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Possibility distribution calculus and the arithmetic of fuzzy numbers

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for the 70-th birthday
of our friend and colleague Eugenio Omodeo

Abstract. *Based on possibility theory and multi-valued logic and taking inspiration from the seminal work in probability theory by A. N. Kolmogorov, we aim at laying a hopefully equally sound foundation for fuzzy arithmetic. A possibilistic interpretation of fuzzy arithmetic has long been known even without taking it to its full consequences: to achieve this aim, in this paper we stress the basic role of the two limit-cases of possibilistic interactivity, namely deterministic equality versus non-interactivity, thus getting rid of weak points which have ridden more traditional approaches to fuzzy arithmetic. Both complete and incomplete arithmetic are covered.*

Keywords: fuzzy arithmetic, fuzzy numbers, possibility theory, interactivity.
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dicebat Bernardus Carnotensis nos esse
quasi nanos gigantium humeris insidentes
Iohannes Saresberiensis
`Metalogicon` (≃1159)

1. Introduction

In the so-called *implicit* approach to probabilities due to the English mathematician sir Harry Raymond Pitt (1914-2005), *random variables* and random numbers X are not explicitly defined, as happens in the *explicit* Kolmogorov's approach, but are rather *operationally* described through their *probability distribution* P_X, i.e. through a 1-normed σ-additive measure on the σ-algebra of Borel sets on $\mathbb{R}$. In the same way random couples (X, Y) or more generally random k-tuples $\underline{X} = X_1 \dots X_k$ rely on bi-dimensional or k-dimensional 1-normed measures $P_{X,Y}$ or $P_{\underline{X}}$ on the Borel sets of $\mathbb{R}^2$ or $\mathbb{R}^k$, respectively. Pitt does not even try to "split" suggestive notations as are e.g. Prob$\{a \leq X \leq b\}$ or

Prob$\{a \leq X \leq b, c \leq Y \leq d\}$ into their components, but simply sees them as inspiring synonyms for the measures (for the probabilities) of the corresponding interval A and the corresponding rectangle B, i.e. for $P_X(A)$ and $P_{X,Y}(B)$, respectively. Needless to say, a *metatheorem* (actually, a quite obvious metatheorem) proves that a statement is true in the implicit approach if and only if it is true in the explicit approach: the two approaches, in a way, are "interchangeable".

Thinking of the quote of John of Salisbury and Bernard de Chartres, even if below we find it more convenient and expedient to mimic Pitt's approach, it is on the shoulders of the Russian giant Andrej N. Kolmogorov (1903-1987) that we shall stand to discuss from a vantage position the *maxitive possibility distribution calculus*, as opposed to the more traditional and mature *additive probability distribution calculus.*

In the final section we shall comment that *fuzzy arithmetic* has much to learn from what was done in probability starting from the letters exchanged by Blaise Pascal and Pierre de Fermat in 1654: *possibility distribution calculus* as covered in Sections 2 to 6 should only serve as a smooth transition tool, meant to cool down polemic reactions on side of the partisans of fuzzy arithmetic in its more traditional *set-theoretic* approach. Fair to say, it is precisely to fuzzy numbers and to a sound mathematical foundation for fuzzy arithmetic that this paper is devoted, even if we shall take our time and reach the point only in Section 6. We will go as far as claiming that old set-theoretic approaches to fuzzy arithmetic should be entirely relinquished in favor of an approach based directly on possibility theory.

Early interpretations of fuzzy arithmetic in a possibilistic key go back to as early as the 80's [3] and have continued following a more and more "radical" view [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 19, 20]; based on this, we are now in a position to present a comprehensive and sound approach to fuzzy arithmetic and its underlying possibility distribution calculus, without any of the drawbacks and snags one was used to fight with; cf. Sections 5 and 6.

Even if ours is basically a position paper, the Pitt-like approach to possibilistic arithmetic, and so to fuzzy arithmetic, is here presented for the first time in a systematic and all-inclusive way, underlying its generality and simplicity: actually, it is precisely trying to speed up computations, as we did in [6, 8, 9, 10, 11, 19, 20], that we were led to deepen the theoretic approach which is here presented; cf. the concluding section.

In Sections 2 to 5 terms as are *fuzzy number* or *fuzzy arithmetic* will be avoided. They will be explicitly used only in Section 6, once we possess all the possibilistic tools which are needed.

