Impact of the wakefields and of an initial energy curvature on a Free Electron Laser

Università degli Studi di Trieste
Sede Amministrativa del Dottorato di Ricerca

Posto di dottorato attivato grazie al contributo della Sincrotrone Trieste S.C.p.A.

XXII ciclo del Dottorato di Ricerca in INGEGNERIA DELL’INFORMAZIONE

Impact of the wakefields and of an initial energy curvature on a Free Electron Laser

Settore scientifico-disciplinare ING-INF/02 – Campi Elettromagnetici

Dottorando
Alberto Lutman

Coordinatore del Dottorato di Ricerca
Chiar.mo Prof. Roberto Vescovo

Relatore
Chiar.mo Prof. Roberto Vescovo
(Università di Trieste)

Tutore
Chiar.mo Prof. Roberto Vescovo
(Università di Trieste)

Tutore esterno
Ing. Paolo Craievich
(Sincrotrone Trieste S.C.p.A.)

Anno Accademico 2008/2009
Abstract

For an X-ray free electron laser (FEL), a high-quality electron bunch with low emittance, high peak current and energy is needed. During the phases of acceleration, bunch compression and transportation, the electron beam is subject to radio frequency curvature and to wakefields effects. Thus, the energy profile of the electron beam can present a parabolic profile, which has important electromagnetic effects on the FEL process. The quality of the electron beam is also degraded by the interaction with the low-gap undulator vacuum chamber. In our work we first analyze this interaction, deriving a formula to evaluate the longitudinal and the transversal wakefields for an elliptical cross section vacuum chamber, obtaining accurate results in the short range. Subsequently within the Vlasov-Maxwell one-dimensional model, we derive the Green functions necessary to evaluate the radiation envelope, having as initial conditions both an energy chirp and curvature on the electrons and eventually an initial bunching, which is useful to treat the harmonic generation FEL cascade configuration. This allows to study the impact of the electron beam energy profile on the FEL performance. Using the derived Green functions we discuss FEL radiation properties such as bandwidth, frequency shift, frequency chirp and velocity of propagation. Finally, we propose a method to achieve ultra-short FEL pulses using a frequency chirp on the seed laser and a suitable electron energy profile.

Sommario

Il funzionamento di un laser ad elettroni liberi operante nella banda dei raggi X richiede l’impiego di un fascio di elettroni di elevata qualità, con bassa emittanza, elevata corrente di picco ed energia. Il fascio di elettroni utilizzato può presentare un profilo di energia parabolico, in quanto durante le fasi di accelerazione, compressione e trasporto, il bunch è soggetto alla curvatura della radiofrequenza ed ai campi scia. La qualità del fascio di elettroni può inoltre essere degradata dall’interazione del fascio medesimo con le pareti della camera da vuoto nell’ondulatore. In questo lavoro, dapprima deriviamo una formula analitica per valutare i campi scia longitudinale e trasversale per una camera da vuoto di sezione ellittica, formula che risulta essere accurata anche per fasci molto corti. Successivamente nell’ambito del modello unidimensionale di Vlasov-Maxwell, deriviamo delle funzioni di Green per valutare l’inviluppo della radiazione, avendo come condizioni iniziali il profilo
parabolico di energia all’ingresso dell’ondulatore ed un eventuale bunching, l’ultima condizione è necessaria per lo studio del FEL a generazione di armoniche. Dette funzioni di Green consentono lo studio dell’impatto del profilo di energia del fascio sulle prestazioni del FEL. In particolare discutiamo le proprietà della radiazione FEL quali l’aumento della banda, lo spostamento della frequenza centrale, il chirp in frequenza e la velocità di propagazione dell’impulso. Infine, proponiamo un metodo per ottenere impulsi ultra-corti utilizzando un chirp in frequenza sul laser di seed ed un opportuno profilo di energia del fascio.
Contents

1 Introduction 7

2 Wakefield effects in the undulator 9
  2.1 Wake functions and impedances .......................... 10
  2.2 Resistive Wall Wakefields .............................. 11
    2.2.1 Circular cross section ................................ 11
    2.2.2 Elliptical cross section .............................. 12
  2.3 Examples and Results ................................ 34
  2.4 Undulator vacuum chamber selection .................... 36

3 1D Fel Theory 43
  3.1 Electron equations ...................................... 43
  3.2 Field equation .......................................... 46
  3.3 Vlasov-Maxwell approach ................................ 48

4 Green Functions with parabolic energy profile 51
  4.1 Vlasov-Maxwell for an initial value problem .......... 52
  4.2 Linear energy chirped exact Green function .......... 56
    4.2.1 Comparison between exact and approximated form ... 61
  4.3 Saddle point approximated Green Function ............ 62
    4.3.1 Properties of the Green function .................. 66
  4.4 Exact Green function with parabolic profile .......... 71
  4.5 Green function with bunching as source .............. 74

5 Impact of the energy profile on the FEL 75
  5.1 Coherent Seed Laser Pulse ............................. 76
    5.1.1 Seed Laser Pulse Duration .......................... 76
  5.2 Time-Frequency FEL characterization ................... 77
5.2.1 FEL expression ................. 79
5.3 Effects of the initial conditions on the FEL ............ 82
  5.3.1 FEL envelope distortions ............. 82
  5.3.2 Frequency shift and chirp of the FEL radiation .... 83
  5.3.3 Velocity of the centroid of the FEL pulse ......... 86
  5.3.4 Impact of chirp and curvature on the cascade configuration ........... 87
5.4 Achieving ultra-short FEL pulses .................. 90

6 Conclusion .......................... 95
Chapter 1

Introduction

The subject of this work is a research carried out in the context of the FERMI@Elettra project [1].

The FERMI single pass Free Electron Laser (FEL) project at the ELETTRA Laboratory of Sincrotrone Trieste is one of the FEL based European projects, designed to provide the international user facility, in Italy, for scientific investigations with ultra high brilliance X-ray pulses, of ultra-fast and ultra-high resolution processes in material science and physical biosciences.

The full FERMI facility will consist of a linear accelerator plus two principal FEL beamlines, in the complex environment of a multi-beamline user facility provided by the ELETTRA synchrotron light source.

The FERMI project is based on the principle of high gain, harmonic generation FEL amplifier employing multiple undulators, up-shifting an initial seed signal in a single pass [2, 3, 4]. The initial seed signal is provided by a pulsed laser. The energy modulation induced by the interaction of the laser with the electron beam in the first undulator, also called modulator, is converted to spatial modulation by sending the beam through the magnetic field of a dispersive section. This gives rise to bunching that further increases the initial bunch modulations at the harmonics of the seed wavelength. Thus, the electrons emit coherent radiation in a second undulator, also called radiator, tuned at a higher harmonic corresponding to the desired FEL output.

In our study, we first consider the interaction of the electron beam with the undulator vacuum chamber. Such interaction leads to an energy loss and to an energy modulation of the electron beam, that can have an impact on the FEL performance. The study has proven to be useful, with reference to the choice of both the shape and the material of the small-gap undulator vacuum
In addition to the energy induced on the electrons in the undulator, the electron energy profile can have an initial non-flat energy profile at the undulator entrance. In fact, during the acceleration, bunch compression and transportation, the electron bunch is subject to the radio frequency curvature and to the wakefields effect. Thus, the energy profile of the electron bunch can undergo modifications, and in particular it can experience a linear energy chirp and a quadratic curvature, which have important electromagnetic effects on the FEL process.

Starting from the FEL equations, we develop a theory to evaluate the impact on the FEL performance of an initial parabolic profile of the electron energy. In the literature, the effect of a linear energy chirp has been studied for the Self Amplified Spontaneous Emission (SASE) [5, 6] and for the seeded FEL process [7]. We extend the work with the analysis of the quadratic effect, giving an integral representation of the FEL light along the undulator, discussing properties of the FEL pulse such as the central frequency shift and the frequency chirp and the velocity of propagation of the FEL light for the seeded FEL configuration. Further, we consider the harmonic generation cascade configuration, and evaluate the effect of the parabolic energy profile at the modulator entrance. Finally, we propose a way to achieve ultra-short FEL pulses using a frequency chirp on the seed laser and an electron bunch with a suitable energy profile.
Chapter 2

Energy Induced on the beam by Wakefields in undulator vacuum chamber

In an accelerator a charged particle beam interacts electromagnetically with a vacuum chamber. In particular, for the Free Electron Laser (FEL) process it is interesting to evaluate the energy induced in the electron beam while it crosses the small-gap undulator vacuum chamber. A full self-consistent treatment for the problem of the charged beam crossing the vacuum chamber should consider the effect of the wake fields on the beam motion. However, for particles travelling through the vacuum chamber close to the velocity of light, it is a good approximation to treat the beam as a rigid one and to consider its motion unaffected by the field that it generates. Furthermore, in the relativistic limit, due to the causality principle there is no electromagnetic beam in front of the beam, thus the field generated by the interaction with the chamber is called wake field. A beam travelling in a perfectly conducting smooth pipe in the center of the cross section does not generate wake fields, but in real cases both the surface roughness and the finite conductivity of the wall chamber are origin of wake fields. Since the energy induced on the beam itself is of interest, the short range effect of the wake field must be evaluated accurately, particularly for short beams with high peak current.
2.1 Wake functions and impedances

Let us refer a vacuum chamber to a coordinate system \( O(u, v, s) \), where \( u, v \) are the transversal coordinates and \( s \) is the longitudinal one. We consider two point charges travelling at the velocity of light, parallel with the axis of the vacuum chamber, eventually with different transversal coordinates. The leading charge \( q \) is followed by the trailing charge \( q_t \) whose displacement from the leading charge is \( z \). The interaction of the leading charge with the vacuum chamber originates the electric field \( E \) and the magnetic field \( B \). The Lorentz force experienced by the trailing charge is

\[
F = q_t \left( E + \hat{z} \times cB \right). \tag{2.1}
\]

The wake function per unit of length is defined by

\[
W = F / qq_t, \tag{2.2}
\]

The effect of the longitudinal wake field \( W_L \) is that of inducing an energy change in the trailing particle, while the effect of the transversal wake field \( W_T \) is that of giving a transversal kick to the particles. The wake function is related to the impedance \( Z \) through the following Fourier transformation:

\[
Z(u, v, k) = \frac{1}{c} \int_{-\infty}^{+\infty} W(z, u, v) e^{-ikz} dz. \tag{2.3}
\]

