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The Euclidean Universe

Vieri Benci

I dedicate this work to my friend Eugenio who more than 40 years ago, when we were two young guys in New York, he gave me a book by Keisler and introduced me to the beauty of Nonstandard Analysis

Abstract. *We introduce a mathematical structure called Euclidean Universe. This structure provides a basic framework for Non-Archimedean Mathematics and Nonstandard Analysis.*

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1. Introduction

In this paper, we introduce a mathematical structure called Euclidean Universe. This structure provides a basic framework for Non-Archimedean Mathematics, namely the Mathematics based on infinite and infinitesimal numbers.

The Euclidean Universe is defined by three axioms which have been chosen in such a way to appear absolutely *natural*. The first axiom introduce the infinite numbers as the *numerosities* of infinite sets in such a way that the V Euclidean common notion

the whole is larger than the part

be preserved. In the second axiom we introduce the Euclidean line with the following peculiarity: if any magnitude can be "represented" by a point on the Euclidean line, then also the infinite (and consequently the infinitesimal) magnitudes have this right. Then the Euclidean line must be larger than the real line. The last axiom is more technical and it is necessary to make the Euclidean line to include (a copy of) the real numbers.

Actually, this paper can be considered a new introduction to the Non-Archimedean Mathematics in the spirit of Veronese [20, 21] and Levi-Civita [18]. Moreover, these axioms are sufficiently strong to include the basic principles of Nonstandard Analysis such as the Leibniz Principle and the Saturation Principle. In particular the Euclidean line incudes, as subfields, infinitely many copies of the hyperreal numbers of any saturation less or equal to the first inaccessible number (see Definition 2.1).

Then, the Euclidean Universe includes many results of Non-Archimedean Mathematics obtained in the last 30 years.

For a recent historical and foundational analysis of the underlying ideas, we refer to [11] and the references therein.

2. The three Axioms

2.1. The Numerosity axiom

The first axiom defines the notion of **numerosity**. The notion of numerosity was introduced in [2, 6, 9] as a generalization of finite cardinality that also applies to infinite sets. The main feature of numerosities is that they preserve the spirit of the ancient Euclidean principle that *the whole is larger than the part*; indeed, the numerosity of a proper subset is strictly smaller than the numerosity of the whole set.

In principle it would be desirable to define the numerosity for the class of all sets; however, in order to develop the theory, it is convenient to work in a "universe" which is a set closed with respect to the main set operations provided that it is very large. In order to do this we recall a well known notion in set theory:

Definition 2.1. *A cardinal number χ is* inaccessible *if it is not a sum of fewer than χ cardinals that are less than χ and $\zeta < \chi$ implies $2^{\zeta} < \chi$.*

χ is strongly inaccessible *if it is inaccessible and uncountable.*

The first inaccessible cardinal number is $\aleph_0$. The first strongly inaccessible cardinal number will be denoted by κ. The existence of sets of strongly inaccessible cardinality is established by the Axiom of Inaccessibility which is independent from ZFC. We will assume this axiom and in particular we will assume that there exists a set of atoms[1] $\mathbb{A}$ having cardinality κ. Then we can define a "universe" Λ defined as follows:

$$\Lambda = \{X \in V(\mathbb{A}) \mid |X| < \kappa\} \tag{1}$$

where for any set A, $V(A)$ denotes the superstructure over A namely

$$V(A) = \bigcup_{n\in\mathbb{N}} V_n(A)$$

with $V_0(A) = A$ and, for every $n \in \mathbb{N}$,

$$V_{n+1}(A) = V_n(A) \cup \wp\left(V_n(A)\right). \tag{2}$$

[1] In set theory, an atom a is any entity that is not a set, namely a is an atom if and only if

$$\forall x,\ x \notin a$$

We will refer to Λ as to the **accessible universe** since its sets have strongly accessible cardinality (and finite rank[2]). Λ can be split as follows

$$\Lambda = \Lambda_S \cup \mathbb{A}$$

where Λ_S is a family of sets. Notice that $\mathbb{A} \notin \Lambda$ since $|\mathbb{A}| = \kappa$.

Now we are ready to state our first axiom:

Axiom 2.2 (Numerosity axiom). *The numerosity is a surjective map*

$$\mathfrak{num} : \Lambda_S \to \mathfrak{N}, \quad \mathfrak{N} \subset \mathbb{A}$$

which satisfies the following properties: if $a, b \in \Lambda$ *and* A,B,A',$B' \in \Lambda_S$,

1. $\mathfrak{num}(\{a\}) = \mathfrak{num}(\{b\})$,
2. *if* $A \subset B$ *strictly, then*
$$\mathfrak{num}(A) < \mathfrak{num}(B) \ ,$$
3. *if* $A \cap B = \varnothing$, $\mathfrak{num}(A) = \mathfrak{num}(A')$, $\mathfrak{num}(B) = \mathfrak{num}(B')$, *then*
$$\mathfrak{num}(A \cup B) = \mathfrak{num}(A' \cup B') \ ,$$
4. *if* $\mathfrak{num}(A) = \mathfrak{num}(A')$, $\mathfrak{num}(B) = \mathfrak{num}(B')$, *then*
$$\mathfrak{num}(A \times B) = \mathfrak{num}(A' \times B') \ ,$$
5. *if* $A \in \Lambda_S$ *and* $b \in \Lambda$, *then* $\mathfrak{num}(A \times \{b\}) = \mathfrak{num}(A)$.

If F and F' are finite sets of the same cardinality, by 2.2.1 and 2.2.3, it follows that $\mathfrak{num}(F) = \mathfrak{num}(F')$; then, by 2.2.5, the numerosities of finite sets can be identified with the natural numbers $\mathbb{N}$. By 2.2.3, $\mathfrak{N}$ can be equipped with an "addition" by setting

$$\sigma + \tau = \mathfrak{num}(A \cup B) \ ,$$

where $\sigma = \mathfrak{num}(A)$, $\tau = \mathfrak{num}(B)$ and $A \cap B = \varnothing$; similarly, by 2.2.4, we can define a "multiplication":

$$\sigma \cdot \tau = \mathfrak{num}(A \times B) \ .$$

Clearly $0 = \mathfrak{num}(\varnothing)$ is the neutral element with respect to the addition and, by 2.2.5, $\mathfrak{num}(\{b\}) = 1$ is the neutral element with respect to the multiplication for any $b \in \Lambda$.

[2]The rank of $\varnothing$ is 0. The rank of a set $E \neq 0$ is the least ordinal number greater than the rank of any element of the set.

2.2. The Euclidean Field Axiom

Our second axioms identifies the Euclidean (straight) line with a field $\mathbb{E}_\kappa \subseteq \mathbb{A}$. Usually the Euclidean line is identified with the real line, however we think that this point of view is too restrictive. In fact, the main intuitive peculiarities of the Euclidean line are the following:

- two oriented segments of the Euclidean line can be added or subtracted;
- if we choose a unitary segment of the Euclidean line, two segments can be multiplied or divided;
- once we have chosen two distinguish points O and U on the line, every magnitude can be posed in a biunivocal correspondence with a point (provided that the unitary magnitude has been defined).

Then, if we take $0 = O$ and $1 = U$, the Euclidean line gets the stucture of ordered field and its points can be identified with numbers. Since the numerosities can be considered magnitudes, the Euclidean line should be richer than the real line. We can formalize these intuitive remarks by the following axiom:

Axiom 2.3 (Euclidean Field Axiom). *There is an ordered field* $\mathbb{E}_\kappa \subset \mathbb{A}$ *such that*

- $\mathbb{E}_\kappa$ *contains the set numerosities* $\mathfrak{N}$ *and the field operations* $+, \cdot$ *coincide with the numerosity operations;*
- *for every* $\xi \in \mathbb{E}_\kappa$, *there exists* $E \in \Lambda$, *such that*

$$|\xi| < \mathfrak{num}\,(E)\,. \tag{3}$$

We will refer to $\mathbb{E}_\kappa$ as the *Euclidean line* or the *Euclidean field* and its elements will be called *Euclidean numbers.* $\mathbb{E}_\kappa$ contains infinite numbers, then it is a non-Archimedean field. Now let us recall some basic definitions relative to non-Archimedean fields. Since $\mathbb{N} \subset \mathbb{E}_\kappa$, the following definition makes sense:

Definition 2.4. *Let* $\xi \in \mathbb{E}_\kappa$. *We say that:*

- ξ *is* infinitesimal *if* $\forall n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ *is* finite *(or bounded) if* $\exists n \in \mathbb{N}$ *such as* $|\xi| < n$;
- ξ *is* infinite *(or unbounded) if* $\forall n \in \mathbb{N}$, $|\xi| > n$.

The following proposition establishes some relations among Euclidean numbers:

PROPOSITION 2.5. *We have the following relations:*

(i) *If ε e δ are infinitesimal, also $\varepsilon+\delta$, $\varepsilon-\delta$ and $\varepsilon\cdot\delta$ are infinitesimal.*

(ii) *If ξ e σ are bounded also $\xi+\sigma$, $\xi-\sigma$ e $\xi\cdot\sigma$ are bounded.*

(iii) *If θ e τ are infinite, also $\theta\cdot\tau$ is infinite; moreover if θ and τ are postive infinite (or negative), also $\omega+\tau$ is postive infinite (or negative).*

(iv) *If ε is infinitesimal and ξ is bouded $\varepsilon\cdot\xi$ is infinitesimal; moreover if $\varepsilon\neq 0$ and ξ is not infinitesimal, ξ/ε is infinite.*

Proof. The proof of this proposition is easy and it is left to the reder. □

DEFINITION 2.6. *We say that two numbers $\xi,\zeta\in\mathbb{E}_\kappa$ are* infinitely close *if $\xi-\zeta$ is infinitesimal. In this case, we write $\xi\sim\zeta$.*

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation. Then the following definition comes naturally

DEFINITION 2.7. *If $\xi\in\mathbb{E}_\kappa$, the monad of ξ is the set of all numbers that are infinitely close to it:*

$$\mathfrak{mon}(\xi)=\{\zeta\in\mathbb{E}_\kappa\mid\xi\sim\zeta\},$$

The galaxy of ξ is the set of all numbers that are finitely close to it:

$$\mathfrak{gal}(\xi)=\{\zeta\in\mathbb{E}_\kappa\mid\xi-\zeta\textit{ is a finite number}\}.$$

2.3. The Center Axiom

The notion of monad allows to state our last axiom:

AXIOM 2.8. *Every monad μ has a distinguished point called* center *of μ and denoted by $Ctr(\mu)$; the set $\mathfrak{C}$ of all the centers is an additive subgroup of $\mathbb{E}_\kappa$ containing $\mathfrak{N}$.*

For every $\xi\in\mathbb{E}_\kappa$, we will use the notation

$$ctr\,(\xi)=Ctr(\mathfrak{mon}\,(\xi));$$

so, every number $\xi\in\mathbb{E}_\kappa$ can be decomposed as follows:

$$\xi=x+\varepsilon \tag{4}$$

where $x=ctr\,(\xi)$ and $\varepsilon\sim 0$. Then $\mathbb{E}_\kappa$ can be decomposed as follows:

$$\mathbb{E}_\kappa=\mathfrak{C}\times\mathfrak{mon}\,(0)$$

and

$$ctr:\mathbb{E}_\kappa\to\mathfrak{C}$$

is a projection.

3. Structure of the Euclidean numbers

In this section we will examine some peculiarities of $\mathbb{E}_\kappa$. In particular, we will see that $\mathbb{E}_\kappa$ contains the ordinal numbers of accessible cardinalities and the real numbers. Moreover we will introduce the notion Λ-limit which, in this context, is a very basic tool.