2. A Pitt-like approach to possibility numbers

A *possibility number* X or more compactly a Π-*number* X is implicitly defined through its *possibility distribution* Π_X on the subsets of $\mathbb{R}$. In its turn a possibility distribution $\Pi = \Pi_X$ on $\mathbb{R}$ is defined by a non-negative *possibility distribution function* $f_X(x) : \mathbb{R} \to [0, 1]$ with the single *normality* constraint that the equation $f_X(x) = 1$ should have at least one solution (cf. however Section 3); the more specific notation Π_X is used beside the generic one Π to stress which Π-number is involved. For any subset A of $\mathbb{R}$ one sets

$$\Pi(A) \ = \Pi_X(A) \ = \ \text{Poss}\{X \in A\} \doteq \sup_{x \in A} f_X(x)\,,$$

where $\doteq$ means "equal by definition". Notice that one has $f_X(x) = \text{Poss}\{X = x\}$, unlike what happens with probability distribution functions.

An observation: we will not discuss the *meaning* of possibilities, their "philosophy", for which we refer e.g. to [4, 18, 15], but rather their mathematical or "technical" maxitive structure. We acknowledge that "possibility" is quite a committal term, possibility theory being a deep and relevant chapter of multi-valued logics.

A further observation: the reader might object that our definitions are too loose and generous: an *event* is *any* subset of $\mathbb{R}$ and the distribution function might be quite pathological. The reason is that *suprema* (generalized maxima), unlike generalized sums (Radon-Nikodym integrals), are quite user-friendly and do not bring about any mathematical snag (devoted mathematicians might add: "*unfortunately*"). We will be as general as possible, quite conscious that in practice only very special subsets and very special distributions functions will be needed, and that relevant theorems might require to impose restrictions on subsets and on distribution functions.

As an example, we will *not* rule out the *vacuous* (uninformative) distribution functions $f_X(x) = 1$, $x \in \mathbb{R}$, which is the analogue of the uniform probability function, with the non-trivial observation that uniform probability functions are defined e.g. on intervals, while a uniform probability on the whole of $\mathbb{R}$ *breaks* the usual σ-additive frame and is allowed only in the very special finitely additive frame of *subjectivist* probability theorists as was Bruno de Finetti (1906-1985). While uniform probability distributions are meant to code *total uncertainty*, uniform possibility distributions are meant to code *total ignorance*, a distinction which is quite familiar in *evidence theory* [21] from which the term "vacuous" is derived.

Further examples, made thinking directly of fuzzy arithmetic, are Π-intervals (u, a, b, v) with $u < a < b < v$, where the possibility distribution function is 0 outside $[u, v]$, grows linearly from 0 to 1 on $[u, a]$, remains equal to 1 on $[a, b]$ and decreases linearly from 1 to 0 on $[b, v]$. Limit cases thereof are Π-triangles $(u, a = b, v)$ when $a = b$, and *crisp* numbers r when $u = a = b = v \doteq r$, $f(r) = 1$

and $f(x) = 0$ for $x \neq r$. Crisp numbers (usual real numbers) are at the other extreme of the vacuous number, crisp knowledge (no ignorance) versus total ignorance.

We move to possibilistic k-tuples $\underline{X}$ and their distribution functions

$$f_{\underline{X}}(x_1, \dots, x_k) : \ \mathbb{R}^k \to [0,1] \, .$$

Actually, to make our point, it will be enough to deal with *couples* (X, Y) whose distribution functions $f_{X,Y}(x, y)$ have domain $\mathbb{R}^2$ and such that for at least one couple $(\xi, \zeta) \in \mathbb{R}^2$ one has $f_{X,Y}(\xi, \zeta) = 1$. Starting from a *joint* (bidimensional) possibility distribution function $f_{X,Y}(x, y)$ one may derive the two corresponding *marginal* (1-dimensional) distribution functions $f_X(x)$ and $f_Y(y)$:

$$f_X(x) \ = \ \mathrm{Poss}\{X = x\} \ = \ \mathrm{Poss}\{X = x, Y \in \mathbb{R}\} \doteq \sup_{y \in \mathbb{R}} f_{X,Y}(x, y)$$

and the analogue for Π_Y. These definitions are in accordance with the maxitive nature of possibilities and remind one of marginalization in probability theory, additive rather than maxitive.