Once the wake function is known, the energy change per unit length induced within a particle bunch is obtained by evaluating the convolution:

\[
\Delta E(z) = -\frac{eQ}{c} \int_{-\infty}^{z} W_L(z - z') \rho(z') dz'. \tag{2.4}
\]

where \( e \) is the electron charge and \( Q \) is the total bunch charge with longitudinal charge distribution \( \rho(z) \). This allows to evaluate the energy induced on different parts of the bunch, thus giving information on how the quality of the electron beam is degraded while travelling through the small-gap undulator vacuum chamber. In a similar way, once the transverse wake function is known, the transverse kick for unit length and for transverse offset, received by the particles in the vertical plane within the bunch, is obtained from the following convolution:

\[
k_T(z) = \frac{eQ}{E_0} \int_{-\infty}^{z} W_T(z - z') \rho(z') dz', \tag{2.5}
\]

where \( E_0 \) is the bunch mean energy.
2.2 Resistive Wall Wakefields

We now consider the interaction between the beam and the vacuum chamber, which is due to the finite conductivity of the metal. The chamber, regardless of the cross-section shape, is modeled as an infinite long cylindrical pipe, the metal wall is considered of infinite thickness and characterized by a conductivity $\sigma$ and an electron relaxation time $\tau$. The leading point charge, travelling along the beam pipe parallel to its axis $s$, is assumed to be ultrarelativistic, and its longitudinal position is $ct$ where $c$ is the velocity of light in vacuum. We introduce the coordinate $z$, which is the longitudinal displacement between the trailing point charge, and the leading point charge defined as $z = s - ct$. The transversal coordinates are defined case by case, depending on the cross section shape. Since the almost flat-top short bunches which will be used for the FERMI@Elettra FEL[1] contain very high frequencies components induced by the residual current spikes, as pointed out by Bane [8], the AC conductivity model has to be used. The AC conductivity model is well discussed in [9], and predicts that the conductivity at higher frequencies depends on the electron relaxation time as

$$\sigma_{AC} = \frac{\sigma_{DC}}{1 - i\omega\tau} \quad (2.6)$$

2.2.1 Circular cross section

For the circular cross section vacuum chamber case, we follow the treatment presented in [10]. The circular cylindrical $(r, \theta, s)$ coordinate system is used, and the source terms for a leading point charge $q$ having coordinates $(a, \theta_0, ct)$ can be written as:

$$\rho = \sum_{m=0}^{\infty} \frac{q}{\pi a(1 + \delta^k_{m,0})} \delta s - ct \delta(r - a) \cos(\theta - \theta_0) \quad (2.7a)$$

$$J = c\rho\delta s. \quad (2.7b)$$

where $\delta^k$ and $\delta$ are the Kronecker delta and the Dirac delta functions, respectively. We introduce the coordinate $z = s - ct$, which is the longitudinal displacement from the point charge, and the electromagnetic fields $(E, B)$ are calculated introducing a Fourier transform, so that:

$$(E, B) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\tilde{E}, \tilde{B}) e^{i k z} dk. \quad (2.8)$$
Where $k$ is the wave number, $\tilde{E}$ and $\tilde{B}$ are the Fourier transform of the electric field and the magnetic field, respectively. Matching the tangential components of the fields on the interface between the vacuum and the conducting wall $r = b$, yields the following expressions for the fields in the vacuum region ($r < b$) in the Fourier complex domain:

\[
\begin{align*}
\tilde{E}_z &= \frac{q}{2\pi \epsilon_0} + \sum_{m=1}^{+\infty} A_r^m \cos m(\theta - \theta_0) \\
\tilde{B}_z &= -\sum_{m=1}^{+\infty} A_r^m \sin m(\theta - \theta_0) \\
\tilde{E}_r &= -\frac{ikr}{2} \frac{q}{2\pi \epsilon_0} + \frac{q}{2\pi \epsilon_0} + \frac{q}{2\pi \epsilon_0} H^+ \\
\tilde{E}_\theta &= -\frac{ikr}{2} \frac{q}{2\pi \epsilon_0} + \frac{q}{2\pi \epsilon_0} + \frac{q}{2\pi \epsilon_0} \frac{H}{2} \\
\tilde{B}_r &= \sum_{m=1}^{+\infty} \left( -\frac{ikA}{2(m+1)} r^{m+1} + \frac{r^{m-1}}{2} \left( -\frac{imA}{k} - \frac{4im_2(1-H)}{a^{2m}} + B \right) + \frac{2imH}{r^{m+1}} \right) \cos m(\theta - \theta_0) \\
\tilde{E}_\theta &= \sum_{m=1}^{+\infty} \left( -\frac{ikA}{2(m+1)} r^{m+1} + \frac{r^{m-1}}{2} \left( -\frac{imA}{k} + \frac{4im_2(1-H)}{a^{2m}} - B \right) + \frac{2imH}{r^{m+1}} \right) \sin m(\theta - \theta_0) \\
\text{where } A &= \frac{q a^m / (\pi \epsilon_0)}{k^{2m+1}(\frac{ikA}{2})^2 - \frac{a^m}{\pi \epsilon_0}^2} \left( \frac{\lambda}{k} \right)^{2m+1}(\frac{sa^m / (\pi \epsilon_0)}{\frac{a^m}{\pi \epsilon_0}^2} - \frac{imA}{ka^{2m}} - \frac{4im_2(1-H)}{a^{2m}} + B \right) + \frac{2imH}{r^{m+1}} \cos m(\theta - \theta_0)
\end{align*}
\]

The expressions in Eq. 2.9 allow to evaluate the longitudinal and the transverse forces acting on a trailing charge $q_t$ per unit of length, in the Fourier domain. Substituting into Eq. 2.1, one obtains:

\[
\begin{align*}
F_\parallel &= q_t E_z, \\
F_\theta &= q_t (E_\theta + B_r), \\
F_r &= q_t (E_r - B_\theta).
\end{align*}
\] (2.10)

Substituting the expressions in Eq. 2.10 into Eq. 2.2 gives the wake fields expressions.

Finally, the wake functions are determined with a numerical inverse Fourier transform.

### 2.2.2 Elliptical cross section

For the FERMI@Elettra FEL undulators an elliptical cross section vacuum chamber has been chosen. For this kind of vacuum chambers, there are references to analytically derived expressions for the low frequency resistive wall coupling impedance [11, 12, 13], where the surface impedance is used to
2.2. RESISTIVE WALL WAKEFIELDS

determine the electromagnetic fields in the vacuum region. The problem of calculating the coupling impedance, including the case of high frequencies, of a resistive beam pipe with arbitrary cross section has been solved by Yokoya [16] with the boundary element method. He applied the method with the aim of numerically calculating the solution in the case of an elliptical pipe, moreover the method can be applied using both AC and DC conductivity models.

In order to have a better insight into the physics of the problem, and to perform faster and reliable calculations, analytical expressions have been derived for the longitudinal and transverse resistive-wall coupling impedance of a vacuum chamber with elliptical cross section, using the field matching method [17].

To describe the elliptical cross section, we denote with \( a \) the major half-axis of the ellipse and with \( b \) the minor one, respectively. Fig. 2.1 shows cross section and longitudinal section of the beam pipe, and the travelling point charge with the elliptic cylindrical \((u, v, s)\) coordinates system, where the \( s\)-axis coincides with the pipe axis. The relations between the Cartesian \((x, y, s)\) and the elliptical coordinates are illustrated in Fig 2.2, and the analytical formulas are the following:

\[
\begin{align*}
  x &= l \cosh u \cos v, \\
  y &= l \sinh u \sin v,
\end{align*}
\]
where \( l = \sqrt{a^2 - b^2} \) is the half focal length.

![Elliptic cylindrical and Cartesian orthogonal coordinates system.](image)

Figure 2.2: Elliptic cylindrical \((u, v, s)\) and Cartesian orthogonal \((x, y, s)\) coordinates system.

The metric coefficients for the elliptic cylindrical coordinates are:

\[
h = h_u = h_v = l \sqrt{\cosh^2 u - \cos^2 v}. \tag{2.13}
\]

The inverse of (2.11) and (2.12) are:

\[
u = \Re \left( \frac{\cosh x + iy}{l} \right), \tag{2.14}
\]

\[
v = \Im \left( \frac{\cosh x + iy}{l} \right). \tag{2.15}
\]

The unit vectors \( \hat{u} \) and \( \hat{v} \) are expressed as:

\[
\hat{u} = \frac{\hat{x} \sinh u \cos v + \hat{y} \cosh u \sin v}{\sqrt{\cosh^2 u - \cos^2 v}}, \tag{2.16}
\]

\[
\hat{v} = \frac{-\hat{x} \cosh u \sin v + \hat{y} \sinh u \cos v}{\sqrt{\cosh^2 u - \cos^2 v}}. \tag{2.17}
\]
The segment $u = 0$ connects the foci, and the couples $(0,v)$ and $(0,-v)$, represent the same point on the cross section. Note that the unit vectors $\hat{u}$ and $\hat{v}$ are discontinuous on the segment $u = 0$. Precisely:

$$\lim_{v \to x} \hat{u}(0,v) = - \lim_{v \to -x} \hat{u}(0,v), \quad (2.18)$$
$$\lim_{v \to x} \hat{v}(0,v) = - \lim_{v \to -x} \hat{v}(0,v). \quad (2.19)$$

The equation $u = u_0$ defines the surface separating the vacuum region from the resistive wall. The vacuum region is specified by $u < u_0$, while the metal region by $u > u_0$. The leading point charge, travelling along the beam pipe, has transverse coordinates $(u_1,v_1)$, with $u_1 < u_0$.

