1, y_2) - (x_1, x_2)\|_W \leqslant r\}.$$

The following stability result is the first main result of this article.

Theorem 3.1. *There exist constants $\delta_0, A, R > 0$ such that for a solution (u, τ) to equation (7) with $\|(u(0), \tau(0))\|_W \leqslant \delta_0$ and any solution (v, μ) to equation (7) with $\alpha \leqslant A$ and initial data $(v(0), \mu(0)) \in \mathcal{B}(u(0), \tau(0), r)$ for $r \leqslant R$, the equality*

$$\lim_{t\to\infty} \|v(t) - u(t)\|_{L^2} = \lim_{t\to\infty} \|\mu(t) - \tau(t)\|_{L^2} = 0$$

holds.

In order to prove Theorem 3.1 we make use of Hölder's and Young's inequality in weak L^p-spaces. For proofs, see e.g., Section 1 of [12]. More specifically, for $1 < p < \infty$ we denote by $L^p_w := L^p_w(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ with norm

$$\|f\|_{p,w} = \sup_{0<|E|<\infty} |E|^{-1+\frac{1}{p}} \int_E |f| ds < \infty, \tag{8}$$

where $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}$.

Lemma 3.2 ([12], Section 1). *Let $p \in [1, \infty)$, $q, r \in (1, \infty)$. Then the following assertins hold.*

a) If $f \in L^p_w, g \in L^q_w$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then $fg \in L^r_w$ and

$$\|fg\|_{r,w} \leqslant C\|f\|_{p,w}\|g\|_{q,w}$$

for some constant C depending only on p and q.

b) If $f \in L^p_w, g \in L^q_w$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $f * g \in L^r_w$ and there is a constant C, depending only on p and q, such that

$$\|f * g\|_{r,w} \leqslant C\|f\|_{p,w}\|g\|_{q,w}.$$

c) If $f \in L^p_w, g \in L^1$, then $f * g \in L^p_w$ and there is a constant C, depending only on p, such that

$$\|f * g\|_{p,w} \leqslant C\|f\|_{p,w}\|g\|_{L_1}.$$

Proof of Theorem 3.1. The strategy of our proof follows to a certain extent the one of Theorem 3.2 in [10]. In the present case, we need to deal, however, with two non trivial solutions to (7).

Let (u,τ) and (v,μ) be two solutions to (7) as in Theorem 3.1. Setting $\tilde{u} := v - u$ and $\tilde{\tau} := \mu - \tau$, we obtain from (7)

$$\left\{\begin{array}{rl} \tilde{u}_t + (\tilde{u}\cdot\nabla)\tilde{u} + (u\cdot\nabla)\tilde{u} + (\tilde{u}\cdot\nabla)u + (1-\alpha)A\tilde{u} & = \mathbb{P}\mathrm{div}\tilde{\tau}, \\ \tilde{\tau}_t + (\tilde{u}\cdot\nabla)\tilde{\tau} + (\tilde{u}\cdot\nabla)\tau + (u\cdot\nabla)\tilde{\tau} + g_a(\tilde{\tau},\nabla\tilde{u}) & \\ +g_a(\tilde{\tau},\nabla u) + g_a(\tau,\nabla\tilde{u}) + \tilde{\tau} & = 2\alpha D(\tilde{u}), \\ \tilde{u}(0) & = v(0) - u(0), \\ \tilde{\tau}(0) & = \mu(0) - \tau(0). \end{array}\right. \tag{9}$$

We first estimate $\tilde{\tau}$ by the second equation of system (9). Denote by $\|\cdot\|$ the norm of $L^2(\Omega)$. Taking the scalar product in the second equation of (9) with $\tilde{\tau}$ we obtain

$$\begin{aligned} &\frac{d}{dt}\|\tilde{\tau}\|^2 + 2\left\langle(\tilde{u}\cdot\nabla)\tau,\tilde{\tau}\right\rangle + 2\left\langle g_a(\tilde{\tau},\nabla\tilde{u}),\tilde{\tau}\right\rangle + 2\left\langle g_a(\tilde{\tau},\nabla u),\tilde{\tau}\right\rangle \\ &\quad +2\left\langle g_a(\tau,\nabla\tilde{u}),\tilde{\tau}\right\rangle + 2\|\tilde{\tau}\|^2 = 4\alpha\left\langle D(\tilde{u}),\tilde{\tau}\right\rangle, \ t\geq 0. \end{aligned}$$

Integrating we obtain by Gronwall's lemma

$$\begin{aligned} \|\tilde{\tau}(t)\|^2 \leqslant{}& e^{-2t}\|\tilde{\tau}(0)\|^2 + 2\int_0^t e^{-2(t-s)}\Big(|\langle g_a(\tilde{\tau}(s),\nabla\tilde{u}),\tilde{\tau}(s)\rangle| \\ &+|\langle g_a(\tilde{\tau}(s),\nabla u(s)),\tilde{\tau}(s)\rangle| + |\langle g_a(\tau(s),\nabla\tilde{u}(s)),\tilde{\tau}(s)\rangle| \\ &+|\langle(\tilde{u}\cdot\nabla)\tau,\tilde{\tau}(s)\rangle| + 2\alpha\,|\langle D(\tilde{u}(s)),\tilde{\tau}(s)\rangle|\Big)ds, \quad t\geq 0. \end{aligned}$$

For $\|u\|_{W_1}\leqslant r$ we thus obtain

$$\begin{aligned} \|\tilde{\tau}(t)\|^2 \leqslant{}& e^{-2t}\|\tilde{\tau}(0)\|^2 + 8rC(|a|+1)\int_0^t e^{-2(t-s)}\|\tilde{\tau}(s)\|^2 ds \\ &+ Cr(4|a|+5)\int_0^t e^{-2(t-s)}\|\tilde{\tau}(s)\|\|\tau(s)\|ds \\ &+ 4\alpha\int_0^t e^{-2(t-s)}\|D(u(s))\|\|\tilde{\tau}(s)\|ds \\ \leqslant{}& e^{-2t}\|\tilde{\tau}(0)\|^2 + 8rC(|a|+1)\int_0^t e^{-2(t-s)}\|\tilde{\tau}(s)\|^2 ds \\ &+ Cr(4|a|+5)\int_0^t e^{-2(t-s)}\big(\tfrac{1}{2}\|\tilde{\tau}(s)\|^2 + \tfrac{1}{2}\|\tau(s)\|^2\big)ds \\ &+ 2\alpha\int_0^t e^{-2(t-s)}(\|D(u(s))\|^2 + \|\tilde{\tau}(s)\|^2)ds, \qquad t\geq 0, \end{aligned}$$

where C denotes the constant in Sobolev's embedding. Therefore,

$$\|\tilde\tau(t)\|^2 \leqslant e^{-2t}\|\tilde\tau(0)\|^2 + \frac{4\alpha+8rC(6|a|+7)}{2}\int_0^t e^{-2(t-s)}\|\tilde\tau(s)\|^2 ds \\ + \int_0^t e^{-2(t-s)}\Big(2\alpha\|D(u(s))\|^2 + \frac{Cr(4|a|+5)}{2}\|\tau(s)\|^2\Big)ds, \quad t\ge 0.$$

Choosing r so small that $K := \frac{4-4\alpha-8rC(6|a|+7)}{2} > 0$, Gronwall's inequality yields for $t \ge 0$

$$\|\tilde\tau(t)\|^2 \leqslant e^{-Kt}\|\tilde\tau(0)\|^2 \\ + \int_0^t e^{-K(t-\xi)}\Big(2\alpha\|D(u(\xi))\|^2 + \frac{Cr(4|a|+5)}{2}\|\tau(s)\|^2\Big)d\xi. \quad (10)$$

In a second step we take the inner product of the first equation in (9) with $\tilde u$ and obtain

$$\frac{1}{2}\frac{d}{dt}\|\tilde u(t)\|^2 + (1-\alpha)\|\nabla\tilde u(t)\|^2 = \langle \mathbb{P}\mathrm{div}\,\tau(t), u(t)\rangle - \langle(\tilde u\cdot\nabla)u,\tilde u\rangle \\ = \langle \mathbb{P}\mathrm{div}\,\tau(t), u(t)\rangle + \langle(\tilde u\cdot\nabla)\tilde u, u\rangle\,.$$

Integrating from s to t yields

$$\|\tilde u(t)\|^2 + 2(1-\alpha)\int_s^t \|\nabla\tilde u(t)\|^2 dt \\ \leqslant \|\tilde u(s)\|^2 + 2\int_s^t \|\tilde\tau(t)\|\|\nabla\tilde u(\xi)\|d\xi + \int_s^t \|(\tilde u(\xi)\cdot\nabla)\tilde u(\xi)\|\|u(\xi)\|d\xi \\ \leqslant \|\tilde u(s)\|^2 + 2\int_s^t \|\tilde\tau(t)\|\|\nabla\tilde u(\xi)\|d\xi \\ + 2\tilde C\int_s^t \|\tilde u(\xi)\|_{L^6}\|\nabla\tilde u(\xi)\|_{L^3}\|u(\xi)\|d\xi \\ \leqslant \|\tilde u(s)\|^2 + 2\int_s^t \|\tilde\tau(t)\|\|\nabla\tilde u(\xi)\|d\xi \\ + 2C\int_s^t \|\nabla\tilde u(\xi)\|\|\nabla\tilde u(\xi)\|^{1/2}\|\nabla^2\tilde u\|^{1/2}\|u(\xi)\|d\xi \\ \leqslant \|\tilde u(s)\|^2 + 2\int_s^t \|\tilde\tau(t)\|\|\nabla\tilde u(\xi)\|d\xi + 2C\|u\|_{C_b}\int_s^t \|\nabla\tilde u(\xi)\|^2_{H^1}d\xi \\ \leqslant \|\tilde u(s)\|^2 + \int_s^t \|\tilde\tau(t)\|^2 d\tau + (1+2C\|u\|_{C_b})\int_s^t \|\nabla\tilde u(\xi)\|^2_{H^1}d\xi$$

where $\tilde{C}$ and C are the constants arising in Gagliardo-Nirenberg and Sobolev inequalities and $\|u\|_{C_b} := \|u\|_{C_b([0,\infty),L^2)}$. Summing up, we obtain

$$\|\tilde{u}(t)\| \leqslant \|\tilde{u}(s)\| + \Big(\int_s^t \|\tilde{\tau}(\xi)\|^2 d\xi\Big)^{1/2} + (1+2C\|u\|_{C_b})^{1/2}\Big(\int_s^t \nabla\tilde{u}(\xi)\|^2_{H^1} d\xi\Big)^{1/2},$$

and integrating with respect to $s \in (0,t)$ yields

$$\|\tilde{u}(t)\| \leqslant \frac{1}{t}\int_0^t \|\tilde{u}(s)\| ds + \Big(\frac{2}{t}\Big)^{1/2} \|\tilde{\tau}\|_{L^2(0,\infty;H^2)} + \Big(\frac{1+2C\|u\|_{C_b}}{t}\Big)^{1/2} \|\nabla\tilde{u}\|_{L^2(0,\infty;H^2)}. \tag{11}$$

Theorem 2.3 yields $\|\tilde{\tau}\|_{L^2(0,\infty;H^2)} < \infty$ as well as $\|\nabla\tilde{u}\|_{L^2(0,\infty;H^2)} < \infty$. Hence, the second and third term on the right-hand side of (11) tend to 0 as $t \to \infty$.

We now turn our attention to the first term on the right hand side of (11) and aim to show that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\tilde{u}(s)\| ds = 0. \tag{12}$$

To this end, we multiply the first line of equation (9) with $\phi \in C(\mathbb{R}_+, H_0^1(\Omega) \cap L^2_\sigma(\Omega)) \cap C^1(\mathbb{R}_+, L^2_\sigma(\Omega))$ and integrate from s to t to obtain

$$\begin{aligned}\langle \tilde{u}(t), \phi(t)\rangle + \int_s^t [(1-\alpha)\,\langle \nabla\tilde{u}, \nabla\phi\rangle + \langle(\tilde{u}\cdot\nabla)\tilde{u}, \phi\rangle + \langle(\tilde{u}\cdot\nabla)u, \phi\rangle \\ + \langle(u\cdot\nabla)\tilde{u}, \phi\rangle] d\xi \\ = \langle \tilde{u}(s), \phi(s)\rangle + \int_s^t [\langle \tilde{u}, \phi'\rangle + \langle \mathbb{P}\mathrm{div}\,\tilde{\tau}, \phi\rangle] d\xi.\end{aligned} \tag{13}$$

Substituting $\phi(\xi) = e^{-(t-\xi)A}\psi$ with $\psi \in C^\infty_{0,\sigma}(\Omega)$ into (13) and setting $s = 0$ we arrive at

$$\begin{aligned}\langle \tilde{u}(t), \psi\rangle = \big\langle e^{-tA}\tilde{u}(0), \psi\big\rangle - \int_0^t \big[< (\tilde{u}\cdot\nabla)\tilde{u}(\xi), e^{-(t-\xi)A}\psi > \\ + < (\tilde{u}\cdot\nabla)u(\xi), e^{-(t-\xi)A}\psi > \big] d\xi \\ + \int_0^t < (u\cdot\nabla)\tilde{u}(\xi), e^{-(t-\xi)A}\psi > d\xi \\ + \alpha\int_0^t < \nabla\tilde{u}(\xi), \nabla e^{-(t-\xi)A}\psi > d\xi \\ + \int_0^t < \tilde{\tau}(\xi), \nabla e^{-(t-\xi)A}\psi > d\xi.\end{aligned}$$

We next note the following estimates for the Stokes semigroup on exterior domains (see e.g. [3], [15])

$$\begin{aligned}
&\|e^{-tA}(w\cdot\nabla)v\| \leqslant Ct^{-1/2}(\|w\|\|v\|)^{1/4}(\|\nabla w\|\|\nabla v\|)^{3/4},\\
&\qquad\qquad\qquad\qquad\qquad\qquad t>0, w,v\in H^1(\Omega)\cap L^2_\sigma(\Omega),\\
&\|\nabla e^{-tA}\psi\| \leqslant Ct^{-1/2}\|\psi\| \quad \text{and} \quad \|\nabla e^{-tA}\psi\|_{L^3} \leqslant Ct^{-3/4}\|\psi\|,\\
&\qquad\qquad\qquad\qquad\qquad\qquad t>0, \psi\in C^\infty_{0,\sigma},
\end{aligned} \tag{14}$$

as well as the Gagliardo-Nirenberg inequality

$$\|\nabla\tilde u(s)\|_{L^{\frac32}} \leqslant C\|\nabla\tilde u(s)\|^{\frac12}\|\nabla^2\tilde u(s)\|^{\frac12} \leqslant C\|\nabla\tilde u(s)\|_{H^1}.$$

Taking the supremum over all $\psi\in C^\infty_{0,\sigma}$ with $\|\psi\|\leqslant 1$ yields

$$\begin{aligned}
\|\tilde u(t)\| &\leqslant \|e^{-tA}\tilde u(0)\| + C\int_0^t (t-s)^{-\frac12}\Big(\|\tilde u(s)\|^{\frac12}\|\nabla\tilde u(s)\|^{\frac32}\\
&\qquad + 2(\|\tilde u(s)\|\|u(s)\|)^{\frac14}(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}\Big)ds\\
&\quad + C\int_0^t (t-s)^{-\frac34}\|\nabla\tilde u(s)\|_{H^1}ds + C\int_0^t (t-s)^{-\frac34}\|\tau(s)\|_{H^1}ds\\
&\leqslant \|e^{-tA}\tilde u(0)\| + Cr^{\frac12}\int_0^t (t-s)^{-\frac12}\|\nabla\tilde u(s)\|^{\frac32}ds\\
&\quad + 2(r\|u\|_{C_b})^{\frac14}\int_0^t (t-s)^{-\frac12}(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}ds\\
&\quad + C\int_0^t (t-s)^{-\frac34}\|\nabla\tilde u(s)\|_{H^1}ds + C\int_0^t (t-s)^{-\frac34}\|\tau(s)\|_{H^1}ds\\
&=: \|e^{-tA}\tilde u(0)\| + I(t) + II(t) + III(t) + IV(t).
\end{aligned} \tag{15}$$

By Theorem 2.3, $\nabla\tilde u\in L^2(\mathbb{R}_+, H^2(\Omega))$ and hence $\|\nabla\tilde u(\cdot)\|^{3/2}\in L^{4/3}(\mathbb{R}_+)$. Setting $h(t):=t^{-1/2}$ and $g_1(t):=\int_0^t h(t-s)\|\nabla\tilde u(s)\|^{3/2}ds$, we see by Lemma 3.2b) that

$$\|g_1\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h\|_{L^2_w(\mathbb{R}_+)}\|\nabla\tilde u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}.$$

Therefore, by (8)

$$\frac1t\int_0^t g_1(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_1\|_{L^4_w(\mathbb{R}_+)} = \frac{C_1}{t^{1/4}}, \quad t>0.$$

for suitable constants $C, C_1>0$. Next, since $\|\nabla u(\cdot)\|$ and $\|\tilde u(\cdot)\|$ belong to $L^2(\mathbb{R}_+)$, Hölder's inequality implies $\|\nabla u(\cdot)\|\|\tilde u(\cdot)\|\in L^1(\mathbb{R}_+)$ and hence $(\|\nabla\tilde u(\cdot)\|\|\nabla u(\cdot)\|)^{\frac34}\in L^{4/3}(\mathbb{R}_+)$. Setting $h(t):=t^{-1/2}$ and $g_2(t):=\int_0^t h(t-s)(\|\nabla\tilde u(s)\|\|\nabla u(s)\|)^{\frac34}ds$ we see that $g_2\in L^4_w(\mathbb{R}_+)$ and satisfies

$$\|g_2\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h\|_{L^2_w(\mathbb{R}_+)}\|\nabla\tilde u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}\|\nabla u\|_{L^2(\mathbb{R}_+;L^2(\Omega))}.$$

Thus, again by (8)

$$\frac{1}{t}\int_0^t g_2(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_2\|_{L^4_w(\mathbb{R}_+)} = \frac{C_2}{t^{1/4}}, \quad t>0.$$

Theorem 2.3 implies $\|\nabla\tilde{u}(\cdot)\|_{H^1} \in L^2(\mathbb{R}_+)$ and hence for h_3 and g_3 given by $h_3(t) := t^{-3/4}$ and $g_3(t) := \int_0^t h_3(t-s)\|\nabla\tilde{u}(s)\|_{H^1}ds$ we obtain

$$\|g_3\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h_3\|_{L^{4/3}_w(\mathbb{R}_+)}\|\nabla\tilde{u}\|_{L^2(\mathbb{R}_+;H^1(\Omega))}.$$

This yields

$$\frac{1}{t}\int_0^t g_3(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_3\|_{L^4_w(\mathbb{R}_+)} = \frac{C_3}{t^{1/4}}, \quad t>0.$$

Similarly, for $IV(t)$ in (15), we have $\|\tilde{\tau}(\cdot)\|_{H^1} \in L^2(\mathbb{R}_+)$. Therefore the function g_4 given by $g_4(t) := \int_0^t (t-s)^{-3/4}\|\nabla\tilde{u}(s)\|_{H^1}ds$ belongs to $L^4_w(\mathbb{R}_+)$ and satisfies

$$\|g_4\|_{L^4_w(\mathbb{R}_+)} \leqslant C\|h_3\|_{L^{4/3}_w(\mathbb{R}_+)}\|\tilde{\tau}\|_{L^2(\mathbb{R}_+;H^1(\Omega))}.$$

As above

$$\frac{1}{t}\int_0^t g_4(s)ds \leqslant \frac{Ct^{3/4}}{t}\|g_4\|_{L^4_w(\mathbb{R}_+)} = \frac{C_4}{t^{1/4}}, \quad t>0.$$

Summing up we see that

$$\frac{1}{t}\int_0^t \|\tilde{u}(s)\|ds \leqslant \frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds + \frac{\tilde{C}}{t^{1/4}}, \quad t>0. \tag{16}$$

Since the Stokes semigroup on exterior domain is strongly stable in the sense that

$$\lim_{t\to\infty}\|e^{-tA}\tilde{u}(0)\| = 0,$$

it follows that $\lim_{t\to\infty}\frac{1}{t}\int_0^t \|\tilde{u}(s)\|ds = 0$. Combining this with estimate (11) we finally obtain

$$\lim_{t\to\infty}\|\tilde{u}(t)\| = 0.$$

Finally, we prove that $\lim_{t\to\infty}\|\tilde{\tau}(t)\| = 0$. To this end, assume that $f, f' \in L^2(0,\infty); L^2(\Omega))$. Then the inequality

$$\|f(t)\|_2^2 \leqslant \|f(t_n)\|_2^2 + 2\Big(\int_{t_n}^t \|f(s)\|_2^2\Big)^{1/2}\Big(\int_{t_n}^t \|f'(s)\|_2^2\Big)^{1/2} \tag{17}$$

yields that $\|f(t)\|_2 \to 0$ as $t\to\infty$ provided $(t_n)\subset(0,\infty)$ is an unbounded sequence satisfying $\|f(t_n)\|_2 \to 0$ as $(t_n)\to\infty$. By Theorem 2.3, the function $\tilde{\tau}$ satisfies (17) and we thus obtain $\|\tilde{\tau}(t)\|_2 \to 0$ as $t\to\infty$. The proof is complete. □

REMARK 3.3: Taking into account that $\tilde{u}(0) \in D(A) \subset H_0^1(\Omega) \subset \mathrm{Rg}(A^{\frac{1}{2}})$ we see that $\frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds$ satisfies a decay rate of the form

$$\frac{1}{t}\int_0^t \|e^{-sA}\tilde{u}(0)\|ds = \frac{1}{t}\int_0^t \|A^{\frac{1}{2}}e^{-sA}(v_0-u_0)\|ds \leqslant \frac{1}{t}\int_0^t \frac{1}{s^{\frac{1}{2}}}\|v_0-u_0\|ds = \frac{C}{t^{\frac{1}{2}}}$$

for $t > 0$. In a similar way we obtain a decay rate for $\tilde{\tau}$ of the form

$$\|\tilde{\tau}(t)\| \leqslant \Big(\frac{C_1}{t^{1/2}} + \frac{C_2}{t^{1/4}}\Big)\|\tau(0) - \mu(0)\|, \ t > 0.$$

Let us also note that combining Theorem 3.1 on the stability of (u, τ) with respect to the $\|\cdot\|_2$-norm with Theorem 2.3 and with the estimate (17) yields a stability result for equation (7) with respect to the $\|\cdot\|_q$-norm for $q \in (2, 6]$. More precisely, the following holds true.

COROLLARY 3.4. *Let $q \in (2, 6]$. Then there exist constants $A, R > 0$ such that any solution (u, τ) to equation (7) with $\alpha \leqslant A$ and with initial data $(u_0, \tau_0) \in B(0, 0, r)$ with $r \leqslant R$ satisfies*

$$\lim_{t\to\infty} \|u(t)\|_{L^q} = \lim_{t\to\infty} \|\tau(t)\|_{L^q} = 0.$$

Proof. Due to Gagliardo-Nirenberg inequality we have

$$\|u\|_q \leqslant c\|\nabla u\|_2^{3(\frac{1}{2}-\frac{1}{q})}\|u\|_2^{\frac{3}{q}-\frac{1}{2}} \text{ for } 2 < q \leqslant 6.$$

By Theorem 2.3, $\nabla u, \nabla u_t \in L^2((0,\infty); L^2(\Omega))$ and hence $\|\nabla u(t)\|_2 \to 0$ as $t \to \infty$ by estimate (17). Since $u \in L^\infty((0,\infty); L^2(\Omega))$, the assertion for u follows. The assertion for $\tilde{\tau}$ follows similarly by noting that $\tilde{\tau}' \in L^\infty((0,\infty); H^1(\Omega))$. □

4. Periodic Solutions

In this section we show that the above stability result, Theorem 3.1, implies also the existence of periodic solutions to (3). More precisely, the following assertion holds.

THEOREM 4.1. *Assume in addition to the assumptions in Theorem 2.3 and 3.1 the function F is time T-periodic for some $T > 0$. Then, if $\|F\|_{L^\infty(H^2)}$ and $\|F'\|_{L^\infty(H^1)}$ are small enough, there exists a T-periodic solution to (3) and this T-periodic solution is stable in the sense of Theorem 3.1.*

Proof. Due to Theorem 2.3, we consider a bounded and small solution

$$(u, \tau) \in C_b([0,\infty); D(A)) \times C_b([0,\infty); H^2(\Omega))$$

of equation (3). In the following, we prove that $\big(u(nT), \tau(nT)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $X := C_b([0,\infty); L^2(\Omega)) \times C_b([0,\infty); L^2(\Omega))$.

To this end, for $m, n \in \mathbb{N}$ with $m > n$ we set

$$(w(t), \mu(t)) := (u(t + (m-n)T), \tau(t + (m-n)T).$$

The periodicity of F implies that $(w(t), \mu(t))$ is again a solution to (3) with the initial data $(w(0), \mu(0)) = (u((m-n)T), \tau((m-n)T)$. Theorem 3.1 and Remark 3.3 imply

$$\|w(t) - u(t)\| + \|\mu(t) - \tau(t)]\| \leqslant \frac{\tilde{C}_1}{t^{1/2}} + \frac{\tilde{C}_2}{t^{1/4}}, \quad t > 0.$$

Hence, by taking $t := nT$ in the above inequality we obtain

$$\|u(mT) - u(nT)\| + \|\mu(mT) - \tau(nT)]\| \leqslant \frac{\tilde{C}_1}{(nT)^{1/2}} + \frac{\tilde{C}_2}{(nT)^{1/4}}.$$

Therefore, $\big(u(nT), \tau(nT))\big)_{n\in\mathbb{N}}$ is a Cauchy sequence in X with limit

$$(u^*, \tau^*) := \lim_{n\to\infty} (u(nT), \tau(nT)) \text{ in } X.$$

Choosing (u^*, τ^*) as initial data, we claim that the solution $(\hat{u}(t), \hat{\tau}(t))$ of equation (3) with $(\hat{u}(0), \hat{\tau}(0)) = (u^*, \tau^*)$ is T-periodic. To this end, for (u, τ) as above and $n \in \mathbb{N}$ we set

$$(v(t), \eta(t)) := (u(t + nT), \tau(t + nT))$$

The periodicity of F implies that $(v(t), \eta(t))$ is a solution of (3) with $(v(0), \eta(0)) = (u(nT), \tau(nT))$. We further see that

$$\begin{aligned}&\|\hat{u}(t) - v(t)\| + \|\hat{\tau}(t) - \eta(t)\| \\ &\qquad \leqslant \Big(\frac{C_1}{t^{\frac{1}{2}}} + \frac{C_2}{t^{\frac{3}{4}}}\Big)\|\hat{u}(0) - v(0))\| + \Big(\frac{C_3}{t^{\frac{1}{2}}} + \frac{C_4}{t^{\frac{1}{4}}}\Big)\|\hat{\tau}(0) - \eta(0)\|.\end{aligned}$$

for $t > 0$. Taking $t = T$ in the above inequality yields

$$\begin{aligned}&\|\hat{u}(T) - u((n+1)T)\| + \|\hat{\tau}(T) - \tau((n+1)T)\| \\ &\qquad \leqslant \Big(\frac{C_1}{T^{\frac{1}{2}}} + \frac{C_2}{T^{\frac{3}{4}}}\Big)\|\hat{u}(0) - u(nT))\| + \Big(\frac{C_3}{T^{\frac{1}{2}}} + \frac{C_4}{T^{\frac{1}{4}}}\Big)\|\hat{\tau}(0) - \tau(nT)\|\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using the fact that $\lim_{n\to\infty}(u(nT), \tau(nT)) = (u^*, \tau^*) = (\hat{u}(0), \hat{\tau}(0))$ in X, we obtain

$$(\hat{u}(T), \hat{\tau}(T)) = (\hat{u}(0), \hat{\tau}(0)).$$

Hence, $(\hat{u}(t), \hat{\tau}(t))$ is $T-$periodic and the proof is complete. $\square$

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Boundedness of solutions to the Cauchy problem for an attraction-repulsion chemotaxis system in two-dimensional space

Toshitaka Nagai and Tetsuya Yamada

Dedicated to Professor Julián López-Gómez on his 60th birthday

Abstract. *We consider the Cauchy problem for an attraction-repulsion chemotaxis system in two-dimensional space. The system consists of three partial differential equations; a drift-diffusion equation incorporating terms for both chemoattraction and chemorepulsion, and two elliptic equations. We denote by β_1 the coefficient of the attractant and by β_2 that of the repellent. The boundedness of nonnegative solutions to the Cauchy problem was shown in the repulsive dominant case $\beta_1 < \beta_2$ and the balance case $\beta_1 = \beta_2$. In this paper, we study the boundedness problem to the Cauchy problem in the attractive dominant case $\beta_1 > \beta_2$.*

Keywords: attraction-repulsion chemotaxis system, attractive dominant case, boundedness of solutions.
MS Classification 2010: 35B45, 35K15, 35K55.

1. Introduction

We consider the Cauchy problem for the following attraction-repulsion chemotaxis system in $\mathbb{R}^2$:

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (\beta_1 u \nabla v_1) + \nabla \cdot (\beta_2 u \nabla v_2), & t > 0,\ x \in \mathbb{R}^2, \\ 0 = \Delta v_j - \lambda_j v_j + u, & t > 0,\ x \in \mathbb{R}^2 \quad (j = 1, 2), \\ u(0, x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{P}$$

where β_j and λ_j $(j = 1, 2)$ are positive constants. We assume that

$$u_0 \geq 0 \text{ on } \mathbb{R}^2,\ u_0 \not\equiv 0,\ u_0 \in L^1 \cap L^\infty, \tag{1}$$

and consider nonnegative solutions to the Cauchy problem (P). Here, $L^p := L^p(\mathbb{R}^2)$ $(1 \le p \le \infty)$ stand for the usual Lebesgue spaces on $\mathbb{R}^2$ with norm $\|\cdot\|_{L^p}$, and in what follows, we denote $\|\cdot\|_{L^p}$ by $\|\cdot\|_p$ for simplicity.

The system (P) is a simplified mathematical model introduced in [19] to describe the aggregation of *Microglia* in the central nervous system. In the system (P), the functions u, v_1 and v_2 denote the density of *Microglia*, the concentration of attractive and repulsive chemical substances, respectively.

In the case $\beta_2 = 0$, the system (P) becomes a minimal version of the classical Keller-Segel model (e.g., [10, 13]):

$$\begin{cases} \partial_t u = \Delta u - \beta_1 \nabla \cdot (u \nabla v_1), & t > 0, x \in \mathbb{R}^2, \\ 0 = \Delta v_1 - \lambda_1 v_1 + u, & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{KS}$$

where λ_1 is a nonnegative constant. The mass conservation for u holds and plays an important role in the existence of nonnegative global solutions to the Cauchy problem (KS). Indeed, in the case $\beta_1 \int_{\mathbb{R}^2} u_0 \, dx \le 8\pi$, the nonnegative solutions exist globally in time (e.g., [4, 5, 6, 21, 22, 29]), meanwhile, in the case $\beta_1 \int_{\mathbb{R}^2} u_0 \, dx > 8\pi$, a nonnegative solution may blow up in finite time (e.g., [2, 5, 15, 29]). The boundedness of nonnegative solutions to the Cauchy problem (KS) was shown under the assumption $\beta_1 \int_{\mathbb{R}^2} u_0 \, dx < 8\pi$ by using rearrangement techniques ([6, 20]). In the critical mass case $\int_{\mathbb{R}^2} u_0 \, dx = 8\pi$ to the Cauchy problem (KS) with $\beta_1 = 1$ and $\lambda_1 = 0$, the boundedness of nonnegative solutions has been studied in [3, 18, 23], and it was shown in [23] that $\sup_{t>0} \|u(t)\|_\infty < \infty$ for the nonnegative *radial* solutions under the assumption $\liminf_{R\to\infty}(R^2 \int_{|x|>R} u_0 \, dx) > 0$. We also remark that $\lim_{t\to\infty} \|u(t)\|_\infty = \infty$ if $\int_{\mathbb{R}^2} |x|^2 u_0(x) \, dx < \infty$ ([4]).

The Cauchy problem (P) has a unique nonnegative smooth solution locally in time for initial data u_0 satisfying (1) ([26]). The nonnegative solutions exist globally in time and are bounded in the repulsive dominant case $\beta_1 < \beta_2$ ([26]) and the balance case $\beta_1 = \beta_2$ ([12, 24]). In the attractive dominant case $\beta_1 > \beta_2$, the nonnegative solutions exist globally in time under the assumption $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx \le 8\pi$ ([24, 25]), whereas there exists a blowing-up solution in finite time under the assumption $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx > 8\pi$ ([26]). We remark that if $\sup_{t>0} \|(u(t), v_1(t), v_2(t))\|_\infty < \infty$, then for all $1 < p \le \infty$,

$$\|(u(t), v_1(t), v_2(t))\|_p \le C(1+t)^{-1+1/p} \quad (t > 0)$$

(see the proof of [26, Theorem 1.3]), and

$$\lim_{t\to\infty} t^{1-1/p} \Big\| u(t) - \int_{\mathbb{R}^2} u_0 \, dx G(t) \Big\|_p = 0,$$

where $G(t,x) = (4\pi t)^{-1} e^{-|x|^2/(4t)}$ is the heat kernel (see the proof of Theorem 1.2 and Remark 1.1 in [12]). Concerning the boundedness problem to the

Cauchy problem for the parabolic system of an attraction-repulsion chemotaxis model, see, e.g., [12] for the balance case.

The boundedness problem to attraction-repulsion chemotaxis systems has been studied on a smooth bounded domain under Neumann boundary conditions (e.g., [7, 11, 16, 17, 28]). When the system (P) is considered on a smooth bounded domain Ω in $\mathbb{R}^2$ under Neumann boundary conditions for u and v_j $(j = 1, 2)$, the boundedness of nonnegative solutions in the attractive dominant case $\beta_1 > \beta_2$ was obtained in [7] under the assumption $(\beta_1 - \beta_2) \int_\Omega u_0 \, dx < 4\pi$ by showing the boundedness of the entropy $\int_\Omega u(t) \log u(t) \, dx$ with respect to $t \in [0, \infty)$. However, the entropy $\int_{\mathbb{R}^2} u(t) \log u(t) \, dx$ on $\mathbb{R}^2$ is not appropriate to get the boundedness of nonnegative solutions to the Cauchy problem (P). The reason is that if $\lim_{t\to\infty} \|u(t)\|_2 = 0$, we observe that

$$\begin{aligned}\int_{\mathbb{R}^2} u(t) \log u(t) \, dx &\le \|u(t)\|_1 \log \Big(\frac{1}{\|u(t)\|_1} \int_{\mathbb{R}^2} u^2(t) \, dx \Big) \\ &= \|u_0\|_1 \big(\log \|u(t)\|_2^2 - \log \|u_0\|_1 \big) \to -\infty \quad (t \to \infty).\end{aligned}$$

Here we used Jensen's inequality for the concave function $\log u$ and $\|u(t)\|_1 = \|u_0\|_1$ $(t > 0)$. For this reason, we introduce the modified entropy $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ in place of $\int_{\mathbb{R}^2} u(t) \log u(t) \, dx$.

For the nonnegative solutions (u, v_1, v_2) to the Cauchy problem (P), the following relation is satisfied ([26, Lemma 3.1]): For $p > 1$,

$$\begin{aligned}&\frac{1}{p} \frac{d}{dt} \|u(t)\|_p^p + \frac{4(p-1)}{p^2} \|\nabla u^{p/2}(t)\|_2^2 + (\beta_2 - \beta_1) \Big(1 - \frac{1}{p}\Big) \|u(t)\|_{p+1}^{p+1} \\ &= -\beta_1 \lambda_1 \Big(1 - \frac{1}{p}\Big) \int_{\mathbb{R}^2} u^p(t) v_1(t) \, dx + \beta_2 \lambda_2 \Big(1 - \frac{1}{p}\Big) \int_{\mathbb{R}^2} u^p(t) v_2(t) \, dx.\end{aligned} \tag{2}$$

In the repulsive dominant case $\beta_1 < \beta_2$, we get the boundedness of $\|u(t)\|_p$ in $t > 0$ from (2) thanks to $\beta_2 - \beta_1 > 0$ in the third term on the left-hand side of (2). In the attractive dominant case $\beta_1 > \beta_2$, we need a smallness condition on initial data to get the boundedness of $\|u(t)\|_p$ in $t > 0$. Hence, we first study the boundedness of the modified entropy $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ in $t > 0$ under the assumption $(\beta_1 - \beta_2) \int_{\mathbb{R}^2} u_0 \, dx < 4\pi$, and then apply (2) to get the boundedness of $\|u(t)\|_p$ in $t > 0$.

The a priori estimate of $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ has been studied for the Keller-Segel model (KS) in [21] and for the Cauchy problem (P) in [24] by applying the Brezis-Merle type inequality established in [21]. However, the a priori bound of $\int_{\mathbb{R}^2} (1 + u(t)) \log(1 + u(t)) \, dx$ for $0 < t < T$ obtained in [21, 24] depends on T, which does not give the uniform boundedness of the solutions on $[0, \infty)$. Another approach from the application of radially symmetric decreasing rearrangement does not seem to work for the Cauchy problem (P) due to the term for chemorepulsion, although it is useful for getting the uniform boundedness of the solutions to the Keller-Segel model (KS) (e.g., [6, 18, 20]). We prove the boundedness of $\int_{\mathbb{R}^2}(1+u(t))\log(1+u(t))\,dx$ on $[0,\infty)$ by applying the sharp form of the Gagliardo-Nirenberg inequality under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<4\pi$, but the uniform boundedness of the solutions is expected under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<8\pi$.

Theorem 1.1. *Let $\beta_1>\beta_2$ and assume that*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<4\pi. \tag{3}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

We next study the boundedness of nonnegative radial solutions to the Cauchy problem (P). For the nonnegative radial initial data u_0 satisfying (1), the uniqueness of solutions to the Cauchy problem (P) ensures that the solution (u,v_1,v_2) for the initial data u_0 is radial in x. Considering the mass function $U(t,s)=\int_0^s\tilde{u}(t,\sigma)\,d\sigma$ of u, where $u(t,x)=\tilde{u}(t,s)$ $(s=\pi|x|^2)$, we reduce the boundedness of u to the following (see Lemma 4.2): There exist $s_0>0$ and $C>0$ such that

$$U(t,s)\le C\sqrt{s}\quad(t\ge0,\ 0\le s\le s_0). \tag{4}$$

Constructing a comparison function and applying the comparison principle for parabolic equations, we show (4) to have the following.

Theorem 1.2. *Let $\beta_1>\beta_2$ and assume that the nonnegative initial data u_0 is radial and*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx<8\pi. \tag{5}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

We lastly study the boundedness problem to the Cauchy problem (P) in the critical mass case $(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx=8\pi$. Using the idea of getting the boundedness of radial solutions to the Cauchy problem (KS) in [23], we have the following theorem under a restricted condition on β_j and λ_j $(j=1,2)$.

Theorem 1.3. *Let $\beta_1>\beta_2$, $\lambda_1\le\lambda_2$ and $\beta_1\lambda_1\ge\beta_2\lambda_2$. Assume that the nonnegative initial data u_0 is radial and*

$$(\beta_1-\beta_2)\int_{\mathbb{R}^2}u_0\,dx=8\pi, \tag{6}$$

$$\liminf_{R\to\infty}\Big(R^2\int_{|x|>R}u_0\,dx\Big)>0. \tag{7}$$

Then, $\sup_{t>0}\|(u(t),v_1(t),v_2(t))\|_p<\infty$ *for all* $1\le p\le\infty$.

The rest of the paper is organized as follows. In Section 2, we mention some properties of nonnegative solutions to the Cauchy problem (P) and give function inequalities on $\mathbb{R}^2$ used in the proof of Theorem 1.1. In Section 3, we give the proof of Theorem 1.1, and in Section 4, the proofs of Theorems 1.2 and 1.3.

Throughout the paper, we use a universal constant C to describe a various constant, and $C(*, \cdots, *)$ when C depends on the quantities appearing in parentheses.

2. Preliminaries

For the nonnegative solutions to the Cauchy problem (P), the conservation of mass is one of important properties, which is obtained by integrating the equations for u and v_j $(j = 1, 2)$ over $\mathbb{R}^2$.

LEMMA 2.1. *Let* (u, v_1, v_2) *be the nonnegative solution to the Cauchy problem* (P) *with nonnegative initial data* u_0 *satisfying* (1). *Then,*

$$\int_{\mathbb{R}^2} u(t)\,dx = \lambda_1 \int_{\mathbb{R}^2} v_1(t)\,dx = \lambda_2 \int_{\mathbb{R}^2} v_2(t)\,dx = \int_{\mathbb{R}^2} u_0\,dx \quad (t > 0).$$

For $\lambda > 0$ and $f \in L^p$ $(1 \leq p \leq \infty)$, we denote by $(\lambda-\Delta)^{-1}f$ the convolution of the Bessel kernel B_λ and f, namely,

$$(\lambda - \Delta)^{-1} f = B_\lambda * f,$$

where

$$B_\lambda(x) = \int_0^\infty e^{-\lambda\sigma} G(\sigma, x)\,d\sigma, \quad x \in \mathbb{R}^2$$

and $G(t, x)$ is the heat kernel given by $G(t, x) = (4\pi t)^{-1} e^{-|x|^2/(4t)}$. For $f \in L^p$ $(1 < p < \infty)$, the function $v := (\lambda - \Delta)^{-1} f$ on $\mathbb{R}^2$ belongs to $W^{2,p}$ and a solution of

$$(\lambda - \Delta)v = f \quad \text{in} \ \ \mathbb{R}^2.$$

By the following estimates

$$\|\partial_x^\alpha B_\lambda\|_p < \infty \text{ for } 1 \leq p < \infty \text{ if } |\alpha| = 0 \text{ and } 1 \leq p < 2 \text{ if } |\alpha| = 1,$$

applying Young's inequality for convolution gives L^p estimates on $(\lambda - \Delta)^{-1} f$ in Lemma 2.2 below, which are often used in the course of the proof of Theorem 1.1. For the Bessel kernel, see, e.g., [9, 27].

LEMMA 2.2. *For $\lambda > 0$, it holds that*

$$\|(\lambda - \Delta)^{-1} f\|_p \le C(\lambda, p, q)\|f\|_q, \ 1 \le q \le p < \infty,$$
$$\|(\lambda - \Delta)^{-1} f\|_\infty \le C(\lambda, q)\|f\|_q, \ 1 < q \le \infty,$$
$$\|\nabla(\lambda - \Delta)^{-1} f\|_\infty \le C(\lambda, q)\|f\|_q, \ 2 < q \le \infty.$$

For later uses, we give some function inequalities on $\mathbb{R}^2$. We begin with the Gagliardo-Nirenberg inequality on $\mathbb{R}^2$ (e.g., [8]): For $1 < p < \infty$, there is a positive constant C depending on p such that for any $f \in L^1$ with $|\nabla f| \in L^2$,

$$\|f\|_p \le C\|\nabla f\|_2^{1-1/p}\|f\|_1^{1/p}. \tag{8}$$

The next inequality is a version of the Gagliardo-Nirenberg inequality on $\mathbb{R}^2$: For any $f \in L^2$ with $|\nabla f| \in L^1$,

$$\|f\|_2 \le \frac{1}{\sqrt{4\pi}}\|\nabla f\|_1. \tag{9}$$

Here, $1/\sqrt{4\pi}$ is the best constant (e.g., [30, Theorem 2.7.4]).

We give two lemmas below, which are proven by applying (9).

LEMMA 2.3. *For $0 < \varepsilon < 1$ and nonnegative functions $g \in L^1 \cap W^{1,2}$,*

$$\int_{\mathbb{R}^2} g^2\,dx \le \frac{1+\varepsilon}{4\pi}\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2} \frac{|\nabla g|^2}{1+g}\,dx\Big) + \frac{2}{\varepsilon}\int_{\mathbb{R}^2} g\,dx. \tag{10}$$

Proof. Let $\alpha > 1$. We have that

$$\begin{aligned}
\int_{\mathbb{R}^2} g^2\,dx &= \int_{g>\alpha} g^2\,dx + \int_{g\le\alpha} g^2\,dx = \int_{g>\alpha} \{(g-\alpha)+\alpha\}^2\,dx + \int_{g\le\alpha} g^2\,dx \\
&= \int_{g>\alpha} (g-\alpha)^2\,dx + 2\alpha\int_{g>\alpha}(g-\alpha)\,dx + \int_{g>\alpha}\alpha^2\,dx + \int_{g\le\alpha} g^2\,dx \\
&\le \int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx + 2\alpha\int_{g>\alpha} g\,dx + \alpha\int_{g\le\alpha} g\,dx \\
&\le \int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx + 2\alpha\int_{\mathbb{R}^2} g\,dx,
\end{aligned}$$

where $(g-\alpha)_+ = \max\{g-\alpha, 0\}$. We estimate $\int_{\mathbb{R}^2}(g-\alpha)_+^2\,dx$ as follows. By the Gagliardo-Nirenberg inequality (9),

$$\begin{aligned}
\int_{\mathbb{R}^2} (g-\alpha)_+^2\,dx &\le \frac{1}{4\pi}\Big(\int_{\mathbb{R}^2} |\nabla(g-\alpha)_+|\,dx\Big)^2 = \frac{1}{4\pi}\Big(\int_{g>\alpha} |\nabla g|\,dx\Big)^2 \\
&= \frac{1}{4\pi}\Big(\int_{g>\alpha} \sqrt{1+g}\frac{|\nabla g|}{\sqrt{1+g}}\,dx\Big)^2 \\
&\le \frac{1}{4\pi}\Big(\int_{g>\alpha}(1+g)\,dx\Big)\Big(\int_{g>\alpha}\frac{|\nabla g|^2}{1+g}\,dx\Big),
\end{aligned}$$

and then,

$$\int_{g>\alpha}(1+g)\,dx = \int_{g>\alpha} dx + \int_{g>\alpha} g\,dx \le \frac{1}{\alpha}\int_{g>\alpha} g\,dx + \int_{g>\alpha} g\,dx$$
$$= \Big(1+\frac{1}{\alpha}\Big)\int_{g>\alpha} g\,dx.$$

Hence,

$$\int_{\mathbb{R}^2}(g-\alpha)_+^2\,dx \le \frac{1}{4\pi}\Big(1+\frac{1}{\alpha}\Big)\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2}\frac{|\nabla g|^2}{1+g}\,dx\Big).$$

Therefore,

$$\int_{\mathbb{R}^2} g^2\,dx \le \frac{1}{4\pi}\Big(1+\frac{1}{\alpha}\Big)\Big(\int_{\mathbb{R}^2} g\,dx\Big)\Big(\int_{\mathbb{R}^2}\frac{|\nabla g|^2}{1+g}\,dx\Big) + 2\alpha\int_{\mathbb{R}^2} g\,dx.$$

By putting $\varepsilon = 1/\alpha$, (10) is derived. □

LEMMA 2.4. *It holds that for any nonnegative function* $g \in L^1 \cap W^{1,2}$,

$$\int_{\mathbb{R}^2} g^3\,dx \le \varepsilon\Big(\int_{\mathbb{R}^2}(1+g)\log(1+g)\,dx\Big)\Big(\int_{\mathbb{R}^2}|\nabla g|^2\,dx\Big) + C(\varepsilon)\int_{\mathbb{R}^2} g\,dx,$$

where ε *is any positive number and* $C(\varepsilon) \to \infty$ $(\varepsilon \to 0)$.

For the proof of Lemma 2.4, see, e.g., [21, Lemma 2.1].

We lastly mention the following interpolation inequality, which is obtained by applying Hölder's inequality: Let $1 \le p_1 < p_2 \le \infty$ and $f \in L^{p_1} \cap L^{p_2}$. Then $f \in L^p$ for all p with $p_1 \le p \le p_2$ and

$$\|f\|_p \le \|f\|_{p_1}^{\lambda}\|f\|_{p_2}^{1-\lambda} \quad \text{where} \quad \frac{1}{p} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2},\ 0 \le \lambda \le 1. \tag{11}$$

3. Boundedness of solutions by entropy estimates

Let (u, v_1, v_2) be the nonnegative solution to the Cauchy problem (P) corresponding to the initial data u_0 satisfying (1). For the proof of Theorem 1.1, we need the following proposition, which is proven in Subsection 3.1.

PROPOSITION 3.1. *Let* $0 < T \le \infty$ *and assume that*

$$E := \sup_{0<t<T}\|(1+u(t))\log(1+u(t))\|_1 < \infty. \tag{12}$$

Then,

$$\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty, E), \quad 0 < t < T.$$

REMARK 3.2: The assumption $\beta_1 > \beta_2$ is not required for proving Proposition 3.1.

We put $\psi = \beta_1 v_1 - \beta_2 v_2$ in the equations of u and v_j $(j = 1, 2)$ in (P). Then,

$$\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi), \quad -\Delta \psi = (\beta_1 - \beta_2) u + h \quad (t > 0,\ x \in \mathbb{R}^2), \tag{13}$$

where $h = \lambda_2 \beta_2 v_2 - \lambda_1 \beta_1 v_1$. As $v_j = (\lambda_j - \Delta)^{-1} u$ $(j = 1, 2)$, applying Lemma 2.2 as $f = u(t)$, we observe that for $j = 1, 2$ and $t > 0$,

$$\|v_j(t)\|_p \le C(p, q) \|u(t)\|_q,\ 1 \le q \le p < \infty, \tag{14}$$

$$\|v_j(t)\|_\infty \le C(q) \|u(t)\|_q,\ 1 < q \le \infty, \tag{15}$$

$$\|\nabla v_j(t)\|_\infty \le C(q) \|u(t)\|_q,\ 2 < q \le \infty. \tag{16}$$

Here and in what follows, we drop λ_j from $C(\lambda_j, p, q)$ and $C(\lambda_j, q)$ for simplicity. In particular, thanks to (14) for $q = 1$ and $\|u(t)\|_1 = \|u_0\|_1$ by Lemma 2.1, we have

$$\|v_j(t)\|_p \le C(p) \|u_0\|_1,\ 1 \le p < \infty. \tag{17}$$

We give the following lemma for the modified entropy.

LEMMA 3.3. *It holds that*

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^2} (1 + u) \log(1 + u)\, dx &+ \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx \\ &= -\int_{\mathbb{R}^2} u \Delta \psi\, dx + \int_{\mathbb{R}^2} \log(1 + u)\, \Delta \psi\, dx,\end{aligned} \tag{18}$$

where $\psi = \beta_1 v_1 - \beta_2 v_2$.

Proof. Using $\partial_t u = \Delta u - \nabla \cdot (u \nabla \psi)$ in (13) and noting $\int_{\mathbb{R}^2} \partial_t u\, dx = 0$, by integration by parts, we have that

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^2} (1 + u) \log(1 + u)\, dx &= \int_{\mathbb{R}^2} \partial_t u \, \log(1 + u)\, dx + \int_{\mathbb{R}^2} \partial_t u\, dx \\ &= \int_{\mathbb{R}^2} \Delta u \, \log(1 + u)\, dx - \int_{\mathbb{R}^2} \nabla \cdot (u \nabla \psi) \, \log(1 + u)\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx + \int_{\mathbb{R}^2} \frac{u}{1 + u} \nabla u \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx + \int_{\mathbb{R}^2} \nabla u \cdot \nabla \psi\, dx - \int_{\mathbb{R}^2} \frac{1}{1 + u} \nabla u \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx - \int_{\mathbb{R}^2} u \Delta \psi\, dx - \int_{\mathbb{R}^2} \nabla \log(1 + u) \cdot \nabla \psi\, dx \\ &= -\int_{\mathbb{R}^2} \frac{|\nabla u|^2}{1 + u}\, dx - \int_{\mathbb{R}^2} u \Delta \psi\, dx + \int_{\mathbb{R}^2} \log(1 + u) \Delta \psi\, dx.\end{aligned}$$

Thus, we derive (18). □

Proof of Theorem 1.1. Since the nonnegative solution exists globally in time under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2} u_0\,dx < 8\pi$ by [24, Theorem 1.1], all we have to do is to show boundedness under the assumption $(\beta_1-\beta_2)\int_{\mathbb{R}^2} u_0\,dx < 4\pi$ by applying Proposition 3.1 as $T=\infty$.

Since $-\Delta\psi = (\beta_1-\beta_2)u + h$ ($h = \lambda_2\beta_2 v_2 - \lambda_1\beta_1 v_1$) by (13), plugging this relation into the right-hand side of (18) yields that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + \int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \\
&\quad = (\beta_1-\beta_2)\int_{\mathbb{R}^2} u^2\,dx + \int_{\mathbb{R}^2} uh\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx \\
&\qquad - \int_{\mathbb{R}^2}\log(1+u)\,h\,dx \qquad (19)\\
&\quad \le (\beta_1-\beta_2)\int_{\mathbb{R}^2} u^2\,dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^2}\left\{u^2 + (\log(1+u))^2\right\}dx \\
&\qquad - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx + C(\varepsilon)\int_{\mathbb{R}^2} h^2\,dx,
\end{aligned}$$

where $0<\varepsilon<1$. By $\log(1+u)\le u$, we have that

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_{\mathbb{R}^2}\left\{u^2 + (\log(1+u))^2\right\}dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\log(1+u)\,dx \\
&\le \varepsilon\int_{\mathbb{R}^2} u^2\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + (\beta_1-\beta_2)\int_{\mathbb{R}^2} u\,dx.
\end{aligned} \qquad (20)$$

Substituting (20) into the right-hand side of (19) and using $\|h\|_2^2 \le C\|u_0\|_1^2$ obtained by (17), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + \int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \\
&\quad \le (\beta_1-\beta_2+\varepsilon)\int_{\mathbb{R}^2} u^2\,dx - (\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx \\
&\qquad + C(\|u_0\|_1,\varepsilon).
\end{aligned} \qquad (21)$$

Applying Lemma 2.3 as $g=u(t)$ yields that

$$\int_{\mathbb{R}^2} u^2(t)\,dx \le \frac{1+\varepsilon}{4\pi}\|u_0\|_1\int_{\mathbb{R}^2}\frac{|\nabla u(t)|^2}{1+u(t)}\,dx + \frac{2}{\varepsilon}\|u_0\|_1.$$

Here we used $\|u(t)\|_1 = \|u_0\|_1$. Plugging this inequality into (21), we have that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx \\ &\qquad + \left\{1-(\beta_1-\beta_2+\varepsilon)\frac{1+\varepsilon}{4\pi}\|u_0\|_1\right\}\int_{\mathbb{R}^2}\frac{|\nabla u|^2}{1+u}\,dx \\ &\le -(\beta_1-\beta_2)\int_{\mathbb{R}^2}(1+u)\log(1+u)\,dx + C(\|u_0\|_1,\varepsilon). \end{aligned} \tag{22}$$

Thanks to $(\beta_1-\beta_2)\|u_0\|_1 < 4\pi$ by assumption (3), we can take $0<\varepsilon<1$ such that

$$1-(\beta_1-\beta_2+\varepsilon)\frac{1+\varepsilon}{4\pi}\|u_0\|_1 \ge 0.$$

Hence, it follows from (22) that

$$\|(1+u(t))\log(1+u(t))\|_1 \le e^{-(\beta_1-\beta_2)t}\|(1+u_0)\log(1+u_0)\|_1 + C(\|u_0\|_1), \quad t>0.$$

Therefore, we conclude the boundedness of $\|u(t)\|_\infty$ on $[0,\infty)$ by Proposition 3.1. □

3.1. Proof of Proposition 3.1

The proof of Proposition 3.1 relies on the following lemma, which is proven by Moser's iteration technique (e.g., [1, 14, 26]).

LEMMA 3.4. *Let $0<T\le\infty$ and assume*

$$A := \sup_{0<t<T}\|\nabla(\beta_1 v_1(t)-\beta_2 v_2(t))\|_\infty < \infty.$$

Then, $\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty, A), \quad 0<t<T.$

To prove Proposition 3.1, we begin with showing

$$\|u(t)\|_2 \le C(\|u_0\|_1, \|u_0\|_2, E), \quad 0<t<T, \tag{23}$$

where $E = \sup_{0<t<T}\|(1+u(t))\log(1+u(t))\|_1$. By (2) for $p=2$, we have that

$$\begin{aligned} &\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 - (\beta_1-\beta_2)\|u(t)\|_3^3 \\ &\quad \le \beta_2\lambda_2\int_{\mathbb{R}^2} u^2 v_2\,dx \le \beta_2\lambda_2\|u(t)\|_3^2\|v_2(t)\|_3 \le \beta_2\|u(t)\|_3^3 + C\|v_2(t)\|_3^3, \end{aligned}$$

from which it follows that

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 - \beta_1\|u(t)\|_3^3 \le C\|v_2(t)\|_3^3 \le C\|u_0\|_1^3.$$

Here we used $\|v_2(t)\|_3 \le C\|u_0\|_1$ by (17). To control $\|u(t)\|_3$, we recall the following inequality on $\mathbb{R}^2$ (see Lemma 2.4): For any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for any nonnegative function $g \in L^1 \cap W^{1,2}$,

$$\|g\|_3^3 \le \varepsilon \|(1+g)\log(1+g)\|_1 \|\nabla g\|_2^2 + C(\varepsilon)\|g\|_1. \tag{24}$$

Thanks to $E = \sup_{0<t<T} \|(1+u(t))\log(1+u(t))\|_1 < \infty$ by assumption (12), applying (24) as $g = u(t)$ and using $\|u(t)\|_1 = \|u_0\|_1$, we have

$$\|u(t)\|_3^3 \le \varepsilon E \|\nabla u(t)\|_2^2 + C(\varepsilon)\|u_0\|_1, \quad 0 < t < T,$$

and hence,

$$\frac{d}{dt}\|u(t)\|_2^2 + (2 - \varepsilon\beta_1 E)\|\nabla u(t)\|_2^2 \le C(\|u_0\|_1, \varepsilon), \quad 0 < t < T,$$

where $0 < \varepsilon < 1$. Take ε such as $2 - \varepsilon\beta_1 E \ge 1$, that is, $0 < \varepsilon \le 1/(\beta_1 E)$. Then,

$$\frac{d}{dt}\|u(t)\|_2^2 + \|\nabla u(t)\|_2^2 \le C(\|u_0\|_1, E), \quad 0 < t < T. \tag{25}$$

Applying (8) as $f = u(t)$ and using $\|u(t)\|_1 = \|u_0\|_1$ yield that

$$\|u(t)\|_2^2 \le C\|\nabla u(t)\|_2 \|u_0\|_1 \le \|\nabla u(t)\|_2^2 + C\|u_0\|_1^2.$$

Substituting this inequality into (25), we have that

$$\frac{d}{dt}\|u(t)\|_2^2 + \|u(t)\|_2^2 \le C(\|u_0\|_1, E), \quad 0 < t < T,$$

from which (23) follows.

We next show that

$$\|u(t)\|_4 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T. \tag{26}$$

By (2) for $p = 4$,

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_4^4 + 3\|\nabla u^2(t)\|_2^2 - 3(\beta_1 - \beta_2)\|u(t)\|_5^5 \\
&\le 3\beta_2\lambda_2 \int_{\mathbb{R}^2} u^4(t)v_2(t)\,dx \le 3\beta_2\lambda_2\|u(t)\|_5^4\|v_2(t)\|_5 \le 3\beta_2\|u(t)\|_5^5 + C\|v_2(t)\|_5^5.
\end{aligned}$$

Putting $w = u^2$ yields that

$$\frac{d}{dt}\|w(t)\|_2^2 + 3\|\nabla w(t)\|_2^2 - 3\beta_1\|w(t)\|_{5/2}^{5/2} \le C\|u_0\|_1^5.$$

Here we used $\|v_2(t)\|_5 \le C\|u_0\|_1$ by (17). Applying the Gagliardo-Nirenberg inequality (8) for $p = 5/2$ and using Young's inequality, we have that

$$\|w(t)\|_{5/2}^{5/2} \le C\|\nabla w(t)\|_2^{3/2}\|w(t)\|_1 \le \eta\|\nabla w(t)\|_2^2 + C(\eta)\|w(t)\|_1^4,$$

where η is a positive number determined later. Hence, for $0 < t < T$,

$$\frac{d}{dt}\|w(t)\|_2^2 + 3(1-\beta_1\eta)\|\nabla w(t)\|_2^2 \le 3\beta_1 C(\eta)\|w(t)\|_1^4 + C\|u_0\|_1^5. \tag{27}$$

Take $\eta > 0$ such that $3(1-\beta_1\eta) \ge 1$ and note that by (23),

$$\|w(t)\|_1 = \|u(t)\|_2^2 \le C(\|u_0\|_1, \|u_0\|_2, E), \quad 0 < t < T.$$

Then, as in the proof of the boundedness of $\|u(t)\|_2$, we derive from (27) that

$$\|u(t)\|_4^4 = \|w(t)\|_2^2 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T.$$

Here we used the fact that $\|u_0\|_2$ is estimated by $\|u_0\|_1$ and $\|u_0\|_4$ by virtue of interpolation inequality (11). Thus, (26) is derived.

By (15) for $q = 2$ and (23),

$$\|v_j(t)\|_\infty \le C\|u(t)\|_2 \le C(\|u_0\|_1, \|u_0\|_2, E), \quad 0 < t < T,$$

and by (16) for $q = 4$ and (26),

$$\|\nabla v_j(t)\|_\infty \le C\|u(t)\|_4 \le C(\|u_0\|_1, \|u_0\|_4, E), \quad 0 < t < T.$$

Hence, since $\|\nabla(\beta_1 v_1(t) - \beta_2 v_2(t))\|_\infty \le C(\|u_0\|_1, \|u_0\|_4, E)$ $(0 < t < T)$, Lemma 3.4 ensures that

$$\|u(t)\|_\infty \le C(\|u_0\|_1, \|u_0\|_\infty, E), \quad 0 < t < T.$$

Thus, we establish the assertion of Proposition 3.1.

4. Boundedness of radial solutions

In this section, we assume that the nonnegative initial data u_0 satisfying (1) is radial in x. Then, by the uniqueness of solutions to the Cauchy problem (P), the nonnegative solution (u, v_1, v_2) corresponding to the initial data u_0 is radial in x.

Define the functions $\tilde{u}(t,s)$ and $\tilde{v}_j(t,s)$ $(j = 1, 2)$ by

$$u(t,x) = \tilde{u}(t,s), \quad v_j(t,x) = \tilde{v}_j(t,s), \quad s = \pi|x|^2$$

and $\tilde{u}_0(s)$ by $u_0(x) = \tilde{u}_0(s)$. We next define U and V_j $(j = 1, 2)$ by

$$U(t,s) = \int_0^s \tilde{u}(t,\sigma)\,d\sigma, \quad V_j(t,s) = \int_0^s \tilde{v}_j(t,\sigma)\,d\sigma \tag{28}$$

and $U_0(s) = \int_0^s \tilde{u}_0(\sigma)\,d\sigma$. By Lemma 2.1, we observe that

$$U(t,\infty) = \int_0^\infty \tilde{u}(t,s)\,ds = 2\pi \int_0^\infty \tilde{u}(t,\pi r^2)\,rdr = \int_{\mathbb{R}^2} u(t,x)\,dx = \int_{\mathbb{R}^2} u_0(x)\,dx,$$

and

$$V_j(t,\infty) = \int_0^\infty \tilde{v}_j(t,s)\,ds = \int_{\mathbb{R}^2} v_j(t,x)\,dx = \frac{1}{\lambda_j}\int_{\mathbb{R}^2} u_0(x)\,dx.$$

Lemma 4.1. *It holds that*

$$\partial_t U = 4\pi s\partial_s^2 U + (\beta_1 - \beta_2)U\partial_s U - (\beta_1\lambda_1 V_1 - \beta_2\lambda_2 V_2)\partial_s U \quad (t > 0,\ s > 0). \tag{29}$$

Proof. Calculating that

$$\begin{aligned}
&\partial_{x_j} u = \partial_s \tilde{u}\partial_{x_j} s = 2\pi x_j \partial_s \tilde{u}, \quad \partial_{x_j}^2 u = 4\pi^2 x_j^2 \partial_s^2 \tilde{u} + 2\pi\partial_s \tilde{u},\\
&\Delta u = 4\pi s\partial_s^2 \tilde{u} + 4\pi\partial_s \tilde{u} = 4\pi\partial_s(s\partial_s \tilde{u}),\\
&\nabla\cdot(u\nabla v_j) = 4\pi s\partial_s(\tilde{u}\partial_s \tilde{v_j}) + 4\pi\tilde{u}\partial_s \tilde{v}_j = 4\pi\partial_s(s\tilde{u}\partial_s \tilde{v}_j),
\end{aligned}$$

we have

$$\partial_t \tilde{u} = 4\pi\partial_s(s\partial_s\tilde{u}) - 4\pi\partial_s(s\tilde{u}\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)), \tag{30}$$

$$0 = 4\pi\partial_s(s\partial_s\tilde{v}_j) - \lambda_j\tilde{v}_j + \tilde{u} \quad (j = 1, 2). \tag{31}$$

Integrating (30) and (31) with respect to s, we have that

$$\begin{aligned}
\partial_t U &= 4\pi s\partial_s\tilde{u} - 4\pi s\tilde{u}\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)\\
&= 4\pi s\partial_s^2 U - \partial_s U\{4\pi s\partial_s(\beta_1\tilde{v}_1 - \beta_2\tilde{v}_2)\},
\end{aligned}$$

$$4\pi s\partial_s\tilde{v}_j = -U + \lambda_j V_j \quad (j = 1, 2).$$

Hence,

$$\partial_t U = 4\pi s\partial_s^2 U + (\beta_1 - \beta_2)U\partial_s U - (\beta_1\lambda_1 V_1 - \beta_2\lambda_2 V_2)\partial_s U.$$

□

To obtain the boundedness of the solution (u, v_1, v_2), by Lemma 3.4 it suffices to show that

$$\sup_{t>0} \|\nabla v_j(t)\|_\infty < \infty \quad (j = 1, 2). \tag{32}$$

Thanks to $4\pi s\partial_s\tilde{v}_j = \lambda_j V_j - U$ and $s = \pi|x|^2$, we have that

$$|\nabla v_j(t,x)| = 2\pi|x||\partial_s\tilde{v}_j(t,s)| = \frac{1}{2\sqrt{\pi s}}|U(t,s) - \lambda_j V_j(t,s)|. \tag{33}$$

By Hölder's inequality we observe that for $0 < \lambda < 1$,

$$\begin{aligned} 0 \le V_j(t,s) &= \int_0^s \tilde{v}_j(t,\sigma)\,d\sigma \le s^\lambda\Big(\int_0^\infty |\tilde{v}_j(t,\sigma)|^{1/(1-\lambda)}\,d\sigma\Big)^{1-\lambda} \\ &= s^\lambda\|v_j(t)\|_{1/(1-\lambda)} \le C(\lambda)\|u_0\|_1 s^\lambda. \end{aligned} \tag{34}$$

Here we used $\|v_j(t)\|_{1/(1-\lambda)} \le C(\lambda)\|u_0\|_1$ $(t > 0)$ by (17). Since $0 \le V_j(t,s) \le C\|u_0\|_1\sqrt{s}$ by (34) for $\lambda = 1/2$, we have the following lemma by virtue of (33).

Lemma 4.2. *If there exist $s_0 > 0$ and $C > 0$ such that*

$$U(t,s) \le C\sqrt{s} \quad (t \ge 0,\ 0 \le s \le s_0),$$

then (32) *is satisfied. Hence,* $\sup_{t>0}\|(u(t), v_1(t), v_2(t))\|_\infty < \infty$.