In the other direction, one might start from the two marginals for X and Y and "stick" them together to obtain an *admissible* joint distribution which would give back the two marginals one had started with. One convenient way to do this is to use a $\top$-*norm* (to be read *tee-norm*), i.e. an "abstract" logical conjunction (e.g., see [4, 11, 12, 15, 18, 20]) $x \top y$, where the two logical values (x, y) belong to the unit square $[0 \leq x \leq 1, \, 0 \leq y \leq 1]$, 0=`false`, 1= `true`.

We shortly recall that $\top$-norms are defined by *axioms* which impose commutativity $x \top y = y \top x$, associativity $(x \top y) \top z = x \top (y \top z)$ and monotony: $y \leq z$ implies $x \top y \leq x \top z$, with 1 as $\top$-identity element $x \top 1 = x$; by using monotony with respect to the $\top$-identity element $1 \geq x$ one soon proves that 0 is a nullific for $\top$, i.e., $x \top 0 = 0$. The negation is simply $\overline{x} \doteq 1 - x$. By resorting to one of the two De Morgan rules one can soon derive a *dual* $\top$-*conorm* $x \bot y \doteq \overline{\overline{x} \top \overline{y}}$, i.e. an abstract disjunction; we recall that the axioms for $\top$-conorms are the same as for $\top$-norms, save that the $\top$-identity element is 0 and so the $\bot$'s nullific is 1, i.e., $x \bot 1 = 1$.

Sticking together two marginals by use of a $\top$-norm, as soon checked, defines an admissible joint distribution. No doubt, the most popular norms are:

- *standard*: $x \top y = \min[x, y] \doteq x \wedge y \, , \ \ x \bot y = \max[x, y] \doteq x \vee y \, ,$
- *Łukasiewicz*: $x \top y = \max[0, x + y - 1] \, , \ \ x \bot y = \min[1, x + y]$,
- *probabilistic*: $x \top y = x \cdot y \, , \ \ x \bot y = x + y - x \cdot y \, .$

The standard norms are those most used in fuzzy logic as made popular by Lotfi A. Zadeh (1921-2017).

Using the definitions of the two marginal distributions Π_X and Π_Y derived from the joint distribution $\Pi_{X,Y}$ one soon proves that, whatever the event A, one has $\Pi_{X,Y}(A) \le \big[\Pi_X \wedge \Pi_Y\big](A)$, that is, in terms of possibility distribution functions, whatever the couple $(x,y) \in \mathbb{R}^2$:

$$f_{X,Y}(x,y) \;\le\; f_X(x) \wedge f_Y(y) \;=\; \min\Big[\sup_{y\in\mathbb{R}} f_{X,Y}(x,y)\,,\, \sup_{x\in\mathbb{R}} f_{X,Y}(x,y)\Big]\,.$$

Thus, to maximize joint possibilities one should use the standard $\top$-norm based on minima. This fact, cf. e.g. [4, 15, 18], has lead possibility theorists to define *non-interactivity*, which is meant to be the appropriate possibilistic analogue of probabilistic *independence*, precisely by means of the standard $\top$-norm: two Π-numbers X and Y are non-interactive when $f_{X,Y}(x,y) = \min[f_X(x), f_Y(y)]$. Unsurprisingly, non-interactivity will play a basic role in what follows; observe that the *product* of probabilistic independence has been replaced by the *minimum* of possibilistic non-interactivity.

Another basic way exists to stick together two marginals, limited to the case when the two Π-numbers X and Y are *equidistributed*, i.e. when $f_X(x) = f_Y(x)$ for all $x \in \mathbb{R}$. We are thinking of *deterministic equality*, where one sets $f_{X,Y}(x,x) = f_X(x) = f_Y(x)$, else $f_{X,Y}(x,y) = 0$.

We stress that one should carefully distinguish between equidistribution $X \approx Y$ i.e. $f_X(x) = f_Y(x)$ and deterministic equality $X = Y$; only in the latter case X and Y are the *same* Π-number and the two symbols X and Y are synonyms (one of the two might be disposed of). Two equidistributed numbers X and Y may be interactive in an infinite variety of ways (unless at least one of the two is a crisp number), going from deterministic equality, indeed a tight form of interactivity, to non-interactivity, when there is no “exchange of information” between them. This observation may sound trivial to probability theorists, but unfortunately it is not always so in the traditional set-theoretic approach to fuzzy arithmetic; cf. our comments in Section 6.