**Fields in the vacuum**

Using the elliptical cylindrical coordinates system $O(u,v,s)$, the Maxwell’s equations in the vacuum region can be written as follows, denoting the electric field components with $E_u$, $E_v$ and $E_s$, and the magnetic field components with $B_u$, $B_v$, and $B_s$:

$$\frac{1}{h^2} \frac{\partial h E_u}{\partial u} + \frac{1}{h^2} \frac{\partial h E_v}{\partial v} + \frac{\partial E_s}{\partial s} = \frac{\rho}{\varepsilon_0}, \quad (2.20a)$$
$$\frac{1}{h^2} \frac{\partial E_u}{\partial v} - \frac{\partial E_v}{\partial s} = -\frac{\partial B_u}{\partial t}, \quad (2.20b)$$
$$\frac{\partial E_u}{\partial s} - \frac{1}{h} \frac{\partial E_s}{\partial u} = -\frac{\partial B_v}{\partial t}, \quad (2.20c)$$
$$\frac{1}{h^2} \frac{\partial h E_v}{\partial u} - \frac{1}{h^2} \frac{\partial h E_u}{\partial v} = -\frac{\partial B_s}{\partial t}, \quad (2.20d)$$
$$\frac{1}{h^2} \frac{\partial h B_u}{\partial u} + \frac{1}{h^2} \frac{\partial h B_v}{\partial v} + \frac{\partial B_s}{\partial s} = 0, \quad (2.20e)$$
$$\frac{1}{h^2} \frac{\partial B_u}{\partial v} - \frac{\partial B_v}{\partial s} = \frac{1}{c^2} \frac{\partial E_u}{\partial t}, \quad (2.20f)$$
$$\frac{\partial B_u}{\partial s} - \frac{1}{h} \frac{\partial B_s}{\partial u} = \frac{1}{c^2} \frac{\partial E_v}{\partial t}, \quad (2.20g)$$
$$\frac{1}{h^2} \frac{\partial h B_v}{\partial u} - \frac{1}{h^2} \frac{\partial h B_u}{\partial v} = \frac{1}{c^2} \frac{\partial E_s}{\partial t} + \mu_0 J. \quad (2.20h)$$
where $h$ is the metric, while the charge and current density are, respectively:

$$
\rho = q \frac{\delta(s - ct) \delta(u - u_1) \delta(v - v_1)}{h^2}, \quad (2.21a)
$$

$$
\mathbf{J} = qc \frac{\delta(s - ct) \delta(u - u_1) \delta(v - v_1)}{h^2} \hat{s}. \quad (2.21b)
$$

In addition, we refer to the coordinate $z = s - ct$, which is the longitudinal displacement from the point charge. Thus it is $z < 0$ behind the leading charge and $z > 0$ ahead of it. Due to the causality principle, all fields must vanish for $z > 0$.

Using the same approach adopted in [10], we write the field vectors $\mathbf{E} = (E_u, E_v, E_z)$ and $\mathbf{B} = (B_u, B_v, B_z)$ in terms of the Fourier transformed vectors $\tilde{\mathbf{E}}, \tilde{\mathbf{B}}$ on the $z$-axis, as in Eq. 2.8. Substituting Eq. (2.21a) into Eq. (2.20a) and Eq. (2.21b) into Eq. (2.20h), one obtains the following system of six equations:

$$
\frac{\partial \tilde{E}_z}{\partial v} = c \frac{\partial \tilde{B}_z}{\partial u}, \quad (2.22a)
$$

$$
\frac{\partial \tilde{E}_z}{\partial u} = -c \frac{\partial \tilde{B}_z}{\partial v}, \quad (2.22b)
$$

$$
\frac{\partial h \tilde{E}_u}{\partial u} - \frac{\partial ch \tilde{B}_u}{\partial v} = \frac{q}{\epsilon_0} \delta(u - u_1) \delta(v - v_1) - \frac{ik h^2 \tilde{E}_z}{k} + \frac{i \partial^2 \tilde{E}_z}{\partial v^2}, \quad (2.22c)
$$

$$
\frac{c \partial \tilde{B}_u}{\partial u} + \frac{\partial h \tilde{E}_u}{\partial v} = -\frac{ik h^2 c \tilde{B}_z}{k} + \frac{i c \partial^2 \tilde{B}_z}{\partial v^2}, \quad (2.22d)
$$

$$
\tilde{E}_v = \frac{i}{hk} \frac{\partial \tilde{E}_z}{\partial v} - c \tilde{B}_u, \quad (2.22e)
$$

$$
\tilde{B}_v = \frac{-ic}{hk} \frac{\partial \tilde{B}_z}{\partial v} + \tilde{E}_u. \quad (2.22f)
$$

The expressions for the longitudinal fields are obtained by solving Eqs. (2.22a) and (2.22b). Performing the derivative of Eq. (2.22a) with respect to $v$ ($u$) and the derivative of Eq. (2.22b) with respect to $u$ ($v$) and then summing (subtracting), we obtain the Laplace equations:

$$
\frac{\partial^2 \tilde{E}_z}{\partial u^2} + \frac{\partial^2 \tilde{E}_z}{\partial v^2} = 0, \quad \frac{\partial^2 \tilde{B}_z}{\partial u^2} + \frac{\partial^2 \tilde{B}_z}{\partial v^2} = 0. \quad (2.23)
$$
2.2. RESISTIVE WALL WAKEFIELDS

Now, let \( F \) be the generic of the two unknowns \( \tilde{E}_z \) and \( \tilde{B}_z \), thus:

\[
\frac{\partial^2 F_z}{\partial u^2} + \frac{\partial^2 F_z}{\partial v^2} = 0.
\]  
(2.24)

Since \( F \) must be \( 2\pi \)-periodic in \( v \), it can be expanded in Fourier series on \( v \) as:

\[
F_z = F_{z0}(u) + \sum_{n=1}^{+\infty} \left( F_{zn}^c(u) \cos nv + F_{zn}^s(u) \sin nv \right).
\]  
(2.25)

Substituting (2.25) into (2.24) yields the equation:

\[
F_z''(u) + \sum_{n=1}^{+\infty} \left[ F_{zn}''(u) - n^2 F_{zn}^c(u) \right] \cos nv + \\
+ \sum_{n=1}^{+\infty} \left[ F_{zn}''(u) - n^2 F_{zn}^s(u) \right] \sin nv = 0.
\]  
(2.26)

where \( F''(u) \) denotes the derivative of the second order on \( u \).

The latter equation is equivalent to the following system of infinitely many equations:

\[
F_{z0}''(u) = 0,
\]  
(2.27)

\[
F_{zn}''(u) - n^2 F_{zn}^c(u) = 0 \quad n = 1, 2, \ldots,
\]  
(2.28)

\[
F_{zn}''(u) - n^2 F_{zn}^s(u) = 0 \quad n = 1, 2, \ldots
\]  
(2.29)

whose solutions are:

\[
F_{z0}(u) = A_0 + A_{0u} u,
\]  
(2.30)

\[
F_{zn}^c(u) = A_n \cosh nu + B_n \sinh nu,
\]  
(2.31)

\[
F_{zn}^s(u) = C_n \cosh nu + D_n \sinh nu.
\]  
(2.32)

Substituting into (2.25), the longitudinal components can be written as:

\[
\tilde{E}_z = A_0 + A_{0u} u + \sum_{n=1}^{+\infty} A_n \cosh nu \cos nv + B_n \sinh nu \sin nv \\
+ E_n \cosh nu \sin nv + F_n \sinh nu \cos nv
\]  
(2.33a)

\[
\tilde{B}_z = B_0 + B_{0u} u + \sum_{n=1}^{+\infty} C_n \cosh nu \cos nv + D_n \sinh nu \sin nv \\
+ G_n \cosh nu \sin nv + H_n \sinh nu \cos nv
\]  
(2.33b)
As the two couples \((0, -v), (0, v)\) specify the same point of the segment \(u = 0\) connecting the foci of the elliptical cross section, we must impose the condition:

\[
F_z(0, v) = F_z(0, -v).
\]  
(2.34)

This implies \(C_n = 0\) and \(H_n = 0\). Taking this into account and substituting (2.33a) and (2.33b) into (2.22a) and (2.22b) yields: \(A_n = -cD_n, B_n = cC_n, E_n = cH_n = 0, F_n = -cG_n = 0, A_{0u} = 0\) and \(B_{0u} = 0\). Thus, (2.33a) and (2.33b) reduce to (2.35a) and (2.35b), respectively. The longitudinal components of the electric and magnetic fields can be written as follows:

\[
\tilde{E}_z = A_0 + \sum_{n=1}^{+\infty} A_n \cosh nu \cos nv + \sum_{n=1}^{+\infty} B_n \sinh nu \sin nv,
\]  
(2.35a)

\[
c\tilde{B}_z = B_0 + \sum_{n=1}^{+\infty} B_n \cosh nu \cos nv - \sum_{n=1}^{+\infty} A_n \sinh nu \sin nv,
\]  
(2.35b)

where \(A_n\) and \(B_n\) are constants to be determined.

The transversal components are obtained by solving Eqs. (2.22c) trough Eqs. (2.22f).

Differentiating and combining Eqs. (2.22c) and (2.22d) yields the equations:

\[
\frac{\partial^2 h \tilde{E}_u}{\partial u^2} + \frac{\partial^2 h \tilde{E}_u}{\partial v^2} = \frac{q}{\varepsilon_0} \delta'(u - u_1)\delta(v - v_1) - ik \frac{\partial h^2 \tilde{E}_z}{\partial u}
\]

\[
+ \frac{i \partial}{k \partial u} \frac{\partial^2 \tilde{E}_z}{\partial v^2} - ik \frac{\partial h^2 \tilde{B}_z}{\partial v} + \frac{i \partial}{k \partial v} \frac{\partial^2 \tilde{B}_z}{\partial u}.
\]  
(2.36a)

\[
\frac{\partial^2 h \tilde{B}_u}{\partial u^2} + \frac{\partial^2 h \tilde{B}_u}{\partial v^2} = -\frac{q}{\varepsilon_0} \delta(u - u_1)\delta'(v - v_1) + ik \frac{\partial h^2 \tilde{E}_z}{\partial v}
\]

\[
- \frac{i \partial}{k \partial v} \frac{\partial^2 \tilde{E}_z}{\partial u} - ik \frac{\partial h^2 \tilde{B}_z}{\partial u} + \frac{i \partial}{k \partial u} \frac{\partial^2 \tilde{B}_z}{\partial v}.
\]  
(2.36b)