Proof of Theorem 1.2. By (29) and $\partial_s U \ge 0$,

$$\partial_t U \le 4\pi s\partial_s^2 U + \beta U\partial_s U + \beta_2\lambda_2 V_2\partial_s U,$$

where $\beta = \beta_1 - \beta_2 > 0$. As $V_2(t,s) \le C(\lambda)\|u_0\|_1 s^\lambda$ for $0 < \lambda < 1$ by (34), putting $B(\lambda) = \beta_2\lambda_2 C(\lambda)\|u_0\|_1$, we have that

$$\partial_t U \le 4\pi s\partial_s^2 U + \beta U\partial_s U + B(\lambda)s^\lambda\partial_s U, \quad t > 0,\ s > 0.$$

In what follows, for simplicity we put

$$\mathcal{N}g = 4\pi s\partial_s^2 g + \beta g\partial_s g + B(\lambda)s^\lambda\partial_s g, \tag{35}$$

where $0 < \lambda < 1$. We then get the following:

$$\begin{cases} \partial_t U \le \mathcal{N}U, \quad t > 0,\ s > 0, \\ U(t,0) = 0,\ U(t,\infty) = \|u_0\|_1, \quad t > 0, \\ U(0,s) = U_0(s), \quad s \ge 0. \end{cases}$$

For $b > 0$ and $\gamma > 0$, we define $W_{b,\gamma}(s)$ by

$$W_{b,\gamma}(s) = \frac{8\pi}{\gamma}\frac{s}{s+b} \quad (s \ge 0).$$

The function satisfies

$$4\pi s\frac{d^2W_{b,\gamma}}{ds^2}+\gamma W_{b,\gamma}\frac{dW_{b,\gamma}}{ds}=0 \quad (s>0). \tag{36}$$

As $\beta\|u_0\|_1<8\pi$ by assumption (5), we can choose γ and λ in (35) such that

$$1<\frac{\gamma}{\beta}<\min\Big\{\frac{8\pi}{\beta\|u_0\|_1},2\Big\}, \quad \Big(\frac{1}{2}<\Big)\frac{\beta}{\gamma}<\lambda<1,$$

and as a comparison function we take

$$W(s)=W_{b,\gamma}(s^\lambda) \quad (s\geq 0).$$

Take $b>0$ so small that

$$s_0:=\Big(\frac{\gamma\|u_0\|_1 b}{8\pi-\gamma\|u_0\|_1}\Big)^{1/\lambda}<1, \quad \frac{\lambda\gamma-\beta}{\gamma}\cdot\frac{8\pi-\gamma\|u_0\|_1}{b}\geq B(\lambda). \tag{37}$$

Here we used $\|u_0\|_1<8\pi/\gamma=W_{b,\gamma}(\infty)$ and $\lambda\gamma>\beta$. Since $W_{b,\gamma}(s)$ is decreasing in b and converges to $8\pi/\gamma$ as $b\to+0$ for each $s>0$ and $W'_{b,\gamma}(0)=8\pi/(\gamma b)$ where $'=d/ds$, we can shorten b such that

$$U_0(s)\leq\|u_0\|_\infty s<W_{b,\gamma}(s) \quad \text{for} \quad 0<s\leq s_0.$$

By $W(s_0)=W_{b,\gamma}(s_0^\lambda)=\|u_0\|_1$, it is apparent that

$$U(t,s_0)\leq\|u_0\|_1=W(s_0) \quad \text{for} \quad t\geq 0.$$

As $W_{b,\gamma}(s)$ is increasing in s and $0<s<s^\lambda$ for $0<s\leq s_0(<1)$, we observe that for $0<s\leq s_0$,

$$U_0(s)<W_{b,\gamma}(s)<W_{b,\gamma}(s^\lambda)=W(s).$$

Since

$$\frac{dW}{ds}=\lambda s^{\lambda-1}\frac{dW_{b,\gamma}}{ds}(s^\lambda), \quad s\frac{d^2W_{b,\gamma}}{ds^2}=-\frac{\gamma}{4\pi}W_{b,\gamma}\frac{dW_{b,\gamma}}{ds},$$

we have

$$\begin{aligned}\frac{d^2W}{ds^2}&=\lambda^2 s^{\lambda-2}\cdot s^\lambda\frac{d^2W_{b,\gamma}}{ds^2}(s^\lambda)-\lambda(1-\lambda)s^{\lambda-2}\frac{dW_{b,\gamma}}{ds}(s^\lambda)\\&=-\frac{\lambda^2\gamma}{4\pi}s^{\lambda-2}W_{b,\gamma}(s^\lambda)\frac{dW_{b,\gamma}}{ds}(s^\lambda)-\lambda(1-\lambda)s^{\lambda-2}\frac{dW_{b,\gamma}}{ds}(s^\lambda)\\&=-\frac{\lambda\gamma}{4\pi}s^{-1}W\frac{dW}{ds}-(1-\lambda)s^{-1}\frac{dW}{ds},\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{N}W &= 4\pi s\frac{d^2W}{ds^2} + \beta W\frac{dW}{ds} + B(\lambda)s^{\lambda}\frac{dW}{ds} \\
&= -(\lambda\gamma-\beta)W\frac{dW}{ds} - 4\pi(1-\lambda)\frac{dW}{ds} + B(\lambda)s^{\lambda}\frac{dW}{ds} \\
&= -4\pi(1-\lambda)\frac{dW}{ds} - s^{\lambda}\left\{(\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s^{\lambda}+b} - B(\lambda)\right\}\frac{dW}{ds}.
\end{aligned}
$$

Let $0 < s < s_0$, where s_0 is given by (37). As $\lambda\gamma > \beta$, we observe that

$$
\begin{aligned}
(\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s^{\lambda}+b} - B(\lambda) &\geq (\lambda\gamma-\beta)\frac{8\pi}{\gamma}\frac{1}{s_0^{\lambda}+b} - B(\lambda) \\
&= \frac{\lambda\gamma-\beta}{\gamma}\cdot\frac{8\pi-\gamma\|u_0\|_1}{b} - B(\lambda) \\
&\geq 0.
\end{aligned}
$$

Hence, $\mathcal{N}W < 0$ $(0 < s < s_0)$ because of $dW/ds > 0$. Therefore,

$$
\begin{cases}
\partial_t U \leq \mathcal{N}U, \quad \mathcal{N}W < 0 \ (t > 0,\ 0 < s < s_0), \\
U(t,0) = W(0) = 0, \quad U(t,s_0) \leq W(s_0) \ (t \geq 0), \\
U(0,s) = U_0(s) \leq W(s) \ (0 \leq s \leq s_0).
\end{cases}
$$

Then, the comparison principle ensures that

$$
U(t,s) \leq W(s) \leq \frac{8\pi}{\gamma}\frac{s^{\lambda}}{b} \quad (t \geq 0,\ 0 \leq s \leq s_0).
$$

Therefore, as $1/2 < \lambda < 1$, we establish Theorem 1.2 thanks to Lemma 4.2. $\square$

Proof of Theorem 1.3. Let $U(t,s)$ and $V_j(t,s)$ $(j = 1,2)$ be the functions defined by (28). We claim that

$$
V_1(t,s) \geq V_2(t,s) \quad (t > 0,\ s \geq 0).
$$

In fact, as $v_j \geq 0$ $(j = 1,2)$ and $\lambda_1 \leq \lambda_2$, by the equations for v_j $(j = 1,2)$,

$$
-\Delta(v_1 - v_2) + \lambda_1(v_1 - v_2) = (\lambda_2 - \lambda_1)v_2 \geq 0 \quad \text{in } \mathbb{R}^2.
$$

By the maximum principle, we have $v_1 \geq v_2$ on $\mathbb{R}^2$. Thus $V_1 \geq V_2$.

By Lemma 4.1,

$$
\partial_t U - 4\pi s\partial_s^2 U - \beta U\partial_s U = (\beta_2\lambda_2 V_2 - \beta_1\lambda_1 V_1)\partial_s U,
$$

where $\beta = \beta_1 - \beta_2$. It follows from $V_1 \geq V_2$ and $\beta_1\lambda_1 \geq \beta_2\lambda_2$ that

$$\beta_2\lambda_2 V_2 - \beta_1\lambda_1 V_1 = \beta_2\lambda_2(V_2 - V_1) + (\beta_2\lambda_2 - \beta_1\lambda_1)V_1 \leq 0.$$

Hence, as $\partial_s U \geq 0$, we have

$$\partial_t U - 4\pi s\partial_s^2 U - \beta U \partial_s U \leq 0 \quad (t > 0,\ s > 0).$$

We note that assumption (7) is equivalent to

$$\liminf_{s\to\infty} \left(s \int_s^\infty \tilde{u}_0 \, d\sigma \right) > 0, \tag{38}$$

where $\tilde{u}_0$ is defined by $u_0(x) = \tilde{u}_0(s)$, $s = \pi|x|^2$. As in the proof of [23, Lemma 3.1.(ii)], by assumption (6) and (38) we can choose $b > 0$ such that

$$U(0,s) = \int_0^s \tilde{u}_0 \, d\sigma \leq W_{b,\beta}(s) = \frac{8\pi}{\beta} \frac{s}{s+b} \quad (s \geq 0).$$

Since $U(t,0) = W_{b,\beta}(0) = 0$, $U(t,\infty) = W_{b,\beta}(\infty) = 8\pi/\beta$ $(t > 0)$ and

$$4\pi s \frac{d^2 W_{b,\beta}}{ds^2} + \beta W_{b,\beta} \frac{dW_{b,\beta}}{ds} = 0 \quad (s > 0)$$

by (36), the comparison principle ensures that

$$U(t,s) \leq W_{b,\beta}(s) = \frac{8\pi}{\beta} \frac{s}{s+b} \quad (t > 0,\ s \geq 0).$$

Therefore, Theorem 1.3 is established by Lemma 4.2. □

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A spatiotemporal model of drug resistance in bacteria with mutations

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"This paper is dedicated to the sixtieth birthday of Professor Julian Lopez-Gomez"

Abstract. *A spatio-temporal dynamics model is presented to study the effects of mutations on the persistence and extinction of bacteria under the antibiotic inhibition. We construct a mixed type Lyapunov functional to prove the global stability of extinction state and coexistence state for the case of forward mutation and forward-backward mutation respectively.*

Keywords: strong maximum principle, Lyapunov functional, invariance principle, drug-resistance,mutations, competitive exclusion, uniform persistence.
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1. Introduction

Antibiotic drug resistance is a global health problems [11]. Today, clinically important bacteria are characterized by their drug resistance to single or multiple drug. Historically penicillin- resistant Staphylococcus aureus are discovered soon after the introduction of penicillin in the 1940s in clinical environments [2] and still up to now the antibiotic drug resistance is still a subject of intense research [6, 7, 8, 9, 15, 16]. Most of the experiments on drug resistance in the laboratory setup are conducted in a well-mixed environment [6, 8]. For mathematical modeling on the subject of drug-resistance of bacteria, the authors [4, 5] constructed a system of ordinary differential equations with impulse conditions to study the selection of drug resistance mutants in a device called "Morbidostat"[4, 5]. In [3] Kishony et al. presented a device for the evolution of bacteria that allows migration and adaption in a large, spatially structured environment. The microbial evolution and growth arena(MEGA)-plate consists of a rectangle acrylic dish 120x60cm, in which successive regions of black-colored agar containing different concentrations of antibiotics are overlaid by soft agar allowing bacterial motility. Motile bacteria inoculated at on location on the plate and spread by chemotaxis to other regions. Only increasing resistant mutants can spread into sections containing higher levels of antibiotic. Interested readers can consult the paper for biological details. Based on their experiments, we shall study the spatiotemporal dynamics of bacteria under antibiotics inhibition by constructing a system of reaction-diffusion equations. The rest of this paper is organized as follows. In Section 2 we describe the mathematical models with forward mutations and forward-backward mutations. In Section 3 we state our main results. Technical proofs are collected in Section 4. We analyze the global stability of the extinction state for the case of forward mutations and the coexistence state for the case of forward-backward mutations respectively. A Lyapunov functional of mixed type is constructed and invariance principle [1] is applied to the establishment of the global stability of the extinction and coexistence state.

2. Description of our models

In the simplest scenario, we formulate the transition from a wild type population $u(x,t)(v_0 := u)$ to N mutant strains $v_i(x,t), i = 1, 2, \ldots, N$ where $x \in \Omega$, Ω is a bounded domain in R^n. Let $P(x)$ be a given distribution of drug inhibitor in Ω and $U = U(x,t) = u(x,t) + \sum_{i=1}^{n} v_i(x,t)$ be the total population in Ω. For the forward mutation model mutant v_i mutates to mutant v_{i+1} with a forward mutation rate q_i. For the forward - backward mutation model, mutant v_i mutates to mutant v_{i+1} with a forward mutation rate q_i, while mutant v_{i+1} mutates to mutant v_i with a backward mutation rate $\tilde{q}_i$. The spatiotemporal dynamics with forward mutation and forward - backward mutation under the influence of the drug inhibition $P(x)$ are given by the following models (1) and (2) respectively.

$$\begin{cases} \frac{\partial u}{\partial t} = d_0 \Delta u + r_0 u (1 - \frac{U}{K}) f_0(P(x)) - q_0 u \\ \frac{\partial v_i}{\partial t} = d_i \Delta v_i + r_i v_i (1 - \frac{U}{K}) f_i(P(x)) + q_{i-1} v_{i-1} - q_i v_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, N-1, \\ \frac{\partial v_N}{\partial t} = d_N \Delta v_N + r_N v_N (1 - \frac{U}{K}) f_N(P(x)) + q_{N-1} v_{N-1} \end{cases} \tag{1}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} = d_0 \Delta u + r_0 u (1 - \frac{U}{K}) f_0(P(x)) - q_0 u + \tilde{q}_0 v_1 \\ \frac{\partial v_i}{\partial t} = d_i \Delta v_i + r_i v_i (1 - \frac{U}{K}) f_i(P(x)) + q_{i-1} v_{i-1} - (\tilde{q}_{i-1} + q_i) v_i + \tilde{q}_i v_{i+1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, N-1, \\ \frac{\partial v_N}{\partial t} = d_N \Delta v_N + r_N v_N (1 - \frac{U}{K}) f_N(P(x)) + q_{N-1} v_{N-1} - \tilde{q}_{N-1} v_N \end{cases} \tag{2}$$

The initial conditions and boundary conditions for both of (1) and (2) are given below in (3) and (4) respectively. The initial conditions are

$$\begin{cases} u(x,0) = u_0(x) \le K, & x \in \Omega \\ v_i(x,0) = v_{i0}(x) \equiv 0, & i = 1, 2, \ldots, N, \ x \in \Omega \end{cases} \tag{3}$$

and the boundary conditions are

$$\begin{cases} \frac{\partial u}{\partial n}(x,t) = 0, & x \in \partial\Omega,\ t > 0 \\ \frac{\partial v_i}{\partial n}(x,t) = 0, & i = 1, 2, \dots, N,\ x \in \partial\Omega,\ t > 0, \end{cases} \tag{4}$$

where $\frac{\partial}{\partial n}$ denotes the differentiation along the outward normal n to $\partial\Omega$. In (1) and (2), we assume that the wild type population $v_0 := u$ and the mutant population $v_i, i = 1, 2, \dots, N$ share the same carrying capacity K and have intrinsic growth rate $r_i, i = 0, 1, \dots, N$. In (1) and (2), $d_i > 0$ is the diffusion coefficient for species v_i; the mutation rate q_i and $\tilde{q}_i$ are assumed to be small. The effect of the drug inhibition is described by $f_0(p)$ and $f_i(p)$ which satisfies $f_i(0) = 1, i = 0, 1, \dots, N$ and $f_i'(p) < 0, p > 0$.
$f_i'(p) < 0$ means a larger drug concentration leads to stronger inhibition of the bacteria species i. Because the mutants have stronger resistance to the inhibition than wild type, we have the following assumption:

$$(H1) \qquad f_0(p) < f_1(p) < \cdots < f_N(p).$$

The example of $f_i(p)$ take the form of Hill function of order L, which are:

$$f_i(p) = \frac{1}{1 + (\frac{p}{K_i})^L},\ i = 0, 1, 2, \dots, N.$$

Thus, (H1) becomes $K_0 < K_1 < \cdots < K_N$.

It is generally accepted that the bacterial drug resistance comes at the cost of lower reproductive fitness. The classical trade off is that in the absence of drug inhibition the wild type has the competitive advantage (hypothesis (H2) below), whereas when the drug in present, the advantage shifts to the resistant types (hypothesis (H1)). Thus in addition to hypothesis (H1), we assume that the intrinsic growth rates $r_i, i = 0, \dots, N$ satisfy :

$$(H2) \qquad r_0 > r_1 > \cdots > r_N.$$

Furthermore it is reasonable to assume that the wild type and mutants have the same diffusion coefficient, i.e.,

$$(H3) \qquad d_0 = d_1 = \cdots = d_N =: d.$$

Now we present the main result of this paper.

THEOREM 2.1. *Suppose that the assumptions* (H1)*,* (H2)*,* (H3) *hold and the initial function $u(x, 0)$ is nontrivial. Then*

(i) *the solutions $u(x,t)$ and $v_i(x,t)$ of* (1)*,* (3)*,* (4) *satisfy* $\lim_{t\to\infty} u(x,t) = 0$, $\lim_{t\to\infty} v_i(x,t) = 0, i = 1, \cdots, N-1$ *and* $\lim_{t\to\infty} v_N(x,t) = K$; *and*

(ii) *The solutions* $u(x,t)$ *and* $v_i(x,t)$ *of* (2), (3), (4) *satisfy*

$$\lim_{t\to\infty} u(x,t) = u^* := v_0^* > 0, \ \lim_{t\to\infty} v_i(x,t) = v_i^* > 0, i = 1, \cdots, N$$

where

$$v_N^* = \frac{K}{\frac{\tilde{q}_0\tilde{q}_1\cdots\tilde{q}_{N-1}}{q_0 q_1\cdots q_{N-1}} + \frac{\tilde{q}_1\cdots\tilde{q}_{N-1}}{q_1\cdots q_{N-1}} + \cdots + \frac{\tilde{q}_{N-1}}{q_{N-1}} + 1},$$

$$v_{N-1}^* = \frac{\tilde{q}_{N-1}}{q_{N-1}} v_N^*, \quad v_{N-2}^* = \frac{\tilde{q}_{N-2}\tilde{q}_{N-1}}{q_{N-2}q_{N-1}} v_N^*, \quad \ldots$$

$$\ldots, v_1^* = \frac{\tilde{q}_1\cdots\tilde{q}_{N-1}}{q_1\cdots q_{N-1}} v_N^*, \quad u^* := v_0^* = \frac{\tilde{q}_0\tilde{q}_1\cdots\tilde{q}_{N-1}}{q_0 q_1\cdots q_{N-1}} v_N^*.$$

REMARK 2.2: The result is independent of the drug distribution $P(x)$.

3. Proof of the main result

Let R_+^{N+1} denote the nonnegative orthant of R^{N+1} and $C(\bar{\Omega}, R_+^{N+1})$ the nonnegative value continuous functions space. Set

$$\Lambda := \Big\{ v \in R_+^{N+1} : U := \sum_{i=0}^N v_i \le K \Big\}$$
$$\text{and } X_\Lambda := \Big\{ \phi \in C(\bar{\Omega}, R_+^{N+1}) : \phi(x) \in \Lambda, x \in \bar{\Omega} \Big\}.$$

For $\phi := (\phi_0, \phi_1, \cdots, \phi_N) \in C(\bar{\Omega}, R_+^{N+1})$, we denote $\Phi_t(\phi)$ the solution of (1) or (2) with Neumann boundary condition (4) passing through ϕ. Then we first prove that both $C(\bar{\Omega}, R_+^{N+1})$ and X_Λ are positively invariant.

PROPOSITION 3.1. *Suppose that the assumptions* (H1), (H2) *and* (H3) *hold. Then both* $C(\bar{\Omega}, R_+^{N+1})$ *and* X_Λ *are positively invariant for the solution semiflow* $\Phi_t(\phi)$ *of both models of* (1) *and* (2) *with Neumann boundary condition* (4). *Furthermore,* $v_i(x,t) > 0$ *for all* $x \in \bar{\Omega}, t > 0$ *and* $i = 0, 1, \cdots, N$ *if* $\phi \in X_\Lambda$ *with* $\phi_0 \not\equiv 0$.

Proof. Let $w(x,t) := (u(x,t), v_1(x,t), \cdots, v_N(x,t))$ and denote the reaction term of (1) or (2) by $F(x,w)$. Then $F : \bar{\Omega} \times R_+^{N+1}$ satisfies

$$F_i(x,w) \ge 0 \textit{ whenever } x \in \bar{\Omega} \textit{ and } w \in R_+^{N+1}, w_i = 0$$

for $i = 0, 1, \cdots, N$. Applying Corollary 3.2 in [13, p.129], we obtain that $v_i(x,t) \ge 0$ for $t > 0, x \in \Omega$ and $i = 0, 1, \cdots, N$, that is, $C(\bar{\Omega}, R_+^{N+1})$ is positively invariant for the solution semiflow $\Phi_t(\phi)$ of both models of (1) and (2) with Neumann boundary condition (4).

Let $U(x,t) := \sum_{i=0}^{N} v_i(x,t) \geq 0$ and

$$K(x,t) := K^{-1} \sum_{i=0}^{N} r_i f_i(P(x)) v_i(x,t) \geq 0\,.$$

Assume that $U(x,0) = \sum_{i=0}^{N} \phi_i(x) \leq K$ for $x \in \Omega$. Then $U(x,t)$ satisfies

$$\begin{cases} \frac{\partial U}{\partial t} = d\Delta U + K(x,t)(K-U),\ x \in \Omega \\ \frac{\partial U}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0 \\ U(x,0) \leq K,\ x \in \Omega. \end{cases} \tag{5}$$

It is easy to see that the constant function K is a solution of the equation in (5). Let $V(x,t) := U(x,t) - K$. Then V satisfies

$$\begin{cases} \frac{\partial V}{\partial t} = d\Delta V - K(x,t)V,\ x \in \Omega, t > 0 \\ \frac{\partial V}{\partial n} = 0,\ x \in \partial\Omega,\ t > 0 \\ V(x,0) \leq 0,\ x \in \Omega. \end{cases} \tag{6}$$

We claim that $V(x,t) \leq 0,\ x \in \bar{\Omega},\ t \geq 0$. Suppose not. Then there exist $\bar{x} \in \bar{\Omega},\ \bar{t} > 0$ such that $V(\bar{x},\bar{t}) > 0$. Denote M^* the maximal value of the function $V(x,t)$ on $\bar{\Omega} \times [0,\bar{t}]$ and let $M^* = V(x^*,t^*)$ with $x^* \in \bar{\Omega},\ t^* \leq \bar{t}$. Then $M^* > 0$. If $x^* \in \Omega$, then it follows from Theorem 2.5 and Remark 2.1 [13, p.126-127] or Theorem 15, Chapter 3 [12], that $V(x,t) \equiv M^*$ for all $x \in \bar{\Omega}$ and $t \leq t^*$. In particular, $M^* = V(x,0) \leq 0$, a contradiction. This proves $x^* \in \partial\Omega$. Applying Theorem 2.5 and Remark 2.1 [13, p.126-127] again, we obtain that $\frac{\partial V}{\partial n}(x^*,t^*) < 0$, this contradicts the Neumann boundary condition of (6).

Similarly, we may prove that $u(x,t) > 0$ for all $x \in \bar{\Omega}, t > 0$ if $\phi \in X_\Lambda$ with $\phi_0 \not\equiv 0$. In the following, we only consider the system (1), the proof of the system (2) is similar.

From (1) and the positive invariance of X_Λ, we get that

$$d\Delta v_1 - \frac{\partial v_1}{\partial t} - q_1 v_1 = -r_1 v_1 \left(1 - \frac{U}{K}\right) f_1(P(x)) \leq 0.$$

We assert that $v_1(x,t) > 0$ for all $x \in \bar{\Omega}, t > 0$. Otherwise, there exist $\bar{x} \in \bar{\Omega},\ \bar{t} > 0$ such that $v_1(\bar{x},\bar{t}) = 0$. Theorem 2.5 and Remark 2.1 [13, p.126-127] and the Neumann boundary condition imply that $\bar{x} \in \Omega$, and hence $v_1(x,t) \equiv 0$ for all $x \in \bar{\Omega}, t \leq \bar{t}$. From the second equation of (1) it follows that $u(x,t) = 0$ for all $x \in \bar{\Omega}, t \leq \bar{t}$, a contradiction. Inductively, we can prove that $v_i(x,t) > 0$ for all $x \in \bar{\Omega}, t > 0$ for $i = 2, \cdots, N$. This completes the proof. $\square$

PROPOSITION 3.2. (i) *The steady state* $E_N = (0, 0, \cdots, K)$ *is locally asymptotically stable for the system* (1), (3), (4) (*Forward mutation model*).

(ii) *The steady state* $E^* = (u^*, v_1^*, \cdots, v_N^*)$ *is locally asymptotically stable for the system* (2), (3), (4) (*Forward-backward mutation model*).

Proof. (i) Let $w_i = v_i,\ i = 0, 1, \cdots, N-1,\ w_N = v_N - K$. Then for (1) we have

$$\begin{cases} \frac{\partial w_0}{\partial t} = d\Delta w_0 + r_0 w_0 \left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_0(P(x)) - q_0 w_0,\ x \in \Omega \\ \frac{\partial w_i}{\partial t} = d\Delta w_i + r_i w_i \left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_i(P(x)) + q_{i-1} w_{i-1} - q_i w_i, \\ \qquad\qquad i = 1, \cdots, N-1,\ x \in \Omega, \\ \frac{\partial w_N}{\partial t} = d\Delta w_N + r_N (w_N + K) \Big(-\frac{\sum_{k=0}^N w_k}{K}) f_N(P(x)\Big) + q_{N-1} w_{N-1}, \\ \qquad\qquad x \in \Omega, \\ \frac{\partial w_i}{\partial n}(x,t) = 0,\ x \in \partial\Omega,\ t > 0,\ i = 0, 1, 2, \cdots, N \end{cases} \tag{7}$$

And the linearized system of (7) around E_N is

$$\begin{cases} \frac{\partial w_0}{\partial t} = d\Delta w_0 - q_0 w_0,\ x \in \Omega \\ \frac{\partial w_i}{\partial t} = d\Delta w_i + q_{i-1} w_{i-1} - q_i w_i,\ i = 1, \cdots, N-1,\ x \in \Omega. \\ \frac{\partial w_N}{\partial t} = d\Delta w_N - r_N \sum_{k=0}^N w_k f_N(P(x)) + q_{N-1} w_{N-1},\ x \in \Omega \\ \frac{\partial w_i}{\partial n}(x,t) = 0,\ x \in \partial\Omega,\ t > 0,\ i = 0, 1, 2, \cdots, N \end{cases} \tag{8}$$

Let $w_i(x,t) = e^{\lambda t} \varphi_i(x),\ i = 0, 1, 2, , \cdots, N$. Then it follows that

$$\begin{cases} \lambda\varphi_0 = d\Delta\varphi_0 - q_0\varphi_0,\ x \in \Omega \\ \lambda\varphi_i = d\Delta\varphi_i + q_{i-1}\varphi_{i-1} - q_i\varphi_i,\ x \in \Omega. \\ \lambda\varphi_N = d\Delta\varphi_N - r_N \sum_{k=0}^N \varphi_k f_N(P(x)) + q_{N-1}\varphi_{N-1},\ x \in \Omega \\ \frac{\partial \varphi_i}{\partial n}(x) = 0,\ x \in \partial\Omega \end{cases} \tag{9}$$

Then the principal eigenvalue is

$$\begin{aligned} \lambda &= \inf_{\substack{\varphi_0 \in H^1(\Omega) \\ \varphi_0 \neq 0}} \frac{d \int_\Omega \varphi_0 \Delta\varphi_0 \, dx - q_0 \int_\Omega \varphi_0^2(x) \, dx}{\int_\Omega \varphi_0^2(x) \, dx} \\ &= \inf_{\substack{\varphi_0 \in H^1(\Omega) \\ \varphi_0 \neq 0}} \frac{-d \int_\Omega |\nabla\varphi_0(x)|^2 \, dx - q_0 \int_\Omega \varphi_0^2(x) \, dx}{\int_\Omega \varphi_0^2(x) \, dx} < 0. \end{aligned}$$

Hence, E_N is locally asymptotically stable for the system (1), (2), (3).

(ii) Let $E^* = (v_0^*, v_1^*, \cdots, v_N^*)$, $w_i = v_i - v_i^*$, $i = 0, 1, 2, \cdots, N$. Then from (2) we have

$$\begin{cases} \frac{\partial w_0}{\partial t} = d\Delta w_0 + r_0(w_0 + v_0^*)\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_0(P(x)) \\ \qquad\qquad -q_0 w_0 + \tilde{q}_0 w_1,\ x \in \Omega \\ \frac{\partial w_i}{\partial t} = d\Delta w_i + r_i(w_i + v_i^*)\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_i(P(x)) + q_{i-1} w_{i-1} \\ \qquad\qquad -(\tilde{q}_{i-1} + q_i) w_i + \tilde{q}_i w_{i+1},\ i = 1, \cdots, N-1,\ x \in \Omega \\ \frac{\partial w_N}{\partial t} = d\Delta w_N + r_N(w_N + v_N^*)\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_N(P(x)) \\ \qquad\qquad +q_{N-1} w_{N-1} - \tilde{q}_{N-1} w_N,\ x \in \Omega \\ \frac{\partial w_i}{\partial n}(x, t) = 0,\ x \in \partial\Omega,\ t > 0,\ i = 0, 1, 2, \cdots, N \end{cases} \tag{10}$$

The linearized system of (10) around E^* is

$$\begin{cases} \frac{\partial w_0}{\partial t} = d\Delta w_0 + r_0 v_0^*\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_0(P(x)) - q_0 w_0 + \tilde{q}_0 w_1,\ x \in \Omega \\ \frac{\partial w_i}{\partial t} = d\Delta w_i + r_i v_i^*\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_i(P(x)) + q_{i-1} w_{i-1} \\ \qquad\qquad -(\tilde{q}_{i-1} + q_i) w_i + \tilde{q}_i w_{i+1},\ i = 1, \cdots, N-1,\ x \in \Omega \\ \frac{\partial w_N}{\partial t} = d\Delta w_N + r_N v_N^*\left(-\frac{\sum_{k=0}^N w_k}{K}\right) f_N(P(x)) + q_{N-1} w_{N-1} \\ \qquad\qquad -\tilde{q}_{N-1} w_N,\ x \in \Omega \\ \frac{\partial w_i}{\partial n}(x, t) = 0,\ x \in \partial\Omega,\ t > 0,\ i = 0, 1, 2, \cdots, N \end{cases} \tag{11}$$

Let $w_i(x, t) = e^{\lambda t}\varphi_i(x)$, $i = 0, 1, 2, , \cdots, N$. Then it follows that

$$\begin{cases} \lambda\varphi_0 = d\Delta\varphi_0 + \frac{r_0}{K} v_0^*\left(-\sum_{k=0}^N \varphi_k\right) f_0(P(x)) - q_0\varphi_0 + \tilde{q}_0\varphi_1,\ x \in \Omega \\ \lambda\varphi_i = d\Delta\varphi_i + \frac{r_i}{K} v_i^*\left(-\sum_{k=0}^N \varphi_k\right) f_i(P(x)) + q_{i-1}\varphi_{i-1} \\ \qquad\qquad -(\tilde{q}_{i-1} + q_i)\varphi_i + \tilde{q}_i\varphi_{i+1},\ x \in \Omega \\ \lambda\varphi_N = d\Delta\varphi_N + \frac{r_N}{K} v_N^*\left(-\sum_{k=0}^N \varphi_k\right) f_N(P(x)) + q_{N-1}\varphi_{N-1} \\ \qquad\qquad -\tilde{q}_{N-1}\varphi_N,\ x \in \Omega \\ \frac{\partial \varphi_i}{\partial n}(x) = 0,\ x \in \partial\Omega,\ i = 0, 1, 2, \cdots, N \end{cases} \tag{12}$$

Adding the equations in (12) yields

$$\lambda \sum_{k=0}^N \varphi_k = d\Delta\left(\sum_{k=0}^N \varphi_k\right) + \left(-\sum_{k=0}^N \varphi_k\right)\left(\sum_{i=0}^N \frac{r_i}{K} v_i^* f_i(P(x))\right).$$

Let $\Phi(x) = \sum_{k=0}^{N} \varphi_k(x)$. From above we have

$$\lambda\Phi(x) = d\Delta\Phi(x) + (-\Phi(x))\left(\sum_{i=0}^{N} \frac{r_i}{K} v_i^* f_i(P(x))\right),$$

and

$$\lambda = \inf_{\substack{\Phi\in H^1(\Omega)\\ \Phi\neq 0}} \frac{-d\int_\Omega |\nabla\Phi|^2\,dx - \int_\Omega(\Phi^2(x))\left(\sum_{i=0}^{N} \frac{r_i}{K} v_i^* f_i(P(x))\right)dx}{\int_\Omega \Phi^2(x)\,dx} < 0.$$

Hence, $E^* = (v_0^*, \cdots, v_N^*)$ is locally asymptotically stable for the system (2), (3), (4). □

Proof of Theorem 2.1. Let $w(x,t) = (u(x,t), v_1(x,t), \cdots, v_N(x,t))$. Introduce Lyapunov functional

$$V(w(\cdot,t)) = \int_\Omega \left(U(x,t) - K - K\ln\frac{U(x,t)}{K}\right)dx,$$

where $U(x,t) = u(x,t) + v_1(x,t) + \cdots + v_N(x,t)$. Then

$$\begin{aligned}
\dot V(w(\cdot,t)) &= \frac{d}{dt}V(w(\cdot,t)) = \int_\Omega \frac{\partial U}{\partial t}\cdot\frac{U(x,t)-K}{U(x,t)}\,dx\\
&= \int_\Omega \sum_{i=0}^{N}(v_i)_t(x,t)\cdot\frac{U(x,t)-K}{U(x,t)}\,dx\\
&= \int_\Omega \left\{d\Delta U + \sum_{i=0}^{N} r_i f_i(P(x))v_i\left(1-\frac{U}{K}\right)\right\}\frac{U-K}{U}\,dx\\
&= \int_\Omega \left(d\Delta U\left(1-\frac{K}{U}\right) - \frac{(U-K)^2}{KU}\sum_{i=0}^{N} r_i f_i(P(x))v_i\right)dx\\
&= d\left[\int_{\partial\Omega}\frac{\partial U}{\partial v}\left(1-\frac{K}{U}\right)dS - \int_\Omega |\nabla U|^2\frac{K}{U^2}dx\right]\\
&\qquad\qquad - \int_\Omega \frac{(U-K)^2}{KU}\sum_{i=0}^{N} r_i f_i(P(x))v_i dx\\
&= -d\int_\Omega |\nabla U|^2\frac{K}{U^2}\,dx - \int_\Omega r_N f_0(P(x))\frac{(U-K)^2}{K}\,dx \le 0.
\end{aligned}$$

For the systems (1) and (2), by invariance principle [1] the $\omega-$limit set lies on the simplex $S = \{(u, v_1, \cdots, v_N) : \ U = u + v_1 + \cdots + v_N = K\} \subset X_\Lambda$.

(i) Forward mutation.
For system (1), (3), (4), the dynamics on S satisfies

$$\begin{cases} u_t = d\Delta u - q_0 u \\ (v_i)_t = d\Delta v_i + q_{i-1}v_{i-1} - q_i v_i, \ i = 1, 2, \cdots, N-1. \\ (v_N)_t = d\Delta v_N + q_{N-1}v_{N-1} \end{cases} \tag{13}$$

Introduce Lyapunov functional on the simplex S,

$$V(w(\cdot, t)) = \int_\Omega \big(u + v_1 + \cdots + v_{N-1} + v_N - K - K \ln \frac{v_N}{K}\big)\, dx.$$

Then

$$\begin{aligned} \dot{V}(w(\cdot, t)) &= \frac{d}{dt} V(w(\cdot, t)) \\ &= -\int_\Omega \Big(|\nabla v_N|^2 \frac{dK}{v_N^2} + q_{N-1}K \Big)\, dx < 0. \end{aligned}$$

It follows that $E_N = (0, 0, \cdots, K)$ is globally asymptotically stable in S. Since the $\omega-$limit set of the solution of (1), (3), (4) lies in the maximal invariant set M in S, from Proposition 3.2(i), E_N is locally asymptotically stable for the system (1), (3), (4), thus E_N is globally asymptotically stable for the system (1), (3), (4). Hence we complete the proof of Theorem 2.1(i).

(ii) Forward - backward mutation.

For the system (2), (3), (4), the dynamics on the simplex S satisfies

$$\begin{cases} u_t = d\Delta u - q_0 u + \tilde{q}_0 v_1 \\ (v_i)_t = d\Delta v_i + q_{i-1}v_{i-1} - (\tilde{q}_{i-1} + q_i)v_i + \tilde{q}_i v_{i+1}, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \cdots, N-1 \\ (v_N)_t = d\Delta v_N + q_{N-1}v_{N-1} - \tilde{q}_{N-1}v_N \end{cases} \tag{14}$$

It is easy to see the linear system(14) is monotone and irreducible and it possesses the invariant function (see [10]) $U = \sum_{i=0}^N v_i$ which determines a family of $d-$hypersurfaces [14]: $U \equiv c, c \in R$. Applying Theorem 6.3 of [10], we know that every solution of (14) is convergent to a steady state in $L := \{\mu E^* : \mu \in R\}$. Precisely, every solution on the invariant $d-$hypersurface : $U \equiv c$ converges to the unique steady state $\frac{c}{K}E^*$ lying on this $d-$hypersurface. In particular, on the simplex $S : U \equiv K$, all solutions on the simplex $S : U \equiv K$

tend to the steady state E^*. This means that E^* is the unique compact invariant set on the S.

Since the $\omega-$limit set of the solution of (2), (3), (4) lies in the maximal invariant M in S, from Proposition 3.2 (ii), E^* is locally asymptotically stable for the system (2), (3), (4), thus E^* is globally asymptotically stable for the system (2), (3), (4). Hence we complete the proof of Theorem 2.1(ii). $\square$

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Influence of the spatial heterogeneities in the existence of positive solutions of logistic BVPs with sublinear mixed boundary conditions

Santiago Cano-Casanova

It is my pleasure to contribute with this paper in this special issue on the occasion of the 60th birthday of Professor J. López-Gómez. I am very grateful with him for getting involved me in a research field which passionates me, and I am indebted to him for all the scientific knowledge he has conveyed me. Clearly, Science is a great part of him and He is pivotal for Science. With all my affection and gratitude.

Abstract. *In this paper we analyze the influence of the spatial heterogeneities in the existence of positive solutions of Logistic problems with heterogeneous sublinear boundary conditions. We will show that the relative positions of the vanishing sets of the potentials in front of the nonlinearities, in the PDE and on the boundary conditions, play a crucial role as for the amplitude of the range of values of the bifurcation parameter for which the problems possess positive solutions. We will compare the cases of the logistic problem with linear and nonlinear boundary conditions. Also, we will show the global bifurcation diagram of positive solutions of the logistic problem with heterogeneous nonlinear boundary conditions, considering the amplitude of the nonlinearity in the boundary conditions as bifurcation-continuation parameter.*

Keywords: principal eigenvalues, positive solutions, nonlinear mixed boundary conditions, spatial heterogeneities, logistic problems.
MS Classification 2010: 35J65, 35J25, 35B09, 35B35, 35B40.

1. Introduction and Main Result

In this paper we consider the logistic elliptic boundary value problem with sublinear mixed boundary conditions and spatial heterogeneities given by

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega\,, \quad p > 1\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = -b(x)u^q & \text{on } \Gamma_1\,, \quad q > 1\,, \end{cases} \tag{1}$$

where:

i) Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are two disjoint components of the boundary,

ii) $-\Delta$ stands for the minus Laplacian operator in $\mathbb{R}^N$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter,

iii) The potential $a \in \mathcal{C}(\bar{\Omega})$ with $a > 0$ measures the spatial heterogeneities in Ω and satisfies that

$$\Omega_0 := int\,\{x \in \Omega : a(x) = 0\} \neq 0\,, \qquad \Omega_0 \in \mathcal{C}^2$$

and

$$a \text{ is bounded away from zero in any compact subset of } \Omega \setminus \bar{\Omega}_0\,. \tag{2}$$

In some case, when it is pointed out, we will assume that

$$a \text{ is bounded away from zero in any compact subset of } (\Omega\setminus\bar{\Omega}_0)\cup\Gamma_1 \tag{3}$$

instead of (2)

iv) The potential $b \in \mathcal{C}(\Gamma_1)$ with $b > 0$ measures the spatial heterogeneities on Γ_1.

v) $\partial_\nu u(x)$ stands for the outward normal derivative of u at each $x \in \Gamma_1$.

By a positive solution of (1) for the value λ of the parameter we mean a strong positive solution, that is, any positive function $u \in W^2_r(\Omega)$ for some $r > N$ which satisfies (1) a.e. in Ω for such a value λ of the parameter.

This kind of elliptic problems has been widely analyzed under linear boundary conditions in some previous works (cf. [4, 6, 13, 14, 15, 18]) and under nonlinear boundary conditions (cf. [7, 9, 11, 16, 21, 22]).

The main goal of this work is to analyze the existence of positive solutions of (1) and to ascertain the global bifurcation diagram of positive solutions of it, depending on the nodal behavior of the spatial heterogeneities a and b, in the domain and on the boundary conditions, respectively. Namely, as for the nodal behavior of the potential a we will distinguish the cases

$$\Gamma_1 \subset \partial\Omega_0\,, \qquad dist(\partial\Omega_0 \cap \Omega, \Gamma_1) > 0 \tag{4}$$

and

$$\bar{\Omega}_0 \subset \Omega \cup \Gamma_0\,, \tag{5}$$

and as for the profile of the potential b we will distinguish the case when

$$b(x) \geq \underline{b} > 0 \qquad \text{for all } x \in \Gamma_1 \tag{6}$$

and the case when

$$b(x) = 0 \quad \forall\, x \in \Gamma_1^0 \qquad \text{and} \qquad b(x) > 0 \quad \forall\, x \in \Gamma_1^+ , \tag{7}$$

being Γ_1^0 and Γ_1^+ two disjoint connected pieces of Γ_1, closed and open, respectively as $N-1$ dimensional manifolds, such that $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$. Hereafter, assuming that Γ_1^0 and Γ_1^+ satisfy the previous assumptions, we will denote

$$\mathcal{C}^+(\Gamma_1^+) := \left\{ V \in \mathcal{C}(\Gamma_1) \,:\, V(x) = 0 \;\forall\, x \in \Gamma_1^0 \text{ and } V(x) > 0 \quad \forall\, x \in \Gamma_1^+ \right\}. \tag{8}$$

In Figures 1 and 2 we show two possible configurations of the subdomain Ω_0 with respect to Γ_1, satisfying (4) in Figure 1 and satisfying (5) in Figure 2. In

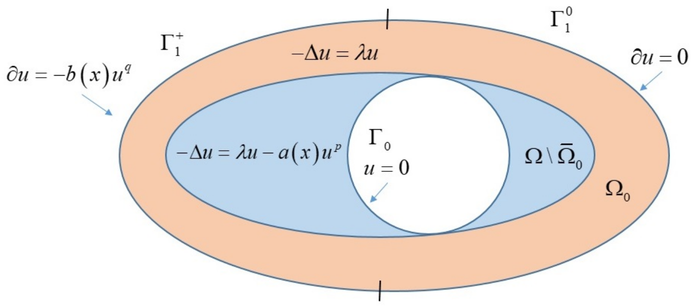

Figure 1: $\Gamma_1 \subset \partial\Omega_0, \quad b \in \mathcal{C}^+(\Gamma_1^+)$.

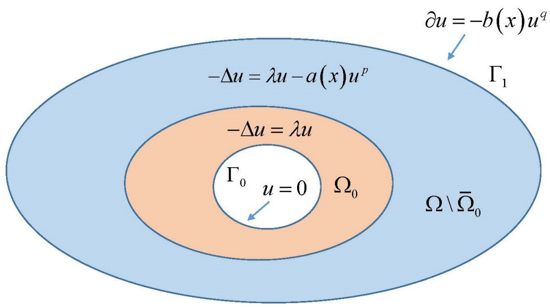

Figure 2: $\bar{\Omega}_0 \subset \Omega \cup \Gamma_0, \quad b \in \mathcal{C}(\Gamma_1)$.

[7, 9] it was analyzed, among other results, the existence of positive solutions of (1), in the particular case when $\bar{\Omega}_0 \subset \Omega$ and the potential a is bounded away from zero in any compact subset of $(\Omega \setminus \bar{\Omega}_0) \cup \Gamma_1$. In [10] it was analyzed

the particular special case when $\Omega = \Omega_0$ and (6) holds. The results obtained in this work extend the previous ones obtained in [7, 9], to cover the case when either (4) or (5) hold, and either (6) or (7) hold. Moreover, it should be noted that since we assume that (2) holds instead of (3) (as it was assumed in [7, 9]), now when $\mathrm{dist}(\bar{\Omega}_0, \Gamma_1) > 0$ we let that a vanishes on Γ_1 or in some subregion of Γ_1 and therefore, in this case a is not bounded away from zero in a neighborhood of Γ_1. The extensions carried out in this work are not straight with respect to the previous results, mainly when (4) and (7) hold, because to obtain them it is necessary to apply a great variety of very sharp results about principal eigenvalues. To obtain the new results under conditions (4) and (7) it is necessary to work with a family of singular boundary eigenvalue problems which possess Dirichlet and Neumann boundary conditions on the component Γ_1 of $\partial\Omega$ in a non-separated way. In this way, the results about principal eigenvalues recently obtained in [5] play a crucial role to develop our analysis.

Hereafter we denote by $\sigma_0^*[b, \Omega_0]$ the principal eigenvalue defined

$$\sigma_0^*[b, \Omega_0] := \begin{cases} \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (5) and (6) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (5) and (3) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}] & \text{if (4) and (6) hold,} \\ \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)] & \text{if (4) and (7) hold,} \end{cases} \tag{9}$$

where $\mathcal{D}$ stands for the Dirichlet boundary operator, $\mathfrak{B}^*(\Gamma_1^0)$ denotes the boundary operator defined

$$\mathfrak{B}^*\left(\Gamma_1^0\right)\varphi = \begin{cases} \varphi & \text{on } \Gamma_0\,, \\ \partial_\nu \varphi & \text{on } \Gamma_1^0\,, \\ \varphi & \text{on } \Gamma_1^+\,, \end{cases} \qquad \Gamma_1^+ = \Gamma_1 \setminus \Gamma_1^0 \tag{10}$$

(cf. (7)) and where $\sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]$ and $\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]$ stand for the principal eigenvalues of the problems $(-\Delta, \Omega_0, \mathcal{D})$ and $\left(-\Delta, \Omega_0, \mathfrak{B}^*(\Gamma_1^0)\right)$, respectively. It must be pointed out that, as show (9) and (41), when (4) and (7) hold, then

$$\sigma_0^*[b, \Omega_0] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)] < \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\,.$$

In (9) we can observe the dependence of $\sigma_0^*[b, \Omega_0]$ with respect to the potential b, since $\Gamma_1^0 = b^{-1}(0)$, and with respect to the relative position of the vanishing set Ω_0 of the potential a with respect to Γ_1. When (4) and (7) hold, the dependence of $\sigma_0^*(b, \Omega_0)$ with respect to b, is not with respect to the size of b but with respect to the amplitude of the piece Γ_1^0 where b vanishes. That is, $\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$ is decreasing with respect to the amplitude of Γ_1^0 and however, if $b_i \in \mathcal{C}^+(\Gamma_1^+)$, $i = 1, 2$, then we have that $\sigma_0^*[b_1, \Omega_0] = \sigma_0^*[b_2, \Omega_0] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$, independently of the size of them.

Let consider the logistic boundary value problem with linear mixed boundary conditions given by

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u = 0 & \text{on } \Gamma_1\,. \end{cases} \tag{11}$$

In the sequel we denote by $\Lambda_{NL}(\Omega_0, b)$ and $\Lambda_L(\Omega_0)$ the ranges of values of λ for which (1) and (11) possess positive solutions, respectively. Also we denote

$$\sigma_1 := \sigma_1^{\Omega}[-\Delta, \mathfrak{B}(0)]\,, \qquad \sigma_0^* := \sigma_0^*[b, \Omega_0]\,,$$

and we will say that a positive function u is strongly positive in Ω, and we will denote it by $u \gg 0$, if

$$u(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu u(x) < 0 \quad \forall\, x \in \Gamma_0 \quad \text{with} \quad u(x) = 0\,.$$

In the sequel we denote

$$W^2(\Omega) := \bigcap_{p>1} W_p^2(\Omega)\,, \qquad W^2_{\mathfrak{B}(V)} := \left\{ u \in W^2(\Omega) \,:\, \mathfrak{B}(V)u = 0 \right\}\,,$$

$$\mathcal{C}^\infty_{\Gamma_0 \cup \Gamma_1^+}(\Omega) := \left\{ \phi \in \mathcal{C}^\infty(\Omega) \,:\, \mathit{supp}\,\phi \subset \bar{\Omega} \setminus (\Gamma_0 \cup \Gamma_1^+) \right\}$$

and by $H^1_{\Gamma_0 \cup \Gamma_1^+}(\Omega)$ the clousure in $H^1(\Omega)$ of the set of functions $\mathcal{C}^\infty_{\Gamma_0 \cup \Gamma_1^+}(\Omega)$.

The following is the main result of this work. It gives the structure of the global bifurcation diagram of positive solutions of (1) and it compares $\Lambda_L(\Omega_0)$ with $\Lambda_{NL}(\Omega_0, b)$ depending on the nodal behavior and profiles of the potentials a and b.

Theorem 1.1. *Under any pair of assumptions of* (9)*, the following assertions are true:*

i) (1) *possesses a positive solution if, and only if*

$$\sigma_1 < \lambda < \sigma_0^*\,. \tag{12}$$

Moreover, the positive solution if it exists, it is unique and strongly positive in Ω. We will denote it by u_λ. Moreover,

$$u_\lambda \in W^2(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega})\,, \quad \forall \alpha \in (0,1)\,. \tag{13}$$

ii) The following hold:

a) If (5) *and* (6) *hold, then*

$$\Lambda_L(\Omega_0) = \Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big)\,. \tag{14}$$

b) If (4) *and* (6) *hold, then*

$$\Lambda_L(\Omega_0) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(0,\Omega_0)]\Big)\,, \tag{15}$$

$$\Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big) \tag{16}$$

and therefore,

$$\Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)\,. \tag{17}$$

c) If (4) *and* (7) *hold, then*

$$\Lambda_L(\Omega_0) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(0,\Omega_0)]\Big)\,, \tag{18}$$

$$\Lambda_{NL}(\Omega_0, b) = \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]\Big) \tag{19}$$

and therefore,

$$\Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)\,. \tag{20}$$

iii) Each positive solution u_λ *of* (1) *is linearly asymptotically stable, i.e., the principal eigenvalue of the linearization of* (1) *around* (λ, u_λ) *is positive. Moreover, the function*

$$\dot{u}_\lambda := \frac{d\,u_\lambda}{d\lambda} \gg 0 \qquad \text{in } \Omega \tag{21}$$

and in particular, for each $x \in \Omega \cup \Gamma_1$ *the map* $(\sigma_1, \sigma_0^*) \to (0, \infty)$ *defined*

$$\lambda \to u_\lambda(x)$$

is strictly increasing.

iv) There exists uniform $L^\infty(\Omega)$ *bounds for the positive solutions of* (1) *in any compact interval* I *of values of* λ *with* $I \subset [\sigma_1, \sigma_0^*)$.

v) The positive solutions of (1) *belong to a differentiable continuum* $\mathfrak{C}^+(\sigma_1)$ *of positive solutions. It emanates supercritically from the trivial branch* $(\lambda, u) = (\lambda, 0)$ *at the unique bifurcation value to positive solutions of* (1) $\lambda = \sigma_1$, *bifurcates from infinity at the unique bifurcation value to positive solutions from infinity* $\lambda = \sigma_0^*$ *and it is increasing in* $\|\cdot\|_{L^\infty(\Omega)}$ *with respect to the* λ*-parameter. In particular,*

$$\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right) = [\sigma_1, \sigma_0^*)\,, \tag{22}$$

and

$$\lim_{\lambda \downarrow \sigma_1} \|u_\lambda\|_{L_\infty(\Omega)} = 0, \qquad \lim_{\lambda \uparrow \sigma_0^*} \|u_\lambda\|_{L_\infty(\Omega)} = \infty\,,$$

where $\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right)$ *denotes the* λ*-projection of the continuum* $\mathfrak{C}^+(\sigma_1)$ *over the* λ*-axis.*

vi) *Let b_1, $b_2 \in \mathcal{C}(\Gamma_1)$ be such that*

$$b_2 > b_1 > 0\,, \tag{23}$$

let λ be satisfying

$$\lambda \in \Lambda_{NL}(\Omega_0, b_1) \cap \Lambda_{NL}(\Omega_0, b_2) \tag{24}$$

and let u_i, $i = 1, 2$ denote the unique positive solution of (1) *for $b = b_i$, $i = 1, 2$. Then,*

$$u_1 - u_2 \gg 0 \qquad \text{in } \Omega\,, \tag{25}$$

that is,

$$u_1(x) > u_2(x) \quad \forall\, x \in \Omega \cup \Gamma_1 \qquad \text{and} \qquad \partial_\nu u_1(x) < \partial_\nu u_2(x) \quad \forall\, x \in \Gamma_0.$$

vii) *Let $b \in \mathcal{C}(\Gamma_1)$ be such that $b > 0$, let $\lambda \in \Lambda_L(\Omega_0)$ be and let $\tilde{u}$, u_0 be the unique positive solution of* (1) *and* (11), *respectively, for such a value λ of the parameter. Then,*

$$u_0 > \tilde{u} \qquad \text{in } \Omega\,. \tag{26}$$

viii) *Assume that* (4) *holds and let b_1, $b_2 \in \mathcal{C}^+(\Gamma_1^+)$ be satisfying* (23). *Let $\lambda \in \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]\Big)$ be and let u_i, $i = 1, 2$ denote the unique positive solution of* (1) *for $b = b_i$, $i = 1, 2$. Then,* (25) *holds.*

ix) *Assume that* (4) *or* (5) *holds and let b_1, $b_2 \in \mathcal{C}(\Gamma_1)$ be bounded away from zero satisfying* (23). *Let $\lambda \in \Big(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\Big)$ be and let u_i, $i = 1, 2$ denote the unique positive solution of* (1) *for $b = b_i$, $i = 1, 2$. Then,* (25) *holds.*

Taking into account the results of Theorem 1.1, Figure 3 shows the global bifurcation diagrams of positive solutions of (1) (red dashed curve) and (11) (blue curve), constituted by the global continuum $\mathcal{C}^+(\sigma_1)$ of positive solutions emanating from $\lambda = \sigma_1$, where $\Lambda_L(\Omega_0) = (\sigma_1, \sigma_0)$ and $\Lambda_{NL}(\Omega_0, b) = (\sigma_1, \sigma_0^*)$, with $\sigma_0 \le \sigma_0^*$. Also, Figure 4 shows the global bifurcation diagrams of positive solutions of (1) for b_1, $b_2 \in \mathcal{C}^+(\Gamma_1^+)$ satisfying (23). The blue curve stands for the continuum $\mathcal{C}^+(\sigma_1)$ of (1) for $b = b_1$ and the the red dashed curve the global continuum $\mathcal{C}^+(\sigma_1)$ of (1) for $b = b_2$.

Following similar arguments to the given in the previous works [6, 7, 9, 12, 14], the results obtained in this paper may be generalized to ascertain the global bifurcation diagram of positive solutions of the following nonlinear elliptic weighted boundary value problem

$$\begin{cases} -\Delta u = \lambda W(x) u - a(x) f(x,u) u & \text{in } \Omega\,, \\ u = 0 & \text{on } \Gamma_0\,, \\ \partial_\nu u + V(x) u = -b(x) g(x,u) u & \text{on } \Gamma_1\,, \end{cases}$$

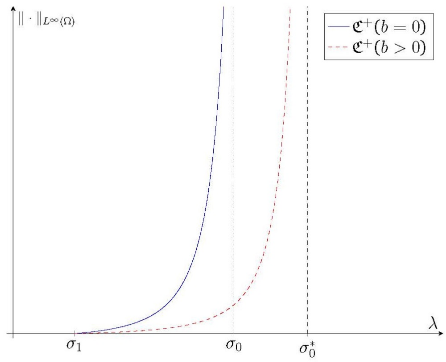

Figure 3: $\mathcal{C}^+(\sigma_1)$ for $b = 0$ and $b > 0$.

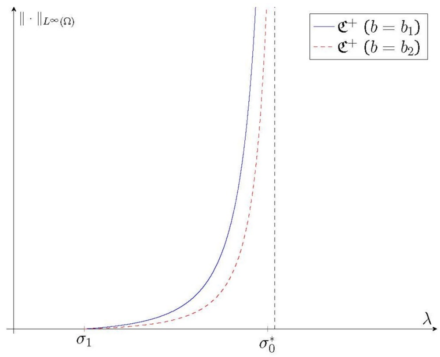

Figure 4: $\mathcal{C}^+(\sigma_1)$ for $b_1,\, b_2 \in \mathcal{C}^+(\Gamma_1^+),\; b_2 > b_1$.

where:

- $W \in L^\infty(\Omega)$ and $V \in \mathcal{C}(\Gamma_1)$ possess arbitrary sign in each point of Ω and Γ_1, respectively.
- The function $f \in \mathcal{C}^1(\bar{\Omega} \times [0,\infty); \mathbb{R})$ satisfies the following assumptions:

$$f(x,0) = 0\,, \qquad \frac{\partial f}{\partial u}(x,u) > 0 \quad (x,u) \in \Omega \times (0,\infty)$$

and

$$\lim_{u \uparrow \infty} f(x,u) = +\infty \text{ uniformly in } \bar{\Omega}\,.$$

- The function $g \in \mathcal{C}^1(\Gamma_1 \times [0,\infty); \mathbb{R})$ satisfies the following assumptions:

$$g(x,0) = 0\,, \qquad \frac{\partial g}{\partial u}(x,u) > 0 \quad (x,u) \in \Gamma_1 \times (0,\infty)$$

and

$$\lim_{u\uparrow\infty} g(x,u) = +\infty \text{ uniformly on } \Gamma_1 \,.$$

- The piece of the boundary Γ_0 possesses finitely many components satisfying

$$\Gamma_0 = \bigcup_{k=1}^{l} \Gamma_0^k \bigcup_{k=l+1}^{m} \Gamma_0^k \,,$$

with

$$\Gamma_0^k \cap \partial\Omega_0 = \emptyset \quad k = 1, \ldots, l \,, \qquad \Gamma_0^k \cap \partial\Omega_0 \neq \emptyset \quad k = l+1, \ldots, m \,.$$

- The piece of the boundary Γ_1 possesses finitely many components, some of them where the potential b in the nonlinear boundary condition is bounded away from zero, and the rest where b vanishes in some subregions of them.

Also, following the arguments and taking into account the results given in [13], the results of this paper may be generalized to cover the very general case when the vanishing set Ω_0 of the potential a is not a nice subdomain of Ω with $\Omega_0 \in \mathcal{C}^2$, but a very general set with no special restriction on its structure.

The main technical tools used to develop our analysis are bifurcation and monotonicity techniques.

The distribution of the rest of this paper is the following. Section 2 contains, without proofs, all the previous results about principal eigenvalues coming from [5, 12, 17, 20] that we will need to prove the main result. Section 3 contains the proof of Theorem 1.1. Finally, Section 4 includes without proof, the main result coming from [11] about the global structure of the diagram of positive solutions of (1) for a fixed λ in a suitable interval, considering the amplitude of the potential b on the boundary conditions as bifurcation-continuation parameter.

2. Preliminaries results about principal eigenvalues

In this section we collect the main results about principal eigenvalues coming from [5, 12, 17, 20] that are going to be used throughout the rest of this paper.

Hereafter, for each $k \in L^\infty(\Omega)$, $\mathcal{L}_k$ stands for the linear second order differential operator

$$\mathcal{L}_k : -\Delta + k(x) \,,$$

$\mathcal{D}$ stands for the Dirichlet boundary operator and for each $V \in \mathcal{C}(\Gamma_1)$, $\mathfrak{B}(V)$ denotes the boundary operator defined

$$\mathfrak{B}(V)\varphi = \begin{cases} \varphi & \text{on } \Gamma_0 \,, \\ \partial_\nu \varphi + V\varphi & \text{on } \Gamma_1 \,, \end{cases}$$

where $\partial_\nu \varphi$ stands for the outward normal derivative on Γ_1. It is known that for each $r > 1$

$$\mathfrak{B}(V) \in \mathcal{L}\left(W^2_r(\Omega), W_r^{2-\frac{1}{r}}(\Gamma_0) \times W_r^{1-\frac{1}{r}}(\Gamma_1)\right)$$

(cf. [2]). Also, given any proper subdomain Ω_0 of Ω of class $\mathcal{C}^2$ with

$$dist(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0\,, \tag{27}$$

we denote $\mathfrak{B}(V, \Omega_0)$ the boundary operator built from $\mathfrak{B}(V)$ through by

$$\mathfrak{B}(V,\Omega_0)\varphi := \begin{cases} \varphi & \text{on } \partial\Omega_0 \cap \Omega\,, \\ \mathfrak{B}(V)\varphi & \text{on } \partial\Omega_0 \cap \partial\Omega\,. \end{cases}$$

It should be pointed out that when $\bar{\Omega}_0 \subset \Omega \cup \Gamma_0$, then $\mathfrak{B}(V,\Omega_0) = \mathcal{D}$.

By *principal eigenvalue* of an eigenvalue problem we mean any eigenvalue of it which possesses a one-signed eigenfunction, and in particular a positive eigenfunction.

It follows from [2, Theorem 12.1] that the eigenvalue problem

$$\begin{cases} \mathcal{L}_k \varphi = \sigma\varphi & \text{in } \Omega\,, \\ \mathfrak{B}(V)\varphi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{28}$$

possesses a unique principal eigenvalue, denoted in the sequel by $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$, which is simple and the least eigenvalue of (28). Moreover, the positive eigenfunction φ^* associated to it, unique up multiplicative constant, is strongly positive in Ω, that is,

$$\varphi^*(x) > 0 \quad \forall\, x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu \varphi^*(x) < 0 \quad \forall\, x \in \Gamma_0\,, \tag{29}$$

and in addition

$$\varphi^* \in W^2_{\mathfrak{B}(V)}(\Omega) \subset \mathcal{C}^{1+\alpha}(\bar{\Omega}) \quad \text{for all } \alpha \in (0,1)\,. \tag{30}$$

The following result collects all the monotonicity properties of $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$ coming from [17, Proposition 3.2], [12, Propositions 3.1, 3.2, 3.3 and 3.5] and [20, Chapter 8] that we will use to develop our analysis.

Proposition 2.1. *The following monotonicity properties hold:*

i) Let k_1, $k_2 \in L^\infty(\Omega)$ and $V \in \mathcal{C}(\Gamma_1)$ be such that $k_1 < k_2$. Then

$$\sigma_1^\Omega[\mathcal{L}_{k_1}, \mathfrak{B}(V)] < \sigma_1^\Omega[\mathcal{L}_{k_2}, \mathfrak{B}(V)]\,. \tag{31}$$

ii) Let $V_1,\, V_2 \in \mathcal{C}(\Gamma_1)$ and $k \in L^\infty(\Omega)$ be such that $V_1 < V_2$. Then

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_1)] < \sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_2)]\,. \tag{32}$$

iii) For any $V \in \mathcal{C}(\Gamma_1)$ and $k \in L^\infty(\Omega)$ the following holds

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)] < \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}]\,. \tag{33}$$

iv) Let Ω_0 be a proper subdomain of Ω of class $\mathcal{C}^2$ satisfying (27). Then, for any $k \in L^\infty(\Omega)$

$$\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)] < \sigma_1^{\Omega_0}[\mathcal{L}_k, \mathfrak{B}(V, \Omega_0)]\,. \tag{34}$$

Let Ω_0 be a subdomain of Ω of class $\mathcal{C}^2$ with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ such that $\Gamma_0^0 \cap \Gamma_1 = \emptyset$, where $\Gamma_0^0 = \partial\Omega_0 \cap \Omega$, and Ω_n, $n \geq 1$, a sequence of bounded domains of $\mathbb{R}^N$ with boundary $\partial\Omega_n = \Gamma_0^n \cup \Gamma_1$ of class $\mathcal{C}^2$ such that $\Gamma_0^n \cap \Gamma_1 = \emptyset$, $n \geq 1$, where $\Gamma_0^n = \partial\Omega_n \cap \Omega$. It is said that Ω_n converges to Ω_0 from the exterior if for each $n \geq 1$

$$\Omega_0 \subset \Omega_{n+1} \subset \Omega_n\,, \qquad \bigcap_{n\geq 1} \bar{\Omega}_n = \bar{\Omega}_0\,. \tag{35}$$

The following result collects all the asymptotic behaviors of $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V)]$, coming from [17, Theorems 4.2 and 5.1], [12, Theorems 7.1, 8.2, 9.1 and 10.1] and [20, Chapter 8], that we will need later.

Proposition 2.2. *Let $k \in L^\infty(\Omega)$ be. Then the following hold:*

i) Let $B_1 := \{x \in \mathbb{R}^N : |x| < 1\}$ be, where $|\cdot|$ stands for the Lebesgue measure of $\mathbb{R}^N$, then

$$\liminf_{|\Omega|\downarrow 0} \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}] \geq |B_1|^{\frac{2}{N}} \sigma_1^{B_1}[-\Delta, \mathcal{D}] |\Omega|^{-\frac{2}{N}}\,. \tag{36}$$

ii) For any sequence $V_n \in \mathcal{C}(\Gamma_1)$, $n \geq 1$ satisfying

$$\lim_{n\uparrow\infty} \min_{x\in\Gamma_1} V_n(x) = \infty$$

yields

$$\lim_{n\to\infty} \sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}(V_n)] = \sigma_1^\Omega[\mathcal{L}_k, \mathcal{D}]\,. \tag{37}$$

iii) Let Ω_0 be a subdomain of Ω with boundary $\partial\Omega_0 = \Gamma_0^0 \cup \Gamma_1$ such that $\Gamma_0^0 \cap \Gamma_1 = \emptyset$ where $\Gamma_0^0 = \partial\Omega_0 \cap \Omega$, and let Ω_n, $n \geq 1$ be any sequence of *bounded domains of* $\mathbb{R}^N$ *of class* $\mathcal{C}^2$ *converging to* Ω_0 *from the exterior in the sense of* (35). *Then*

$$\lim_{n\to\infty} \sigma_1^{\Omega_n}[\mathcal{L}_k, \mathfrak{B}_n(V)] = \sigma_1^{\Omega_0}[\mathcal{L}_k, \mathfrak{B}(V)]\,, \tag{38}$$

where $\mathfrak{B}_n(V)$ *denotes the boundary operator defined*

$$\mathfrak{B}_n(V)u := \begin{cases} u & \text{on } \Gamma_0^n\,, \\ \partial_\nu u + Vu & \text{on } \Gamma_1\,, \end{cases} \qquad \Gamma_0^n := \partial\Omega_n \cap \Omega\,.$$

iv) Let $V_n \in \mathcal{C}(\Gamma_1)$*,* $n \geq 1$*, be an arbitrary sequence satisfying*

$$\lim_{n\to\infty} \|V_n - V\|_{L^\infty(\Gamma_1)} = 0\,,$$

with $V \in \mathcal{C}(\Gamma_1)$*. Then,*

$$\lim_{n\to\infty} \sigma_1^{\Omega}[\mathcal{L}_k, \mathfrak{B}(V_n)] = \sigma_1^{\Omega}[\mathcal{L}_k, \mathfrak{B}(V)]\,. \tag{39}$$

Now, let consider the boundary operator $\mathfrak{B}^*(\Gamma_1^0)$ defined in (10), where Γ_1^0 and Γ_1^+ are two disjoint connected pieces of Γ_1, closed and open, respectively as $N-1$ dimensional manifolds, such that $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$

The following result collects all the properties about the principal eigenvalue of the problem $\left(\mathcal{L}_k, \Omega, \mathfrak{B}^*\left(\Gamma_1^0\right)\right)$, coming from [5, Th. 1.1, Prop. 3.2, Cor. 3.4 and 3.5], that we will use in the sequel.

Proposition 2.3. *Let* $k \in L^\infty(\Omega)$ *be and let consider the eigenvalue problem*

$$\begin{cases} \mathcal{L}_k\varphi = \sigma\varphi & \text{in } \Omega\,, \\ \mathfrak{B}^*(\Gamma_1^0)\varphi = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{40}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ *and* $\Gamma_1 = \Gamma_1^0 \cup \Gamma_1^+$*, being* Γ_1^0 *and* Γ_1^+ *two disjoint connected pieces of* Γ_1*, closed and open, respectively as* $N-1$ *dimensional manifolds. Then,* (40) *possesses a unique principal eigenvalue, denoted in the sequel by* $\sigma_1^\Omega[\mathcal{L}_k, \mathfrak{B}^*(\Gamma_1^0)]$*, which is simple and the smallest eigenvalue of all other eigenvalues of* (40)*. Moreover, any eigenfunction of* (40) *associated to the principal eigenvalue is one-signed in* Ω *and if we denote by* $\varphi_1 \in H^1_{\Gamma_0\cup\Gamma_1^+}(\Omega)$ *the positive eigenfunction associated to it, unique up multiplicative constant, yields*

$$\varphi_1(x) > 0 \quad a.e. \text{ in } \quad \Omega\,.$$

In addition:

i) If $\psi^{\mathcal{D}}$ denotes the principal eigenfunction associated to $\sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]$ normalized so that $\|\psi^{\mathcal{D}}\|_{L^\infty(\Omega)}=1$, then

$$\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)]=\sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]+\frac{\int_{\Gamma_1^0}\partial_\nu\psi^{\mathcal{D}}\varphi_1}{\int_\Omega\psi^{\mathcal{D}}\varphi_1}<\sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]\,. \tag{41}$$

ii) For each $V\in\mathcal{C}^+(\Gamma_1^+)$ the following hold

$$\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(0)]<\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(V)]<\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)]<\sigma_1^\Omega[\mathcal{L}_k,\mathcal{D}]\,. \tag{42}$$

iii) The following characterization holds

$$\sigma_1^\Omega\left[\mathcal{L}_k,\mathfrak{B}^*(\Gamma_1^0)\right]=\sup_{V\in\mathcal{C}^+(\Gamma_1^+)}\sigma_1^\Omega[\mathcal{L}_k,\mathfrak{B}(V)]\,. \tag{43}$$

3. Proof of Theorem 1.1

In this section we prove some previous results that we need to prove Theorem 1.1 and finally we prove it.

PROPOSITION 3.1. *Let u_λ be a positive solution of (1) for the value λ of the parameter. Then,*

$$\lambda=\sigma_1^\Omega\left[-\Delta+a(x)u_\lambda^{p-1},\mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)\right]\,, \tag{44}$$

$$\lambda>\sigma_1^\Omega[-\Delta,\mathfrak{B}(0)] \tag{45}$$

and

$$u_\lambda\gg 0 \text{ in } \Omega\,,\quad u_\lambda\in W^2(\Omega)\subset\mathcal{C}^{1+\alpha}(\bar\Omega)\quad\forall\,\alpha\in(0,1)\,. \tag{46}$$

Moreover:

i) If either (5) is satisfied or (4) and (6) are satisfied, then

$$\lambda<\sigma_1^{\Omega_0}[-\Delta,\mathcal{D}]\,. \tag{47}$$

ii) If (4) and (7) are satisfied, then

$$\lambda<\sigma_1^{\Omega_0}[-\Delta,\mathfrak{B}^*(\Gamma_1^0)]\,. \tag{48}$$

Proof. Let u_λ be a positive solution of (1) for the value λ of the parameter. Then, $u_\lambda\in W_r^2(\Omega)$ for some $r>N$ and since $a\in\mathcal{C}(\bar\Omega)$ and $b\in\mathcal{C}(\Gamma_1)$, the following hold

$$\begin{cases}\left(-\Delta+a(x)u_\lambda^{p-1}\right)u_\lambda=\lambda u_\lambda & \text{in } \Omega\,,\\ \mathfrak{B}\left(b(x)u_\lambda^{q-1}\right)u_\lambda=0 & \text{on } \partial\Omega\,,\end{cases} \tag{49}$$

where

$$a(x)u_\lambda^{p-1} \in \mathcal{C}(\bar{\Omega}) , \qquad b(x)u_\lambda^{q-1} \in \mathcal{C}(\Gamma_1) .$$

Then, (49) fits into the framework of (28) and u_λ is a positive eigenfunction of (49) associated to the eigenvalue λ. Thus, (44) and (46) follow owing to the existence and uniqueness of the principal eigenvalue of (49), joint with the strongly positivity and regularity of its principal eigenfunction (cf.(28), (29) and (30)).