3. A detour to incomplete distributions

One might consider also possibility distribution functions which are *incomplete* or *sub-normal*, i.e. for which $\sup_{x\in\mathbb{R}} f(x)$ might be strictly less than 1. Unlike incomplete probabilities, an odd notion indeed, incomplete possibilities do pop up in important contexts as is fuzzy control, as based on the fuzzy extension of the logical syllogism called *modus ponens*: the reader is referred e.g. to [4, 15, 18] for an exhaustive discussion; however, to facilitate self-readability, we have provided the Remark below.

A limit-case of incomplete distributions is the all-0 distribution function; in fuzzy control, such an unpleasant mathematical object is obtained when the logical premise and the logical implication are at contradiction with one another, cf. Remark 3.1. So, while the vacuous (all-1 and so complete) distribution describes *total ignorance*, cf. Section 2, one might be tempted to go as far as claiming that the all-0 distribution is an adequate description of a *totally self-contradictory* state of knowledge about the possibilistic quantity X. Managing contradiction in logic is a hard nut, indeed: once again, however, we notice that similar situations are found in evidence theory [21], where they have been amply discussed and commented upon. As for the link between incompleteness and the representation of logical contradiction in evidence theory, cf. [22].

REMARK 3.1 (Incompleteness). In *modus ponens*, or rather in its fuzzy extension which we shall shortly cover "adapting" it to the possibilistic terminology used so far in this paper, one has two possibilistic quantities X and Y, $X \sim f_X(x)$ which is well-known, the *premise* of the syllogism, and $Y \sim g_Y(y)$ which is unknown, to be computed in the *conclusion* (the symbol $\sim$ refers here to the respective distribution functions). One has also the *inference rule*: "were $X \sim \phi_X(x)$ then one would have $Y \sim \psi_Y(y)$" (the two "hypothetical" distributions $\phi_X(x)$ and $\psi_Y(y)$ are known and can be used in calculations):

- *premise*: $X \sim f_X(x)$
- *inference rule*: if $X \sim \phi_X(x)$ then $Y \sim \psi_Y(y)$
- *conclusion*: $Y \sim g_Y(y)$ to be computed

The logical operations are those standard in fuzzy logic, maximum for disjunction $x \vee y$, minimum $x \wedge y$ for conjunction and implication (we are using the so-called *Mamdani implication* as is standard in *fuzzy control*, cf. e.g. [4, 15, 18]; x and y are two logical values in the interval [0,1]; as is usual, the symbols $\vee$ and $\wedge$ are "double-use", both logical and numerical). Computations show that

$$g_Y(y) \;=\; \psi_Y(y) \;\wedge\; \sup_{x \in \mathbb{R}} \Big[f_X(x) \wedge \phi_X(x) \Big]$$

Even if the three distributions one starts with are complete, the resulting distribution $g_Y(y)$ is *not* complete unless the supremum is equal to 1, i.e. unless ξ exists such that $f_X(\xi) \wedge \phi_X(\xi) = 1$, a case which fuzzy logicians and fuzzy-control people consider scarcely interesting and rarely met in applications $\big($in this case one would have $g_Y(y) = \psi_Y(y)\big)$. Even more rarely met in applications is the case when the supremum is 0 because the two X-supports, the one in the premise and the other in the inference rule, are *disjoint*: in such an unfortunate situation of conflicting distributions one would have an all-0 distribution function for Y in the conclusion.

4. Possibility distribution calculus

We deal with deterministic functions of Π-numbers $Z = \phi(X)$ and with deterministic functions of a Π-couple, i.e. with binary operations $Z = X \circ Y$, where $\circ$ denotes a generic binary function, $x \circ y = \Phi(x, y)$. Below, probability theorists will readily recognize what one does in probability distribution calculus.

It will be enough to be able to compute the distribution functions $f_Z = f_{\phi(X)}$ and $f_Z = f_{X \circ Y}$ starting from $f_X(x)$ and $f_Y(y)$. Mimicking what we already did when marginalizing, one has:

$$f_Z(z) \;=\; \text{Poss}\{Z = z\} \;=\; \text{Poss}\{\phi(X) = z\} \;=\; \sup_{x:\phi(x)=z} f_X(x)$$

and

$$f_Z(z) \;=\; \text{Poss}\{Z = z\} \;=\; \text{Poss}\{X \circ Y = z\} \;=\; \sup_{x,y:\, x \circ y = z} f_{X,Y}(x, y)\,,$$

where we set, as it should be, $f_Z(z) = 0$ when the minimization set where one takes the supremum is void.