3.1. The ordinal numerosities

In this subsection we introduce a set $\mathbf{Ord} \subset \mathfrak{N}$ that is isomorphic (in a sense specified below) to the initial segment of length κ of ordinal numbers.

Definition 3.1. *The set* $\mathbf{Ord} \subset \mathfrak{N}$ *of ordinal numerosities is defined as follows:* $\tau \in \mathbf{Ord}$ *if and only if*

$$\tau = \mathfrak{num}\left(\Omega_\tau\right),$$

where

$$\Omega_\tau = \{x \in \mathbf{Ord} \mid x < \tau\}.$$

It is easy to see by transfinite induction that this is a good recursive definition. In fact, it is immediate to check that

- $0 \in \mathbf{Ord}$;
- if $\tau \in \mathbf{Ord}$, then $\tau + 1 = \mathfrak{num}\left(\Omega_\tau \cup \{\tau\}\right) \in \mathbf{Ord}$ (and hence $\mathbb{N} \subset \mathbf{Ord}$).
- if $\tau_k = \mathfrak{num}\left(\Omega_k\right)$, $k \in K$, $(|K| < \kappa)$ are ordinal numerosities, then

$$\tau := \mathfrak{num}\left(\bigcup_{k\in K} \Omega_k\right) \in \mathbf{Ord}.$$

In particular, $\omega = \mathfrak{num}\left(\mathbb{N}\right)$ is an ordinal. This construction of the ordinal numbers is similar to the construction of Von Neumann. However, whilst a Von Neumann ordinal τ is the set of all the Von Neumann ordinals contained in τ, in our construction an ordinal τ is the numerosity of the set of ordinals smaller than τ. Hence, here, an ordinal number, as any other numerosity, is an atom in $\mathbb{E}_\kappa$.

It is easy to see that $\mathbf{Ord}$ is well ordered and hence it is isomorphic to the initial segment of length κ of the full class of the ordinals. Obviously, not all numerosities are ordinals: for example, $\omega - 1 = \mathfrak{num}\left(\mathbb{N}^+\right) = \mathfrak{num}\left(\mathbb{N}\backslash\{0\}\right)$ is not an ordinal.

The most remarkable thing in this theory is that the numerosity operations $+$ and $\cdot$, correspond to the *natural* (or Hessenberg) operations between ordinals. We refer to [13] for an in-depth analysis of this topic.

Remark 3.2. Notice that the existence of $\mathbf{Ord}$ depends only on Axiom 1.

3.2. The Λ-limit theorem

We set
$$\mathfrak{L} = \{\lambda \in \Lambda \mid \lambda \text{ is a finite set}\}$$
and
$$\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa) = \left\{\varphi \in \mathbb{E}_\kappa^{\mathfrak{L}} \mid \exists A \in \Lambda, \forall \lambda \in \mathfrak{L},\ \varphi(\lambda) = \varphi(\lambda \cap A)\right\}.$$
Since $(\mathfrak{L}, \subseteq)$ is a directed sets, the elements of $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ are nets. The set $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ is a partially ordered commutative algebra over $\mathbb{E}_\kappa$ with the operations
$$\begin{aligned}(\varphi + \psi)(\lambda) &= \varphi(\lambda) + \psi(\lambda), \\ (\varphi \cdot \psi)(\lambda) &= \varphi(\lambda) \cdot \psi(\lambda).\end{aligned}$$

Theorem 3.3 (Λ-limit theorem). *There is a unique ring homomorphism*
$$J : \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa) \to \mathbb{E}_\kappa$$
such that,
$$\forall A \in \Lambda,\ J(\psi_A) = \mathfrak{num}(A),$$
where
$$\psi_A(\lambda) = |A \cap \lambda|. \tag{5}$$

Proof. Let $\mathfrak{F}_q(\mathfrak{L}, \mathbb{E}_\kappa)$ be the $\mathbb{E}_\kappa$-subalgebra of $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ *generated* by $\{\psi_A \mid A \in \Lambda\}$ namely the subset of the elements φ of $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ which can be written as follows:
$$\varphi(\lambda) = \frac{\sum_{A \in \mathcal{A}} a_A \psi_A(\lambda)}{\sum_{B \in \mathcal{B}} b_A \psi_B(\lambda)},$$
where $\mathcal{A}, \mathcal{B}$ are finite subsets of Λ, $a_A, b_A \in \mathbb{E}_\kappa$, ψ_A, ψ_B are defined by (5) and $\forall \lambda \in \mathfrak{L},\ \sum_{B \in \mathcal{B}} b_A \psi_B(\lambda) \neq 0$. We define a field homomorphism
$$J_q : \mathfrak{F}_q(\mathfrak{L}, \mathbb{E}_\kappa) \to \mathbb{E}_\kappa$$
as follows:
$$J_q(\varphi) = \frac{\sum_{A \in \mathcal{A}} a_A \cdot \mathfrak{num}(A)}{\sum_{B \in \mathcal{B}} b_A \cdot \mathfrak{num}(B)}.$$
Since $\mathrm{Im}(J_q) = \mathbb{E}_\kappa$ is a field, $\ker(J_q)$ is a maximal ideal in $\mathfrak{F}_q(\mathfrak{L}, \mathbb{E}_\kappa)$; hence, the set
$$\mathcal{U}_0 = \left\{Q \in \mathfrak{L} \mid \exists \psi \in \ker(J_q),\ Q = \psi^{-1}(0)\right\}$$
is a filter over $\mathfrak{L}$. We denote by $\mathcal{U}$ an ultrafilter such that $\mathcal{U}_0 \subseteq \mathcal{U}$. Then also
$$\mathfrak{I} := \{\psi \in \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa) \mid \exists Q \in \mathcal{U},\ \forall \lambda \in Q,\ \psi(\lambda) = 0\}$$
is a maximal ideal in $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ and hence
$$\mathbb{F} := \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa) / \mathfrak{I}$$

is a field. We will see that $\mathbb{F}$ is isomorphic to $\mathbb{E}_\kappa$. We denote by $[\varphi]$ a generic element of $\mathbb{F}$ and we claim that

$$\forall [\varphi] \in \mathbb{F},\ \exists \bar{\mu} \in \mathfrak{L},\ \exists Q \in \mathcal{U},\ \forall \lambda \in Q,\ \varphi(\lambda) = \varphi(\bar{\mu}), \tag{6}$$

namely

$$[\varphi] = \left[C_{\varphi(\bar{\mu})}\right], \tag{7}$$

where $\lambda \mapsto C_\xi(\lambda)$ denotes the net identically equal to ξ. In order to prove (7) we set

$$R^- := \left\{\mu \in \mathfrak{L} \mid \left[C_{\varphi(\mu)}\right] < [\varphi]\right\}, \tag{8}$$

$$R^0 := \left\{\mu \in \mathfrak{L} \mid \left[C_{\varphi(\mu)}\right] = [\varphi]\right\}, \tag{9}$$

$$R^+ := \left\{\mu \in \mathfrak{L} \mid \left[C_{\varphi(\mu)}\right] > [\varphi]\right\}. \tag{10}$$

By (8), if $R^- \neq \varnothing,\ \forall \mu \in R^-,\ \exists Q_\mu^- \in \mathcal{U}$ such that

$$\forall \lambda \in Q_\mu^-,\ C_{\varphi(\mu)}(\lambda) < \varphi(\lambda)$$

then,

$$\mu \in R^- \cap Q_\mu^- \Rightarrow\ C_{\varphi(\mu)}(\mu) < \varphi(\mu)$$

and since, by definition $C_{\varphi(\mu)}(\mu) = \varphi(\mu)$, it follows that

$$\forall \mu \in R^-,\ R^- \cap Q_\mu^- = \varnothing$$

and hence,

$$R^- \notin \mathcal{U}.$$

By (10), arguing in the same way, we have that

$$R^+ \notin \mathcal{U}.$$

Since $(R^- \cup R^+) \cup R^0 = \mathfrak{L}$, it follows that $R^0 \in \mathcal{U}$ and hence $R^0 \neq \varnothing$. Now, if you take $\bar{\mu}$ in R^0, there is $Q^0 \in \mathcal{U}$ such that

$$\forall \lambda \in Q^0,\ \varphi(\lambda) = C_{\varphi(\bar{\mu})}(\lambda) = \varphi(\bar{\mu}),$$

namely (7) is satisfied. Now, we can extend J_q to $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$; given $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$, using (6) we set

$$J(\varphi) = \left[C_{\varphi(\bar{\mu})}\right].$$

So every function φ in $\mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ is eventually constant in the sense that

$$\exists \xi \in \mathbb{E}_\kappa, \exists Q \in \mathcal{U}, \forall \lambda \in Q,\ \varphi(\lambda) = \xi.$$

Then $\mathbb{F}$ is isomorphic to $\mathbb{E}_\kappa$.

It remains to prove the uniqueness of J. Let us assume that J_1 and J_2 extend J_q to all $\mathfrak{F}(\mathfrak{L},\mathbb{E}_\kappa)$. We have to prove that for every $\varphi \in \mathfrak{F}(\mathfrak{L},\mathbb{E}_\kappa)$

$$J_1(\varphi) = J_2(\varphi).$$

We set $c_1 = J_1(\varphi)$; $c_2 = J_2(\varphi)$,

$$\begin{aligned} A_\varphi &= \{J_q(\psi) \mid \psi \in \mathfrak{F}_q(\mathfrak{L},\mathbb{E}_\kappa),\ \exists Q \in \mathcal{U}_0,\ \forall \lambda \in Q,\ \psi(\lambda),\ \psi \le \varphi\}; \\ B_\varphi &= \{J_q(\psi) \mid \psi \in \mathfrak{F}_q(\mathfrak{L},\mathbb{E}_\kappa)\ \ \exists Q \in \mathcal{U}_0,\ \forall \lambda \in Q,\ \psi(\lambda),\ \psi > \varphi\}. \end{aligned}$$

Clearly $\forall a \in A_\varphi,\ \forall b \in B_\varphi$,

$$a \le c_1 \le b \quad \text{and} \quad a \le c_2 \le b. \tag{11}$$

and hence, assuming that $c_1 \le c_2$

$$\forall a \in A_\varphi,\ \forall b \in B_\varphi,\ 0 \le c_2 - c_1 \le b - a.$$

Since $A_\varphi \cup B_\varphi = \mathfrak{F}_q(\mathfrak{L},\mathbb{E}_\kappa)$ contains $\frac{1}{\mathfrak{num}(E)}$ for any set $E \in \Lambda_S \setminus \{0\}$, we have that

$$0 \le c_2 - c_1 \le \frac{1}{\mathfrak{num}(E)}$$

and so, by (3), $c_2 = c_1$. $\square$

Definition 3.4. *The number $J(\varphi)$ is called Λ-limit of the net φ and will be denoted by*

$$J(\varphi) = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda).$$

The reason of this name and notation is that the operation

$$\varphi \mapsto \lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$$

satisfies some of the properties of the usual limit over a net:

- If eventually $\varphi(\lambda) \ge \psi(\lambda)$, then
$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \ge \lim_{\lambda \uparrow \Lambda} \psi(\lambda).$$
- If $\forall q \in \mathbb{Q}$, $C_q(\lambda) = q$, then
$$\lim_{\lambda \uparrow \Lambda} C_q(\lambda) = q.$$
- For all $\varphi, \psi \in \mathfrak{F}(\mathfrak{L},\mathbb{E}_\kappa)$
$$\begin{aligned} \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) + \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda)), \\ \lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) &= \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda)). \end{aligned}$$

In this framework, Λ can be regarded as the "point at infinity" of $\mathfrak{L}$. The Λ-limit is not a limit in a topological sense, in fact there are also strong differences with a topological limit; we list some of them:

- Every net $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$ has a limit $L \in \mathbb{E}_\kappa$.
- If $\forall \lambda \in \mathfrak{L}$, $\varphi(\lambda) \neq 0$, then $\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \neq 0$; in fact,
$$\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \cdot \lim_{\lambda\uparrow\Lambda} \frac{1}{\varphi(\lambda)} = 1$$
and hence $\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \neq 0$.
- If, $\xi \in \mathbb{E}_\kappa \backslash \mathbb{R}$,
$$\lim_{\lambda\uparrow\Lambda} C_\xi(\lambda) \neq \xi\,.$$

For example, take
$$\omega := \lim_{\lambda\uparrow\Lambda} |\lambda \cap \mathbb{N}|\,;$$
then $\forall \lambda \in \mathfrak{L},\ |\lambda \cap \mathbb{N}| < \omega$, so
$$0 > \lim_{\lambda\uparrow\Lambda} (|\lambda \cap \mathbb{N}| - \omega) = \lim_{\lambda\uparrow\Lambda} |\lambda \cap \mathbb{N}| - \lim_{\lambda\uparrow\Lambda} \omega = \omega - \lim_{\lambda\uparrow\Lambda} \omega$$
and hence
$$\omega < \lim_{\lambda\uparrow\Lambda} \omega.$$

The last statement suggests the following notation: for any $\xi \in \mathbb{E}_\kappa$, we set
$$\xi^* = \lim_{\lambda\uparrow\Lambda} \xi. \tag{12}$$

3.3. The real numbers

We remark that the notion and the first properties of the Λ-limit do not depend on Axiom 2.8. In this section we will see that also Axiom 2.8 is very relevant.