Differentiating Eqs. (2.22a) and (2.22b) yields:

\[
\frac{\partial^2 \partial \tilde{E}_z}{\partial v^2 \partial u} + \frac{\partial^2 \partial \tilde{c}B_z}{\partial v^2 \partial v} = 0,
\]  
(2.37a)

\[
\frac{\partial^2 \partial \tilde{c}B_z}{\partial v^2 \partial u} - \frac{\partial^2 \partial \tilde{E}_z}{\partial v^2 \partial v} = 0.
\]  
(2.37b)
2.2. RESISTIVE WALL WAKEFIELDS

Furthermore, using Eqs. (2.22a) and (2.22b) the following equalities can be verified:

\[
\begin{align*}
\frac{\partial h^2 \tilde{E}_z}{\partial u} + \frac{\partial h^2 c \tilde{B}_z}{\partial v} &= c \tilde{B}_z \frac{\partial h^2}{\partial u} + \tilde{E}_z \frac{\partial h^2}{\partial u}, \\
\frac{\partial h^2 \tilde{E}_z}{\partial v} - \frac{\partial h^2 c \tilde{B}_z}{\partial u} &= -c \tilde{B}_z \frac{\partial h^2}{\partial v} + \tilde{E}_z \frac{\partial h^2}{\partial v}.
\end{align*}
\] (2.38a)

Finally, substituting (2.38a) into (2.36a) and (2.38b) into (2.36b), and using (2.37a), (2.37b) and (2.13) we obtain:

\[
\begin{align*}
\frac{\partial^2 h \tilde{E}_u}{\partial u^2} + \frac{\partial^2 h \tilde{E}_u}{\partial v^2} &= \frac{q}{\epsilon_0} \delta'(u - u_1) \delta(v - v_1) + \\
&-ikl^2 c \tilde{B}_z \sin 2v - ikl^2 \tilde{E}_z \sinh 2u, \\
\frac{\partial^2 h c \tilde{B}_u}{\partial u^2} + \frac{\partial^2 h c \tilde{B}_u}{\partial v^2} &= -\frac{q}{\epsilon_0} \delta(u - u_1) \delta'(v - v_1) + \\
&-ikl^2 c \tilde{B}_z \sinh 2u + ikl^2 \tilde{E}_z \sin 2v.
\end{align*}
\] (2.39a)

The homogeneous equations associated with (2.39a) and (2.39b) are Laplace equations. Their solutions are denoted by \( K_{\tilde{E}_u} \) and \( K_{\tilde{B}_u} \), respectively, and have the form in (2.33) with \( \tilde{E}_z \) and \( \tilde{B}_z \) replaced by \( h \tilde{E}_u \) and \( h \tilde{B}_u \), respectively.

In order to find particular solutions of Eqs. (2.39a) and (2.39b) in the presence of only the longitudinal fields in the right-hand sides, we substitute \( \tilde{E}_z \) and \( c \tilde{B}_z \) (given by (2.35a) and (2.35b) respectively) into (2.39a) and (2.39b). Then the right hand sides of (2.39a) and (2.39b) become linear combinations of the forcing terms in Table 2.1, so that the problem reduces to that of finding particular solutions of the equations:

\[
\frac{\partial^2 F(u, v)}{\partial u^2} + \frac{\partial^2 F(u, v)}{\partial v^2} = g(u, v),
\] (2.40)

where \( g(u, v) \) is the generic forcing term. For each forcing term a particular solution is reported in Table 2.1. Particular solutions related to the longitudinal fields are given by (2.50c), (2.50d), (2.50e) and (2.50f).

Substituting the solutions \( K_{\tilde{E}_u} \), \( K_{\tilde{B}_u} \) and (2.50c)-(2.50f) into the first order system of Eqs. (2.22c) and (2.22d), yields the conditions: \( A_{0u} = 0, B_{0u} = 0, \)
Table 2.1: The terms related to the longitudinal components in the right hand sides of Eqs. (2.39) are proportional to the functions in the Forcing term column. The corresponding particular solutions are reported in the other column.

<table>
<thead>
<tr>
<th>Forcing term</th>
<th>Particular Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sinh 2u$</td>
<td>$\sinh 2u$</td>
</tr>
<tr>
<td>$\sinh 2u \cosh u \cos v$</td>
<td>$\left[ \frac{\sinh 3u}{16} + \frac{1}{4} u \cosh u \right] \cos v$</td>
</tr>
<tr>
<td>$\sinh 2u \cosh nu \cos nv$</td>
<td>$\left[ \frac{\sinh (n+2)u}{8n+8} + \frac{\sinh (n-2)u}{8n-8} \right] \cos nv$</td>
</tr>
<tr>
<td>$\sinh 2u \sinh u \sin v$</td>
<td>$\left[ \frac{\cosh 3u}{16} - \frac{1}{4} u \sinh u \right] \sin v$</td>
</tr>
<tr>
<td>$\sinh 2u \sinh nu \sin nv$</td>
<td>$\left[ \frac{\sinh (n+2)u}{8n+8} + \frac{\cosh (n-2)u}{8n-8} \right] \sin nv$</td>
</tr>
<tr>
<td>$\sin 2v$</td>
<td>$\left[ \frac{-\sin 3u}{16} \cosh u + \frac{1}{4} u \sinv \right] \sinh u$</td>
</tr>
<tr>
<td>$\sin 2v \cosh u \cos v$</td>
<td>$\left[ \frac{-\sin (n+2)u}{8n+8} - \frac{\sin (n-2)u}{8n-8} \right] \cosh nu$</td>
</tr>
<tr>
<td>$\sin 2v \cosh nu \cos nv$</td>
<td>$\left[ \frac{-\cos 3u}{16} \sinh u + \frac{1}{4} \cosh v \cos u \right] \sinh nu$</td>
</tr>
<tr>
<td>$\sin 2v \sinh u \sin v$</td>
<td>$\left[ \frac{\cos (n+2)u}{8n+8} + \frac{\cos (n-2)u}{8n-8} \right] \sinhv$</td>
</tr>
</tbody>
</table>

$A_n = cD_n$, $B_n = -cC_n$, $E_n = -cH_n - \frac{ni}{k} B_n$, $F_n = cG_n - \frac{ni}{k} A_n$. Furthermore, the following conditions of continuity of the vectors $\hat{h}\tilde{E}_u\hat{u}$ and $\hat{h}\tilde{B}_u\hat{u}$ must be imposed on the segment $u = 0$:

\[
\begin{align*}
    \hat{h}\tilde{B}_u(0, v) &= -\hat{h}\tilde{B}_u(0, -v), \\
    \hat{h}\tilde{E}_u(0, v) &= -\hat{h}\tilde{E}_u(0, -v).
\end{align*}
\]  

(2.41)
(2.42)

The latter conditions lead to impose: $A_0 = 0$, $B_0 = 0$, $A_n = 0$, and $C_n = 0$. In conclusion, the solutions $K\tilde{E}_u$ and $K\tilde{B}_u$ of the homogeneous Eqs. (2.39a) and (2.39b) are given by (2.50a) and (2.50b), respectively. The particular solutions of (2.39a) and (2.39b), related only to the charge $q$, are denoted with $S^q_{\tilde{E}_u}$ and $S^q_{\tilde{B}_u}$ respectively, and are obtained by starting from the following Fourier series expansion of the $u-$components of the fields and of the source terms, as follows:

\[
S^q_{\tilde{E}_u} = E_0(u) + \sum_{n=1}^{+\infty} E_n^c(u) \cos nv + \sum_{n=1}^{+\infty} E_n^s(u) \sin nv,
\]  

(2.43a)
2.2. RESISTIVE WALL WAKEFIELDS

\[ S_{Bu} = B_0(u) + \sum_{n=1}^{\infty} B_c^n(u) \cos n\nu + \sum_{n=1}^{\infty} B_s^n(u) \sin n\nu, \quad (2.43b) \]

\[ \delta'(u - u_1)\delta(v - v_1) = \delta'(u - u_1)\left[ \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\nu \cos n\nu_1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \sin n\nu \sin n\nu_1 \right], \quad (2.44a) \]

\[ \delta(u - u_1)\delta'(v - v_1) = \frac{\delta(u - u_1)}{\pi} \times \left[ \sum_{n=1}^{\infty} n \cos n\nu \sin n\nu_1 - \sum_{n=1}^{\infty} n \sin n\nu \cos n\nu_1 \right]. \quad (2.44b) \]

Substituting (2.43a) and (2.44a) into (2.39a), (2.43b) and (2.44b) into (2.39b), and assuming the presence, in the right-hand sides, of only the terms related to the charge \( q \), yields the following systems of equations:

\[
\begin{cases}
E_0''(u) = \frac{1}{2\pi}\frac{q}{\epsilon_0} \delta'(u - u_1) \\
E_c''(u) - n^2 E_c^n(u) = \frac{q}{\epsilon_0} \delta'(u - u_1) \frac{1}{\pi} \cos n\nu_1 \\
E_s''(u) - n^2 E_s^n(u) = \frac{q}{\epsilon_0} \delta'(u - u_1) \frac{1}{\pi} \sin n\nu_1,
\end{cases} \quad (2.45)\]

\[
\begin{cases}
B_0''(u) = 0 \\
B_c''(u) - n^2 B_c^n(u) = -\frac{nq}{\epsilon_0} \delta(u - u_1) \frac{1}{\pi} \sin n\nu_1 \\
B_s''(u) - n^2 B_s^n(u) = \frac{nq}{\epsilon_0} \delta(u - u_1) \frac{1}{\pi} \cos n\nu_1,
\end{cases} \quad (2.46)\]

for all positive integers \( n \). A possible solution of system (2.45) is given by:

\[ E_0 = \begin{cases} 
0 & u < u_1 \\
\frac{q}{2\pi\epsilon_0} & u > u_1,
\end{cases} \]

\[ E_c^n(u) = \frac{q}{\pi\epsilon_0} \cos n\nu_1 \begin{cases} 
\sinh n\nu_1 \sinh nu & u < u_1 \\
\cosh n\nu_1 \cosh nu & u > u_1,
\end{cases} \quad (2.47)\]

\[ E_s^n(u) = -\frac{q}{\pi\epsilon_0} \sin n\nu_1 \begin{cases} 
\cosh n\nu_1 \cosh nu & u < u_1 \\
\sinh n\nu_1 \sinh nu & u > u_1.
\end{cases} \]
A possible solution of system (2.46) is given by:

\[ B_0 = 0, \]

\[ B^c_n(u) = \frac{q}{\pi \epsilon_0} \sin n\nu_1 \begin{cases} \cosh n\nu_1 \sinh n\nu & u < u_1 \\ \sinh n\nu_1 \cosh n\nu & u > u_1, \end{cases} \]

\[ B^s_n(u) = \frac{q}{\pi \epsilon_0} \cos n\nu_1 \begin{cases} \sinh n\nu_1 \cosh n\nu & u < u_1 \\ \cosh n\nu_1 \sinh n\nu & u > u_1. \end{cases} \]

Substituting (2.47) and (2.48) into (2.43a) and (2.43b) yields (2.50g) and (2.50h).