Now, since $a > 0$ and $b > 0$, owing to (31) and (32) it follow from (44) that

$$\lambda = \sigma_1^\Omega \left[-\Delta + a(x)u_\lambda^{p-1}, \mathfrak{B}\left(b(x)u_\lambda^{q-1} \right) \right] > \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] ,$$

which proves (45). Also, owing to the monotonicity of the principal eigenvalue with respect to the domain (cf.(34)) it follows from (44) that

$$\lambda = \sigma_1^\Omega \left[-\Delta + a(x)u_\lambda^{p-1}, \mathfrak{B}\left(b(x)u_\lambda^{q-1} \right) \right] < \sigma_1^{\Omega_0} \left[-\Delta, \mathfrak{B}\left(b(x)u_\lambda^{q-1}, \Omega_0 \right) \right] . \tag{50}$$

We now prove (47). Indeed, let assume that (5) holds. Then,

$$\mathfrak{B}\left(b(x)u_\lambda^{q-1}, \Omega_0 \right) = \mathcal{D} \tag{51}$$

and hence, (50) and (51) imply (47) under condition (5). In the same way, let assume now that (4) and (6) are satisfied. Then, since $b(x)u_\lambda^{q-1} \in \mathcal{C}(\Gamma_1)$, (47) follows from (50) owing to (33).

Finally we now prove (48). Indeed, let assume that (4) and (7) are satisfied. Then, since $u_\lambda(x) > 0$ for all $x \in \Gamma_1$ and $b \in \mathcal{C}^+(\Gamma_1^+)$ (cf. (8)), we have that $b(x)u_\lambda^{q-1} \in \mathcal{C}^+(\Gamma_1^+)$ and hence, (48) follows from (50) owing to (42).

This completes the proof. □

Proposition 3.2. *For each*

$$\lambda > \sigma_1^\Omega \left[-\Delta, \mathfrak{B}(0) \right] \tag{52}$$

(1) *possesses a positive strict subsolution arbitrarily small and strongly positive in* Ω.

Proof. Let λ be satisfying (52). Owing to (32) and (39) we have that for each $\varepsilon > 0$

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)] ,$$

and

$$\lim_{\varepsilon \downarrow 0} \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)] = \sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] ,$$

and therefore since (52) holds, there exists $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ the following hold

$$\sigma_1^\Omega[-\Delta, \mathfrak{B}(0)] < \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)] < \lambda\,. \tag{53}$$

Fix $\varepsilon \in (0, \varepsilon_1]$ satisfying (53) and let us denote $\sigma_1^\varepsilon := \sigma_1^\Omega[-\Delta, \mathfrak{B}(\varepsilon)]$ and φ_ε the principal eigenfunction associated to σ_1^ε normalized so that

$$\|\varphi_\varepsilon\|_{L^\infty(\Omega)} = 1\,. \tag{54}$$

Now, let us consider the function

$$\underline{u} = \alpha\varphi_\varepsilon\,, \tag{55}$$

where $\alpha > 0$ is an small constant to determine later.

Since φ_ε is strongly positive in Ω, to complete the proof it remains to prove that there exists $\tilde{\alpha} > 0$ small enough such that for each $\alpha \in (0, \tilde{\alpha})$ the function (55) provides us with a positive strict subsolution of (1). Indeed, pick up $\tilde{\alpha}$ satisfying

$$0 < \tilde{\alpha} < \min\left\{ \left(\frac{\lambda - \sigma_1^\varepsilon}{\|a\|_{L^\infty(\Omega)}} \right)^{\frac{1}{p-1}}, \left(\frac{\varepsilon}{\|b\|_{L^\infty(\Gamma_1)}} \right)^{\frac{1}{q-1}} \right\}.$$

Then, taking into account (53) and (54), we find that for each $\alpha \in (0, \tilde{\alpha}]$ the following estimate is satisfied in Ω

$$\begin{array}{rcl} -\Delta\underline{u} - \lambda\underline{u} + a(x)\underline{u}^p & = & \alpha\varphi_\varepsilon \left(\sigma_1^\varepsilon - \lambda + a(x)\alpha^{p-1}\varphi_\varepsilon^{p-1}\right) \\ & < & \alpha\varphi_\varepsilon \left(\sigma_1^\varepsilon - \lambda + \|a\|_{L^\infty(\Omega)}\tilde{\alpha}^{p-1}\right) < 0\,. \end{array} \tag{56}$$

Also, by construction the following estimate is satisfied on Γ_1

$$\begin{array}{rcl} \partial_\nu\underline{u} + b(x)\underline{u}^q & = & \alpha\varphi_\varepsilon \left(-\varepsilon + b(x)\alpha^{q-1}\varphi_\varepsilon^{q-1}\right) \\ & < & \alpha\varphi_\varepsilon \left(-\varepsilon + \|b\|_{L^\infty(\Gamma_1)}\tilde{\alpha}^{q-1}\right) < 0\,. \end{array} \tag{57}$$

Finally the following holds on Γ_0

$$\underline{u} = \alpha\varphi_\varepsilon = 0 \qquad \text{on } \Gamma_0\,. \tag{58}$$

Therefore, (56)-(58) prove that $\underline{u}$ provides us with a positive strict subsolution of (1) for each $\alpha \in (0, \tilde{\alpha}]$, which by construction is strongly positive in Ω.

This completes the proof. □

Proposition 3.3. *Assume that either*

i) (4) *and* (7), *or*

ii) (4) *and* (6)*, or*

iii) (5) *and* (6)*, or*

iv) (5) *and* (3)

hold. Then, for each

$$\lambda < \sigma_0^*[b, \Omega_0]\,, \tag{59}$$

(1) *possesses a positive strict supersolution, arbitrarily large and strongly positive in* Ω.

Proof. Taking into account the definition of $\sigma_0^*[b,\Omega_0]$ (cf. (9)), we have that under condition *i*), (59) becomes

$$\lambda < \sigma_1^{\Omega_0}\left[-\Delta, \mathfrak{B}^*(\Gamma_1^0)\right] \tag{60}$$

and under conditions *ii*), *iii*) or *iv*), (59) becomes

$$\lambda < \sigma_1^{\Omega_0}\left[-\Delta, \mathcal{D}\right]\,. \tag{61}$$

We now prove the result under conditon *i*).

Let us denote $\sigma_0^* := \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)]$ and let λ be satisfying (60).

Necessarily, either

$$\partial\Omega_0 \cap \Gamma_0 = \emptyset \tag{62}$$

or

$$\partial\Omega_0 \cap \Gamma_0 \neq \emptyset\,. \tag{63}$$

Assume (62) holds. Since (7) is satisfied we have that $b \in \mathcal{C}^+(\Gamma_1^+)$ and owing to (42) and (43) it follows from (60) that there exists $V \in \mathcal{C}^+(\Gamma_1^+)$ such that

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^*\,. \tag{64}$$

Set for each interval $I \subset (0,\infty)$

$$\Gamma_1^I = \left\{x \in \Gamma_1^+ \,:\, \mathrm{dist}_{\Gamma_1}(x, \Gamma_1^0) \in I\right\}\,,$$

where $\mathrm{dist}_{\Gamma_1}(\cdot, \Gamma_1^0)$ stands for the $N-1$ dimensional minimal distance along Γ_1. Now, for each $\varepsilon > 0$ sufficiently small, let us take a continuous perturbation $V_\varepsilon \in \mathcal{C}^+(\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]})$ of V satisfying

$$V_\varepsilon(x) \le V(x) \qquad \text{for all } x \in \Gamma_1$$

and

$$\lim_{\varepsilon\to 0} \|V - V_\varepsilon\|_{L^\infty(\Gamma_1)} = 0\,. \tag{65}$$

By construction we have that

$$V_\varepsilon(x) = 0 \quad \forall x \in \Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]} \qquad \text{and} \qquad V_\varepsilon(x) > 0 \quad \forall x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]} \tag{66}$$

and

$$V_\varepsilon < V \qquad \text{on } \Gamma_1 \,. \tag{67}$$

Owing to (67) and (32) we find that

$$\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] \tag{68}$$

and owing to (65), it follows from (39) that

$$\lim_{\varepsilon \downarrow 0} \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] \,. \tag{69}$$

Then, (64), (68) and (69) imply the existence of $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1]$ the following hold

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^* \,. \tag{70}$$

Fix $\varepsilon \in (0, \varepsilon_1]$ satisfying (70). Also, since $b \in \mathcal{C}^+(\Gamma_1^+)$, there exists a constant $\beta_\varepsilon > 0$ such that

$$b(x) \geq \beta_\varepsilon > 0 \quad \forall x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon)} \,. \tag{71}$$

On the other hand, for each $\delta > 0$ sufficiently small, let consider the δ-neighborhoods

$$\Omega_\delta := (\Omega_0 + B_\delta) \cap \Omega \,, \qquad \mathcal{N}_\delta := (\Gamma_0 + B_\delta) \cap \Omega \,, \tag{72}$$

where $B_\delta \subset \mathbb{R}^N$ denotes the ball of radius δ centered at the origin, and set

$$\Gamma_\delta := \partial\Omega_\delta \cap \Omega \,.$$

Then, $\partial\Omega_\delta = \Gamma_\delta \cup \Gamma_1$. Since $\Gamma_0 \cap \Gamma_1 = \emptyset$ and (62) holds, there exists $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$

$$\bar{\Omega}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset \,. \tag{73}$$

By construction we have that Ω_0 is a proper subdomain of Ω_δ and Ω_δ converges to Ω_0 from the exterior in the sense of (35). Then, it follows from (34) and (38) that

$$\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] \,, \quad 0 < \delta < \delta_0 \tag{74}$$

and

$$\lim_{\delta \downarrow 0} \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] \tag{75}$$

and therefore, (70), (74) and (75) imply the existence of $\delta_1 \in (0, \delta_0)$ such that

$$\lambda < \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V_\varepsilon)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(V)] < \sigma_0^* \tag{76}$$

for each $\delta \in (0, \delta_1)$. Let us denote in the sequel $\sigma_1^{\delta,\varepsilon} := \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(V_\varepsilon)]$. Also, since $\lim_{\delta\downarrow 0} |\mathcal{N}_\delta| = 0$, it follows from (36) the existence of $\delta_2 \in (0, \delta_1)$ such that for each $\delta \in (0, \delta_2)$

$$\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] > \sigma_0^* \,. \tag{77}$$

Now, fix $\delta \in (0,\delta_2)$ satisfying (76) and (77) and let $\varphi_\delta^\varepsilon$ and η_δ denote the principal eigenfunctions associated with the principal eigenvalues $\sigma_1^{\delta,\varepsilon}$ and $\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}]$, respectively, normalized so that $\|\varphi_\delta^\varepsilon\|_{L^\infty(\Omega_\delta)} = 1$ and $\|\eta_\delta\|_{L^\infty(\mathcal{N}_\delta)} = 1$.

Then, let consider now the positive function

$$\bar{u} := K\Phi \,,$$

where $K > 0$ is a sufficiently large constant to be determined later and $\Phi : \bar{\Omega} \to [0, \infty)$ is defined by

$$\Phi := \begin{cases} \varphi_\delta^\varepsilon & \text{in } \bar{\Omega}_{\frac{\delta}{2}} \,, \\ \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}} \,, \\ \xi_\delta^\varepsilon & \text{in } \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,, \end{cases}$$

where ξ_δ^ε is any regular positive extension of $\varphi_\delta^\varepsilon$ and η_δ from $\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ which is bounded away from zero in $\bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right)$. The existence of ξ_δ^ε is guaranteed since the functions

$$\varphi_\delta^\varepsilon|_{\Gamma_{\frac{\delta}{2}}} \,, \qquad \eta_\delta|_{\partial\mathcal{N}_{\frac{\delta}{2}} \cap \Omega}$$

are bounded away from zero. Let $\mu_\delta > 0$ be such that

$$\xi_\delta^\varepsilon(x) \geq \mu_\delta > 0 \qquad \forall\, x \in \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,. \tag{78}$$

Also, since a is bounded away from zero in any compact subset of $\Omega \setminus \bar{\Omega}_0$, there exists $\underline{a}_\delta > 0$ such that

$$a(x) \geq \underline{a}_\delta > 0 \qquad \forall\, x \in \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,. \tag{79}$$

To complete the proof it remains to show that there exists $\kappa > 0$ sufficiently large such that $\bar{u} = K\Phi$ provides us with a positive strict supersolution of (1) for each $K \geq \kappa$. Indeed, since $a > 0$ it follows from (76) that in $\Omega_{\frac{\delta}{2}}$ the following estimate is satisfied for each $K > 0$

$$\begin{aligned} -\Delta\bar{u} - \lambda\bar{u} + a(x)\bar{u}^p &= K\varphi_\delta^\varepsilon \left(\sigma_1^{\delta,\varepsilon} - \lambda + a(x)K^{p-1}(\varphi_\delta^\varepsilon)^{p-1} \right) \\ &\geq K\varphi_\delta^\varepsilon \left(\sigma_1^{\delta,\varepsilon} - \lambda \right) > 0 \,. \end{aligned} \tag{80}$$

Similarly, owing to (70) and (77) and since $a > 0$, the following estimate is satisfied in $\mathcal{N}_{\frac{\delta}{2}}$ for each $K > 0$

$$\begin{array}{rcl} -\Delta \bar{u} - \lambda \bar{u} + a(x)\bar{u}^p & = & K\eta_\delta \left(\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] - \lambda + a(x)K^{p-1}\eta_\delta^{p-1} \right) \\ & \geq & K\eta_\delta \left(\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] - \lambda \right) > 0 \,. \end{array} \tag{81}$$

Also, owing to (78) and (79) there exists $\kappa_1 > 0$ such that for any $K \geq \kappa_1 > 0$ the following estimate is satisfied in $\Omega \setminus \left(\Omega_{\frac{\delta}{2}} \cup \mathcal{N}_{\frac{\delta}{2}} \right)$

$$\begin{array}{rcl} -\Delta \bar{u} - \lambda \bar{u} + a(x)\bar{u}^p & \geq & K \left(-\Delta \xi_\delta^\varepsilon - \lambda \xi_\delta^\varepsilon + a(x) \left(\xi_\delta^\varepsilon \right)^p K^{p-1} \right) \\ & \geq & K \left(-\Delta \xi_\delta^\varepsilon - \lambda \xi_\delta^\varepsilon + \underline{a}_\delta \mu_\delta^p \kappa_1^{p-1} \right) \\ & \geq & K \left(-\|\Delta \xi_\delta^\varepsilon + \lambda \xi_\delta^\varepsilon\|_{L^\infty} + \underline{a}_\delta \mu_\delta^p \kappa_1^{p-1} \right) > 0 \,. \end{array} \tag{82}$$

As for the boundary conditions, on Γ_1 we will distinguish two different subregions, $\Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]}$ and $\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$. Since

$$V_\varepsilon(x) = 0, \qquad b(x) \geq 0 \qquad \forall\, x \in \Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]},$$

we find that by construction the following estimate is satisfied for any $K > 0$ on $\Gamma_1^0 \cup \Gamma_1^{(0,\varepsilon]}$

$$\begin{array}{rcl} \partial_\nu \bar{u} + b(x)\bar{u}^q & = & K\partial_\nu \varphi_\delta^\varepsilon + b(x)K^q \left(\varphi_\delta^\varepsilon \right)^q \\ & = & -KV_\varepsilon \varphi_\delta^\varepsilon + b(x)K^q \left(\varphi_\delta^\varepsilon \right)^q \\ & = & b(x)K^q \left(\varphi_\delta^\varepsilon \right)^q \geq 0 \,. \end{array} \tag{83}$$

Also, since $\varphi_\delta^\varepsilon$ is strongly positive in Ω_δ yields

$$m_\delta^\varepsilon := \min_{x \in \Gamma_1} \varphi_\delta^\varepsilon(x) > 0 \,. \tag{84}$$

Then, owing to the fact that $V_\varepsilon(x) > 0$ for all $x \in \Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$ and (71) and (84) hold, we find that there exists $\kappa_2 \geq \kappa_1 > 0$ such that the following estimate is satisfied on $\Gamma_1^+ \setminus \Gamma_1^{(0,\varepsilon]}$ for each $K \geq \kappa_2 > 0$

$$\begin{array}{rcl} \partial_\nu \bar{u} + b(x)\bar{u}^q & = & K\varphi_\delta^\varepsilon \left[-V_\varepsilon(x) + b(x)K^{q-1} \left(\varphi_\delta^\varepsilon \right)^{q-1} \right] \\ & \geq & K\varphi_\delta^\varepsilon \left[-\|V_\varepsilon\|_{L^\infty(\Gamma_1)} + \beta_\varepsilon \kappa_2^{q-1} \left(m_\delta^\varepsilon \right)^{q-1} \right] > 0 \,. \end{array} \tag{85}$$

Finally, by construction

$$\bar{u}|_{\Gamma_0} = K\eta_\delta|_{\Gamma_0} = 0 \,. \tag{86}$$

Then, (80)-(82) and (83)-(86) prove that, under condition (62), $\bar{u}$ provides us with a positive strict supersolution of (1) for each $K \geq \kappa_2 > 0$, which by construction is strongly positive in Ω.

This completes the proof of the result under condition (62).

Now, let assume that (63) holds. Then, pick up $\delta > 0$, let denote

$$\tilde{\Omega} := \Omega \cup B_\delta(\Gamma_0)\,, \qquad \tilde{\Gamma}_0 := \partial\tilde{\Omega} \setminus \Gamma_1\,.$$

where $B_\delta(\Gamma_0) \subset \mathbb{R}^N$ stands for a δ-neighborhood of Γ_0, let consider the auxiliary potential

$$\tilde{a} = \begin{cases} 1 & \text{in } \tilde{\Omega} \setminus \Omega\,, \\ a & \text{in } \Omega\,, \end{cases}$$

the auxiliary boundary operator

$$\tilde{\mathfrak{B}}(b) := \begin{cases} \mathcal{D} & \text{on } \tilde{\Gamma}_0\,, \\ \partial_\nu + b & \text{on } \Gamma_1\,, \end{cases}$$

and the associated boundary value problem

$$\begin{cases} -\Delta u = \lambda u - \tilde{a}(x)u^p & \text{in } \tilde{\Omega}\,, \\ u = 0 & \text{on } \tilde{\Gamma}_0\,, \\ \partial_\nu u = -b(x)u^q & \text{on } \Gamma_1\,. \end{cases} \tag{87}$$

By construction

$$\tilde{\Omega}_0 = \Omega_0\,, \quad \tilde{\Gamma}_0 \cap \bar{\tilde{\Omega}}_0 = \emptyset\,, \quad \sigma_1^{\tilde{\Omega}_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)] = \sigma_0^*$$

and (60) becomes

$$\lambda < \sigma_1^{\tilde{\Omega}_0}[-\Delta, \mathfrak{B}^*(\Gamma_1^0)]\,. \tag{88}$$

Then, (87) satisfies condition (62), and hence, since (88) holds, we find by the above arguments that (87) possesses a positive strict supersolution $\tilde{u}$ arbitrarily large and strongly positive in $\tilde{\Omega}$ for each λ satisfying (60). Now, it is straightforward to prove that the function

$$\bar{u} := \tilde{u}|_{\bar{\Omega}}$$

provides us with, for each λ satisfying (60), a positive strict supersolution of (1) under condition (63), which is arbitrarily large and strongly positive in Ω.

This completes the proof of the result under condition i).

We now prove the result under condition ii). Let us denote $\sigma_0 := \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]$ and let λ be satisfying (61). Whithout lost of generality we will assume that (62) holds. On the contrary we would argue as in case i) when (63) holds. Owing to (33) and (37) we have that

$$\sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \quad \forall\, n \in \mathbb{N}\,, \qquad \lim_{n\uparrow\infty} \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] = \sigma_0\,. \tag{89}$$

Then, owing to (61) and (89), there exists $n \in \mathbb{N}$ large enough such that

$$\lambda < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \,. \tag{90}$$

Fix $n \in \mathbb{N}$ satisfying (90). Now, let consider the δ-neighborhoods Ω_δ and $\mathcal{N}_\delta$ defined in (72). Using the same arguments than in case i), it follows the existence of $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$ (73) holds and moreover

$$\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)]$$

and

$$\lim_{\delta \downarrow 0} \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] = \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] \,.$$

Thus, taking into account (90), there exists $\delta_1 \in (0, \delta_0)$ such that for each $\delta \in (0, \delta_1)$ the following hold

$$\lambda < \sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)] < \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}(n)] < \sigma_0 \,. \tag{91}$$

Also, arguing as in case i), there exists $\delta_2 \in (0, \delta_1)$ such that for each $\delta \in (0, \delta_2)$

$$\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}] > \sigma_0 \,. \tag{92}$$

Now, fix $\delta \in (0, \delta_2)$ satisfying (91) and (92), let φ_δ^n and η_δ denote the principal eigenfunctions associated with the principal eigenvalues $\sigma_1^{\Omega_\delta}[-\Delta, \mathfrak{B}(n)]$ and $\sigma_1^{\mathcal{N}_\delta}[-\Delta, \mathcal{D}]$ normalized so that $\|\varphi_\delta^n\|_{L^\infty(\Omega_\delta)} = 1$ and $\|\eta_\delta\|_{L^\infty(\mathcal{N}_\delta)} = 1$ Now, let consider the positive function

$$\bar{u} = K\Phi \,,$$

where $K > 0$ is a sufficiently large constant to be determined later and $\Phi : \bar{\Omega} \to [0, \infty)$ is defined by

$$\Phi := \begin{cases} \varphi_\delta^n & \text{in } \bar{\Omega}_{\frac{\delta}{2}} \,, \\ \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}} \,, \\ \xi_\delta^n & \text{in } \bar{\Omega} \setminus \left(\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}} \right) \,, \end{cases}$$

being ξ_δ^n any regular positive extension of φ_δ^n and η_δ from $\bar{\Omega}_{\frac{\delta}{2}} \cup \bar{\mathcal{N}}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ which is bounded away from zero. Now, taking into account (91) and (92) and the fact that φ_δ^n is strongly positive in Ω_δ and arguing as in case i), it is not hard to prove that there exists $\tilde{\kappa} > 0$ such that for each $K \geq \tilde{\kappa}$, $\bar{u}$ is a positive strict supersolution of (1), which by construction is strongly positive in Ω.

This completes the proof of the result under condition ii).

Now, taking the notations of the previous cases, the proof of the result under condition iii) follows arguing as in cases i) and ii), taking $\bar{u} = K\Phi$, where $K>0$ is sufficiently large and

$$\Phi := \begin{cases} \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}}\,, \\ \varphi_\delta & \text{in } \bar{\Omega}_{\frac{\delta}{2}}\,, \\ \psi_\delta & \text{in } \bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega\,, \\ \xi_\delta & \text{in } \bar{\Omega} \setminus \Big(\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}} \cup (\bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega) \Big)\,, \end{cases}$$

being $\Omega_\delta := \bar{\Omega}_0 + B_\delta$ and $\mathcal{A}_\delta := \Gamma_1 + B_\delta$, for $\delta > 0$ small enough so that

$$\bar{\Omega}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset\,, \qquad \bar{\Omega}_\delta \cap \bar{\mathcal{A}}_\delta = \emptyset\,, \qquad \bar{\mathcal{A}}_\delta \cap \bar{\mathcal{N}}_\delta = \emptyset\,,$$

and where ψ_δ stands for the principal eigenfunction associated to $\sigma_1^{\mathcal{A}_\delta}[-\Delta, \mathcal{D}]$, normalized with L^∞-norm equals 1 in its domain, and ξ_δ is any positive regular extension of η_δ, φ_δ and ψ_δ from $\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}} \cup (\bar{\mathcal{A}}_{\frac{\delta}{2}} \cap \Omega)$ to $\bar{\Omega}$, bounded away from zero.

Finally, taking the notations of the previous cases, the proof of the result under condition *iv*) follows arguing in a similar way taking $\bar{u} = K\Phi$ for $K>0$ sufficiently large and

$$\Phi := \begin{cases} \eta_\delta & \text{in } \bar{\mathcal{N}}_{\frac{\delta}{2}}\,, \\ \varphi_\delta & \text{in } \bar{\Omega}_{\frac{\delta}{2}}\,, \\ \xi_\delta & \text{in } \bar{\Omega} \setminus \Big(\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}} \Big)\,, \end{cases}$$

where now ξ_δ is any positive regular extension of η_δ and φ_δ from $\bar{\mathcal{N}}_{\frac{\delta}{2}} \cup \bar{\Omega}_{\frac{\delta}{2}}$ to $\bar{\Omega}$ bounded away from zero, satisfying

$$\partial_\nu \xi_\delta(x) \geq 0 \qquad \forall\, x \in \Gamma_1\,, \tag{93}$$

whose existence is guaranteed by construction. This completes the proof. $\square$

We now prove Theorem 1.1

Proof of Theorem 1.1: We are going to prove *i*). Indeed, let u_λ be a positive solution of (1) for the value λ of the parameter. Then, taking into account the definition of $\sigma_0^*[b, \Omega_0]$ (cf.(9)), the necessary condition (12) for the existence of positive solution follows from (45), (47) and (48).

To prove the sufficient condition (12) for the existence of positive solution of (1) we will use the sub-supersolution method (cf. [1]). Let λ be satisfying (12). Then owing to Proposition 3.2, (1) possesses a positive strict subsolution $\underline{u}_\lambda$, arbitrarily small and strongly positive in Ω. On the other hand, it follows from Proposition 3.3 that for each λ satisfying (12), (1) possesses a positive strict supersolution $\bar{u}_\lambda$, arbitrarily large and strongly positive in Ω.

Then, since both of them, the subsolution $\underline{u}_\lambda$ and the supersolution $\bar{u}_\lambda$, are strongly positive in Ω, it is possible to take them satisfying $\underline{u}_\lambda < \bar{u}_\lambda$ in Ω, and owing to the sub-supersolution method we find that (1) possesses a positive solution u_λ, with $\underline{u}_\lambda < u_\lambda < \bar{u}_\lambda$, for each λ satisfying (12).

The proof of the uniqueness of positive solution, if it exists, is obtained following the same arguments than in [9, Theorem 3.1].

The fact that any positive solution u_λ of (1) is strongly positive in Ω and that (13) holds, follow from (46).

We now prove *ii)* The results about the structure of $\Lambda_L(\Omega_0)$ in (14), (15) and (18) follow from [14, Theorem 3.5] and [6, Theorem 3.4]. The results about the structure of $\Lambda_{NL}(\Omega_0, b)$ in (14), (16) and (19) follow from (12), taking into account the definition of $\sigma_0^*[\Omega_0, b]$. Finally, (17) and (20) follow from (33) and (42).

We now prove *iii)* Let u_λ be a positive solution of (1) for the value λ of the parameter. Then, (12) and (44) hold and differentiating (1) with respect to λ we find that $\dot{u}_\lambda := \frac{du_\lambda}{d\lambda}$ satisfies the following problem

$$\begin{cases} \left(-\Delta + pa(x)u_\lambda^{p-1} - \lambda\right)\dot{u}_\lambda = u_\lambda > 0 & \text{in } \Omega\,, \\ \mathfrak{B}\left(qb(x)u_\lambda^{q-1}\right)\dot{u}_\lambda = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{94}$$

Also, since $a > 0$ and $u_\lambda > 0$ in Ω, $b > 0$ on Γ_1 and $p, q > 1$, owing to (31) and (32) it follows from (44) that

$$\sigma_1^\Omega\left[-\Delta + pau_\lambda^{p-1} - \lambda, \mathfrak{B}(qbu_\lambda^{q-1})\right] > \sigma_1^\Omega\left[-\Delta + au_\lambda^{p-1} - \lambda, \mathfrak{B}(bu_\lambda^{q-1})\right] = 0\,, \tag{95}$$

that is, u_λ is linearly asymptotically stable. Moreover, owing to the Characterization of the Strong Maximum Principle given by H. Amann and J. López-Gómez in [3, Theorem 2.4], it follows from (95) that (94) satisfies the strong maximum principle, and hence (21) holds.

We now prove *iv)*. Let $I = [\alpha, \beta]$ be, with $\beta > \sigma_1$, a compact interval with $I \subset [\sigma_1, \sigma_0^*)$ and let u_β the unique positive solution of (1) for $\lambda = \beta$, whose existence and uniqueness are guaranteed by *i)*. Then, owing to (21) we have that $u_\lambda \leq u_\beta$ for all $\lambda \in I$ and therefore, $\|u_\lambda\|_{L^\infty(\Omega)} \leq \|u_\beta\|_{L^\infty(\Omega)}$ for all $\lambda \in I$ which proves *iv)*.

We now prove *v)* The fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (1) from the trivial branch $(\lambda, u) = (\lambda, 0)$, and the existence of a differentiable continuum $\mathfrak{C}(\sigma_1)$ of solutions of (1) emanating from the trivial branch at the value $\lambda = \sigma_1$, follow from [8, Theorem 1.1]. Now, let denote by $\mathfrak{C}^+(\sigma_1)$ the maximal subcontinuum of $\mathfrak{C}(\sigma_1)$ constituted by the positive solutions of (1) emanating from the trivial branch at $(\lambda, u) = (\sigma_1, 0)$ and $\mathcal{P}_\lambda\left(\mathfrak{C}^+(\sigma_1)\right)$ its projection on the λ axis. The fact that $\mathfrak{C}^+(\sigma_1)$ emanates supercritically from the trivial branch follows from (12) or [8, Theorem 1.1].

Let φ_1 denote the principal eigenfunction associated to the principal eigenvalue σ_1, normalized such that $\|\varphi_1\|_{L^\infty(\Omega)} = 1$. It is known that $\varphi_1 \gg 0$ in Ω. Since $\lambda = \sigma_1$ is a simple eigenvalue of the linearization of (1) around $(\lambda, u) = (\sigma_1, 0)$ and owing to the fact that $\lambda = \sigma_1$ is the unique bifurcation value to positive solutions of (1) from the trivial branch, it follows from the updated version of the Global Alternative of P.H. Rabinowitz [23, Theorem 1.27] given by J. López-Gómez in [19, Theorem 6.4.3] that either $\mathfrak{C}^+(\sigma_1)$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}$, or it contains a pair $(\tilde{\lambda}, \tilde{u})$ with $\tilde{u}$ strongly positive in Ω satisfying $\int_\Omega \tilde{u}\varphi_1 = 0$, which is impossible since φ_1 is strongly positive in Ω. Then, we get that $\mathfrak{C}^+(\sigma_1)$ is unbounded in $\mathbb{R} \times \mathcal{C}^1_{\Gamma_0}(\bar{\Omega})$ and since (12) holds, we find that $\mathfrak{C}^+(\sigma_1)$ is bounded in $\mathbb{R}$ and unbounded in $L^\infty(\Omega)$. Now, the existence of uniform $L^\infty(\Omega)$ bounds for the positive solutions of (1) in compact intervals of values of λ contained in $[\sigma_1, \sigma_0^*)$, guaranteed by *iv*), joint with the fact that $\mathfrak{C}^+(\sigma_1)$ is unbounded in $L^\infty(\Omega)$, imply that $\lambda = \sigma_0^*$ is the unique bifurcation value to positive solutions of (1) from infinity and that $\mathfrak{C}^+(\sigma_1)$ bifurcates from infinity to positive solutions at $\lambda = \sigma_0^*$. In particular, owing to (12) and since $\mathfrak{C}^+(\sigma_1)$ is connected and it bifurcates to positive solutions from the trivial branch at $\lambda = \sigma_1$ and from infinity at $\lambda = \sigma_0^*$, we find that (22) holds.

Finally, since (12) and (22) hold and taking into account the structure of $\mathfrak{C}^+(\sigma_1)$, the fact that any positive solution of (1) belongs to $\mathfrak{C}^+(\sigma_1)$ follows from the uniqueness of positive solution of (1) for any λ satisfying (12). The fact that $\mathfrak{C}^+(\sigma_1)$ is increasing in $\|\cdot\|_{L^\infty(\Omega)}$ with respect to the λ-parameter follows from (21) and the uniqueness of positive solution of (1) for each λ satisfying (12).

We now prove *vi*) Since (24) holds, the existence and uniqueness of u_i, $i = 1, 2$, follow from (24) and *i*). Owing to (44) the following holds

$$\sigma_1^\Omega[-\Delta - \lambda + a(x)u_i^{p-1}, \mathfrak{B}(b_i u_i^{q-1})] = 0\,, \qquad i = 1, 2\,. \tag{96}$$

Now, let denote $\Theta = u_1 - u_2$. By construction, Θ satisfies the following problem

$$\begin{cases} (-\Delta - \lambda + a(x)F(x))\,\Theta = 0 & \text{in } \Omega\,, \\ \Theta = 0 & \text{on } \Gamma_0\,, \\ (\partial_\nu + b_1(x)G(x))\,\Theta = (b_2 - b_1)u_2^q > 0 & \text{on } \Gamma_1\,, \end{cases} \tag{97}$$

where $F \in \mathcal{C}(\bar{\Omega})$ and $G \in \mathcal{C}(\Gamma_1)$ are defined by

$$F(x) := \begin{cases} \dfrac{u_1(x)^p - u_2(x)^p}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x)\,, \\ p u_1^{p-1}(x) & \text{if } u_1(x) = u_2(x)\,, \end{cases}$$

and

$$G(x) := \begin{cases} \dfrac{u_1(x)^q - u_2(x)^q}{u_1(x) - u_2(x)} & \text{if } u_1(x) \neq u_2(x)\,, \\ q u_1^{q-1}(x) & \text{if } u_1(x) = u_2(x)\,. \end{cases}$$

By construction, and since $p, q > 1$ and $u_1 \gg 0$ in Ω we have that

$$F(x) > u_1^{p-1}, \qquad G(x) > u_1^{q-1}$$

and hence, since $a > 0$, $b_1 > 0$ and owing to (31), (32) and (96) we find that

$$\sigma_1^{\Omega}[-\Delta - \lambda + a(x)F(x), \mathfrak{B}(b_1 G(x))] \\ > \sigma_1^{\Omega}[-\Delta - \lambda + a(x)u_1^{p-1}, \mathfrak{B}(b_1 u_1^{q-1})] = 0 . \quad (98)$$

Then, owing to the Characterization of the Strong Maximum Principle [3, Theorem 2.4] it follows from (98) that (97) satisfies the strong maximum principle and therefore $\Theta := u_1 - u_2 \gg 0$ in Ω, which proves (25) and completes the proof of *vi*).

We now prove *vii*). The existence and uniqueness of u_0 follows from the fact that $\lambda \in \Lambda_L(\Omega_0)$ and [6, Theorem 1.5]. The existence of $\tilde{u}$ follows from the fact that owing to *ii*), we have that $\lambda \in \Lambda_L(\Omega_0) \subset \Lambda_{NL}(\Omega_0, b)$. The uniqueness of $\tilde{u}$ follows from *i*). Finally (26) follows arguing exactly as in *vi*), taking into account that $b > 0$ instead of (23).

We now prove *viii*). Since (4) holds and $b_i \in \mathcal{C}^+(\Gamma_1^+)$, $i = 1, 2$, it follows from (19) that

$$\Lambda_{NL}(\Omega_0, b_1) = \Lambda_{NL}(\Omega_0, b_2) = (\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathfrak{B}^*\left(\Gamma_1^0\right)] ,$$

and hence,

$$\lambda \in \Lambda_{NL}(\Omega_o, b_i) , \quad i = 1, 2 . \quad (99)$$

Then, the existence and uniqueness of u_i, $i = 1, 2$ is guaranteed by (99) and *i*). Now the result follows from *vi*).

We now prove *ix*). The result follows from *vi*), arguing as in *viii*), taking into account that now

$$\Lambda_{NL}(\Omega_0, b_1) = \Lambda_{NL}(\Omega_0, b_2) = \left(\sigma_1, \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\right) .$$

This completes the proof. □

4. The amplitude of the nonlinearity in the boundary conditions as bifurcation parameter

In order to complete the exposition, in this section we consider, under the general assumptions of this paper, the bi-parameter logistic elliptic problem

$$\begin{cases} -\Delta u = \lambda u - a(x)u^p & \text{in } \Omega , \quad p > 1 , \\ u = 0 & \text{on } \Gamma_0 , \\ \partial_\nu u = \gamma b(x) u^q & \text{on } \Gamma_1 , \quad q > 1 , \end{cases} \quad (100)$$

where λ, $\gamma \in \mathbb{R}$, $a > 0$ in Ω and $b > 0$ on Γ_1. The main goal of this section is to ascertain the global structure of the set of positive solutions of (100) considering γ as the bifurcation-continuation parameter, for some λ fixed in a suitable interval. We focus in the particular case when $\bar{\Omega}_0 \subset \Omega$ and (3) holds, and we denote

$$\sigma_1 := \sigma_1^{\Omega}[-\Delta, \mathfrak{B}(0)]\,, \qquad \tilde{\sigma}_0 := \sigma_1^{\Omega}[-\Delta, \mathcal{D}]\,, \qquad \sigma_0^* := \sigma_1^{\Omega_0}[-\Delta, \mathcal{D}]\,,$$

where $\sigma_1 < \tilde{\sigma}_0 < \sigma_0^*$.

The following result collects the main findings about the global structure of the set of positive solutions of (100) considering γ as bifurcation-continuation parameter, for some λ fixed in a suitable interval. It is one of the main results of [11]. We include it without proof and we remit to [11, Theorem 1.2] for the details of its proof.

THEOREM 4.1. *Assume $\lambda \in (\sigma_1, \sigma_0^*)$, and let u_0 denote the unique positive solution of* (100) *for $\gamma = 0$. Then:*

i) For each $\gamma \leq 0$, (100) *possesses a unique positive solution u_γ, which is linearly asymptotically stable. Moreover, the map*

$$\begin{array}{ccc} (-\infty, 0] & \longrightarrow & \mathcal{C}^1_{\Gamma_0}(\bar{\Omega}) \\ \gamma & \rightarrow & u_\gamma \end{array}$$

is differentiable and $\dot{u}_\gamma := \dfrac{du_\gamma}{d\gamma} \gg 0$ *in* Ω, *and in particular, the map*

$$\begin{array}{ccc} (-\infty, 0] & \mapsto & \mathcal{C}_{\Gamma_0}(\bar{\Omega}) \\ \gamma & \mapsto & u_\gamma \end{array}$$

is strictly increasing.

ii) If (6) *holds and $\lambda \in (\sigma_1, \tilde{\sigma}_0)$, then there exists $D(\lambda) > 0$ such that*

$$\|u_\gamma\|_{L^\infty(\Omega)} \leq D(\lambda) \left(\frac{1}{\underline{b}\tilde{\gamma}}\right)^{\frac{1}{q-1}}\,, \qquad \textit{for all } \gamma < 0\,.$$

In particular

$$\lim_{\gamma \downarrow -\infty} \|u_\gamma\|_{L^\infty(\Omega)} = 0\,,$$

that is, the problem exhibits bifurcation to positive solutions from the trivial branch $(\gamma, u) = (\gamma, 0)$ when $\gamma \downarrow -\infty$.

iii) There exists $\varepsilon_0 > 0$ and a differentiable map

$$\begin{array}{cccc} u\; : & (-\varepsilon_0, \varepsilon_0) & \rightarrow & \mathcal{C}^1_{\Gamma_0}(\bar{\Omega}) \\ & \gamma & \mapsto & u^*_\gamma \end{array}$$

such that $u^*_\gamma = u_\gamma$ *for all* $\gamma \in (-\varepsilon_0, 0]$, *there exists a neighborhood* $\mathfrak{U}$ *of* $(\gamma, u) = (0, u_0)$ *in* $(-\varepsilon_0, \varepsilon_0) \times \mathcal{C}^1_{\Gamma_0}(\bar\Omega)$ *such that if* $(\gamma, \tilde u) \in \mathfrak{U}$ *is a positive solution of* (100), *then* $\tilde u = u^*_\gamma$, *and in addition* (γ, u^*_γ) *is a positive linearly asymptotically stable solution of* (100) *for all* $\gamma \in (-\varepsilon_0, \varepsilon_0)$. *Moreover,*

$$\dot u^*_\gamma := \frac{du^*_\gamma}{d\gamma} \gg 0 \qquad in\ \Omega\,,$$

and in particular the map

$$\begin{array}{ccc} (-\varepsilon_0, \varepsilon_0) & \to & \mathcal{C}_{\Gamma_0}(\bar\Omega) \\ \gamma & \mapsto & u^*_\gamma \end{array}$$

is strictly increasing.

iv) Any positive solution $\hat u_\gamma$ *of* (100) *for* $\gamma > 0$ *satisfies* $\hat u_\gamma \gg u_0$.

v) If $p > 2q - 1$, *then the following hold:*

a) For each $\gamma > 0$, (100) *possesses at least a positive solution.*

b) For each $\gamma > 0$, (100) *possesses a minimal positive solution* u^{min}_γ *satisfying* $u^{min}_\gamma \gg u_0$ *and* $u^{min}_\gamma = u^*_\gamma$ *for* $\gamma \in (0, \varepsilon_0)$, *where* ε_0 *and* u^*_γ *are defined by iii).*

c) There exist uniform $L^\infty(\Omega)$ *bounds for the positive solutions of* (100) *in compact intervals of values of* γ.

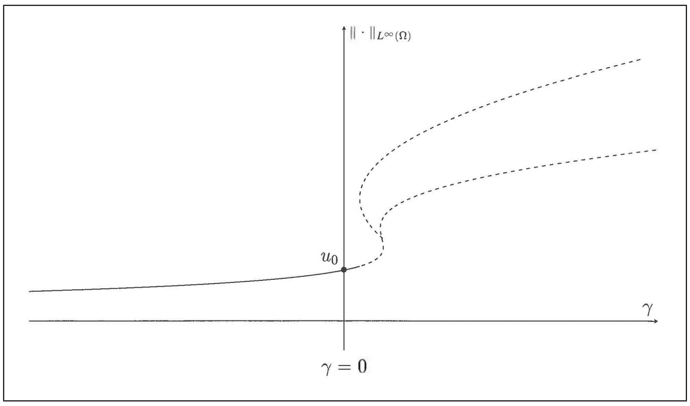

Figure 5: Global bifurcation diagram of positive solutions of (100) in the γ parameter ($\lambda \in (\sigma_1, \sigma_0^*)$).

Theorem 4.1 establishes that, for each fixed $\lambda \in (\sigma_1, \sigma_0^*)$, the global bifurcation diagram in the γ-parameter of the positive solutions of (100) should be like shown by Figure 5, where the continuous line stands for the exact structure of the set of positive solutions for $\gamma < \varepsilon_0$ and the dashed line stands for a possible configuration of the set of positive solutions of (100) for $\gamma > \varepsilon_0$.

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A note on a class of double well potential problems

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"Dedicated to Julian Lopez-Gomez on the occasion of his 60th birthday"

Abstract. *It is well known that under appropriate conditions on a double well potential, the associated Hamiltonian system possesses a pair of heteroclinic solutions joining the minima of the potential in addition to infinitely many other homoclinics and heteroclinics that oscillate between these minima. This paper studies the effect on such solutions of replacing the temporal domain, $\mathbb{R}$, by a finite but long time interval.*

Keywords: double well potential, variational methods, nondegeneracy condition, heteroclitic solutions, homoclinic solutions, multitransition solutions.
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1. Introduction

Consider the Hamiltonian system:

$$-\ddot{q} + V_q(t,q) = 0,\ t \in \mathbb{R} \tag{HS}$$

where V is a double well potential. Several papers, [8, 9, 11, 12, 15, 19, 34, 35] have used variational methods to treat the existence and multiplicity of solutions of (HS) that are heteroclinic or homoclinic to the the points a^- and a^+ corresponding to the bottoms of the potential wells. See also [1–7, 10, 13, 14, 16–18, 20–33] for the use of such methods for related problems. The main goal of this note is to study (i) the extent to which these solutions persist qualitatively if (HS) is replaced by a large time boundary value problem with a^- and a^+ as boundary states and (ii) the behavior of these finite time solutions as the time interval tends to $\mathbb{R}$. To be more precise, suppose that V satisfies

(V_1) $V \in C^1(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ and is 1-periodic in $t \in \mathbb{R}$.

(V_2) There are points $a^-, a^+ \in \mathbb{R}^m$ such that $V(t,q) > V(t,a^\pm) = 0$ for any $t \in \mathbb{R}$ and $q \in \mathbb{R}^m \setminus \{a^-, a^+\}$.

(V_3) There is a constant, $V_0 > 0$, such that $\liminf_{|q|\to+\infty} V(t,q) \geq V_0$.

Associated with (HS) is the Lagrangian, $L(q) = \frac{1}{2}|\dot{q}|^2 + V(t,q)$, and the functional

$$I(q) = \int_{\mathbb{R}} L(q)\, dt.$$

For $i \in \mathbb{Z}$, let $T_i = [i, i+1]$. Set

$$E \equiv \left\{ q \in W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m) \mid \int_{\mathbb{R}} |\dot{q}|^2\, dt + \int_{T_0} |q|^2\, dt < \infty \right\}.$$

E is a Hilbert space under the inner product associated with the norm

$$\|q\|^2 = \int_{\mathbb{R}} |\dot{q}|^2\, dt + \int_{T_0} |q|^2\, dt.$$

Consider I on E and set

$$\Gamma(a^-, a^+) = \left\{ q \in E \mid q(\pm\infty) = a^\pm \right\}$$

where by $q(\pm\infty) = a^\pm$ is meant $\lim_{t\to\pm\infty} q(t) = a^\pm$. In the present setting, this condition is equivalent to requiring that, as in [15], $\lim_{i\to\pm\infty} \|q - a^\pm\|_{L^2(T_i, \mathbb{R}^m)} = 0$. Define

$$c(a^-, a^+) = \inf_{q \in \Gamma(a^-, a^+)} I(q). \tag{1}$$

Let

$$\mathcal{M}(a^-, a^+) = \left\{ q \in \Gamma(a^-, a^+) \mid I(q) = c(a^-, a^+) \right\}.$$

It was shown in [15] that $\mathcal{M}(a^-, a^+) \neq \emptyset$ and any $Q \in \mathcal{M}(a^-, a^+)$ is a C^2 solution of (HS) heteroclinic from a^- to a^+. Likewise reversing the roles of a^- and a^+ in $\Gamma(a^-, a^+), c(a^-, a^+)$ and $\mathcal{M}(a^-, a^+)$ yields solutions of (HS) heteroclinic from a^+ to a^-.

It was further shown in [15] that there are many other heteroclinics joining a^- and a^+ as well as homoclinic solutions to a^- and to a^+ provided that the sets, $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$ are not too degenerate. Indeed, when the corresponding nondegeneracy condition is satisfied, for any $k \in \mathbb{N}$, there are infinitely many solutions that oscillate k times between small neighborhoods of a^- and a^+, the solutions being distinguished by the amount of time they spend near the intermediate equilibria. Similar statements apply to the other possibilities for such connecting orbits. The nondegeneracy requirements will be described more fully in Section 2. These requirements lead to new multi-transition solutions that are obtained as local minima of I.

Turning now to our main goal, let $\sigma = (\sigma^-, \sigma^+)$. The analogue of (HS) that will be studied here is

$$-\ddot{q} + V_q(t,q) = 0,\ t \in \sigma,\ \ q(\sigma^-) = a^-, q(\sigma^+) = a^+. \tag{2}$$

The corresponding functional is

$$I_\sigma(q) = \int_\sigma L(q)\, dt$$

where

$$q \in \Gamma_\sigma(a^-, a^+) \equiv \left\{ q \in E \mid q(t) = a^- \text{ for } t \le \sigma^-;\ q(t) = a^+ \text{ for } t \ge \sigma^+ \right\}.$$

Due to the periodicity of V in t, the problem (2) is equivalent to the analogous one on the translated interval $\sigma + k$ for any $k \in \mathbb{Z}$. Thus, without loss of generality, we can normalize the choice of the interval, σ, by assuming that its center belongs to $[0, 1)$. With this choice, when $|\sigma| > 1$, we have $\sigma^- < 0 < \sigma^+$.

Let

$$c_\sigma(a^-, a^+) = \inf_{q \in \Gamma_\sigma(a^-,a^+)} I_\sigma(q) = \inf_{q \in \Gamma_\sigma(a^-,a^+)} I(q). \tag{3}$$

Thus $\Gamma_\sigma(a^-, a^+) \subset \Gamma(a^-, a^+)$ and $c_\sigma(a^-, a^+) \ge c(a^-, a^+)$.

In Section 2, it will be shown that for any $\sigma \subset \mathbb{R}$, there is a global minimizer, $Q_\sigma \in \Gamma_\sigma(a^-, a^+)$, of I_σ. In addition, under the same nondegeneracy condition on $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$ that leads to the infinitude of local minima of I, it will be proved that there are also local minimizers of I_σ whenever $|\sigma|$ is sufficiently large. These local minimizers are near elements of $\mathcal{M}(a^-, a^+)$ since, as will be proved, the local minimizers converge along subsequences to members of $\mathcal{M}(a^-, a^+)$ as $|\sigma| \to +\infty$. Then in Section 3, the same nondegeneracy assumption leads to analogous results in the setting of multitransition local minima solutions. In particular as $\sigma^+ - \sigma^-$ increase, there appear more and more local minima of I_σ and associated multitransition solutions of (2). Moreover as $\sigma^+, -\sigma^- \to \infty$, again any corresponding sequence of such solutions has a subsequence converging to a solution of the same type of (HS).

2. One transition local minima of I_σ

In this section the existence of minima of I_σ and their behavior for large σ will be studied.

Lemma 2.1. *For all $\sigma = (\sigma^-, \sigma^+)$ with $\sigma^+ > \sigma^-$, there is a $Q_\sigma \in \Gamma_\sigma$ such that $I_\sigma(Q_\sigma) = c_\sigma(a^-, a^+)$. Any such minimizer is a (classical) solution of* (HS).

Proof. The existence is immediate since I_σ is weakly lower semicontinuous and Γ_σ is weakly closed. That the minimizer is C^2 and satisfies (HS) follows from standard arguments. □

To study the behavior of the solutions, Q_σ of (2) as $\sigma^+, -\sigma^- \to \infty$, some a priori bounds for these functions will be obtained. For convenience, suppose that $\sigma^+ \geq 1$ and $\sigma^- \leq 0$. For $t \in [0,1]$, set $\varphi(t) = ta^+ + (1-t)a^-$. Extend the domain of φ to $\mathbb{R}$ via $\varphi(t) = a^-$ for $t \leq 0$ and $\varphi(t) = a^+$ for $t \geq 1$. Thus $\varphi \in \Gamma_\sigma$ for all such σ and

$$c_\sigma(a^-, a^+) \leq I_\sigma(\varphi) = \int_0^1 L(\varphi)\, dt \equiv M_0 \tag{4}$$

independently of σ.

Proposition 2.2. *There is a constant $M > 0$ such that*

$$\|Q_\sigma\|_{W^{1,2}(T_i,\mathbb{R}^m)} \leq M \tag{5}$$

independently of i and σ.

Proof. From (4), we find

$$\|\dot{Q}_\sigma\|^2_{L^2(\sigma,\mathbb{R}^m)} \leq 2M_0. \tag{6}$$

With this initial estimate, the argument of the proof of Proposition 2.24 of [15] can be followed yielding (5). □

Taking advantage of Proposition 2.2, a natural approach to study the behavior of the minima, c_σ, and minimizers, Q_σ, of I_σ for large σ is to begin by taking any sequence, $\sigma_k = (\sigma_k^-, \sigma_k^+)$ with $-\sigma_k^-, \sigma_k^+ \to \infty$ as $k \to \infty$ together with corresponding sequences, c_{σ_k} and Q_{σ_k}. The minima, c_{σ_k} are easy to deal with:

Proposition 2.3. *Suppose that $\sigma_k \subset \sigma_{k+1}$ and $\sigma_k^+, -\sigma_k^- \to \infty$ as $k \to \infty$. Then $c_{\sigma_k} \geq c_{\sigma_{k+1}} \to c(a^-, a^+)$ as $k \to \infty$.*

Proof. Let $Q_{\sigma_k} \in \Gamma_{\sigma_k}$ such that $I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k}$. Then, since $\Gamma_{\sigma_k} \subset \Gamma_{\sigma_{k+1}}$,

$$c_{\sigma_k} = I_{\sigma_{k+1}}(Q_{\sigma_k}) \geq c_{\sigma_{k+1}}.$$

To show that $c_{\sigma_k} \to c(a^-, a^+)$ as $k \to \infty$, let $q^* \in \mathcal{M}(a^-, a^+)$ and $\varepsilon > 0$. Choosing $s \in \mathbb{N}$, define q_s^* where

$$q_s^*(t) = \begin{cases} a^-, & t \leq -s-1, \\ (-s-t)a^- + (t+s+1)q^*(t), & -s-1 \leq t \leq -s, \\ q^*(t), & -s \leq t \leq s, \\ (t-s)a^+ + (s+1-t)q^*(t), & s \leq t \leq s+1, \\ a^+, & s+1 \leq t. \end{cases}$$

Then $q_s^* \in \Gamma_\sigma$ for any σ for which $-\sigma^-, \sigma^+ \geq s+1$. For $s = s(\varepsilon)$ sufficiently large, it can be assumed that

$$\int_{s \leq |t| \leq s+1} L(q_s^*)\, dt \leq \varepsilon. \tag{7}$$

Choose k so that $-\sigma_k^-, \sigma_k^+ \geq s(\varepsilon)+1$. Then $q_{\sigma_k}^* \in \Gamma_{\sigma_k}$ and by (7)

$$\begin{aligned} c_{\sigma_k} \leq I_{\sigma_k}(q_{s_k}^*) = \int_{|t|<s_k} L(q^*)\, dt + \int_{s_k \leq |t| \leq s_k+1} L(q_{s_k}^*)\, dt \\ \leq I(q^*) + \varepsilon = c(a^-, a^+) + \varepsilon \end{aligned} \tag{8}$$

and the Proposition follows from (8) and the fact that $c_{\sigma_k} \geq c(a^-, a^+)$ $\square$

Next we would like to show that a subsequence of the functions, Q_{σ_k}, converges to a member of $\mathcal{M}(a^-, a^+)$. The bounds of (5) imply there is a function, $Q \in E$ such that along a subsequence, Q_{σ_k} converges to Q weakly in E. Unfortunately it may be the case that $Q = a^-$ or $Q = a^+$. This possibility was excluded in the proof in [15] showing that $I(q)$ has a minimizer in $\Gamma(a^-, a^+)$ by exploiting the fact that $\Gamma(a^-, a^+)$ is invariant under the family of integer phase shifts $q(t) \to q(t+j)$ for $j \in \mathbb{Z}$. This invariance property no longer holds for $\Gamma_\sigma(a^-, a^+)$. Nevertheless as the next result shows, more can be said about the convergence of the sequence, Q_{σ_k}. For $z \in \mathbb{R}$, let $[z]$ denote the integer part of z.

Proposition 2.4. *Suppose that $\sigma_k \subset \sigma_{k+1}$ and $\sigma_k^+, -\sigma_k^- \to \infty$ as $k \to \infty$. Let $Q_{\sigma_k} \in \Gamma_{\sigma_k}$ be such that $I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k}$. Then there is a $\tau_k \in \sigma_k$ for each $k \in \mathbb{N}$, and there is a $Q \in \mathcal{M}(a^-, a^+)$ such that along a subsequence, $Q_{\sigma_k}(\cdot + [\tau_k]) - Q \to 0$ in E as $k \to \infty$.*

Remark 2.5: Note that $Q_{\sigma_k} \in \Gamma_{\sigma_k} \subset \Gamma(a^-, a^+)$ and by Proposition 2.3, $I(Q_{\sigma_k}) = I_{\sigma_k}(Q_{\sigma_k}) = c_{\sigma_k} \to c(a^-, a^+)$. Hence the sequence (Q_{σ_k}) is a minimizing sequence for I on $\Gamma(a^-, a^+)$. Consequently the conclusion of Proposition 2.4 can be interpreted as a variant of the Palais-Smale condition for minimizing sequences in the current setting. Similar conclusions have been obtained in related settings. See e.g. Proposition 2.50 of [31] or Theorem 2.7 of [24].

Proof of Proposition 2.4. As has just been noted, the sequence (Q_{σ_k}) is a minimizing sequence for I on $\Gamma(a^-, a^+)$. By Proposition 2.2, for any $i \in \mathbb{Z}$, $\|Q_{\sigma_k}\|_{W^{1,2}((i,i+1),\mathbb{R}^m)} \leq M$.

Choose $\tau_k \in \sigma_k$ so that $|Q_{\sigma_k}(\tau_k) - a^-| = 1/2|a^+ - a^-|$. Then via (V_1),

(i) $q_k \equiv Q_{\sigma_k}(\cdot + [\tau_k]) \in \Gamma(a^-, a^+)$,

(ii) $|q_k(\tau_k - [\tau_k]) - a^-| = |Q_{\sigma_k}(\tau_k) - a^-| = 1/2|a^+ - a^-|$,

(iii) $I(q_k) = I(Q_{\sigma_k}) = c_{\sigma_k} \to c(a^-, a^+)$,

(iv) $\|q_k\|_{W^{1,2}((i,i+1),\mathbb{R}^m)} = \|Q_{\sigma_k}\|_{W^{1,2}((i+[\tau_k],i+1+[\tau_k]),\mathbb{R}^m)} \le M$ for any $i \in \mathbb{Z}$.

Therefore as in the paragraph before this Proposition, there exists a $Q \in E$ such that along a subsequence still denoted by (q_k), as $k \to \infty, q_k \to Q$ weakly in $W^{1,2}(T, \mathbb{R}^m)$ for any bounded interval $T \subset \mathbb{R}$. Item (iv) and the fact that $\int_{\mathbb{R}} |\dot{q}_k|^2\, dt \le 2I(q_k)$, show (q_k) is bounded in E. Hence $q_k \to Q$ weakly in E. We claim

$$Q \in \Gamma(a^-, a^+). \tag{9}$$

Assuming (9) for the moment, the rest of Proposition 2.4 follows. Indeed (9) implies $I(Q) \ge c(a^-, a^+)$. By the weak lower semicontinuity of I,

$$I(Q) \le \liminf_{k\to+\infty} I(q_k) = c(a^-, a^+).$$

Thus $I(Q) = c(a^-, a^+)$, and $Q \in \mathcal{M}(a^-, a^+)$.

Next to show that $q_k - Q \to 0$ in E, it suffices to verify that

$$\int_{\mathbb{R}} |\dot{q}_k|^2\, dt \to \int_{\mathbb{R}} |\dot{Q}|^2\, dt \tag{10}$$

as $k \to \infty$. Towards this end, observe that by weak lower semicontinuity again,

$$\int_{\mathbb{R}} V(t, Q)\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} V(t, q_k)\, dt$$

and

$$\int_{\mathbb{R}} |\dot{Q}|^2\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt.$$

Thus combining these estimates gives

$$\begin{aligned}\limsup_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt &= \lim_{k\to+\infty} 2I(q_k) - 2\liminf_{k\to+\infty} \int_{\mathbb{R}} V(t, q_k)\, dt \\ &\le 2I(Q) - 2\int_{\mathbb{R}} V(t, Q)\, dt = \int_{\mathbb{R}} |\dot{Q}|^2\, dt \le \liminf_{k\to+\infty} \int_{\mathbb{R}} |\dot{q}_k|^2\, dt\end{aligned}$$

from which (10) follows.

It remains to prove (9). To do so, let $B_r(a^\pm)$ denote the open ball of radius r in $\mathbb{R}^m$ centered at $a^\pm$. Let $r_0 \in (0, \frac{1}{2}|a^+ - a^-|)$ be such that

$$\max\{V(t, \xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_{r_0}(a^+) \cup \bar{B}_{r_0}(a^-)\} < V_0.$$

For $r \in (0, r_0)$ set

$$\underline{\omega}_r = \min\{V(t, \xi) \mid t \in \mathbb{R}, \xi \in \mathbb{R}^m \setminus (B_r(a^+) \cup B_r(a^-))\} \text{ and}$$

$$\overline{\omega}_r = \max\{V(t,\xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_r(a^+) \cup \bar{B}_r(a^-)\}.$$

By $(V_1) - (V_3)$, $\underline{\omega}_r > 0$ and $\overline{\omega}_r \to 0$ as $r \to 0$. Moreover if $(\alpha,\beta) \subset \mathbb{R}$ is such that $q_k(t) \in \mathbb{R}^m \setminus (B_r(a^+) \cup B_r(a^-))$ for any $t \in (\alpha,\beta)$, then

$$\begin{aligned} I_{(\alpha,\beta)}(q_k) &= \tfrac{1}{2}\|\dot{q}_k\|^2 + \int_{(\alpha,\beta)} V(t,q_k)\,dt \\ &\geq \tfrac{1}{2(\beta-\alpha)}|q_k(\beta) - q_k(\alpha)|^2 + \underline{\omega}_r(\beta-\alpha) \geq \sqrt{2\underline{\omega}_r}|q_k(\beta) - q_k(\alpha)|. \end{aligned} \tag{11}$$

Set $\omega = \underline{\omega}_{|a^+-a^-|/4}$ and define a constant, Δ, by

$$\Delta = \sqrt{2\omega}\,|a^+ - a^-|/8.$$

Since $\overline{\omega}_\varepsilon \to 0$ as $\varepsilon \to 0$, ε can be chosen in $(0, |a^+ - a^-|/8)$ so that

$$\varepsilon^2 + 2\overline{\omega}_\varepsilon < 2\Delta. \tag{12}$$

Set $C = \max_{k\in\mathbb{N}} I(q_k)$ and $T_\varepsilon = 2C/\underline{\omega}_\varepsilon$. Due to (ii), $q_k(\tau_k - [\tau_k]) \notin B_\varepsilon(a^+) \cup B_\varepsilon(a^-)$. Suppose $q_k(t) \notin B_\varepsilon(a^+) \cup B_\varepsilon(a^-)$ for any $t \in (\tau_k - [\tau_k], \tau_k - [\tau_k] + T_\varepsilon)$. Then by (11),

$$I_{(\tau_k-[\tau_k],\tau_k-[\tau_k]+T_\varepsilon)}(q_k) \geq \underline{w}_\varepsilon T_\varepsilon \geq 2C. \tag{13}$$

which is not possible by the definition of C. Hence for any $k \in \mathbb{N}$,

$$\text{there is an } \ell_k^+ \in (\tau_k - [\tau_k], \tau_k - [\tau_k] + T_\epsilon) \text{ with } q_k(\ell_k^+) \in B_\epsilon(a^+) \cup B_\epsilon(a^-). \tag{14}$$

Similarly for any $k \in \mathbb{N}$

$$\text{there is an } \ell_k^- \in (\tau_k - [\tau_k] - T_\epsilon, \tau_k - [\tau_k]) \text{ with } q_k(\ell_k^-) \in B_\epsilon(a^+) \cup B_\epsilon(a^-). \tag{15}$$

The next step in proving (9) is to verify that for any $k \in \mathbb{N}$,

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \geq \ell_k^+ \text{ and} \tag{16}$$

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \leq \ell_k^-. \tag{17}$$

Their proofs being the same, only (16) will be proved. Recall that by definition, $q_k(t) = a^+$ for any $t \geq \sigma_k^+ - [\tau_k]$ and by (ii), $|q_k(\tau_k - [\tau_k]) - a^-| = \frac{1}{2}|a^+ - a^-|$, for any $k \in \mathbb{N}$. Arguing indirectly, assume that for some $k \in \mathbb{N}$, there exists an $\ell_0 \in [\ell_k^+, \sigma_k^+ - [\tau_k])$ for which $q_k(\ell_0) \in B_\epsilon(a^-)$. Since $|q_k(\tau_k - [\tau_k]) - a^-| = \frac{1}{2}|a^+ - a^-|$ and $q_k(\ell_0) \in B_\epsilon(a^-)$, there is an interval $(\alpha,\beta) \subset (\tau_k - [\tau_k], \ell_0)$ such that $q_k(t) \notin B_{|a^+-a^-|/4}(a^-) \cup B_{|a^+-a^-|/4}(a^+)$ for any $t \in (\alpha,\beta)$ and $|q(\beta) - q(\alpha)| \geq |a^+ - a^-|/4$. Then, by the definition of ω, $V(t, q_k(t)) \geq \omega$ for any $t \in (\alpha,\beta)$ so by (11),

$$I_{(\alpha,\beta)}(q_k) \geq \sqrt{2\omega}\,|q_k(\beta) - q_k(\alpha)| \geq \sqrt{2\omega}\,|a^+ - a^-|/4 = 2\Delta. \tag{18}$$

Thus if $q_k(\ell_0) \in B_\epsilon(a^-)$, (18) provides a positive lower bound for $I_{(\alpha,\beta)}(q_k)$. Next it will be shown that the same is true for $I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k)$. Indeed, consider the function

$$\bar{q}_k(t) = \begin{cases} a^- & t \le \ell_0 - 1, \\ (\ell_0 - t)a^- + (t + 1 - \ell_0)q_k(\ell_0) & \ell_0 - 1 \le t \le \ell_0, \\ q_k(t) & \ell_0 \le t. \end{cases}$$

Then $\bar{q}_k \in \Gamma_{\sigma_k+[\tau_k]}(a^-, a^+)$ and $\bar{Q}_k(t) \equiv \bar{q}_k(t - [\tau_k]) \in \Gamma_{\sigma_k}(a^-, a^+)$. Hence $I(\bar{q}_k) = I_{\sigma_k}(\bar{Q}_k) \ge c_{\sigma_k}$. Since $I(\bar{q}_k) \ge I_{(\ell_0-1,\ell_0)}(\bar{q}_k) + I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k)$, it follows that

$$I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge c_{\sigma_k} - I_{(\ell_0-1,\ell_0)}(\bar{q}_k). \tag{19}$$

Using the definition of $\bar{q}_k$ on $(\ell_0 - 1, \ell_0)$ and that $q_k(\ell_0) \in B_\epsilon(a^-)$ gives

$$I_{(\ell_0-1,\ell_0)}(\bar{q}_k) \le \int_{\ell_0-1}^{\ell_0} \tfrac{1}{2}|q_k(\ell_0) - a^-|^2 + \max_{(t,\xi)\in[\ell_0-1,\ell_0]\times B_\epsilon(a^-)} V(t,\xi)\,dt \le \tfrac{1}{2}\epsilon^2 + \overline{\omega}_\epsilon.$$

This estimate together with (12) and (19) implies

$$I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge c_{\sigma_k} - \Delta. \tag{20}$$

Combining (iii), (18) and (20) then yields

$$c_{\sigma_k} = I(q_k) \ge I_{(\alpha,\beta)}(q_k) + I_{(\ell_0,\sigma_k^+-[\tau_k])}(q_k) \ge 2\Delta + c_{\sigma_k} - \Delta,$$

a contradiction. Thus (16) is proved.

Since $-T_\epsilon - 1 < \ell_k^- < \tau_k - [\tau_k] < \ell_k^+ < T_\epsilon + 1$, by (16) and (17),

$$q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \ge T_\epsilon + 1 \text{ and } q_k(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \le -T_\epsilon - 1.$$

The convergence of $q_k(t)$ to $Q(t)$ for any $t \in \mathbb{R}$ then shows

$$Q(t) \in \mathbb{R}^m \setminus B_\epsilon(a^-) \text{ for } t \ge T_\epsilon + 1 \\ \text{and } Q(t) \in \mathbb{R}^m \setminus B_\epsilon(a^+) \text{ for } t \le -T_\epsilon - 1. \tag{21}$$

But $I(Q) < +\infty$ so by Proposition 2.3 of [15], there are points, $\varphi^\pm \in \{a^-, a^+\}$ such that $Q(\pm\infty) = \varphi^\pm$. Consequently (21) shows $\varphi^\pm = a^\pm$. Then (9) follows and the proposition is proved. $\square$

In general the sequence (τ_k) given by Proposition 2.4 may not be bounded. Thus the sequence Q_{σ_k} may not converges in E to a $Q \in \mathcal{M}(a^-, a^+)$. In other words we cannot guarantee that the problem (2) has solutions which approximate fixed elements of $\mathcal{M}(a^-, a^+)$ as $|\sigma| \to +\infty$. We do not know if it is essential, but to get around this difficulty, we require a further condition. As was shown in [15], in order to obtain multitransition solutions of (HS), some nondegeneracy conditions are required for $\mathcal{M}(a^-,a^+) \cup \mathcal{M}(a^+,a^-)$ and they also suffice to overcome the present difficulty. To introduce them, some results from [15] will be recalled. Set

$$\mathcal{S}(a^-,a^+) = \{q|_{T_0} \mid q \in \mathcal{M}(a^-,a^+)\}.$$

The subset $\mathcal{S}(a^-,a^+)$ of $W^{1,2}(T_0,\mathbb{R}^m)$ possesses the following properties:

- $\bar{\mathcal{S}}(a^-,a^+) = \mathcal{S}(a^-,a^+) \cup \{a^-\} \cup \{a^+\}$,
- $\bar{\mathcal{S}}(a^-,a^+)$ is compact in $W^{1,2}(T_0,\mathbb{R}^m)$.

For the details, see [15].

Let $\mathcal{C}_{a^-}(a^-,a^+)$ be the component of $\bar{\mathcal{S}}(a^-,a^+)$ containing a^- and let $\mathcal{C}_{a^+}(a^-,a^+)$ be the component of $\bar{\mathcal{S}}(a^-,a^+)$ containing a^+. Then from e.g. [15], we have

Proposition 2.6. *Either*

(i) $\mathcal{C}_{a^-}(a^-,a^+) = \mathcal{C}_{a^+}(a^-,a^+)$, *or*

(ii) $\mathcal{C}_{a^-}(a^-,a^+) = \{a^-\}$ *and* $\mathcal{C}_{a^+}(a^-,a^+) = \{a^+\}$.

If (ii) holds, there exist nonempty disjoint compact sets,

$$K_{a^-}(a^-,a^+), K_{a^+}(a^-,a^+) \subset \bar{\mathcal{S}}(a^-,a^+)$$

such that

a) $a^- \in K_{a^-}(a^-,a^+)$, $a^+ \in K_{a^+}(a^-,a^+)$,

b) $\bar{\mathcal{S}}(a^-,a^+) = K_{a^-}(a^-,a^+) \cup K_{a^+}(a^-,a^+)$,

c) $\mathrm{dist}(K_{a^-}(a^-,a^+), K_{a^+}(a^-,a^+)) \equiv 5r(a^-,a^+) > 0$.

Remark 2.7: The splitting, $K_{a^-}(a^-,a^+), K_{a^+}(a^-,a^+)$, of $\bar{\mathcal{S}}(a^-,a^+)$ is not unique. Indeed subjecting each of the functions, q that make up these sets to the same integer phase shift produces a new such splitting. For what follows, we fix the choice of this splitting.

Remark 2.8: Reversing the roles of a^- and a^+ yields $\mathcal{C}_{a^+}(a^+,a^-)$, $\mathcal{C}_{a^-}(a^+,a^-)$, $K_{a^+}(a^+,a^-)$, $K_{a^-}(a^+,a^-)$, namely the analogous sets for heteroclinics from a^+ to a^- of what we have obtained for heteroclinics from a^- to a^+.