Generalizing to the case of a k-argument deterministic function

$$Z = \phi(X_1, \ldots, X_k)\,,\ \ k \geq 1\,,$$

where the possibilistic k-tuple $(X_1, \ldots, X_k)$ is defined by the possibility distribution function $f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)$, one has:

$$f_Z(z) \;=\; \text{Poss}\{Z = z\} \;=\; \sup_{x_1,\ldots,x_k:\ \phi(x_1,\ldots,x_k)=z} f_{X_1,\ldots,X_k}(x_1, \ldots, x_k)\,,$$

with the by now usual convention $f_Z(z) = 0$ when the minimization set is void.

The additive mathematics of measure theory is more complicated than the often unassuming mathematics of suprema, indeed; actually, convenient tools could be developed to shorten calculations, the most relevant being possibly *irrelevance* first introduced in [19] and then used in [6, 7, 8, 20]. We shall shortly mention this tool in the last Section, Remark 6.1.

We stress that completeness was *never* used in this section 4; on the other hand, as soon checked, if one starts with complete distributions, completeness of the resulting distributions is ensured.

5. Montecatini lemma

Lemma 5.1 (Montecatini lemma [20]). *The equality*

$$f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$$

is an identity for crisp numbers if and only if the two possibilistic quantities $Z_1 \doteq f(X_1, \ldots, X_n)$ *and* $Z_2 \doteq g(X_1, \ldots, X_n)$ *are deterministically equal whatever the joint distribution of* $X_1, \ldots, X_n$*.*

E.g., since $x(y+z) = xy + xz$ for any crisp numbers x, y and z, one has $X(Y+Z) = XY + XZ$ for *any* possibilistic quantities X, Y and Z, whatever their joint distribution. Since $\log xy = \log x + \log y$ for positive x and y one has $\log XY = \log X + \log Y$ for *any* possibilistic quantities X and Y whatever their joint distribution with positive support.

Proving the lemma is trivial in both directions. To prove the equidistribution of Z_1 and Z_2 just observe that one is taking the supremum of the same function over two sets, $\{\underline{x} : f(\underline{x}) = z\}$ and $\{\underline{x} : g(\underline{x}) = z\}$, which are however equal (are the very same set). As for deterministic equality $Z_1 = Z_2$, one cannot have $f(\underline{x}) \neq g(\underline{x})$, and so the joint distribution of the possibilistic couple (Z_1, Z_2) is zero outside the main diagonal $z_1 = z_2$.

Remark 5.2. Completeness is never used and so is not requested, but it is preserved whenever present at the start.

Remark 5.3. The lemma is stated in terms of arbitrary possibilistic distribution functions; if one insists on certain properties, e.g. upper continuity or unimodality, one should of course check *stability*, i.e check whether the result $Z = f(X_1, \ldots, X_n) = g(X_1, \ldots, X_n)$ still verifies those properties.

Remark 5.4. The curious name of the lemma (Montecatini is a resort spa in Tuscany) simply reminds of a lively discussion held at an INdAM-GNCS meeting held there. However trivial the lemma might be, its consequences on fuzzy arithmetic, as we shall now argue, are quite remarkable.

6. Does possibility distribution calculus offer a sound basis for fuzzy arithmetic?

In fuzzy arithmetic a *fuzzy number* X is once more described through a function $f_X(x) : \mathbb{R} \to [0,1]$ but this function is rather seen as the *membership function* of a fuzzy set, which explains why we are calling this approach the set-theoretic approach to fuzzy arithmetic. This different interpretation will be of no special consequence in the following; what matters is that we are dealing with a non-negative function $f_X(x)$ which takes on the value 1 at least once (the corresponding fuzzy set must be *normal*). Usually a lot of restrictions are imposed on the function $f_X(x)$, which has so to be quite “regular”. We shall come to this moot point later; for the moment being *no* additional restriction will be imposed on $f_X(x)$ and to stress this fact we find it convenient to replace the rather ambitious term “number” with the less committal term “quantity”. Thus a *fuzzy quantity* is described, at least formally, in exactly the same way as the Π-numbers of Section 2.