The expressions for the transverse fields \( \tilde{E}_u \) and \( \tilde{B}_u \) can be finally written as:

\[ h\tilde{E}_u(u,v) = K_{E_u} + S^c_{E_u} + S^s_{E_u} + S^q_{E_u}, \]

\[ h\tilde{B}_u(u,v) = K_{B_u} + S^c_{B_u} + S^s_{B_u} + S^q_{B_u}, \]

where denoting with \( H(u) \) the Heaviside step function and letting \( W = ik(a^2 - b^2) \) we can write:

\[ K_{E_u} = \sum_{n=1}^{+\infty} \left( E_n - \frac{n^2}{k} A_n \right) \cosh n\nu \sin n\nu + \sum_{n=1}^{+\infty} F_n \sinh n\nu \cos n\nu, \]

\[ K_{B_u} = \sum_{n=1}^{+\infty} \left( F_n + \frac{n^2}{k} A_n \right) \cosh n\nu \sin n\nu - \sum_{n=1}^{+\infty} E_n \sinh n\nu \cos n\nu, \]

\[ S^c_{E_u} = -A_0 W \sinh \frac{2u}{4} + \frac{A_1 W}{16} \left( \sinh u \cos 3v - \sinh 3u \cos v \right) + W \sum_{n=2}^{+\infty} A_n \frac{\sinh nu}{8(n + 1)} \cos (n + 2)v - W \sum_{n=2}^{+\infty} A_n \left[ \frac{\sinh (n + 2)u}{8(n + 1)} + \frac{\sinh (n - 2)u}{8(n - 1)} \right] \cos n\nu, \]

\[ S^s_{E_u} = +B_0 W \sin \frac{2v}{4} + \frac{B_1 W}{16} \left( \cosh u \sin 3v - \cosh 3u \sin v \right) + W \sum_{n=2}^{+\infty} B_n \frac{\cosh nu}{8(n + 1)} \sin (n + 2)v + W \sum_{n=2}^{+\infty} B_n \frac{\cosh nu}{8(n - 1)} \sin (n - 2)v - W \sum_{n=2}^{+\infty} B_n \left[ \frac{\cosh (n + 2)u}{8(n + 1)} + \frac{\cosh (n - 2)u}{8(n - 1)} \right] \sin n\nu, \]
2.2. RESISTIVE WALL WAKEFIELDS

\[
S_{Bu}^c = -B_0 W \sinh \frac{2u}{4} + \frac{B_1 W}{16} (\sinh u \cos 3v - \sinh 3u \cos v) + \\
W \sum_{n=2}^{+\infty} B_n \frac{\sinh nu}{8(n+1)} \cos (n+2)v + W \sum_{n=2}^{+\infty} B_n \frac{\sinh nu}{8(n-1)} \cos (n-2)v \\
- W \sum_{n=2}^{+\infty} B_n \left[ \frac{\sinh (n+2)u}{8(n+1)} + \frac{\sinh (n-2)u}{8(n-1)} \right] \cos nv, 
\]

(2.50e)

\[
S_{Bu}^s = -A_0 W \frac{\sin 2v}{4} + \frac{A_1 W}{16} (\cosh 3u \sin v - \cosh u \sin 3v) \\
- W \sum_{n=2}^{+\infty} A_n \frac{\cosh nu}{8(n+1)} \sin (n+2)v + W \sum_{n=2}^{+\infty} A_n \frac{\cosh nu}{8(n-1)} \sin (n-2)v \\
+ W \sum_{n=2}^{+\infty} A_n \left[ \frac{\cosh (n+2)u}{8(n+1)} + \frac{\cosh (n-2)u}{8(n-1)} \right] \sin nv, 
\]

(2.50f)

\[
S_{Eu}^q = \frac{q}{2\pi \epsilon_0} H(u - u_1) \\
+ \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos nv_1 \cos nvH(u - u_1) \cosh nu_1 \cosh nu \\
+ \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos nv_1 \cos nvH(u_1 - u) \sinh nu_1 \sinh nu \\
- \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \sin nv_1 \sin nvH(u - u_1) \sinh nu_1 \sinh nu \\
- \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \sin nv_1 \sin nvH(u_1 - u) \cosh nu_1 \cosh nu, 
\]

(2.50g)

\[
S_{Bu}^q = \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \sin nv_1 \cos nvH(u - u_1) \sinh nu_1 \cosh nu \\
+ \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \sin nv_1 \cos nvH(u_1 - u) \cosh nu_1 \sinh nu \\
+ \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos nv_1 \sin nvH(u - u_1) \cosh nu_1 \sinh nu \\
+ \sum_{n=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos nv_1 \sin nvH(u_1 - u) \sinh nu_1 \cosh nu. 
\]

(2.50h)
The field components $\tilde{E}_v$ and $\tilde{B}_v$ can be directly obtained from Eqs. (2.22e) and (2.22f) using the solutions (2.35) and (2.49), yielding:

$$h\tilde{E}_v = K_{\tilde{E}_v} + S_{\tilde{E}_v}^c + S_{\tilde{E}_v}^s + S_{\tilde{E}_v}^q,$$

(2.51a)

$$hc\tilde{B}_v = K_{\tilde{B}_v} + S_{\tilde{B}_v}^c + S_{\tilde{B}_v}^s + S_{\tilde{B}_v}^q.$$

(2.51b)

where:

$$K_{\tilde{E}_v} = -K_{\tilde{B}_v},$$

(2.52a)

$$S_{\tilde{E}_v}^s = -S_{\tilde{B}_v}^s + \sum_{n=1}^{+\infty} \frac{ni}{k} A_n \cosh nu \sin nv,$$

(2.52b)

$$S_{\tilde{E}_v}^c = -S_{\tilde{B}_v}^c - \sum_{n=1}^{+\infty} \frac{ni}{k} B_n \sinh nu \cos nv,$$

(2.52c)

$$S_{\tilde{E}_v}^q = -S_{\tilde{B}_v}^q,$$

(2.52d)

$$K_{\tilde{B}_v} = +K_{\tilde{E}_v},$$

(2.52e)

$$S_{\tilde{B}_v}^s = +S_{\tilde{E}_v}^s + \sum_{n=1}^{+\infty} \frac{ni}{k} B_n \cosh nu \sin nv,$$

(2.52f)

$$S_{\tilde{B}_v}^c = +S_{\tilde{E}_v}^c + \sum_{n=1}^{+\infty} \frac{ni}{k} A_n \sinh nu \cos nv,$$

(2.52g)

$$S_{\tilde{B}_v}^q = +S_{\tilde{E}_v}^q.$$

(2.52h)

**Fields in the resistive wall**

The constants $A_n$, $B_n$, $E_n$ and $F_n$ will be determined by imposing the boundary conditions at the surface separating the vacuum from the resistive-wall. To this aim we need to calculate the expressions of the fields inside the wall ($u > u_0$). In the conductor we assume that:

$$\rho = 0,$$

(2.53a)

$$J = \sigma E,$$

(2.53b)
2.2. RESISTIVE WALL WAKEFIELDS

where \( \sigma \) is the conductivity of the metal. Furthermore, the Maxwell equations can be written as:

\[
\frac{1}{h^2} \frac{\partial h E_u}{\partial u} + \frac{1}{h^2} \frac{\partial h E_v}{\partial v} + \frac{\partial E_s}{\partial s} = 0, \quad (2.54a)
\]

\[
\frac{1}{h^2} \frac{\partial E_s}{\partial u} - \frac{1}{h^2} \frac{\partial E_v}{\partial v} = -\frac{\partial B_u}{\partial t}, \quad (2.54b)
\]

\[
\frac{\partial E_u}{\partial s} - \frac{1}{h} \frac{\partial E_s}{\partial u} = -\frac{\partial B_v}{\partial t}, \quad (2.54c)
\]

\[
\frac{1}{h^2} \frac{\partial h E_v}{\partial u} - \frac{1}{h^2} \frac{\partial h E_u}{\partial v} = -\frac{\partial B_s}{\partial t}, \quad (2.54d)
\]

\[
\frac{1}{h^2} \frac{\partial B_u}{\partial u} + \frac{1}{h^2} \frac{\partial B_v}{\partial v} + \frac{\partial B_s}{\partial s} = 0, \quad (2.54e)
\]

\[
\frac{1}{h^2} \frac{\partial B_s}{\partial u} - \frac{1}{h^2} \frac{\partial B_u}{\partial v} = \frac{1}{c^2} \frac{\partial E_u}{\partial t} + \mu_0 \sigma E_u, \quad (2.54f)
\]

\[
\frac{1}{h^2} \frac{\partial B_v}{\partial u} - \frac{1}{h^2} \frac{\partial B_u}{\partial v} = \frac{1}{c^2} \frac{\partial E_v}{\partial t} + \mu_0 \sigma E_v, \quad (2.54g)
\]

\[
\frac{1}{h^2} \frac{\partial B_s}{\partial u} - \frac{1}{h^2} \frac{\partial B_v}{\partial v} = \frac{1}{c^2} \frac{\partial E_s}{\partial t} + \mu_0 \sigma E_s. \quad (2.54h)
\]

Using the variable \( z = s - ct \) and manipulating, Eqs. (2.54) yields the equations:

\[
\frac{\partial^2}{\partial u^2} \tilde{E}_z + \frac{\partial^2}{\partial v^2} \tilde{E}_z + h^2 \lambda^2 \tilde{E}_z = 0, \quad (2.55a)
\]

\[
\frac{\partial^2}{\partial u^2} c \tilde{B}_z + \frac{\partial^2}{\partial v^2} c \tilde{B}_z + h^2 \lambda^2 c \tilde{B}_z = 0, \quad (2.55b)
\]

\[
h \tilde{E}_u = \frac{ik}{\lambda^2} \frac{\partial}{\partial u} \tilde{E}_z + \frac{ik}{\lambda^2} \frac{\partial}{\partial v} c \tilde{B}_z, \quad (2.55c)
\]

\[
h \tilde{E}_v = \frac{ik}{\lambda^2} \frac{\partial}{\partial v} \tilde{E}_z - \frac{ik}{\lambda^2} \frac{\partial}{\partial u} c \tilde{B}_z, \quad (2.55d)
\]

\[
 ch \tilde{B}_u = -\left[ \frac{ik}{\lambda^2} + \frac{i}{k} \right] \frac{\partial}{\partial v} \tilde{E}_z + \frac{ik}{\lambda^2} \frac{\partial}{\partial u} c \tilde{B}_z, \quad (2.55e)
\]

\[
 ch \tilde{B}_v = \left[ \frac{ik}{\lambda^2} + \frac{i}{k} \right] \frac{\partial}{\partial u} \tilde{E}_z + \frac{ik}{\lambda^2} \frac{\partial}{\partial v} c \tilde{B}_z. \quad (2.55f)
\]

where \( \lambda^2 = ik Z_0 \sigma \) and \( \lambda \) is chosen with positive imaginary part.