The nondegeneracy conditions that we impose are those of alternative (ii) of Proposition 2.6. They will be used to construct a subset of $\Gamma_\sigma(a^-, a^+)$ in which a new family of local minima of I_σ will be found that have the convergence properties that we were unable to verify for the functions, Q_σ.

To carry out the new construction, select a $\delta \in (0, r(a^-, a^+))$ and let $q^* \in \mathcal{M}(a^-, a^+)$. Then with $K_{a^-}(a^-, a^+), K_{a^+}(a^-, a^+)$ as in Proposition 2.6, there is an $s_0 \in \mathbb{N}$ depending on δ and q^* such that for all $i \in \mathbb{Z}$ with $|i| \geq s_0$,

$$\|q^* - K_{a^-}(a^-, a^+)\|_{L^2(T_{-i})}, \|q^* - K_{a^+}(a^-, a^+)\|_{L^2(T_i)} \leq \delta. \tag{22}$$

Fix such an i and choose σ so that $[-i, i+1] \subset \sigma$. Define

$$\Gamma_{\sigma,i}(a^-, a^+) = \{q \in \Gamma_\sigma(a^-, a^+) \mid q \text{ satisfies (22)}\}.$$

Then $\Gamma_{\sigma,i}(a^-, a^+) \neq \emptyset$. Set

$$c_{\sigma,i}(a^-, a^+) = \inf_{q \in \Gamma_{\sigma,i}(a^-, a^+)} I_\sigma(q) \tag{23}$$

The existence of a minimizer, $Q_{\sigma,i}$, in (23) follows as in Lemma 2.1 and standard regularity arguments imply it is a solution of (2) except possibly in the constraint intervals, $T_{-i} \cup T_i$. Letting σ_k be as earlier, we will show that for large k, there is strict inequality for $Q_{\sigma,i}$ in (22). Towards that end, an analogue of results from [24] or [15] is needed. Set

$$\begin{aligned}\Lambda(a^-, a^+) = \{q \in \Gamma(a^-, a^+) \mid \ & \|q - K_{a^-}(a^-, a^+)\|_{L^2(T_{-i})} = \delta \\ & \text{or } \|q - K_{a^+}(a^-, a^+)\|_{L^2(T_i)} = \delta\}.\end{aligned}$$

Note that $\Lambda(a^-, a^+)$ also depends on δ. Define

$$d(a^-, a^+) = \inf_{q \in \Lambda(a^-, a^+)} I(q). \tag{24}$$

Then the arguments of Proposition 2.47 of [24] show

$$d(a^-, a^+) > c(a^-, a^+). \tag{25}$$

Now we have:

THEOREM 2.9. *Suppose that* $\sigma_k \subset \sigma_{k+1}$ *and* $\sigma_k^+, -\sigma_k^- \to \infty$ *as* $k \to \infty$. *Then*

$$c_{\sigma_k,i}(a^-, a^+) \geq c_{\sigma_{k+1},i}(a^-, a^+) \to c(a^-, a^+) \text{ as } k \to \infty. \tag{26}$$

Moreover for any k *for which* $c_{\sigma_k,i}(a^-, a^+) < d(a^-, a^+)$ *and in particular for large* k, *any minimizer of* I_{σ_k} *in* $\Gamma_{\sigma_k,i}(a^-, a^+)$ *is a solution of* (2). *In addition, there is a* $Q \in \mathcal{M}(a^-, a^+)$ *such that along a subsequence,* $Q_{\sigma_k,i} \to Q$ *in* $W^{1,2}_{loc}(\mathbb{R}, \mathbb{R}^m)$.

Proof. The argument of Proposition 2.3 shows that (26) holds. To show that the constraints are satisfied with strict inequality provided that k is sufficiently large, arguing indirectly, suppose that

$$\|Q_{\sigma_k,i} - K_{a^-}(a^-,a^+)\|_{L^2(T_{-i})} = \delta \text{ or } \|Q_{\sigma_k,i} - K_{a^+}(a^-,a^+)\|_{L^2(T_i)} = \delta.$$

Then $Q_{\sigma_k,i} \in \Lambda(a^-,a^+)$ and by (23) and (25),

$$c_{\sigma_k,i}(a^-,a^+) = I_\sigma(Q_{\sigma_k,i}) = I(Q_{\sigma_k,i}) \geq d(a^-,a^+). \tag{27}$$

But (25) and (26) show (27) is not possible.

Thus for any k for which $c_{\sigma_k,i}(a^-,a^+) < d(a^-,a^+)$ and in particular for large k, any minimizer of I_{σ_k} in $\Gamma_{\sigma_k,i}(a^-,a^+)$ is a solution of (2).

It remains to establish the convergence of the solutions, $Q_{\sigma_k,i}$, along a subsequence. By the argument of Proposition 2.24 of [15] again, $\|Q_{\sigma_k,i}\|_{W^{1,2}(T_j,\mathbb{R}^m)}$ is bounded independently of k and j. As in Corollary 2.42 of [15], this leads to a k-independent bound for $\|Q_{\sigma_k,i}\|_{L^\infty(\sigma_k,\mathbb{R}^m)}$ and then via (2), a k-independent bound for $\|Q_{\sigma_k,i}\|_{C^2(\sigma_k,\mathbb{R}^m)}$. Thus by the Arzela-Ascoli Theorem and (2), as $k \to \infty$, for any subsequence of $Q_{\sigma_k,i}$, there is a solution, Q of (HS) such that along a further subsequence, $Q_{\sigma_k,i}$ converges to Q in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$. Moreover restricting ourselves to this subsequence, for any $p \in \mathbb{N}$, by (26),

$$\begin{aligned}\int_{-p}^{p} L(Q)\,dt \leq \liminf_{k\to\infty} \int_{-p}^{p} L(Q_{\sigma_k,i})\,dt &\leq \liminf_{k\to\infty} I_{\sigma_k}(Q_{\sigma_k,i}) \\ &= \liminf_{k\to\infty} c_{\sigma_k,i}(a^-,a^+) = c(a^-,a^+).\end{aligned}$$

Letting $p \to \infty$ further shows

$$I(Q) \leq c(a^-,a^+) < \infty. \tag{28}$$

Thus by (28) and Proposition 2.3 of [15], there are points, $\varphi^\pm \in \{a^-,a^+\}$ such that $Q(\pm\infty) = \varphi^\pm$.

We claim that $\varphi^\pm = a^\pm$. It then follows that $Q \in \Gamma(a^-,a^+)$ with $I(Q) = c(a^-,a^+)$ so $Q \in \mathcal{M}(a^-,a^+)$. Towards proving that $\varphi^\pm = a^\pm$, let

$$\mathcal{P} = \{Q_{\sigma,i} \mid c_{\sigma,i}(a^-,a^+) < d(a^-,a^+)\}.$$

Then we have

Proposition 2.10. *There is a $\beta = \beta(\delta) > 0$ such that*

$$\beta = \inf_{q\in\mathcal{P}} \int_{-i}^{i+1} L(q)\,dt. \tag{29}$$

Proof. As was noted earlier, the set of functions, $\mathcal{P}$ is a bounded subset of $C^2([-i,i+1],\mathbb{R}^m)$. Choose a minimizing sequence (q_l) for (29). Therefore as $l \to \infty$,

$$\int_{-i}^{i+1} L(q_l)\,dt \to \beta.$$

By the Arzela-Ascoli Theorem, there is a function , $\hat{q} \in C^1([-i,i+1],\mathbb{R}^m)$, such that along a subsequence, as $l \to \infty$,

$$\int_{-i}^{i+1} L(q_l)\,dt \to \int_{-i}^{i+1} L(\hat{q})\,dt = \beta.$$

Moreover $\hat{q}$ is a solution of (HS) on $[-i,i+1]$ and satisfies the constraints in (22). If $\beta = 0$, $\hat{q} \equiv a^-$ or $\hat{q} \equiv a^+$. But by the choice of δ, the function $\hat{q}$ cannot satisfy both constraints. Therefore $\beta > 0$. □

Conclusion of the proof of Theorem 2.9. Returning to our claim that $\varphi^\pm = a^\pm$, and arguing indirectly, suppose that the pair of equalities is not satisfied. Then there are three possibilities: (i) $\varphi^- = a^+$ and $\varphi^+ = a^+$, (ii) $\varphi^- = a^+$ and $\varphi^+ = a^-$, or (iii) $\varphi^- = a^-$ and $\varphi^+ = a^-$. Now a comparison argument will be employed. Suppose e.g. that (i) occurs. For $l \in \mathbb{Z}$, set $X_l = \cup_{j=l-1}^{l+1} T_j$. Pick an $\varepsilon > 0$. Then there is a $p = p(\varepsilon) \in \mathbb{N}$ with $p \le \min(-\sigma_k^-, \sigma_k^+)$ such that for all large k in our subsequence, $\|Q_{\sigma_k,i} - a^+\|_{C^2(X_{-p},\mathbb{R}^m)} \le \varepsilon$. Define $v_k \in \Gamma(a^-,a^+)$ via modifying $Q_{\sigma_k,i}$ in X_{-p}:

$$v_k(t) = \begin{cases} Q_{\sigma_k,i}(t), & t \le -p-1, \\ (-p-t)Q_{\sigma_k,i}(-p-1) + (t+p+1)a^+, & -p-1 \le t \le -p, \\ a^+, & -p \le t. \end{cases}$$

Then

$$\int_{X_p} L(v_k)dt \le \kappa(\varepsilon) \tag{30}$$

where $\kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now using (30) and Proposition 2.10 yields

$$\begin{aligned} c_{\sigma_k,i} = I(Q_{\sigma_k,i}) = \int_{-\infty}^{-p+1} L(v_k)\,dt - \int_{-p-1}^{-p+1} L(v_k)\,dt + \int_{-p-1}^{\infty} L(Q_{\sigma_k,i})\,dt \\ \ge c(a^-,a^+) - \kappa(\varepsilon) + \beta(\delta). \quad (31) \end{aligned}$$

Choose ε so that $\kappa(\varepsilon) < \frac{1}{2}\beta(\delta)$. Since the first assertion of this theorem shows $c_{\sigma_k,i} \to c(a^-,a^+)$ as $k \to \infty$, (31) and case (i) are not possible. A similar argument excludes case (ii) and likewise case (iii) is excluded by doing the cutting and pasting near $t = \infty$. Thus Theorem 2.9 is proved. □

Remark 2.11: There is an analogous result on interchanging the roles of a^- and a^+.

Remark 2.12: There is another approach we could have taken to the material in this section. Replacing $\mathcal{S}(a^-, a^+)$ by

$$\mathcal{T}(a^-, a^+) = \{q(0) \mid q \in \mathcal{M}(a^-, a^+)\},$$

then $\bar{\mathcal{T}}(a^-, a^+) = \mathcal{T}(a^-, a^+) \cup \{a^-, a^+\}$ and $\bar{\mathcal{T}}$ is compact in $\mathbb{R}^m$. Moreover the analogue of Proposition 2.6 with $\mathcal{T}$ replacing $\mathcal{S}$ holds for this new setting. Then in (22) and the definition of Λ, replace the L^2 norm by the $\mathbb{R}^m$ norm leading to a variant of Theorem 2.9 (although one can no longer invoke [15] for part of the proof). A similar approach using pointwise constraints can likewise be made the in the next section where local minima of I that are multitransition solutions of (HS) are obtained. However unfortunately this replacement can no longer be made when dealing with mountain pass solutions of (HS) and its finite time relative, (2). The reason it fails is that for mountain pass solutions, again an appropriate version of Proposition 2.6 is needed. To obtain it, one has to work with the map from the set of solutions of (HS) in say $\Gamma(a^-, a^+)$ with $I(q) \leq d$ to

$$\mathcal{T}^d(a^-, a^+) = \{q(0) \mid q \in \Gamma(a^-, a^+), \text{ satisfies (HS), and } I(q) \leq d\}$$

where d is greater than the mountain pass minimax value (see [25, 26]). Unfortunately, unlike the case where $d = c(a^-, a^+)$, this map is not one to one causing the earlier proof of Proposition 2.6 to break down. This failure does not occur when working with

$$\mathcal{S}^d(a^-, a^+) = \{q|_{[0,1]} \mid q \in \Gamma(a^-, a^+), \text{ satisfies (HS), and } I(q) \leq d\}.$$

The analogues of the results of this paper for mountain pass solutions of (2) will be explored in a future publication.

3. Multitransition local minima

In this section, the existence and multiplicity of solutions of (2) that undergo multiple transitions will be studied. As in Section 2, the results here will follow with the aid of comparison arguments involving multitransition local minima for (HS). Such results were obtained in [15] where it was shown that there is an infinitude of $k-$transition solutions of (HS) for each $k \in \mathbb{N}$. In [15], the same ideas were employed to treat $k = 2$ as for general $k > 2$. This is also the case in the current setting. In [15], the main concern was the solution of a PDE problem in a cylindrical domain for which (HS) occurs as a degenerate special case. We begin here by stating a slightly stronger version of the result of [15] specialized to (HS) when $k = 2$, show how to use it to obtain a related result for (2) and then discuss the case of $k > 2$.

To formulate the result for (HS), choose $\mathbf{m} = (m_1, \cdots, m_4) \in \mathbb{Z}^4$ and $l \in \mathbb{N}$ so that

$$m_1 + 2l < m_2 - 2l < m_2 + 2l < m_3 - 2l < m_3 + 2l < m_4 - 2l. \tag{32}$$

The integers m_i and l will depend on a parameter, ε, that will be introduced in the next theorem. For $r > 0$ and $A \subset W^{1,2}(T_0, \mathbb{R}^m)$, let

$$N_r(A) \equiv \{q \in W^{1,2}(T_0, \mathbb{R}^m) \mid \mathrm{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(q, A) \le r\}.$$

Set $\delta = \min(\rho, r(a^-, a^+), r(a^+, a^-))$ where $\rho \in (0, \frac{1}{4}|a^- - a^+|)$. As the class of admissible functions in which local minima of I will be sought, take

$$\mathcal{A}_2 = \mathcal{A}_2(\mathbf{m}, l) = \{q \in E \mid q \text{ satisfies (33)}\}$$

where

$$q(\cdot + j)|_{T_0} \in \begin{cases} N_\delta(K_{a^-}(a^-, a^+)), & j < m_1 + l, \\ N_\delta(K_{a^+}(a^-, a^+)), & m_2 - l \le j < m_2 + l, \\ N_\delta(K_{a^+}(a^+, a^-)), & m_3 - l \le j < m_3 + l, \\ N_\delta(K_{a^-}(a^+, a^-)), & m_4 - l \le j. \end{cases} \tag{33}$$

Define

$$b_2 = b_2(\varepsilon) = \inf_{q \in \mathcal{A}_2} I(q). \tag{34}$$

This setting was studied in Section 5 of [15]. As a somewhat more quantitative version of the result there, we have:

Theorem 3.1. *Let $(V_1) - (V_3)$ and the four conditions $\mathcal{C}_{a^\pm}(a^-, a^+) = \{a^\pm\}$, $\mathcal{C}_{a^\pm}(a^+, a^-) = \{a^\pm\}$ be satisfied. For any $\varepsilon \in (0, \delta/16)$, there exists an $m_0 = m_0(\varepsilon) \in \mathbb{N}$, an $l = l(\varepsilon) \in \mathbb{N}$, $l \ge m_0$ and a $\zeta_0 = \zeta_0(\varepsilon) > 0$ with $\zeta_0 \to 0$ as $\varepsilon \to 0$ such that*

1^o *for each $\mathbf{m} = (m_1, m_2, m_3, m_4)$ satisfying*

$$m_{j+1} - m_j - 6l \ge m_0 \text{ for } j = 1, 2, 3, \tag{35}$$

the set

$$\mathcal{M}(b_2) \equiv \{q \in \mathcal{A}_2 \mid I(q) = b_2\} \ne \emptyset.$$

2^o *Any $Q \in \mathcal{M}(b_2)$ satisfies the constraints, (33), with strict inequality and is a classical solution of* (HS) *that is homoclinic to a^-.*

3^o *If* $Q \in \mathcal{M}(b_2)$,

$$\|Q - a^-\|_{L^\infty((-\infty,m_1-l],\mathbb{R}^m)} < \zeta_0\,, \|Q - a^+\|_{L^\infty([m_2+l,m_3-l],\mathbb{R}^m)} < \zeta_0$$
$$\text{and } \|Q - a^-\|_{L^\infty([m_4-l,+\infty),\mathbb{R}^m)} < \zeta_0.$$

4^o *As* $\varepsilon \to 0$, $b_2(\varepsilon) \to c(a^-, a^+) + c(a^+, a^-)$.

Remark 3.2: The statement of Theorem 3.1 is rather technical due to the presence of so many parameters, but the geometrical content of the result, both for $k = 2$ and more generally, is simple. Heuristically the result says that if the constraint regions are far enough apart, there exists an interior minimizer of the associated variational problem and this minimizer is a classical solution of (2). Moreover in the region between each pair of constraint regions involving the same point, the point being a^+ for Theorem 3.1, the solution remains close to that point.

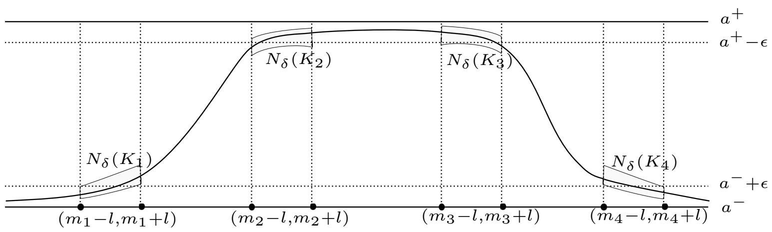

Figure 1: *In the diagram* $K_1 = K_{a^-}(a^-, a^+), K_2 = K_{a^+}(a^-, a^+), K_3 = K_{a^+}(a^+, a^-), K_4 = K_{a^-}(a^+, a^-)$

Proof of Theorem 3.1. Fix $\mathbf{m}$ satisfying (35). Both 1^o and 2^o are either part of the statement or the proof of Theorem 5.16 of [15]. To verify 3^o, note first that for m_0 is sufficiently large, one can choose members of $\mathcal{M}(a^-, a^+)$ and $\mathcal{M}(a^+, a^-)$, modify them slightly to obtain a member of $\mathcal{A}_2$, and use it to find an upper bound for b_2 as in (5.19) of [15]:

$$b_2 < c(a^-, a^+) + c(a^+, a^-) + 2 \tag{36}$$

independently of $\mathbf{m}$. Next we will show that in any interval, $\mathcal{I}$, of length at least m_0, there is a subinterval, $X_i = \cup_{k=i-2}^{i+2} T_k \subset \mathcal{I}$ such that either

$$\|Q - a^-\|_{L^\infty(T_j,\mathbb{R}^m)} < \varepsilon \text{ or } \|Q - a^+\|_{L^\infty(T_j,\mathbb{R}^m)} < \varepsilon \tag{37}$$

for $T_j \in X_i$. Indeed if both inequalities in (37) fail for some T_j, by (2.12) $I_{T_j}(Q) \geq \min_{(t,\xi)\in T_j \times (\mathbb{R}^m \setminus B_\varepsilon(a^-) \cup B_\varepsilon(a^+))} V(t, \xi) \equiv \gamma(\varepsilon) > 0$. Thus if there were no such $X_i \subset \mathcal{I}$,

$$b_2 = I(Q) \geq \frac{1}{5} m_0(\varepsilon)\gamma(\varepsilon), \tag{38}$$

which is contrary to the upper bound for b_2 for m_0 sufficiently large. It is here that the dependence of m_0 (and hence l) on ε first enters.

Since $l \geq m_0$, applying this observation to the constraint regions associated with m_1, m_2, m_3 and m_4 yields intervals, $X_1 \subset [m_1 - l, m_1]$, $X_2 \subset [m_2, m_2 + l]$, $X_3 \subset [m_3 - l, m_3]$ and $X_4 \subset [m_4, m_4 + l]$ in which (37) holds. In fact due to the definition of δ in the constraint, (37) can be strengthened to

$$\|Q(t) - a^-\|_{L^\infty(X_i,\mathbb{R}^m)} < \varepsilon,\ i = 1, 4,\ \|Q(t) - a^+\|_{L^\infty(X_i,\mathbb{R}^m)} < \varepsilon,\ i = 2, 3. \quad (39)$$

Now we verify 3^o, i.e. there is a $\zeta_0 = \zeta_0(\varepsilon)$ such that $\zeta_0 \to 0$ as $\varepsilon \to 0$ for which

(i) $\|Q - a^-\|_{L^\infty((-\infty,m_1-l],\mathbb{R}^m)} < \zeta_0$,

(ii) $\|Q - a^+\|_{L^\infty([m_2+l,m_3-l],\mathbb{R}^m)} < \zeta_0$, and

(iii) $\|Q - a^-\|_{L^\infty([m_4-l,+\infty),\mathbb{R}^m)} < \zeta_0$.

We will only prove (ii). The proofs of properties (i) and (iii) are similar and simpler and will be omitted. Set

$$\bar\alpha = \sup X_2,\ \bar\beta = \inf X_3,$$

and the function

$$\bar Q(t) = \begin{cases} Q(t) & t \leq \bar\alpha - 1, \\ Q(\bar\alpha - 1) + (t + 1 - \bar\alpha)(a^+ - Q(\bar\alpha - 1)) & \bar\alpha - 1 \leq t \leq \bar\alpha, \\ a^+ & \bar\alpha \leq t \leq \bar\beta \\ a^+ + (t - \bar\beta)(Q(\bar\beta + 1) - a^+) & \bar\beta \leq t \leq \bar\beta + 1, \\ Q(t) & t \geq \bar\beta + 1. \end{cases}$$

We claim the function $\bar Q$ satisfies the constraint (33). Indeed on the intervals T_j for $j \leq \bar\alpha - 2$ or $j \geq \bar\beta + 1$, $\bar Q(t) = Q(t)$ so the constraint is satisfied. The same is true when $\bar\alpha \leq j \leq \bar\beta - 1$ since on the corresponding intervals T_j, $\bar Q(t) = a^+$. If $j = \bar\alpha - 1$, then $\bar Q(t) = Q(\bar\alpha - 1) + (t + 1 - \bar\alpha)(a^+ - Q(\bar\alpha - 1))$ for $t \in T_j$. Thus by (39)

$$\begin{aligned} \mathrm{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(\bar Q(\cdot + j), K_{a^+}(a^-, a^+))^2 &\leq \|\bar Q(t) - a^+\|^2_{W^{1,2}(T_j,\mathbb{R}^m)} \\ &= \int_{T_j} |a^+ - Q(\bar\alpha - 1)|^2 + |t - \bar\alpha|^2 |Q(\bar\alpha - 1) - a^+|^2 \leq 2\varepsilon^2 < \delta^2. \end{aligned}$$

Likewise another application of (39) shows $\|\bar Q(t) - a^+\|^2_{W^{1,2}(T_j,\mathbb{R}^m)} \leq 2\varepsilon^2 < \delta^2$ for $j = \beta$. Hence all the inequalities in (33) hold for the function $\bar Q$ so it belongs to the class $\mathcal{A}_2$. Consequently, since Q minimizes I on $\mathcal{A}_2$,

$$0 \leq I(\bar Q) - I(Q) \leq \int_{\bar\alpha - 1}^{\bar\alpha} L(\bar Q)\, dt + \int_{\bar\beta}^{\bar\beta + 1} L(\bar Q)\, dt - I_{(\bar\alpha,\bar\beta)}(Q) \quad (40)$$

and so

$$I_{(\bar{\alpha},\bar{\beta})}(Q) \leq \int_{\bar{\alpha}-1}^{\bar{\alpha}} L(\bar{Q})\,dt + \int_{\bar{\beta}}^{\bar{\beta}+1} L(\bar{Q})\,dt. \tag{41}$$

To conclude (ii) from (41), the definition of $\bar{Q}$ on the intervals $[\bar{\alpha}-1,\bar{\alpha}]$ and $[\bar{\beta},\bar{\beta}+1]$ as well as (39) will be used. Recalling the function, $\overline{\omega}_\varepsilon$, in the proof of Proposition 2.4:

$$\overline{\omega}_\varepsilon = \max\{V(t,\xi) \mid t \in \mathbb{R}, \xi \in \bar{B}_\varepsilon(a^+) \cup \bar{B}_\varepsilon(a^-)\},$$

by (39) we have

$$\begin{aligned}\int_{\bar{\alpha}-1}^{\bar{\alpha}} L(\bar{Q})\,dt \leq \int_{\bar{\alpha}-1}^{\bar{\alpha}} \tfrac{1}{2}|a^+ - Q(\bar{\alpha}-1)|^2 + \max_{(t,\xi)\in[\bar{\alpha}-1,\bar{\alpha}]\times B_\varepsilon(a^+)} V(t,\xi)\,dt \\ \leq \tfrac{1}{2}\varepsilon^2 + \overline{\omega}_\varepsilon.\end{aligned}$$

Similarly

$$\int_{\bar{\beta}}^{\bar{\beta}+1} L(\bar{Q})\,dt \leq \tfrac{1}{2}\varepsilon^2 + \overline{\omega}_\varepsilon$$

so by (41)

$$I_{(\bar{\alpha},\bar{\beta})}(Q) \leq \varepsilon^2 + 2\overline{\omega}_\varepsilon. \tag{42}$$

To conclude the proof of property (ii), recall the function, $\underline{\omega}_\zeta$, defined for $\zeta > 0$:

$$\underline{\omega}_\zeta \equiv \min\{V(t,\xi) \mid t \in \mathbb{R}, \mathrm{dist}(\xi, \{a^-, a^+\}) \geq \zeta\}.$$

The function $\underline{\omega}_\zeta$ is increasing and continuous on $[0, \frac{1}{4}|a^+ - a^-|)$. Define ζ_0 by

$$\zeta_0 = \zeta_0(\varepsilon) = \min\{\zeta \in [2\varepsilon, \tfrac{1}{4}|a^+ - a^-|) \mid \sqrt{2\underline{\omega}_{\zeta/2}}\,\zeta \geq 2(\varepsilon^2 + 2\overline{\omega}_\varepsilon)\}.$$

Since $\overline{\omega}_\varepsilon \to 0$ as $\varepsilon \to 0$, ζ_0 is well defined for small ε. Note that $\zeta_0(\epsilon) \to 0$ as $\epsilon \to 0$. Finally to verify Property (ii), i.e.

$$\|Q - a^+\|_{L^\infty([m_2+l, m_3-l],\mathbb{R}^m)} < \zeta_0, \tag{43}$$

we argue indirectly. Assume that there exists a $\mu \in [m_2+l, m_3-l]$ for which $|Q(\mu) - a^+| \geq \zeta_0$. See Figure 3. Since by (39), $|Q(\bar{\alpha}) - a^+| < \varepsilon$, there exists an interval $(\alpha,\beta) \subset (\bar{\alpha},\mu)$ such that $Q(t) \notin B_{\zeta_0/2}(a^-) \cup B_{\zeta_0/2}(a^+)$ for any $t \in (\alpha,\beta)$ and $|Q(\beta) - Q(\alpha)| \geq \zeta_0$. Then $V(t,Q(t)) \geq \underline{\omega}_{\zeta_0/2}$ for any $t \in (\alpha,\beta)$ and by (11),

$$I_{(\alpha,\beta)}(Q) \geq \sqrt{2\underline{\omega}_{\zeta_0/2}}\,|Q(\beta) - Q(\alpha)| \geq \sqrt{2\underline{\omega}_{\zeta_0/2}}\,\zeta_0. \tag{44}$$

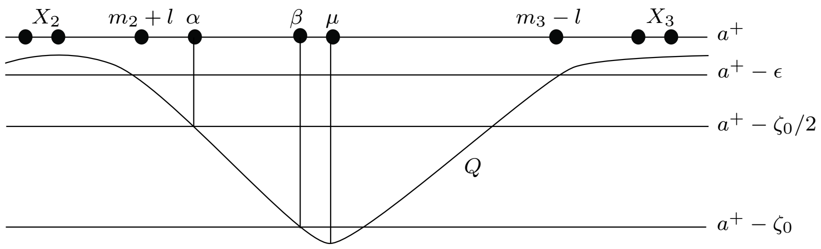

Figure 2: *The indirect argument*

Hence, by (42), (44), $\sqrt{2\underline{\omega}_{\zeta_0/2}}\,\zeta_0 \le I_{(\alpha,\beta)}(Q) \le I_{(\bar\alpha,\bar\beta)}(Q) \le \varepsilon^2 + 2\overline{\omega}_\varepsilon$ which is not possible by the definition of ζ_0. This completes the proof of (ii) and of 3^o.

It remains to prove 4^o. For m_0 possibly still larger, there is a $q^- \in \mathcal{M}(c(a^-,a^+))$ such that q^- satisfies the m_1 and m_2 constraints in (33) and there is a $q^+ \in \mathcal{M}(c(a^+,a^-))$ such that q^+ satisfies the m_3 and m_4 constraints in (33). Moreover it can be assumed that

$$\|q^- - a^+\|_{L^\infty([m_2+l,\infty),\mathbb{R}^m)} < \varepsilon \text{ and } \|q^+ - a^+\|_{L^\infty((-\infty,m_3-l],\mathbb{R}^m)} < \varepsilon.$$

Therefore appropriately modifying q^- for $t > m_2 + l$ and q^+ for $t < m_3 - l$ yields a function $q_2 \in \mathcal{A}_2$ satisfying the improved version of (36):

$$I(q_2) \le c(a^-,a^+) + c(a^+,a^-) + \kappa_1(\varepsilon)$$

where $\kappa_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence

$$b_2(\varepsilon) \le c(a^-,a^+) + c(a^+,a^-) + \kappa_1(\varepsilon). \tag{45}$$

To obtain a lower bound for $b_2(\varepsilon)$, let $Q \in \mathcal{M}(b_2(\varepsilon))$. Define a function, $\bar{Q}$ as in the proof of 3^o where now $\bar\alpha$ and $\bar\beta$ are replaced respectively by α_1 and α_1+1 where these points are integers interior to (m_2+l, m_3-l). By its definition, $I(\bar{Q}) \ge c(a^-,a^+) + c(a^+,a^-)$. Thus in the spirit of (30) and (40), there is a function, $\kappa_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ with

$$c(a^-,a^+) + c(a^+,a^-) - b_2(\varepsilon) \le I(\bar{Q}) - b_2(\varepsilon) = I(\bar{Q}) - I(Q) \le \kappa_2(\varepsilon)$$

or

$$c(a^-,a^+) + c(a^+,a^-) - \kappa_2(\varepsilon) \le b_2(\varepsilon). \tag{46}$$

Combining (45) and (46) then gives 4^o and completes the proof of Theorem 3.1. □

Now the results just mentioned for (HS) can be used as a tool to obtain 2-transition solutions of (2). Continuing with $\mathbf{m}$ as in Theorem 3.1, let $\sigma^- < m_1$ and $\sigma^+ > m_4$, and define

$$\mathcal{A}_{2,\sigma} = \mathcal{A}_{2,\sigma}(\mathbf{m}, l) = \{q \in \mathcal{A}_2 \mid q(t) = a^- \text{ for } t \notin \sigma\}.$$

Then for $\sigma^+, -\sigma^-$ sufficiently large, $\mathcal{A}_{2,\sigma} \neq \emptyset$. Define

$$b_{2,\sigma} = b_{2,\sigma}(\mathbf{m}, l) = \inf_{q \in \mathcal{A}_{2,\sigma}} I(q). \tag{47}$$

Then parallelling Theorem 3.1, we have:

THEOREM 3.3. *Suppose the hypotheses of Theorem 3.1 are satisfied. Then*

1^o *For $\sigma^+, -\sigma^-$ sufficiently large,*

$$\mathcal{M}(b_{2,\sigma}) \equiv \{q \in \mathcal{A}_{2,\sigma} \mid I(q) = b_{2,\sigma}\} \neq \emptyset.$$

2^o *Any $Q_\sigma \in \mathcal{M}(b_{2,\sigma})$ is a solution of* (2).

Proof. The existence of $Q_\sigma \in \mathcal{M}(b_{2,\sigma})$ follows as in the proof of Lemma 2.1. As in earlier arguments, it is a solution of (2) in any of the intervals where there is no constraint and also in any constraint interval in which the constraint is satisfied with strict inequality. Thus to complete the proof of item 2^o, it must be shown that strict inequality holds in the 4 constraint regions. The argument involving (36)-(39) also holds in the current setting so for $\varepsilon < \delta$, again there are intervals, $X_j = \cup_{k=i_j-2}^{i_j+2} T_k$ for $j = 2, 3$ and $i_j \subset [m_j - l, m_j + l]$, in which

$$\|Q_\sigma - a^+\|_{L^\infty(T_i, \mathbb{R}^m)} < \varepsilon \text{ for } T_i \in X_j. \tag{48}$$

Now suppose that Q_σ satisfies one of the m_1 or m_2 constraints with equality. Cutting and pasteing in X_3 yields a pair of functions,

$$f(t) = \begin{cases} Q_\sigma(t), & t \leq i_3 - 1, \\ (i_3 - t)Q_\sigma(i_3 - 1) + (t - i_3 + 1)a^+, & i_3 - 1 \leq t \leq i_3, \\ a^+, & t \geq i_3, \end{cases}$$

$$g(t) = \begin{cases} a^+, & t \leq i_3 + 1, \\ (i_3 + 2 - t)a^+ + (t - i_3 - 1))Q_\sigma(i_3 + 2), & i_3 + 1 \leq t \leq i_3 + 2, \\ Q_\sigma(t), & t \geq i_3 + 2. \end{cases}$$

Note that $f \in \Gamma(a^-, a^+)$, $g \in \Gamma(a^+, a^-)$ and since Q_σ satisfies one of the m_1 or m_2 constraints with equality, $f \in \Lambda(a^-, a^+)$. Hence $I(g) = I_{(i_3+1,+\infty)}(g) \geq c(a^+, a^-)$ and by (25), $I(f) = I_{(-\infty,i_3)}(f) \geq d(a^-, a^+)$. Moreover by (48), arguing as in the proof of Theorem 3.1 for the function $\bar{Q}$ shows

$$I_{(i_3-1,\leq i_3)}(f) \leq e_1(\varepsilon) \text{ and } I_{(i_3+1,i_3+2)}(g) \leq e_1(\varepsilon)$$

with $e_1(\varepsilon) \to 0$ as $\varepsilon \to 0$.

By the above observations

$$I_{(-\infty,i_3-1)}(Q_\sigma) = I_{(-\infty,i_3-1)}(f) \geq I(f) - e_1(\varepsilon) \geq d(a^-,a^+) - e_1(\varepsilon) \text{ and}$$
$$I_{(i_3+2,+\infty)}(Q_\sigma) = I_{(i_3+2,+\infty)}(g) \geq I(g) - e_1(\varepsilon) \geq c(a^+,a^-) - e_1(\varepsilon)$$

so that

$$I(Q_\sigma) \geq I_{(-\infty,i_3-1)}(Q_\sigma) + I_{(i_3+2,+\infty)}(Q_\sigma) \geq d(a^-,a^+) + c(a^+,a^-) - 2e_1(\varepsilon). \quad (49)$$

On the other hand, by 4^o of Theorem 3.1, as $\varepsilon \to 0$, $b_2 \to c(a^-,a^+)+c(a^+,a^-)$. Thus for small ε and $-\sigma^-,\sigma^+$ sufficiently large, there is a function, $e_2(\varepsilon)$ with $e_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$I(Q_\sigma) \leq c(a^-,a^+) + c(a^+,a^-) + e_2(\varepsilon). \quad (50)$$

But (49) and (50) are incompatible for small ε since $d(a^-,a^+) > c(a^-,a^+)$. A similar argument establishes the result if Q_σ satisfies one of the m_3 or m_4 constraints with equality and item 2^o is proved. □

The existence of the 2−transition solutions having been established, now their behavior as $-\sigma^-,\sigma^+ \to \infty$ will be studied. We will show

Theorem 3.4. *Let the hypotheses of Theorem 3.3 be satisfied for a fixed admissible ε. If $-\sigma_i^-,\sigma_i^+ \to \infty$ as $i \to \infty$, then $b_{2,\sigma_i} \to b_2$. Moreover if $Q_{\sigma_i} \in \mathcal{M}(b_{2,\sigma_i})$, then there is a $Q \in \mathcal{M}(b_2)$, such that along a subsequence, $Q_{\sigma_i} \to Q$ in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$ as $i \to \infty$.*

Proof. Since $\mathcal{M}(b_{2,\sigma_i}) \subset \mathcal{M}(b_2)$, $b_2 \leq b_{2,\sigma_i}$. Let $\bar\varepsilon > 0$ be small. Choose any $Q \in \mathcal{M}(b_2)$. Then Q can be modified near $t = \pm\infty$ to produce $Q_{\bar\varepsilon} \in \mathcal{M}(b_{2,\sigma_i})$ for all large $|\sigma_i^\pm|$ and $b_{2,\sigma_i} \leq I(Q_{\bar\varepsilon}) \leq b_2 + \bar\varepsilon$. Thus the first assertion of the theorem follows. To prove the second assertion, since $Q_{\sigma_i} \in \mathcal{A}_2$, by earlier arguments, there is an $M > 0$ such that $\|Q_{\sigma_i}\|_{C^2(\sigma_i,\mathbb{R}^m)} \leq M$ for all $i \in \mathbb{N}$. Thus by the Arzela-Ascoli Theorem, a subsequence of Q_{σ_i} converges in $C^2_{loc}(\mathbb{R},\mathbb{R}^m)$ to a function $Q \in C^2_{loc}(\mathbb{R},\mathbb{R}^m) \cap \mathcal{A}_2$ so $I(Q) \geq b_2$, But along our subsequence, for any $p \in \mathbb{N}$,

$$\int_{-p}^{p} L(Q)\,dt \leq \liminf_{i\to\infty} \int_{-p}^{p} L(Q_{\sigma_i})\,dt \leq \liminf_{i\to\infty} I(Q_{\sigma_i}) = \liminf_{i\to\infty} b_{2,\sigma_i} = b_2.$$

Thus $I(Q) = b_2$ and $Q \in \mathcal{M}(b_2)$. □

Next the case of $k > 2$ transitions will be discussed briefly. See [24] for a detailed argument in a related case. Again one takes $l \in \mathbb{N}$ and now chooses

$\mathbf{m} = (m_1, \cdots, m_{2k}) \in \mathbb{Z}^{2k}$ with $m_j - m_{j-1} > 4l$ for $j = 2, \cdots, 2k$. (We note at this point a typo in the first line of page 1763 of [15] where $> 2l$ is written rather than $> 4l$). To describe the analogue of the condition, (33), choose $\{a_1, \ldots, a_{2k}\} \in \{a^-, a^+\}^{2k}$ so that

$$a_1 \neq a_2 = a_3 \neq \ldots \neq a_{2k-2} = a_{2k-1} \neq a_{2k}$$

and define the family of sets $\{K_1, \ldots, K_{2k}\}$ by

$$K_{2j-1} = K_{a_{2j-1}}(a_{2j-1}, a_{2j}) \text{ and } K_{2j} = K_{a_{2j}}(a_{2j-1}, a_{2j}), \quad j = 1, \ldots, k.$$

Then the class of admissible functions for the $k-$transition problem is

$$\mathcal{A}_k = \mathcal{A}_k(\mathbf{m}, l) = \{q \in E \mid q \text{ satisfies (51)}\}$$

where

$$q(\cdot + p)|_{T_0} \in \begin{cases} N_\delta(K_1), & p \in (-\infty, m_1 + l) \cap \mathbb{Z}, \\ N_\delta(K_j), & p \in [m_j - l, m_j + l) \cap \mathbb{Z},\ 2 \leq j \leq 2k-1, \\ N_\delta(K_{2k}), & p \in [m_{2k} - l, +\infty) \cap \mathbb{Z}. \end{cases} \tag{51}$$

Now set

$$b_k = b(k, \mathbf{m}, l) = \inf_{q \in \mathcal{A}(k, \mathbf{m}, l)} I(q). \tag{52}$$

Then we have

THEOREM 3.5. *Under the hypotheses of Theorem 3.1,*

$$\mathcal{M}(b_k) \equiv \{Q \in \mathcal{A}(k, \mathbf{m}, l) \mid I(Q) = b(k, \mathbf{m}, l)\} \neq \emptyset$$

and any $Q \in \mathcal{M}(b_k)$ is a classical solution of (PDE) satisfying (BC).

REMARK 3.6: There are also analogues of $3^o - 4^o$ of Theorem 3.1.

To state the result corresponding to Theorem 3.5 for (2), let $\sigma^- < m_1$ and $\sigma^+ > m_{2k}$, and set

$$\mathcal{A}_{k,\sigma} = \mathcal{A}_{k,\sigma}(\mathbf{m}, l) = \{q \in \mathcal{A}_k \mid q(t) = a^- \text{ for } t \notin \sigma\}.$$

As earlier for $\sigma^+, -\sigma^-$ sufficiently large, $\mathcal{A}_{k,\sigma} \neq \emptyset$. Define

$$b_{k,\sigma} = b_{k,\sigma}(\mathbf{m}, l) = \inf_{q \in \mathcal{A}_{k,\sigma}} I(q). \tag{53}$$

Then we have:

THEOREM 3.7. *Suppose the hypotheses of Theorem 3.5 are satisfied. Then*

1^o *For* $\sigma^+, -\sigma^-$ *sufficiently large,*

$$\mathcal{M}(b_{k,\sigma}) \equiv \{q \in \mathcal{A}_{k,\sigma} \mid I(q) = b_{k,\sigma}\} \neq \emptyset.$$

2^o *Any* $Q_\sigma \in \mathcal{M}(b_{k,\sigma})$ *is a solution of* (2).

The proof is quite similar to that of Theorem 3.3, relying on (25) and a cutting and pasteing argument.

Remark 3.8: In conclusion, we note that the natural version of Theorem 3.4 holds in the $k > 2$ setting.

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Past and recent contributions to indefinite sublinear elliptic problems

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Dedicated to Professor J. López-Gómez on the occasion of his 60th birthday

Abstract. *We review the indefinite sublinear elliptic equation $-\Delta u = a(x)u^q$ in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with Dirichlet or Neumann homogeneous boundary conditions. Here $0 < q < 1$ and a is continuous and changes sign, in which case the strong maximum principle does not apply. As a consequence, the set of nonnegative solutions of these problems has a rich structure, featuring in particular both dead core and/or positive solutions. Overall, we are interested in sufficient and necessary conditions on a and q for the existence of positive solutions. We describe the main results from the past decades, and combine it with our recent contributions. The proofs are briefly sketched.*

Keywords: elliptic sublinear problem, indefinite, strong maximum principle.
MS Classification 2010: 35J15, 35J25, 35J61.

1. Introduction

Let $N \geq 1$, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and Δ the usual Laplace operator. This article is devoted to the semilinear equation

$$-\Delta u = a(x)u^q \quad \text{in} \quad \Omega, \tag{1}$$

under the condition

(AQ) $\qquad a$ changes sign and $0 < q < 1$.

This is a prototype of *indefinite* (due to the change of sign of a) and sublinear (with respect to u) elliptic pde, which is motivated by the porous medium type equation [22, 44]

$$w_t = \Delta(w^m) + a(x)w, \quad m > 1,$$

after the change of variables $u = w^m$ and $q = 1/m$. Indefinite elliptic problems have attracted considerable attention since the 70's, mostly in the linear ($q = 1$)

and superlinear ($q > 1$) cases [2, 4, 7, 13, 18, 24, 37, 39, 40, 43]. We intend here to give an overview of the main results known in the *sublinear* case. For the sign-definite case $a \geq 0$ we refer to [3, 11, 10, 35, 36, 42].

We shall consider (1) under Dirichlet and Neumann homogeneous boundary conditions, i.e. the problems

$$(P_{\mathcal{D}}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(P_{\mathcal{N}}) \qquad \begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ∂_ν is the exterior normal derivative.

Throughout this article, we assume that $a \in C(\overline{\Omega})$. By a *solution* of $(P_{\mathcal{D}})$ we mean a *strong* solution $u \in W^{2,r}_{\mathcal{D}}(\Omega)$ for some $r > N$, where

$$W^{2,r}_{\mathcal{D}}(\Omega) := \{u \in W^{2,r}(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

Note that $u \in C^1(\overline{\Omega})$, and so the boundary condition is satisfied in the usual sense. A similar definition holds for $(P_{\mathcal{N}})$. We say that a solution u is *nontrivial* if $u \not\equiv 0$, and *positive* if $u > 0$ in Ω. Among positive solutions of $(P_{\mathcal{D}})$, we are interested in *strongly positive* solutions (denoted by $u \gg 0$), namely, solutions in

$$\mathcal{P}^\circ_{\mathcal{D}} := \left\{u \in C^1_0(\overline{\Omega}) : u(x) > 0 \text{ in } \Omega, \text{ and } \partial_\nu u(x) < 0 \text{ on } \partial\Omega\right\}.$$

For $(P_{\mathcal{N}})$, a solution is *strongly positive* if it belongs to

$$\mathcal{P}^\circ_{\mathcal{N}} := \left\{u \in C^1(\overline{\Omega}) : u(x) > 0 \text{ on } \overline{\Omega}\right\}.$$

In case that *every* nontrivial solution of $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$) is *strongly positive* we say that this problem has the *positivity property.*

The condition (AQ) gives rise to the main feature of this class of problems, namely, the fact that the strong maximum principle (shortly **SMP**) does *not* apply. Let us recall the following version of this result (for a proof, see e.g. [38, Theorem 7.10]):

Strong maximum principle: Let $u \in W^{2,r}(\Omega)$ for some $r > N$ be such that $u \geq 0$ and $(-\Delta + M)u \geq 0$ in Ω, for some constant $M \geq 0$. Then either $u \equiv 0$ or $u > 0$ in Ω and $\partial_\nu u(x) < 0$ for any $x \in \partial\Omega$ such that $u(x) = 0$.

Given u satisfying (1), we see that under (AQ) we can't find in general some $M > 0$ such that $(-\Delta + M)u = a(x)u^q + Mu \geq 0$ in Ω, which prevents us to apply the **SMP**, unlike when $a \geq 0$ (the *definite* case) or $q \geq 1$ (the *linear* and *superlinear* cases). This fact is reinforced by a simple example of a nontrivial solution u (of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$) violating the conclusion of the **SMP** (see Example C below), which shows that the positivity property may fail. Moreover, such example also provides us with nontrivial *dead core* solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, i.e. solutions vanishing in some open subset of Ω. The formation of dead cores has already been investigated by Diaz in [16, Proposition 1.11] for a more general class of problems.

To the best of our knowledge, the study of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ in the indefinite and sublinear case was launched in the late 80's by Bandle, Pozio and Tesei [5, 6, 41]. These works were then followed by the contributions of Hernández, Mancebo and Vega [23], Delgado and Suarez [15], and Godoy and Kaufmann [20, 21]. We shall review the main results of these papers in the next section and complement it with our main recent results from [27, 28, 30] in the subsequent sections. Since the proofs can be found in the aforementioned articles, in most cases we shall only sketch them here.

2. First results

Let us recall the first existence and uniqueness results for the problems above. For the Neumann problem, the following condition on a plays an important role:

$$(\mathbf{A.0}) \qquad \int_\Omega a < 0.$$

Indeed, we shall see that (A.0) is *necessary* for the existence of a positive solution of $(P_{\mathcal{N}})$, and *sufficient* for the existence of a nontrivial solution, for any $q \in (0, 1)$. As for the uniqueness results, some merely technical conditions (see also the beginning of Section 7) on the set

$$\Omega_+ := \{x \in \Omega : a(x) > 0\}$$

shall be used, namely:

$(\mathbf{A.1})$ $\qquad \Omega_+$ has *finitely* many connected components,

$(\mathbf{A.2})$ $\qquad \partial\Omega_+$ satisfies an inner sphere condition with respect to Ω_+.

The following results were proved by Bandle, Pozio and Tesei [5, 6], and Delgado and Suárez [15]. Although [5, 6] require that $a \in C^\theta(\overline{\Omega})$ for some $0 < \theta < 1$, one can easily see from the proofs that these results still hold for strong solutions assuming that $a \in C(\overline{\Omega})$.

THEOREM A. *(i) The Dirichlet case:*

(a) $(P_{\mathcal{D}})$ *has at most one positive solution [15, Theorem 2.1]. Moreover, if (A.1) and (A.2) hold then* $(P_{\mathcal{D}})$ *has at most one solution positive in* Ω_+ *[5, Theorem 2.1].*

(b) $(P_{\mathcal{D}})$ *has at least one nontrivial solution [5, Theorem 2.2].*

(ii) The Neumann case:

(a) $(P_{\mathcal{N}})$ *has at most one solution in* $\mathcal{P}_{\mathcal{N}}^{\circ}$ *[6, Lemma 3.1]. Moreover, if (A.1) and (A.2) hold then* $(P_{\mathcal{N}})$ *has at most one solution positive in* Ω_+ *[6, Theorem 3.1].*

(b) If (A.0) holds then $(P_{\mathcal{N}})$ *has at least one nontrivial solution. Conversely, if* $(P_{\mathcal{N}})$ *has a positive solution then (A.0) holds [6, Theorem 2.1].*

Sketch of the proof. The uniqueness assertions rely on the following change of variables: if $u > 0$ and $-\Delta u = a(x)u^q$ in Ω then $v := (1-q)^{-1}u^{1-q}$ solves $-\Delta v = qu^{q-1}|\nabla v|^2 + a(x)$ in Ω. Let u_1, u_2 be positive solutions of (P_D) and assume that $\tilde{\Omega} := \{x \in \Omega : u_1(x) > u_2(x)\}$ is nonempty. We set $v_i := (1-q)^{-1}u_i^{1-q}$ for $i = 1, 2$, so that $\Phi := v_1 - v_2 > 0$ in $\tilde{\Omega}$. In addition,

$$-\Delta\Phi = q\left(u_1^{q-1}|\nabla v_1|^2 - u_2^{q-1}|\nabla v_2|^2\right) < qu_1^{q-1}\left(|\nabla v_1|^2 - |\nabla v_2|^2\right),$$

i.e.

$$-\Delta\Phi - qu_1^{q-1}\nabla(v_1 + v_2)\nabla\Phi < 0 \quad \text{in} \quad \tilde{\Omega}. \tag{2}$$

Since $\Phi = 0$ on $\partial\tilde{\Omega}$, we obtain a contradiction with the maximum principle. This shows that $(P_{\mathcal{D}})$ has at most one positive solution. Now, if $u_1, u_2 \in \mathcal{P}_{\mathcal{N}}^{\circ}$ solve $(P_{\mathcal{N}})$ then Φ satisfies (2) and for any $x \in \partial\tilde{\Omega}$ we have either $\Phi(x) = 0$ or $\partial_\nu\Phi(x) = 0$. By the maximum principle, we infer that Φ is constant in $\tilde{\Omega}$, which contradicts (2). The proof of the uniqueness of a solution of $(P_{\mathcal{D}})$ positive in Ω_+ (respect. a solution of $(P_{\mathcal{N}})$ in $\mathcal{P}_{\mathcal{N}}^{\circ}$) uses the same change of variables, but is more involved. We refer to [5, 6] for the details.

The existence results can be proved either by a variational argument or by the sub-supersolutions method. In the first case, it suffices to show that the functional

$$I_q(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{q+1}a(x)|u|^{q+1}\right)$$

has a negative global minimum in $H_0^1(\Omega)$ or $H^1(\Omega)$. In the latter case the condition (A.0) is crucial. The second approach consists in taking a ball $B \subset \Omega_+$ and a sufficiently small first positive eigenfunction of $-\Delta$ on $H_0^1(B)$ extended by zero to Ω, to find a (nontrivial) subsolution of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$. An arbitrary large supersolution of $(P_{\mathcal{D}})$ is given by kz, where z is the unique solution of $-\Delta z = a^+$ in Ω, $z = 0$ on $\partial\Omega$, and $k > 0$ is large enough (as usual, we write $a = a^+ - a^-$, with $a^\pm := \max(\pm a, 0)$). The construction of a suitable supersolution of $(P_{\mathcal{N}})$ under (A.0) is more delicate, and we refer to [6] for the details.

Finally, if $(P_{\mathcal{N}})$ has a positive solution u then, multiplying the equation by $(u+\varepsilon)^{-q}$ (with $0 < \varepsilon < 1$) and integrating by parts, we find that

$$\int_\Omega a \left(\frac{u}{u+\varepsilon}\right)^q = -q \int_\Omega (u+\varepsilon)^{-(q+1)} |\nabla u|^2 < -q \int_\Omega (u+1)^{-(q+1)} |\nabla u|^2 < 0.$$

Letting $\varepsilon \to 0$ we can check that $\int_\Omega a < 0$. □

Although not stated explicitly in [5, 6], the next corollary follows almost directly from the existence and uniqueness results in these papers.

Corollary B. *Let Ω_+ be connected and satisfy (A.2). Then $(P_{\mathcal{D}})$ has a unique nontrivial solution. The same conclusion holds for $(P_{\mathcal{N}})$ assuming in addition (A.0).*

Sketch of the proof. It is based on the fact that a nontrivial solution u of $(P_{\mathcal{D}})$ or $(P_{\mathcal{N}})$ satisfies $u \not\equiv 0$ in Ω_+, which follows from the inequality $0 < \int_\Omega |\nabla u|^2 \le \int_\Omega a^+(x) u^{q+1}$. Since Ω_+ is connected, by the maximum principle we find that $u > 0$ in Ω_+. And there is only one solution having this property, by Theorem A. □

Remark 2.1: (i) Let us remark that the nontrivial solutions provided by Theorem A (i-b) and (ii-b) are *not* necessarily unique, see e.g. [5, 6].

(ii) Regarding Theorem A (ii-b), it is worth pointing out that (A.0) is *not* necessary for the existence of a nontrivial solution of $(P_{\mathcal{N}})$ for some $q \in (0, 1)$, cf. [6, Section 4] and [28, Remark 4.3].

Let us now give an example of a nontrivial solution $u \not\gg 0$ of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$. It is essentially due to [20], where the case $q = \frac{1}{2}$ was considered (see Figure 1).

Example C. Let $\Omega := (0, \pi)$ and $q \in (0, 1)$. We choose

$$r = r_q := \frac{2}{1-q} \in (2, \infty), \quad a(x) = a_q(x) := r^{1-\frac{2}{r}} \left(1 - r\cos^2 x\right) \quad \text{for } x \in \overline{\Omega}.$$

Then $u(x) := \frac{\sin^r x}{r} \in C^2(\overline{\Omega})$ satisfies

$$\begin{cases} -u'' = a(x) u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = u' = u'' = 0 & \text{on } \partial\Omega. \end{cases}$$

The above example also provides *dead core* solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ both. Indeed, it suffices to consider any bounded open interval Ω' with $\Omega' \supset \overline{\Omega}$, and extend u by zero and a in any way to Ω'. Then u is a nontrivial dead core solution of both $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, considered now in Ω'.

Since the **SMP** does not apply and *dead core* solutions may exist, obtaining a positive solution for these problems is a delicate issue which has been given little consideration. Let $\varphi \in W_{\mathcal{D}}^{2,r}(\Omega)$ be the unique solution of the Poisson equation

$$\begin{cases} -\Delta\varphi = a(x) & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\mathcal{S} : L^r(\Omega) \to W_{\mathcal{D}}^{2,r}(\Omega)$ be the corresponding *solution operator*, i.e. $\mathcal{S}(a) = \varphi$. In [23] Hernández, Mancebo and Vega showed that the condition

$$(\mathbf{A.3}) \qquad \mathcal{S}(a) \gg 0$$

implies the existence of a positive solution of $(P_{\mathcal{D}})$ for all $q \in (0,1)$. Later on Godoy and Kaufmann [20, 21] provided other sufficient conditions, namely, that a^- is small enough, or q is close enough to 1 (for some particular choices of N, Ω, and a). We shall state a simplified version of these results in the sequel, and refer to [23, Theorem 4.4], [21, Theorems 3.1 and 3.2], and [20, Theorems 2.1 (i) and 3.2] for the precise statements.

Theorem D. *(i) If a satisfies (A.3) then $(P_{\mathcal{D}})$ has a positive solution for every $q \in (0,1)$.*

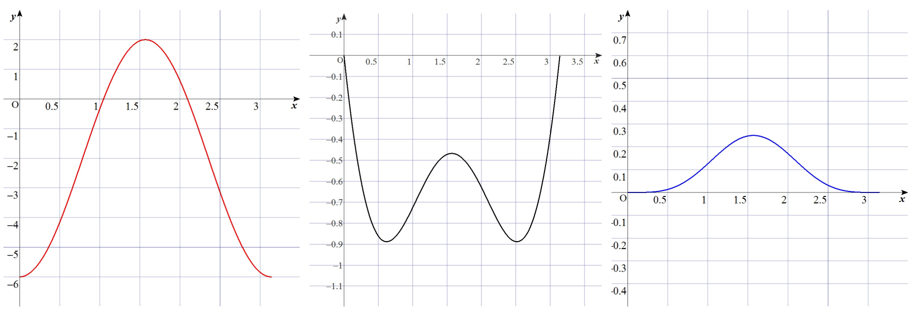
(ii) Let q and a^+ be fixed. Then there exists a constant $C > 0$ such that $(P_{\mathcal{D}})$ has a positive solution if $\|a^-\|_{C(\overline{\Omega})} < C$.

(iii) If either $N = 1$ or Ω is a ball, a is radial, and $0 \not\equiv a \geq 0$ in some smaller ball, then there exists $\overline{q} = \overline{q}(a)$ such that $(P_{\mathcal{D}})$ has a positive solution for $\overline{q} < q < 1$.

Remark 2.2: Let us mention that Theorem D (i) is still true for a linear second order elliptic operator with nonnegative zero order coefficient. On the other side, it may happen that $\mathcal{S}(a) < 0$ *everywhere* in Ω and yet $(P_{\mathcal{D}})$ admits a positive solution for some $q \in (0,1)$. Indeed, if we take $q = \frac{1}{2}$ in Example C then $\mathcal{S}(a_q) = x^2 - \pi x + 1 - \cos 2x < 0$ in $(0,\pi)$, see Figure 1 (ii). Note also that (A.3) is *not* compatible with the existence of a positive solution for $(P_{\mathcal{N}})$, since it implies $\int_\Omega a > 0$, contradicting (A.0), which is necessary by Theorem A (ii-b).

Sketch of the proof. All assertions follow by the well known sub-supersolutions method. Let us note that (*unlike* for $(P_{\mathcal{N}})$) it is easy to provide arbitrary big supersolutions for $(P_{\mathcal{D}})$. Indeed, a few computations show that $k\mathcal{S}(a^+)$ is a supersolution of $(P_{\mathcal{D}})$ for all $k > 0$ large enough. So the only task is to provide a *positive* subsolution. In (i), after some computations one can check that $[(1-q)\,\mathcal{S}\,(a)]^{1/(1-q)}$ is the desired subsolution.

In both (ii) and (iii), the subsolution is constructed by splitting the domain in two parts (a ball B in which $0 \not\equiv a \geq 0$, and $\Omega \setminus B$), constructing "subsolutions" in each of them, and checking that they can be glued appropriately to get a subsolution in the entire domain (see [8]). This fact depends on obtaining estimates for the normal derivatives of these subsolutions on the boundaries of the subdomains. In (iii) these bounds can be computed rather explicitly using the symmetry of a and the fact that Ω is a ball, while in (ii) the key tool is an estimate due to Brezis and Cabré [9, Lemma 3.2]. The proof of both (ii) and (iii) involve several computations, and we refer to [20, 21] for the details. $\square$



(i) (ii) (iii)

Figure 1: (i) The indefinite weight $a_{\frac{1}{2}}$; (ii) $\mathcal{S}(a_{\frac{1}{2}})$; (iii) The positive solution $u \not\gg 0$ for $a_{\frac{1}{2}}$.

Godoy and Kaufmann [21] also proved that when a is too negative in a ball there are no positive solutions of $(P_{\mathcal{D}})$ (see also Remark 2.3 (i) below). This result, which has been proved in [16, Proposition 1.11] in a more general setting, can also be seen as a first step towards the construction of dead core solutions.

Theorem E. *Let q and a^+ be fixed. Given a ball $B = B_R(x_0) \subset \Omega \setminus \Omega_+$ there exists a constant $C = C(\Omega, N, q, R, a^+) > 0$ such that any solution of $(P_{\mathcal{D}})$ vanishes at x_0 if $\min_{\overline{B}} a^- > C$.*

Sketch of the proof. We use a comparison argument: let u be a nontrivial solution of $(P_{\mathcal{D}})$, and $\underline{a} := \min_{\overline{B}} a^-$. Set

$$C_{N,q} := \frac{(1-q)^2}{2\,(N\,(1-q)+2q)} \quad \text{and} \quad w(x) := \left(C_{N,q}\underline{a}\,|x-x_0|^2\right)^{\frac{1}{1-q}}.$$

One can check that $\Delta w \leq a^- w^q$ in B. On the other hand, note that $\Delta u = a^- u^q$ in B and $\|u\|_\infty \leq (\|\mathcal{S}\| \, \|a^+\|_\infty)^{\frac{1}{1-q}}$, and so $u \leq w$ on ∂B if

$$\underline{a} \geq \frac{\|\mathcal{S}\| \, \|a^+\|_\infty}{R^2 C_{N,q}}. \tag{3}$$

It follows then from the comparison principle that $u \leq w$ in B. In particular, $u\,(x_0) = 0$. □

REMARK 2.3: (i) The latter proof can be adapted for the Neumann problem, taking into account the following *a priori* bound: Under (A.0), there exists $C > 0$ (independent of a^-) such that $\|u\|_{C(\overline{\Omega})} \leq C$ for every subsolution of $(P_\mathcal{N})$.

(ii) Note that $C_{N,q} \to 0$ as $q \to 1^-$, i.e. the closer is q to 1, the larger is the right-hand side in (3), and the more negative a needs to be in $B_R(x_0)$ to satisfy (3). This fact is consistent with Theorem D (ii) and (iii).

3. Recent results

Let us now briefly describe our main contributions to the study of $(P_\mathcal{D})$ and $(P_\mathcal{N})$, which can be found in [27, 28, 30]:

(I) We determine the values of $q \in (0,1)$ for which $(P_\mathcal{D})$ and $(P_\mathcal{N})$ have the positivity property. In other words, we provide a characterization of the following *positivity sets*:

$$\mathcal{A}_\mathcal{D} = \mathcal{A}_\mathcal{D}(a) := \{q \in (0,1) : \ u \gg 0 \text{ for any nontrivial solution } u \text{ of } (P_\mathcal{D})\},$$
$$\mathcal{A}_\mathcal{N} = \mathcal{A}_\mathcal{N}(a) := \{q \in (0,1) : \ u \gg 0 \text{ for any nontrivial solution } u \text{ of } (P_\mathcal{N})\}.$$

Thanks to a continuity argument inspired by Jeanjean [25], and based on the fact that the **SMP** applies when $q = 1$, we shall see in Theorem 4.1 that under (A.1) we have $\mathcal{A}_\mathcal{D} = (q_\mathcal{D}, 1)$ and, assuming additionally (A.0), $\mathcal{A}_\mathcal{N} = (q_\mathcal{N}, 1)$, for some $q_\mathcal{D}, q_\mathcal{N} \in [0,1)$ (see also Corollary 4.3 and Theorem 4.2).

Note that in view of the existence and uniqueness results in Theorem A, the sets $\mathcal{A}_\mathcal{D}$ and $\mathcal{A}_\mathcal{N}$ can also be expressed as follows:

$$\mathcal{A}_\mathcal{D} = \{q \in (0,1) : (P_\mathcal{D}) \text{ has a unique nontrivial solution } u, \text{ and } u \gg 0\},$$
$$\mathcal{A}_\mathcal{N} = \{q \in (0,1) : (P_\mathcal{N}) \text{ has a unique nontrivial solution } u, \text{ and } u \gg 0\}.$$

We also obtain some positivity properties for the *ground state* solution of $(P_\mathcal{D})$.

(II) By the previous discussion we deduce that $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$, under (A.0)) has a solution $u \gg 0$ for $q \in \mathcal{A}_{\mathcal{D}}$ (respect. $q \in \mathcal{A}_{\mathcal{N}}$). Thus, setting

$$\mathcal{I}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}}(a) := \{q \in (0,1) : (P_{\mathcal{D}}) \text{ has a solution } u \gg 0\},$$
$$\mathcal{I}_{\mathcal{N}} = \mathcal{I}_{\mathcal{N}}(a) := \{q \in (0,1) : (P_{\mathcal{N}}) \text{ has a solution } u \gg 0\},$$

we observe that $\mathcal{A}_{\mathcal{D}} \subseteq \mathcal{I}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}} \subseteq \mathcal{I}_{\mathcal{N}}$. We will further investigate $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{I}_{\mathcal{N}}$) and analyze how close $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{A}_{\mathcal{N}}$ and $\mathcal{I}_{\mathcal{N}}$) can be to each other, see Theorems 5.3, 5.8, and also Proposition 4.4 and Remark 5.2 (i).

Note that Corollary B tells us that if Ω_+ is connected and satisfies (A.2), then $\mathcal{A}_{\mathcal{D}} = \mathcal{I}_{\mathcal{D}}$, and if additionally (A.0) holds, then $\mathcal{A}_{\mathcal{N}} = \mathcal{I}_{\mathcal{N}}$. Assuming moreover (A.3), we find by Theorem 5.1 (iv-c) that $\mathcal{A}_{\mathcal{D}} = (0,1)$.

(III) We consider $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ via a bifurcation approach, looking at q as a bifurcation parameter and taking advantage of the fact that $(P_{\mathcal{D}})$ has a trivial line of strongly positive solutions when $q = 1$, see Theorems 5.1 and 5.5 for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, respectively. We also analyze the structure of the nontrivial solutions set (with respect to q) of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ via variational methods and the construction of sub and supersolutions, see Theorem 5.1 and Remark 5.9 for $(P_{\mathcal{D}})$; Remarks 5.7 and 5.9 for $(P_{\mathcal{N}})$. In particular, we describe the asymptotic behaviors of nontrivial solutions as $q \to 0^+$ and $q \to 1^-$.

(IV) Finally, in Section 6 we present, without proofs, two further kind of results. On the one hand, we provide *explicit* sufficient conditions for the existence of positive solutions for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$, see Theorems 6.1 and 6.2. And on the other hand, in Theorem 6.3 we state sufficient conditions for the existence of dead core solutions for $(P_{\mathcal{N}})$.

The above issues will be developed in the forthcoming sections. In the last section we include some final remarks and list some open questions.

4. The positivity property

The next theorem extends Theorem D (iii) under (A.1), showing that $(P_{\mathcal{D}})$, as well as $(P_{\mathcal{N}})$ under $(A.0)$, has a positive solution (and no other nontrivial solution) if q is close enough to 1. In other words, we show that under (A.1) the positivity property holds for such values of q [27, Theorems 1.3 and 1.7]:

Theorem 4.1. *Assume (A.1). Then:*

(i) $\mathcal{A}_{\mathcal{D}} = (q_{\mathcal{D}}, 1)$ *for some* $q_{\mathcal{D}} \in [0,1)$.

(ii) If (A.0) holds then $\mathcal{A}_{\mathcal{N}} = (q_{\mathcal{N}}, 1)$ for some $q_{\mathcal{N}} \in [0, 1)$.

Sketch of the proof. First we show that $\mathcal{A}_{\mathcal{D}}$ is nonempty. We proceed by contradiction, assuming that $q_n \to 1^-$ and u_n are nontrivial solutions of $(P_{\mathcal{D}})$ with $q = q_n$ and $u_n \not\gg 0$. We know that $u_n \not\equiv 0$ in Ω_+, and thanks to (A.1) we can assume that, for every $n \in \mathbb{N}$, $u_n > 0$ in some fixed connected component of Ω_+. If $\{u_n\}$ is bounded in $H_0^1(\Omega)$ then, by standard compactness arguments, up to a subsequence, we have $u_n \to u_0$ in $H_0^1(\Omega)$ and u_0 solves $-\Delta u_0 = a(x)u_0$. Moreover, we can show that $\{u_n\}$ is away from zero, so that $u_0 \not\equiv 0$. By the **SMP** we get that $u_0 \gg 0$. Finally, by standard elliptic regularity, we find that $u_n \to u_0$ in $C^1(\overline{\Omega})$, up to a subsequence. Thus $u_n \gg 0$ for n large enough, and we have a contradiction. If $\{u_n\}$ is unbounded in $H_0^1(\Omega)$ then, normalizing it, we obtain a sequence v_n converging to some $v_0 \not\equiv 0$ that solves an eigenvalue problem. Once again, the **SMP** implies that $v_0 \gg 0$, a contradiction. A similar argument shows that $\mathcal{A}_{\mathcal{D}}$ is open. Indeed, assume to the contrary that there exist $q_0 \in \mathcal{A}_{\mathcal{D}}$ and $q_n \notin \mathcal{A}_{\mathcal{D}}$ such that $q_n \to q_0$. We take nontrivial solutions $u_n \not\gg 0$ of $(P_{\mathcal{D}})$ with $q = q_n$. It is easily seen that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, $u_n \to u_0$ in $C^1(\overline{\Omega})$, where u_0 is a nontrivial solution of $(P_{\mathcal{D}})$ with $q = q_0$. Since $q_0 \in \mathcal{A}_{\mathcal{D}}$, we have $u_0 \gg 0$, and so $u_n \gg 0$ for n large enough, which is a contradiction. Thus $\mathcal{A}_{\mathcal{D}}$ is open. The proof of the connectedness of $\mathcal{A}_{\mathcal{D}}$ is more technical, and we refer to [27] for the details. The proof of (ii) follows similarly, see also [27]. □

Following a similar strategy, we show that the positivity property also holds in the Dirichlet case if a^- is small enough (assuming now that $q \in (0, 1)$ is fixed), which extends Theorem D (ii) under (A.1). Let us add that this theorem is also true for some non-powerlike nonlinearities [27, Theorem 1.1].

THEOREM 4.2. *Assume (A.1). Then there exists $\delta > 0$ (possibly depending on q and a^+) such that every nontrivial nonnegative solution u of $(P_{\mathcal{D}})$ satisfies that $u \gg 0$ if $\|a^-\|_{C(\overline{\Omega})} < \delta$.*

Note that since (A.0) is necessary for the existence of positive solutions of $(P_{\mathcal{N}})$, we can't expect an analogue of the above theorem for this problem.