Definition 3.5. *An Euclidean number is called* standard *if it is finite and it is the center of a monad. The set of standard points will be denoted by* $\mathbb{R}$*, namely*
$$\mathbb{R} := \mathfrak{C} \cap \mathfrak{gal}(0).$$
If ξ is a finite number, then $ctr(\xi)$ is called standard part *of x and will be denoted also by $st(\xi)$.*

First let us examine some (obvious) properties of the function $st(\cdot)$.

Proposition 3.6. *Let ξ and ζ be finite numbers, then*

1. *$\xi \in \mathbb{R} \Leftrightarrow st(\xi) = \xi$;*
2. *$\xi \leq \zeta \Rightarrow st(\xi) \leq st(\zeta)$;*
3. *$st(\xi + \zeta) = st(\xi) + st(\zeta)$;*
4. *$st(\xi \cdot \zeta) = st(\xi) \cdot st(\zeta)$;*
5. *if $st(\zeta) \neq 0$, then $st\left(\frac{\xi}{\zeta}\right) = \frac{st(\xi)}{st(\zeta)}$.*

Proof. The first four statements trivially descend from (4) and Proposition 2.5. In order to prove 3.6.4, we put

$$\xi = r + \varepsilon \,, \qquad \zeta = s + \theta \,,$$

where $r, s \in \mathbb{R}$, $\varepsilon \sim \theta \sim 0$. Then,

$$st(\xi \cdot \zeta) = st\left[(r+\varepsilon)(s+\theta)\right] = st\left[rs + (\varepsilon s + \theta r + \varepsilon\theta)\right] \,.$$

Since $\varepsilon s + \theta r + \varepsilon\theta \sim 0$, we have that $st(\xi \cdot \zeta) = rs = st(\xi) \cdot st(\zeta)$. Let us prove 3.6.5; by 3.6.4 we have that

$$st(\zeta) \cdot st\left(\frac{\xi}{\zeta}\right) = st\left(\zeta \cdot \frac{\xi}{\zeta}\right) = st(\xi) \,;$$

hence

$$st\left(\frac{\xi}{\zeta}\right) = \frac{st(\xi)}{st(\zeta)}. \qquad \square$$

Theorem 3.7. *The set of standard numbers $\mathbb{R}$ is isomorphic to the set of real numbers.*

Proof. We will prove that every Cauchy sequence of rationals is convergent to some $L \in \mathbb{R}$ with respect to the metric topology. Let x_n be a Cauchy sequence in $\mathbb{Q}$. We set

$$\varphi(\lambda) := x_{|\mathbb{N} \cap \lambda|}$$

and

$$L = st\left(\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)\right).$$

Then, by Proposition 3.6.1, $L \in \mathbb{R}$. We have to prove that L is the Cauchy limit of x_n. We choose a number $\varepsilon \in \mathbb{Q}^+$; then, there exists n_0 such that $\forall n, m \geq n_0$,

$$|x_n - x_m| < \varepsilon \,.$$

Now take $\lambda_0 \in \mathfrak{L}$ such that $|\mathbb{N} \cap \lambda_0| \geq n_0$; thus, $\forall \lambda \supset \lambda_0$, we have that $|\mathbb{N} \cap \lambda| \geq n_0$. Then

$$|\varphi(\lambda) - x_m| = \left|x_{|\mathbb{N} \cap \lambda|} - x_m\right| < \varepsilon$$

and taking the Λ-limit, we get the conclusion:

$$\varepsilon > \lim_{\lambda\uparrow\Lambda} |\varphi(\lambda) - x_m| = \left| \lim_{\lambda\uparrow\Lambda} \varphi(\lambda) - x_m \right| = |L - x_m| . \qquad \square$$

From now on, the set $\mathbb{R}$ of standard numbers will be identified with the set of real numbers, namely the real number will be considered "special" points on the Euclidean line.

Given a net $\varphi : \Lambda \to \mathbb{R}$, since Λ is a directed set, also the Cauchy limit is well defined:

$$L = \lim_{\lambda\to\Lambda} \varphi(\lambda) \Leftrightarrow \forall\varepsilon \in \mathbb{R}^+, \exists\lambda_0 \in \mathfrak{L}, \forall\lambda \supset \lambda_0, |\varphi(\lambda) - L| \le \varepsilon . \tag{13}$$

Notice that in order to distinguish the Cauchy limit (13) from the Λ-limit, we have used the symbols "$\lambda \to \Lambda$" and "$\lambda \uparrow \Lambda$" respectively.

The standard part of a number is related to the Cauchy notion of limit. If a real net x_λ admits the Cauchy limit, the relation with the Λ-limit is given by the following identity:

$$\lim_{\lambda\to\Lambda} x_\lambda = st\left(\lim_{\lambda\uparrow\Lambda} x_\lambda\right) . \tag{14}$$

Another important relation between the two limits is the following:

PROPOSITION 3.8. *If*

$$\lim_{\lambda\uparrow\Lambda} x_\lambda = \xi \in \mathbb{E}_\kappa$$

and ξ is bounded, then there exist a sequence $\lambda_n \in \mathfrak{L}$ such that

$$\lim_{n\to\infty} x_{\lambda_n} = st(\xi).$$

Proof. Set $x_0 = st(\xi)$ and for every $n \in \mathbb{N}$, take λ_n such that $x_{\lambda_n} \in [x_0 - 1/n,\ x_0 + 1/n]$. $\square$

REMARK 3.9. As we have remarked in the intruduction, the Centrum Axiom is necessary to prove Theorem 3.7. Actually it is not difficult to prove that the Centrum Axiom is equivalent to the following:

$\mathbb{E}_\kappa$ contains a subfield isomorphic to the field of real numbers.

The notion of "standard entity" can be extended from numbers (i.e. the real numbers) to other elements of the universe by the following definition:

DEFINITION 3.10. *An element $E \in V(\mathbb{R})$ is called* standard *and $V(\mathbb{R})$ is called* standard universe*; $V(\mathbb{E}_\kappa)$ is called* Euclidean universe*.*

Notice that

$$V(\mathbb{R}) \subset \Lambda \subset V(\mathbb{E}_\kappa).$$

Also the second inclusion is strict since $V(\mathbb{E}_\kappa)$ contains sets of inaccessible cardinality such as $\mathbb{E}_\kappa$.

4. The Euclidean universe

In this section we will inestigate the structure of the Euclidean universe $V(\mathbb{E}_\kappa)$.

4.1. Λ-limit of sets

By Definition 3.4, the Λ-limit has been defined for every net $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{E}_\kappa)$; next we will extend this notion to the nets of sets in

$$\mathfrak{F}(\mathfrak{L}, V_n(\mathbb{E}_\kappa)) = \left\{\Phi \in V_n(\mathbb{E}_\kappa)^{\mathfrak{L}} \mid \exists A \in \Lambda, \forall \lambda \in \mathfrak{L},\ \Phi(\lambda) = \Phi(\lambda \cap A)\right\}$$

for every $n \in \mathbb{N}$. In the following, in order to simplify the notation, a net of sets $\Phi \in \mathfrak{F}(\mathfrak{L}, V_n(\mathbb{E}_\kappa))$ will be denoted by $\{E_\lambda\}$ where $E_\lambda = \Phi(\lambda)$.

We define the Λ-limit of sets by induction over n. If $n = 0$, $\lim_{\lambda\uparrow\Lambda} \Phi(\lambda)$ is a net of numbers defined by Definition 3.4; if $n > 0$, we set

$$E_\Lambda = \lim_{\lambda\uparrow\Lambda} E_\lambda := \left\{\lim_{\lambda\uparrow\Lambda} \Psi(\lambda) \mid \Psi \in \mathfrak{F}(\mathfrak{L}, V_{n-1}(\mathbb{E}_\kappa)), \forall\lambda\in\mathfrak{L} : \Psi(\lambda) \in E_\lambda\right\}. \tag{15}$$

Clearly, by (1), $E_\Lambda \in V(\mathbb{E}_\kappa)$.

Definition 4.1. *A set E obtained as Λ-limit of a net of sets*

$$\{E_\lambda\} \in \mathfrak{F}(\mathfrak{L}, V_n(\mathbb{E}_\kappa))$$

is called internal. *If not it, is called* external.

For example the set $\mathbb{R}$ is external.

If $C_A(\lambda) = A \in \Lambda_S$ is a constant net, we set

$$A^* := \lim_{\lambda\uparrow\Lambda} C_A(\lambda) = \left\{\lim_{\lambda\uparrow\Lambda} \Psi(\lambda) \mid \Psi \in \mathfrak{F}(\mathfrak{L}, V_{n-1}(\mathbb{E}_\kappa)), \forall\lambda\in\mathfrak{L} : \Psi(\lambda) \in A\right\}; \tag{16}$$

then, if $A \in V_n(\mathbb{E}_\kappa)$, also $A^* \in V_n(\mathbb{E}_\kappa)$. This definition extends (12) to all the elements of $\Lambda = \Lambda_S \cup \mathbb{E}_\kappa$. A^* will be called the $*$-*transform* of A.

The $*$-transform allows to build a family $\{\mathbb{E}_j\}_{j\in\mathbf{Ord}}$ of subsets of $\mathbb{E}_\kappa$ as follows:

- $\mathbb{E}_0 = \mathbb{R}$;
- $\mathbb{E}_{j+1} = \mathbb{E}_j^*$;
- if $j \leq \kappa$ is a limit ordinal, then $\mathbb{E}_j = \bigcup_{k<j} \mathbb{E}_k$.

4.2. Λ-limit of functions

Since in set theory a function f can be identified with its graph Γ_f,

$$f_\Lambda := \lim_{\lambda\uparrow\Lambda} f_\lambda$$

is well defined. However, it is not immediate to see that f_Λ is function. For this reason, we will analyze this situation explicitly.