Each of Eqs. (2.55a) and (2.55b) can be solved by separating the variables \( u \) and \( v \), that is, assuming the unknown is proportional to the product between two functions \( U(u) \) and \( V(v) \), in order to determine the longitudinal fields in
the conductor. Substituting the product $U(u)V(v)$ yields:

\[
\frac{d^2}{dv^2}V + \left( a - 2Q \cos 2v \right) V = 0, \tag{2.56a}
\]

\[
\frac{d^2}{du^2}U - \left( a - 2Q \cosh 2u \right) U = 0, \tag{2.56b}
\]

where $Q = \ell^2 \chi^2 / 4$ and $a$ is a separation constant.

Eqs. (2.56a) and (2.56b) are called the Mathieu angular and radial equations, respectively [18]. On the other hand, the electric and magnetic fields, as well as the function $V(v)$, must be $2\pi$-periodic in $v$. There is a countable infinity of values of the constant $a$ that allow $2\pi$-periodic solutions in $v$. Such values are called the Mathieu Characteristic Numbers (MCNs), and can be calculated with the algorithms in [19]. In particular, for imaginary values of $Q$ (i.e., in the case of DC conductivity), an useful algorithm can be found in [20].

Adopting the standard notation for the MCNs [21], the constants $a_{2n}$ and $a_{2n+1}$ ($n \geq 0$) produce even $\pi$-periodic and $2\pi$-periodic solutions, respectively, while the constants $b_{2n}$ ($n \geq 1$) and $b_{2n+1}$ ($n \geq 0$) produce odd $\pi$-periodic and $2\pi$-periodic solutions, respectively.

Furthermore the possible solutions $V(v)$ can be written as:

\[
V_{2n}(v) = \sum_{m=0}^{+\infty} A_{2m}^{2n} \cos 2mv, \tag{2.57a}
\]

\[
V_{2n+1}(v) = \sum_{m=0}^{+\infty} A_{2m+1}^{2n+1} \cos (2m + 1)v, \tag{2.57b}
\]

\[
V_{b_{2n}}(v) = \sum_{m=1}^{+\infty} B_{2m}^{2n} \sin 2mv, \tag{2.57c}
\]

\[
V_{b_{2n+1}}(v) = \sum_{m=0}^{+\infty} B_{2m+1}^{2n+1} \sin (2m + 1)v, \tag{2.57d}
\]

where the Fourier coefficients can be calculated by recursion formulas [21].

It is worth noting that, for each of the solutions in (2.57), the other linearly independent solution of the Mathieu angular equation, for a fixed MCN, is not periodic in $v$ [18], thus can be discarded.

The Mathieu radial equation has two linearly independent solutions for every
2.2. RESISTIVE WALL WAKEFIELDS

MCN. They are called the first and the second kind radial Mathieu functions, respectively, and can be expressed as series of products of Bessel functions [19].

Since we are interested in damped solutions inside the conductor, a proper linear combination of the solutions of the first and second kind has to be chosen. The damped solutions required for a complex $Q$ are:

\[
U_{a2n}(u, Q) = \sum_{m=0}^{+\infty} (-1)^{m+n} A_{2m}^{a} \times \left[ J_{m-n}(w_1) H_{m+n}^{(1)}(w_2) + J_{m+n}(w_1) H_{m-n}^{(1)}(w_2) \right],
\]

(2.58a)

\[
U_{a2n+1}(u, Q) = \sum_{m=0}^{+\infty} (-1)^{m+n} A_{2m+1}^{a} \times \left[ J_{m-n}(w_1) H_{m+n+1}^{(1)}(w_2) + J_{m+n+1}(w_1) H_{m-n}^{(1)}(w_2) \right],
\]

(2.58b)

\[
U_{b2n}(u, Q) = \sum_{m=1}^{+\infty} (-1)^{m+n} B_{2m}^{b} \times \left[ J_{m-n}(w_1) H_{m+n}^{(1)}(w_2) - J_{m+n}(w_1) H_{m-n}^{(1)}(w_2) \right],
\]

(2.58c)

\[
U_{b2n+1}(u, Q) = \sum_{m=0}^{+\infty} (-1)^{m+n} B_{2m+1}^{b} \times \left[ J_{m-n}(w_1) H_{m+n+1}^{(1)}(w_2) - J_{m+n+1}(w_1) H_{m-n}^{(1)}(w_2) \right].
\]

(2.58d)

where $J_n$ is the Bessel function of the first kind of $n$-th order, $H_n^{(1)}$ is the Hankel function of the first kind of $n$-th order, $w_1 = \sqrt{Q} e^{-u}$ and $w_2 = \sqrt{Q} e^{u}$, $\sqrt{Q}$ having positive imaginary part.

Being proportional to $U(u)V(v)$, the unknowns $\tilde{E}_z$ and $c\tilde{B}_z$, can be finally expressed as:

\[
\tilde{E}_z = \sum_{n=0}^{+\infty} C_{a_n}^{E} U_{a_n}(u) \sum_{m=0}^{+\infty} A_{m}^{a_n} \cos mv + \sum_{n=1}^{+\infty} D_{b_n}^{E} U_{b_n}(u) \sum_{m=1}^{+\infty} B_{m}^{b_n} \sin mv (2.59a)
\]

\[
c\tilde{B}_z = \sum_{n=0}^{+\infty} C_{a_n}^{B} U_{a_n}(u) \sum_{m=0}^{+\infty} A_{m}^{a_n} \cos mv + \sum_{n=1}^{+\infty} D_{b_n}^{B} U_{b_n}(u) \sum_{m=1}^{+\infty} B_{m}^{b_n} \sin mv (2.59b)
\]

where the constants $C_{a_n}^{E}$, $D_{b_n}^{E}$, $C_{a_n}^{B}$, $D_{b_n}^{B}$ are to be determined.

The transverse fields in the conductor can be directly calculated by substi-
tuting Eqs. (2.59a) and (2.59b) into Eqs. (2.55c), (2.55d), (2.55e) and (2.55f). As in the following only tangential fields are involved, we give only the expressions for $h\tilde{E}_v$ and $c\tilde{B}_v$:

$$
\begin{align*}
    h\tilde{E}_v &= \sum_{n=0}^{\infty} \left[ -\frac{ik}{\lambda^2} C_{an} U_{an} \sum_{m=1}^{\infty} mA_m^a \cos mv - \frac{ik}{\lambda^2} C_{an} U_{an} \sum_{m=0}^{\infty} A_m^a \cos mv \right] \\
    &+ \sum_{n=1}^{\infty} \left[ \frac{ik}{\lambda^2} D_{bn}^a U_{bn} \sum_{m=1}^{\infty} mA_m^a \cos mv - \frac{ik}{\lambda^2} D_{bn}^a U_{bn} \sum_{m=0}^{\infty} B_m^b \sin mv \right],
\end{align*}
$$

(2.60a)

$$
\begin{align*}
    c\tilde{B}_v &= \sum_{n=0}^{\infty} \left[ \left( \frac{ik}{\lambda^2} + \frac{i}{\delta} \right) C_{an} U_{an} \sum_{m=1}^{\infty} A_m^a \cos mv - \frac{ik}{\lambda^2} C_{an} U_{an} \sum_{m=1}^{\infty} A_m^a \cos mv \right] \\
    &+ \sum_{n=1}^{\infty} \left[ \left( \frac{ik}{\lambda^2} + \frac{i}{\delta} \right) D_{bn}^a U_{bn} \sum_{m=1}^{\infty} B_m^b \sin mv + \frac{ik}{\lambda^2} D_{bn}^a U_{bn} \sum_{m=1}^{\infty} B_m^b \sin mv \right],
\end{align*}
$$

(2.60b)

Evaluation of the constants

The constants involved in the field expressions are determined by satisfying the boundary conditions at the surface equation $u = u_0$, which separates the vacuum from the resistive-wall.

Precisely, the continuity of the tangential components of $\tilde{E}_z$, $h\tilde{E}_v$, $c\tilde{B}_z$ and $hc\tilde{B}_v$ is imposed on the surface $u = u_0$, is imposed. As a consequence, the constants $A_{2n}$, $A_{2n+1}$, $B_{2n}$ and $B_{2n+1}$ can be calculated by solving four independent tri-diagonal linear systems.

Observing the Fourier series expansion of the fields (Eqs.(2.50) and (2.52a)), it can be noticed that there is no relation between even and odd constants, because the $n$-th component of the expansion depends on the $(n-2)$-th and on the $(n+2)$-th components in the vacuum, while Eq. (2.57) shows that angular the Mathieu functions are sums of either $\pi$-periodic functions (even MCNs) or $2\pi$-periodic functions (odd MCNs). Considering, for $\tilde{E}_z$ and $c\tilde{B}_v$, only the terms proportional to cosines and for $\tilde{B}_z$ and $h\tilde{E}_v$ only the terms proportional to sines, only the subset of constants $A_n$, $C_{an}E_n^a$, $D_{bn}^a$ is involved, while considering for $\tilde{E}_z$ and $h\tilde{B}_v$ only the terms proportional to sines and for $\tilde{B}_z$ and $c\tilde{B}_v$ only the terms proportional to cosines, only the subset of constants $B_n$, $E_nC_{an}^B$, $D_{bn}^E$ is involved. This shows that the whole system can be divided into four independent subsystems.