As an immediate consequence of Theorem 4.1 and Corollary B, we infer:

COROLLARY 4.3. *Assume that Ω_+ is connected and satisfies (A.2), and let u_q be the unique nontrivial solution of $(P_{\mathcal{D}})$. Then $u_q \not\gg 0$ for all $q \in (0, q_{\mathcal{D}}]$ and $u_q \gg 0$ for all $q \in (q_{\mathcal{D}}, 1)$. A similar result holds for $(P_{\mathcal{N}})$ assuming, in addition, (A.0).*

Let us mention that, if in addition to the assumptions of Corollary 4.3, Ω_+ includes a *tubular neighborhood of* $\partial\Omega$ (i.e., a set of the form $\{x \in \Omega : d(x, \partial\Omega) < \rho\}$, for some $\rho > 0$) then the **SMP** shows that the solution u_q above satisfies either $u_q \gg 0$ or $u_q = 0$ somewhere in Ω, see Figure 4.

Although Theorem 4.1 claims that under (A.1) the sets $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}}$ are *always* nonempty, by Example C we see that given *any* $q \in (0,1)$, we may find $a = a_q$ satisfying (A.1) and such that $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ have a nontrivial solution $u \not\gg 0$. In view of Theorem 4.1, this fact shows that $\mathcal{A}_{\mathcal{D}}$ and $\mathcal{A}_{\mathcal{N}}$ can be arbitrarily small for a suitable a.

The next result (cf. [28, Theorem 1.4 (i)], [30, Proposition 5.1 (i)]) shows that for any $q \in (0,1)$, we may find a such that $q \in \mathcal{I}(a) \setminus \mathcal{A}(a)$ (and so, in general, $\mathcal{A} \subsetneq \mathcal{I}$).

Proposition 4.4. *(i) Given $\Omega \subset \mathbb{R}$ and $q \in (0,1)$, there exists $a \in C(\overline{\Omega})$ such that $q \in \mathcal{I}_{\mathcal{N}} \setminus \mathcal{A}_{\mathcal{N}}$.*

(ii) Given $\Omega \subset \mathbb{R}$ and $q \in (0,1)$, there exists $a \in C(\Omega) \cap L^r(\Omega)$, $r > 1$, such that $q \in \mathcal{I}_{\mathcal{D}} \setminus \mathcal{A}_{\mathcal{D}}$.

4.1. The ground state solution

Recall that the Dirichlet eigenvalue problem

$$(E_{\mathcal{D}}) \qquad \begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

has a first positive eigenvalue $\mu_{\mathcal{D}}(a)$, which is principal and simple, and a positive eigenfunction $\phi_{\mathcal{D}}(a) \gg 0$ associated with $\mu_{\mathcal{D}}(a)$ and normalized by $\int_\Omega \phi_{\mathcal{D}}^2 = 1$.

Let $I_q : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$I_q(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q+1}\int_\Omega a(x)|u|^{q+1}$$

for $q \in [0,1)$. It is well-known that nonnegative critical points (in particular minimizers) of I_q are solutions of $(P_{\mathcal{D}})$. By a *ground state* of I_q we mean a global minimizer of this functional.

Proposition 4.5. *I_q has a unique nonnegative ground state U_q for every $q \in (0,1)$. In addition:*

(i) $U_q > 0$ in Ω_+ and $q \mapsto U_q$ is continuous from $(0,1)$ to $W^{2,r}_{\mathcal{D}}(\Omega)$.

(ii) There exists $q_0 \in (0,1)$ such that $U_q \gg 0$ for $q \in (q_0, 1)$.

(iii) As $q \to 1^-$ we have $U_q \to 0$ in $C_0^1(\overline{\Omega})$ if $\mu_{\mathcal{D}}(a) > 1$, whereas $\|U_q\|_{C(\overline{\Omega})} \to \infty$ if $\mu_{\mathcal{D}}(a) < 1$.

(iv) If $q_n \to 0^+$ then, up to a subsequence, $U_{q_n} \to U_0$ in $C_0^1(\overline{\Omega})$, where U_0 is a nonnegative global minimizer of I_0. In particular, if $0 \not\equiv \mathcal{S}(a) \geq 0$ in Ω, then $U_q \to \mathcal{S}(a)$ in $C_0^1(\overline{\Omega})$ as $q \to 0^+$.

Sketch of the proof. By a standard minimization argument, one may easily prove the existence of a global minimizer of I_q. Moreover, there is a 1 to 1 correspondence between global minimizers of I_q and minimizers of $\int_\Omega |\nabla u|^2$ over the C^1 manifold $\left\{u \in H_0^1(\Omega) : \int_\Omega a(x)|u|^{q+1} = 1\right\}$. By [34, Theorem 1.1], we infer that if U_q and V_q are global minimizers of I_q then $U_q = tV_q$ for some $t > 0$. But since U_q and V_q solve $(P_{\mathcal{D}})$, we deduce that $t = 1$, i.e. U_q is the unique nonnegative global minimizer of I_q. If $U_q(x) = 0$ for some $x \in \Omega_+$ then, by the **SMP**, U_q vanishes is some ball $B \subset \Omega_+$. We choose a non-trivial and smooth $\psi \geq 0$ supported in B and extend it by zero to Ω. Then $I_q(U_q + t\psi) = I_q(U_q) + I_q(t\psi) < I_q(U_q)$ if t is small enough, which yields a contradiction. Using standard compactness arguments and the uniqueness of U_q, we can show that $U_q \to U_{q_0}$ in $W^{2,r}_{\mathcal{D}}(\Omega)$ as $q \to q_0$, for any $q_0 \in (0,1)$. Arguing as in the proof of Theorem 4.1 we prove that $U_q \gg 0$ for q close to 1, and $U_q \to 0$ in $C_0^1(\overline{\Omega})$ if $\mu_{\mathcal{D}}(a) > 1$. If $\mu_{\mathcal{D}}(a) < 1$ and $\{u_n\}$ is bounded in $H_0^1(\Omega)$, where $u_n := U_{q_n}$ and $q_n \to 1^-$, then again as in the proof of Theorem 4.1, we find that $u_n \to u_0$ and $u_0 \geq 0$ solves $-\Delta u_0 = a(x)u_0$ in Ω, $u_0 = 0$ on $\partial\Omega$. Using the fact that u_n are ground state solutions, we can show that $u_0 \not\equiv 0$, so that $\mu_{\mathcal{D}}(a) = 1$, a contradiction. Finally, we refer to [30] for the proof of (iv). ☐

REMARK 4.6: (i) It is not hard to show that under (A.0) the functional I_q, considered now in $H^1(\Omega)$, has a ground state, which is positive in Ω_+, and strongly positive for q close enough to 1.

(ii) Proposition 4.5 (ii) extends Theorem D and Theorem 4.1(i) (as long as the existence of a positive solution is concerned) without assuming (A.1).

5. Structure of the positive solutions set

This section is devoted to a further investigation of the set $\mathcal{I}_{\mathcal{D}}$ (respect. $\mathcal{I}_{\mathcal{N}}$), which provides a rather complete description of the positive solutions set of $(P_{\mathcal{D}})$ (respect. $(P_{\mathcal{N}})$). From Theorem 4.1 we observe that $(q_{\mathcal{D}}, 1) \subseteq \mathcal{I}_{\mathcal{D}}$ and $(q_{\mathcal{N}}, 1) \subseteq \mathcal{I}_{\mathcal{N}}$. Taking advantage of the ground state solution, constructing suitable sub and supersolutions, and also using a bifurcation approach, we analyze the asymptotic behavior of the positive solutions as $q \to 1^-$ and $q \to 0^+$.

5.1. The Dirichlet problem

Let us consider $(P_{\mathcal{D}})$, with $q \in (0,1)$ as a bifurcation parameter. To this end, we introduce two further conditions on a. The first one slightly weakens (A.3) requiring that

$$(\mathbf{A.3'}) \qquad \mathcal{S}(a) > 0 \ \text{ in } \ \Omega,$$

whereas the second one is a technical decay condition near $\partial\Omega$:

$$(\mathbf{A.4}) \qquad |a(x)| \leq Cd(x,\partial\Omega)^{\eta} \text{ a.e. in } \Omega_{\rho_0}, \text{ for some } \rho_0 > 0 \text{ and } \eta > 1 - \frac{1}{N},$$

where

$$\Omega_\rho =: \{x \in \Omega : d(x,\partial\Omega) < \rho\} \tag{4}$$

is the tubular neighborhood of $\partial\Omega$. It turns out that (A.3′) is sufficient to deduce the conclusion of Theorem D (i), i.e. that $(P_{\mathcal{D}})$ has a positive solution for every $q \in (0,1)$. In addition, we shall use (A.3′) to show that this solution converges to $\mathcal{S}(a)$ as $q \to 0^+$. On the other hand, (A.4) is needed to obtain solutions of $(P_{\mathcal{D}})$ bifurcating from $t\phi_{\mathcal{D}}$, for some $t > 0$, when $\mu_{\mathcal{D}}(a) = 1$. Since $\phi_{\mathcal{D}} = 0$ on $\partial\Omega$, we assume (A.4) to ensure that $a\,\phi_{\mathcal{D}}^{q-2}$ has the appropriate integrability to carry out this bifurcation procedure, see Subsection 5.1.1.

Denoting by $u_{\mathcal{D}}(q)$ the unique positive solution of $(P_{\mathcal{D}})$ for $q \in (0,1)$ whenever it exists, we see from Proposition 4.5 (ii) that $U_q = u_{\mathcal{D}}(q)$ for q close to 1, so that Proposition 4.5 (iii) provides the asymptotics of $u_{\mathcal{D}}(q)$ when $\mu_{\mathcal{D}}(a) \neq 1$. We treat now the case $\mu_{\mathcal{D}}(a) = 1$ and also provide the asymptotic behavior of $u_{\mathcal{D}}(q)$ as $q \to 0^+$, as well as sufficient conditions to have $u_{\mathcal{D}}(q) \gg 0$ for *every* $q \in (0,1)$. Under these conditions, we obtain a rather complete description of the positive solutions set $\{(q, u_{\mathcal{D}}(q)) : q \in (0,1)\}$ of $(P_{\mathcal{D}})$, see Figure 2. We shall present here a simplified version of these results. For the precise assumptions required in each of following items we refer to [30, Theorems 1.2 and 1.4, Corollary 1.6]. Under (A.4), let us set

$$t^*_{\mathcal{D}} := \exp\left[-\frac{\int_\Omega a(x)\phi_{\mathcal{D}}^2 \log \phi_{\mathcal{D}}}{\int_\Omega a(x)\phi_{\mathcal{D}}^2}\right]. \tag{5}$$

Theorem 5.1. *Let $r > N$. Assume (A.1), (A.2), (A.3′) and (A.4). Then $u_{\mathcal{D}}(q) = U_q > 0$ in Ω for every $q \in (0,1)$. In addition, if we set $u_{\mathcal{D}}(0) := \mathcal{S}(a)$ then $q \mapsto u_{\mathcal{D}}(q)$ is continuous from $[0,1)$ to $W^{2,r}_{\mathcal{D}}(\Omega)$. The asymptotic behavior of $u_{\mathcal{D}}(q)$ as $q \to 1^-$ is characterized as follows:*

(i) If $\mu_{\mathcal{D}}(a) \geq 1$ and we set

$$u_{\mathcal{D}}(1) := \begin{cases} t^*_{\mathcal{D}}\,\phi_{\mathcal{D}}, & \text{if } \mu_{\mathcal{D}}(a) = 1, \\ 0, & \text{if } \mu_{\mathcal{D}}(a) > 1 \text{ (bifurcation from zero)}, \end{cases}$$

then $q \mapsto u_{\mathcal{D}}(q)$ is left continuous at $q = 1$ (see Figure 2 (i), (ii)).

(ii) If $\mu_{\mathcal{D}}(a) < 1$ then the curve $\{(q, u_{\mathcal{D}}(q)) : q \in [0,1)\}$ bifurcates from infinity at $q = 1$ (see Figure 2 (iii)).

Finally, as for the strong positivity of $u_{\mathcal{D}}(q)$, we have the following two assertions:

(iii) If (A.3) holds then $u_{\mathcal{D}}(q) \gg 0$ *for* q *close to* 0 *or* 1.

(iv) In the following cases, we have $u_{\mathcal{D}}(q) \gg 0$ *for all* $q \in (0,1)$ *(and so,* $\mathcal{I}_{\mathcal{D}} = (0,1)$*):*

(a) $a \geq 0$ *in* Ω_{ρ_0} *for some* $\rho_0 > 0$,

(b) Ω *is a ball and* a *is radial,*

(c) (A.3) holds and Ω_+ *is connected.*

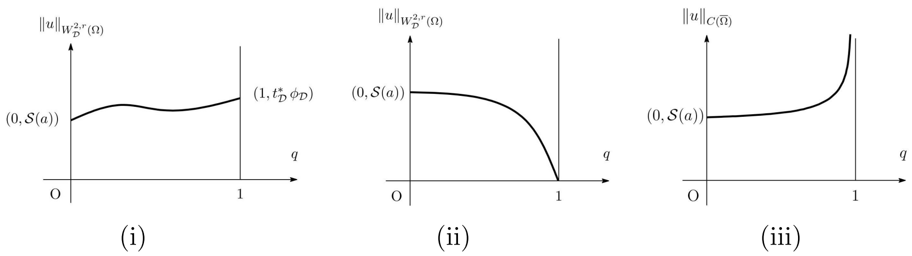


Figure 2: The curve of positive solutions emanating from $(0, \mathcal{S}(a))$: Cases (i) $\mu_{\mathcal{D}}(a) = 1$, (ii) $\mu_{\mathcal{D}}(a) > 1$, (iii) $\mu_{\mathcal{D}}(a) < 1$.

Remark 5.2: (i) From Theorem A and Proposition 4.5 (ii), it suffices to assume (A.1) and (A.2) to have $U_q = u_{\mathcal{D}}(q)$ whenever $u_{\mathcal{D}}(q)$ exists. Moreover, under these conditions,

$$\mathcal{I}_{\mathcal{D}} = \{q \in (0,1) : U_q \gg 0\}, \tag{6}$$

and $\mathcal{I}_{\mathcal{D}}$ is open.

(ii) The assertion in Theorem 5.1 (i) when $\mu_{\mathcal{D}}(a) = 1$ also gives a better asymptotic estimate for U_q as $q \to 1^-$ if (A.4) holds and $\mu_{\mathcal{D}}(a) \neq 1$. Indeed, a rescaling argument yields that

$$U_q \sim \mu_{\mathcal{D}}(a)^{-\frac{1}{1-q}}\, t^*_{\mathcal{D}}\, \phi_{\mathcal{D}} \quad \text{as} \quad q \to 1^-,$$

i.e.

$$\mu_{\mathcal{D}}(a)^{\frac{1}{1-q}} U_q \to t^*_{\mathcal{D}}\, \phi_{\mathcal{D}} \ \text{ in } \ W^{2,r}_{\mathcal{D}}(\Omega) \ \text{ as } \ q \to 1^-.$$

(iii) As already stated, under (A.3′) we have a positive solution for every $q \in (0,1)$. Assuming additionally (A.3), we can deduce the conclusion of Theorem 5.1 (iii), which extends Theorem D (i). Let us add that in some cases, by Theorem 6.1 below, the condition $\int_\Omega a \geq 0$ (which is weaker than (A.3′)) is also sufficient to have a positive solution of $(P_{\mathcal{D}})$ for all $q \in (0,1)$.

(iv) Under the assumptions of Theorem 5.1 (iv-c), we infer from Corollary B that $\mathcal{A}_{\mathcal{D}} = (0, 1)$.

Next we consider the linearized stability of a solution in $\mathcal{P}_{\mathcal{D}}^{\circ}$ of $(P_{\mathcal{D}})$ for $q \in \mathcal{I}_{\mathcal{D}}$. Let us recall that a solution $u \gg 0$ of $(P_{\mathcal{D}})$ is said to be *asymptotically stable* if $\gamma_1(q, u) > 0$, where $\gamma_1(q, u)$ is the first eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\varphi = qa(x)u^{q-1}\varphi + \gamma\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Observe that under the decay condition (A.4), given $q \in [0, 1)$ and $u \gg 0$, we have $au^{q-1} \in L^t(\Omega)$ for some $t > N$, so that $\gamma_1(q, u)$ is well defined.

The implicit function theorem (**IFT** for short) provides us with the following result [30, Theorem 1.5]:

THEOREM 5.3. *If (A.4) holds then $\mathcal{I}_{\mathcal{D}}$ is open, and $u_{\mathcal{D}}(q)$ is asymptotically stable for $q \in \mathcal{I}_{\mathcal{D}}$.*

5.1.1. Local bifurcation analysis in the case $\mu_{\mathcal{D}}(a) = 1$

Let us give a sketch of the proof of Theorem 5.1 (i) when $\mu_{\mathcal{D}}(a) = 1$. In this case, $(P_{\mathcal{D}})$ has the trivial line of strongly positive solutions:

$$\Gamma_1 := \{(q, u) = (1, t\phi_{\mathcal{D}}) : t > 0\}.$$

For $q \simeq 1$, where q is a bifurcation parameter, we shall construct solutions of $(P_{\mathcal{D}})$ bifurcating at certain $(1, t\phi_{\mathcal{D}}) \in \Gamma_1$ in $\mathbb{R} \times W_{\mathcal{D}}^{2,\xi}(\Omega)$, for some fixed $\xi > N$. This bifurcation result (Proposition 5.4 below) complements Proposition 4.5 (iii).

Under (A.4), choose $\sigma_0 > 0$ such that $\eta > 1 + \sigma_0 - \frac{1}{N}$ and set $J_0 := (1 - \frac{\sigma_0}{2}, 1 + \frac{\sigma_0}{2})$. We fix then $\xi \in (N, r)$, depending only on N and σ_0, in such a way that $\xi(\eta + q - 2) > -1 + \frac{\sigma_0 N}{4}$ for $q \in J_0$. Following the Lyapunov-Schmidt procedure, we reduce $(P_{\mathcal{D}})$ to a bifurcation equation. Set $A := -\Delta - a(x)$ with domain $D(A) := W_{\mathcal{D}}^{2,\xi}(\Omega)$. Then $\mathrm{Ker}A = \{t\phi_{\mathcal{D}} : t \in \mathbb{R}\}$ and $\mathrm{Im}A = \left\{f \in L^{\xi}(\Omega) : \int_\Omega f\phi_{\mathcal{D}} = 0\right\}$. Let Q be the projection of $L^{\xi}(\Omega)$ to $\mathrm{Im}A$, given by $Q[f] := f - \left(\int_\Omega f\phi_{\mathcal{D}}\right)\phi_{\mathcal{D}}$. As long as we consider solutions $u \gg 0$, $(P_{\mathcal{D}})$ is equivalent to the following coupled equations: for $u = t\phi_{\mathcal{D}} + w \in D(A) = \mathrm{Ker}A + X_2$ with $t = \int_\Omega u\phi_{\mathcal{D}}$ and $X_2 = \{u \in D(A) : \int_\Omega u\phi_{\mathcal{D}} = 0\}$,

$$Q\left[A(t\phi_{\mathcal{D}} + w)\right] = Q\left[a(x)\left((t\phi_{\mathcal{D}} + w)^q - (t\phi_{\mathcal{D}} + w)\right)\right], \tag{8}$$

$$(1 - Q)\left[A(t\phi_{\mathcal{D}} + w)\right] = (1 - Q)\left[a(x)\left((t\phi_{\mathcal{D}} + w)^q - (t\phi_{\mathcal{D}} + w)\right)\right]. \tag{9}$$

Given $t_0 > 0$, first we solve (8) with respect to w at $(q, t, w) = (1, t_0, 0)$, where $(1, t_0, 0)$ is a solution of (8). Note that (A.4) gives that (8) is C^2 for

$(q,t,w) \simeq (1,t_0,0)$, since the choice of ξ ensures that $a(t\phi_{\mathcal{D}}+w)^{q-2} \in L^{\xi}(\Omega)$ for such (q,t,w). An **IFT** argument shows the existence of a unique $w = w(q,t)$ for every $(q,t) \simeq (1,t_0)$ such that (q,t,w) solves (8). We plug $w(q,t)$ into (9), and thus, deduce the desired bifurcation equation

$$\Phi(q,t) := \int_{\Omega} a(x)\{(t\phi_{\mathcal{D}} + w(q,t))^q - (t\phi_{\mathcal{D}} + w(q,t)\}\phi_{\mathcal{D}} = 0, \quad (q,t) \simeq (1,t_0),$$

where we note that Φ is C^2 for $(q,t) \simeq (1,t_0)$.

As an application of the **IFT**, we find that if $(1, t_0\phi_{\mathcal{D}})$ is a bifurcation point on Γ_1 then

$$\frac{\partial \Phi}{\partial q}(1,t_0) = t_0 \left\{ (\log t_0) \int_{\Omega} a(x)\phi_{\mathcal{D}}^2 + \int_{\Omega} a(x)\phi_{\mathcal{D}}^2 \log \phi_{\mathcal{D}} \right\} = 0,$$

so that $t_0 = t^*_{\mathcal{D}}$, given by (5). Conversely, since direct computations [30, Lemma 4.3] provide

$$\frac{\partial \Phi}{\partial t}(1,t^*_{\mathcal{D}}) = \frac{\partial^2 \Phi}{\partial t^2}(1,t^*_{\mathcal{D}}) = 0, \quad \frac{\partial^2 \Phi}{\partial t \partial q}(1,t^*_{\mathcal{D}}) = \int_{\Omega} a\,(x)\,\phi_{\mathcal{D}}^2 > 0,$$

the Morse Lemma [17, Theorem 4.3.19] yields the following existence result [30, Proposition 4.4]:

PROPOSITION 5.4. *Suppose (A.4) with $\mu_{\mathcal{D}}(a) = 1$. Then the set of solutions of $(P_{\mathcal{D}})$ near $(1, t^*_{\mathcal{D}}\phi_{\mathcal{D}})$ consists of two continuous curves in $\mathbb{R} \times W^{2,\xi}_{\mathcal{D}}(\Omega)$ intersecting only at $(1, t^*_{\mathcal{D}}\phi_{\mathcal{D}})$ transversally, given by $\Gamma_1 \cup \Gamma_2$, where Γ_2 for $q < 1$ represents the ground state solution U_q.*

Let us mention that Proposition 5.4 remains true in $\mathbb{R} \times W^{2,r}_{\mathcal{D}}(\Omega)$ by elliptic regularity.

5.2. The Neumann problem

Under (A.0), the Neumann eigenvalue problem

$$(E_{\mathcal{N}}) \qquad \begin{cases} -\Delta\phi = \mu a(x)\phi & \text{in } \Omega, \\ \partial_\nu \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

has a first positive eigenvalue $\mu_{\mathcal{N}}(a)$, which is principal and simple, and an eigenfunction $\phi_{\mathcal{N}}(a) \gg 0$ associated to $\mu_{\mathcal{N}}(a)$ and satisfying $\int_\Omega \phi_{\mathcal{N}}^2 = 1$.

The bifurcation scheme from the previous subsection also applies to $(P_{\mathcal{N}})$, with the advantage of *not* requiring the decay condition (A.4), since $\phi_{\mathcal{N}} > 0$ on $\overline{\Omega}$. We look at q as a bifurcation parameter in $(P_{\mathcal{N}})$. Similarly as in the

Dirichlet case, if $\mu_{\mathcal{N}}(a) = 1$ then $u = t\phi_{\mathcal{N}}$ solves $(P_{\mathcal{N}})$ with $q = 1$, i.e., $(P_{\mathcal{N}})$ has the trivial line

$$\Gamma_1 := \{(q, u) = (1, t\phi_{\mathcal{N}}) : t > 0\}.$$

We shall obtain, for q close to 1, a curve of solutions $u \gg 0$ bifurcating from Γ_1 (see Figure 3).

The definition of *asymptotically stable* for solutions $u \gg 0$ of $(P_{\mathcal{N}})$ is similar to the one for $(P_{\mathcal{D}})$, see (7). Setting

$$t^*_{\mathcal{N}} := \exp\left[-\frac{\int_\Omega a(x)\phi^2_{\mathcal{N}} \log \phi_{\mathcal{N}}}{\int_\Omega a(x)\phi^2_{\mathcal{N}}}\right], \tag{10}$$

we have the following result [28, Theorem 1.2].

THEOREM 5.5. *Assume (A.0) and $r > N$. Then there exists $q_0 = q_0(a) \in (0, 1)$ such that $(P_{\mathcal{N}})$ has a solution $u_q \gg 0$ for $q_0 < q < 1$. Moreover, u_q is asymptotically stable and satisfies*

$$u_q \sim \mu_{\mathcal{N}}(a)^{-\frac{1}{1-q}}\, t^*_{\mathcal{N}}\, \phi_{\mathcal{N}} \quad \text{as} \quad q \to 1^-,$$

i.e. $\mu_{\mathcal{N}}(a)^{\frac{1}{1-q}} u_q \to t^*_{\mathcal{N}}\, \phi_{\mathcal{N}}$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$. *More specifically (see Figure 3):*

(i) If $\mu_{\mathcal{N}}(a) = 1$, *then* $u_q \to t^*_{\mathcal{N}}\, \phi_{\mathcal{N}}$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$.

(ii) If $\mu_{\mathcal{N}}(a) > 1$, *then* $u_q \to 0$ *in* $W^{2,r}(\Omega)$ *as* $q \to 1^-$.

(iii) If $\mu_{\mathcal{N}}(a) < 1$, *then* $\min_{\overline{\Omega}} u_q \to \infty$ *as* $q \to 1^-$.

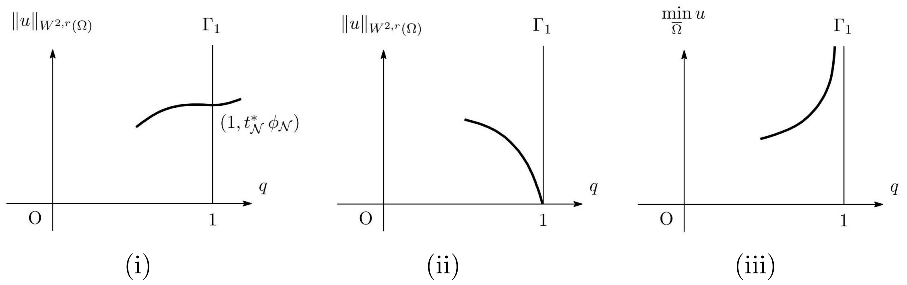


Figure 3: Bifurcating solutions $u \gg 0$ (i) from Γ_1 at $(1, t^*_{\mathcal{N}}\, \phi_{\mathcal{N}})$ in case $\mu_{\mathcal{N}}(a) = 1$; (ii) from zero in case $\mu_{\mathcal{N}}(a) > 1$; (iii) from infinity in case $\mu_{\mathcal{N}}(a) < 1$.

Let us point out that, in general, it is hard to give a lower estimate for $q_0(a)$, as one can see from Example C. As a direct consequence of Theorem 5.5, we complement Theorem A (ii-b) showing that (A.0) is also sufficient for the existence of a positive solution of $(P_{\mathcal{N}})$, for some $q \in (0,1)$:

COROLLARY 5.6. *$(P_{\mathcal{N}})$ has a positive solution (or a solution $u \gg 0$) for some $q \in (0,1)$ if and only if (A.0) holds.*

REMARK 5.7: Differently from the Dirichlet case, under (A.0) and (A.1) one may deduce the existence of a *dead core* limit function for nontrivial solutions of $(P_{\mathcal{N}})$ as $q \to 0^+$. Indeed, thanks to an *a priori* bound [28, Proposition 2.1], we may assume that a nontrivial solution u_n of $(P_{\mathcal{N}})$ with $q = q_n \to 0^+$ converges to $u_0 \geq 0$ in $C^1(\overline{\Omega})$. We claim that u_0 *vanishes somewhere* in Ω. Indeed, if $u_0 > 0$ in Ω then Lebesgue's dominated convergence theorem shows that $\int_\Omega \nabla u_0 \nabla v = \int_\Omega a(x)v$ for all $v \in C^1(\overline{\Omega})$, i.e. u_0 is a nontrivial solution of $(P_{\mathcal{N}})$ with $q = 0$, implying $\int_\Omega a = 0$, a contradiction. This situation does not occur in $(P_{\mathcal{D}})$ under (A.3′) (see Theorem 5.1).

The final result of this section is a characterization of the set $\mathcal{I}_{\mathcal{N}}$, which is proved by combining the **IFT** and the sub-supersolutions method [28, Theorem 1.4 (i)]. Also, using the **IFT** approach developed by Brown and Hess [12, Theorem 1], we have a stability result analogous to the one in Theorem 5.3.

THEOREM 5.8. *Assume (A.0). Then $\mathcal{I}_{\mathcal{N}} = (\hat{q}_{\mathcal{N}}, 1)$ for some $\hat{q}_{\mathcal{N}} \in [0,1)$. Moreover, for $q \in \mathcal{I}_{\mathcal{N}}$, the unique solution in $\mathcal{P}^\circ_{\mathcal{N}}$ is asymptotically stable.*

In addition to the local result given by Theorem 5.5, we can give a *global* description (i.e. for all $q \in (0,1)$) of the nontrivial solutions set of $(P_{\mathcal{N}})$ when Ω_+ is connected and satisfies (A.2):

REMARK 5.9: If Ω_+ is connected and satisfies (A.2), then Corollary 4.3 yields that $u_{\mathcal{D}}(q) \gg 0$ for $q \in (q_{\mathcal{D}}, 1)$, and the unique nontrivial solution of $(P_{\mathcal{D}})$ does not belong to $\mathcal{P}^\circ_{\mathcal{D}}$ for $q \in (0, q_{\mathcal{D}}]$. Moreover, if additionally Ω_+ includes a tubular neighborhood of $\partial\Omega$, then this solution vanishes somewhere in Ω. Note that the asymptotic behavior of $u_{\mathcal{D}}(q)$ as $q \to 1^-$, i.e. assertions (i) and (ii) of Theorem 5.1, remain valid, assuming additionally (A.4), see Figure 4. A similar result holds for $(P_{\mathcal{N}})$ if we assume, in addition, (A.0). In this case, the asymptotic behavior of the solution $u_q \gg 0$ as $q \to 1^-$, i.e. assertions (i)-(iii) of Theorem 5.5, also remain valid without assuming (A.4).

6. Some further results

In this section we present some results (without proofs) on the two following issues:

- *Explicit* sufficient conditions for the existence of positive solutions for $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$.
- Sufficient conditions for the existence of dead core solutions for $(P_{\mathcal{N}})$.

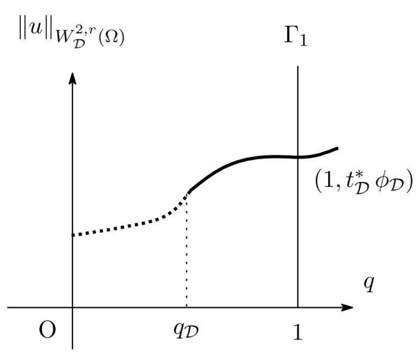


Figure 4: The bifurcation curve of the unique nontrivial solution in the case $\mu_{\mathcal{D}}(a) = 1$, assuming that Ω_+ is connected, satisfies (A.2), and includes a tubular neighborhood of $\partial\Omega$. Here the full curve represents $u_{\mathcal{D}}(q) \gg 0$, whereas the dotted curve represents solutions vanishing somewhere in Ω.

Given $0 < R_0 < R$, we write $B_{R_0} := \left\{x \in \mathbb{R}^N : |x| < R_0\right\}$. When $\Omega = B_R$ and a is radial, we shall exhibit some explicit conditions on q and a so that $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ admit a positive solution. In Theorem 6.1 below we consider the case that supp a^+ is contained in B_{R_0} and give a condition that guarantees the existence of a positive solution u (not necessarily $\gg 0$), while in Theorem 6.2 we consider the case that supp a^- is contained in B_{R_0} and provide a solution $u \gg 0$. These theorems are based on a sub-supersolutions approach and are inspired in the proofs of [20, Section 3] (see also the proof of Theorem D (iii)). If f is a radial function, we shall write $f(x) := f(|x|) := f(r)$, and we also set $A_{R_0,R} := \left\{x \in \mathbb{R}^N : R_0 < |x| < R\right\}$.

Theorem 6.1. *Let $a \in C(\overline{B_R})$ be a radial function such that*

- *$a \geq 0$ in B_{R_0} and $a \leq 0$ in $A_{R_0,R}$;*
- *$r \to a(r)$ is differentiable and nonincreasing in (R_0, R), and*

$$\frac{1-q}{1+q}\int_{A_{R_0,R}} a^- \leq \int_{B_{R_0}} a^+. \tag{11}$$

Then, $(P_{\mathcal{D}})$ has a positive solution. If, in addition, (A.0) holds, then $(P_{\mathcal{N}})$ has a positive solution.

Note that (11) holds for all $q \in (0,1)$ if $\int_{B_R} a \geq 0$, and this condition can also be formulated as

$$\frac{-\int_\Omega a}{\int_\Omega |a|} \leq q < 1. \tag{12}$$

In particular, we see that (11) is satisfied if q is close enough to 1. Note that if we replace a by

$$a_\delta = a^+ - \delta a^-, \quad \text{with} \quad \delta > \delta_0 := \frac{\int_\Omega a^+}{\int_\Omega a^-},$$

then the left-hand side in (12) approaches 1 as $\delta \to \infty$, so that this condition becomes very restrictive for a_δ as $\delta \to \infty$. On the other side, $\int_\Omega a_\delta \to 0^-$ as $\delta \to \delta_0^+$, so that (12) becomes much less constraining for a_δ as $\delta \to \delta_0^+$.

We denote by ω_{N-1} the surface area of the unit sphere ∂B_1 in $\mathbb{R}^N$.

THEOREM 6.2. *Let $a \in C(\overline{B_R})$ be a radial function satisfying (A.0). Assume that $a \geq 0$ in $A_{R_0,R}$ and*

$$\frac{1-q}{2q+N(1-q)}\omega_{N-1}R_0^N \left\|a^-\right\|_{C(\overline{B_{R_0}})} < \int_{A_{R_0,R}} a^+. \tag{13}$$

Then $(P_\mathcal{N})$ has a solution $u \gg 0$.

Unlike in Theorem 6.1, we observe that no differentiability nor monotonicity condition is imposed on a^- in Theorem 6.2. Note again that (13) is also clearly satisfied if q is close enough to 1.

Finally, we consider the existence of nontrivial dead core solutions of $(P_\mathcal{N})$. From [5, 6] we recall that the set $\{x \in \Omega : u(x) = 0\}$ is called the *dead core* of a nontrivial solution u of $(P_\mathcal{N})$ if it contains an interior point. Recall that in Theorem E we have already given sufficient conditions for the existence of a nontrivial solution of $(P_\mathcal{N})$ vanishing somewhere in Ω. We proceed now with the construction of dead cores for solutions of $(P_\mathcal{N})$. To this end, let us first introduce the following assumption:

$$0 \leq b_1, b_2 \in C(\overline{\Omega}) \quad \text{and} \quad \operatorname{supp} b_1 \cap \{x \in \Omega : b_2(x) > 0\} = \emptyset. \tag{14}$$

Given a nonempty open subset $G \subseteq \Omega$ and $\rho > 0$, we set

$$G^\rho := \{x \in G : \operatorname{dist}(x, \partial G) > \rho\}. \tag{15}$$

The following result is based on a comparison argument from [19]:

THEOREM 6.3. *Let $a_\delta := b_1 - \delta b_2$, with $b_1, b_2 \not\equiv 0$ satisfying (14), and $\delta > 0$. If we set $G := \{x \in \Omega : b_2(x) > 0\}$ then, given $0 < \overline{q} < 1$ and $\rho > 0$, there exists $\delta_0 = \delta_0(\rho, \overline{q}) > 0$ such that any nontrivial solution of $(P_\mathcal{N})$ with $a = a_\delta$ and $q \in (0, \overline{q}]$ vanishes in G^ρ if $\delta \geq \delta_0$.*

Theorem 6.3 holds also for the Dirichlet problem $(P_\mathcal{D})$. In particular, it complements Theorem 4.2 as follows: given $q \in (0,1)$ there exist $0 < \delta_1 < \delta_0$ such that every nontrivial solution u of $(P_\mathcal{D})$ with $a = a_\delta$ satisfies $u \gg 0$ for $\delta < \delta_1$, whereas u has a nonempty dead core for $\delta > \delta_0$.

7. Final remarks

Several conditions in this paper are assumed for the sake of presentation or technical reasons. As a matter of fact, the results in Sections 4 and 5 remain true more generally for $a \in L^r(\Omega)$ with $r > N$. In this situation, we assume, instead of (A.1), that

$$\begin{cases} \Omega_+ \text{ is the largest open subset of } \Omega \text{ where } a > 0 \text{ a.e.,} \\ \text{satisfies } |(\operatorname{supp} a^+) \setminus \Omega_+| = 0 \text{ and has a finite number} \\ \text{of connected components,} \end{cases}$$

where supp is the support in the measurable sense.

It is also important to highlight that the uniqueness results in Theorem A hold without assuming (A.1) and (A.2). Indeed, one may prove that the ground state solution U_q is the only solution of $(P_{\mathcal{D}})$ being positive in Ω_+, and a similar result applies to $(P_{\mathcal{N}})$ under (A.0), see [33]. A similar situation occurs in Theorem 5.1: without (A.1) and (A.2) the solution $u_{\mathcal{D}}(q)$ still exists for every $q \in (0,1)$, and satisfies assertions (i)-(iv) in Theorem 5.1 (cf. Remark 5.2 (i)).

Also, let us mention that some of the results in this paper can be extended to the Robin problem

$$\begin{cases} -\Delta u = a(x)u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \partial_\nu u = \alpha u & \text{on } \partial\Omega, \end{cases} \tag{16}$$

with $\alpha \in \mathbb{R}$. Some work in this direction has already been done in [31, 32]. Let us note that there are striking differences between (16) and the problems considered here. For instance, under (A.0)–(A.2) and some additional assumptions, for any $q \in \mathcal{I}_{\mathcal{N}}$ fixed, there exists some $\overline{\alpha} > 0$ such that (16) has *exactly two* strongly positive solutions for $\alpha \in (0, \overline{\alpha})$, *one* strongly positive solution for $\alpha = \overline{\alpha}$, and *no* strongly positive solutions for $\alpha > \overline{\alpha}$ [32, Theorem 1.3].

It is also worth pointing out that the positivity results in Section 4 can be applied to the study of positive solutions for indefinite concave-convex equations of the form $-\Delta u = a(x)u^q + b(x)u^p$, where $0 < 1 < q < p$, see [27, 29]. Finally, let us mention that several results presented here can be extended to problems involving a class of fully nonlinear homogeneous operators [14].

We conclude now with some interesting questions that remain open in the context of this paper:

(i) Is the set $\mathcal{I}_{\mathcal{D}}$ connected?

(ii) Is there some a such that $\mathcal{I}_{\mathcal{N}} = (0,1)$? Let us note that we can construct a sequence $a_n \in L^\infty(\Omega)$ such that $\mathcal{I}_{\mathcal{N}}(a_n) = (q_n, 1)$ with $q_n \searrow 0$ [28, Remark 4.5].

(iii) Assume $\mathcal{I}_{\mathcal{N}}(a) = (0,1)$. Can we characterize the limiting behavior of the solution $u_q \gg 0$ of $(P_{\mathcal{N}})$ as $q \to 0^+$?

(iv) By Theorem E, we see that we may have $q_{\mathcal{D}} > 0$ or $q_{\mathcal{N}} > 0$. On the other side, Theorem 5.1 (iv-c) shows a situation in which $q_{\mathcal{D}} = 0$. Can we have $q_{\mathcal{N}} = 0$ (i.e., $\mathcal{A}_{\mathcal{N}} = (0,1)$) ?

(v) Can we obtain explicit sufficient conditions for the existence of positive solutions of $(P_{\mathcal{D}})$ and $(P_{\mathcal{N}})$ (as e.g. the ones in Theorems 6.1 and 6.2 for $(P_{\mathcal{N}})$; or the ones in Theorem 6.1 and [20, Theorem 3.2 (i)] for $(P_{\mathcal{D}})$) without assuming that Ω is a ball and a is radial?

(vi) Is it possible to extend the results in this paper to a general operator of the form

$$Lu = -\operatorname{div}(A(x)\nabla u) + \langle b(x), \nabla u\rangle + c\,(x)\,u,$$

under suitable assumptions on the coefficients? Let us note that if $b \not\equiv 0$ variational techniques do not apply. Furthermore, the size of the coefficient c plays an important role: in the one-dimensional Dirichlet case no positive solutions exist if $c > 0$ is large enough, cf. [26, Theorem 3.11]. Let us add that the Neumann case with $A \equiv 1$, $b \equiv 0$, and c constant has been treated in [1, 32].

(vii) We believe that many of the resuts and techniques reviewed here also apply to the corresponding p-Laplacian equation

$$-\Delta_p u = a(x)u^q,$$

with $p > 1$ and $0 < q < p - 1$. Some progress in this direction has been achieved in [33].

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Nonlinear boundary value problems relative to the one dimensional heat equation

Laurent Véron

To Julian with high esteem and sincere friendship

Abstract. *We consider the problem of existence of a solution u to $\partial_t u - \partial_{xx} u = 0$ in $(0,T)\times\mathbb{R}_+$ subject to the boundary condition $-u_x(t,0) + g(u(t,0)) = \mu$ on $(0,T)$ where μ is a measure on $(0,T)$ and g a continuous nondecreasing function. When $p > 1$ we study the set of self-similar solutions of $\partial_t u - \partial_{xx} u = 0$ in $\mathbb{R}_+\times\mathbb{R}_+$ such that $-u_x(t,0) + u^p = 0$ on $(0,\infty)$. At end, we present various extensions to a higher dimensional framework.*

Keywords: nonlinear heat flux, singularities, Radon measures, Marcinkiewicz spaces.
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1. Introduction

Let $g:\mathbb{R}\mapsto\mathbb{R}$ be a continuous nondecreasing function. Set $Q^T_{\mathbb{R}_+} = (0,T)\times\mathbb{R}_+$ for $0 < T \leq \infty$ and $\partial_\ell Q^T_{\mathbb{R}_+} = \overline{\mathbb{R}_+}\times\{0\}$. The aim of this article is to study the following 1-dimensional heat equation with a nonlinear flux on the parabolic boundary

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^T_{\mathbb{R}_+}\\ -u_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) &= \mu && \text{in } [0,T)\\ u(0,\,\cdot\,) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \tag{1}$$

where ν,μ are Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A related problem in $Q^\infty_{\mathbb{R}_+}$ for which there exist explicit solutions is the following,

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+}\\ -u_x(t,0) + |u|^{p-1}u(t,0) &= 0 && \text{for all } t > 0\\ \lim_{t\to 0} u(t,x) &= 0 && \text{for all } x > 0, \end{aligned} \tag{2}$$

where $p > 1$. Problem (2) is invariant under the transformation T_k defined for all $k > 0$ by

$$T_k[u](t,x) = k^{\frac{1}{p-1}} u(k^2 t, kx). \tag{3}$$

This leads naturaly to look for existence of self-similar solutions under the form

$$u_s(t,x) = t^{-\frac{1}{2(p-1)}} \omega\left(\frac{x}{\sqrt{t}}\right). \tag{4}$$

Putting $\eta = \frac{x}{\sqrt{t}}$, ω satisfies

$$\begin{aligned} -\omega'' - \frac{1}{2}\eta\omega' - \frac{1}{2(p-1)}\omega = 0\,, \quad \text{in } \mathbb{R}_+\,, \\ -\omega'(0) + |\omega|^{p-1}\omega(0) = 0\,, \\ \lim_{\eta\to\infty} \eta^{\frac{1}{p-1}}\omega(\eta) = 0\,. \end{aligned} \tag{5}$$

Self-similar solutions of non-linear diffusion equations such as porous-media or fast-diffusion equation were discovered long time ago by Kompaneets and Zeldovich and a thourougful study was made by Barenblatt, reducing the study to the one of integrable ordinary differential equations with explicit solutions. Concerning semilinear heat equation Brezis, Terman and Peletier opened the study of self-similar solutions of semilinear heat equations in proving in [5] the existence of a positive strongly singular function satisfying

$$u_t - \Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \tag{6}$$

and vanishing at $t = 0$ on $\mathbb{R}^n \setminus \{0\}$. They called it the *very singular solution*. Their method of construction is based upon the study of an ordinary differential equation with a phase space analysis. A new and more flexible method based upon variational analysis has been provided by [7]. Other singular solutions of (6) in different configurations such as boundary singularities have been studied in [13]. We set $K(\eta) = e^{\eta^2/4}$ and

$$L^2_K(\mathbb{R}_+) = \left\{ \phi \in L^1_{loc}(\mathbb{R}_+) : \int_{\mathbb{R}_+} \phi^2 K dx := \|\phi\|^2_{L^2_K} < \infty \right\},$$

and, for $k \geq 1$,

$$H^k_K(\mathbb{R}_+) = \left\{ \phi \in L^2_K(\mathbb{R}_+) : \sum_{\alpha=0}^{k} \left\| \phi^{(\alpha)} \right\|^2_{L^2_K} := \|\phi\|^2_{H^k_K} < \infty \right\}.$$

Let us denote by $\mathcal{E}$ the subset of $H^1_K(\mathbb{R}_+)$ of weak solutions of (5) that is the set of functions satisfying

$$\int_0^\infty \left(\omega'\zeta' - \frac{1}{2(p-1)}\omega\zeta\right) K(\eta)d\eta + \left(|\omega|^{p-1}\omega\zeta\right)(0) = 0,$$

and by $\mathcal{E}_+$ the subset of nonnegative solutions. The next result gives the structure of $\mathcal{E}$.

THEOREM 1.1. *1. If $p \geq 2$, then $\mathcal{E} = \{0\}$.*

2. If $1 < p \leq \frac{3}{2}$, then $\mathcal{E}_+ = \{0\}$

3. If $\frac{3}{2} < p < 2$ then $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$ where ω_s is the unique positive solution of (5). Furthermore there exists $c > 1$ such that

$$c^{-1}\eta^{\frac{1}{p-1}-1} \leq e^{\frac{\eta^2}{4}}\omega_s(\eta) \leq c\eta^{\frac{1}{p-1}-1} \quad \textit{for all} \quad \eta > 0. \tag{7}$$

Whenever it exists the function u_s defined in (4) is the limit, when $\ell \to \infty$ of the positive solutions $u_{\ell\delta_0}$ of

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in} \;\; Q^{\infty}_{\mathbb{R}_+} \\ -u_x(t,\,\cdot\,) + |u|^{p-1}u(t,\,\cdot\,) &= \ell\delta_0 \quad \text{in} \;\; [0,T) \\ \lim_{t\to 0} u(t,x) &= 0 \quad \text{for all} \;\; x \in \mathbb{R}_+. \end{aligned}$$

When such a function u_s does not exits the sequence $\{u_{\ell\delta_0}\}$ tends to infinity. This is a charateristic phenomenon of an underlying fractional diffusion associated to the linear equation

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in} \;\; Q^{\infty}_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) &= \mu \quad \text{in} \;\; [0,\infty) \\ u(0,\,\cdot\,) &= 0 \quad \text{in} \;\; \mathbb{R}_+. \end{aligned}$$

More generaly we consider problem (1). We define the set $\mathbb{X}(Q^T_{\mathbb{R}_+})$ of test functions by

$$\mathbb{X}(Q^T_{\mathbb{R}_+}) = \left\{\zeta \in C^{1,2}_c([0,T) \times [0,\infty)) : \zeta_x(t,0) = 0 \;\; \text{for} \; t \in [0,T]\right\}.$$

DEFINITION 1.2. *Let ν, μ be Radon measures in $\mathbb{R}_+$ and $[0,T)$ respectively. A function u defined in $\overline{Q^T_{\mathbb{R}_+}}$ and belonging to $L^1_{loc}(\overline{Q^T_{\mathbb{R}_+}}) \cap L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ such that $g(u) \in L^1(\partial_\ell Q^T_{\mathbb{R}_+}; dt)$ is a weak solution of (1) if for every $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$ there holds*

$$\begin{aligned} -\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})udxdt + \int_0^T (g(u)\zeta)\,(t,0)dt \\ = \int_0^\infty \zeta d\nu(x) + \int_0^T \zeta(t,0)d\mu(t). \end{aligned} \tag{8}$$

We denote by $E(t,x)$ the Gaussian kernel in $\mathbb{R}_+\times\mathbb{R}$. The solution of

$$\begin{aligned} v_t - v_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -v_x &= \delta_0 \quad \text{in } \overline{\mathbb{R}_+} \\ v(0,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+, \end{aligned}$$

has explicit expression

$$v(t,x) = 2E(t,x) = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

If $x,y>0$ and $s<t$ we set $\tilde{E}(t-s,x,y) = E(t-s,x-y) + E(t-s,x+y)$. When $\nu\in\mathfrak{M}^b(\mathbb{R}_+)$ and $\mu\in\mathfrak{M}^b(\overline{\mathbb{R}_+})$ the solution of

$$\begin{aligned} v_t - v_{xx} &= 0 \quad \text{in } Q^{\infty}_{\mathbb{R}_+} \\ -v_x(\,\cdot\,,0) &= \mu \quad \text{in } \overline{\mathbb{R}_+} \\ u(0,\cdot\,) &= \nu \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{9}$$

is given by

$$\begin{aligned} v_{\nu,\mu}(t,x) &= \int_0^\infty \tilde{E}(t,x,y)d\nu(y) + 2\int_0^t E(t-s,x)d\mu(s) \\ &= \mathcal{E}_{\mathbb{R}_+}[\nu](t,x) + \mathcal{E}_{\mathbb{R}_+\times\{0\}}[\mu](t,x) = \mathcal{E}_{Q^{\infty}_{\mathbb{R}_+}}[(\nu,\mu)](t,x). \end{aligned} \tag{10}$$

We prove the following existence and uniqueness result.

THEOREM 1.3. *Let $g:\mathbb{R}\mapsto\mathbb{R}$ be a continuous nondecreasing function such that $g(0)=0$. If g satisfies*

$$\int_1^\infty (g(s)-g(-s))s^{-3}ds < \infty, \tag{11}$$

then for any bounded Borel measures ν in $\mathbb{R}_+$ and μ in $[0,T)$, there exists a unique weak solution $u := u_{\nu,\mu}\in L^1(Q^T_{\mathbb{R}_+})$ of (1). Furthermore the mapping $(\nu,\mu)\mapsto u_{\nu,\mu}$ is nondecreasing.

When $g(s)=|s|^{p-1}s$, condition (11) is satisfied if

$$0<p<2.$$

The above result is still valid under minor modifications if $\mathbb{R}_+$ is replaced by a bounded interval $I:=(a,b)$, and problem (1) by

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^T_I \\ u_x(\,\cdot\,,b) + g(u(\,\cdot\,,b)) &= \mu_1 \quad \text{in } [0,T) \\ -u_x(\,\cdot\,,a) + g(u(\,\cdot\,,a)) &= \mu_2 \quad \text{in } [0,T) \\ u(0,\cdot\,) &= \nu \quad \text{in } (a,b), \end{aligned}$$

where ν, μ_j $(j = 1, 2)$ are Radon measures in I and $(0, T)$ respectively.

In the last section we present the scheme of the natural extensions of this problem to a multidimensional framework

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) = \mu \quad & \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0, \cdot\,) = \nu \quad & \text{in } \mathbb{R}^n_+, \end{aligned}$$

The construction of solutions with measure data can be generalized but there are some difficulties in the obtention of self-similar solutions. The equation with a source flux

$$\begin{aligned} u_t - \Delta u = 0 \quad & \text{in } Q^T_{\mathbb{R}^n_+} \\ u_{x_n} + g(u) = 0 \quad & \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0, \cdot\,) = \nu \quad & \text{in } \mathbb{R}^n_+, \end{aligned} \tag{12}$$

has been studied by several authors, in particular Fila, Ishige, Kawakami and Sato [8, 10, 11]. Their main concern deals with global existence of solutions.

2. Self-similar solutions

2.1. The symmetrization

We define the operator $\mathcal{L}_K$ in $C^2_0(\mathbb{R})$ by

$$\mathcal{L}_K(\phi) = -K^{-1}(K\phi')'.$$

The operator $\mathcal{L}_K$ has been thouroughly studied in [7]. In particular

$$\inf\left\{\int_{-\infty}^{\infty} \phi'^2 K(\eta)\eta : \int_{-\infty}^{\infty} \phi^2 K(\eta) d\eta = 1\right\} = \frac{1}{2}.$$

The above infimum is achieved by $\phi_1 = (4\pi)^{-\frac{1}{2}} K^{-1}$ and $\mathcal{L}_K$ is an isomorphism from $H^1_K(\mathbb{R})$ onto its dual $(H^1_K(\mathbb{R}))' \sim H^{-1}_K(\mathbb{R})$. Finally $\mathcal{L}^{-1}_K$ is compact from $L^2_K(\mathbb{R})$ into $H^1_K(\mathbb{R})$, which implies that $\mathcal{L}_K$ is a Fredholm self-adjoint operator with

$$\sigma(\mathcal{L}_K) = \left\{\lambda_j = \tfrac{1+j-1}{2} : j = 1, 2, \ldots\right\},$$

and

$$ker\,(\mathcal{L}_K - \lambda_j I_d) = span\left\{\phi_1^{(j)}\right\}.$$

If ϕ is defined in $\mathbb{R}_+$, $\tilde{\phi}(x) = \phi(-x)$ is the symmetric with respect to 0 while $\phi^*(x) = -\phi(-x)$ is the antisymmetric with respect to 0. The operator $\mathcal{L}_K$ restricted to $\mathbb{R}_+$ is denoted by $\mathcal{L}^+_K$. The operator $\mathcal{L}^{+,N}_K$ with Neumann condition at $x = 0$ is again a Fredholm operator. This is also valid for the operator $\mathcal{L}_K^{+,D}$ with Dirichlet condition at $x = 0$. Hence, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,N}$, then $\tilde{\phi}$ is an eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$. Similarly, if ϕ is an eigenfunction of $\mathcal{L}_K^{+,D}$, then ϕ^* is an eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$. Conversely, any even (resp. odd) eigenfunction of $\mathcal{L}_K$ in $L^2_K(\mathbb{R})$ satisfies Neumann (resp. Dirichlet) boundary condition at $x = 0$. Hence its restiction to $L^2_K(\mathbb{R}_+)$ is an eigenfunction of $\mathcal{L}_K^{+,N}$ (resp. $\mathcal{L}_K^{+,D}$). Since $\phi_1^{(j)}$ is even (resp. odd) if and only if j is even (resp. odd), we derive

$$H_K^{1,0}(\mathbb{R}_+) = \bigoplus_{\ell=1}^{\infty} span\left\{\phi_1^{(2\ell+1)}\right\},$$

and

$$H_K^{1}(\mathbb{R}_+) = \bigoplus_{\ell=0}^{\infty} span\left\{\phi_1^{(2\ell)}\right\}.$$

Note that $\phi \in H^1_K(\mathbb{R}_+)$ such that $\phi_x(0) = 0$ (resp. $\phi(0) = 0$) implies $\tilde{\phi} \in H^1_K(\mathbb{R})$ (resp. $\phi^* \in H^1_K(\mathbb{R})$). Furthermore, ϕ_1 is an eigenfunction of $\mathcal{L}_K^+$ in $H^1_K(\mathbb{R}^n_+)$ with Neumann boundary condition on $\partial\mathbb{R}^n_+$ while $\partial_{x_n}\phi_1$ is an eigenfunction of $\mathcal{L}_K^+$ in $H^1_K(\mathbb{R}^n_+)$ with Dirichlet boundary condition on $\partial\mathbb{R}^n_+$. We list below two important properties of $H^1_K(\mathbb{R}_+)$ valid for any $\beta > 0$. Actually they are proved in [7, Prop. 1.12] with $H^1_{K^\beta}(\mathbb{R})$ but the proof is valid with $H^1_{K^\beta}(\mathbb{R}_+)$.

$$\begin{array}{ll} (i) & \phi \in H^1_{K^\beta}(\mathbb{R}_+) \Longrightarrow K^{\frac{\beta}{2}}\phi \in C^{0,\frac{1}{2}}(\mathbb{R}_+) \\ (ii) & H^1_{K^\beta}(\mathbb{R}_+) \hookrightarrow L^2_{K^\beta}(\mathbb{R}_+) \quad \text{is compact for all } n \geq 1. \end{array}$$

2.2. Proof of Theorem 1.1-(i)-(ii)

Assume $p \geq 2$, then $\frac{1}{2(p-1)} \leq \frac{1}{2}$. If ω is a weak solution, then

$$\int_0^\infty \left(\omega'^2 - \frac{1}{2(p-1)}\omega^2\right) K d\eta + |\omega|^{p+1}(0) = 0.$$

If $\frac{1}{2} > \frac{1}{2(p-1)}$ we deduce that $\omega = 0$. Furthermore, when $\frac{1}{2} = \frac{1}{2(p-1)}$ then

$$|\omega|^{p+1}(0) = 0.$$

If ω is nonzero, it is an eigenfunction of $\mathcal{L}_K^{+,D}$. Since the first eigenvalue is 1 it would imply $1 = \frac{1}{2(p-1)} \leq \frac{1}{2}$, contradiction.

Assume $1 < p \leq \frac{3}{2}$ and ω is a nonnegative weak solution. We take $\zeta(\eta) = \eta e^{-\frac{\eta^2}{4}} = -2\phi_1'(\eta)$, then

$$\int_0^\infty \left(-\zeta'' - \frac{1}{2(p-1)}\zeta\right)\omega K(\eta)d\eta + \zeta'(0)\omega^p(0) = 0.$$

Since $-\zeta'' = \zeta\lfloor_{\mathbb{R}_+} > 0$ and $\zeta'(0) = \phi_1(0) = 1$, we derive $\omega\zeta = 0$ if $1 > \frac{1}{2(p-1)}$ and $\omega(0) = 0$ if $1 = \frac{1}{2(p-1)}$. Hence $\omega'(0) = 0$ by the equation and $\omega \equiv 0$ by the Cauchy-Lipschitz theorem.

2.3. Proof of Theorem 1.1-(iii)

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi) = \frac{1}{2}\int_0^\infty \left(\phi'^2 - \frac{1}{2(p-1)}\phi^2\right) K d\eta + \frac{1}{p+1}|\phi(0)|^{p+1}.$$

LEMMA 2.1. *The functional J is lower semicontinuous in $H^1_K(\mathbb{R}_+)$. It tends to infinity at infinity and achieves negative values.*

Proof. We write

$$J(\psi) = J_1(\psi) - J_2(\psi) = J_1(\psi) - \frac{1}{2(p-1)} \|\psi\|^2_{L^2_K}.$$

Clearly J_1 is convex and J_2 is continuous in the weak topology of $H^1_K(\mathbb{R}_+)$ since the imbedding of $H^1_K(\mathbb{R}_+)$ into $L^2_K(\mathbb{R}_+)$ is compact. Hence J is weakly semicontinuous in $H^1_K(\mathbb{R}_+)$.

Let $\epsilon > 0$, then

$$J(\epsilon\phi_1) = \left(\frac{1}{4} - \frac{1}{4(p-1)}\right)\frac{\epsilon^2\sqrt{\pi}}{2} + \frac{\epsilon^{p+1}}{p+1}.$$

Since $1 < p < 2$, $\frac{1}{4} - \frac{1}{4(p-1)} < 0$. Hence $J(\epsilon\phi_1) < 0$ for ϵ small enough, thus J achieves negative values on $H^1_K(\mathbb{R}_+)$.

If $\psi \in H^1_K(\mathbb{R}_+)$ it can be written in a unique way under the form $\psi = a\phi_1 + \psi_1$ where $a = 2\sqrt{\pi}\psi(0)$ and $\psi_1 \in H^{1,0}_K(\mathbb{R}_+)$. Hence, for any $\epsilon > 0$,

$$\begin{aligned}
J(\psi) &= \frac{1}{2}\int_0^\infty \left(\psi_1'^2 - \frac{1}{2(p-1)}\psi_1^2\right) K d\eta + \frac{a^2}{2}\int_0^\infty \left(\phi_1'^2 - \frac{1}{2(p-1)}\phi_1^2\right) K d\eta \\
&\quad + a\int_0^\infty \left(\psi_1'\phi_1' - \frac{1}{2(p-1)}\psi_1\phi_1\right) K d\eta + \frac{1}{p+1}|a|^{p+1} \\
&\geq \frac{2p-3}{4(p-1)}\int_0^\infty \psi_1'^2 K d\eta - \frac{a\epsilon}{2}\int_0^\infty \left(\psi_1'^2 + \frac{1}{2(p-1)}\psi_1^2\right) K d\eta \\
&\quad + \frac{a^2(p-2)\sqrt{\pi}}{4(p-1)} - \frac{ap\sqrt{\pi}}{4(p-1)\epsilon} + \frac{1}{p+1}|a|^{p+1}.
\end{aligned}$$

Note that $\|\psi\|^2_{H^1_K} \leq 4\left(\|\psi_1'\|^2_{L^2_K} + a^2\right)$. Since $2p-3>0$, we can take $\epsilon>0$ small enough in order that

$$\lim_{\|\psi\|_{H^1_K}\to\infty} J(\psi) = \infty.$$

□

By Lemma 2.1 the functional J achieves its minimum in $H^1_K(\mathbb{R}_+)$ at some $\omega_s \neq 0$, and ω_s can be assumed to be nonnegative since J is even. By the strong maximum principle $\omega_s > 0$, and by the method used in the proof of [15, Proposition 1] is is easy to prove that positive solutions belong to $H^2_K(\mathbb{R}_+)$. Assume that $\tilde{\omega}$ is another positive solution, then

$$\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde{\omega}_s')'}{\tilde{\omega}_s}\right)(\omega_s^2 - \tilde{\omega}_s^2)d\eta = 0.$$

Integration by parts, easily justified by regularity, yields

$$\begin{aligned}
&\int_0^\infty \left(\frac{(K\omega_s')'}{\omega_s} - \frac{(K\tilde{\omega}_s')'}{\tilde{\omega}_s}\right)(\omega_s^2 - \tilde{\omega}_s^2)d\eta \\
&\quad= \left[K\omega_s'\left(\omega_s - \frac{\tilde{\omega}_s^2}{\omega_s}\right) - K\tilde{\omega}_s'\left(\frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s\right)\right]_0^\infty \\
&\qquad - \int_0^\infty \left(\omega_s - \frac{\tilde{\omega}_s^2}{\omega_s}\right)' K\omega_s' d\eta + \int_0^\infty \left(\frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s\right)' K\omega_s' d\eta \\
&\quad= -\left(\omega_s^{p-1} - \tilde{\omega}_s^{p-1}\right)\left(\omega_s^2 - \tilde{\omega}_s^2\right)(0) \\
&\qquad - \int_0^\infty \left(\left(\frac{\omega_s'\tilde{\omega}_s - \omega_s\tilde{\omega}_s'}{\tilde{\omega}_s}\right)^2 + \left(\frac{\omega_s\tilde{\omega}_s' - \tilde{\omega}_s\omega_s'}{\omega_s}\right)^2\right) d\eta.
\end{aligned}$$

This implies that $\omega_s = \tilde{\omega}_s$. The proof of (7) is similar as the proof of estimate (2.5) in [13, Theorem 4.1].

2.4. The explicit approach

This part is an adaptation to our problem of what has been done in [9] concerning the blow-up problem in equation (12). Let ω be a solution of

$$\omega'' + \frac{1}{2}\eta\omega' + \frac{1}{2(p-1)}\omega = 0 \qquad \text{in } \mathbb{R}_+. \tag{13}$$

We set

$$r = \frac{\eta^2}{4} \text{ and } \omega(\eta) = r^{-\frac{1}{4}} e^{-\frac{r}{2}} Z(r).$$

Then Z satisfies the Whittaker equation (with the standard notations)

$$Z_{rr} + \left(-\frac{1}{4} + \frac{k}{r} + \frac{1-4\mu^2}{4r^2} \right) Z = 0$$

where $k = \frac{1}{2(p-1)} - \frac{1}{4}$ and $\mu = \frac{1}{4}$. Notice that the only difference with the expression in [9, Lemma 3.1] is the value of the coefficient k. This equation admits two linearly independent solutions

$$Z_1(r) = e^{-\frac{r}{2}} r^{\frac{1}{2}+\mu} U\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right),$$

and

$$Z_2(r) = e^{-\frac{r}{2}} r^{\frac{1}{2}+\mu} M\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right).$$

The functions U and M are the Whittaker functions which play an important role not only in analysis but also in group theory. They have the following asymptotic expansion as $r \to \infty$ (see e.g. [1]),

$$U\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right) = r^{k-\mu-\frac{1}{2}}\left(1 + O(r^{-1})\right) = r^{\frac{1}{2(p-1)}-1}\left(1 + O(r^{-1})\right),$$

and

$$\begin{aligned} M\left(\tfrac{1}{2} + \mu - k, 1 + 2\mu, r\right) &= \frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu-k)} e^r r^{-(\mu+\frac{1}{2}+k)}\left(1 + O(r^{-1})\right) \\ &= \frac{\Gamma(\frac{3}{2})}{\Gamma(1-\frac{1}{2(p-1)})} e^r r^{-\frac{p}{2(p-1)}}\left(1 + O(r^{-1})\right). \end{aligned}$$

Then

$$Z_1(r) = r^{\frac{1}{2(p-1)}-\frac{1}{4}} e^{-\frac{r}{2}}\left(1 + O(r^{-1})\right),$$

and

$$Z_2(r) = \frac{\Gamma(\frac{3}{2})}{\Gamma(1-\frac{1}{2(p-1)})} r^{\frac{1}{4}-\frac{1}{2(p-1)}-} e^{\frac{r}{2}}\left(1 + O(r^{-1})\right).$$

To this correspond the two linearly independent solutions ω_1 and ω_2 of (13) with the following behaviour as $\eta \to \infty$,

$$\begin{aligned} &(i) \qquad \omega_1(\eta) = c_1 \eta^{\frac{1}{p-1}-1} e^{-\frac{\eta^2}{4}}\left(1 + O(\eta^{-2})\right), \\ &(ii) \qquad \omega_2(\eta) = c_2 \eta^{-\frac{1}{p-1}}\left(1 + O(\eta^{-2})\right). \end{aligned}$$

Clearly only ω_1 satisfies the decay estimate $\omega(\eta) = o(\eta^{-\frac{1}{p-1}})$ as $\eta \to \infty$. Hence the solution ω is a multiple of ω_1 and the multiplicative constant c is adjusted in order to fit the condition $\omega'(0) = \omega^p(0)$.

3. Problem with measure data

3.1. The regular problem

Set $G(r) = \int_0^r g(s)ds$. We consider the functional J in $L^2(\mathbb{R}_+)$ with domain $D(J) = H^1(\mathbb{R}_+)$ defined by

$$J(u) = \frac{1}{2}\int_0^\infty u_x^2 dx + G(v(0)).$$

It is convex and lower semicontinuous in $L^2(\mathbb{R}_+)$ and its subdifferential ∂J sastisfies

$$\int_0^\infty \partial J(u)\zeta dx = \int_0^\infty u_x \zeta_x dx + g(u(0))\zeta(0),$$

for all $\zeta \in H^1(\mathbb{R}_+)$. Therefore

$$\int_0^\infty \partial J(u)\zeta dx = -\int_0^\infty u_{xx}\zeta dx + (g(u(0)) - u_x(0))\zeta(0).$$

Hence

$$\partial J(u) = -u_{xx} \text{ for all } u \in D(\partial J) = \{v \in H^1(\mathbb{R}_+) : v_x(0) = g(v(0))\}.$$

The operator ∂J is maximal monotone, hence it generates a semi-group of contractions. Furthermore, for any $u_0 \in L^2(\mathbb{R}_+)$ and $F \in L^2(0,T; L^2(L^2(\mathbb{R}_+))$ there exists a unique strong solution to

$$\begin{aligned} U_t + \partial J(U) &= F \quad \text{a.e. on } (0,T) \\ U(0) &= u_0. \end{aligned}$$

Proposition 3.1. *Let $\mu \in H^1(0,T)$ and $\nu \in L^2(\mathbb{R}_+)$. Then there exists a unique function $u \in C([0,T]; L^2(\mathbb{R}_+)$ such that $\sqrt{t}u_{xx} \in L^2((0,T) \times \mathbb{R}_+)$ which satisfies* (14). *The mapping $(\mu,\nu) \mapsto u := u_{\mu,\nu}$ is non-decreasing and u is a weak solution in the sense that it satisfies* (8).

Proof. Let $\eta \in C_0^2([0,\infty))$ such that $\eta(0) = 0$, $\eta'(0) = 1$. If $f \in H^1(0,T)$, $\nu \in L^2(\mathbb{R}_+)$, and u is a solution of

$$\begin{aligned} u_t - u_{xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) &= \mu(t) \quad \text{in } [0,T) \\ u(0,\,\cdot\,) &= \nu \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{14}$$

where $\nu \in L^2(\mathbb{R}_+)$, then the function $v(t,x) = u(t,x) - \mu(t)\eta(x)$ satisfies

$$\begin{aligned} v_t - v_{xx} &= F \quad \text{in } Q^T_{\mathbb{R}_+} \\ -v_x(\,\cdot\,,0) + g(v(\,\cdot\,,0)) &= 0 \quad \text{in } [0,T) \\ v(0,\,\cdot\,) &= \nu - \mu(0)\eta \quad \text{in } \mathbb{R}_+, \end{aligned}$$

with $F(t,x) = -(\mu'(t)\eta(x) + \mu(t)\eta''(x))$. The proof of the existence follows by using [3, Theorem 3.6].
Next, let $(\tilde{\mu},\tilde{\nu}) \in H^1(0,T) \times L^2(\mathbb{R}_+)$ such that $\tilde{\mu} \le \mu$ and $\tilde{\nu} \le \nu$ and let $\tilde{u} = u_{\tilde{\mu},\tilde{\nu}}$, then

$$\frac{1}{2}\frac{d}{dt}\int_0^\infty (\tilde{u}-u)_+^2 dx + \int_0^\infty \left(\partial_x(\tilde{u}-u)_+\right)^2 dx - (\tilde{\mu}(t)-\mu(t))\,(\tilde{u}(t,0)-u(t,0))_+ \\ + (g(\tilde{u}(t,0)) - g(u(t,0))))(\tilde{u}(t,0)-u(t,0)) = 0.$$

Then

$$\int_0^\infty (\tilde{u}-u)_+^2 dx\lfloor_{t=0} \Longrightarrow \int_0^\infty (\tilde{u}-u)_+^2 dx = 0 \quad \text{on } [0,T].$$

We can also use (10) to express the solution of (14):

$$u(t,x) = \int_0^\infty \tilde{E}(t,x,y)\nu(y)dy + 2\int_0^t E(t-s,x)(\mu(s) - g(u(s,0)))ds.$$

In particular, if $g(0) = 0$, then

$$|u(t,x)| \le \int_0^\infty \tilde{E}(t,x,y)|\nu(y)|dy + 2\int_0^t E(t-s,x)|\mu(s)|ds.$$

The proof of (8) follows since u is a strong solution. □

Next, we prove that the problem is well-posed if $\mu \in L^1(0,T)$.