Theorem 4.2. *Given a net of functions* $\{f_\lambda\}$

$$f_\lambda : A_\lambda \to B_\lambda, \quad A_\lambda, B_\lambda \in V_n(\mathbb{E}_\kappa),$$

then $f_\Lambda : A_\Lambda \to B_\Lambda$ *defined by*

$$f_\Lambda\left(\lim_{\lambda\uparrow\Lambda} x_\lambda\right) = \lim_{\lambda\uparrow\Lambda} f_\lambda\left(x_\lambda\right); \tag{17}$$

is a function and we have that

$$\Gamma_{f_\Lambda} = \left(\Gamma_f\right)_\Lambda .$$

Proof. First, we will prove that (17) is a good definition, namely that $f_\Lambda(\xi)$ does not depend on the net x_λ which defines ξ. We set

$$\xi = \lim_{\lambda\uparrow\Lambda} x_\lambda = \lim_{\lambda\uparrow\Lambda} y_\lambda$$

and we have to prove that

$$\lim_{\lambda\uparrow\Lambda} f\left(x_\lambda\right) = \lim_{\lambda\uparrow\Lambda} f\left(y_\lambda\right).$$

We take

$$\chi\left(\lambda\right) = \begin{cases} 1 & if \quad x_\lambda = y_\lambda \\ 0 & if \quad x_\lambda \neq y_\lambda \end{cases} .$$

Hence $\forall\lambda$, $\chi\left(\lambda\right) + \left(x_\lambda - y_\lambda\right) \neq 0$ and so

$$\lim_{\lambda\uparrow\Lambda} \left[\chi\left(\lambda\right) + \left(x_\lambda - y_\lambda\right)\right] \neq 0,$$

then,

$$\begin{aligned} \lim_{\lambda\uparrow\Lambda} \chi\left(\lambda\right) &= \lim_{\lambda\uparrow\Lambda} \chi\left(\lambda\right) + \lim_{\lambda\uparrow\Lambda} x_\lambda - \lim_{\lambda\uparrow\Lambda} y_\lambda \\ &= \lim_{\lambda\uparrow\Lambda} \left[\chi\left(\lambda\right) + \left(x_\lambda - y_\lambda\right)\right] \neq 0. \end{aligned}$$

Moreover, we have that

$$\forall\lambda,\ \chi\left(\lambda\right) \cdot \left[f_\lambda\left(x_\lambda\right) - f_\lambda\left(y_\lambda\right)\right] = 0;$$

then

$$0=\lim_{\lambda\uparrow\Lambda}\left(\chi\left(\lambda\right)\cdot\left[f\left(x_{\lambda}\right)-\ f\left(y_{\lambda}\right)\right]\right)=\lim_{\lambda\uparrow\Lambda}\chi\left(\lambda\right)\cdot\lim_{\lambda\uparrow\Lambda}\left[f_{\lambda}\left(x_{\lambda}\right)-f_{\lambda}\left(y_{\lambda}\right)\right].$$

Since $\lim_{\lambda\uparrow\Lambda}\chi\left(\lambda\right)\neq 0$, we have that

$$0=\lim_{\lambda\uparrow\Lambda}\left[f_{\lambda}\left(x_{\lambda}\right)-\ f_{\lambda}\left(y_{\lambda}\right)\right]=\lim_{\lambda\uparrow\Lambda}\ f_{\lambda}\left(x_{\lambda}\right)-\lim_{\lambda\uparrow\Lambda}\ f_{\lambda}\left(y_{\lambda}\right).$$

Finally, it is immediate to check that f^{*} is the graph of the function (17), in fact

$$\begin{aligned}
\left(\Gamma_{f}\right)_{\Lambda} &= \left\{\lim_{\lambda\uparrow\Lambda}\left(x_{\lambda},f_{\lambda}\left(x_{\lambda}\right)\right)\ \mid\forall\lambda,\ \left(x_{\lambda},f_{\lambda}\left(x_{\lambda}\right)\right)\in\Gamma_{f_{\lambda}}\right\} \\
&= \left\{\left(\lim_{\lambda\uparrow\Lambda}\ x_{\lambda},\lim_{\lambda\uparrow\Lambda}f_{\lambda}\left(x_{\lambda}\right)\right)\ \mid\forall\lambda,\ x_{\lambda}=f_{\lambda}\left(x_{\lambda}\right)\right\} \\
&= \left\{\left(\xi,\ f_{\Lambda}\left(\xi\right)\right)\ \mid\xi=f_{\Lambda}\left(\xi\right)\right\}=\Gamma_{f_{\Lambda}}.
\end{aligned}$$

□

DEFINITION 4.3. *A function f obtained as Λ-limit of a net of functions f_{λ} is called* internal. *Otherwise is called* external.

If $\{f\}$ is a constant net, we set

$$f^{*}=\lim_{\lambda\uparrow\Lambda}\ f; \tag{18}$$

then, if $f:A\rightarrow B$, then $f^{*}:A^{*}\rightarrow B^{*}$ and f^{*} will be called the $*$-*transform* of f.

4.3. Hyperfinite sets

Another fundamental notion in Euclidean calculus is the following:

DEFINITION 4.4. *We say that a set $F\in V\left(\mathbb{E}_{\kappa}\right)$ is* hyperfinite *if there is a net $\left\{F_{\lambda}\right\}_{\lambda\in\Lambda}$ of finite sets such that*

$$F=\lim_{\lambda\uparrow\Lambda}\ F_{\lambda}=\left\{\lim_{\lambda\uparrow\Lambda}\ x_{\lambda}\mid x_{\lambda}\in F_{\lambda}\right\}.$$

The hyperfinite sets share many properties of finite sets. For example, a hyperfinite set $F\subset\mathbb{E}_{\kappa}$ has a maximum x_{M} and a minimum x_{m} respectively given by

$$x_{M}=\lim_{\lambda\uparrow\Lambda}\max F_{\lambda};\quad x_{m}=\lim_{\lambda\uparrow\Lambda}\min F_{\lambda}.$$

Moreover, it is possible to "add" the elements of an hyperfinite set of numbers. If F is an hyperfinite set of numbers, the *hyperfinite sum* of the elements of F is defined as follows:

$$\sum_{x\in F} x = \lim_{\lambda\uparrow\Lambda} \sum_{x\in F_\lambda} x.$$

One peculiarity of Euclidean analysis the possibility to associate a unique hyperfinite set $E^{\circledcirc}$ to any set $E \in V(\mathbb{E}_\kappa)$ according to the following definition:

DEFINITION 4.5. *Given a set* $E \in \Lambda_S$, *the set*

$$E^{\circledcirc} := \lim_{\lambda\uparrow\Lambda} \ (E \cap \lambda)$$

is called hyperfinite extension *of* E.

If $F = \lim_{\lambda\uparrow\Lambda} \ F_\lambda$ is a hyperfinite set, its *hypercardinality* is given by

$$|F|^* = \lim_{\lambda\uparrow\Lambda} \ |F_\lambda| \ ,$$

where $|\cdot|^*$ is the $*$-tranform of the fuction "cardinality" defined on finite sets.

Notice that, by vitue of (5), the hypercardinality of $E^{\circledcirc}$, given by

$$\left|E^{\circledcirc}\right|^* = \lim_{\lambda\uparrow\Lambda} \ |E \cap \lambda| \, ,$$

is the numerosity of E as it has been defined by Axiom 2.2.

If we put

$$E^\sigma = \{x^* \mid x \in E\} \, ,$$

we can associate the sets E^σ, $E^{\circledcirc}$ and E^* to any set $E \in \Lambda$. They are ordered as follows:

$$E^\sigma \subseteq E^{\circledcirc} \subseteq E^*;$$

in particular, if $E \subseteq \mathbb{R}$, $E^\sigma = E$. The hyperfinite analysis is very relevant in the applications and the operator "$\circledcirc$" plays a special role. You can see some examples of this fact in Section 6.3.

5. Nonstandard Analysis

Even if some notions and definitions of Nonstandard Analysis have already been introduced in the previous sections, now we will treat this topic in details. In this section, we assume that the reader is familiar with the basic notions on NSA.

5.1. Nonstandard theories

In this subsection, we recall the basic notion of Nonstandard Analysis how have been developed in the *superstructure* approach. Following Keisler (see [17]), we give the following definition:

DEFINITION 5.1. *A nonstandard theory is a triple* $(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet)$ *such that*[3]

- $V(\mathbb{R})$ *is a superstructure over* $\mathbb{R}$ *called standard universe;*
- $\mathbb{R}^\bullet$ *is a set such that* $\mathbb{R} \subset \mathbb{R}^\bullet$ *which is called field of the* ($\bullet$)*-hyperreal numbers;*
- $V(\mathbb{R}^\bullet)$ *is a superstructure over* $\mathbb{R}^\bullet$ *called nonstandard universe;*
- *the map*
$$\bullet\ : V(\mathbb{R}) \to V(\mathbb{R}^\bullet)$$
satisfies the Leibniz principle and
$$\forall r \in \mathbb{R}, r = r^\bullet. \tag{19}$$

We recall the notion of Leibniz (or transfer) Principle. It is well known that the map $\bullet$ transforms any elementary sentence $P(a_1, a_2, ..., a_n)$ to a elementary sentence $P(a_1^\bullet, a_2^\bullet, ..., a_n^\bullet)$ in $V(\mathbb{R}^\bullet)$ where $a_1, a_2, ..., a_n$ are constants in $V(\mathbb{R})$. The adjective elementary refers to the fact that the quantifiers in elementary sentences are of the form $(\forall x \in y)$ or $(\exists x \in y)$ where x is a variable and y is a constant or a variable. The Leibniz principle states that $P(a_1, a_2, ..., a_n)$ is true if an only if $P(a_1^\bullet, a_2^\bullet, ..., a_n^\bullet)$ is true. For details, see e.g. [8] or [17].

DEFINITION 5.2. *Given two sets* $\mathbb{A}$ *and* $\mathbb{S}$*, a superstructure embedding is a triple* $(V(\mathbb{A}), V(\mathbb{S}), \bullet)$ *where* $\bullet : V(\mathbb{A}) \to V(\mathbb{S})$ *is a injective map such that*
$$\mathbb{A}^\bullet = \mathbb{S},$$
$$\forall x, y \in V(\mathbb{A}),\ x \in y \Leftrightarrow x^\bullet \in y^\bullet.$$

The following fact is well known:

THEOREM 5.3. *If* $(V(\mathbb{A}), V(\mathbb{S}), \bullet)$ *is superstructure embedding then the map* $\bullet$ *satisfies the Leibniz principle.*

Proof. This result can be proved by induction over the complexity of the sentences; see e.g. [8] Th. 5.8. or [17]. □

---

[3] To be precise, Keisler calls $(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet)$ *nonstandard universe* while we use the espression *nonstandard universe* to denote the set $V(\mathbb{R}^\bullet)$.

By the above theorem, we get the following result:

Corollary 5.4. *If* $(V(\mathbb{R}), V(\mathbb{S}), \bullet)$ *is a superstructure embedding such that* $\mathbb{R} \neq \mathbb{S}$*, then,* $(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet)$ *is a nonstandard theory with* $\mathbb{R}^\bullet = \mathbb{S}$.

An isomorphism beetwen nonstandard theories is defined as follows:

Definition 5.5. *Two nonstandard theories*

$$(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet) \text{ and } (V(\mathbb{R}), V(\mathbb{R}^\star), \star)$$

are isomorphic if there is a map

$$h : V(\mathbb{R}^\bullet) \to V(\mathbb{R}^\star)$$

such that

1. $\forall r \in \mathbb{R},\ h(r) = r$;
2. *h maps* $\mathbb{R}^\bullet$ *one to one onto* $\mathbb{R}^\star$;
3. *for each* $A \in V(\mathbb{R}^\bullet) \backslash \mathbb{R}^\bullet$,
$$h(A) = \{h(a) \mid a \in A\},$$
4. *for each* $A \in V(\mathbb{R})$, $h(A^\bullet) = A^\star$.