Assuming the metal as a good conductor (like copper or aluminium) allows us to use the following asymptotic approximation for the Mathieu radial
2.2. RESISTIVE WALL WAKEFIELDS

functions (which has been validated numerically):

\[ U(u) \approx e^{i(w_2-w_1)} = e^{i2\sqrt{Q}\sinh u}, \tag{2.61} \]

where \( w_1 = \sqrt{Q}e^{-u} \) and \( w_2 = \sqrt{Q}e^{u} \). \( Q \) is a complex quantity with large modulus, and its square root has positive imaginary part. From Eq. (2.61), the derivative of \( U(u) \) on \( u = u_0 \) is given by:

\[ \frac{\partial}{\partial u} U'(u_0) = 2i\sqrt{Q} \cosh u_0 U(u_0) = R(u_0)U(u_0). \tag{2.62} \]

In the sequel the subsystems will be solved, denoting with \( (E^<, B^<) \) the fields in the vacuum region, and with \( (E^>, B^>) \) the fields in the metal. Considering the components in cos \( mv \) of \( \widetilde{E}_z \), and imposing the continuity condition

\[ \widetilde{E}_z^< \big|_{u_0} = \widetilde{E}_z^> \big|_{u_0}, \]

we obtain:

\[ \sum_{m=0}^{+\infty} A_m \cosh mu_0 \cos mv = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} C^E_{an} U_{an}(u_0) A^a_m \cos mv. \tag{2.63} \]

Thus, for each \( m \) we can write:

\[ \sum_{n=0}^{+\infty} C^E_{an} U_{an}(u_0) A^a_m = A_m \cosh mu_0. \tag{2.64} \]

Considering the components in sin \( mv \) of \( c\widetilde{B}_z \) and imposing the continuity condition

\[ c\widetilde{B}_z^< \big|_{u_0} = c\widetilde{B}_z^> \big|_{u_0}, \]

we obtain:

\[ -\sum_{m=1}^{+\infty} A_m \sinh mu_0 \sin mv = \sum_{b_n}^{+\infty} D^B_{bn} U_{bn}(u_0) \sum_{m=1}^{+\infty} A^b_m \sin mv. \tag{2.65} \]

Thus, for each \( m \) we can write:

\[ \sum_{b_n} D^B_{bn} U_{bn}(u_0) B^b_m = -A_m \sinh mu_0. \tag{2.66} \]
We now consider the components in \( \sin mv \) of \( hE_v \) and the components of \( chB_v \) in \( \cos mv \) on \( u = u_0 \); further, using Eqs. (2.60), (2.62), (2.64) and (2.66) we write the components in \( \cos mv \) of \( hE_v^> \) and the \( \sin mv \) of \( hcB_v^> \), in terms of the constants \( A_n \), obtaining:

\[
\begin{align*}
    hE_v^< &= -\sum_{m=1}^{+\infty} F_m \cosh mu_0 \sin mv + S_{Ev}^s - \\
            &\quad \sum_{m=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos mv_1 \cosh mu_1 \sinh mu_0 \sin mv,
    \\
    hcB_v^< &= \sum_{m=1}^{+\infty} F_m \sinh mu_0 \cos mv + S_{Bv}^c + \frac{q}{2\pi \epsilon_0} + \\
            &\quad \sum_{m=1}^{+\infty} \frac{q}{\pi \epsilon_0} \cos mv_1 \cosh mu_1 \cosh mu_0 \sin mv,
    \\
    hE_v^> &= -\sum_{m=1}^{+\infty} m \frac{ik}{\lambda^2} A_m \cosh mu_0 \sin mv + \\
            &\quad \sum_{m=1}^{+\infty} \frac{ik}{\lambda^2} R(u_0) A_m \sinh mu_0 \sin mv,
    \\
    hcB_v^> &= \sum_{m=0}^{+\infty} \left[ \frac{ik}{\lambda^2} + \frac{i}{k} \right] R(u_0) A_m \cosh mu_0 \cos mv \\
            - \sum_{m=1}^{+\infty} m \frac{ik}{\lambda^2} A_m \sinh mu_0 \cos mv.
\end{align*}
\]

Finally, imposing the continuity of \( hcB_v \) and \( hE_v \) on \( u = u_0 \), the tridiagonal linear systems (2.71) and (2.72) are obtained.

Using the same arguments, the tridiagonal linear systems (2.73) and (2.74) are obtained.

Truncating every tri-diagonal linear system at the \((M+1)\)-th order we obtain:

\[
\begin{pmatrix}
    d_0 & z_0 \\
    s_0 & d_1 & z_1 \\
    & s_1 & \ddots & \ddots \\
    & & \ddots & \ddots & \ddots \\
    & & & s_{M-1} & \ddots & \ddots \\
    & & & & s_{M-1} & d_M
\end{pmatrix}
\begin{pmatrix}
    X_0 \\
    X_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    X_M
\end{pmatrix}
= 
\begin{pmatrix}
    t_0 \\
    t_1 \\
    \vdots \\
    \vdots \\
    \vdots \\
    t_M
\end{pmatrix}
\]

(2.67)

Since the coefficients on the diagonals are not zero the system can be easily determined by Gaussian elimination without pivoting. Thus, the following
recursive formulas can be used to obtain the coefficients:

\[ X_M = \frac{T_M^c}{C_M}, \]  
\[ X_n = \frac{T_n^c - z_n X_{n+1}}{C_n}. \]

where:

\[
C_n = \begin{cases} 
  d_0 & n = 0 \\
  d_n - s_{n-1} z_{n-1} z_n & n = 1..M
\end{cases}
\]

\[
D_n = \frac{s_{n-1}}{C_{n-1}} \quad n = 1..M
\]

\[
T_n^c = \begin{cases} 
  t_0 & n = 0 \\
  t_n - T_{n-1}^c D_n & n = 1..M
\end{cases}
\]

Introducing the quantities

\[
l = \sqrt{a^2 - b^2}, \ W = i k l^2, \ \lambda^2 = i k Z_0 \sigma, \ R(u_0) = i l \lambda \cosh u_0, \] the coefficients \( s_n, d_n, z_n \) and \( t_n \) on the diagonals are given by the following relations for each system:

**System involving the constants \( A_{2n} \)**

\[
d_0 = -\frac{W}{4} \sinh 2u_0 - R(u_0) \left[ \frac{ik}{\lambda^2} + \frac{i}{k} \right],
\]

\[
d_n = -W \left[ \frac{\sinh(2n+2)u_0}{(16n+8) \sinh 2u_0} + \frac{\cosh(2n+2)u_0}{(16n+8) \cosh 2u_0} \right]
- \left[ \frac{\sinh(2n-2)u_0}{(16n-8) \sinh 2u_0} + \frac{\cosh(2n-2)u_0}{(16n-8) \cosh 2u_0} \right]
- \frac{i k}{\lambda^2} \left[ R(u_0) \left( \tanh 2nu_0 + \coth 2nu_0 \right) - 4n \right]
+ \frac{2n}{k^2} - \frac{i}{\lambda^2} R(u_0) \coth 2nu_0 \]
\[ n \geq 1, \]  

\[
z_0 = \frac{W}{4} \sinh 2u_0,
\]

\[
z_n = \frac{W}{16n + 8} \left[ \frac{\sinh(2n+2)u_0}{\sinh 2u_0} + \frac{\cosh(2n+2)u_0}{\cosh 2u_0} \right] \quad n \geq 1,
\]

\[
s_0 = \frac{W}{4} \cosh 2u_0,
\]

\[
s_n = \frac{W}{16n - 8} \left[ \frac{\sinh(2n-2)u_0}{\sinh 2u_0} + \frac{\cosh(2n-2)u_0}{\cosh 2u_0} \right] \quad n \geq 1,
\]

\[
t_0 = -\frac{q}{2\pi \epsilon_0},
\]

\[
t_n = \frac{W}{\pi \epsilon_0} \cos(2n+1) \cosh 2u_0 \left[ \tanh 2nu_0 - \coth 2nu_0 \right] \quad n \geq 1.
System involving the constants $A_{2n+1}$

\begin{align}
  d_0 &= -\frac{W}{16} \left[ \sinh 3u_0 \cosh 3w_0 + \cosh 3w_0 \sinh u_0 \right] + \frac{ik}{k} - \frac{i}{k} R(u_0) \coth u_0 \\
  d_n &= -\frac{W}{16(n+1)} \left[ \sinh (2n+2)u_0 \cosh (2n+2)w_0 + \sinh (2n+1)u_0 \cosh (2n+1)w_0 \right] \\
  &\quad + \frac{W}{16(n-8)} \left[ \sinh (2n-2)u_0 \cosh (2n-2)w_0 + \sinh (2n-1)u_0 \cosh (2n-1)w_0 \right] \\
  &\quad - \frac{2n+1}{k} + \frac{i}{k} R(u_0) \coth 2nu_0 + \frac{2n+1}{k} R(u_0) \left( \tanh 2nu_0 + \coth 2nu_0 \right) + 4n \\
  &\quad \quad n \geq 1, \\
  z_{n-1} &= \frac{W}{16} \left[ \sinh (2n-1)u_0 \cosh (2n-1)w_0 + \cosh (2n-1)u_0 \sinh (2n-1)w_0 \right] \\
  s_{n-1} &= i \frac{W}{16} \left[ \sinh (2n-1)u_0 \cosh (2n-1)w_0 + \cosh (2n-1)u_0 \sinh (2n-1)w_0 \right] \\
  t_{n-1} &= \frac{W}{16} \left[ \cos (2n-1)u_1 \cosh (2n-1)w_1 \tanh (2n-1)u_0 - \coth (2n-1)u_0 \right]. \tag{2.72}
\end{align}