PROPOSITION 3.2. *Assume* $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ *and* $\{\mu_n\} \subset C^1([0,T])$ *are Cauchy sequences in* $L^1(\mathbb{R}_+)$ *and* $L^1(0,T)$ *respectively. Then the sequence* $\{u_n\}$ *of solutions of*

$$\begin{aligned} u_{n\,t} - u_{n\,xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -u_{n\,x}(\,\cdot\,,0) + g(u_n(\,\cdot\,,0)) &= \mu_n(t) \quad \text{in } [0,T) \\ u_n(0,\,\cdot\,) &= \nu_n \quad \text{in } \mathbb{R}_+, \end{aligned} \tag{15}$$

converges in $C([0,T];L^1(\mathbb{R}_+)$ *to a function* u *which satisfies* (8).

Proof. For $\epsilon > 0$ let p_ϵ be an odd C^1 function defined on $\mathbb{R}$ such that $p'_\epsilon \ge 0$ and $p_\epsilon(r) = 1$ on $[\epsilon,\infty)$, and put $j_\epsilon(r) = \int_0^r p_\epsilon(s)ds$. Then

$$\frac{d}{dt}\int_0^\infty j_\epsilon(u_n - u_m)dx + \int_0^\infty (u_{n\,x} - u_{m\,x})^2 p'_\epsilon(u_n - u_m)dx \\ + (g(u_n(t,0)) - g(u_m(t,0)))\,p_\epsilon(u_n(t,0) - u_m(t,0)) \\ = (\mu_n(t) - \mu_m(t))\,p_\epsilon(u_n(t,0) - u_m(t,0)).$$

Hence

$$\int_0^\infty j_\epsilon(u_n-u_m)(t,x)dx+(g(u_n(t,0))-g(u_m(t,0)))\,p_\epsilon(u_n(t,0)-u_m(t,0))$$
$$\leq\int_0^\infty j_\epsilon(\nu_n-\nu_m)dx+(\mu_n(t)-\mu_m(t))\,p_\epsilon(u_n(t,0)-u_m(t,0)).$$

Letting $\epsilon\to 0$ implies $p_\epsilon\to sgn_0$, hence for any $t\in[0,T]$,

$$\int_0^\infty |u_n-u_m|(t,x)dx+|g(u_n(t,0))-g(u_m(t,0)|$$
$$\leq\int_0^\infty|\nu_n-\nu_m|dx+|\mu_n(t)-\mu_m(t)|.$$

Therefore $\{u_n\}$ and $\{g(u_n(\,\cdot\,,0)\}$ are Cauchy sequences in $C([0,T];L^1(\mathbb{R}_+))$ and $C([0,T])$ respectively with limit u and $g(u)$ and $u=u_{\nu,\mu}$ satisfies (8). If we assume that $(\nu,\tilde\nu)$ and $(\mu,\tilde\mu)$ are couples of elements of $L^1(\mathbb{R}_+)$ and $L^1(0,T)$ respectively and if $u=u_{\nu,\mu}$ and $\tilde u=u_{\tilde\nu,\tilde\mu}$, there holds by the above technique,

$$\int_0^\infty |u-\tilde u|(t,x)dx+|g(u(t,0))-g(\tilde u(t,0)|$$
$$\leq\int_0^\infty|\tilde\nu-\tilde\nu|dx+|\tilde\mu(t)-\tilde\mu(t)|\quad\text{for all }\ t\in[0,T].\tag{16}$$

□

The following lemma is a parabolic version of an inequality due to Brezis.

Lemma 3.3. *Let $\nu\in L^1(\mathbb{R}_+)$ and $\mu\in L^1(0,T)$ and v be a function defined in $[0,T)\times\mathbb{R}_+$, belonging to $L^1(Q^T_{\mathbb{R}_+})\cap L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and satisfying*

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})v\,dxdt=\int_0^T\zeta(\,\cdot\,,0)\mu dt+\int_0^\infty\nu\zeta dx.\tag{17}$$

Then for any $\zeta\in\mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta\geq 0$, there holds

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})|v|dxdt\leq\int_0^\infty\zeta(\,\cdot\,,0)sign(v)\mu dt+\int_0^\infty|\nu|\zeta dx.\tag{18}$$

Similarly

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})v_+dxdt\leq\int_0^\infty\zeta(\,\cdot\,,0)sign_+(v)\mu dt+\int_0^\infty\nu_+\zeta dx.\tag{19}$$

Proof. Let p_ϵ be the approximation of $sign_0$ used in Proposition 3.2 and η_ϵ be the solution of

$$\begin{aligned} -\eta_{\epsilon\, t} - \eta_{\epsilon\, xx} = p_\epsilon(v) \quad & \text{in } Q^T_{\mathbb{R}_+} \\ \eta_{\epsilon\, x}(\,\cdot\,, 0) = 0 \quad & \text{in } [0,T] \\ \eta_\epsilon(0,\,\cdot\,) = 0 \quad & \text{in } \mathbb{R}_+. \end{aligned}$$

Then $|\eta_\epsilon| \leq \eta^*$ where η^* satisfies

$$\begin{aligned} -\eta^*_t - \eta^*_{xx} = 1 \quad & \text{in } Q^T_{\mathbb{R}_+} \\ \eta^*_x(\,\cdot\,, 0) = 0 \quad & \text{in } [0,T] \\ \eta^*(0,\,\cdot\,) = 0 \quad & \text{in } \mathbb{R}_+. \end{aligned}$$

Although η_ϵ does not belong to $\mathbb{X}(Q^T_{\mathbb{R}_+})$ (it is not in $C^{1,2}([0,T) \times \mathbb{R}_+)$), it is an admissible test function and we deduce that there exists a unique solution to (17). Thus v is given by expression (10).

In order to prove (18), we can assume that μ and ν are smooth, $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$, $\zeta \geq 0$ and set $h_\epsilon = p_\epsilon(v)\zeta$ and $w_\epsilon = vp_\epsilon(v)$, then

$$\begin{aligned} \int_0^\infty h_{\epsilon\, xx} v dx &= \int_0^\infty \left(2p'_\epsilon(v)v_x\zeta_x + p_\epsilon(v)\zeta_{xx} + \zeta(p_\epsilon(v))_{xx}\right) v dx \\ &= \int_0^\infty \left(2vp'_\epsilon(v)v_x\zeta_x - w_{\epsilon\, x}\zeta_x - (v\zeta)_x(p_\epsilon(v))_x\right) dx \\ &\qquad - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0) \\ &= -\int_0^\infty \left(\zeta_x(j_\epsilon(v))_x + \zeta p'(v)_\epsilon v_x^2\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0) \\ &= -\int_0^\infty \left(\zeta p'(v)_\epsilon v_x^2 - j_\epsilon(v)\zeta_{xx}\right) dx - \zeta(t,0)v(t,0)p'_\epsilon(v(t,0))v_x(t,0), \end{aligned}$$

and

$$\int_0^T h_{\epsilon\, t} v dt = \int_0^T (p_\epsilon(v)\zeta_t + p'_\epsilon(v)\zeta v_t) v dt.$$

Since v is smooth

$$\begin{aligned} 0 &= \int_0^T \int_0^\infty (v_t - v_{xx}) h_\epsilon dx dt \\ &= -\int_0^T \int_0^\infty (h_{\epsilon\, t} + h_{\epsilon\, xx}) v dx dt - \int_0^\infty h_\epsilon(0,x)\nu(x) dx \\ &\qquad - \int_0^T \left[p_\epsilon(v(t,0)) - v(t,0)p'_\epsilon(v(t,0))\right] \zeta(t,0)\mu(t) dt. \end{aligned}$$

Therefore, using (18) and (19),

$$-\int_0^T\int_0^\infty (j_\epsilon v)\zeta_{xx} + vp_\epsilon(v)\zeta_t)\,dxdt$$
$$+\int_0^T\int_0^\infty \left(\zeta p'_\epsilon(v)v_x^2 - vp'_\epsilon(v)v_t\zeta\right)dxdt$$
$$= \int_0^\infty h_\epsilon(0,x)\nu(x)dx + \int_0^T h_\epsilon(t,0)\mu(t)dt. \quad (20)$$

Put $\ell_\epsilon(s) = \int_0^s rp'_\epsilon(r)dr$, then $|\ell_\epsilon(s) \le c\epsilon^{-1}s^2\chi_{[-\epsilon,\epsilon]}(s)|$. Since

$$\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_t dxdt = -\int_0^\infty \ell_\epsilon(v(0,x))\zeta(x)dx - \int_0^T\int_0^\infty \zeta_t\ell_\epsilon(v)dxdt,$$

and ζ has compact support, it follows that

$$\lim_{\epsilon\to 0}\int_0^T\int_0^\infty \zeta vp'_\epsilon(v)v_t dxdt = 0.$$

Letting $\epsilon \to 0$ in (20), we derive (18) for smooth v. Using Proposition 3.2 completes the proof of (18). The proof of (19) is similar. □

Remark 3.4: Inequalities (18) and (19) hold if $\zeta(t,x)$ does not vanish if $|x| \ge R$ for some R but if it satisfies

$$\lim_{x\to\infty}\sup_{t\in[0,T]}(\zeta(t,x) + |\zeta_x(t,x)|) = 0. \quad (21)$$

The proof follows by replacing $\zeta(t,x)$ by $\zeta(t,x)\eta_n(x)$ where $\eta_n \in C_c^\infty(\mathbb{R}_+)$ with $0 \le \eta_n \le 1$, $\eta_n(x) = 1$ on $[0,n]$, $\eta_n(x) = 0$ on $[n+1,\infty)$, $|\eta'_n| \le 2$, $|\eta''_n| \le 4$. Then $\eta_n\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$ by letting $n \to \infty$ and the proof follows by letting $n \to \infty$.

3.2. Proof of Theorem 1.3

We give first some *heat-ball* estimates relative to our problem. For $r > 0$, $x \in \mathbb{R}_+$ and $t \in \mathbb{R}$ we set

$$e(t,x;r) = \left\{(s,y) \in (0,T)\times\mathbb{R}_+ : s \le t,\ \tilde{E}(t-s,x,y) \ge r\right\}.$$

Since

$$e(t,x;r) \subset [t - \tfrac{1}{4\pi er^2}, t] \times [x - \tfrac{1}{r\sqrt{\pi e}}, x + \tfrac{1}{r\sqrt{\pi e}}],$$

there holds

$$|e(t,x;r)| \le \frac{1}{2r^3(\pi e)^{\frac{3}{2}}},$$

and if
$$e^*(t;r) = \{s \in (0,T) : s \le t,\, E(t-s,0,0) \ge r\}\,,$$
then we have
$$e^*(t;r) \subset [t - \tfrac{1}{4\pi e r^2}, t] \Longrightarrow |e^*(t;r)| \le \frac{1}{4r^2\pi e}. \tag{22}$$
If G is a measured space, λ a positive measure on G and $q > 1$, $M^q(G,\lambda)$ is the Marcinkiewicz space of measurable functions $f : G \mapsto \mathbb{R}$ satisfying for some constant $c > 0$ and all measurable set $E \subset G$,
$$\int_E |f| d\lambda \le c\,(\lambda(E))^{\frac{1}{p'}}\,,$$
and
$$\|f\|_{M^q(G,\lambda)} = \inf\{c > 0 \text{ s.t. (22) holds}\}.$$

Lemma 3.5. *Assume μ,ν are bounded measure in $\overline{\mathbb{R}_+}$ and $\mathbb{R}_+$ respectively and u is the solution of* (9) *given by* (10) *and $v_{\nu,\mu}$ is the solution of* (9)*. Then*
$$\|v_{\nu,\mu}\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\|v_{\nu,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \le c\left(\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})} + \|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}\right).$$

Proof. First we consider $v_{0,\mu}$
$$v_{0,\mu}(t,x) = 2\int_0^t E(t-s,x)d\mu(s).$$
If $F \subset [0,T]$ is a Borel set, than for any $\tau > 0$
$$\begin{aligned}\int_F E(t-s,0)ds &= \int_{F\cap\{E\le\tau\}} E(t-s,0)ds + \int_{F\cap\{E>\tau\}} E(t-s,0)ds\\ &\le \tau|F| + \int_{\{E>\tau\}} E(t-s,0)ds\\ &\le \tau|F| - \int_\tau^\infty \lambda d|e^*(t,\lambda)|\\ &\le \tau|F| + \int_\tau^\infty \lambda d|e^*(t,\lambda)|\\ &\le \tau|F| + \frac{1}{4\pi e\tau}.\end{aligned}$$
If we choose $\tau^2 = \frac{1}{4\pi e|F|}$, we derive
$$\int_F E(t-s,0)ds \le \frac{|F|^{\frac{1}{2}}}{\sqrt{\pi e}}.$$

If $F \subset (0,T)$ is a Borel set then

$$\left|\int_F v_{0,\mu}(t,0)dt\right| = 2\left|\int_0^t\int_F E(t-s,0)dtd\mu(s)\right| \leq \frac{2|F|^{\frac{1}{2}}}{\sqrt{\pi e}}\left\|\mu\right\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

This proves that

$$\left\|v_{0,\mu}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\left\|\mu\right\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

Similarly, if $G \subset [0,T]\times[0,\infty)$ is a Borel set, then

$$\int_G \tilde{E}(t-s,x,0)ds \leq \frac{2|G|^{\frac{1}{3}}}{\sqrt{\pi e}},$$

and

$$\left\|v_{0,\mu}\right\|_{M^3(Q^T_{\mathbb{R}_+})} \leq c\left\|\mu\right\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})}.$$

In the same way we prove that

$$\left\|v_{\nu,0}\right\|_{M^3(Q^T_{\mathbb{R}_+})} + \left\|v_{\nu,0}\lfloor_{\partial Q^T_{\mathbb{R}_+}}\right\|_{M^2(\partial Q^T_{\mathbb{R}_+})} \leq c\left\|\nu\right\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}.$$

This ends the proof of the lemma. □

Proof of Theorem 1.3. Uniqueness. Assume u and $\tilde{u}$ are solutions of (1), then $w = u - \tilde{u}$ satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 \quad \text{in } Q^T_{\mathbb{R}_+} \\ -w_x(\,\cdot\,,0) + g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)) &= 0 \quad \text{in } [0,T) \\ w(0,\,\cdot\,) &= 0 \quad \text{in } \mathbb{R}_+. \end{aligned}$$

Applying (18), we obtain

$$-\int_0^T\int_0^\infty(\zeta_t+\zeta_{xx})|w|dxdt + \int_0^\infty(g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)))sign(w)\zeta(t,0)dt \leq 0,$$

for any $\zeta \in \mathbb{X}^T_{\mathbb{R}_+}$ with $\zeta \geq 0$. Let $\theta \in C^1_c(Q^T_{\mathbb{R}_+})$, $\eta \geq 0$, we take ζ to be the solution of

$$\begin{aligned} -\zeta_t - \zeta_{xx} &= \theta \quad \text{in } (0,T)\times\mathbb{R}_+ \\ \zeta_x(\,\cdot\,,0) &= 0 \quad \text{in } (0,T) \\ \zeta(T,\,\cdot\,) &= 0 \quad \text{in } (0,\infty). \end{aligned}$$

Then ζ satisfies (21), hence

$$\int_0^T\int_0^\infty \theta|w|dxdt + \int_0^\infty(g(u(\,\cdot\,,0)) - g(\tilde{u}(\,\cdot\,,0)))sign(w)\zeta(t,0)dt \leq 0.$$

This implies $w = 0$.

Existence. Without loss of generality we can assume that μ and ν are nonnegative. Let $\{\nu_n\} \subset C_c(\mathbb{R}_+)$ and $\{\mu_n\} \subset C_c([\mathbb{R}_+]0,T))$ converging to ν and μ in the sense of measures and let u_n be the solution of (15). Then from (16),

$$\int_0^T \int_0^\infty |u_n| dx dt + \int_0^T |g(u_n(t,0))| dt \le T \int_0^\infty |\nu_n| dx + \int_0^T |\mu_n| dt.$$

Therefore u_n and $g(u_n(\,\cdot\,,0))$ remain bounded respectively in $L^1(Q^T_{\mathbb{R}_+})$ and in $L^1(0,T)$. Furthermore, by Lemma 3.5, u_n remains bounded in $M^3(Q^T_{\mathbb{R}_+})$ and in $M^2(\partial Q^T_{\mathbb{R}_+})$. We can also write u_n under the form

$$\begin{aligned} u_n(t,x) &= \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy + 2\int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds \\ &= A_n(t,x) + B_n(t,x). \end{aligned} \tag{23}$$

Since we can perform the even reflexion through $y = 0$, the mapping

$$(t,x) \mapsto A_n(t,x) := \int_0^\infty \tilde{E}(t,x,y)\mu_n(y)dy,$$

is relatively compact in $C^m_{loc}(\overline{Q^T_{\mathbb{R}_+}})$ for any $m \in \mathbb{N}^*$. Hence we can extract a subsequence $\{u_{n_k}\}$ which converges uniformly on every compact subset of $(0,T] \times [0,\infty)$, hence a.e. on $(0,T]$ for the 1-dimensional Lebesque measure. Concerning the boundary term

$$(t,x) \mapsto B_n(t,x) := \int_0^t E(t-s,x)(\nu_n(t) - g(u_n(t,0)))ds,$$

it is relatively compact on every compact subset of $[0,T] \times (0,\infty)$. If $x = 0$, then

$$B_n(t,0) = \int_0^t (\nu_n(t) - g(u_n(t,0)))\frac{ds}{\sqrt{\pi(t-s)}}.$$

Since $\|\nu_n(\,\cdot\,) - g(u_n(\,\cdot\,,0))\|_{L^1(0,T)}$, $t \mapsto B_n(t,0)$ is uniformly integrable on $(0,T)$, hence relatively compact by the Frechet-Kolmogorov Theorem. Therefore there exists a subsequence, still denoted by $\{n_k\}$ such that $B_{n_k}(t,0)$ converges for almost all $t \in (0,T)$. This implies that the sequence of function $\{u_{n_k}\}$ defined by (23) converges in $\overline{Q^T_{\mathbb{R}_+}}$ up to a set $\Theta \cup \Lambda$ where $\Theta \subset Q^T_{\mathbb{R}_+}$ is neglectable for the 2-dimensional Lebesgue measure and $\Lambda \subset \partial_\ell Q^T_{\mathbb{R}_+}$ neglectable for the 1-dimensional Lebesgue measure.

From Lemma 3.5, $(u_{n,k}\lfloor_{Q^T_{\mathbb{R}_+}}, u\lfloor_{\partial_\ell Q^T_{\mathbb{R}_+}})$ converges in $L^1_{loc}(Q^T_{\mathbb{R}_+}) \times L^1(\partial_\ell Q^T_{\mathbb{R}_+})$ and the convergence of each of the components holds also almost everywhere (up to a subsequence). Since $u_{n,k}$ is a weak solution, it satisfies for any $\zeta \in \mathbb{X}(Q^T_{\mathbb{R}_+})$

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})u_{n,k}dxdt + \int_0^T (g(u_{n,k})\zeta)(t,0)dt = \int_0^\infty \zeta\nu_{n,k}(x)dx + \int_0^T \zeta(t,0)\mu_{n,k}(t)dt.$$

In order to prove the convergence of $g(u_{n,k}(t,0))$, we use Vitali's convergence theorem and the assumption (11). Let $F \subset [0,T]$ be a Borel set. Using the fact that $0 \leq u_{n,k} \leq v_{\nu_{n,k},\mu_{n,k}}$ and the estimate of Lemma 3.5, we have for any $\lambda > 0$,

$$\begin{aligned}\int_F |g(u_{n,k}(t,0))|dt &\leq \int_{F\cap\{u_{n,k}(t,0)\leq\lambda\}} |g(u_{n,k}(t,0))|dt \\ &\quad + \int_{\{u_{n,k}(t,0)>\lambda\}} |g(u_{n,k}(t,0))|dt \\ &\leq g(\lambda)|F| - \int_\lambda^\infty \sigma d|\{t : |g(u_{n,k}(t,0))| > \sigma\}| \\ &\leq g(\lambda)|F| + c\int_\lambda^\infty |g(\sigma)|\sigma^{-3}ds,\end{aligned}$$

where c depends of $\|\mu\|_{\mathfrak{M}(\partial Q^T_{\mathbb{R}_+})} + \|\nu\|_{\mathfrak{M}(Q^T_{\mathbb{R}_+})}$. For $\epsilon > 0$ given, we chose λ large enough so that the integral term above is smaller than ϵ and then $|F|$ such that $g(\lambda)|F| + \leq \epsilon$. Hence $\{g(u_{n,k}(\,\cdot\,,0))\}$ is uniformly integrable. Therefore up to a subsequence, it converges to $g(u(\,\cdot\,,0))$ in $L^1(0,T)$. Clearly u satisfies

$$-\int_0^T\int_0^\infty (\zeta_t + \zeta_{xx})udxdt + \int_0^T (g(u)\zeta)(t,0)dt = \int_0^\infty \zeta\nu(x)dx + \int_0^T \zeta(t,0)\mu(t)dt,$$

which ends the existence proof.

Monotonicity. If $\nu \geq \tilde\nu$ and $\mu \geq \tilde\mu$; we can choose the approximations such that $\nu_n \geq \tilde\nu_n$ and $\mu_n \geq \tilde\mu_n$. It follows from (19) that $u_{\nu_n,\mu_n} \geq u_{\tilde\nu_n,\tilde\mu_n}$. Choosing the same subsequence $\{n_k\}$, the limits u, $\tilde u$ are in the same order. The conclusion follows by uniqueness. □

3.3. The case $g(u) = |u|^{p-1}u$

Condition (11) is satisfied if $p < 2$. If this condition holds there exists a solution $u_{\ell\delta_0} = u_{0,\ell\delta_0}$ and the mapping $\ell \mapsto u_{\ell\delta_0}$ is increasing.

THEOREM 3.6. *(i) If* $1 < p \leq \frac{3}{2}$, $u_{\ell\delta_0}$ *tends to* ∞ *when* $k \to \infty$.
(ii) If $\frac{3}{2} < p < 2$, $u_{\ell\delta_0}$ *converges to* U_{ω_s} *defined by*

$$U_{\omega_s}(t,x) = t^{-\frac{1}{2(p-1)}}\omega_s(\tfrac{x}{\sqrt{t}}),$$

when $k \to \infty$.

Proof. By uniqueness and using (3), there holds

$$T_k[u_{\ell\delta_0}] = u_{k^{\frac{2-p}{p-1}}\ell\delta_0},$$

for any $k, \ell > 0$. Since $\ell \mapsto u_{\ell\delta_0}$ is increasing, its limit u_∞, when $\ell \to \infty$, satisfies

$$T_k[u_\infty] = u_\infty.$$

Hence u_∞ is a positive self-similar solution of (2), provided it exists. Hence $u_\infty = U_{\omega_s}$ if $\frac{3}{2} < p < 2$. If $1 < p \leq \frac{3}{2}$, $u_{k\delta_0}$ admits no finite limit when $k \to \infty$ which ends the proof. □

REMARK 3.7: As a consequence of this result, no a priori estimate of Brezis-Friedman type (parabolic Keller-Osserman) exists for a nonnegative function $u \in C^{2,1}(\overline{Q^\infty_{\mathbb{R}_+}} \setminus \{(0,0)\})$ solution of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q^\infty_{\mathbb{R}_+} \\ -u_x(\,\cdot\,,0) + |u|^{p-1}u(\,\cdot\,,0) &= 0 && \text{for all } t > 0 \\ u(0,\,\cdot\,) &= 0 && \text{for all } x > 0. \end{aligned}$$

when $1 < p \leq \frac{3}{2}$. When $\frac{3}{2} < p < 2$ it is expected that

$$u(t,x) \leq \frac{c}{(|x|^2 + t)^{\frac{1}{2(p-1)}}}.$$

The type of phenomenon (i) in Theorem 3.6 is characteristic of fractional diffusion. It has already been observed in [6, Theorem 1.3] with equations

$$\begin{aligned} u_t + (-\Delta)^\alpha u + t^\beta u^p &= 0 && \text{in } \mathbb{R}_+ \times \mathbb{R}^N \\ u(0,\,\cdot\,) &= k\delta_0 && \text{in } \mathbb{R}^N, \end{aligned}$$

when $0 < \alpha < 1$ is small and $p > 1$ is close to 1.

4. Extension and open problems

The natural extension is to replace a one dimensional domain by a mutidimenional one. The main open problem is the question of a priori estimate as stated in the last remark above.

4.1. Self-similar solutions

Let $\eta = (\eta_1, \ldots, \eta_n)$ be the coordinates in $\mathbb{R}^n$ and denote

$$\mathbb{R}^n_+ = \{\eta = (\eta_1, \ldots, \eta_n) = (\eta', \eta_n) : \eta_n > 0\}.$$

We set $K(\eta) = e^{\frac{|\eta|^2}{4}}$ and $K'(\eta') = e^{\frac{|\eta'|^2}{4}}$. Similarly to Section 2 we define $\mathcal{L}_K$ in $C^2_0(\mathbb{R}^n)$ by

$$\mathcal{L}_K(\phi) = -K^{-1} div(K\nabla\phi). \tag{24}$$

If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$. We denote by ϕ_1 the function K^{-1}. Then the set of eigenvalues of $\mathcal{L}_K$ is the set of numbers $\left\{\lambda_k = \frac{n+k}{2} : k \in \mathbb{N}\right\}$ with corresponding set of eigenspaces

$$N_k = span\,\{D^\alpha\phi_1 : |\alpha| = k\}.$$

The operators $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are defined accordingly in $H^1_K(\mathbb{R}^n_+)$ and $H^{1,0}_K(\mathbb{R}^n_+)$ respectively, and

$$\sigma(\mathcal{L}^{+,N}_K) = \left\{\frac{n+k}{2} : k \in \mathbb{N}\right\} \quad \text{and} \quad \sigma(\mathcal{L}^{+,D}_K) = \left\{\frac{n+k}{2} : k \in \mathbb{N}^*\right\}.$$

Furthermore

$$N_{k,N} = ker\left(\mathcal{L}^{+,N}_K - \tfrac{n+k}{2} I_d\right) = span\,\{D^\alpha\phi_1 : |\alpha| = k, \alpha_n = 2\ell,\ \ell \in \mathbb{N}\},$$

and

$$N_{k,D} = ker\left(\mathcal{L}^{+,D}_K - \tfrac{n+k}{2} I_d\right) = span\,\{D^\alpha\phi_1 : |\alpha| = k, \alpha_n = 2\ell+1,\ \ell \in \mathbb{N}\}.$$

Since $\mathcal{L}^{+,N}_K$ and $\mathcal{L}^{+,D}_K$ are Fredholm operators,

$$H^1_K(\mathbb{R}^n_+) = \bigoplus_{k=0}^{\infty} N_{k,N} \quad \text{and} \quad H^{1,0}_K(\mathbb{R}^n_+) = \bigoplus_{k=1}^{\infty} N_{k,D}.$$

We define the following functional on $H^1_K(\mathbb{R}^n_+)$

$$J(\phi) = \frac{1}{2}\int_{\mathbb{R}^n_+} \left(|\nabla\phi|^2 - \frac{1}{2(p-1)}\phi^2\right) K d\eta + \frac{1}{p+1}\int_{\partial\mathbb{R}^n_+} |\phi|^{p+1} K' d\eta'.$$

The critical points of J satisfies

$$\begin{aligned} -\Delta\omega - \frac{1}{2}\eta.\nabla\omega - \frac{1}{2(p-1)}\omega = 0 \quad &\text{in } \mathbb{R}^n_+ \\ -\omega_{\eta_n} + |\omega|^{p-1}\omega = 0 \quad &\text{in } \partial\mathbb{R}^n_+. \end{aligned} \tag{25}$$

If ω is a solution of (25), the function

$$u_\omega(t,x) = t^{-\frac{1}{2(p-1)}}\,\omega\left(\frac{x}{\sqrt{t}}\right)$$

satisfies

$$\begin{aligned} u_{\omega\,t} - \Delta u_\omega &= 0 && \text{in } Q^{\infty}_{\mathbb{R}^n_+} := (0,\infty)\times\mathbb{R}^n_+ \\ -u_{\omega\,x_n} + |u_\omega|^{p-1}u_\omega &= 0 && \text{in } \partial_\ell Q^{\infty}_{\mathbb{R}^n_+} := (0,\infty)\times\partial\mathbb{R}^n_+. \end{aligned}$$

Here we have set $\mathbb{R}^n_+ = \{x = (x_1,\dots,x_n) = (x', x_n) : x_n > 0\}$. We denote by $\mathcal{E}$ the subset $H^1_K(\mathbb{R}^n_+)\cap L^p(\partial\mathbb{R}^n_+; d\eta')$ of solutions of (25) and by $\mathcal{E}_+$ the subset of positive solutions. As for the case $n = 1$ we have the following non-existence result

Proposition 4.1. *1. If $p \geq 1 + \frac{1}{n}$, then $\mathcal{E} = \{0\}$.*

2. If $1 < p \leq 1 + \frac{1}{n+1}$, then $\mathcal{E}_+ = \{0\}$.

The proof is similar to the one of Theorem 1.1. Hence the existence is to be found in the range $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$.

Conjecture 4.2. *Assume $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$, then the functional J is bounded from below in $H^1_K(\mathbb{R}^n_+)\cap L^p_{K'}(\partial\mathbb{R}^n_+)$. Furthermore $J(\phi)$ tends to infinity when $\|\phi\|_{H^1_K(\mathbb{R}^n_+)} + \left\|\phi\lfloor_{\partial\mathbb{R}^n_+}\right\|_{L^{p+1}_{K'}(\partial\mathbb{R}^n_+)}$ tends to infinity.*

4.2. Problem with measure data

The method for proving Theorem 1.3 can be adapted to prove the following n-dimensional result

Theorem 4.3. *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a nondecreasing continuous function such that $g(0) = 0$ and*

$$\int_1^\infty (g(s) - g(-s))s^{-\frac{2n+1}{n}}\,ds < \infty,$$

then for any bounded Radon measures ν in $\mathbb{R}^n_+$ and μ in $(0,T)\times\partial\mathbb{R}^n_+$, there exists a unique Borel function $u := u_{\nu,\mu}$ defined in $\overline{Q^{\mathbb{R}^n_+}_T} := [0,T]\times\mathbb{R}^n_+$ such that $u \in L^1(Q^{\mathbb{R}^n_+}_T)$, $u\lfloor_{(0,T)\times\partial\mathbb{R}^n_+} \in L^1((0,T)\times\partial\mathbb{R}^n_+)$ and $g(u) \in L^1((0,T)\times\partial\mathbb{R}^n_+)$ solution of

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } Q^T_{\mathbb{R}^n_+} \\ -u_{x_n} + g(u) &= \mu && \text{in } \partial_\ell Q^T_{\mathbb{R}^n_+} \\ u(0,\cdot) &= \nu && \text{in } \mathbb{R}^n_+, \end{aligned}$$

in the sense that

$$\int\int_{Q^T_{\mathbb{R}^n_+}}(-\partial_t\zeta-\Delta\zeta)udxdt+\int\int_{\partial_\ell Q^T_{\mathbb{R}^n_+}}g(u)\zeta dx'dt = \int_{\mathbb{R}^n_+}\zeta d\nu+\int\int_{\partial_\ell Q^T_{\mathbb{R}^n_+}}\zeta d\mu,$$

for all $\zeta\in C^{1,2}_c(\overline{Q^T_{\mathbb{R}^n_+}})$ *such that* $\zeta_{x_n}=0$ *on* $(0,T)\times\partial\mathbb{R}^n_+$ *and* $\zeta(T,\cdot)=0$. *Furthermore* $(\nu,\mu)\mapsto u_{\nu,\mu})$ *is nondecreasing.*

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Existence of attractors when diffusion and reaction have polynomial growth

Shair Ahmad and Dung Le

Dedicated to our 60 years young friend, Julian Lopez Gomez, whose accomplishments have well exceeded his age.

Abstract. *We study an interesting model, with reaction terms of Lotka-Volterra type, where diffusion and reaction have polynomial growth of any order. We establish existence of global attractors as well as exponential attractors. In the sequel we study the long time dynamics of an appropriate semigroup and show that it possesses a global attractor (and exponential attractors) in a certain Banach space.*

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1. Introduction

Consider the following model introduced in [16]:

$$\begin{cases} u_t = \mathrm{div}[\nabla(a_1 u + \alpha_{11}u^2 + \alpha_{12}uv) + b_1 u\nabla\Phi(x)] + f_1(u,v), \\ v_t = \mathrm{div}[\nabla(a_2 v + \alpha_{21}uv + \alpha_{22}v^2) + b_2 v\nabla\Phi(x)] + f_2(u,v), \end{cases} \tag{1}$$

where $f_i(u,v)$ are reaction terms of Lotka-Volterra type and quadratic in u, v. The unknowns $u(x,t), v(x,t)$ denote the densities of two species at time t and location $x \in \Omega$, a bounded domain in $\mathbb{R}^2$. Dirichlet or Neumann boundary conditions were usually assumed for (1). This model was used to describe the population dynamics of the species u, v which move under the influence of population pressures and of the environmental potential $\Phi(x)$.

Under suitable assumptions on the coefficients in (1), Yagi [18] proved the global existence of strong solutions (their first derivatives are bounded and their second spatial derivatives exist) to the above system for a planar domain Ω (i.e. $n = 2$).

Let us consider the following system of m equations ($m \geq 2$)

$$u_t = \Delta(\mathcal{P}(u)) + \hat{f}(u, Du), \qquad (x,t) \in Q = \Omega \times (0,T), \tag{2}$$

where $u : \Omega \to \mathbb{R}^m$, $\mathcal{P} : \mathbb{R}^m \to \mathbb{R}^m$ and $f : \mathbb{R}^m \times \mathbb{R}^{nm} \to \mathbb{R}^m$ are vector valued functions. It is clear that (1) is a special case of the above when $m = 2$, the components of $\mathcal{P}(u), \hat{f}(u, Du)$ are *quadratic* in u and the terms with the potential $\Phi(x)$ are incorporated in $\hat{f}(u, Du)$, which has *linear* growth in Du.

For simplicity, we assume the homogeneous Dirichlet boundary condition for u (see Section 6 for other boundary conditions) and rewrite (2) as

$$\begin{cases} u_t = \operatorname{div}(A(u)Du) + \hat{f}(u, Du) & (x,t) \in Q = \Omega \times (0,T), \\ u(x,0) = U_0(x) & x \in \Omega\,, \\ u = 0 & \text{on } \partial\Omega \times (0,T)\,. \end{cases} \tag{3}$$

Here, Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^2$; $A(u) = \partial_u \mathcal{P}(u)$ is a full matrix $m \times m$ and $\hat{f} : \mathbb{R}^m \times \mathbb{R}^{2m} \to \mathbb{R}^m$. The initial data U_0 is given in $W^{1,p_0}(\Omega)$ for some $p_0 > 2$, the dimension of Ω. In this paper we will allow $\mathcal{P}, \hat{f}$ to have *polynomial growth of any order* in u. We then refer to the above system as the *generalized (1) system.*

We will show that (3) defines a global semiroup $\{\mathcal{S}(t)\}_{t\ge 0}$ in the Banach space $X = W^{1,p_0}(\Omega)$, namely

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x,t)$$

is defined for all $t > 0$ with u being the solution of (3). Moreover, this semigroup possesses a global attractor and exponential attractors – a result that, to the best of our knowledge, is established for the first time for (SKT) systems under such general setting.

Let us recall the definition of a global attractor. The notion and conditions for the existence of exponential attractors in Hilbert spaces were introduced in [4]: a set $\mathcal{A} \subset X$ is an exponential attractor if 1) $\mathcal{A}$ is a positively invariant set ($S(t)\mathcal{A} \subset \mathcal{A}$, $\forall t \ge 0$), 2) For any $U_0 \in X$, $S(t)U_0$ converges expontiallly to $\mathcal{A}$ as $t \to \infty$. We refer the readers to [12] for the notion and existence of exponential attractors of semigroups in Banach spaces.

First of all, we need to establish the global existence result for (3) in order to verify that the associated semigroup is global. In the last few decades, papers concerning strongly coupled parabolic systems like (3) usually relied on a result of Amann in [1, 2] which showed that a solution to (3) exists globally if its $W^{1,p_0}(\Omega)$ norm does not blow up in finite time. This requires the existence of a continuous function $\mathcal{C}$ on $(0,\infty)$ such that for $p_0 > n$, the dimension of Ω,

$$\|u(\cdot,t)\|_{W^{1,p_0}(\Omega)} \le \mathcal{C}(t)\,, \qquad \forall t \in (0, T_0). \tag{4}$$

The checking of (4) is very difficult and equivalently requires Hölder continuity of the solution u. The latter is a hard problem in the theory of PDEs as known techniques for the regularity of solutions to scalar equations could not be extended to systems and counterexamples were available. Maximum or comparison principles for systems are also unavailable so that the boundedness of solutions to (3) are generally unknown. The conditions for comparison principles in [14, 15] do not apply to the structure of (2) or even (1). In a recent work by the second author [11], he considered (3) on a domain in $\mathbb{R}^n$ $(n \geq 2)$ and was able to relax the condition (4) by

$$\|u(\cdot,t)\|_{W^{1,n}(\Omega)} \leq \mathcal{C}(t)\,, \qquad \forall t \in (0,T_0). \tag{5}$$

By checking this condition when $n = 2$ in [13], he established the global existence of classical solutions for the generalized (SKT) systems (3) on bounded planar domains. Obviously, (5) does not imply that $|u|$ is bounded so that (3) is not necessarily regularly elliptic, i.e. eigenvalues of $A(u)$ can be unbounded. In fact, in [11] we allowed $A(u)$ to have a polynomial growth of *any* order and assumed that the eigenvalues of $A(u)$ grow like $(\lambda_0 + |u|)^k$ for *any* positive reals λ_0, k.

In this paper, we study long time dynamics of the semigroup defined in (2), a special case of (3), and show that it possesses a global attractor (and exponential attractors) in the Banach space $X = W^{1,p_0}(\Omega)$, for some $p_0 > 2$, if λ_0 is sufficiently large.

To that aim we will make use of the following well known result (e.g., see [4]).

THEOREM 1.1. *Let X be a Banach space. The semigroup $\{\mathcal{S}(t)\}_{t\geq 0}$ on X possesses a global attractor $\mathcal{A}$ if*

i) there exists an absorbing ball $\mathcal{B}_0$ contained in X for $\{\mathcal{S}(t)\}_{t\geq 0}$. That is, $\mathcal{B}_0$ is bounded and for any bounded subset $\mathcal{B}$ of X there exists $T(\mathcal{B})$ such that $\mathcal{S}(t)(\mathcal{B}) \subset \mathcal{B}_0$ for $t \geq T(\mathcal{B})$,

ii) for some $t_1 > 0$, $\mathcal{S}(t_1) : X \to X$ is compact.

Our paper is organized as follows. In Section 2 we introduce some notations, assumptions, and the statement of our main result on the existence of global attractors. Section 3 establishes uniform estimates for various weighted norms of the solutions in $W^{1,2}(\Omega)$. Since we have to work in the space $X = W^{1,p_0}(\Omega)$ with $p_0 > 2$, we establish similar uniform estimates for the norm in X; thus giving the existence of an absorbing ball required by i) of Theorem 1.1. The proof is fairly technical and will be presented in Section 4. Finally, we establish the required compactness in ii) and prove our main result in Section 5. We also show that the global existence result for (3) can be obtained by modifying some of our argument.

2. Preliminaries and Main Results

Throughout this paper, Ω is a bounded domain with smooth boundary in $\mathbb{R}^2$. For any smooth (vector valued) function u defined on $\Omega \times (0,T)$, $T > 0$, its temporal and spatial derivatives are denoted by u_t, Du respectively. If A is a C^1 function in u then we also abbreviate $\frac{\partial A}{\partial u}$ by A_u. In the sequel, we will write $a \sim b$ if there are two generic positive constants C_1, C_2 such that $C_1 b \le a \le C_2 b$.

As usual, $W^{1,p}(\Omega, \mathbb{R}^m)$, $p \ge 1$, will denote the standard Sobolev spaces whose elements are vector valued functions $u : \Omega \to \mathbb{R}^m$ with finite norm

$$\|u\|_{W^{1,p}(\Omega,\mathbb{R}^m)} = \|u\|_{L^p(\Omega)} + \|Du\|_{L^p(\Omega)}.$$

We assume the following structural conditions.

(A) $A(u)$ is C^1 in u. Moreover, there are constants $\lambda_0, C, k > 0$ and a scalar C^1 function $\lambda(u)$ such that $\lambda(u) \sim (\lambda_0 + |u|)^k$ for all $u \in \mathbb{R}^m$. Furthermore, for any $\zeta \in \mathbb{R}^{nm}$

$$\lambda(u)|\zeta|^2 \le \langle A(u)\zeta, \zeta\rangle \text{ and } |A(u)| \le C\lambda(u). \tag{6}$$

We also assume $|A_u| \le C|\lambda_u|$ and

$$|\lambda_u(u)| \sim C(\lambda_0 + |u|)^{k-1}. \tag{7}$$

For the sake of simplicity, we will consider first the case where the reaction term $\hat{f}$ in (3) does not depend on Du. That is $\hat{f}(u, Du) = f(u)$, satisfying the following growth condition.

(F) There are positive constants ε_0, C and nonnegative C^1 functions P, F : $\mathbb{R}^m \to \mathbb{R}^+$ satisfying $F(0) = P(0) = 0$ and for all $u \in \mathbb{R}^m$

$$|F_u(u)| \le C\lambda^{\frac{1}{2}}(u), \tag{8}$$

$$|P_u(u)| \le C\lambda(u) \tag{9}$$

such that

$$|f(u)||u| \le \varepsilon_0 F^2(u) + C, \tag{10}$$

$$\frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1} \le C(F(u)+1). \tag{11}$$

More generally, we can replace $f(u)$ by a function $\hat{f}$ depending on u, Du and satisfying a linear growth in Du. Namely, we will assume the following.

(F') There exist a constant C and a function $f(u)$ satisfying (F) such that

$$|\hat{f}(u, Du)| \le C\lambda^{\frac{1}{2}}(u)|Du| + f(u), \tag{12}$$

$$|f_u(u)| \le C\lambda(u). \tag{13}$$

Our main result is the following.

THEOREM 2.1. *Assume that (A) and (F') hold. Then there exists* $p_0 > 2$ *such that (3) defines a semigroup* $\{\mathcal{S}(t)\}_{t\geq 0}$ *on* $X = W^{1,p_0}(\Omega)$*, namely*

$$\mathcal{S}(0)U_0 = U_0, \quad \mathcal{S}(t)U_0(x) = u(x,t)$$

and (u(x, t), the solution of (3), is defined for all $t > 0$*. Furthermore, if we assume that either* ε_0 *(see (10)) or the diameter* $d(\Omega)$ *is small, then for* λ_0 *is sufficiently large in term of the geometry of* Ω *and other parameters in (A) and (F), this semigroup possesses a global attractor in* X*.*

Let us discuss some situations where the conditions (A) and (F) can be verified for the generalized (SKT) model, with $A(u), f(u)$ having polynomial growths in u. In many applications, in particular if the components of u are all nonnegative densities of species or chemicals, we have nonnegative numbers k and $\lambda_0 > 0$ such that $\lambda(u) \sim (\lambda_0 + |u|)^k$ and $|f(u)| \sim (\lambda_0 + |u|)^{k+1}$. It is reasonable to assume the growth of $|\lambda_u|$ to be like (7) in (A). Concerning (F), we can take $F(u) = |u|^{\frac{k+2}{2}}$ and $P(u) = |u|^{k+1}$. It is clear that (8) and (9) hold for such choices of F, P. We also have $|f(u)||u| \leq (\lambda_0 + |u|)^{k+2} \sim (F(u) + 1)^2$. Thus, (10) is satisfied with ε_0 being the coefficient of the highest power of u in $f(u)$. The (SKT) system

$$\begin{cases} u_t = \mathrm{div}[\nabla(a_1 u + \alpha_{11} u^2 + \alpha_{12} uv) + b_1 u \nabla\Phi(x)] + f_1(u,v), \\ v_t = \mathrm{div}[\nabla(a_2 v + \alpha_{21} uv + \alpha_{22} v^2) + b_2 v \nabla\Phi(x)] + f_2(u,v) \end{cases}$$

introduced in the Introduction section is a special case when $k = 1$.

We easily see that (11) is satisfied because

$$\begin{aligned} \frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1} &\leq C\frac{(1+|u|)^{\frac{k}{2}+k+1}}{(1+|u|)^{k+1}} \sim (1+|u|)^{k+1-\frac{k}{2}-1} \\ &\leq C(1+|u|)^{\frac{k}{2}+1} = C(F(u)+1). \end{aligned}$$

Hence, it is clear that the main assumptions in (F) and (13) in (F') are verified.

3. Uniform Estimates in $W^{1,2}(\Omega)$

In this section, we will consider a classical solution to (3) that exists in its maximal time interval $(0, T_0)$. Refering to the results in [13], or Section 5 of this work, we see that $T_0 = \infty$ under the assumptions (A) and (F').

In the proof, when there is no ambiguity C, C_i will denote universal constants that can change from line to line in our argument. Furthermore, $C(\cdots)$ is used to denote quantities which are bounded in terms of theirs parameters.

For any $T_1, T_2, T_3 > 0$ such that $0 < T_1 < T_2 < T_3$ we will say that η is a cutoff function for $[T_1, T_2]$ and $[T_1, T_3]$ if η is a nonnegative C^1 function satisfying $\eta(s) \in [0, 1]$ for all s and

$$\eta(s) = \begin{cases} 0 & s \le T_1, \\ 1 & s \ge T_2, \end{cases} \quad \text{and } |\eta'(s)| \le \frac{1}{T_2 - T_1}. \tag{14}$$

In particular, for $T_2 = (T_1 + T_3)/2$ we simply say that such η is a cutoff function for $[T_1, T_3]$.

LEMMA 3.1. *Assume that F satisfies (8) and (10)); and let $T, \tau_0 > 0$. If either ε_0 or $d(\Omega)$ is sufficiently small, then then there is a constant $C(|\Omega|, r)$, which depends also on the parameters in (A) and (F) but not on λ_0, such that*

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} \lambda(u)|Du|^2 \, dz \le C(|\Omega|, \tau_0), \tag{15}$$

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} \lambda^2(u)|Du|^2 \, dz \le C(|\Omega|, \tau_0)(\lambda_0^k + 1). \tag{16}$$

Proof. For any $l \in [0, k]$ we multiply the $i^t h$ equation of the system for u with $|u|^l u_i \eta^p(t)$, where η is a cutoff function for $[T, T + 2\tau_0]$ and $p > 1$ to be determined later, and integrate over $Q = \Omega \times [T, T + 2\tau_0]$. Integrating by parts in x, adding the results and using the fact that $|\eta_t| \le 1/\tau_0$, we easily obtain

$$\begin{aligned} \frac{2}{l+2} \sup_{t\in[T+\tau_0, T+2\tau_0]} \int_\Omega |u|^{l+2} \, dx + \iint_Q \left\langle A(u)Du, D(|u|^l u)\right\rangle \eta^p \, dz \\ \le C \iint_Q \left[\left\langle f(u), |u|^l u\right\rangle \eta^p + \frac{1}{\tau_0}|u|^{l+2}\eta^{p-1}\right] dz. \end{aligned} \tag{17}$$

Since $n = 2$, we have $\left\langle A(u)Du, D(|u|^l u)\right\rangle \ge C(l)\lambda(u)|u|^l|Du|^2$ for some positive constant $C(l)$ (see also (37) below).

By (8) and (10) of (F) we can take $F(u) \sim |u|^{\frac{k+2}{2}}$ and then find a constant C such that

$$\int_\Omega \left\langle f(u), |u|^l u\right\rangle \eta^p \, dx \le \varepsilon_0 \int_\Omega |u|^{k+l+2}\eta^p \, dx + C|\Omega|.$$

Here, $|\Omega|$ is the Lebesgue measure of Ω. Using Poincaré's inequality for $|u|^{\frac{k+l}{2}+1}$ and the fact that $\lambda(u) \sim (\lambda_0 + |u|)^k$, we have

$$\begin{aligned} \int_\Omega |u|^{k+l+2}\eta^p \, dx &\le Cd^2(\Omega) \int_\Omega |u|^{k+l}|Du|^2\eta^p \, dx \\ &\le Cd^2(\Omega) \int_\Omega |u|^l \lambda(u)|Du|^2\eta^p \, dx. \end{aligned} \tag{18}$$

On the other hand, since $k > 0$, if we now fix a p such that $k+2 < (p-1)k$ then $l+2 < (p-1)k$ for all $l \in [0,k]$. For such p, it is clear that $p-1 > p\frac{l+2}{k+l+2}$ so that we can write $p-1 = p\frac{l+2}{k+l+2} + \varepsilon(p)\frac{k}{k+l+2}$ for some $\varepsilon(p) > 0$. Hence, we can use Young's inequality to find some positive constant $C(\varepsilon_0, k, \tau_0)$ such that

$$\frac{1}{\tau_0}|u|^{l+2}\eta^{p-1} \le \varepsilon_0|u|^{k+l+2}\eta^p + C(\varepsilon_0, k, \tau_0)\eta^{\varepsilon(p)} \le \varepsilon_0|u|^{k+l+2}\eta^p + C(\varepsilon_0, k, \tau_0).$$

The integral of the first term on the right can be treated by Poincaré's inequality as before.

Therefore, if either ε_0 or $d(\Omega)$ is sufficiently small then we can deduce from (17) the following estimate.

$$\sup_{t\in[t+\tau_0, t+2\tau_0]} \int_\Omega |u|^{l+2}\, dx + \iint_Q |u|^l \lambda(u)|Du|^2\eta^p\, dz \le C(|\Omega|, \tau_0). \tag{19}$$

For $l = 0$ the above implies (15) of the lemma, using the property of η. Multipyling (19) when $l = 0$ with λ_0^k and add the result to (19) with $l = k$, we get

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} (\lambda_0^k + |u|^k)\lambda(u)|Du|^2\, dz \le C(|\Omega|, \tau_0)(\lambda_0^k + 1).$$

Because $\lambda(u) \sim (\lambda_0 + |u|)^k$, we can use Young's inequality to see that $\lambda(u) \le C(\lambda_0^k + |u|^k)$ for some constant C depending on k. The above then yields

$$\iint_{\Omega\times[T+\tau_0, T+2\tau_0]} \lambda^2(u)|Du|^2\, dz \le C(|\Omega|, \tau_0)(\lambda_0^k + 1).$$

This gives (16) and the proof is then complete. $\square$

To proceed, we need the following elementary fact. If $U = 0$ on the boundary $\partial\Omega$ then the Sobolev's imbedding theorem for planar domains gives

$$\left(\int_\Omega |U|^4\, dx\right)^{\frac{1}{2}} \le \int_\Omega |DU|^2\, dx. \tag{20}$$

Hence, if U, V vanish on the boundary $\partial\Omega$ then

$$\begin{aligned}\int_\Omega |U|^2|V|^2\, dx &\le \left(\int_\Omega |U|^4\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |V|^4\, dx\right)^{\frac{1}{2}} \\ &\le C\int_\Omega |DU|^2\, dx \int_\Omega |DV|^2\, dx. \end{aligned}\tag{21}$$

We also note that $\lambda(u)$ is the smallest eigenvalue of $(A+A^T)/2$ and $\Lambda(u)$ is the smallest eigenvalue of A^TA. Thus, if $\mu(u)$ is the eigenvalue of A with smallest real part then $\lambda(u)=\Re(\mu(u))$ and $\Lambda(u)=|\mu(u)|^2$. Therefore,

$$|A(u)\zeta|^2=\left\langle A^T(u)A(u)\zeta,\zeta\right\rangle\geq\Lambda(u)|\zeta|^2\geq\lambda^2(u)|\zeta|^2. \tag{22}$$

LEMMA 3.2. *Under the assumptions of Lemma 3.1, there exists a constant $C(|\Omega|,\tau_0)$, which depends also on the parameters in (A) and (F) but not on λ_0, such that*

$$\int_{\Omega\times\{T\}}\lambda^2(u)|Du|^2\,dx\leq C(|\Omega|,\tau_0)(\lambda_0^k+1)\,,\qquad\forall T>2\tau_0. \tag{23}$$

Proof. For any $t>0$ we test the system for u by $A(u)u_t$ (i.e. multiplying the i^{th} equation of (3) by $\sum_j a_{ij}(u)(u_j)_t$, $A(u)=(a_{ij}(u))$, integrating over Ω and summing the results) and integrate by parts to get

$$\int_\Omega\left(\langle A(u)u_t,u_t\rangle+\langle A(u)Du,D(A(u)u_t)\rangle\right)\,dx=\int_\Omega\langle f(u),A(u)u_t\rangle\,dx. \tag{24}$$

Because $A(u)=\mathcal{P}_u$, we have $\langle A(u)Du,D(A(u)u_t)\rangle=\frac{1}{2}\frac{\partial}{\partial t}|D\mathcal{P}(u)|^2$. Thus, we can rewrite (24) as

$$\int_\Omega\left(\langle A(u)u_t,u_t\rangle+\frac{1}{2}\frac{\partial}{\partial t}|ADu|^2\right)\,dx=\int_\Omega\langle f(u),A(u)u_t\rangle\,dx.$$

The ellipticity of $A(u)$ then gives

$$\int_\Omega\lambda(u)|u_t|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_\Omega|ADu|^2\,dx\leq C\int_\Omega\lambda(u)|f(u)||u_t|\,dx.$$

Using Young's inequality, we find a constant $C(\varepsilon)$ such that for any $\varepsilon>0$

$$|f(u)|\lambda(u)|u_t|\leq\varepsilon\lambda(u)|u_t|^2+C(\varepsilon)\lambda(u)|f(u)|^2.$$

For sufficiently small and fixed ε we then have

$$\frac{d}{dt}\int_\Omega|ADu|^2\,dx\leq C\int_\Omega\lambda(u)|f(u)|^2\,dx. \tag{25}$$

Now, let $U=P(u)$ be the function described in (F) and $V=\frac{\lambda^{\frac{1}{2}}(u)|f(u)|}{P(u)+1}$.

Then (11) gives $|V| \le CF(u)$ ($F(u)$ was also defined in (A)). We observe that

$$\begin{aligned}\int_\Omega \lambda(u)|f(u)|^2\,dx &\le C\int_\Omega (U^2+1)(F(u)^2+1)\,dx \\ &= C\int_\Omega P(u)^2F(u)^2\,dx + C\int_\Omega P(u)^2\,dx + C\int_\Omega F(u)^2\,dx + C(|\Omega|) \\ &\le C\int_\Omega |DP(u)|^2\,dx \int_\Omega |DF(u)|^2\,dx \\ &\qquad\qquad + C\int_\Omega (|DP(u)|^2 + |DF(u)|^2)\,dx + C(|\Omega|),\end{aligned}$$

where we used (21) and then Poincaré's inequality for $P(u), F(u)$ in the last estimate, noting that $P(u), F(u)$ vanish on the boundary of Ω. By (9) and (22), we have

$$\begin{aligned}|DP(u)|^2 \le |P_u(u)|^2|Du|^2 &\le C\lambda^2(u)|Du|^2 \\ &\le C\left\langle A^T(u)A(u)Du, Du\right\rangle = C|A(u)Du|^2.\end{aligned}$$

Since $|DF(u)|^2 \le C\lambda(u)|Du|^2$ by (8), we can use the above estimates in (25) to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |ADu|^2\,dx \le C\left[\int_\Omega \lambda(u)|Du|^2\,dx + 1\right]\int_\Omega |A(u)Du|^2\,dx \\ + \int_\Omega \lambda(u)|Du|^2\,dx + C(|\Omega|). \quad (26)\end{aligned}$$

We now set

$$y(t) = \int_\Omega |A(u)Du(x,t)|^2\,dx,\ \alpha(t) = \int_\Omega \lambda(u)|Du(x,t)|^2\,dx + C(|\Omega|),$$

and

$$\beta(t) = \int_\Omega \lambda(u)|Du(x,t)|^2\,dx + 1.$$

From (26) we obtain

$$y'(t) \le \alpha(t) + C\beta(t)y(t) \quad \forall t \in (0,\infty).$$

By (15) and (16), we have, for any $\tau_0 > 0$ and $t > \tau_0$, the followings

$$\int_t^{t+\tau_0} \beta(s)ds \le a_1,\ \int_t^{t+\tau_0} \alpha(s)ds \le a_2,\ \text{and}\ \int_t^{t+\tau_0} y(s)ds \le a_3,$$

where $a_1 = a_2 = C(|\Omega|, \tau_0)$ and $a_3 = C(|\Omega|, \tau_0)(\lambda_0^k + 1)$.

The uniform Gronwall inequality (see [17, Lemma 1.1] which was used in [5] in the context of Navier Stokes equation) then gives

$$y(t+\tau_0) \le \left[\frac{a_3}{\tau_0} + a_2\right] \exp(a_1) \le C(|\Omega|, \tau_0)(\lambda_0^k + 1)\,, \qquad \forall t > \tau_0.$$

Using the definition of $y(t+r)$ and (22) we complete the proof. □

REMARK 3.3: If we asume (F') and replace $f(u)$ by $\hat{f}(u, Du)$ satisfying

$$|\hat{f}(u, Du)| \le C\lambda^{\frac{1}{2}}(u)|Du| + f(u)$$

then the result continues to hold. Firstly, by Young's inequality and (F')

$$\left\langle \hat{f}(u, Du), |u|^l u \right\rangle \le \varepsilon\lambda(u)|u|^l|Du|^2 + C(\varepsilon)|u|^{l+2} + |f(u)||u|^{l+1}.$$

For suficiently small ε, there will be an extra term $|u|^{l+2}$ in the last integral of (17) in the argument in the proof of Lemma 3.1. Since $k > 0$ we can use Young's inequality again to have $|u|^{l+2} \le \varepsilon|u|^{k+l+2} + C(\varepsilon)$ and to obtain (19) again. The proof continues as before.

Next,

$$\begin{aligned} |\hat{f}(u, Du)|\lambda(u)|u_t| &\le C\lambda^{\frac{1}{2}}(u)\lambda(u)|Du||u_t| + C|f(u)|\lambda(u)|u_t| \\ &\le \varepsilon\lambda(u)|u_t|^2 + C(\varepsilon)\lambda^2(u)|Du|^2 + C|f(u)|\lambda(u)|u_t|. \end{aligned}$$

As $f(u)$ satisfies (F), for small ε in the above, the proof of Lemma 3.2 can continue.

4. Absorbing Balls in $W^{1,p_0}(\Omega)$

In this section we will establish a uniform bound for the $W^{1,p_0}(\Omega)$ norms of solutions when t is sufficiently large. To begin, let us fix a number $R > 0$ such that

(C) Ω can be covered by finitely many balls $B_{\frac{R}{4}}(x_i)$, $i = 1, \dots, n(R)$, with the property that either $x_i \in \partial\Omega$ or $x_i \in \Omega$ and $B_{2R}(x_i) \subset \Omega$.

The main result of this section is the following.

PROPOSITION 4.1. *For any $T, r_0 > 0$ and $p > 1$, if p is close to 1 and λ_0 is sufficiently large (in terms of $|\Omega|, r_0$ and the parameters in (A) and (F)) then there is a constant $C(\Omega, r_0, p)$ such that*

$$\sup_{t\in[T,T+r_0]} \int_\Omega |Du|^{2p}\, dx + \iint_{\Omega\times[T,T+r_0]} \lambda(u)|Du|^{2p-2}|D^2u|^2\, dz \le C(\Omega, r_0, p). \quad (27)$$

The constant $C(\Omega, r_0, p)$ depends on the geometry of Ω as well, namely, the numbers R and $n(R)$ in (C).

We will establish local estimates for the gradients of our solutions in these balls and then add up the results to obtain their global estimates. In the proof, we will only consider the case when $B_{2R}(x_i) \subset \Omega$. The boundary case ($x_i \in \partial\Omega$) is similar, invoking a reflection argument and using the fact that $\partial\Omega$ is smooth.

In the sequel, we will denote $\Phi(u) = \frac{|\lambda_u(u)|^2}{\lambda(u)}$. Before going to the proof of the proposition, we need some estimates for the integral of $\Phi(u)|Du|^4$.

LEMMA 4.2. *Assume (7) in (A). For any $R, r > 0$ and any nonnegative function $\psi \in C_0^1(B_R)$ there is a constant $C(|\Omega|, r)$, as in (23) of Lemma 3.2, such that for $t > r$*

$$\int_{B_R} \Phi(u)|Du|^4\psi^4 \, dx \leq \frac{C(|\Omega|, r)}{\lambda_0^{k+2}} \int_{B_R} (\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2) \, dx + \frac{C(|\Omega|, r)}{\lambda_0^{k+2}} \int_{B_R} \lambda(u)|D\psi|^2|Du|^2 \, dx. \quad (28)$$

Furthermore,

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4 \, dx \leq C(|\Omega|, r) \int_\Omega \big[\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\big] \, dx. \quad (29)$$

Proof. We establish (29). First we recall Ladyzhenskaya's inequality

$$\left(\int_\Omega |U|^4 \, dx\right)^{\frac{1}{2}} \leq C \left(\int_\Omega |U|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |DU|^2 \, dx\right)^{\frac{1}{2}}, \quad (30)$$

if $U = 0$ on $\partial\Omega$. Using the above with $\Omega = B_R$ and $U = \lambda^{\frac{1}{2}}(u)\psi Du$ we have

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4 \, dx = \int_{B_R} |U|^4 \, dx \leq C \int_{B_R} |U|^2 \, dx \int_{B_R} |DU|^2 \, dx \leq C \int_{B_R} \lambda(u)|Du|^2\psi^2 \, dx \int_{B_R} |D(\lambda^{\frac{1}{2}}(u)\psi Du)|^2 \, dx.$$

It is clear that there is a constant C_2 such that

$$|D(\lambda^{\frac{1}{2}}(u)\psi Du)|^2 \leq C_2 \left[\lambda(u)|D^2u|^2\psi^2 + \frac{|\lambda_u(u)|^2}{\lambda(u)}|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\right].$$

Since $\lambda(u) \sim (\lambda_0 + |u|)^k$ and $|\lambda_u(u)| \sim (\lambda_0 + |u|)^{k-1}$, we have $\lambda(u) \leq C\lambda_0^{-k}\lambda^2(u)$, and (23) implies

$$\int_{B_R} \lambda(u)|Du|^2\psi^2 \, dx \leq C\lambda_0^{-k} \int_{B_R} \lambda^2(u)|Du|^2\psi^2 \, dx \leq C(|\Omega|, r) \quad (31)$$

for all $t > r$. We then obtain

$$\int_{B_R} \lambda^2(u)|Du|^4\psi^4\ dx \\ \le C(|\Omega|, r)\int_\Omega \left[\lambda(u)|D^2u|^2\psi^2 + \Phi(u)|Du|^4\psi^2 + \lambda(u)|D\psi|^2|Du|^2\right]\ dx.$$

The above is (29). We also note that

$$\Phi(u) = \frac{|\lambda_u(u)|^2}{\lambda(u)} = \frac{|\lambda_u(u)|^2}{\lambda^3(u)}\lambda^2(u) \le \frac{C}{\lambda_0^{k+2}}\lambda^2(u).$$

Using (29) and the above estimate, we obtain (28) and prove the lemma. □

In the sequel, let $B_{2R}(x_0)$ be a fixed ball in the condition (C). For any s, t such that $0 \le s < t \le 2R$, let ψ be a cutoff function for two balls B_s, B_t centered at x_0. That is, ψ is nonnegative, $\psi \equiv 1$ in B_s and $\psi \equiv 0$ outside B_t with $|D\psi| \le 1/(t-s)$. We also let $r_0 > 0$ be a positive constant. For any $T > r_0$ let η be a cutoff function for $[T - r_0, T + r_0]$, see (14).

For any $p \ge 1$ and $t > 0$ we denote $Q_t = B_t \times [T - r_0, T + r_0]$ and

$$\mathcal{A}_p(t) = \sup_{\tau\in[T-r_0,T+r_0]}\int_{B_t} |Du|^{2p}\eta\ dx,\ \mathcal{H}_p(t) \\ = \iint_{Q_t} \lambda(u)|Du|^{2p-2}|D^2u|^2\eta\ dz, \tag{32}$$

$$\mathcal{B}_p(t) = \iint_{Q_t} \Phi(u)||Du|^{2p+2}\eta\ dz,\ \mathcal{G}_p(t) \\ = \iint_{Q_t} \lambda(u)|Du|^{2p}\ dz,\ \mathcal{J}_p = \frac{1}{r_0}\iint_{Q_t} |Du|^{2p}\ dz. \tag{33}$$

We then have the following local energy estimate result.

Lemma 4.3. *For any $p \ge 1$ and sufficiently close to 1 such that (36) holds, there is a constant C_1 depending on p such that*

$$\mathcal{A}_p(s) + \mathcal{H}_p(s) \le C_1\mathcal{B}_p(t) + \frac{C_1}{(t-s)^2}\mathcal{G}_p(t) + C_1\mathcal{J}_p(t)\,,\quad 0 < s < t \le 2R. \tag{34}$$

Proof. Since u is a strong solution, we can differentiate (3) with respect to x to get

$$(Du)_t = \mathrm{div}((A(u)D^2u + A_u(u)DuDu) + Df(u, Du). \tag{35}$$

By the uniform ellipticity of $A(u)$, we can find a constant C such that $|A(u)\zeta| \le C\lambda(u)|\zeta|$. Thus, for some $p > 1$ and sufficiently close to 1 there is

$\delta \in (0,1)$ such that $\alpha = 2p-2$ satisfies

$$\frac{\alpha}{2+\alpha} = \frac{2p-2}{2p} = \delta C^{-1}. \tag{36}$$

We then recall the following simple algebraic fact in [3, Lemma 2.1]. If A is a matrix satisfying $\lambda_0|\zeta|^2 \le \langle A\zeta, \zeta\rangle$ and $|A\zeta| \le \Lambda_0|\zeta|$ then for any $\alpha > 0$ if the number $\delta_\alpha := \frac{\alpha}{2+\alpha}\frac{\Lambda_0}{\lambda_0} \in (0,1)$ then

$$\langle AD\zeta, D(|\zeta|^\alpha\zeta)\rangle \ge \widehat{\lambda}|\zeta|^\alpha|D\zeta|^2, \quad \hat{\lambda} = (1-\delta_\alpha^2)\lambda_0. \tag{37}$$

We test (35) with $|Du|^{2p-2}Du\psi^2\eta$ and integration by parts in x. By (36) and (37), with $\zeta = Du$, it is standard to see that there is a positive constant $C(p)$ such that for $Q = \Omega \times [T-r_0, T+r_0]$

$$\begin{aligned}
&\sup_{\tau\in(T-r_0,T+r_0)}\int_\Omega |Du|^{2p}\psi^2\eta\,dx + C(p)\iint_Q \lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2\eta\,dz\\
&\le \iint_Q |A(u)||D^2u||Du|^{2p-1}\psi|D\psi|\eta\,dz\\
&\quad - \iint_Q A_u(u)DuDuD(|Du|^{2p-2}Du\psi^2)\eta\,dz\\
&\qquad + \iint_Q D\hat{f}(u,Du)|Du|^{2p-2}Du\psi^2\eta\,dz + \frac{1}{r_0}\iint_Q |Du|^{2p}\psi^2\,dz.
\end{aligned}$$

For simplicity, we will assume in the sequel that $\hat{f} \equiv 0$. The presence of $\hat{f}$ will be discussed later in Remark 4.4. For any given positive ε we use Young's inequality to find a constant $C(\varepsilon)$ such that

$$\begin{aligned}
&|A(u)||D^2u||Du|^{2p-1}\psi|D\psi|\\
&\qquad \le \varepsilon\lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2 + C(\varepsilon)\lambda(u)|Du|^{2p}|D\psi|^2,\\
&|A_u(u)DuDuD(|Du|^{2p-2}Du\psi^2)|\\
&\qquad \le |A_u(u)||Du|^{2p}|D^2u|\psi^2 + |A_u(u)||Du|^{2p+1}\psi|D\psi|\\
&\qquad \le \varepsilon\lambda(u)|Du|^{2p-2}|D^2u|^2 + C(\varepsilon)\frac{|A_u|^2}{\lambda(u)}|Du|^{2p+2}\psi^2\\
&\qquad\quad + C(\varepsilon)\lambda(u)|Du|^{2p}|D\psi|^2.
\end{aligned}$$

Therefore, taking ε small and using the above two inequalities in the previous one, we easily deduce

$$\sup_{\tau\in[T-r_0,T+r_0]}\int_{B_s}|Du|^{2p}\eta\,dx+\iint_{Q_s}\lambda(u)|Du|^{2p-2}|D^2u|^2\eta\,dz$$
$$\leq C_1\iint_{Q_t}\Phi(u)|Du|^{2p+2}\psi^2\eta\,dz$$
$$+C_1\iint_{Q_t}\left(\frac{1}{(t-s)^2}\lambda(u)+\frac{1}{r_0}\right)|Du|^{2p}\,dz. \quad (38)$$

Here, we used the definition of ψ and $\Phi(u)$ and the fact that $|A_u|\sim\lambda_u$. From the notations (32) and (33), the above gives the lemma. □

Remark 4.4: If we replace $f(u)$ by a function $\hat{f}$ depending on u, Du and satisfying a linear growth in Du then the proof can go on with minor modification. Namely, there exist a constant C and a function $f(u)$ satisfying (F) such that

$$|\hat{f}(u,Du)|\leq C\lambda^{\frac{1}{2}}(u)|Du|+f(u).$$

Formally, we can assume that $|D\hat{f}(u,Du)|\leq C|D(\lambda^{\frac{1}{2}}(u)|Du|+|f_u(u)||Du|$ so that by Young's inequality ($\Phi(u)=\frac{|\lambda_u(u)|^2}{\lambda(u)}$)

$$|D\hat{f}(u,Du)|\leq C\lambda^{\frac{1}{2}}|D^2u|+C\Phi^{\frac{1}{2}}(u)|Du|^2+|f_u(u)||Du|.$$

Therefore, the extra term $|D\hat{f}(u,Du)||Du|^{2p-1}\psi^2$ in the proof can be handled by using the following estimates, which are the results of a simple use of Young's inequality.

$$|D\hat{f}(u,Du)||Du|^{2p-1}\leq C[\lambda^{\frac{1}{2}}|D^2u|+C\Phi^{\frac{1}{2}}(u)|Du|^2+|f_u(u)||Du|]|Du|^{2p-1}$$
$$\leq\varepsilon\lambda|Du|^{2p-2}|D^2u|^2+C(\varepsilon)\lambda|Du|^{2p}$$
$$+C\Phi(u)|Du|^{2p+2}+C|Du|^{2p}+C|f_u||Du|^{2p}.$$

We can then assume that $|f_u|\leq C\lambda(u)$ for some constant C and see that the proof can continue to obtain the energy estimate (38).