Definition 5.6. *A nonstandard theory* $(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet)$ *is called* ***saturated*** *if any family of sets* $\mathfrak{S} \in V_n(\mathbb{R})^\bullet$ *with cardinality smaller than* $\mathbb{R}^\bullet$ *and with the finite intersection property has non empty intersection; namely if*

$$S_1 \cap ... \cap S_n \neq \varnothing, \quad S_i \in \mathfrak{S};\ |\mathfrak{S}| < |\mathbb{R}^\bullet|,$$

then

$$\bigcap \mathfrak{S} \neq \varnothing.$$

Among all the nonstandard theories there is a privileged one which is unique up to isomorphisms.

Theorem 5.7. *A saturated nonstandard theory* $(V(\mathbb{R}), V(\mathbb{R}^\bullet), \bullet)$ *with*

$$|\mathbb{R}^\bullet| = \kappa$$

is unique up to isomorphism. In this case, $V(\mathbb{R}^\bullet)$ *will be called Keisler universe.*

Proof. See [17]. □

5.2. The Normal Universe

According to the theory of the previous section we give following

DEFINITION 5.8. *If "$*$" is the map defined by (16), $V(\mathbb{R}^*)$ will be called* normal universe*; $\mathbb{R}^*$ will be called* normal Euclidean field *and from now on, it will be simply denoted by $\mathbb{E}$.*

THEOREM 5.9. *The triple $(V(\mathbb{R}), V(\mathbb{E}), *)$ is a nonstandard theory and $V(\mathbb{E})$ is a Keisler universe.*

Proof. By the definition of $V(\mathbb{E})$, it follows that $(V(\mathbb{R}), V(\mathbb{E}), *)$ is a superstructure embedding. Then by Cor. 5.4, we have to prove that $|\mathbb{E}| = \kappa$ and that $(V(\mathbb{R}), V(\mathbb{E}), *)$ is saturated. Since

$$\mathbb{E} = \mathbb{R}^* = \left\{ \lim_{\lambda \to \Lambda} x_\lambda \mid x_\lambda \in \mathfrak{F}(\mathfrak{L}, \mathbb{R}) \right\}$$

we have that

$$|\mathbb{E}| \leq |\mathfrak{F}(\mathfrak{L}, \mathbb{R})| .$$

Moreover

$$\mathfrak{F}(\mathfrak{L}, \mathbb{R}) = \left\{ \varphi \in \mathbb{R}^{\mathfrak{L}} \mid \exists A \in \Lambda, \forall \lambda \in \mathfrak{L},\ \varphi(\lambda) = \varphi(\lambda \cap A) \right\}$$

and hence

$$|\mathfrak{F}(\mathfrak{L}, \mathbb{R})| = \left| \bigcup_{A \in \Lambda} \mathbb{R}^A \right|$$

and since $\left|\mathbb{R}^A\right| < \kappa$ and $|\Lambda| = \kappa$, we have that

$$|\mathbb{E}| \leq \left| \bigcup_{A \in \Lambda} \mathbb{R}^A \right| = \kappa \,.$$

Also we have that

$$|\mathbb{E}| \geq |\mathfrak{N}| \geq |\mathbf{Ord}| = \kappa.$$

Then $|\mathbb{E}| = \kappa$. It remains to prove that it is saturated.

If $\mathfrak{S} \in V_n(\mathbb{R})^*$, then

$$\mathfrak{S} = \{E_\mu \mid \mu \in H\} ,$$

where H is a set of indices with $|H| < \kappa$. Since $|H| < \kappa$, it is not restrictive to assume that $H \subset \mathfrak{L}$. For every $\mu \in H$, let $\varphi_\mu(\lambda)$ be a net such that

$$\lim_{\lambda \uparrow \Lambda} \varphi_\mu(\lambda) = E_\mu \,.$$

For any fixed λ, pick an element

$$\psi(\lambda) \in \bigcap_{\mu \subseteq \lambda} \varphi_\mu(\lambda)$$

if this intersection is nonempty. Otherwise, pick

$$\psi(\lambda) \in \bigcap_{\mu \subseteq \lambda \backslash \{x_1\}} \varphi_\mu(\lambda);\ x_1 \in \lambda$$

if this intersection is nonempty, and continue in this manner until the element $\psi(\lambda)$ is defined. In case that this intersection is always empty, we set $\psi(\lambda) = \varnothing$. As a consequence of this definition, the following property holds:

$$\bigcap_{\mu \subseteq \tau} \varphi_\mu(\lambda) \neq \varnothing \Rightarrow \forall \lambda \supseteq \tau,\ \psi(\lambda) \in \bigcap_{\mu \subseteq \tau} \varphi_\mu(\lambda) \tag{20}$$

Now let $\tau \in H$ be fixed. By the finite intersection property,

$$\varnothing \neq \bigcap_{\mu \subseteq \tau} E_\mu = \bigcap_{\mu \subseteq \tau} \lim_{\lambda \uparrow \Lambda} \varphi_\mu(\lambda) = \lim_{\lambda \uparrow \Lambda} \left(\bigcap_{\mu \subseteq \tau} \varphi_\mu(\lambda) \right).$$

Then, there exists a set $Q \in \mathcal{U}$ ($\mathcal{U}$ is defined in the proof of Th. 3.3) such that,

$$\forall \lambda \in Q,\ \bigcap_{\mu \subseteq \tau} \varphi_\mu(\lambda) \neq \varnothing;$$

and hence, by (20), it follows that

$$\forall \lambda \in Q,\ \psi(\lambda) \in \bigcap_{\mu \subseteq \tau} \varphi_\mu(\lambda) \neq \varnothing$$

and taking the Λ-limit

$$\lim_{\lambda \uparrow \Lambda} \psi(\lambda) \in \bigcap_{\mu \subseteq \tau} \lim_{\lambda \uparrow \Lambda} \varphi_\mu(\lambda) = \bigcap_{\mu \subseteq \tau} E_\mu$$

and in particular,

$$\lim_{\lambda \uparrow \Lambda} \psi(\lambda) \in E_\tau\,.$$

As this holds for every $\tau \in H$, we conclude that

$$\lim_{\lambda \uparrow \Lambda} \psi(\lambda) \in \bigcap_{\tau \in H} E_\tau = \bigcap \mathfrak{S}. \qquad \square$$

$V(\mathbb{E})$ is not the only Keisler universe contained in $V(\mathbb{E}_\kappa)$. For example, using the notations at the end of Section 4.1, we have that also

$$(V(\mathbb{R}), V(\mathbb{E}_2), **), \ (V(\mathbb{R}), V(\mathbb{E}_3), ***), \ \ etc.$$

are saturated nonstandard theories and hence we have infinite Keisler universes included in $V(\mathbb{E}_\kappa)$. More in general, for $0 < j \leq \kappa$, it is possible to define a Keisler universe

$$\left(V(\mathbb{R}), V(\mathbb{E}_j), *^j\right)$$

where the map $*^j$ is defined $\forall A \in V(\mathbb{R})$ as follows:

- if $j = 1$, $A^{*^1} = A^* \subset \mathbb{E}_1 (= \mathbb{E})$,
- if $j = k + 1$, $A^{*^{j+1}} = \left(A^{*^j}\right)^* \subset \mathbb{E}_{j+1}$,
- if k is a limit ordinal, and $A \in V(\mathbb{E}_k)$, then $A \in V(\mathbb{E}_j)$ for some $j < k$ and hence $A^{*^k} = A^{*^j}$.

In particular we have that all the fields $\mathbb{E}_j$, $j \leq \kappa$ are isomorphic; for $j < \kappa$, the map

$$* \ : \mathbb{E}_j \to \mathbb{E}_{j+1}$$

is a field homomorphism and if k is a limit ordinal, the map

$$* \ : \mathbb{E}_k \to \mathbb{E}_k$$

is a field isomorphism. In any case, the only fixed points of $*$ are the real numbers. The spaces $\mathbb{E}_j$'s differ from each other by the way they are embedded in $\mathbb{A}$.

Moreover in a Euclidean universe there are other interesting superstructure embeddings which can be useful in some application. For example,

$$(V(\mathbb{E}_j), V(\mathbb{E}_{j+1}), *)$$

is a superstructure embedding; however is not a nonstandard theory, since $\mathbb{E}_j \neq \mathbb{R}$; moreover $(V(\mathbb{E}_j), V(\mathbb{E}_{j+1}), *)$ violates an other important request of Definition 5.1, namely

$$\exists \xi \in \mathbb{E}_j, \ \xi \neq \xi^*$$

in contrast with (19). Nevertheless, by Th. 5.3, $(V(\mathbb{E}_j), V(\mathbb{E}_{j+1}), *)$ satisfies the Leibniz principle. Then the following fact follows straightforwardly:

Theorem 5.10. $(V(\mathbb{E}_\kappa), V(\mathbb{E}_\kappa), *)$ *is a superstructure embedding that satisfies the Leibniz principle.*

5.3. The α-theory

The α-theory has been introduced in [7] and it represents an elementary approach to nonstandard analysis particularly suitable for some application; see e.g [5, 8, 12] and references. Actually, the Euclidean universe contains many non-standard universes which can be easily defined and, therefore, are more suitable for the elementary applications of practitioners. The α-theory is one of them and it can be constructed using the notion of α-limit where $\alpha = \mathfrak{num}\left(\mathbb{N}^+\right)$, $\mathbb{N}^+ = \mathbb{N}\backslash\left\{0\right\}$.

Definition 5.11. *Given a sequence $\varphi : \mathbb{N} \to V_n(\mathbb{E}_\kappa)$, we set*

$$\lim_{n\uparrow\alpha} \varphi(n) := \lim_{\lambda\uparrow\Lambda} \varphi\left(\left|\lambda \cap \mathbb{N}^+\right|\right).$$

If $C_b(n)$ is the constant sequence with value $b \in V(\mathbb{E}_\kappa)$, we set

$$b^{*\alpha} = \lim_{n\uparrow\alpha} C_b(n).$$

Then, by Th. 5.3, it follows that

$$\left(V\left(\mathbb{R}\right), V\left(\mathbb{R}^{*\alpha}\right), *_\alpha\right)$$

is a nonstandard theory.