Nothing like a joint distribution is present, however: in a way, a fuzzy quantity X *is the same* as its corresponding function $f_X(x)$. In particular, two equidistributed numbers are the very same number: equidistribution boils down to strict equality, is indistinguishable from it. Given a binary operation $Z = X \circ Y$ fuzzy arithmetic states that

$$f_Z(z) \;=\; f_{X\circ Y} \doteq \sup_{x,y:\, x\circ y=z} \min\big[f_X(x), f_Y(y)\,,\big]$$

with $f_Z(z) = 0$ if the minimization set is void. By the way, the functions involved are usually so regular that the supremum is almost unavoidably a maximum. Even if the definition is usually justified by means of the so-called *extension principle* of fuzzy-set theory (cf. e.g. [4, 18]), the reader will recognize the corresponding definition for Π-numbers in the special case of *non-interactivity*, when $f_{X,Y}(x,y) = \min\big[f_X(x), f_Y(y)\big]$.

In fuzzy arithmetic as built on this basis, several unpleasant facts occur, which seem to contradict Montecatini lemma, for example:

$$X(Y+U) \neq XY + XU\;,\; X - X \neq 0\;,\; \frac{1}{X} \neq X^{-1}\,.$$

In the last case we are of course assuming $f_X(0) = 0$; the first side of $\frac{1}{X} \neq X^{-1}$ refers to the binary operation of division, while the second side to the deterministic (1-dimensional) function of inversion; for the crisp number $X = 1$ seen as a fuzzy number recall that one has $f_X(1) = 1$, else 0.

As a first comment: even in probability theory one would have nasty claims as the ones above, were one so clumsy as to confuse equidistribution with deterministic equality. Everything becomes quite clear as soon as one would write:

even if X and $\tilde{X}$ are equidistributed

$$X(Y+U) \neq XY + \tilde{X}U\;,\; X - \tilde{X} \neq 0\;,\; \frac{1}{X} \neq \tilde{X}^{-1}$$

as happens also in probability theory, by the way. In accordance with Montecatini lemma, one has three equalities as soon as X and $\tilde{X}$ are not only equidistributed but also deterministically equal, and only in this case the symbol $\tilde{X}$ can be disposed of and be replaced by X. This implies that even in set-theoretic fuzzy arithmetic one should not give up the precious distinction between equidistribution and deterministic equality, and therefore one should not use a single symbol X to denote two equidistributed fuzzy number which are *not* the same number.

By the way, observe that when dealing with *crisp* numbers the distinction between equidistribution and deterministic equality would still make sense (the first and the second co-ordinate of the point (3,3) of the Cartesian plane are equal at a certain level of abstraction, but different at another level, proof be the fact that the first co-ordinate, unlike the second, possesses the qualification "`abscissa`" and can be increased without having to increase the second); however, in the crisp case the distinction is of no consequence in calculations: 3+3=6 whatever the level of abstraction. We deem that building fuzzy arithmetic by taking inspiration from crisp arithmetic rather than random arithmetic has been a sore mistake.

We come to another stumbling block of set-theoretic fuzzy arithmetic: what should a fuzzy number be, when does a fuzzy quantity deserve to be called a fuzzy number?

Everybody agrees that (piecewise) linear fuzzy triangles (u, a, v) as in Section 2 do qualify for appropriate fuzzy numbers, but the increasing portion on $[u, a]$ and the decreasing one on $[a, v]$ should be allowed to be non-linear, even if "regular enough". Definitions of fuzzy numbers found in the literature are slightly at divergence from one another, but the following requests appear to be typical, thinking of a triangular number (u, a, v) as a starting point:

outside $[u, v]$ *the function* $f_X(x)$ *is zero,* $f_X(a) = 1$*,* $f_X(x)$ *is strictly increasing on* $[u, a]$ *and strictly decreasing on* $[a, v]$*; to ensure that the suprema of marginalizations are also maxima (are actually achieved) one further imposes upper semicontinuity*

We make some critical remarks: definitions like this rule out fuzzy intervals $[u, a, b, v]$ but then should one be allowed to sum an appropriate fuzzy number with a fuzzy quantity which is *not* a number, even if the addition rule is available and readily usable? More seriously: the support, i.e. the subset of $\mathbb{R}$ where $f_X(x) \neq 0$, has to be an interval: what about the inverse $Y = X^{-1}$ of the quite appropriate linear triangle $X = (0, 1, 2)$? The rule to compute $Y = \phi(X) = X^{-1}$ is available and usable, but it gives back a quantity whose support is a half-line; the unpleasant consequence would be that the inverse of a number, be it fuzzy, is *not* a number, be it fuzzy. All this entails, no wonder, that the word *unstable* pops up quite frequently in set-theoretic fuzzy arithmetic. Unfortunately, unassuming fuzzy quantities, which might work as a sort of "escape route", are usually ruled out.