Next, we also need the following elementary iteration result (e.g., see [8, Lemma 6.1, p.192]).

Lemma 4.5. *Let f, g, h be bounded nonnegative functions in the interval $[\rho, R]$ with g, h being increasing. Assume that for $\rho\leq s<t\leq R$ we have*

$$f(s)\leq[(t-s)^{-\alpha}g(t)+h(t)]+\varepsilon f(t)$$

with $\alpha>0$ and $0\leq\varepsilon<1$. Then

$$f(\rho)\leq c(\alpha,\varepsilon)[(R-\rho)^{-\alpha}g(R)+h(R)].$$

The constant $c(\alpha,\varepsilon)$ can be taken to be $(1-\nu)^{-\alpha}(1-\nu^{-\alpha}\nu_0)^{-1}$ for any ν satisfying $\nu \in (0,1)$ and $\nu^{-\alpha}\nu_0 < 1$.

We then have

LEMMA 4.6. *If λ_0 is sufficiently large such that for some $\mu_0 \in (0,1)$*

$$C_1 \frac{C(|\Omega|, r_0)}{\lambda_0^{k+2}} \le \frac{\mu_0}{2}, \tag{39}$$

where $C_1, C(|\Omega|, r_0)$ are the constants in (34) and (28) (with $r = r_0$), then there is a constant C such that

$$\iint_{Q_{\frac{R}{2}}} \Phi^2(u)|Du|^4 \, dz \le \frac{C}{\lambda_0^4} \iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz\,, \quad \forall T > 2r_0. \tag{40}$$

Here, $Q_{R/2} = B_{R/2} \times [T - r_0, T + r_0]$ and $Q'_{2R} = B_{2R} \times [T - 2r_0, T + r_0]$.

Proof. Let ψ be the cutoff function for $B_s(x_0), B_t(x_0)$ in Lemma 4.2. Fix a number $\mu_0 \in (0,1)$. Multiplying (28) by η and integrating the result over $[T - r_0, T + r_0]$ we see that if λ_0 satisfies (39) then, with the notations (32) and (33), we have

$$C_1 \mathcal{B}_1(s) \le \frac{\mu_0}{2} \left(\mathcal{H}_1(t) + \mathcal{B}_1(t) + \frac{1}{(t-s)^2}\mathcal{G}_1(t)\right) \tag{41}$$

for all s, t such that $0 < s < t < R$.

For $p = 1$, (34) gives

$$\mathcal{H}_1(s) \le C_1 \mathcal{B}_1(t) + \frac{C_1}{(t-s)^2}\mathcal{G}_1(t) + C_1 \mathcal{J}_1(t), \quad 0 < s < t < R.$$

Let $t_1 = (s+t)/2$ and use (41) with s being t_1 and the above with t being t_1 to obtain

$$\mathcal{H}_1(s) \le \frac{\mu_0}{2}[\mathcal{H}_1(t) + \mathcal{B}_1(t)] + \frac{C_2}{(t-s)^2}\mathcal{G}_1(t) + C_2 \mathcal{J}_1(t). \tag{42}$$

Obviously, we can assume that $C_1 \ge 1$. Thus, we can add (41) and (42) to have

$$\mathcal{H}_1(s) + \mathcal{B}_1(s) \le \mu_0[\mathcal{H}_1(t) + \mathcal{B}_1(t)] + \frac{C_3}{(t-s)^2}\mathcal{G}_1(t) + C_3 \mathcal{J}_1(t), \quad 0 < s < t < R.$$

Since $\mu_0 \in (0,1)$, we can use Lemma 4.5 with $f(t) = \mathcal{H}_1(t) + \mathcal{B}_1(t)$, $h(t) = \mathcal{J}_1(t)$, $g(t) = \mathcal{G}_1(t)$ and $\alpha = 2$ to obtain a constant C_4 depending on μ_0, C_3 such that

$$\mathcal{H}_1(s) + \mathcal{B}_1(s) \le \frac{C_4}{(t-s)^2}\mathcal{G}_1(t) + C_4 \mathcal{J}_1(t), \quad 0 < s < t < R.$$

For $s = R$ and $t = 2R$ the above gives

$$\mathcal{H}_1(R) + \mathcal{B}_1(R) \le C_4 \iint_{Q_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz. \tag{43}$$

Now, if $T > 2r_0$ and η is a cutoff function for $[T-2r_0, T]$ ($\eta \equiv 1$ in $[T-r_0, T]$) then the above and the definitions (32) and (33) give the estimate

$$\iint_{B_R \times [T-r_0, T+r_0]} [\lambda(u)|D^2u|^2 + \Phi(u)|Du|^4] \, dz \\ \le 2C_4 \iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz,$$

where $Q'_{2R} = B_{2R} \times [T - 2r_0, T + r_0]$.

Integrating (29) over $[T - r_0, T + r_0]$ and using the above estimate in the result with $s = R/2$ and $t = R$, we have

$$\iint_{Q_{\frac{R}{2}}} \lambda^2(u)|Du|^4 \, dz \le 2C_5 \iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right) |Du|^2 \, dz\,, \quad \forall T > 2r_0. \tag{44}$$

Because $\Phi^2(u) = \frac{|\lambda_u(u)|^4}{\lambda^2(u)} = \frac{|\lambda_u(u)|^4}{\lambda^4(u)}\lambda^2(u) \le \frac{C}{\lambda_0^4}\lambda^2(u)$, we have

$$\iint_{Q_{\frac{R}{2}}} \Phi^2(u)|Du|^4 \, dz \le \frac{C}{\lambda_0^4} \iint_{Q_{\frac{R}{2}}} \lambda^2(u)|Du|^4 \, dz.$$

Combining the above with (44), we obtain the lemma. □

Remark 4.7: We can choose $r_0 = R^2$ to obtain a uniform bound for the integral on the right of (40).

We are now giving the uniform estimate for $W^{1,p}(\Omega)$ of our solutions.

Proof of Proposition 4.1. For any $p > 1$ we have by Hölder's inequality

$$\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2\eta \, dz \\ \le \left(\iint_{Q_t} \Phi^2(u)|Du|^4 \, dz\right)^{\frac{1}{2}} \left(\iint_{Q_t} |Du|^{4p}\psi^4\eta^2 \, dz\right)^{\frac{1}{2}}. \tag{45}$$

Using Ladyzhenskaya's inequality (30) with $U = |Du|^{p-1}Du\psi$, multiplying the result with η^2 and integrating over $[T - r_0, T + r_0]$, we have

$$\iint_{Q_t} |Du|^{4p}\psi^4\eta^2 \, dz \le C \sup_{\tau \in [T-r_0, T+r_0]} \int_{B_t} |Du|^{2p}\psi^2\eta \, dx \iint_{Q_t} |DU|^2\eta \, dz.$$

Since $\lambda(u) \sim (\lambda_0 + |u|)^k$ there is a constant C such that

$$|DU|^2 \le \frac{C}{\lambda_0^k}[\lambda(u)|Du|^{2p-2}|D^2u|^2\psi^2 + \lambda(u)|Du|^{2p}|D\psi|^2].$$

Therefore,

$$\begin{aligned}\iint_{Q_t} |Du|^{4p}\psi^4\eta^2\, dz &\le \frac{C}{\lambda_0^k}\mathcal{A}_p(t)\left[\mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right] \\ &\le \frac{C}{\lambda_0^k}\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right]^2.\end{aligned}$$

Here, Cauchy's inequality was used in the last inequality.

We now assume $T > 2r_0$, $p > 1$ sufficiently close to 1, and λ_0 is sufficiently large as in Lemma 4.6. Applying the above inequality in (45) for $t \in (0, R/2)$ we derive, using (40) of Lemma 4.6 to estimate the first factor on the right of (45),

$$\begin{aligned}&\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2\, dz \\ &\qquad \le C\left(\frac{C_*(u,R,r_0)}{\lambda_0^4}\right)^{\frac{1}{2}} \lambda_0^{\frac{-k}{2}}\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right], \qquad (46)\end{aligned}$$

where we denote

$$C_*(u,R,r_0) = C\iint_{Q'_{2R}} \left(\frac{1}{R^2}\lambda(u) + \frac{1}{r_0}\right)|Du|^2\, dz.$$

By (15) of Lemma 3.1 we can find a constant $C(|\Omega|, r_0)$ such that

$$C_*(u,R,r_0) \le \max\left\{\frac{1}{R^2}, \frac{1}{r_0}\right\} C(|\Omega|, r_0).$$

Recall that $C(|\Omega|, r_0)$ does not depend on λ_0. Hence, for any given $\mu_1 \in (0,1)$ and $k \ge 0$ if λ_0 is sufficiently large such that, with C_1 being the constant in (38),

$$C_1 C\left(\max\left\{\frac{1}{R^2}, \frac{1}{r_0}\right\} C(|\Omega|, r_0)\right)^{\frac{1}{2}} \lambda_0^{\frac{-k}{2}-2} \le \mu_1, \qquad (47)$$

then (46) gives

$$\begin{aligned}&C_1\iint_{Q_t} \Phi(u)|Du|^{2p+2}\psi^2\, dz \\ &\qquad \le \mu_1\left[\mathcal{A}_p(t) + \mathcal{H}_p(t) + \frac{1}{(t-s)^2}\mathcal{G}_p(t)\right] \quad 0 < s < t < \frac{R}{2}. \qquad (48)\end{aligned}$$

If $p > 1$ and satisfies (36), we then have from (38) and the above inequality the following.

$$\mathcal{A}_p(s)+\mathcal{H}_p(s) \le \mu_1(\mathcal{A}_p(t)+\mathcal{H}_p(t))+\frac{C_2}{(t-s)^2}\mathcal{G}_p(t)+C_2\mathcal{J}_p(t), \quad 0 < s < t < \frac{R}{2}.$$

For $f(t) = \mathcal{A}_p(t) + \mathcal{H}_p(t)$, $h(t) = \mathcal{J}_p(t)$, $g(t) = \mathcal{G}_p(t)$ and $\alpha = 2$ we can use Lemma 4.5, as $\mu_1 \in (0,1)$, to obtain

$$f(s) \le \frac{C_3(\mu_1)}{(t-s)^2}\mathcal{G}_p(t) + C_3(\mu_1)\mathcal{J}_p(t), \quad 0 < s < t < \frac{R}{2}.$$

For $s = R/4$ and $t = R/2$ the above yields, recalling $\eta \equiv 1$ in $[T, T + r_0]$,

$$\sup_{t\in[T,T+r_0]} \int_{B_{\frac{R}{4}}} |Du|^{2p}\, dx + \mathcal{H}_p(\tfrac{R}{4}) \le \frac{C_3(\mu_1)}{R^2}\mathcal{G}_p(\tfrac{R}{2}) + C_3(\mu_1)\mathcal{J}_p(\tfrac{R}{2}). \quad (49)$$

If $2p < 4$ then a simple use of Hölder's inequality and the uniform bound in Lemma 4.6 for $\|Du\|_{L^4(B_R\times[T-r_0,T+r_0])}$ show that the right hand side can be bounded by a constant depending only on R, r_0. Thus, for some p such that $p \in (1, 2)$ a finite covering of Ω with balls $B_{R/4}$ as in (C) yields

$$\sup_{t\in[T,T+r_0]} \int_{\Omega} |Du|^{2p}\, dx + \iint_{\Omega\times[T,T+r_0]} \lambda(u)|Du|^{2p-2}|D^2u|^2\, dz \le C(\Omega, R, r_0).$$

This is (27) of the proposition and completes the proof. □

5. Proof of the Main Result

We are now ready to present the proof of our main result, Theorem 2.1, by verifying the conditions of Theorem 1.1 giving the existence of global attractors.

Proof of Theorem 2.1. By Amann's results in [1, 2], (3) defines a local semi-group on $W^{1,p}(\Omega)$ for any $p > 2$. By the proof of Proposition 4.1, the norm $\|u\|_{W^{1,p}(\Omega)}$ never blows up in any finite interval $(0, T)$ so that u exists globally . Thus, under the assumption (A) and (F'), (3) defines a semigroup $\{\mathcal{S}(t)\}_{t\ge0}$ on $X = W^{1,p_0}(\Omega)$, with $p_0 = 2p$ for some $p > 1$.

The existence of an absorbing ball in i) of Theorem 1.1 is also established by Proposition 4.1. Indeed, as the solutions of (3) exist globally, we can choose a uniform $r_0 = R^2$, where R is described in (C). Hence, the key assumption on the existence of μ_0, μ_1 in (39) and (47) can be guaranteed if λ_0 is sufficiently large in terms the parameters in (A), (F) and R, or the geometry of Ω. Therefore, for any solution u of (3) Proposition 4.1 shows that there is a uniform constant

C depending only on the geometry of Ω such that $\|u(\cdot,t)\|_{W^{1,p_0}(\Omega)} \le C$ if t is sufficiently large. We have proved the existence of an absorbing ball in X.

Concerning the compactness of the $\mathcal{S}(t_1)$ in ii) of Theorem 1.1, we fix a $t_1 > 0$. By (27), with $p = 1$ and $p > 1$, we have the bound for the following quantities

$$\sup_{t\in[t_1,2t_1]} \int_\Omega |Du|^2\ dx,\ \sup_{t\in[t_1,2t_1]} \int_\Omega |Du|^{2p}\ dx,\ \iint_{\Omega\times[t_1,2t_1]} \lambda(u)|D^2u|^2\ dz \le C(\Omega, t_1, p).$$

From the system for u we can estimate $|u_t|^p$ in terms of $|Du|^{2p}, |D^2u|^p$ and powers of $|u|$. Since $2p > 2$, the above and Sobolev's imbedding theorem show that $u(\cdot,t)$ is Hölder continuous in x and thus bounded. Therefore, by the above estimates, u_t is in $L^p(Q)$. It is now standard to show that u is Hölder in (x,t) and then Du is Hölder continuous (see [6]). The compactness of $\mathcal{S}(t_1)$ then follows. □

6. Mixed and Neumann Boundary Conditions

We notice that the only place in the proof we need the boundary condition $u = 0$ is the validity of the Poincaré inequality

$$\int_\Omega |U|^2\ dx \le Cd^2(\Omega)\int_\Omega |DU|^2\ dx, \quad U \in W_0^{1,2}(\Omega), \tag{50}$$

in the proof of Lemma 3.1. The above inequality also yields Sobolev's inequality of the form (20) and Lemma 3.2 to obtain the key uniform Gronwall inequality (26).

It is well known that (50) continues to hold if $U = 0$ on a nonempty relatively open set $\partial\Omega_1$ of $\partial\Omega$ and Ω is starshaped with respect to $\partial\Omega_1$. This comes from a simple modification of the proof of [7, Lemma 7.14] and then the use of Riesz's potential estimates in [7, Lemma 7.12]. Hence, we can assume mixed boundary conditions for (3). Namely, u satifies the homogenous Dirichlet condition on a nonempty relatively open set $\partial\Omega_1$ of $\partial\Omega$, Ω is starshaped with respect to $\partial\Omega_1$ and the homogeneous Neumann condition on $\partial\Omega \setminus \partial\Omega_1$. We then see that our results still hold in this case.

On the other hand, if u satifies the homogenous Neumann condition on the boundary $\partial\Omega$ then we have to assume that the semigroup defined by (3) *has an absorbing ball in* $L^1(\Omega)$. That is, for any bounded set $B \subset W^{1,p_0}(\Omega)$ there are $T(B) > 0$ and a universal constant C such that

$$\forall U_0 \in B,\ \|S(t)U_0\|_{L^1(\Omega)} \le C \quad t \ge T(B). \tag{51}$$

Indeed, by the compactness of the imbedding $W^{1,2}(\Omega) \to L^p(\Omega)$ $(n = 2, p > 1)$ and a simple argument by contradiction shows that for any positive reals ε, q there is a constant $C(\varepsilon, q)$ such that

$$\left(\int_\Omega |U|^p\, dx\right)^{\frac{1}{p}} \le \varepsilon \left(\int_\Omega |DU|^2\, dx\right)^{\frac{1}{2}} + C(\varepsilon, p, q) \left(\int_\Omega |U|^q\, dx\right)^{\frac{1}{q}}. \qquad (52)$$

Therefore, if U_0 belongs to a bounded set in $W^{1,p_0}(\Omega)$ and U is a polynomial in u, with u being the solution to (3), then by choosing q small in the above and using (51) we can easily see that

$$\left(\int_\Omega |U|^p\, dx\right)^{\frac{1}{p}} \le \varepsilon \left(\int_\Omega |DU|^2\, dx\right)^{\frac{1}{2}} + C(\varepsilon, p, q, B). \qquad (53)$$

Using the above for $p = k + l + 2$ and $U = u$ in the proof of Lemma 3.1 we see that the proof can go on. Similarly, we use the above for $p = 4$ and U being $P(u)$ or $F(u)$ in the proof of Lemma 3.2, the uniform Gronwall inequality (26).

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Numerical global bifurcation diagrams for a superlinear indefinite problem with a parameter appearing in the domain

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To Julián López-Gómez, on the occasion of his 60th birthday, with deep gratitude and my best wishes for many more years of mathematical creativity, in the art of moving parameters

Abstract. *We consider a superlinear indefinite problem with homogeneous Neumann boundary conditions and a parameter appearing in the domain of the differential equation. Such a problem is an extension of the one studied in [33], in the sense that also negative values of the parameter are allowed.*
First, we show how to discretize the problem in a way that is suitable to perform numerical continuation methods and obtain the associated bifurcation diagrams. Then, we analyze the results of the simulations, also studying the stability of the solutions.

Keywords: Superlinear indefinite problems, numerical global bifurcation diagrams, high multiplicity of positive solutions, stability, Neumann boundary conditions.
MS Classification 2010: 65L10, 65L60, 65P30, 65L07.

1. Introduction

The term *superlinear indefinite problems* is used in the literature to refer to nonlinear boundary value problems of elliptic type which are characterized by the presence of a sign-changing nonlinearity. These problems have attracted the attention of many researches in the last decades, since they have revealed a wide phenomenology of multiplicity of positive solutions. We refer to [1, 3–6, 16, 20, 21, 29, 31] for some pioneering works, to the book [22] for some related results (see, in particular, Chapter 9), and to the monograph [10] for an extended review on the existing literature, up to the most recent one.

In [28], we have considered, together with J. López-Gómez and F. Zanolin, the following one-dimensional problem with Dirichlet boundary conditions

$$\begin{cases} -u'' = \lambda u + a(t)u^p, & t \in (0,1), \\ u(0) = u(1) = M, \end{cases} \tag{1}$$

with $\lambda < 0$, $p > 1$ (which makes the problem superlinear), and the weight $a(t)$ defined as the piecewise constant function

$$a(t) = \begin{cases} -c, & \text{for } t \in (0,\alpha) \cup (1-\alpha,1), \\ b, & \text{for } t \in (\alpha, 1-\alpha), \end{cases} \tag{2}$$

where $\alpha \in \left(0, \frac{1}{2}\right)$ and $b, c > 0$. Thus, it is apparent that problem is indefinite. As for the boundary condition, M is taken in $(0, +\infty]$ and, when $M = +\infty$, the condition is understood in the limiting sense and gives rise to the so called *large* or *blow-up solutions*.

The main result of [28] is that problem (1) admits, for certain values of the parameters, an arbitrarily high number of positive solutions. We mention that a different mechanism for obtaining high multiplicity of positive solutions for superlinear indefinite problems had been previously observed numerically in [16], and analytically proved - in a different setting - in [7, 8, 12, 13, 15]. Nonetheless, these results substantially differ from the one of [28], since the multiplicity was originated by the high number of positive parts of the weight. Instead, [28] is the first work where a high multiplicity result has been obtained with weights having only one positive part (cf. (2)).

In addition, the fact of considering the piecewise constant weight (2) allowed us in [28] to determine the structure of the global bifurcation diagrams, which become more and more complex (i.e., they exhibit an increasing number of turning points and secondary bifurcations) as the number of solutions increases. Such bifurcation diagrams have been obtained analytically, and the value of the weight in the positive part, b, has been used as a main continuation parameter, an idea that originally goes back to [21].

To complete the reference to previous results related to problem (1), we mention [27], where with J. López-Gómez we considered, in place of (2), an asymmetric weight, which entailed a break-up of the bifurcation diagrams into several connected components, and [32], where we studied the relation between the symmetric and asymmetric case in a neighborhood of the bifurcation points. Finally, in [24], together with J. López-Gómez and M. Molina-Meyer, we considered the same problem in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and obtained general (minimal) multiplicity results in that context, also studying the stability of the solutions.

The numerical computation of the complex bifurcation diagrams arising in such situation has turned to be an intricate question, since the increasing number of singular points (secondary bifurcations and turning point) which were closer and closer one to each other required some refinements in the algorithms. We refer to [23, 25, 26] for these numerical aspects.

In [33], instead, we have considered the superlinear indefinite problem with homogeneous Neumann boundary conditions

$$\begin{cases} -u'' = \lambda u + a(t)u^p, & \text{for } t \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{3}$$

This change made the use of b as a continuation parameter no longer possible. Instead, we used the parameter α, which measures the *amplitude* of the positive part of the weight; this idea was suggested by [18]. The main results of [33] can be summarized as follows (see also Figure 1, which is taken from [33]).

THEOREM 1.1 ([33], Theorem 5.1). *Let* $\lambda_n := -\frac{(n\pi)^2}{p-1}$, $n \in \mathbb{N}$, *and assume that* $\lambda \in [\lambda_{n+1}, \lambda_n)$ *for some* $n \in \mathbb{N}$. *Then:*

(i) if $n = 0$, *the minimal bifurcation diagram in* α *for problem* (3) *consists of a curve starting from* $\{\alpha = 0\}$ *and bifurcating from* $+\infty$ *at* $\alpha = 1/2$. *Such a curve, that contains symmetric solutions, will be referred to as* principal curve *(see Figure 1(A));*

(ii) if $n = 1$, *the minimal bifurcation diagram in* α *for problem* (3) *consists of one component containing the principal curve with two additional branches, containing asymmetric solutions, that start from* $\{\alpha = 0\}$ *and merge in a bifurcation point on the principal curve (see Figure 1(B));*

(iii) if $n = 2k + 1$, $k \in \mathbb{N}^*$, *the minimal bifurcation diagram in* α *for problem* (3) *consists of* $k + 1$ *components: one, as in (ii), containing the principal curve with two branches bifurcating from it and reaching the axis* $\{\alpha = 0\}$, *plus* k *additional bounded components, each formed by four branches (two with symmetric solutions and two with asymmetric solutions) that start from the axis* $\{\alpha = 0\}$ *and three of which merge in a bifurcation point, while (at least) two of them merge in a subcritical turning point (see Figure 1(D));*

(iv) if $n = 2k$, $k \in \mathbb{N}^*$, *the minimal bifurcation diagram in* α *for problem* (3) *consists of* $k+1$ *components: one, as in (ii), containing the principal curve with two branches bifurcating from it and reaching the axis* $\{\alpha = 0\}$, $k - 1$ *bounded components as in (iii), each consisting of four branches (two with symmetric solutions and two with asymmetric solutions) that start from* $\{\alpha = 0\}$ *and form a subcritical turning point and a bifurcation point, and an additional bounded component formed by two branches of symmetric solutions that start from the axis* $\{\alpha = 0\}$ *and merge in a subcritical turning point (see Figure 1(C));*

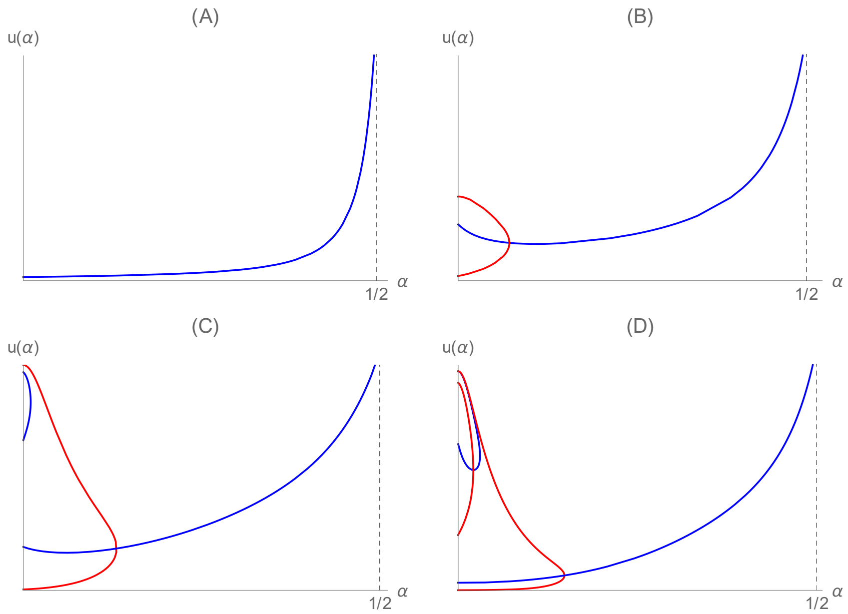

Figure 1: Bifurcation diagrams in α for problem (3) corresponding to the following cases: (A) $\lambda \in [\lambda_1, \lambda_0)$, (B) $\lambda \in [\lambda_2, \lambda_1)$, (C) $\lambda \in [\lambda_3, \lambda_2)$, (D) $\lambda \in [\lambda_4, \lambda_3)$. The blue branches are formed by symmetric solutions, the red ones by asymmetric solutions.

In particular, the previous result establishes that the bifurcation diagrams in α, with $\alpha \geq 0$, are always non connected for sufficiently negative λ's (precisely, for $\lambda < \lambda_2$), which was not the case in [28]. However, one may think that this non-connectedness is only apparent and is due to the fact that in [33] we only considered $\alpha \geq 0$. Indeed, one can extend the problem also for negative α's, and might think that the several branches combine to form a connected diagram, like the ones of [28].

One of the main goals of this work is to give numerical evidence that this does not happen, and the global diagrams of the extended problem remain disconnected. In addition, with our numerical study, we will analyze the stability of the solutions of the problem.

In order to extend problem (3) for $\alpha < 0$, first of all, we extend the weight as follows

$$a_1(t) = \begin{cases} -c, & \text{for } t \in (\alpha, 0) \cup (1, 1-\alpha), \\ b, & \text{for } t \in (0, 1), \end{cases} \qquad \alpha < 0.$$

Then, we impose the Neumann boundary conditions on the boundary of the new domain, arriving at

$$\begin{cases} -u'' = \lambda u + a_1(t)u^p, & t \in (\alpha, 1-\alpha), \\ u'(\alpha) = 0 = u'(1-\alpha), \end{cases} \qquad \alpha < 0. \tag{4}$$

A motivation for considering such a boundary condition arises from the biological interpretation of superlinear indefinite problems, which can be used to describe the stationary states of the evolution of the density of a population, taking into account intra-specific competition, according to the classical logistic model, in the regions where the weight is negative, and intra-specific facilitative effects in the region where the weight is positive. Under this perspective, Neumann boundary conditions describe the fact that the habitat is isolated and no inner or outer flux of individuals takes place.

One of the main issues to numerically deal with problems (3) and (4) is the fact that the bifurcation parameter does not appear explicitly in the differential equations, but only implicitly in its domain. Indeed, for the numerical continuation methods one has to differentiate the approximating problem with respect to the bifurcation parameter. For this to be possible, one cannot use the collocation procedure, used in [23, 25, 26] to compute the bifurcation diagrams in b for problem (1), which is very efficient from the computational point of view. Instead, as suggested by [30], one can use a Galerkin method which makes the differentiation with respect to α treatable, but entails the disadvantage of being much slower for the computations.

In Section 2 we present such a method to discretize problem (4) and in Section 3 we present the results of the numerical experiments that we have performed, using the obtained discretization. Finally, in Section 4, we present some remarks on two different extensions of problem (3) for $\alpha < 0$.

2. Discretization of problem (4)

One of the main difficulties to study problem (4) as $\alpha < 0$ varies is the fact that the domain grows without restrictions as α becomes more and more negative. For this reason, first of all, we perform the change of variable $x = \frac{t-\alpha}{1-2\alpha}$ to transform (4) into the following equivalent problem, which is set in a domain of fixed size:

$$\begin{cases} -\frac{1}{(1-2\alpha)^2}u'' = \lambda u + a_2(x)u^p, & x \in (0,1), \\ u'(0) = 0 = u'(1), \end{cases} \tag{5}$$

where the $'$ now indicates derivatives with respect to x and

$$a_2(x) := a_1(t(x)) = \begin{cases} -c, & \text{for } x \in \left(0, \frac{-\alpha}{1-2\alpha}\right) \cup \left(\frac{1-\alpha}{1-2\alpha}, 1\right), \\ b, & \text{for } x \in \left(\frac{-\alpha}{1-2\alpha}, \frac{1-\alpha}{1-2\alpha}\right), \end{cases} \qquad \alpha < 0.$$

We observe that problem (5) shares some features with (3): both are superlinear indefinite problems set in $(0,1)$ with homogeneous Neumann boundary conditions, and in both cases the position of the points of discontinuity of the weight varies with α. Moreover, for $\alpha = 0$, both problems reduce to the same purely superlinear one, i.e., with a positive weight. Nonetheless, their behavior with respect to α might be substantially different, since the coefficient in front of the second derivative in (5) also depends on α, while it is constant in (3). Equivalently, (5) can be seen as a variant of (3) in which λ and the values of the weight depend on α. In the light of Theorem 1.1, the number of solutions of (3) depends on the value of λ, thus it is not easy to relate one problem to the other.

Since the analysis is not easy, we perform numerical simulations to get insight into problem (5) and, hence, on the equivalent problem (4). To do so, we have to discretize (5). We consider $p = 2$ and, as suggested by [30], we apply a Fourier–Galerkin method. It consists in approximating a solution $u(x)$ by the truncated Fourier series

$$\overline{u}(x) = \sum_{j=1}^{n} u_j \phi_j(x), \qquad \text{where } \phi_j(x) := \cos((j-1)\pi x), \tag{6}$$

in multiplying the differential equation of (5) by the i-th element of the Fourier basis $\phi_i(x)$, and in integrating over $(0,1)$. In this way, we obtain the i-th equation of the discretized problem, which will be denoted by F_i, and the unknown is now the vector of Fourier coefficients $\mathbf{u} = (u_j)_{j=1}^n \in \mathbb{R}^n$, considered as a column vector. The fact of taking only cosine terms in (6) is due to the boundary conditions in (5). Hence, the equation F_i, $1 \leq i \leq n$, reads

$$\begin{aligned} &\sum_{j=1}^{n} \left(\lambda - \left(\frac{(j-1)\pi}{1-2\alpha} \right)^2 \right) u_j \int_0^1 \phi_j(x)\phi_i(x)\,dx \\ &- c \int_0^{\frac{-\alpha}{1-2\alpha}} \left(\sum_{j=1}^{n} u_j \phi_j(x) \right)^2 \phi_i(x)\,dx + b \int_{\frac{-\alpha}{1-2\alpha}}^{\frac{1-\alpha}{1-2\alpha}} \left(\sum_{j=1}^{n} u_j \phi_j(x) \right)^2 \phi_i(x)\,dx \\ &\qquad\qquad - c \int_{\frac{1-\alpha}{1-2\alpha}}^{1} \left(\sum_{j=1}^{n} u_j \phi_j(x) \right)^2 \phi_i(x)\,dx. \end{aligned} \tag{7}$$

On the one hand, the orthogonality conditions give

$$\int_0^1 \phi_j(x)\phi_i(x)\,dx = \begin{cases} 1 & \text{if } i = j = 1, \\ 1/2 & \text{if } i = j > 1, \\ 0 & \text{otherwise,} \end{cases}$$

on the other one we observe that

$$\left(\sum_{j=1}^{n} u_j\phi_j(x)\right)^2 = \left(\sum_{j=1}^{n} u_j\phi_j(x)\right)\left(\sum_{k=1}^{n} u_k\phi_k(x)\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} u_j u_k \phi_j(x)\phi_k(x),$$

and that

$$\begin{aligned} 4\phi_i(x)\phi_j(x)\phi_k(x) = {} & \cos((i+j+k-3)\pi x) + \cos((i+j-k-1)\pi x) \\ & + \cos((i-j+k-1)\pi x) + \cos((i-j-k+1)\pi x); \end{aligned}$$

thus, if we set

$$\begin{aligned} h_{jk}^{i,1}(x) &= \begin{cases} x & \text{if } i{+}j{+}k{=}3, \\ \frac{\sin((i+j+k-3)\pi x)}{(i+j+k-3)\pi} & \text{otherwhise,} \end{cases} \\ h_{jk}^{i,2}(x) &= \begin{cases} x & \text{if } i{+}j{-}k{=}1, \\ \frac{\sin((i+j-k-1)\pi x)}{(i+j-k-1)\pi} & \text{otherwhise,} \end{cases} \\ h_{jk}^{i,3}(x) &= \begin{cases} x & \text{if } i{-}j{+}k{=}1, \\ \frac{\sin((i-j+k-1)\pi x)}{(i-j+k-1)\pi} & \text{otherwhise,} \end{cases} \\ h_{jk}^{i,4}(x) &= \begin{cases} x & \text{if } i{-}j{-}k{=}{-}1, \\ \frac{\sin((i-j-k+1)\pi x)}{(i-j-k+1)\pi} & \text{otherwhise,} \end{cases} \end{aligned}$$

we have that a primitive of $\phi_i(x)\phi_j(x)\phi_k(x)$ is

$$h_{jk}^{i}(x) := \frac{1}{4}\left(h_{jk}^{i,1}(x) + h_{jk}^{i,2}(x) + h_{jk}^{i,3}(x) + h_{jk}^{i,4}(x)\right).$$

As a consequence, (7) reads

$$\begin{aligned} &\sum_{j=1}^{n}\left(\lambda - \left(\frac{(j-1)\pi}{1-2\alpha}\right)^2\right) u_j \int_0^1 \phi_j(x)\phi_i(x)\,dx \\ &\quad - c\left(\sum_{j=1}^{n}\sum_{k=1}^{n} u_j\left(h_{jk}^{i}\left(\frac{-\alpha}{1-2\alpha}\right) - h_{jk}^{i}(0)\right) u_k\right) \\ &\quad + b\left(\sum_{j=1}^{n}\sum_{k=1}^{n} u_j\left(h_{jk}^{i}\left(\frac{1-\alpha}{1-2\alpha}\right) - h_{jk}^{i}\left(\frac{-\alpha}{1-2\alpha}\right)\right) u_k\right) \\ &\quad - c\left(\sum_{j=1}^{n}\sum_{k=1}^{n} u_j\left(h_{jk}^{i}(1) - h_{jk}^{i}\left(\frac{1-\alpha}{1-2\alpha}\right)\right) u_k\right) \end{aligned}$$

and, observing that $h^i_{jk}(0) = 0$, it reduces to

$$\sum_{j=1}^{n} \left(\lambda - \left(\frac{(j-1)\pi}{1-2\alpha} \right)^2 \right) u_j \int_0^1 \phi_j(x)\phi_i(x)\,dx$$
$$+ (b+c) \left(\sum_{j=1}^{n} \sum_{k=1}^{n} u_j \left(h^i_{jk}\left(\frac{1-\alpha}{1-2\alpha} \right) - h^i_{jk}\left(\frac{-\alpha}{1-2\alpha} \right) \right) u_k \right)$$
$$- c \sum_{j=1}^{n} \sum_{k=1}^{n} u_j h^i_{jk}(1) u_k,$$

which leads to

$$F_1 = \lambda u_1 + (b+c)\,\mathbf{u}^t \left(H^1\left(\frac{1-\alpha}{1-2\alpha} \right) - H^1\left(\frac{-\alpha}{1-2\alpha} \right) \right) \mathbf{u} - c\,\mathbf{u}^t\, H^1(1)\,\mathbf{u},$$

and, for $i > 1$,

$$F_i = \left(\lambda - \left(\frac{(i-1)\pi}{1-2\alpha} \right)^2 \right) \frac{u_i}{2} + (b+c)\,\mathbf{u}^t \left(H^i\left(\frac{1-\alpha}{1-2\alpha} \right) - H^i\left(\frac{-\alpha}{1-2\alpha} \right) \right) \mathbf{u}$$
$$- c\,\mathbf{u}^t\, H^i(1)\,\mathbf{u},$$

where we have set $H^i(x) := (h^i_{jk}(x))^n_{j,k=1}$ and $\mathbf{u}^t$ denotes the transposed of vector $\mathbf{u}$. As a consequence, the discretized problem is

$$F(\mathbf{u}, \alpha) = 0, \qquad F = (F_i)^n_{i=1}.$$

Before concluding this section, we remark that, for the numerical bifurcation algorithms that we use in our simulations, it is necessary to differentiate the discretized equations also with respect to the parameter α. The advantage of employing the Fourier–Galerkin method described above consists in the fact that such a derivative can be easily computed from (7) by means of the fundamental theorem of Calculus.

3. Results of the numerical experiments

In this section we show the results of the experiments performed by applying numerical continuation methods to the discretized problem obtained in Section 2. We send the interested reader to [2, 9, 17, 19] for general references on numerical continuation methods, and to [25, 26] for more recent references where some improvements to the algorithms are performed in order to be able to compute complex bifurcation diagrams as those appearing in this work.

Structure of the bifurcation diagrams. For our numerical experiments, we have used the following values of the parameters:

$$b = c = 1, \qquad p = 2,$$

and $\lambda \in \{-4, -25, -60, -120\}$, which are the same values used to obtain the diagrams in Figure 1. The choice of the number of discretization points has been $n = 300$ with the aim of achieving a good precision (measured by the size of the Fourier coefficients u_j, $n - 10 \leq j \leq n$, whose modulus in all our simulation is smaller than 10^{-4} and, in many cases, smaller than 10^{-8}) in a reasonable computational time. The resulting bifurcation diagrams are represented in Figure 2.

In particular we observe that for $\alpha = 0$ problems (3) and (4) coincide, thus the number of solutions is necessarily the same, and the branches are continuous. Moreover, the global patterns of the branches for $\alpha < 0$ are the same as those for $\alpha > 0$ described in Theorem 1.1, the only difference being that the principal curve seems to be continuable for all $\alpha < 0$.

To understand this difference, we observe that a necessary condition for (3) and (4) to possess positive solutions, which can be easily obtained by integrating the differential equation and using the boundary conditions, is that the weight has to change sign. This condition is no longer true for problem (3) when $\alpha = \frac{1}{2}$, thus all the solutions are lost before such a value of the parameter is reached. Instead, the weight in (4) changes sign for all $\alpha < 0$, thus no restrictions exist on α, and actually our simulations suggest that existence occurs for all $\alpha < 0$.

Moreover, we observe that the diagrams are non-connected for sufficiently negative λ's and the number of connected components increases as λ becomes more and more negative.

This is not a priori evident, since, as commented above, problem (5) can be equivalently written as

$$-u'' = \tilde{\lambda}(\alpha)u + (1 - 2\alpha)^2 a_2(x) u^p, \qquad x \in (0, 1),$$

with $\tilde{\lambda}(\alpha) = (1 - 2\alpha)^2 \lambda$, and we have the following opposite trends coinciding as $\alpha \to -\infty$:

i) first, as above, the positive part of the weight, whose size is $\frac{1}{1-2\alpha}$, becomes smaller and smaller, thus the necessary condition for the existence of solutions - the change of sign of the weight - tends to be violated, though it is so only in the limiting case $\alpha = -\infty$. This makes one infer that the solutions are lost as $\alpha < 0$ decreases;

ii) contrastingly, the value of $\tilde{\lambda}(\alpha)$ goes to $-\infty$ as $\alpha \to -\infty$, thus, if all the other parameters were fixed, Theorem (1.1) would guarantee the existence of an increasing number of solutions. Nonetheless, the values of α for which such a high number of solution is present depends on λ: essentially, one should study the dependence of the turning points on λ, which is a very interesting open problem both from the analytical and the numerical point of view;

iii) finally, the values of the weight, both in the negative and the positive part, go to $+\infty$ as $\alpha \to -\infty$, and the overall effect is not clear in this case (cf. [11] for a similar problem).

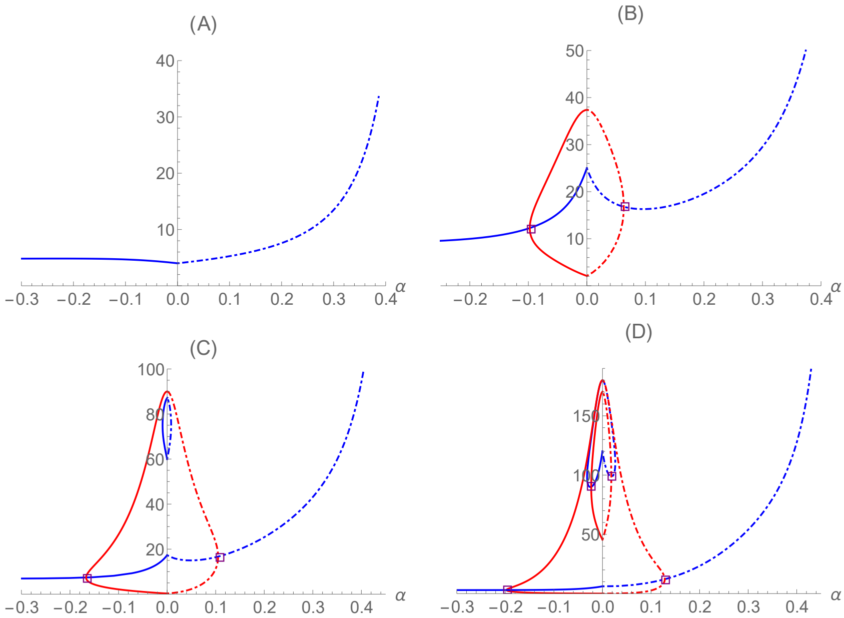

Figure 2: Numerical bifurcation diagrams in α obtained for problem (3) ($\alpha > 0$, with dash-dotted line) and problem (4) ($\alpha < 0$, with continuous line) corresponding to the following values of λ:

(A) $\lambda = -4$, (B) $\lambda = -25$, (C) $\lambda = -60$, (D) $\lambda = -120$.

On the vertical axes we plot the values of $u(\alpha)$ for $\alpha > 0$ and $u(0)$ for $\alpha < 0$, i.e. we represent the value of the solution where the weight changes sign for the first time. As in Figure 1, the blue branches are formed by symmetric solutions, while the red ones by asymmetric solutions. The bifurcation points have been marked with squares.

REMARK 3.1: A natural question, that arises after the comments performed above on the general properties of the bifurcation diagram, is whether the branches are differentiable, with respect to α, at $\alpha = 0$. By using a finite difference method, the results that we obtain for the approximation of the left and right derivatives of the several branches appearing in the diagrams of Figure 2 have been gathered in Table 1.

λ	Point	Right derivative	Left derivative
-4	$(0, 4.00000)$	10.2952	-10.2952
-25	$(0, 2.09423)$	63.8469	-42.0332
-25	$(0, 25.0000)$	-333.815	333.815
-25	$(0, 37.3889)$	-4.71944	2.46041
-60	$(0, 0.311947)$	14.5182	-9.67665
-60	$(0, 17.3092)$	-129.934	290.669
-60	$(0, 60.0000)$	1040.64	-1040.64
-60	$(0, 87.1313)$	-312.416	262.173
-60	$(0, 89.9989)$	10.861	-10.8944
-120	$(0, 0.0251730)$	1.72774	-1.17619
-120	$(0, 6.14466)$	-7.37161	76.2526
-120	$(0, 45.5856)$	1197.07	-765.859
-120	$(0, 120.000)$	-2503.6	2503.6
-120	$(0, 170.378)$	-432.671	262.252
-120	$(0, 179.797)$	16.8713	-21.3551
-120	$(0, 180.000)$	43.8456	-43.8475

Table 1: Values of the derivatives with respect to α of the branches in the bifurcation diagrams of Figure 2, evaluated at $\alpha = 0$.

As we can see from the previous table, the right derivative always has the opposite sign of the left one; thus, our numerical simulations suggest that none of the branches is differentiable at $\alpha = 0$. Moreover, we see that, at the points $(0, -\lambda)$, with $\lambda \in \{-4, -25, -60, -120\}$, the right and the left derivative have the same absolute value. In view of this, we conjecture that, for all $\lambda < 0$, the following relation holds true:

$$\left.\frac{d}{d\alpha}\hat{u}_\lambda(0;\alpha)\right|_{\alpha=0^+} = -\left.\frac{d}{d\alpha}\hat{u}_\lambda(0;\alpha)\right|_{\alpha=0^-}, \tag{8}$$

where $\hat{u}_\lambda(\cdot;\alpha)$ is the unique solution of (3) for $\alpha > 0$ and (4) for $\alpha < 0$ such that $u_\lambda(0;0) = -\lambda$. To try to prove this relation, one could perform some asymptotic expansions for $\alpha \sim 0$, in the spirit of the ones - carried out in a completely different context - of [11, Section 7], but this goes outside the scope of this work, and we leave it as an open question.

Stability of the solutions. As established in [14, Theorem 3.8], problems (3) and (4) do not admit any positive stable solutions, since $\lambda < 0 = \sigma_0$ (we use the same notation of [14] and denote by σ_0 the principal eigenvalue of the linearized problem at $u = 0$).

Here, we study the linear stability of the solutions of (5) following [9], i.e. by considering the parabolic counterpart of (5)

$$u_t - \frac{1}{(1-2\alpha)^2} u_{xx} = \lambda u + a_2(x) u^p, \qquad t > 0, \quad x \in (0,1) \tag{9}$$

taking time-dependent approximating functions

$$\overline{u}(t,x) = \sum_{j=1}^{n} u_j(t) \phi_j(x),$$

and obtaining, by reasoning as in Section 2, a system of ordinary differential equations for the unknown functions $u_j(t)$. This nonlinear system is then linearized around a steady state of (9), i.e. a solution of (5), and the dimension of the unstable manifold of such a steady state corresponds to the number of eigenvalues of the linearization having positive real part.

The observed stability patterns can be summarized as follows and are illustrated in Figure 3 (we use the notation of Theorem 1.1 and assume that $\lambda \in [\lambda_{n+1}, \lambda_n)$ for some $n \in \mathbb{N}$):

- for $\alpha = 0$, problem (5) has $2n + 1$ solutions. We denoted them by $u^{(i)}$, $i = 1, 2, \ldots, 2n + 1$, (we use superscripts in order not to confuse them with the coefficients of the Fourier expansions used above) so that

  $$u^{(1)}(0) < u^{(2)}(0) < \ldots < u^{(2n+1)}(0).$$

  For all $i = 1, 2, \ldots, n + 1$, the dimension of the unstable manifold of the solution $u^{(i)}$ coincides with the one of the solution $u^{(2n+2-i)}$ and equals i;

- on the branches of asymmetric solutions (represented in red in Figure 3), the dimension of the unstable manifold of the solution does not change;

- on the branches of symmetric solutions (represented in blue in Figure 3), the dimension of the unstable manifold changes by 1 as a bifurcation or a turning point is crossed, monotonically on each branch. Moreover, the unique solution of the problem for $\alpha \to -\infty$, which lies on the principal branch, has a 1-dimensional unstable manifold.

We point out that the observed stability patterns for problem (3) with $\alpha > 0$ are exactly the same.

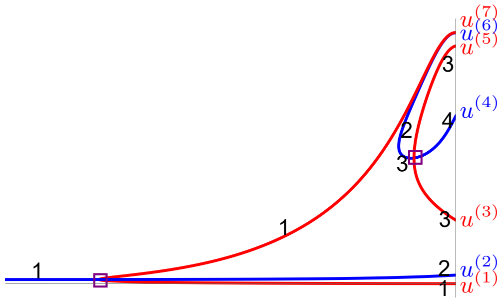

Figure 3: Dimensions of the unstable manifold for the solutions of problem (4) with $\lambda = -120 \in [\lambda_4, \lambda_3)$ (case (D) of Figure 2).

Profiles of the solutions. To conclude the presentation of the results of our numerical experiments, we plot in Figure 4 the profiles of the solutions in a case of high multiplicity, corresponding to the values of the parameters that give rise to the bifurcation diagram of Figure 2(D).

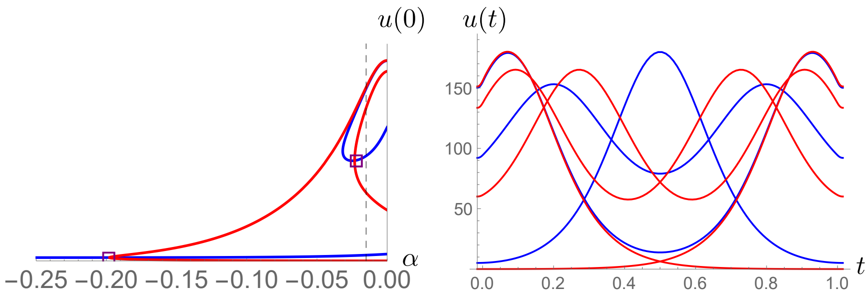

Figure 4: Bifurcation diagram in α for problem (4) with $\lambda = -120$ (left) and profiles of the seven solutions of the problem for $\alpha = -0.015$ (right). The level $\alpha = -0.015$ has been marked in the bifurcation diagram with a dashed line. Observe that the position of each solution of the right plot can be determined in the bifurcation diagram, at the level $\alpha = -0.015$, from its value at $t = 0$ and its symmetry.

Moreover, in order to make apparent that the behavior of the solutions is similar for positive and negative α's, we now present a description of the behavior of the solutions along each of the branches of the bifurcation diagram. Once again, we present the plots corresponding to the bifurcation diagram of Figure 2(D), since it is the most illustrative one.

Figure 5 shows the plots of some solutions on the upper blue branch in

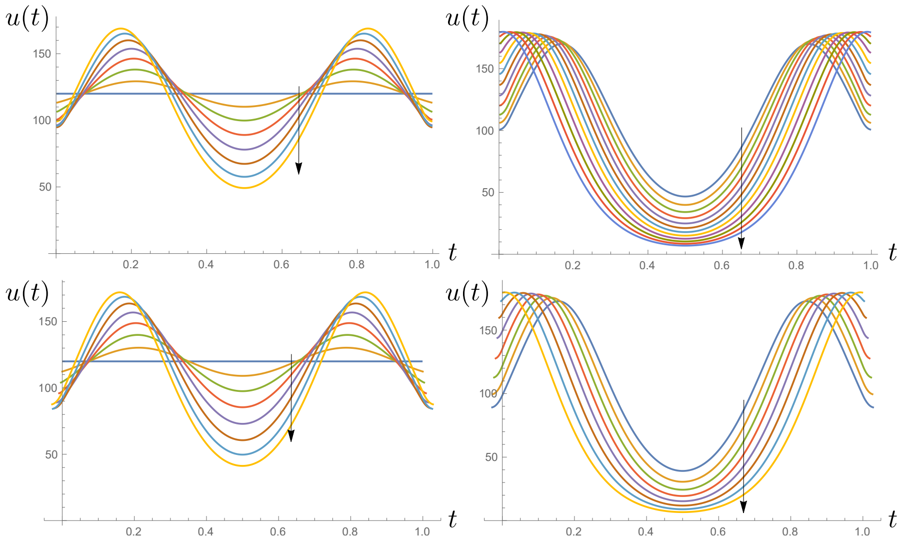

Figure 5: Plots of some solutions on the upper blue branch in Figure 2(D): upper row for $\alpha > 0$, lower row for $\alpha < 0$. The arrows indicate the direction in which the bifurcation diagram has been gone through, according to the description in the text.

Figure 2(D), that connects the point $(0, 120)$ to the point $(0, 179.797)$ in the bifurcation diagram. All the solution on this branch are symmetric. In the upper row of Figure 5 we represent the solutions for $\alpha > 0$: in the left plot we start from the constant solution corresponding to the point $(0, 120)$ in the bifurcation diagram and arrive to the turning point, which occurs at $(0.0263530, 110.425)$, while in the right plot the solutions go from the turning point to the upper point $(0, 179.797)$. In the lower row of the figure, instead, we represent the solutions for $\alpha < 0$ according to the same pattern: in the left plot from the point $(0, 120)$ to the turning point $(-0.0316540, 98.6296)$, and in the right plot from the turning point to the upper point $(0, 179.797)$. The arrows in the figure visually indicate the direction along which the solutions evolve on the bifurcation diagram, following the starting and the endpoint specified in the previous description.

Figure 6 shows the plots of some solutions on the lower blue branch in Figure 2(D), starting from $(0, 6.14466)$: the upper plot is for $\alpha > 0$ and the lower ones for $\alpha < 0$. All the solutions are, again, symmetric. In the lower left plot we represent the solutions after the change of variables that transforms the domain $(\alpha, 1-\alpha)$, which varies with α, in the fixed domain $(0, 1)$, while in the right plot we use the original domain of problem (4). This has been done

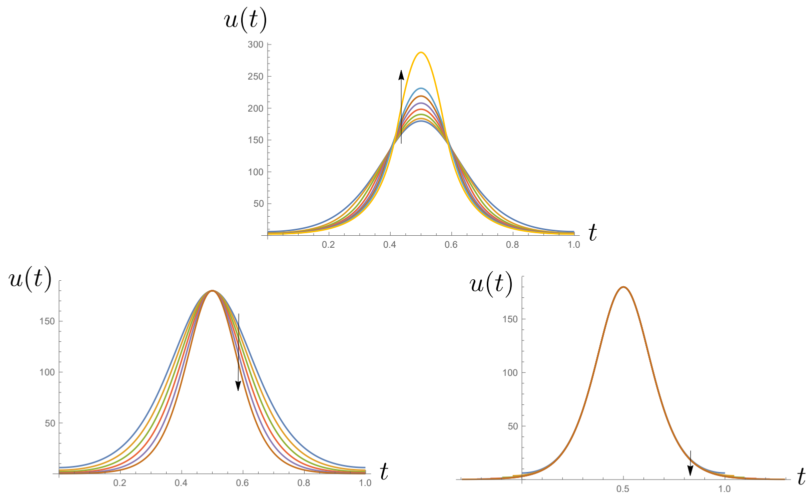

Figure 6: Plots of some solutions on the lower blue branch in Figure 2(D), starting from $(0, 6.14466)$. The upper plot is for $\alpha > 0$, the lower left plot for $\alpha < 0$, working with fixed domain $(0, 1)$, and the lower right plot for $\alpha > 0$ with the original domain of problem (4).

because the difference between the different solutions is amplified in the fixed domain.

In Figure 7 we represent the solutions that lie on the red branch of Figure 2(D) starting from $(0, 0.0251730)$ and arriving at $(0, 180.000)$: on the upper row the ones for $\alpha > 0$ and on the lower row the ones for $\alpha < 0$. All the solutions, apart from the ones on the bifurcation points $(-0.197821, 3.03203)$ and $(0.128325, 12.0364)$, are asymmetric. The left plots go from the starting point $(0, 0.0251730)$ on the bifurcation diagram up to the bifurcation point, while the ones on the right go from the bifurcation point to the ending point $(0, 180.000)$.

Finally, in Figure 8 we represent the solutions that lie on the other red branch of Figure 2(D), starting from $(0, 45.5856)$, arriving at $(0, 170.378)$ and following the same patterns used in Figure 7: top left for $\alpha > 0$ up to the bifurcation point $(0.0194360, 98.8542)$, top right $\alpha > 0$ starting from the bifurcation point, bottom left for $\alpha < 0$ up to the bifurcation point $(-0.0234266, 90.0263)$ and bottom right for $\alpha < 0$ starting from the bifurcation point.

4. Final remarks

To conclude this work, we observe that we may extend problem (3) also in other different ways than the one considered above. A first possibility consists in maintaining the condition on the derivatives at the fixed points $t = 0$ and

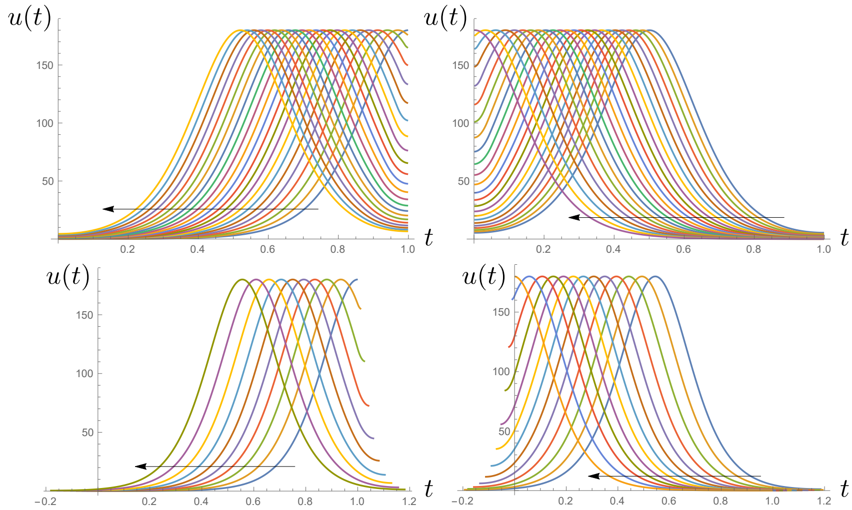

Figure 7: Plots of some solutions on the red branch in Figure 2(D) starting from $(0, 0.0251730)$. The upper plots are for $\alpha > 0$, the lower ones for $\alpha < 0$. The left plots represent, in the direction of the arrows, the solutions from the starting point to the bifurcation points, where they become symmetric, while in the right ones we start from the bifurcation points and arrive at $(0, 180.000)$.

$t = 1$, obtaining

$$\begin{cases} -u'' = \lambda u + a_1(t)u^p, & t \in (\alpha, 1-\alpha), \\ u'(0) = 0 = u'(1), \end{cases} \qquad \alpha < 0. \tag{10}$$

Doing so, we no longer have to deal with a boundary value problem, but with an "intermediate" value problem.

This problem is less interesting, since one readily observes that its solutions are in 1-1 correspondence with the solutions of the purely superlinear Neumann problem

$$\begin{cases} -u'' = \lambda u + bu^p, & t \in (0,1), \\ u'(0) = 0 = u'(1). \end{cases} \tag{11}$$

Indeed, one takes any solution $\overline{u}$ of (11) and extends it to $(\alpha, 0)$ and $(1, 1-\alpha)$ with the unique solutions of the initial value problems

$$\begin{cases} -u'' = \lambda u - cu^p, & t \in (\alpha, 0), \\ u(0) = \overline{u}(0), \\ u'(0) = 0, \end{cases} \qquad \begin{cases} -u'' = \lambda u - cu^p, & t \in (1, 1-\alpha), \\ u(1) = \overline{u}(1), \\ u'(1) = 0, \end{cases} \tag{12}$$

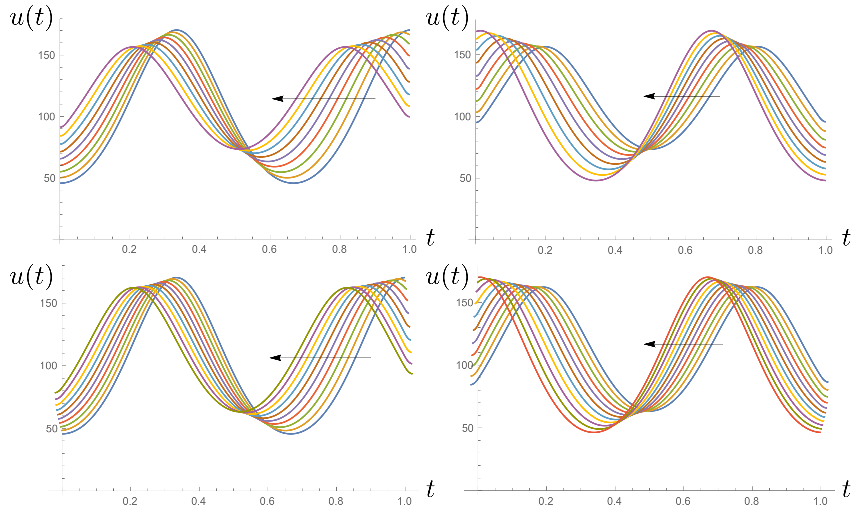

Figure 8: Plots of some solutions on the red branch in Figure 2(D) starting from $(0, 45.5856)$. The upper plots are for $\alpha > 0$, the lower ones for $\alpha < 0$. The left plots represent, in the direction of the arrows, the solutions from the starting point to the bifurcation points, where they become symmetric, while in the right ones we start from the bifurcation points and arrive at $(0, 170.378)$.

respectively. In this way, a solution of (10) is obtained. For this reason, we can say that the extension (10) makes the problem lose its indefinite nature. Nonetheless, we remark that the existence of global solutions for problems (12) depends on the values of α, since the solutions blow up in finite time, which has to be compared with α. By studying such a blow-up time, one can also construct the bifurcation diagrams in α of problem (10). This can be done with the elements developed in [33].

A second possible extension is the following one

$$\begin{cases} -u'' = \lambda u + a_3(t)u^p, & t \in (2\alpha, 1-2\alpha), \\ u'(2\alpha) = 0 = u'(1-2\alpha), \end{cases} \qquad \alpha < 0, \tag{13}$$

with

$$a_3(t) := \begin{cases} -c, & \text{for } t \in (2\alpha, \alpha) \cup (1-\alpha, 1-2\alpha), \\ b, & \text{for } t \in (\alpha, 1-\alpha). \end{cases}$$

This problem cannot be directly related to the one extensively studied above, since, here, both the size of the positive part of the weight and of the negative one vary with α.

To study this problem numerically, we first had to slightly modify the discretization performed in Section 3. Once that done, we have computed the corresponding bifurcation diagrams and the related profiles of the solutions. The results of our computations have been represented in Figures 9 and 10.

The left plot of Figure 9 shows the bifurcation diagram of problem (13) with $\lambda = -120$. By comparing it with the right part of the diagram in Figure 2(D), which corresponds to the same value of λ, we observe that the qualitative structure of the bifurcation diagrams is similar for the two extensions.

Nevertheless, a closer look at it (see the right plot of Figure 9) shows that some differences arise in the quantitative behavior of the bifurcation diagram. Indeed, some of the branches are not monotone, which does not occur for the corresponding ones in Figure 2(D). Moreover, in the left plot of Figure 9 we have marked only one bifurcation point, while in Figure 2(D) there were two of them. We think that this is uniquely due to the fact that we have not been able to perform the simulations for sufficiently negative values of α, since the solutions on the three branches are very close to each other apart from being very small. We conjecture, that the qualitative shape of the diagram for problem (13) is exactly as for problem (4) and that, if one is able to continue the simulations for more negative α's, the bifurcation point should arise. In order to do so, one may try to apply the treatment of narrow turning points developed in [26].

Finally, if we compare the plots of the solutions of problem (13), which are shown in Figure 10, with the corresponding ones of problem (4) (see Figures 5–8), we observe that they also follow the same qualitative patterns.

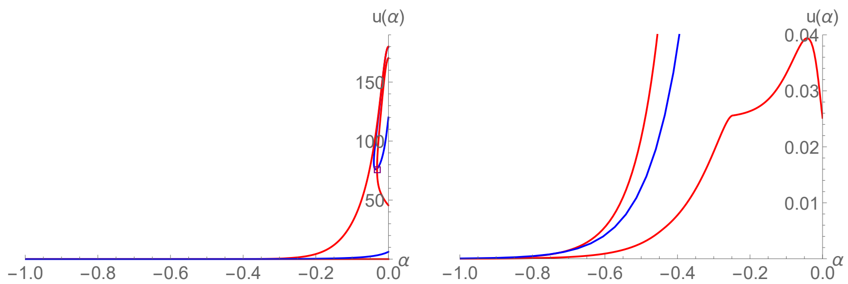

Figure 9: Bifurcation diagram of problem (13) for $\lambda = -120$ (left) and a zoom of its lower part (right).

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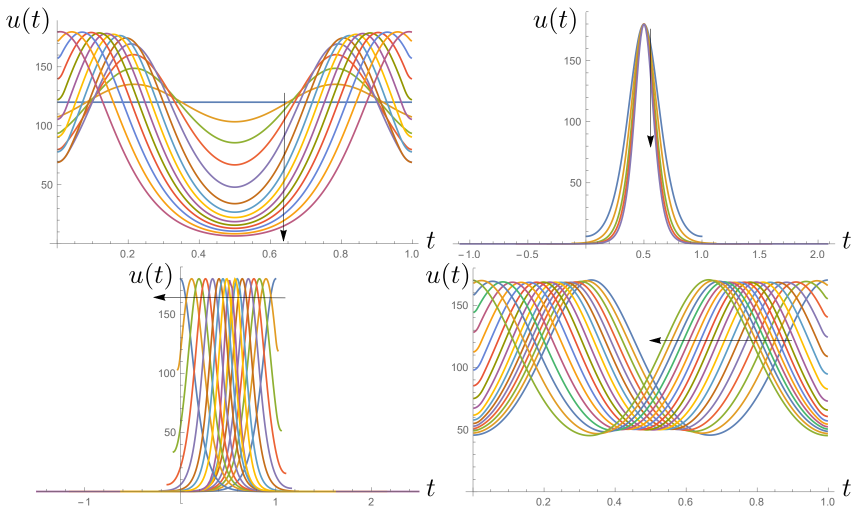

Figure 10: Plots of some solutions related to the left diagram of Figure 9: on the upper blue branch (top left), on the lower blue branch (top right), on the biggest red branch (bottom left) and on the smallest red branch (bottom right). The arrows indicate how the solutions evolve as the bifurcation diagrams are gone though, as described in the text of Section 3 and as marked in the corresponding plots of the solutions of problem (4).

improvement of this work: in particular, the suggestions related to the figures and to consider the extended problem (13).

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A collocation-spectral method to solve the bi-dimensional degenerate diffusive logistic equation with spatial heterogeneities in circular domains

Marcela Molina-Meyer and
Frank Richard Prieto Medina

"Cambia, todo cambia... Pero no cambia mi amor. " Marcela.

"Profesor Julián, usted me enseñó que la enseñanza es más que impartir conocimiento, es inspirar el cambio. Por esta razón, le dedico este artículo. Felicidades en su sexágesimo aniversario." Frank.

Abstract. *In this paper we simulate positive solutions, large solutions and metasolutions of the heterogeneous logistic equation in a disk and an annulus. The numerical methods introduced in this paper are extremely innovative because they make unnecessary determining any previous lifting and solving any decoupled system of ordinary differential equations. Moreover, they can be used to solve non-radially symmetric problems. The models are of a huge interest in Spatial Ecology because they enable us to analyse the effects of the spatial heterogeneity on the evolution of the terrestrial ecosystems. The large solutions and the metasolutions have been computed by the first time in this paper.*

Keywords: Heterogeneous logistic equation, unequal distribution of resources, spectral methods, collocation methods, numerical simulation of large solutions and metasolutions.
MS Classification 2010: 35J61, 35J70, 65N35, 65P30, 92D25.

1. Introduction

As a consequence of the unequal distribution of resources, populations distribute themselves in habitats of different size and quality. Algae, cyanobacteria and mountain pine beetles, see [1, 17, 29], grow and reproduce rapidly in some concrete habitats, having extraordinary and dramatic impact in some ecosystems, as changing food webs, decreasing biodiversity and altering ecosystem conditions. Inspired by Section 1.2 of López-Gómez [21], we propose the diffusive heterogeneous logistic equation to model the disproportionate growth of a population.

Definitely, modelling the heterogeneous distribution of populations in patches of the landscape with different population densities is crucial in conservation planning. Using mathematical models where the habitat is assumed to be spatially homogeneous becomes a tight restriction that leads too often to numerical results that do not match up with the collected field data. At the same time, modelling with reaction-diffusion systems with constant coefficients may also result in inaccurate predictions. The key issue is to implement variable coefficients in reaction-diffusion equations.

Moreover, it is incredibly important to assign correct values to the parameters, in this case, the proliferation rate λ that depends on the size of the patches. There are critical values of this parameter for which the species can survive and grow in each patch as we are going to see below in this paper.

In contrast to spatial structure population models, we use a simpler model that is more tractable and easier to interpret. We solve numerically for the first time the following master equation in Spatial Ecology in an habitat Ω to be considered circular, in the presence of spatial heterogeneity,

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } \Omega, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Δ is the Laplacian, $\Omega \in \{B_R((0,0)), A(R_0,R_1)\}$, with

$$B_R((x_0,y_0)) := \{(x,y) \in \mathbb{R}^2 : \|(x-x_0, y-y_0)\| < R\},$$

$$A(R_0,R_1) := \{(x,y) \in \mathbb{R}^2 : 0 < R_0 < \|(x,y)\| < R_1\},$$

and either

$$\mathcal{B}u = \mathcal{D}u = u - f$$

(general Dirichlet boundary conditions), whith $f \geq 0$ or

$$\mathcal{B}u = \frac{\partial u}{\partial \eta} = 0$$

(homogeneous Neumann boundary conditions), where η stands for the outward unit normal vector-field on $\partial\Omega$, $\lambda \in \mathbb{R}$ is a constant, f are the prescribed values of u along the boundary $\partial\Omega$, and $m \geq 0$, $m \neq 0$, is a function of class $\mathcal{C}^\mu(\overline{\Omega})$, for some $\mu \in (0,1]$, satisfying the following hypotheses:

(A) The set

$$\Omega_+ := \{x \in \Omega : m(x,y) > 0\}$$

is a subdomain of Ω with $\overline{\Omega}_+ \subset \Omega$, whose boundary, $\partial\Omega_+$, is of class $\mathcal{C}^3$, and the open set

$$\Omega_0 := \Omega \backslash \overline{\Omega}_+$$

consists of two components $\Omega_{0,i}$, $i \in \{1,2\}$, such that

$$\overline{\Omega}_{0,1} \cap \overline{\Omega}_{0,2} = \emptyset,$$

and

$$\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]. \tag{2}$$

Throughout this paper, for any given regular subdomain D of Ω, we denote by $\lambda_1[-\Delta, \mathcal{D}, D]$ the principal eigenvalue of $-\Delta$ in D under homogeneous Dirichlet boundary conditions. As a consequence of the *Maximum Principle*,

$$\lambda_1[-\Delta, \mathcal{D}, D_2] < \lambda_1[-\Delta, \mathcal{D}, D_1] \quad \text{if} \ \ D_1 \subsetneq D_2$$

(see [20] for any further required details). So, roughly speaking, (2) entails $\Omega_{0,1}$ to be larger than $\Omega_{0,2}$, but not exactly, as the principal eigenvalue also dependes on some hidden geometrical properties of the underlying domains. Figure 1 shows some of spatial configurations of $m(x,y)$ treated in this paper. Problem (1) is considered degenerate, always that $\Omega_0 \neq \emptyset$.