Definition 5.12. *The nonstandard theory $\left(V\left(\mathbb{R}\right), V\left(\mathbb{R}^{*\alpha}\right), *_\alpha\right)$ is called α-theory.*

If

$$i : \mathbb{N} \to \mathbb{E}_\kappa;\ i(n) = n$$

then taking the α-limit we get that

$$\lim_{n\uparrow\alpha} i(n) = \lim_{\lambda\uparrow\Lambda} i\left(\left|\lambda \cap \mathbb{N}^+\right|\right) = \mathfrak{num}(\mathbb{N}^+) = \alpha.$$

Hence, the set of $(*_\alpha)$-hyperreal numbers $\mathbb{R}^{*\alpha}$ can be characterized as follows:

$$\mathbb{R}^{*\alpha} = \left\{\lim_{n\uparrow\alpha} \varphi(n) \mid \varphi : \mathbb{N} \to \mathbb{R}\right\}$$

namely, every $(*_\alpha)$-hyperreal numbers is the α-limit of real sequence. For example, we have that ω is a $(*_\alpha)$-hyperreal number since

$$\begin{aligned} \omega &= \mathfrak{num}(\mathbb{N}) = \mathfrak{num}(\mathbb{N}^+) + \mathfrak{num}(\{0\}) \\ &= \alpha + 1 = \lim_{n\uparrow\alpha} (n+1). \end{aligned}$$

As we will see in Section 7.1, the construction of a model of the theory, there is an ultrafilter $\mathcal{U}$ which plays a central role (see definition (23)). By choosing $\mathcal{U}$ in a suitable way, then we get the following result:

THEOREM 5.13. *It is compatible with axioms 1-3 that the number α satisfies the following properties:*

- DIVISIBILITY PROPERTY *: For every $k \in \mathbb{N}$, the number α is a multiple of k and the numerosity of the set of multiples of k:*

$$\mathfrak{num}(\{k, 2k, 3k, ..., nk, ...\}) = \frac{\alpha}{k}$$

- ROOT PROPERTY*: For every $k \in \mathbb{N}$, the number α is a k-th power and the numerosity of the set of k-th powers:*

$$\mathfrak{num}(\{1^k, 2^k, 3^k, ..., n^k, ...\}) = \sqrt[k]{\alpha}$$

- POWER PROPERTY*: If we set $\wp_{fin}(A) = \{F \in \wp(A) \mid F$ is a finite set$\}$, then*

$$\mathfrak{num}(\wp_{fin}(\mathbb{N}^+)) = 2^\alpha$$

- INTEGER NUMBERS PROPERTY*:*

$$\mathfrak{num}(\mathbb{Z}) = 2\alpha + 1$$

- RATIONAL NUMBERS PROPERTY*: For every $q \in \mathbb{Q}$,*

$$\mathfrak{num}((q, q+1] \cap \mathbb{Q}) = \mathfrak{num}((0,1] \cap \mathbb{Q}) = \alpha$$

and

$$\mathfrak{num}(\mathbb{Q}) = 2\alpha^2 + 1.$$

Proof. See [8, Sections 16.6 and 16.7]. □

5.4. About the idea of continuum

The idea of (linear) continuum is described or modeled by the geometric line. In classical Euclidean geometry, lines and segments are not considered sets of points; on the contrary, in the last two centuries the reductionist attitude of modern mathematics has described Euclidean geometry through a set interpretation. In the last century, the geometric continuum has been identified with the Dedekind continuum and the geometric line has been identified with the set of real numbers (once the origin O and a unitary segment OU have been fixed). Even is this identification, today, is almost universally accepted, we have seen that also the Euclidean line, as defined by Axiom 2.3, has some right to represent the geometric continuum. In this section we will compare $\mathbb{R}$ and $\mathbb{E}$ ($\cong \mathbb{E}_\kappa$) with respect to the idea of geometric continuum.

In our naive intuition, we think of a linear continuum as a linearly ordered set without interruptions, that is, without holes between one part and the other. Let's make this definition rigorous. Contrary to our intuition, a set X satisfying the following property

$$\forall a, b \in X,\ a < b,\ \exists c \in X,\ a \leq c \leq b$$

it cannot be considered a continuum: this notion, satisfied for example by the set of rational numbers, is not a good candidate for a continuum as the set of rationals is full of holes represented by irrational numbers.

So we are led to discuss the notion of κ-saturation and to the Eudoxus Principle:

DEFINITION 5.14. *A linearly ordered set X is called* κ-saturated *if it satisfies following property: given two sets $A, B \subset X$, such that*

$$\begin{aligned} &\left|A\right|, \left|B\right| < \kappa\,, \\ &\forall a \in A,\ \forall b \in B,\ a < b\,, \end{aligned} \tag{21}$$

then $\exists c \in X$,

$$\forall a \in A,\ \forall b \in B,\ a \leq c \leq b.$$

DEFINITION 5.15 (Eudoxus Principle). *A linearly ordered Abelian group $\mathbb{F}$ satisfies the Eudoxus Principle if given two sets $A, B \subset \mathbb{F}$ such that*

$$\begin{aligned} &\forall a \in A,\ \forall b \in B,\ a < b\,, \\ &\forall \varepsilon \in \mathbb{F}^+,\ \exists a \in A,\ \exists b \in B,\ b - a < \varepsilon. \end{aligned}$$

then $\exists c \in X$,

$$\forall a \in A,\ \forall b \in B,\ a \leq c \leq b.$$

Using these notions, we can characterize $\mathbb{R}$ and $\mathbb{E}$ as follows:

THEOREM 5.16. *The field of the real numbers $\mathbb{R}$ is the only field $\mathbb{F}$ such that:*

(i) satisfies the Eudoxus Principle,

(ii) satisfies the Archimedes' Axiom, namely :

$$\forall a, b \in \mathbb{F}^+,\ \exists n \in \mathbb{N},\ na > b\,.$$

Proof. Well known. □

The request (ii) is necessary; in fact, for example, the field of rational functions with a suitable order structure[4] satisfy (i) but not (ii).

[4]For example the field of rational fuction $\mathbb{F}$ can be equipped with an order structure by setting

$$\mathbb{F}^+ = \left\{ \frac{r_n x^n + r_{n-1}x^{n-1} + ... + r_0}{w_m x^m + w_{m-1}x^{m-1} + ... + w_0} \mid \frac{r_n}{w_m} > 0 \right\}.$$

THEOREM 5.17. *The field of the Euclidean numbers $\mathbb{E}$ is the smallest field that:*

(i) is κ-saturated,

(ii) is a real closed field, namely every polynomial of odd degree has at least one root.

Proof. By Theorem 5.9, it is easy to check that $\mathbb{E}$ is κ-saturated according to Definition 5.14. Moreover, since $\mathbb{E}$ is hyperreal, it is real closed. All the real closed fields of cardinality κ are isomorphic and $|\mathbb{E}| = \kappa$; hence $\mathbb{E}$ is the smallest of such fields . □

Notice that the request (ii) is necessary; in fact $\mathbb{Q}^*$ is a κ-saturated field, but it does not satisfy (i) since the equation $x^3 = 2$ does not have any solution in $\mathbb{Q}^*$. We observe that the request (ii) fits well the idea of continuity, in fact, a polynomial of odd degree must take positive and negative values and hence, by continuity, it must have some 0's.

The above discussion suggests the following definitions of continuum:

DEFINITION 5.18. *A linearly ordered Abelian group $\mathbb{F}$ is a* Dedekind continuum *if it satisfies the following property: given two sets $A, B \subset X$ such that*

$$A, B \neq \varnothing\,, \qquad \forall a \in A,\ \forall b \in B,\ a < b\,, \tag{22}$$

then $\exists c \in X$,

$$\forall a \in A,\ \forall b \in B,\ a \leq c \leq b;$$

A linearly ordered ordered field $\mathbb{F}$ is an absolute continuum[5] *if it is saturated and real closed.*

With these notions of continuity $\mathbb{R}$ and $\mathbb{E}$ have the following characterization: $\mathbb{R}$ is the only Dedekind continuum field; $\mathbb{E}$ is the smallest absolutely continuum field.

6. Euclidean Calculus

In this section we pretend to not know the classical calculus and we will define the basic notion of calculus, derivative and integral, in the most natural way provided that you are equipped with infinitesimal and infinite numbers. Hence, these definitions are very similar to those of the XVIII century. With these definitions, we will discover that every function is both integrable and it has a

[5]This notion of *absolute contiuum* has been introduced by Ehrlich in [15]. However in his definition $\mathbb{F}$ is a class in the sense of Von Neumann–Bernays–Gödel set theory.

left and right derivative. Of course, if a function is differentiable, the *Euclidean derivative* corresponds to the usual derivative and if it is Lebesgue-integrable, the *Euclidean integral* corresponds to the usual Lebesgue integral. We limit this game to the *normal* functions as defined below.

Of course this game could be extended to other notions and to a larger class of functions and it might have some interest for the foundations and the philosophy of Mathematics.

The idea to work directly in a nonstandard unverse is not new; we recall [1, 16, 19]. This section can be considered an other experiment in that direction.

6.1. Normal functions and sets

In most application, the space $\mathbb{E}_\kappa$ and the Euclidean universe $V(\mathbb{E}_\kappa)$ are too large and hence might imply useless technicalities. It is more convenient to work in the normal Euclidean field $\mathbb{E} = \mathbb{R}^*$ and in the normal universe $V(\mathbb{E})$. So we are lead to the following definition:

DEFINITION 6.1. *A function* $f : \mathbb{E} \to \mathbb{E}$ *is called* normal *if*

$$f = h^*$$

where h *is a standard real function, i.e.* $f \in \mathbb{R}^E,\ E \subseteq \mathbb{R}$.

If f is normal then $\forall x \in \mathbb{R},\ f(x) \in \mathbb{R}$.

DEFINITION 6.2. *A subset* $N \subset \mathbb{E}$ *is called* normal *if* $N = A^*$ *for some set* $A \subset \mathbb{R}$.

REMARK 6.3. In the nonstandard analysis community there is the habit to call standard both functions and sets of the form f^*, A^* and functions and sets in $V(\mathbb{R})$. Here we call standard the elements of $V(\mathbb{R})$ and normal their counterpart defined as above.

The usual functions used in the applications of mathematics can be regarded as normal functions and not as standard functions. The advantadge of this point of view is that the main notions of infinitesimal analysis can be defined using the "actual infinitesimal" in a natural way and hence they assume a different meaning.

6.2. The notion of derivative

Since the normal functions are in a biunivocal correspondence with the real functions, sometimes we will denote both with the same symbol. The same we will do with the intervals.

Now let us introduce some notions of Euclidean calculus. In order to intoduce a "Euclidean derivative", we will take the advantage to have a distinguished infinite number, namely α; then we can define a distinguished infinitesimal number as follows:

$$\eta := \frac{1}{\alpha}.$$

Definition 6.4. *The* right derivative *of a normal function* $f : (a, b) \to \mathbb{E}$ *in a standard point* $x_0 \in (a, b)$*, in the sense of Euclidean Calculus, is defined as follows:*

$$D^+ f(x_0) = ctr\left(\frac{f(x_0 + \eta) - f(x_0)}{\eta}\right);$$

similarly the left $\mathbb{E}$-derivative *is defined as follows:*

$$D^- f(x_0) = ctr\left(\frac{f(x_0) - f(x_0 - \eta)}{\eta}\right);$$

the mean $\mathbb{E}$-derivative *is defined as follows:*

$$Df(x_0) = \frac{1}{2}\left[D^+ f(x_0) + D^- f(x_0)\right] = ctr\left(\frac{f(x_0 + \eta) - f(x_0 - \eta)}{2\eta}\right).$$

We say that a function is derivable in a point $x_0 \in (a, b)$ *if* $Df(x_0) = D^+ f(x_0)$ *and* $Df(x_0) \in \mathbb{R}$*. In this case,*

$$Df(x_0) = st\left(\frac{f(x_0 + \eta) - f(x_0)}{\eta}\right)$$

is called generalized derivative in the sense of Euclidean Calculus or simply $\mathbb{E}$-derivative.

It is easy to check that given a real function f differentiable in a point $x_0 \in (a, b) \cap \mathbb{R}$, then

$$Df(x_0) = f'(x_0)$$

but the converse is not true; for example the function $f(x) = x \sin \frac{1}{x^2}$ is not differentiable for $x = 0$, but

$$D^+ f(0) = ctr\left(\sin \frac{1}{\eta^2}\right) = ctr\left(\sin \frac{1}{(-\eta)^2}\right) = D^- f(0)$$

and hence the $\mathbb{E}$-derivative, given by

$$\left[D\left(x \sin \frac{1}{x^2}\right)\right]_{x=0} = st\left(\sin \frac{1}{\eta^2}\right),$$

is well defined.

The notion of $\mathbb{E}$-derivative is more general and consequently the fact that if f is $\mathbb{E}$-derivable does not imply that f (resticted to $\mathbb{R}$) is continuous. For example the Dirichlet function

$$f_D(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q}^* \\ 0 & \text{if } x \in \mathbb{R}^* \backslash \mathbb{Q}^* \end{cases}$$

is $\mathbb{E}$-derivable in every point with $Df_D = 0$ (remember that by Th. 5.13, $\eta \in \mathbb{Q}^*$), but it is not continuous.