Once more we prefer to mimic probability theory and its authoritative century-old history. The definition of *random variables*, both in the explicit Kolmogorov's approach and in the implicit Pitt's approach, are as ample as the mathematics of σ-additive measures allows it to be, even if this gives obviously room to "monsters" one will never use in practice. Important theorems require that the definition be restricted, for example one might require that the variance of X does exist, or even, as happens in statistics, that X is normal (gaussian). In our opinion the same should be done in fuzzy arithmetic: the definition should be as large as the maxitive mathematics of suprema allow it to be, precisely as we did in Section 2 with Π-numbers. Needless to say, important theorems may require that the fuzzy number is very regular, for example that it fits the definition with upper semicontinuity given above.

We are strongly convinced that fuzzy arithmetic should be done along the generous lines that we have settled for possibilistic arithmetic. To resume:

- a fuzzy number X should be defined by a non-negative function $f_X(x)$ without any restriction, save for normality if one is interested only in complete arithmetic
- one may ignore interactivity, provided one is able to distinguish at least between equidistribution $X \approx Y$, i.e. $f_X(x) = f_Y(x)$, and deterministic equality $X = Y$

Remark 6.1 (Irrelevance). A theorem known very early in fuzzy arithmetic, for which cf. e.g. [3, 4, 7, 8, 18, 20], deals with generalized fuzzy triangles X and Y whose defining functions $f_X(x)$ and $f_Y(x)$ on the common support $[u, v]$ are bound to be convex-cup, which certainly is the case in the piecewise linear situation of an actual triangle. The generalized triangles X and Y are assumed to be equidistributed, $f_X(x) = f_Y(x)$. Think now of a binary operation $X \circ Y$ "regular enough" which might be the sum $X + Y$ or the product $X \cdot Y$ (in the latter case one has however to assume also $u \geq 0$, i.e. a non-negative support). This is a proto-example of *irrelevance*, because it can be proven, using our terminology, that the distribution function $f_{X \circ Y}(x)$ of $X \circ Y$ is the *same* both under non-interactivity and under deterministic equality; the choice of the joint distribution out of these two is irrelevant. This relevant fact, in itself quite fortunate, has probably had a bad consequence, since it has masqued the importance of accurately distinguishing between two different ways of sticking together marginal distributions: irrelevance is not always at hand.

7. Conclusions and future work

We did mathematics rather than philosophy: the discussion remains open whether a possibilistic number is related or not to a fuzzy number thinking of the *meaning* of fuzziness and logical possibilities. Our point is that they should be tackled in the same way insofar as mathematical calculations are involved: after all, even in probability theory, or rather in probability theor*ies* [5], the same calculations can be interpreted in quite different ways by objectivists (empiricists, followers of Ludwig von Mises, 1881-1973) versus subjectivists (neo-Bayesians, followers of Bruno de Finetti, already mentioned, and Leonard J. Savage, 1917-1971).

The message is: everything works if, beside non-interactivity (which is always there, even if implicit or even "hidden"), one considers also deterministic equality. Crisp arithmetic may ignore the difference, but random and fuzzy arithmetic cannot.

The theoretic framework which has been presented above is best appreciated in contexts where heavy computations are needed, a meaningful example being our coding-theoretic paper [9], to which the reader is referred. As for future work, one might ask: should a Pitt-like approach to possibilistic and fuzzy arithmetic be relinquished in favor of an explicit Kolmogorov-like approach? At present, this remains a moot point: in probability the advantages of the explicit approach are quite obvious when one moves to complicated stochastic processes, not even necessarily ergodic, but an analog theory of possibilistic or *dynatic* processes is at the moment almost non-existing; cf. however [16, 17] and also [9], where the possibilistic processes envisaged are actually quite unassuming.

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