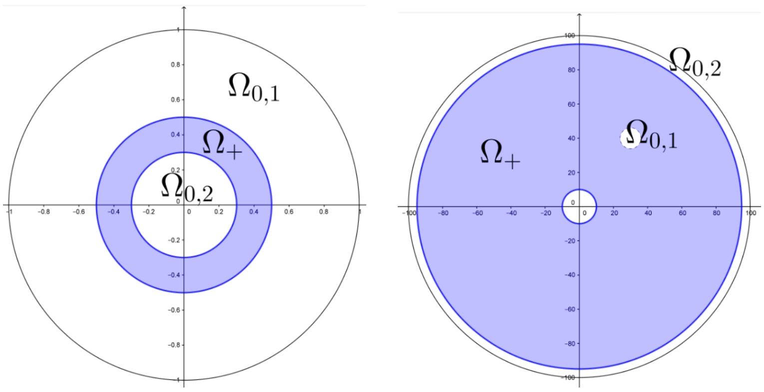

Figure 1: Spatial configuration of $m(x,y)$ in the disk $B_R((0,0))$ and the annulus $A(10,100)$.

This problem is used in Spatial Ecology to model the evolution of the distribution of a single species, u, randomly dispersed in the inhabiting area, Ω. In this context, it is very important to obtain the solutions of (1) because, at least in case $f = 0$, they provide us with the limiting profiles as $t \to \infty$ of the solutions of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \lambda u - m(x,y)u^2 & \text{in } \Omega \times (0,\infty), \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 > 0 & \text{in } \Omega. \end{cases} \tag{3}$$

From the point of view of the applications, allowing $m(x,y)$ to vanish on a subdomain of Ω, enables us to model the three different possible behaviors of the solution of (1), with $f=0$ and $\mathcal{B}=\mathcal{D}$, according to three distinct ranges of the parameter λ. Precisely, according to López-Gómez [18]:

- The inhabiting region Ω cannot support the species u if $\lambda \le \lambda_1[-\Delta,\mathcal{D},\Omega]$.
- The species u grows according to the Verhulst law if $\lambda_1[-\Delta,\mathcal{D},\Omega] < \lambda < \lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}]$.
- The species u grows according to the Malthus law in $\overline{\Omega}_{0,1}$, while it has a logistic behavior in $\Omega\backslash\overline{\Omega}_{0,1}$ if $\lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}] \le \lambda < \lambda_1[-\Delta,\mathcal{D},\Omega_{0,2}]$.
- The species u grows according to the Verhulst law in Ω_+, while it exhibits Malthusian growth in $\Omega\backslash\overline{\Omega}_+$ if $\lambda \ge \lambda_1[-\Delta,\mathcal{D},\Omega_{0,2}]$.

Therefore, as the previous results establish that, for the appropriate ranges of values of the parameter λ, the metasolutions provide us with the limiting profiles of the positive solutions of the evolution problem, from the point of view of the applications it is imperative to design efficient numerical algorithms to compute all the solutions and *metasolutions* of (1). A function $\mathcal{M}:\Omega\to[0,\infty]$ is said to be a metasolution of (1) supported in D, $D\in\{\Omega\backslash\overline{\Omega}_{0,1},\Omega_+\}$ if there exists a solution (*large solution*) L of

$$\begin{cases} -\Delta L = \lambda L - m(x,y)L^2 & \text{in } D,\\ L=0 & \text{on } \partial D\cap\partial\Omega, \end{cases}$$

satisfying

$$\lim_{\operatorname{dist}((x,y),\partial D\backslash\partial\Omega)\downarrow 0} L(x,y)=\infty,$$

for which

$$\mathcal{M}=\begin{cases} \infty & \text{in } \Omega\backslash D,\\ L & \text{in } D. \end{cases}$$

Computing the positive solutions, the large solutions and the metasolutions is the main goal of this paper, where, for the first time, the degenerate logistic equation in circular domains, without radial symmetries on the coefficient $m(x,y)$, has been solved numerically. Our numerical schemes and methods enjoy a great versatility, as it will become apparent later.

From the point of view of numerical analysis, our main contribution here consists in developing a number of, really necessary, algebraic manipulations on the differentiation matrix L_Δ of the Laplace operator in polar coordinates in order to impose either general inhomogeneous Dirichlet boundary conditions, or homogeneous Neumann ones, both in arbitrary disks and circular annuli.

From a theoretical point of view, in the problem with $\Omega = B_1((0,0))$ the most common pseudo-spectral method available is based on the expansion of u in terms of eigenfunctions of the Laplace operator and it can be expressed as

$$u(r,\theta) \approx \sum_{m=0}^{M}\sum_{n=1}^{N} a_{mn} J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta) + \sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta),$$

where N, M are positive integers, the J_m's denote the Bessel functions of first kind, λ_{mn} are the eigenvalues of $-\Delta$ in $B_1((0,0))$ under Dirichlet boundary conditions, and a_{mn}, b_{mn} are the (unknown) coefficients of the expansion, that are determined in this paper through the collocation points, (r_i, θ_j), which are the Chebyshev–Gauss–Lobatto points in the r-direction and the equidistant spaced points in the θ-direction. Unfortunately, in the case of the logistic equation, this paradigmatic classical scheme becomes unstable for

$$\lambda > \lambda[-\Delta, \mathcal{D}, B_1((0,0))] + \epsilon$$

if $\epsilon > 0$, being precisely this range of values of λ the one for which the large solutions and metasolutions of the model play a significant role in describing the dynamics of the evolution problem (3).

As a by-product, during the last several years a variety of methods have been developed to approximate the solutions of the Poisson equation in a disk. The monograph of Boyd and Fu Yu [3] collects a rather complete review of them comparing some of the main available schemes to solve the Poisson equation in a disk through the Zernike and the Logan–Shepp ridge polynomials, the Chebyshev–Fourier series, the cylindrical Robert functions, the Bessel–Fourier expansions, the square-to-disk conformal mapping, and the radial basis functions. But yet none of these schemes can be directly applied to compute the large solutions and the metasolutions of our problem. Very recently, the authors obtained in [24] the differentiation matrices of the Laplace equation in polar coordinates subjected to non homogeneous Robin boundary conditions and also, the differentiation matrix of the biharmonic equation subjected to non-homogeneous boundary conditions. More references concerning pseudospectral methods in the disk can also be found in [24]. The paradigmatic monographs of e.g., Gottlieb–Orszag [13], Fornberg [10], Boyd [2], Peyret [26], Canuto et al [6], and Shen–Tang–Wang [32] reveal the great importance of using spectral and pseudo-spectral methods to solve a huge variety of partial differential equations.

Although some sophisticated numerical calculations of radially symmetric classical solutions for (1), as well as some explosive solutions that do not belong to $\cup_{p=1}^{\infty} L^p_{loc}(\Omega)$, were carried out by Gómez-Reñasco and López-Gómez [12], this paper solves for the first time (1) without imposing any radial symmetry on the coefficients. Actually, the numerics of [12] where utterly one-dimensional using ODE's techniques.

Collocation-spectral methods are some of the most versatile methods for treating non-linear problems as well as simulating solutions of partial differential equations with variable coefficients, as it will be seen in this work. Furthermore, solving problem (1) in the unit disk is the first necessary step to solve the same problem on a more complicated geometry via a conformal mapping. But this analysis will be accomplished in an upcoming work and will appear elsewhere.

The organization of this paper is as follows. In Section 2 we apply the underlying collocation spectral method to simulate numerically the classical positive solutions, large solutions and metasolutions of the heterogeneous logistic equation in the unit disk and in a circular annulus for both the Dirichlet and the Neumann problems. In Appendixes A and B we obtain the discretization matrices of the Laplace operator in polar coordinates for homogeneous and inhomogeneous Dirichlet boundary conditions, as well as for homogeneous Neumann boundary conditions.

2. The Logistic Equation with Spatial Heterogeneity.

In this section, we apply the collocation spectral method developed in the Appendix to approximate the positive solutions of (1). As a consequence of the presence of the weight function $m(x, y)$ in front of the non-linear term, the richness of the set of positive solutions of (1) increases extraordinarily. Actually the model can exhibit classical positive solution, large positive solutions and metasolutions of (1). Subsequently, we will compute all these types of solutions.

It should be emphasized that, without a deep previous knowledge of the analytical results of López-Gómez [18] and [21], the numerical resolution of (1) would be an extremely hard task, by the lack of a priori bounds in L^∞ for the gradients of all these classical and non-classical solutions, which might become infinity even in some open sub-domains of the underlying domain. When necessary, we will refer to [18] for the available theoretical results about (1).

2.1. Classical solutions and metasolutions in $B_1((0,0))$ under Dirichlet boundary conditions

In this section we consider the problem (1) with homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } B_1((0,0)), \\ u = 0 & \text{on } \partial B_1((0,0)), \end{cases} \tag{4}$$

where $m : B_1((0,0)) \to [0,\infty)$ is given by:

$$m(x,y) = \begin{cases} -\left(\sqrt{x^2+y^2}-0.5\right)\left(\sqrt{x^2+y^2}-0.3\right) & \text{if } (x,y) \in A(0.3,0.5), \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Figure 2 shows a plot of $m(x,y)$ for this choice.

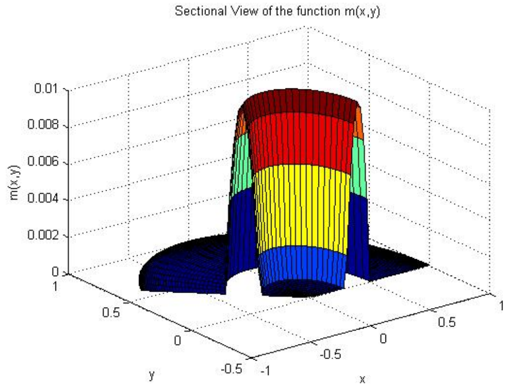

Figure 2: Plot of $m(x,y)$ for the choice (5).

In polar coordinates, the problem (4) becomes into

$$\begin{cases} -\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta)u^2 & \text{in } (0,1)\times[0,2\pi) \\ u(1,\theta) = 0 & \text{on } [0,2\pi), \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [0,1]\times(-\infty,\infty), \end{cases} \quad (6)$$

where $m : [0,1]\times[0,2\pi) \to [0,\infty)$ is given by

$$m(r,\theta) = \begin{cases} -(r-0.5)(r-0.3) & \text{if } (r,\theta) \in (0.3,0.5)\times[0,2\pi), \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Naturally, this model fits into the abstract setting of this paper with

$$\Omega_+ = A(0.3,0.5), \qquad \Omega_0 = B_{0.3}((0,0)) \cup A(0.5,1).$$

Table 1 collects the theoretical and numerical values of the principal eigenvalue λ_1 of $-\Delta$ in the most relevant subdomains of Ω from the point of view of describing the dynamics of (3). Namely, Ω and each of the two components of

Ω_0. We call *theoretical* λ_1 the value of the approximation obtained by using Bessel function and *computed* λ_1 the value calculated through the Inverse Power Method applied to the differentiation matrices approximating the Laplace operator with $N+1$ nodes in the r-direction and N_θ nodes in the θ-direction.

Subdomain	Theoretical λ_1	Computed λ_1 N=17, $N_\theta = 40$	Computed λ_1 N=42, $N_\theta = 40$
$B_1((0,0))$	5.783185962946	5.783185962959	5.783185962956
A(0.5,1)	39.013288499083	39.013288499012	39.013288498923
$B_{0.3}((0,0))$	64.257621810519	64.257621810502	64.257621810334

Table 1: The principal eigenvalues in the relevant subdomains.

Thanks to Table 1, if we take

$$\Omega_{0,1} = A(0.5,1), \qquad \Omega_{0,2} = B_{0.3}((0,0)),$$

then $\lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}] < \lambda_1[-\Delta,\mathcal{D},\Omega_{0,2}]$. The existence of classical positive solutions of (1) is guaranteed from the following theorem borrowed from [18]. As all the remaining results going back to [18] and [12], it is collected here by the sake of completeness.

THEOREM 2.1. *Suppose $m(x,y)$ satisfies* **(A)**. *Then,*

1. *The problem (4) possesses a classical positive solution if, and only if,*

$$\lambda_1[-\Delta,\mathcal{D},\Omega] < \lambda < \lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}]. \tag{8}$$

Moreover, it is unique if it exists.

2. *Suppose (8) and let θ_λ denote the unique classical positive solution of (4). Then*

$$\lim_{\lambda\downarrow\lambda_1[-\Delta,\mathcal{D},\Omega]} ||\theta_\lambda||_{L^\infty(\Omega)} = 0, \tag{9}$$

and

$$\lim_{\lambda\uparrow\lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}]} ||\theta_\lambda||_{L^\infty(\Omega)} = \infty \tag{10}$$

uniformly in $(\Omega_{0,1}\cup\Omega_{0,2})\backslash\partial\Omega$.

3. *The mapping $\lambda\to\theta_\lambda$ is point-wise increasing and, if we regard to it as a mapping from $(\lambda_1[-\Delta,\mathcal{D},\Omega],\lambda_1[-\Delta,\mathcal{D},\Omega_{0,1}])$ into $C^{1,\nu}(\Omega)$, $0<\nu<1$, then it is differentiable and $\frac{\partial\theta_\lambda}{\partial\lambda}\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ for all $p>1$.*

In order to compute some distinguished solutions along the global curve of classical positive solutions of (4) we apply the collocation spectral method already described in Appendix A to obtain a nonlinear system of equations that we solve using the Newton method. Succeeding in the choice of an appropriate initial data for the Newton method is utterly based on a good knowledge of the available analytical results.

Figure 3 shows some of the classical positive solutions that we have computed with our method. The value $\lambda_1[-\Delta, \mathcal{D}, B_1((0,0))] = 5.783185$ is the unique value of λ for which bifurcation to positive solutions from $u = 0$ occurs. It should be noted how these solutions grow in $\overline{\Omega_{0,1}}$, while, in strong apparent contrast, they stabilize in $B_1((0,0))\backslash\overline{\Omega_{0,1}}$, as λ increases.

As λ moves up from $\lambda_1[-\Delta, \mathcal{D}, B_1((0,0))] = 5.783185$, the principal eigenvalue of the linearization around the positive solutions grows from zero up to reach its maximum value critical λ, where it becomes decreasing for any further value λ up to approach the critical value where the bifurcation from infinity takes place, where it converges to zero. As this feature, was not previously observed in the specialized literature, we conjecture that

$$\lim_{\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]} \lambda_1[-\Delta + 2\, m(x,y)\, \theta_\lambda - \lambda, \mathcal{D}, \Omega] = 0.$$

Table 2 collects some representative values of λ together with the L^∞-norms of the corresponding positive solutions and the principal eigenvalues of their linearizations (p.e.l.).

Value of λ	p.e.l.	$\|\|u\|\|_\infty$
30.0	1.9279	2.3520e+005
32.5	1.0511	1.3294e+006
33.6	0.7035	5.0819e+006
34.0	0.4480	1.0785e+007

Table 2: The principal eigenvalues of the linearizations.

Now, we will show the results of our numerical experiments for computing the metasolutions of (4). First, we need to introduce some concepts going back to [12].

Definition 2.2. *Consider the problem*

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } D, \\ u = \infty & \text{on } \partial D, \end{cases} \tag{11}$$

where D is un proper subdomain of Ω. A function $u \in \mathcal{C}^{2+\mu}(D)$ is said to be a large (or explosive) solution *of (11) if it satisfies the differential equation in*

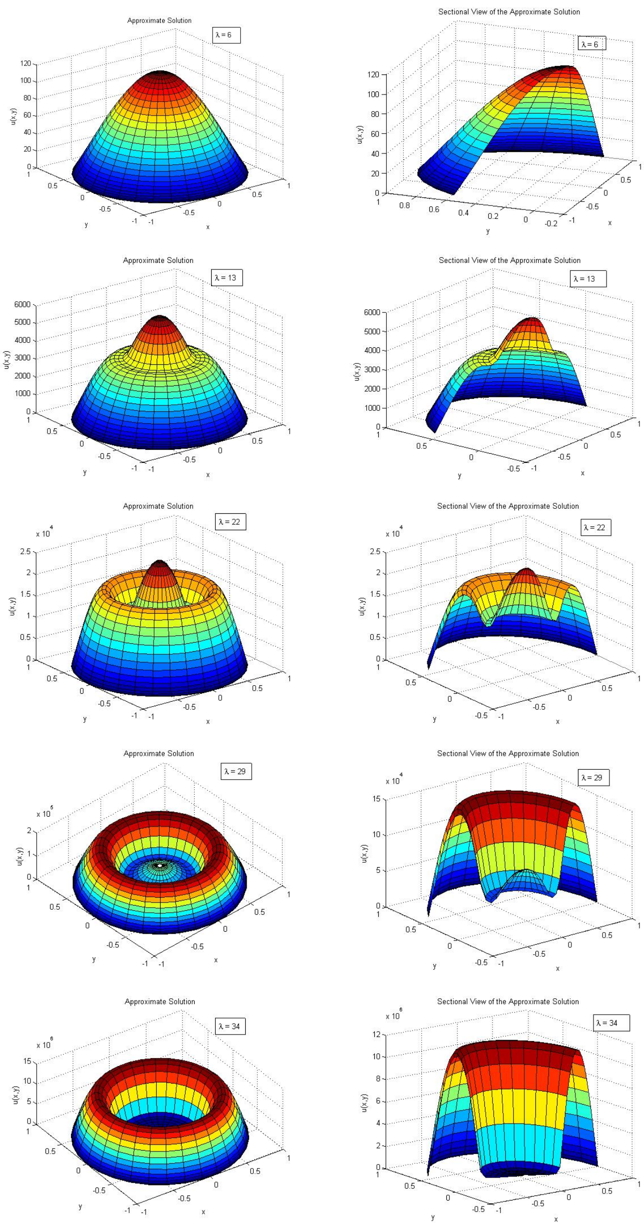

Figure 3: Plots of the classical positive solutions of (4) for $\lambda \in \{6, 13, 22, 29, 34\}$.

D, $u = 0$ on $\partial D \cap \partial\Omega$, and

$$\lim_{\mathrm{dist}((x,y),\partial D\setminus\partial\Omega)\downarrow 0} u(x,y) = \infty.$$

DEFINITION 2.3. *Consider (11) with $D \in \{\Omega\setminus\overline{\Omega_{0,1}}, \Omega_+\}$. Then, a function $\mathcal{M} : \Omega \to [0,\infty]$ is said to be a* metasolution *of (11) supported in D if there exists a large solution L of (11) in D for which*

$$\mathcal{M} = \begin{cases} \infty & \textit{in } \Omega\setminus D, \\ L & \textit{in } D. \end{cases} \tag{12}$$

According to López-Gómez [18] and [21] , it is known that:

- If $\lambda_1[-\Delta, \mathcal{D}, \Omega] \le \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$, the problem (4) admits a classical positive solution.
- If $\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] \le \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, the problem (4) admits a metasolution supported in $\Omega\setminus\overline{\Omega_{0,1}}$.
- If $\lambda \ge \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, the problem (4) admits a metasolution supported in Ω_+.

Moreover, the minimal metasolutions in these ranges describe the limiting profiles of all positive solutions of the evolution problem (3), when the initial data u_0 is a subsolution of problem (1), see Theorem 5.2 in [21]. So, the importance of computing them from the point of view of the design, or restoration, of spatially heterogeneous ecosystems. According to the previous analytical results, (1) possesses a metasolution supported in $\Omega\setminus\overline{\Omega_{0,1}}$ if

$$\lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}] \backsimeq 39.013288 \le \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}] \backsimeq 64.257622. \tag{13}$$

To compute this metasolution, we first computed the large solution u of

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } B_{0.5}((0,0)), \\ u = \infty & \text{on } \partial B_{0.5}((0,0)). \end{cases} \tag{14}$$

The most natural strategy to approximate the large solution of (14) is to compute the unique positive solution of

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } B_{0.5}((0,0)), \\ u = \beta & \text{on } \partial B_{0.5}((0,0)), \end{cases} \tag{15}$$

for sufficiently large β. Figure 4 shows some numerical solutions of (15) with $\beta = 3 * 10^5$. Our numerics reveal that the metasolutions supported in $\Omega\setminus\overline{\Omega_{0,1}}$

are point-wise increasing in $\Omega\backslash\overline{\Omega_{0,1}}$ with respect to λ. They grow at a faster rate in $A(0.5, 1)$, where $m = 0$, than in $B_{0.5}((0, 0))$, where $m > 0$. Each of these metasolutions takes the value β on $\partial B_{0.5}((0, 0))$. As $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}]$, the corresponding metasolution exhibits a complete blow-up in $B_{0.3}((0, 0))$, while it stabilizes in $A(0.3, 0.5)$. Finally, to obtain the metasolution supported in Ω_+ for $\lambda \geq \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,2}])$, we computed the large solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(0.3, 0.5), \\ u = \infty & \text{on } \partial A(0.3, 0.5), \end{cases} \tag{16}$$

approximating it by the unique solution of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(0.3, 0.5), \\ u = \beta & \text{on } \partial A(0.3, 0.5), \end{cases} \tag{17}$$

for β sufficiently large. Figure 5 shows some plots of these metasolutions.

Since the problems (14) and (17) are radially symmetric, the positive large solution of each of these problems is unique, by, e.g., Theorem 7.1 of J. López-Gómez [21] (see also [19]). Moreover, due to Theorem 4.7 of [21], we already know that the positive solutions of (14) and (17) approximate these unique large solutions as $\beta \uparrow \infty$. For uniqueness results in more general settings, the reader is sent to the more recent paper of J. López-Gómez and L. Maire [22]. Figure 6 shows a zoom of the profiles of the positive solutions of (15) for $\lambda = 40$, as well as the profiles of the positive solutions of (17) for $\lambda = 70$ and $\beta \in \{3 * 10^5, 4 * 10^5, 5 * 10^5\}$.

2.2. Classical positive solutions in $A(R_0, R_1)$ under Dirichlet boundary conditions

In this subsection, we compute numerically some classical positive solutions of

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(10, 100), \\ u = 0 & \text{on } \partial A(10, 100), \end{cases} \tag{18}$$

where $m : A(10, 100) \to [0, \infty)$ is defined by:

$$m(x, y) = \begin{cases} 0 & \text{if } (x, y) \in A(95, 100), \\ 10^{-11} p(x, y)(x^2 + y^2 - 10^2)(95^2 - x^2 - y^2) & \\ \qquad\qquad \text{if } (x, y) \in A(10, 95)\backslash B_6((30, 40)), & \\ 0 & \text{if } (x, y) \in B_6((30, 40)). \end{cases} \tag{19}$$

where $p(x, y) = (x - 30)^2 + (y - 40)^2 - 36$. Figure 7 shows a plot of $m(x, y)$ defined in (19).

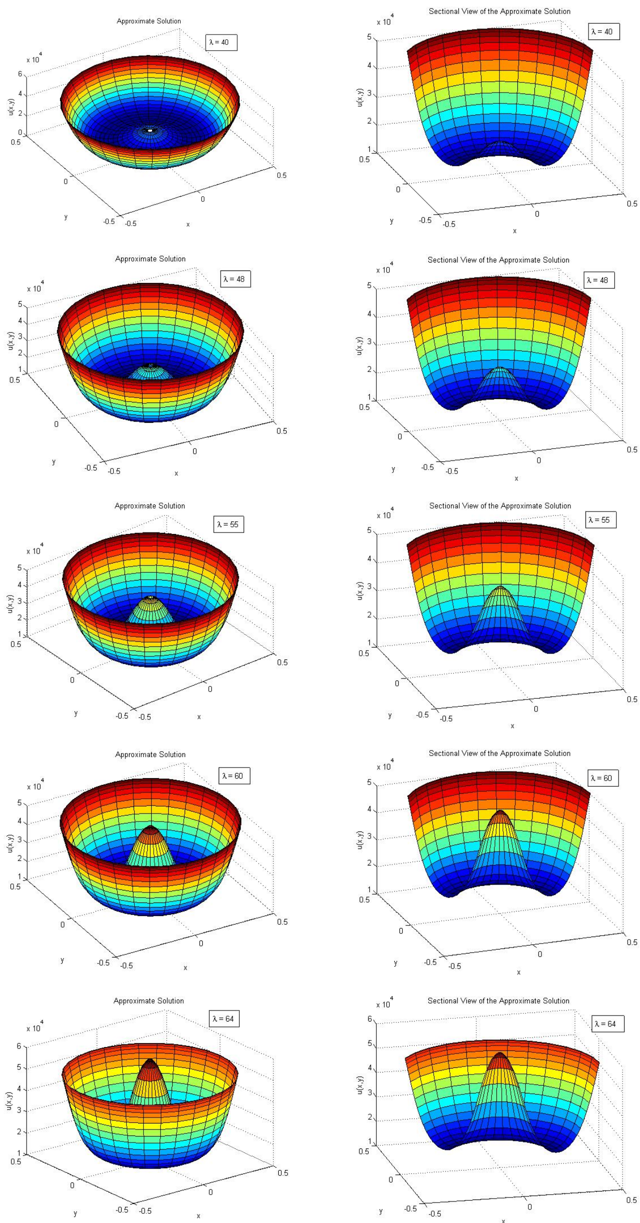

Figure 4: Plots of the solutions of (15) in $B_1(0)\backslash\overline{A(0.5,1)}$ for $\lambda \in \{40, 48, 55, 60, 64\}$.

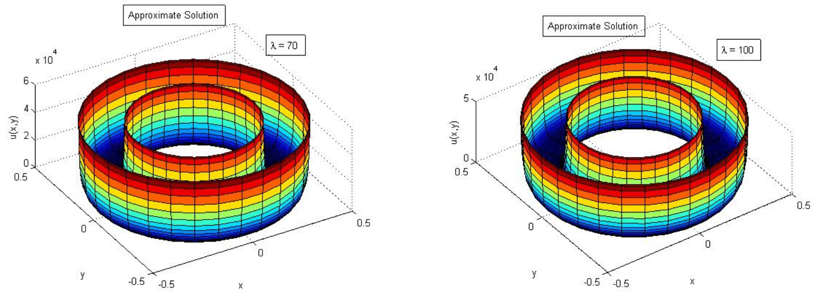

Figure 5: Plots of the solutions of (17) in $A(0.3, 0.5)$ for $\lambda \in \{70, 100\}$.

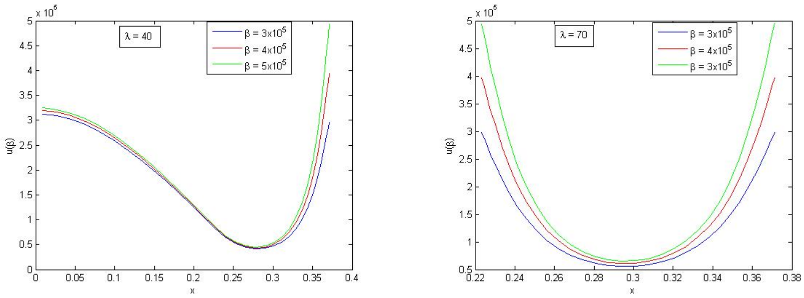

Figure 6: Profiles approximating the large solution of (14) for $\lambda = 40$ and, of (16) for $\lambda = 70$.

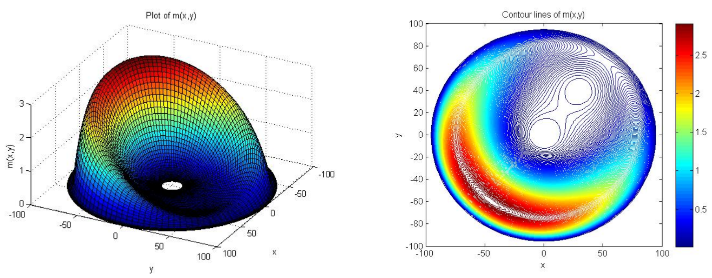

Figure 7: Plots of $m(x, y)$ and its contour lines for the choice (19).

Consequently, the problem is far from being radially symmetric. The existence of classical positive solutions of (18) is guaranteed by Theorem 2.1. Problem (18) can be rewritten as:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta)u^2 & \text{in } (10,100)\times[0,2\pi) \\ u(10,\theta) = 0 & \text{on } [0,2\pi) \\ u(100,\theta) = 0 & \text{on } [0,2\pi) \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [10,100]\times(-\infty,\infty). \end{cases} \tag{20}$$

In Table 3 we are giving the theoretical and numerical values of the principal eigenvalue of $-\Delta$ in some of the relevant subdomains of Ω. The theoretical value is calculated from the estimate 2.4048 for the first zero of the Bessel function J_0.

Subdomain	Theoretical λ_1	Computed λ_1
A(10,100)	0.001097	0.001098
$B_6((30,40))$	0.160640	0.160644
A(95,100)	0.394757	0.394757

Table 3: The principal eigenvalues in some relevant subdomains.

The corresponding model fits into the general setting of this paper with

$$\Omega_+ = A(10,95)\backslash\overline{B_6((30,40))}, \quad \Omega_{0,1} = B_6((30,40)) \quad \Omega_{0,2} = A(95,100).$$

Figure 8 shows some of the classical positive solutions that we have computed. These solutions grow in $\overline{B_6((30,40))}$, while they stabilize in the set $A(10,100)\backslash\overline{B_6((30,40))}$, as λ increases. As λ increases from 0.001098 approximating the principal eigenvalue in $B_6((30,40))$, which is given by 0.160644, the solutions blow up in $B_6((30,40))$ as $\lambda \uparrow 0.160644$.

2.3. Classical positive solutions in $B_1((0,0))$ under Neumann conditions

In this subsection we compute the classical positive solution of

$$\begin{cases} -\Delta u = \lambda u - m(x,y)u^2 & \text{in } B_1((0,0)), \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial B_1((0,0)), \end{cases} \tag{21}$$

using the collocation spectral method described in the Appendixes. Here, η stands for the outward unit normal along $\partial B_1((0,0))$. So, $\eta(x,y) = (x,y)$

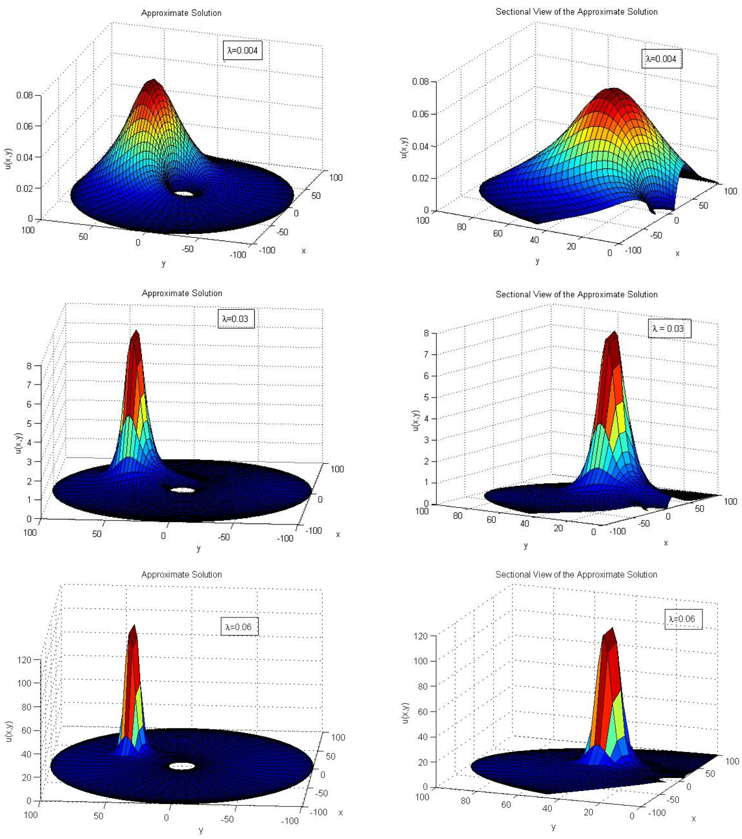

Figure 8: Plots of the classical solutions of (18) in $A(10, 100)$ for $\lambda \in \{0.004, 0.03, 0.06\}$.

for all $(x, y) \in \partial B_1((0,0))$. The existence of solutions of (21) for any domain $\Omega \subset \mathbb{R}^2$ is guaranteed by the next theorem going back to Ouyang [25]. The case of general boundary operators on $\partial\Omega$ was first considered by J. M. Fraile et al. [11], where the open set Ω_0 consists of a single component with $\overline{\Omega}_0 \subset \Omega$. Nevertheless, in this paper we will investigate numerically some cases where Ω_0 consists of two disjoint components. Our numerical experiments show that the positive classical solutions of (21) tend to infinity in $\Omega_{0,1}$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$.

THEOREM 2.4. *Assume that $m \geq 0$ ($\not\equiv 0$) is a smooth function in Ω.*

1. *If $\Omega_0 = \emptyset$, then for every $\lambda > 0$ there exists a unique solution $u(\lambda)$ of problem (21).*
2. *If $\Omega_0 \neq \emptyset$, then for any $\lambda \in (0, \lambda_1[-\Delta, \mathcal{D}, \Omega_0])$ there exists a unique solution of (21), whereas (21) cannot admit a positive solution if $\lambda \geq \lambda_1[-\Delta, \mathcal{D}, \Omega_0]$.*

Moreover

$$\lim_{\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_0]} ||u(\lambda)||_{L^2(\Omega)} = \infty. \tag{22}$$

Note that problem (21) can be written as:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta)u^2 & \text{in } (0,1) \times [0, 2\pi), \\ \dfrac{\partial u}{\partial r}(1, \theta) = 0 & \text{on } [0, 2\pi), \\ u(r,\theta) = u(r, \theta + \pi) & \text{in } [0,1] \times (-\infty, \infty). \end{cases} \tag{23}$$

In order to show the excellent accuracy of the numerical method, we are taking $m \equiv 1$ in the first simulation. In this case, the corresponding model fits into the abstract setting of Theorem 2.4, with $\Omega_+ = B_1((0,0))$ and $\Omega_0 = \emptyset$. Thus, the problem (21) has a unique positive solution for all $\lambda > 0$. In this case, we know that the solution of (21) is $u \equiv \lambda$. Figure 9 shows the plots of some classical positive solutions computed through the spectral collocation method introduced in this paper, and the distribution of the error $E(x, y) = |u(x, y) - \lambda|$ in $B_1((0,0))$ for $\lambda = 8$ and $\lambda = 100$. Note that the maximum value of the error is of order 10^{-13} for $\lambda = 8$ and 10^{-12} for $\lambda = 100$.

Finally, for the last simulation, we take m as in (5). It should be remember that for this choice the model fits into the abstract setting of this paper with $\Omega_+ = A(0.3, 0.5)$, $\Omega_0 = B_{0.3}((0,0)) \cup A(0.5, 1)$, $\Omega_{0,1} = A(0.5, 1)$ and $\Omega_{0,2} = B_{0.3}((0,0)$. In this case, combining the abstract theory of Fraile et al. [11] with López-Gómez [21, Ch. 4], it becomes apparent that (21) has a classical positive solution if, and only if, $0 < \lambda < \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$. Actually, this is a rather direct consequence of Daners and López-Gómez [7, Th. 1.1].

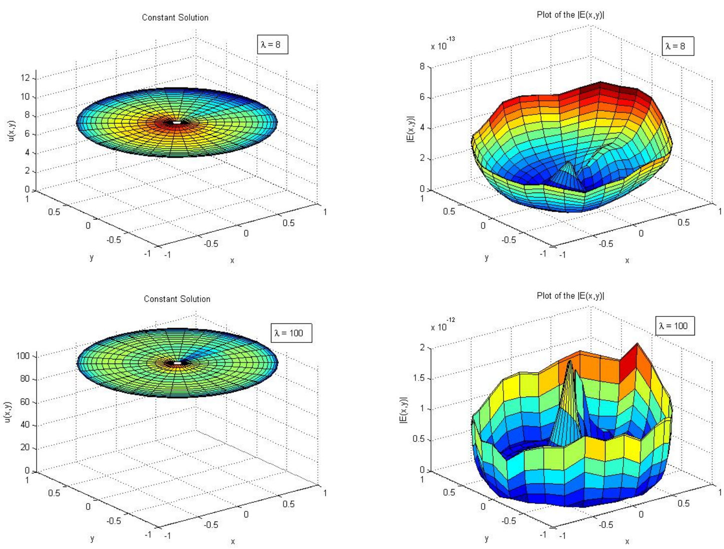

Figure 9: Numerical solution of (21) for $\lambda \in \{8, 100\}$ with $m \equiv 1$ and the corresponding errors $E(x, y)$.

Figure 10 shows the plots of the numerical solutions of (21) for $\lambda \in \{0.0003, 5\}$. Although it is well known that the solutions are point-wise increasing in Ω with respect to λ, our experiments suggest that they grow at a faster rate on $\Omega_{0,1}$. Actually, these solutions grow up to infinity on $\bar{\Omega}_{0,1}$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$.

2.4. Case Neumann II : Numerical Computation of Classical Positive Solutions in the circular annulus $\Omega = A(R_0, R_1)$

Firstly, we consider the problem

$$\begin{cases} -\Delta u = \lambda u - m(x, y)u^2 & \text{in } A(4, 10), \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } \partial A(4, 10), \end{cases} \tag{24}$$

where η is the unit outward vector on $\partial A(4, 10)$ and $m \equiv 1$. The existence of solutions of (24) is guaranteed by Theorem 2.4. The problem (24) in polar

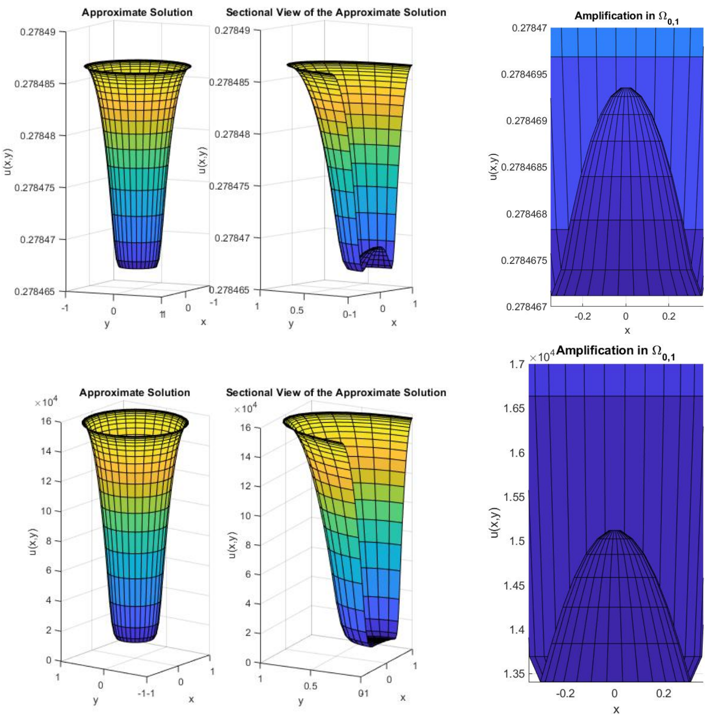

Figure 10: Plots of the classical solutions of (23) with m as in (5) for $\lambda \in \{0.0003, 5\}$.

coordinates can be rewritten as:

$$\begin{cases} -\dfrac{\partial^2 u}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial u}{\partial r} - \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = \lambda u - m(r,\theta)u^2 & \text{in } (4,10)\times[0,2\pi), \\ -\dfrac{\partial u}{\partial r}(4,\theta) = 0 & \text{on } [0,2\pi), \\ \dfrac{\partial u}{\partial r}(10,\theta) = 0 & \text{on } [0,2\pi), \\ u(r,\theta) = u(r,\theta+\pi) & \text{in } [4,10]\times(-\infty,\infty). \end{cases} \tag{25}$$

Since $\Omega_+ = A(4,10)$ and $\Omega_0 = \emptyset$, by the Theorem 2.4, there exists a classical positive solution for all $\lambda > 0$. Naturally, as in the previous section, the solutions of (24) must be $u \equiv \lambda$. Figure 11 shows some of the numerical solutions that we computed.

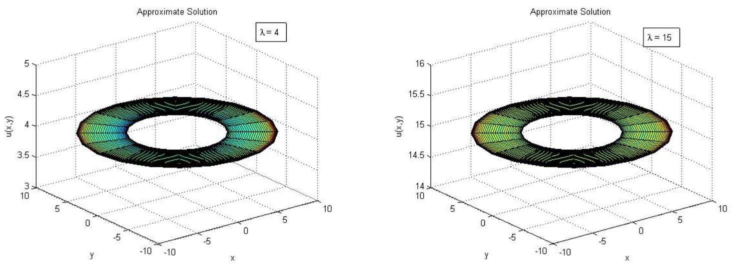

Figure 11: Numerical solution of (24) in $A(4,10)$ for $\lambda \in \{4,15\}$ with $m \equiv 1$.

To end this paper, we consider (24) in $A(4,10)$ with two different coefficients $m : A(4,10) \to [0,\infty)$ defined by

$$m(x,y) = \begin{cases} (\sqrt{x^2+y^2} - \gamma)(9 - (\sqrt{x^2+y^2}) & \text{if } (x,y) \in A(\gamma,9), \\ 0 & \text{if } (x,y) \in A(4,\gamma)\cup A(9,10). \end{cases} \tag{26}$$

where $\gamma \in \{4.9, 5.5\}$. Figure 12 shows a plot of $m(x,y)$ for $\gamma = 5.5$. For this choice, $\Omega_+ = A(\gamma,9)$ and $\Omega_0 = A(4,\gamma)\cup A(9,10)$. Table 4 provides the numerical values of the principal eigenvalues of $-\Delta$ in some relevant subdomains of Ω. These values have been computed applying the Inverse Power Method to the discretization matrix of the Laplace operator, taking 85 nodes in the r-direction and 60 nodes in the θ-direction.

Although it is possible to give a theoretical value for the underlying principal eigenvalues as in the tables above, in this occasion it is much faster and versatile to compute them through the Inverse Power Method applied to the

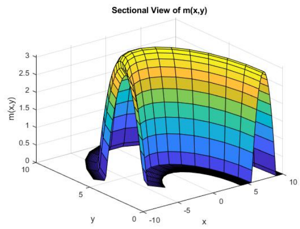

Figure 12: Plot of $m(x, y)$ for the choice (26) with $\gamma = 5.5$.

corresponding differentiation matrix. Actually, our method might be far more accurate than using the available tables.

Subdomain	A(4,10)	A(4,5.5)	A(9,10)	A(4,4.9)
Computed λ_1	0.268642	4.375300	9.866831	12.172021

Table 4: The principal eigenvalue of some relevant subdomains.

Thanks to the values given in Table 4, we have that $\Omega_{0,1} = A(4, 5.5)$ and $\Omega_{0,2} = A(9, 10)$ if $\gamma = 5.5$, since

$$\lambda_1[-\Delta, \mathcal{D}, A(4, 5.5)] < \lambda_1[-\Delta, \mathcal{D}, A(9, 10)],$$

whereas $\Omega_{0,1} = A(9, 10)$ and $\Omega_{0,2} = A(4, 4.9)$ if $\gamma = 4.9$, because in such case

$$\lambda_1[-\Delta, \mathcal{D}, A(9, 10)] < \lambda_1[-\Delta, \mathcal{D}, A(4, 4.9)].$$

So, the relative position of these principal eigenvalues have inter-exchanged.

Figures 13 and 14 show the plots of some positive solutions of (24) with $m(x, y)$ defined by (26); these plots were computed for $\gamma = 5.5$ and $\gamma = 4.9$, respectively. In both cases, as predicted by the theory, the solutions are pointwise increasing with respect to λ. However, these solutions grow faster in $\bar{A}(4, 5.5)$ than in $\bar{A}(9, 10)$ if $\gamma = 5.5$, while they grow faster in $\bar{A}(9, 10)$ than in $\bar{A}(4, 5.5)$ if $\gamma = 4.9$, as expected from the existing theory.

Actually, these solutions grow to infinity in $\bar{A}(4, 5.5)$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$ if $\gamma = 5.5$, stabilizing to some fixed profile in $\bar{A}(9, 10)$, whereas they grow-up to infinity in $\bar{A}(9, 10)$ as $\lambda \uparrow \lambda_1[-\Delta, \mathcal{D}, \Omega_{0,1}]$ if $\gamma = 4.9$, staying bounded in its complement.

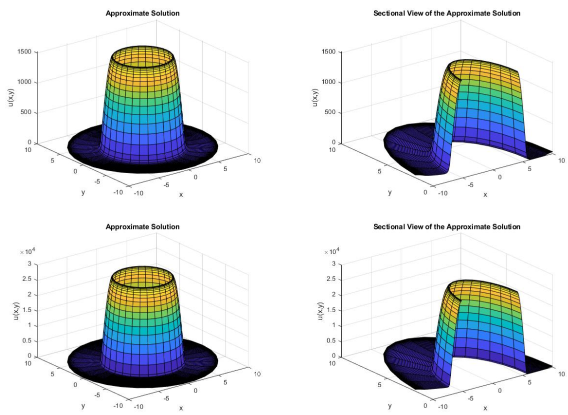

Figure 13: Numerical solution of (24) in $A(4, 10)$ for $\lambda \in \{1, 1.1\}$ with $m(x, y)$ given by (26) with $\gamma = 5.5$.

2.5. Final remarks

Table 5 collects the number of collocation points used in the simulations presented in this paper. When the domain is a disk, it equals $\left(\frac{N_r+1}{2}\right) N_\theta$, while it is given by $(N_r + 1)N_\theta$ if, instead, it is an annulus.

As illustrated in Table 5, for obtaining Figure 8, in order to capture the fastest growth of the solution in $\Omega_{0,1} = B_6((30, 40))$, we had to increase the number of collocations points up to 2300.

In the simulations sketched by Figure 5, we have taken more collocations points than in the simulations of Figures 3-4 to approximate the growth of the solution on $\partial A(0.3, 0.5)$. Finally, note that, in order to get Figures 13 and 14, where $\Omega_{0,1} \neq \emptyset$, we have used more collocation points than in the simulations necessary to get Figure 11, where $\Omega_0 = \emptyset$.

Appendix

A. Construction of the differentiation matrices in the unit disk

The main goal of this appendix is to discretize the Laplace operator in polar coordinates in the unit disk $B_1((0, 0))$ in order to impose inhomogeneous Dirichlet and homogeneous Neumann conditions. First, we will discretize the disk spectrally by taking a periodic Fourier grid in θ and a nonperiodic Chebyshev grid in r. Note that, when performing the radial interpolation, as the

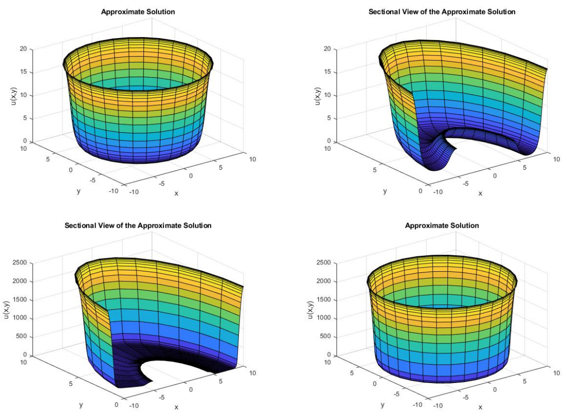

Figure 14: Numerical solution of (24) in $A(4,10)$ for $\lambda \in \{1, 1.8\}$ with $m(x,y)$ given by (26) with $\gamma = 4.9$.

radius is positive, the collocation points (r_i, θ_j) with negative r_i, correspond to those which have the same radius and θ increased by π (see [4, 9, 15, 23]). The collocation points are $(r_i, \theta_j) = \left(\cos\left(\frac{(i-1)\pi}{N_r}\right), \frac{2\pi\, j}{N_\theta}\right)$ for $1 \le i \le N+1$ and $1 \le j \le N_\theta$, where $N = (N_r - 1)/2$. To avoid the inherent loss of regularity at the origin, the grid parameter N_r in the r-direction is taken to be odd, and N_θ must be even to be able to apply the symmetry properties in θ.

Some pioneer results about Chebyshev-Fourier expansion can be found in [2, 5, 8, 28, 30, 31]. In Gottlied, Hussaini and Orszag [14] it was shown that the trigonometric interpolant of a smoothly differentiable function with period

Figure	Domain	N_r	N_θ	Total
3	$B_1((0,0))$	55	30	990
4	$B_{0.5}((0,0))$	55	30	990
5	$A(0.3, 0.5)$	45	30	1380
8	$A(10, 100)$	45	50	2300
9	$B_1((0,0))$	35	30	540
11	$A(4, 10)$	40	30	1230
13	$A(4, 10)$	61	30	1830
14	$A(4, 10)$	61	30	1830

Table 5: Total number of collocations points.

2π, $g(\theta)$ can be written as

$$g(\theta) = \sum_{l=1}^{N_\theta} g(\theta_l) S_{N_\theta}(\theta - \theta_l)$$

where S_{N_θ} is the periodic sinc function:

$$S_{N_\theta}(\theta) = \frac{\sin\left(\frac{N_\theta \theta}{2}\right)}{N_\theta \tan\left(\frac{\theta}{2}\right)}.$$

Thus, let us consider

$$u_{N+1,N_\theta}(r,\theta) = \sum_{k=1}^{N_r} L_k(r)\, P_k(\theta)$$

where L_k's are the Lagrange polynomials $L_k(r) = \prod_{i\neq k}(r - r_i)/(r_k - r_i)$ and

$$P_k(\theta) = \sum_{l=1}^{N_\theta} a_{k,l} S_{N_\theta}(\theta - \theta_l)$$

is the trigonometric interpolant of $u(r_k,\theta)$ in the points θ_l, $l = 1,\dots,N_\theta$. Then

$$u_{N+1,N_\theta}(r,\theta) = \sum_{k=1}^{N_r+1} \sum_{l=1}^{N_\theta} a_{k,l}\, S_{N_\theta}(\theta - \theta_l)\, L_k(r). \tag{27}$$

Note that the approximate solution used in Huang and Sloan [16] coincides with the expression in (27) but there, the collocation points in the radial direction are of the form $\frac{1-\cos\left(\frac{(i-1)\pi}{N_r}\right)}{2}$ for $1 \le i \le N+1$.

Taking into account that

$$r_{N_r+2-i} = -r_i \text{ and } \theta_{j+\frac{N_\theta}{2}} = \theta_j + \pi, \text{ for } 1 \le j \le \frac{N_\theta}{2} \text{ and } 1 \le i \le \frac{N_r+1}{2},$$

we can conclude that

$$u(r_{N_r+2-i}\, \cos\theta_j, r_{N_r+2-i}\, \sin\theta_j) = u(r_i\, \cos\theta_{j+\frac{N_\theta}{2}}, r_i\, \sin\theta_{j+\frac{N_\theta}{2}}). \tag{28}$$

Since

$$a_{N_r+2-i,j} = u(r_{N_r+2-i},\theta_j), \qquad a_{i,j+\frac{N_\theta}{2}} = u(r_i,\theta_{j+\frac{N_\theta}{2}}),$$

from (28) we finally obtain that

$$a_{N_r+2-i,j} = a_{i,j+\frac{N_\theta}{2}}, \quad 1 \le j \le \frac{N_\theta}{2}, \quad 1 \le i \le \frac{N_r+1}{2}. \tag{29}$$

Very recently, using (29) , the authors proved in [24] that $u_{N+1,N_\theta}(r,\theta)$ can be rewritten as:

$$u_{N+1,N_\theta}(r,\theta) = \sum_{k=1}^{\frac{N_r+1}{2}} \sum_{l=1}^{N_\theta} a_{k,l} \left[S_{N_\theta}(\theta - \theta_l)\, L_k(r) + S_{N_\theta}\left(\theta - \theta_{l+\frac{N_\theta}{2}}\right) L_{N_r+2-k}(r) \right]$$

where $a_{k,l} = u(r_k, \theta_l)$.

Therefore, there exist $\left(\frac{N_r+1}{2}\right) N_\theta$ unknowns in $u_{N+1,N_\theta}(r,\theta)$.

Thus, the associated matrix to the Laplacian in polar coordinates on the full grid is an $(N+1)N_\theta \times (N+1)N_\theta$ matrix consisting of Kronecker products where $N = (N_r - 1)/2$. Let us define the differentiation matrices D_1, D_2, E_1, E_2 and $D_\theta^{(2)}$ by

$$\begin{aligned}
(E_1)_{i,j} &= L'_j(r_i); && 1 \le i, j \le N+1,\\
(E_2)_{i,j} &= L'_{N_r+2-j}(r_i); && 1 \le i, j \le N+1,\\
(D_1)_{i,j} &= L''_j(r_i); && 1 \le i, j \le N+1,\\
(D_2)_{i,j} &= L''_{N_r+2-j}(r_i); && 1 \le i, j \le N+1,\\
(D_\theta^{(2)})_{k,l} &= S''_{N_\theta}(\theta_k - \theta_l); && 1 \le l, k \le N_\theta.
\end{aligned} \tag{30}$$

Consequently, the discretization matrix of the Laplacian in polar coordinates, denoted by L_Δ, takes the following form:

$$L_\Delta = (D_1 + RE_1) \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (D_2 + RE_2) \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} + R^2 \otimes D_\theta^{(2)}$$

where I stands for the identity of order $\frac{N_\theta}{2} \times \frac{N_\theta}{2}$ and R is the diagonal matrix $R_{i\,i} = r_i^{-1}$, $i = 1, \ldots, N+1$, see [33] and [27].

Finally, one should extract the N_θ- first rows of L_Δ because they correspond to the discretization of the Laplacian on the boundary points (r_1, θ_j) for $j = 1, \ldots, N_\theta$. So, the discretization matrix of the Laplace operator on the inner collocations points is given by $\tilde{L}$ where $\tilde{L}$ is obtained by stripping L_Δ of its N_θ-first rows, so,

$$L_\Delta = \left(\begin{array}{|c|} \hline \cdots \\ \hline \tilde{L} \\ \hline \end{array} \right).$$

Throughout the rest of this section, we will set:

$$\begin{aligned}
u &= (u(r_1,\theta_1), \ldots, u(r_1,\theta_{N_\theta}), u(r_2,\theta_1), \ldots, u(r_2,\theta_{N_\theta}), \ldots \\
&\qquad \ldots, u(r_{N+1},\theta_1), \ldots, u(r_{N+1},\theta_{N_\theta}))^T,\\
u_0 &= (u(r_1,\theta_1), \ldots, u(r_1,\theta_{N_\theta}))^T,\\
\tilde{u} &= (u(r_2,\theta_1), \ldots, u(r_2,\theta_{N_\theta}), \ldots, u(r_{N+1},\theta_1), \ldots, u(r_{N+1},\theta_{N_\theta}))^T.
\end{aligned}$$

Note that $u = (u_0, \tilde{u})^T$ and $N+1 = \frac{N_r+1}{2}$. Finally,

$$(\tilde{L}\,\tilde{u})_{(i-2)N_\theta+j} = \left(\frac{\partial^2 u_{N+1,N_\theta}}{\partial r^2} + \frac{1}{r}\frac{\partial u_{N+1,N_\theta}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_{N+1,N_\theta}}{\partial \theta^2}\right)\Bigg|_{(r_i,\theta_j)}$$

for every $i = 2,\dots,N+1$ and $j = 1,\dots,N_\theta$. It should be noted that the subsequent analysis depends on the nature of the boundary conditions of the problem we want to solve.

A.1. Inhomogeneous Dirichlet condition ($u = f \not\equiv 0$ on $\partial B_1((0,0))$)

To impose the boundary condition we fix $(u_0)_j = u(r_1,\theta_j) = f(\theta_j)$ for $j = 1,\dots,N_\theta$. Then, we divide $\tilde{L}$ as:

$$\tilde{L} = \left(\begin{array}{|c|c|} \hline L_1 & \quad L_2 \quad \\ \hline \end{array} \right) \tag{31}$$

where L_1 and L_2 are the matrices obtained by stripping $\tilde{L}$ of its NN_θ-last and N_θ-first columns, respectively. Note that

$$\tilde{L}\,u = L_1\,u_0 + L_2\,\tilde{u}.$$

Thus, L_2 provides us with the discretization matrix of the Laplace operator on the inner collocation points.

A.2. Homogeneous Neumann conditions ($\frac{\partial u}{\partial \eta} = 0$ on $\partial B_1((0,0))$)

Let E be the differentiation matrix of $\frac{\partial}{\partial r}$ on the colocation points (r_i,θ_j) for $i = 1,\dots,N$ and $j = 1,\dots,N_\theta$:

$$E := E_1 \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + E_2 \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where E_1 and E_2 are the matrices defined in (30), and I stands for the identity of order $\frac{N_\theta}{2} \times \frac{N_\theta}{2}$. In order to impose the Neumann boundary conditions on the collocation points on $\partial B_1((0,0))$, we are interested in the portion of E that discretizes the first derivative on these points. Thus, we introduce the matrix

$$A = F_1^{E_1} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + F_1^{E_2} \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

where $F_1^{E_1}$ and $F_1^{E_2}$ denote the first row of the matrices E_1 and E_2, respectively. Note that, for any $j = 1, \ldots, N_\theta$ fixed, the discrete partial derivative with respect to r in (r_1, θ_j) corresponds to the j-th row of the matrix A. That is,

$$\left.\frac{\partial}{\partial r}\right|_{(r_1,\theta_j)} = F_j^A \qquad \text{for } j = 1, \ldots, N_\theta.$$

Then, we obtain the following partion of E:

$$E = \left(\begin{array}{|c|} \hline A \\ \hline \cdots \\ \hline \end{array} \right).$$

Next, we break up A as follows:

$$A = \left(\begin{array}{|c|c|} \hline A_1 & \quad A_2 \quad \\ \hline \end{array} \right)$$

where A_1 and A_2 stand for the matrices obtained by stripping of A the NN_θ-last and the N_θ-first columns, respectively. Finally, the homogeneous Neumann boundary conditions

$$\left.\frac{\partial u}{\partial \eta}\right|_{\partial B_1((0,0))} = \left.\frac{\partial u}{\partial r}\right|_{\partial B_1((0,0))} = 0$$

implies that

$$0 = A\, u = A_1\, u_0 + A_2\, \tilde{u}.$$

Thus, u_0 satisfies

$$u_0 = -A_1^{-1}\, A_2\, \tilde{u}.$$

Considering $\tilde{L}$ as in (31), we have:

$$\begin{aligned} \tilde{L}u &= L_1 u_0 + L_2 \tilde{u} \\ &= (-L_1 A_1^{-1} A_2 + L_2)\tilde{u}. \end{aligned}$$

Therefore, the discretization matrix of the Laplacian on the inner collocation points with homogeneous Neumann boundary conditions becomes

$$\tilde{\tilde{L}} = -L_1\, A_1^{-1}\, A_2 + L_2.$$

We claim that A_1 is non-singular. Indeed, since

$$A_1 = (E_1)_{11} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + (E_2)_{11} \otimes \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

using (30) and the well known Chebyshev differentiation matrix (see, e.g., [26], [5] or [33]), it follows that

$$(E_1)_{11} = \frac{2N_r^2+1}{6}, \quad (E_2)_{11} = \frac{1}{2}(-1)^{N_r}.$$

Thus,

$$\det(A_1) = \left[((E_1)_{11})^2 - ((E_2)_{11})^2\right]^{\frac{N_\theta}{2}} \neq 0.$$

B. The differentiation matrices in a circular annulus

The rotational symmetry of $A(R_0, R_1)$ enables us to use polar coordinates. In such case, there is an isomorphism between $A(R_0, R_1)$ and the rectangle $[R_0, R_1] \times [0, 2\pi]$. Hence, we need to take a linear transformation of the Chebyshev grid in the r-direction and a periodic Fourier grid in θ. The grid in the ρ-direction is obtained from the usual Chebyshev grid $r \in [-1, 1]$. So, the collocation points in the annulus are

$$(\rho_i, \theta_j) = \left(\frac{(R_1-R_0)r_i + (R_1+R_0)}{2}, \frac{2\pi\, j}{N_\theta}\right) \quad 1 \le i \le N_r+1,\ 1 \le j \le N_\theta.$$

It should be remembered that $\rho_1 = R_1$ and $\rho_{N_r+1} = R_0$ correspond to the boundary points of the annulus. As a by-product, the discretization of the Laplace operator in polar coordinates in the annulus is the matrix of order $((N_r+1)N_\theta) \times ((N_r+1)N_\theta)$ defined by

$$L_\Delta = (p^2\, D_r^2 + p\, R\, D_r) \otimes I + R^2 \otimes D_\theta^{(2)}$$

where $p = \frac{2}{R_1-R_0}$, I stands for the $N_\theta \times N_\theta$ identity matrix, R is the diagonal matrix with entries $R_{i\,i} = \frac{1}{\rho_i}$ for $i = 1, \ldots, N_r+1$, and D_r is the full Chebyshev differentiation matrix

$$(D_r)_{i,j} = L'_j(r_i); \quad 1 \le i, j \le N_r+1. \tag{32}$$

Note that in this case we are not discarding any blocks of D_r because we need to consider exactly r in the closed interval $[-1, 1]$. Before imposing the boundary conditions on L_Δ, we set

$$\begin{aligned} u_0 &= (u(\rho_1,\theta_1), \ldots, u(\rho_1,\theta_{N_\theta}))^T, \\ \tilde{u} &= (u(\rho_2,\theta_1), \ldots, u(\rho_2,\theta_{N_\theta}), \ldots, u(\rho_{N_r},\theta_1), \ldots, u(\rho_{N_r},\theta_{N_\theta}))^T, \\ u_1 &= (u(\rho_{N_r+1},\theta_1), \ldots, u(\rho_{N_r+1},\theta_{N_\theta}))^T. \end{aligned}$$

So, u is factorized as $(u_0, \tilde{u}, u_1)^T$.

B.1. Homogeneous Dirichlet conditions ($u = 0$ on $\partial A(R_0, R_1)$)

First, set $w := L_\Delta\, u$. Next, we factorize w in the same way as u, so that $w = (w_0, \tilde{w}, w_1)^T$ with $w_0, w_1 \in \mathbb{R}^{N_\theta}$ and $\tilde{w} \in \mathbb{R}^{N_\theta(N_r-1)}$. Then, the procedure scheme adopted here to impose the homogeneous Dirichlet conditions on L_Δ consists in fixing the vectors u_0 and u_1 at zero, and ignoring w_0 and w_1 because, as already mentioned above, the Laplacian is computed in the interior of domain where the differential equation holds. This implies that the N_θ-first and N_θ-last columns of L_Δ have no computational effects, because they correspond to the discretization of the Laplacian at points along the boundary. Accordingly, the discretization matrix for the Laplacian is the matrix $\tilde{L}$ obtained by stripping L_Δ of its N_θ-first and N_θ-last rows and columns.

$$L_\Delta = \left(\quad \boxed{\tilde{L}} \quad \right).$$

B.2. Inhomogeneous Dirichlet conditions ($u = f \not\equiv 0$ on $\partial A(R_0, R_1)$)

We consider w as in the previous subsection. In the present situation, to impose the inhomogeneous Dirichlet condition on L_Δ we first fix u_0 and u_1 at the vectors f_{N_r+1} and f_1, respectively, where $(f_i)_j = f(\rho_i, \theta_j)$ with $i \in \{1, N_r + 1\}$ fixed and $j = 1, \ldots, N_\theta$, and we ignore w_0 and w_1. So, the N_θ-first and N_θ-last rows have no effects and they can be ignored. Accordingly, the matrix L_Δ is split into the three blocks

$$L_\Delta = \left(\begin{array}{|c|} \hline \cdots \\ \hline \overline{L} \\ \hline \cdots \\ \hline \end{array} \right)$$

where $\overline{L}$ is the matrix obtained by stripping L_Δ of its N_θ-first and N_θ-last rows. Consequently, we can discard the top and bottom blocks of L_Δ. Next, we split $\overline{L}$ into another three blocks, as follows

$$\overline{L} = \left(\begin{array}{|c|c|c|} \hline L_1 & L_2 & L_3 \\ \hline \end{array} \right), \tag{33}$$

where

- L_1 is the matrix formed by the first N_θ columns of $\overline{L}$.

- L_2 is obtained by stripping $\overline{L}$ of its N_θ-first and N_θ-last columns.
- L_3 is the matrix formed by the last N_θ columns of $\overline{L}$.

Note that, owing to (33), we have

$$\overline{L}u = L_1 u_0 + L_2 \tilde{u} + L_3 u_1. \tag{34}$$

Naturally, L_2 is the discrete matrix of the Laplacian in polar coordinates on the inner collocation points of the annulus.

B.3. Homogeneous Neumann conditions ($\frac{\partial u}{\partial \eta} = 0$ on $\partial A(R_0, R_1)$)

In this case, we denote by E the corresponding discratization matrix of the first partial derivative with respect to r on the collocation points (ρ_i, θ_j) for $i = 1, \ldots, N_r + 1$ and $j = 1, \ldots, N_\theta$. That is,

$$E = p\, D_r \otimes I$$

where I is the identity matrix of dimension $N_\theta \times N_\theta$.

Now, to impose the Neumann boundary conditions on the collocation points contained in $\partial A(R_0, R_1)$, we are just interested into the portion of E that discretizes the first derivative on the inner and outer components of the boundary of the annulus. Accordingly, we introduce the matrices A and B as follows:

$$\begin{aligned} A &= p\, F_1^{D_r} \otimes I \\ B &= p\, F_{N_r+1}^{D_r} \otimes I \end{aligned} \tag{35}$$

where $F_1^{D_r}$ and $F_{N_r+1}^{D_r}$ denote the first and last rows, respectively, of the matrix D_r. Note that, for any fixed $j = 1, \ldots, N_\theta$, the discrete partial derivative with respect to ρ at (ρ_1, θ_j) and (ρ_{N_r+1}, θ_j) corresponds to the j-th row of the matrices A and B, respectively. That is,

$$\begin{aligned} p\,\frac{\partial}{\partial r}\bigg|_{(\rho_1,\theta_j)} &= F_j^A \qquad \text{for } j = 1, \ldots, N_\theta, \\ p\,\frac{\partial}{\partial r}\bigg|_{(\rho_{N_r+1},\theta_j)} &= F_j^B \qquad \text{for } j = 1, \ldots, N_\theta. \end{aligned}$$

Now, we divide both, A and B, in three blocks

$$A = \left(\begin{array}{|c|c|c|} \hline A_1 & A_2 & A_3 \\ \hline \end{array} \right), \tag{36}$$

$$B = \left(\begin{array}{|c|c|c|} \hline B_1 & B_2 & B_3 \\ \hline \end{array} \right), \tag{37}$$

where:

- A_1 (resp. B_1) is obtained by stripping A (resp. B) of its N_rN_θ-last (resp. -first) columns.
- A_2 (resp. B_2) is obtained by stripping A (resp. B) of its N_θ-first (resp. -last) and N_θ-last (resp. -first) columns.
- A_3 (resp. B_3) is obtained by stripping A (resp. B) of its N_rN_θ-first (resp. -last) columns.

Imposing $0 = \frac{\partial u}{\partial r}(R_0,\theta) = \frac{\partial u}{\partial r}(R_1,\theta)$, yields

$$0 = A\,u = A_1\,u_0 + A_2\,\tilde{u} + A_3\,u_1 \quad\Longrightarrow\quad \boxed{A_1\,u_0 + A_3\,u_1 = -A_2\,\tilde{u}}$$

$$0 = B\,u = B_1\,u_0 + B_2\,\tilde{u} + B_3\,u_1 \quad\Longrightarrow\quad \boxed{B_1\,u_0 + B_3\,u_1 = -B_2\,\tilde{u}}$$

an solving the matricial system

$$\begin{cases} A_1u_0 + A_3u_1 = -A_2\tilde{u} \\ B_1u_0 + B_3u_1 = -B_2\tilde{u}, \end{cases}$$

we obtain

$$\begin{cases} u_0 = -A_1^{-1}(A_2 + A_3(B_3 - B_1A_1^{-1}A_3)^{-1})(B_1A_1^{-1}A_2 - B_2)\tilde{u} \\ u_1 = (B_3 - B_1A_1^{-1}A_3)^{-1}(B_1A_1^{-1}A_2 - B_2)\tilde{u}. \end{cases}$$

We claim that $(B_3 - B_1A_1^{-1}A_3)$ is non-singular. Indeed, from (35), (36) and (37) it becomes apparent that

$$\begin{aligned} A_1 &= (D_r)_{1\,1}I, \\ A_3 &= (D_r)_{1\,N_r+1}I, \\ B_1 &= (D_r)_{N_r+1\,1}I, \\ B_3 &= (D_r)_{N_r+1\,N_r+1}I. \end{aligned}$$

Using (32) and the coefficients of the Chebyshev differentiation matrix, it follows that

$$(D_r)_{1\,1} = -(D_r)_{N_r+1\,N_r+1} = \frac{2N_r^2+1}{6}$$

and that

$$(D_r)_{1\,N_r+1} = -(D_r)_{N_r+1\,1} = \frac{1}{2}(-1)^{N_r}.$$

Therefore,

$$\det(B_3 - B_1A_1^{-1}A_3) = \Big[(D_r)_{N_r+1\,N_r+1} \\ -(D_r)_{N_r+1\,1}\left((D_r)_{1\,1}\right)^{-1}(D_r)_{1\,N_r+1}\Big]^{N_\theta} \neq 0.$$

Consequently, by substituting u_0 and u_1 in (34), we find the discretization matrix of the Laplacian. Namely,

$$\begin{aligned}\tilde{\tilde{L}} = -L_1 A_1^{-1}(A_2 + A_3(B_3 - B_1 A_1^{-1} A_3)^{-1})(B_1 A_1^{-1} A_2 - B_2) \\ + L_2 + L_3(B_3 - B_1 A_1^{-1} A_3)^{-1}(B_1 A_1^{-1} A_2 - B_2).\end{aligned}$$

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