As far, we have defined the derivative of a normal function in a standard point. The following definition extends the notion of derivative to every Euclidean point, namely it defines the function "derivative" in all the points of (a,b) while Definition 6.4 defines it only for $x \in (a,b) \cap \mathbb{R}$.

Definition 6.5. *The $\mathbb{E}$-derivative of a normal derivable function $f : (a,b) \to \mathbb{E}$ is defined by*

$$Df := \left(Df|_{(a,b)\cap\mathbb{R}} \right)^* .$$

Example 6.6. Take $f(x) = x \sin \frac{1}{x^2}$; then

$$D\left(x \sin \frac{1}{x^2} \right) = \begin{cases} \sin \frac{1}{x^2} - \frac{2}{x^2} \cos \frac{1}{x^2} & \text{if } x \in \mathbb{E} \backslash \{0\} \\ st \left(\sin \frac{1}{\eta^2} \right) & \text{if } x = 0 . \end{cases}$$

Obviously, the $\mathbb{E}$-derivability does not imply the differentiability defined as follows:

Definition 6.7. *A normal function $f : (a,b) \to \mathbb{E}$ is said to be differentiable in a point $x_0 \in (a,b)$ if there exists a linear function $t \mapsto df\,(x_0)\,[t]$ such that, for every infinitesimal ε,*

$$f(x_0 + \varepsilon) = f(x_0) + df\,(x_0)\,[\varepsilon] + \varepsilon\varepsilon_1$$

where ε_1 is an infinitesimal (which might depend on ε).

It is immediate to check that a function is differentiable in x_0 if and and only if

$$\forall \varepsilon \in \mathfrak{mon}\,(0) \setminus \{0\}, \quad Df\,(x_0) = st \left(\frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon} \right).$$

Then, if a function is differentiable in $x_0 \in (a,b) \cap \mathbb{R}$, it has the $\mathbb{E}$-derivative in that point and

$$df\,(x_0)\,[t] = Df(x_0) \cdot t.$$

but the converse is not true. The derivability of a function does not coincide with the differentiability; it is well known that in "classic calculus" this phenomenon occurs only in dimension ≥ 2.

REMARK 6.8. With the above definitions, the two classical problems of the "istantaneous velocity" and of the "tangent" get different solutions given by the $\mathbb{E}$-derivative and the differential respectively. They coincide only for continuous functions.

Even if the $\mathbb{E}$-derivative is weaker than the usual one, it is *quite surprising* that the main theorems of calculus remain true. For example, let us consider the Fermat theorem:

THEOREM 6.9 (Fermat theorem). *If a normal function $f : (a, b) \to \mathbb{E}$ achieves a local maximum (or minimum) in a point $x_0 \in (a + \eta, b - \eta)$ and it has the $\mathbb{E}$-derivative in that point, then*

$$Df(x_0) = 0\,.$$

Proof. We have that $D^+f(x_0) \leq 0$, and $D^-f(x_0) \geq 0$. Since $D^+f(x_0) = D^-f(x_0)$, it follows that

$$Df(x_0) = D^+f(x_0) = 0.$$

□

Following the usual procedure, we can prove Rolle theorem, the Lagrange intermediate value theorem and most of the theorems of real calculus for a class of function that are not necessarely differentiable, but have only the $\mathbb{E}$-derivative. We will sketch this fact (see [3] for details). First of all we recall some well known fact in NSA:

DEFINITION 6.10. *A function $f : D \to \mathbb{E}$, $D \subset \mathbb{E}$ is called* continuous in a point $\xi \in D$ *iff*

$$\xi \sim x \Rightarrow f(\xi) \sim f(x)\,;$$

It is called continuous in D *if it is normal and it is continuous in every point $\xi \in D \cap \mathbb{R}$. It is called* uniformly continuous in D *if it is normal and it is continuous in every point $\xi \in D$.*

THEOREM 6.11 (Weierstrass). *Let f be a continuous fuction on an interval $[a, b]$. Then f has a maximum point in $[a, b]$ and it is a standard point.*

Proof. Since the set $[a, b] \cap \mathbb{R}^{\circledcirc}$ is hyperfinite, f restricted to $[a, b] \cap \mathbb{R}^{\circledcirc}$ has a maximum point ξ. We calim that $c = st(\xi)$ is the maximum in $[a, b]$; in fact, since $[a, b] \cap \mathbb{R} \subset [a, b] \cap \mathbb{R}^{\circledcirc}$, $\forall x \in [a, b] \cap \mathbb{R}$

$$f(\xi) \geq f(x)$$

and hence by the continuity of f, $\forall x \in [a,b] \cap \mathbb{R}$

$$f(c) = st[f(\xi)] \geq st[f(x)] = f(x).$$

The inequality above, can be extended to every $\zeta = \lim_{\lambda\uparrow\Lambda} x_\lambda \in [a,b]$, $x_\lambda \in [a,b] \cap \mathbb{R}$. In fact, since $f(c) \geq f(x_\lambda)$,

$$f(c) = \lim_{\lambda\uparrow\Lambda} f(c) = \lim_{\lambda\uparrow\Lambda} f(x_\lambda) = f(\zeta). \qquad \square$$

So we have the following result involving the $\mathbb{E}$-derivative:

LEMMA 6.12 (Rolle). *Let f be a continuous fuction on an interval $[a,b]$ such that $f(a) = f(b)$; then if f is $\mathbb{E}$-derivable in (a,b), there is a point $c \in (a,b)$ such that*

$$Df(c) = 0.$$

Proof. By Fermat's and Weierstrass' theorems, the proof is equal to the usual one. $\square$

THEOREM 6.13 (Lagrange). *Let f be a continuous fuction on an interval $[a,b]$ and $\mathbb{E}$-derivable in (a,b), there is a point $c \in (a,b)$ such that*

$$Df(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. By Rolle's lemma, the proof is equal to the usual one. $\square$

These results show that even if a $\mathbb{E}$-derivable function can be quite wild (think of the Dirichlet function), the continuous $\mathbb{E}$-derivable functions behave quite well. For example, the space of the solutions of the equation

$$Df = 0$$

in general, is not finite-dimensional. However, by the Lagrange theorem it follows that the only continuous functions which solve the above equations are the constants. Among the other consequences of the Lagrange's teorem, we get the following result:

THEOREM 6.14. *A sufficient condition for a function f to be differentiable in $x_0 \in (a,b) \cap \mathbb{R}$ is that both f and Df be continuous in x_0.*

Proof. See [3]. $\square$

This discussion shows that the notion of $\mathbb{E}$-derivability, even if it is essentially irrelevant for the applications, it seems interesting for the foundation of the notion of derivative and its relation with the differentiability.

REMARK 6.15. In the framework of Euclidean calculus there are several other notions of "generalzed derivative" which make sense. For example we can define the right *grid derivative* as follows:

$$D^+f(x_0) = ctr\left(\frac{f(x^+) - f(x)}{x^+ - x}\right)$$

where

$$x^+ = \min\ \left\{y \in \mathbb{R}^{\circledcirc} \mid y > x\right\}$$

and similarly the left grid derivative etc. An other notion of generalized derivative useful for the applications can be found in [4]. In this paper the notion of grid derivative is combined with the notion of weak derivative in such a way to include the derivative of distributions (identified with suitable internal functions).

6.3. The integral

Also the definition of the integral takes advantage of a peculiarity of Euclidean analysis, namely of the operator "$\circledcirc$" introduced by Definition 4.5.

DEFINITION 6.16. *Given a normal function $f : [a,b] \to \mathbb{E}$, we define the $\mathbb{E}$-integral as follows:*

$$\int_a^b f(x)dx := ctr\left(\sum_{x\in[a,b]^{\circledcirc}} f(x)\left(x^+ - x\right)\right)$$

where

$$x^+ = \min\ \left\{y \in \mathbb{R}^{\circledcirc} \mid y > x\right\}.$$

Clearly, if f is Riemann integrable, the $\mathbb{E}$-integral coincides with the Riemann integral. Moreover the $\mathbb{E}$-integral is well defined for every normal function even when $[a,b] = \mathbb{R}$ or/and f is unbounded. However, the most interesting property of the $\mathbb{E}$-integral is given by the following theorem:

THEOREM 6.17. *Let f be a bounded Lebesgue integrable function, then the $\mathbb{E}$-integral is equal to the Lebesgue integral.*

Proof. Assume that f is a bounded Lebesgue integrable function in $[a,b]$ and set

$$f_\lambda(x) := \sum_{z\in[a,b]\cap\lambda} f(z)\chi_{\left[z,z_\lambda^+\right)}(x)$$

where $\chi_{\left[z,z_\lambda^+\right)}$ is the characteristic function of $\left[z, z_\lambda^+\right)$ and

$$z_\lambda^+ = \min\ \{y \in \lambda \mid y > z\}.$$

Now let us denote by $\int_L$ the Lebesgue integral; then

$$\int_L f_\lambda(x)dx = \int f_\lambda(x)dx\,.$$

The net of functions $\{f_\lambda\}$ converges to f in every point $x \in [a,b]\cap\mathbb{R}$ since $[a,b]\cap\mathbb{R} \subset [a,b]^{\circledcirc}$; in fact, eventually we have that $\forall x \in [a,b]\cap\mathbb{R}$

$$f_\lambda(x) = f(x)$$

and hence by (14), $\forall x \in [a,b]\cap\mathbb{R}$,

$$\lim_{\lambda\to\Lambda} f_\lambda(x) = st\,[f(x)] = f(x)$$

where $\lim_{\lambda\to\Lambda}$ is the usual Cauchy limit. By the Dominated Convergence Theorem,

$$\lim_{\lambda\to\Lambda}\int_L f_\lambda(x)dx = \int_L \lim_{\lambda\to\Lambda} f_\lambda(x)dx = \int_L f(x)dx\,.$$

On the other hand,

$$\begin{aligned}\lim_{\lambda\uparrow\Lambda}\int f_\lambda(x)dx &= \lim_{\lambda\uparrow\Lambda}\left[\sum_{z\in[a,b]\cap\lambda} f(z)\chi_{[z,z_\lambda^+)}(x)\right]\\ &= \sum_{x\in[a,b]^{\circledcirc}} f(x)\left(x^+ - x\right)\,.\end{aligned}$$

Then, by (14),

$$\int_L f(x)dx \sim \sum_{x\in[a,b]^{\circledcirc}} f(x)\left(x^+ - x\right)$$

and hence

$$\int_L f(x)dx = st\left(\sum_{x\in[a,b]^{\circledcirc}} f(x)\left(x^+ - x\right)\right) = \int f(x)dx. \qquad \square$$

Remark 6.18. All the normal functions are $\mathbb{E}$-integrable; however the functions which are not Lebesgue-integrable might have a pathological behavior; for example their integral is not invariant for translations. Nevertheless $\mathcal{L}^1$, the space of the Lebesgue-integrable functions, can be easily characterized as the closure of the continuous functions with compact support with respect to the norm

$$\|f\| = \int |f(x)|\,dx.$$

7. Consistency of the axioms

In this sections, we will prove the consistency of the three axioms introduced in Section 2 by building a model in ZFC+{Axiom of Inaccessibility}.

Remark 7.1. In many models on nonstandard analysis, the axiom of regularity of ZFC may fail for external sets (see e.g. [14, 10]). However, in this paper, both, the standard universe $V(\mathbb{R})$ and the Euclidean universe $V(\mathbb{E}_\kappa)$ contain only sets of finite rank and this peculiarity allows to have a model in ZFC. We are forced to work with sets of finite rank by axiom 2.2. In fact, assuming that Λ contains a set A of infinite rank, we would get a contradiction. Take for instance a set a defined as follows:

$$A = \{b_n \mid n \in \mathbb{N}\}$$

where $b_0 = a$ and

$$b_{n+1} = (b_n, a)$$

namely

$$A = \{a, (a,a), (a,a,a), (a,a,a,a), , ...\}.$$

Then, setting

$$B = A \times \{a\} = \{(a,a), (a,a,a), (a,a,a,a), ...\} = \left\{b_n \mid n \in \mathbb{N}^+\right\}$$

we have that $B \subset A$ and hence, by Axiom 2.2.2

$$\mathfrak{num}(B) < \mathfrak{num}(A),$$

while, by Axiom 2.2.5

$$\mathfrak{num}(B) = \mathfrak{num}(A \times \{a\}) = \mathfrak{num}(A) \cdot \mathfrak{num}(\{a\}) = \mathfrak{num}(A) \cdot 1 = \mathfrak{num}(A).$$

Contradiction!

7.1. The construction of the field $\mathbb{E}$

We assume that $\mathbb{A}$ is a set of atoms having cardinality $|\mathbb{A}| = \kappa$ and that it contains a set $\mathbb{R}$ isomorphic to the real numbers. Moreover, we assume that Λ_S, Λ and $\mathfrak{L}$ be sets as defined in Section 2.1.

If $n_0 \in \mathbb{N}$, and $\lambda_0 \in \mathfrak{L}$, we set

$$Q(n_0, \lambda_0) = \{V_n(\lambda) \in \mathfrak{L} \mid n \in \mathbb{N},\ n \geq n_0;\ \lambda \in \mathfrak{L},\ \lambda \supseteq \lambda_0\}.$$

We have that $Q(n_0, \lambda_0) \subset \mathfrak{L}$ and

$$Q(n_0, \lambda_0) \cap Q(m_0, \mu_0) = Q(\max\{n_0, m_0\},\ \lambda_0 \cup \mu_0).$$

Hence there exists an ultrafilter $\mathcal{U}$ over $\mathfrak{L}$ such that $\forall n \in \mathbb{N}$, $\forall \lambda \in \mathfrak{L}$,

$$Q(n, \lambda) \in \mathcal{U}. \tag{23}$$

Now, for any $j < \kappa$, we define by transfinite induction a sequence of ordered fields $\mathbb{K}_j \subset \mathbb{A}$ such that $|\mathbb{K}_j| < \kappa$ and

$$\mathbb{K}_j \subset \mathbb{K}_k \quad \text{ if } \ j < k.$$

For $j = 0$, we set

$$\mathbb{K}_0 = \mathbb{R}$$

and for every ordinal $j < \kappa$, we set

$$\mathfrak{F}_{j+1} := \mathfrak{F}(\mathfrak{L}, \mathbb{K}_j) / I_j \,, \tag{24}$$

where

$$\mathfrak{F}_j(\mathfrak{L}, \mathbb{K}_j) = \left\{ \varphi \in (\mathbb{K}_j)^{\mathfrak{L}} \ \mid \forall \lambda \in \mathfrak{L}, \ \varphi(\lambda) = \varphi(\lambda \cap \mathbb{K}_j) \right\}$$

and $I_j \subset \mathfrak{F}_j(\mathfrak{L}, \mathbb{K}_j)$ is the maximal ideal defined as follows:

$$I_j := \{ \varphi \in \mathfrak{F}_j(\mathfrak{L}, \mathbb{K}_j) \ \mid \exists Q \in \mathcal{U}, \ \forall \lambda \in Q, \ \varphi(\lambda) = 0 \} \,.$$

Then $\mathfrak{F}_{j+1}$ is a field and the projection

$$\Pi_j : \mathfrak{F}_j(\mathfrak{L}, \mathbb{K}_j) \to \mathfrak{F}_{j+1}$$

defined by

$$\Pi_j(\varphi) = [\varphi]_{\mathcal{U}} := \varphi + I_j$$

is a surjective ring homomorphism. Now, since $|\mathbb{K}_j| < \kappa$, also $|\mathfrak{F}_{j+1}| < \kappa$; then we can define an injetive map

$$\Theta_j : \mathfrak{F}_{j+1} \to \mathbb{A}$$

such that $\forall \xi \in \mathbb{K}_j$,

$$\Theta_j\left([C_\xi]_{\mathcal{U}}\right) = \xi \,,$$

where $\forall \lambda \in \mathfrak{L}$, $C_\xi(\lambda) := \xi \in \mathbb{K}_j$. The set $\mathbb{K}_{j+1} := \mathrm{Im}\Theta_j$ can be equipped with the structure of ordered field by setting, $\forall \xi, \zeta \in \mathbb{K}_{j+1}$

$$\xi + \zeta = \Theta_j\left(\Theta_j^{-1}(\xi) + \Theta_j^{-1}(\zeta)\right) \,,$$
$$\xi \cdot \zeta = \Theta_j\left(\Theta_j^{-1}(\xi) \cdot \Theta_j^{-1}(\zeta)\right) .$$

Then $\mathbb{K}_{j+1}$ is a field which contains the real numbers. Now set

$$J_j = \Theta_j \circ \Pi_j : \mathfrak{F}(\mathfrak{L}, \mathbb{K}_j) \to \mathbb{K}_{j+1} \,.$$

By this construction, J_j is a surjective ring homomorphism.

If $k \leq \kappa$ is a limit ordinal, we set

$$\mathbb{K}_k = \bigcup_{j<k} \mathbb{K}_j$$

and

$$J_k = \varinjlim J_j \ ,$$

namely, $J_k(\varphi) = J_j(\varphi)$ for every j such that $\varphi \in \mathfrak{F}(\mathfrak{L}, \mathbb{K}_j)$. Then, also

$$J_k : \mathfrak{F}(\mathfrak{L}, \mathbb{K}_k) \to \mathbb{K}_k$$

is a surjective ring homomorphism.

In particular, if $j = \kappa$, we have that

$$J := J_\kappa : \mathfrak{F}(\mathfrak{L}, \mathbb{K}_\kappa) \to \mathbb{K}_\kappa \subset \mathbb{A} \tag{25}$$

is a surjective ring homomorphism and $|\mathbb{E}_\kappa| = \kappa$.

If $A \in \Lambda$, by the definition of Λ (see (1)), we have that $|A| < \kappa$, then the net $\{\lambda \mapsto |\lambda \cap A|\} \in \mathfrak{F}(\mathfrak{L}, \mathbb{K}_\kappa)$. Then, we can define the numerosity of A as follows:

$$\mathfrak{num}(A) = J(|\lambda \cap A|) \ , \tag{26}$$

where, with some abuse of notation, $|\lambda \cap A|$ denotes the net $\{\lambda \mapsto |\lambda \cap A|\}$.

7.2. Proof of the consistency of Axioms 1-3

Now we can prove the consistency of our axioms.

THEOREM 7.2. *The numerosity function defined by (26) satisfies the request of Axiom 2.2.*

Proof. 2.2.1 and 2.2.2 follows directly by the definition (26) of $\mathfrak{num}$.

2.2.3 - We have that

$$\begin{aligned}\mathfrak{num}(A \cup B) &= J(|\lambda \cap (A \cup B)|) = J(|(\lambda \cap A) \cup (\lambda \cap B)|) \\ &= J(|(\lambda \cap A)| + |(\lambda \cap B)|) = J(|(\lambda \cap A)|) + J(|(\lambda \cap A)|) \\ &= \mathfrak{num}(A) + \mathfrak{num}(B) .\end{aligned}$$

2.2.4 - Let n_0 be so large that $A, B \in V_{n_0}(\mathbb{E}_\kappa)$, then $A \times B \in V_{n_0+2}(\mathbb{E}_\kappa)$; if we take $V_n(\lambda) \in Q(n_0 + 2, \lambda)$, we have that

$$\begin{aligned}|V_n(\lambda) \cap (A \times B)| &= |(V_n(\lambda) \cap A) \times (V_n(\lambda) \cap B)| \\ &= |V_n(\lambda) \cap A| \cdot |V_n(\lambda) \cap B| \ .\end{aligned} \tag{27}$$

If we set

$$\varphi_E(\lambda) = |V_n(\lambda) \cap E|$$

by (27), we have that $\forall \lambda \in Q(n_0+2, \lambda_0)$,

$$\varphi_{A\times B}(\lambda) = \varphi_A(\lambda) \cdot \varphi_B(\lambda)$$

and since $Q(n_0+2, \lambda_0) \in \mathcal{U}$,

$$J(\varphi_{A\times B}) = J(\varphi_A) \cdot J(\varphi_B) \ .$$

Then

$$\mathfrak{num}(A \times B) = J(\varphi_{A\times B}) = J(\varphi_A) \cdot J(\varphi_B) = \mathfrak{num}(A) \cdot \mathfrak{num}(B) \ . \tag{28}$$

2.2.5 - It follows immediately from (28). □

THEOREM 7.3. *The set $\mathbb{E}_\kappa$ satisfies the request of Axiom 2.3.*

Proof. Trivial by our construction. □

We define the set of the *Euclidean integers* as follows:

$$\mathfrak{Z} := \bigcup_{j<\kappa} \mathfrak{Z}_j \,,$$

where

$$\mathfrak{Z}_0 = \mathbb{Z}$$

and, for $j < \kappa$

$$\mathfrak{Z}_j := J\left(\bigcup_{k<j} \mathfrak{Z}_k\right). \tag{29}$$

Before proceeding we need the following

LEMMA 7.4. *Every number $\xi \in \mathbb{E}_\kappa$ can be decomposed as follows:*

$$\xi = \zeta + \mu \tag{30}$$

where $\zeta \in \mathfrak{Z}$ and $0 \leq \mu < 1$.

Proof. Let $\xi \in \mathbb{E}_j$. We argue by transfinite induction over j. If $j = 0$, (30) holds with $\zeta \in \mathbb{Z}$. If $j > 0$, then,

$$\xi := J(\{\xi_\lambda\})$$

with $\xi_\lambda \in \mathbb{E}_k$ for some $k < j$. By our inductive assumption,

$$\xi_\lambda = \zeta_\lambda + \mu_\lambda, \quad \zeta_\lambda \in \mathfrak{Z} \text{ and } 0 \leq \mu_\lambda < 1.$$

Then

$$\xi := J(\{\zeta_\lambda + \mu_\lambda\}) = J(\{\zeta_\lambda\}) + J(\{\mu_\lambda\})$$

By (29), $\zeta := J(\{\zeta_\lambda\}) \in \mathfrak{Z}$ and if we set $\mu := J(\{\mu_\lambda\})$, we have that $0 \leq \mu < 1$. Then (30) is satisfied. □

We now set

$$\mathfrak{C} := \{\zeta + r \mid \zeta \in \mathfrak{Z},\ r \in \mathbb{R}\}. \tag{31}$$

Clearly $\mathfrak{Z}$ and $\mathbb{R}$ are additive subgroups of $\mathbb{E}_\kappa$ and hence also $\mathfrak{C}$ is an additive group. $\mathfrak{C}$ is the set of the centers.

THEOREM 7.5. *The set $\mathfrak{C}$ defined by (31) satisfies the requests of Axiom 2.8.*

Proof. Since $\mathbb{E}_\kappa$ contains the real numbers, if $\theta \in \mathbb{E}_\kappa$ is bounded, $st(\theta)$ is well defined. If $\xi \in \mathbb{E}_\kappa$, by Lemma 7.4, we can write $\xi = \zeta + \mu$, with $\zeta \in \mathfrak{Z}$ and $0 \leq \mu < 1$; then we set

$$ctr(\zeta + \mu) = \zeta + st(\mu).$$

Then $ctr(\zeta + \mu) \in \mathfrak{C}$. □

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Author's address:

Vieri Benci
Dipartimento di Matematica
Università degli Studi di Pisa
Via Buonarroti 1/c, Pisa, Italy
E-mail: `vieri.benci@unipi.it`

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