System involving the constants $B_{2n}$

\begin{align}
  d_0 &= +\frac{W}{16} \sinh 2u_0 + \frac{W}{16} \sinh (2n+2)u_0 \left( \frac{2n+1}{k} + \frac{i}{k} \right), \\
  d_n &= +\frac{W}{16(n+1)} \left[ \sinh (2n+2)u_0 \cosh (2n+2)w_0 + \sinh (2n+1)u_0 \cosh (2n+1)w_0 \right] \\
  &\quad + \frac{W}{16(n-8)} \left[ \sinh (2n-2)u_0 \cosh (2n-2)w_0 + \sinh (2n-1)u_0 \cosh (2n-1)w_0 \right] \\
  &\quad - \frac{2n+1}{k} + \frac{i}{k} R(u_0) \coth 2nu_0 + \frac{2n+1}{k} R(u_0) \left( \tanh 2nu_0 + \coth 2nu_0 \right) - 4n \\
  &\quad \quad n \geq 1, \\
  z_0 &= -\frac{W}{16} \sinh 2u_0, \\
  z_n &= -\frac{W}{16(n+8)} \left[ \sinh (2n+2)u_0 \cosh (2n+2)w_0 + \cosh (2n+2)u_0 \sinh (2n+2)w_0 \right] n \geq 1, \\
  s_0 &= -\frac{W}{16} \cos 2u_0, \\
  s_n &= -\frac{W}{16(n-8)} \left[ \sinh (2n-2)u_0 \cosh (2n-2)w_0 + \cosh (2n-2)u_0 \sinh (2n-2)w_0 \right] n \geq 1, \\
  t_0 &= 0, \\
  t_n &= \frac{W}{16} \sin 2nu_1 \sinh 2nu_1 \left[ \coth 2nu_0 - \tanh 2nu_0 \right] n \geq 1. \tag{2.73}
\end{align}
2.2. RESISTIVE WALL WAKEFIELDS

System involving the constants $B_{2n+1}$

$$d_0 = \frac{1}{W} \left[ \frac{\sinh 3u_0 + \cosh 3u_0}{\sinh u_0} \right] - \frac{i}{k} + \frac{1}{k} R(u_0) \tanh u_0$$
$$+ \frac{ik}{\lambda} \left[ R(u_0) \left( \tanh u_0 + \coth u_0 \right) - 2 \right].$$

$$d_n = \frac{1}{W} \left[ \frac{\sinh (2n+3)u_0 + \cosh (2n+3)u_0}{\sinh (2n+1)u_0} \right]$$
$$+ \frac{ik}{\lambda} \left[ R(u_0) \left( \tanh (2n+1)u_0 + \coth (2n+1)u_0 \right) - 2(2n+1) \right].$$

$$z_{n-1} = -\frac{1}{W} \left[ \frac{\sinh (2n+1)u_0 + \cosh (2n+1)u_0}{\sinh (2n-1)u_0} \right],$$

$$s_{n-1} = -\frac{1}{W} \left[ \frac{\sinh (2n-1)u_0 + \cosh (2n-1)u_0}{\sinh (2n+1)u_0} \right],$$

$$t_{n-1} = \frac{q \pi \epsilon_0}{2} \sin (2n-1)v_1 \sinh (2n-1)u_1 \times$$
$$\left[ \frac{\coth (2n-1)u_0 - \tanh (2n-1)u_0}{2} \right].$$

(2.74)

Short range wake functions

The longitudinal and the transverse components of $F$ in Eq. (2.1) in the elliptical cross section case are given by:

$$F_L = q_E z,$$

$$F_T = q \left[ \hat{u} (E_u - cB_v) + \hat{v} (E_v + cB_u) \right],$$

where the fields $E_z, E_u, E_v, B_u$ and $B_v$ are evaluated in $(u, v, z)$ and depend on the position $(u_1, v_1, 0)$ of the leading charge $q$. Using the Eqs. (2.22e) and (2.22f), we can write:

$$\tilde{E}_u + c\tilde{B}_u = -\frac{i}{hk} \frac{\partial \tilde{E}_z}{\partial v},$$

$$\tilde{E}_u - c\tilde{B}_u = +\frac{i}{hk} \frac{\partial \tilde{B}_z}{\partial v}.$$

(2.76a)

Thus, the force depends only on the longitudinal components $E_z$ and $B_z$. The longitudinal impedance per unit of length is given by $1/qc$ multiplied by the expression of $\tilde{E}_z(k)$ in Eq. (2.35a):

$$Z_L(u, v, k) = \sum_{n=0}^{+\infty} A_n \frac{q}{c} \cosh nu \cos nv + \sum_{n=1}^{+\infty} B_n \frac{q}{c} \sinh nu \sin nv.$$
while the two components $Z_u$ and $Z_v$ of the transverse impedance $Z_T = Z_u \hat{u} + Z_v \hat{v}$ are calculated using Eqs. (2.75b) and (2.76):

$$Z_u(u, v, k) = -\frac{i}{qck} \sum_{n=1}^{+\infty} n \left( A_n \sinh nu \cos nv + B_n \cosh nu \sin nv \right),$$

$$Z_v(u, v, k) = +\frac{i}{qck} \sum_{n=1}^{+\infty} n \left( A_n \cosh nu \sin nv - B_n \sinh nu \cos nv \right).$$

Thus, the wake functions are obtained by a numerical inverse Fourier transformation of the impedance. The fields vanish for $z > 0$, and since for larger values of $k$ the real part of the impedance drops more quickly than the imaginary part it appears to be more convenient to use the cosine inverse transformation, obtaining:

$$W_L = \frac{2c}{\pi} \int_{0}^{+\infty} \Re(Z_L) \cos (kz) \, dk$$

$$W_T = \frac{2c}{\pi} \int_{0}^{+\infty} \left[ \Re(Z_v) \hat{v} + \Re(Z_u) \hat{u} \right] \cos (kz) \, dk$$

### 2.3 Examples and Results

In this section the impedances and the short wake functions are calculated for several cases, such as cross sections with different aspect ratio values $a/b$ and two different kinds of conductors, aluminium and copper with DC and AC conductivity values. The conductivity $\sigma$ and the relaxation time $\tau$ used for aluminium and copper are listed in Table 2.2. The relative displacement between the leading and the trailing charge is also considered. Basically, the tri-diagonal linear system in Eq. (2.67) must be truncated and when we consider large values for the aspect ratio $a/b$, more sine and cosine components are needed to represent the fields accurately. A particular and interesting case arises when both the leading and the trailing charges, travel along on-axis in the vacuum chamber. In this case, only
2.3. EXAMPLES AND RESULTS

the $A$ even subsystem is excited, and the longitudinal impedance is then simplified as:

\[ Z_L(u, v, k) = \sum_{n=0}^{+\infty} (-1)^n \frac{q_c}{A_{2n}}. \]  

(2.80)

In a small region around the axis the longitudinal wake function remains approximately constant while the transverse wake function depends linearly on the displacements of the leading and the trailing charges. This effect is shown in Fig. 2.3. The longitudinal wake function has been calculated at $z = 0$ with different transverse positions of both the leading and the trailing charges from the axis (offset). Figs. 2.3(a) and 2.3(b) show these values normalized to the value of the wake function on axis. For the transverse wake function, the maximum values divided by the offset of the charges have been calculated, and Figs. 2.3(c) and 2.3(d) show these values normalized with respect to the limit of the wake function for the offset approaching zero. In this way it is shown that, if the offset is sufficiently small, the longitudinal wake function is approximately equal to the value on-axis, while the transverse wake function can be considered as a linear function of the offset. For this reason, whenever the offset is small, transverse wake functions are expressed per unit of length of transverse displacement in V/pC/m$^2$.

In the following examples, we applied our method using vacuum chambers having the short half axis length $b = 3$ mm.

Fig. 2.4 shows the longitudinal impedance with the AC conductivity model for copper and aluminum, as a function of the wave number $k$ and for several values of the aspect ratio $a/b$.

Fig. 2.5 shows the longitudinal wake functions for a vacuum chamber with semi-minor axis $b = 3$ mm and several aspect ratio values $a/b$, as a function of the longitudinal displacements behind the leading charge. It is worthwhile noting that the longitudinal wake functions reduce to those of the circular case when $a/b = 1$ [15] and to those of the parallel plates when $a/b >> 1$ [8], respectively.

Fig. 2.6 shows the transverse wake functions as functions of the longitudinal displacement behind the leading charge, for aluminum. Here we considered three different relative transverse positions between the leading and the trailing charges: both charges with a $y$-offset, the leading charge off-axis and the trailing charge on-axis and vice versa. The semi-minor axis $b$ is 3 mm and the more significant case of AC conductivity model is considered. Fig. 2.6(f) shows that $\frac{\partial}{\partial x} W_x + \frac{\partial}{\partial y} W_x$ vanishes for large values of $a/b$, when the ellipse approaches the parallel plates limit.

When an ultra short bunch is considered, then the transverse wake with the leading and the trailing charges off-axis with the same offset should be considered. Figs. 2.6(e) and 2.6(f) represent this case. As explained in [22], the transverse wake forces near the axis of a bi-symmetric pipe (the elliptical pipe has mirror
simmetry in both $x$ and $y$) have the property $\frac{\partial W_y}{\partial y} = -\frac{\partial W_x}{\partial x}$. This explains why Fig. 2.6(d) is the mirror of Fig. 2.6(c).

Fig. 2.7 shows the convolutions obtained with the wake functions and flat top bunches of charge equal to 1 nC and different lengths.

For the transversal effect, only the vertical plane is considered, because for aspect ratio values $a/b > 3$ the effect of the transverse wake for a horizontal offset can be neglected. Fig. 2.8 shows the convolutions obtained with the wake functions and flat top bunches of charge equal to 1 nC and different lengths. Finally, Fig. 2.9 shows a color map with the wake field intensity at $z = 0$ for different elliptical aspect ratios. This figure shows that for higher elliptical aspect ratios the wake field is negligible far from the axis in the long axis direction.

2.4 Undulator vacuum chamber selection

In our study we considered different bunch configurations, different elliptical aspect ratios for the vacuum chamber and two different materials, aluminum and copper.

An analysis of the wake functions behavior shows that the copper chambers present higher oscillations compared to the aluminum chambers. This is due to the higher electron relaxation time of the copper.

The effect of the oscillations is strong in the short range both for the longitudinal and the transverse wakes, thus it is particularly important for the medium and the short bunch configurations, while it has almost no effect in case of long bunches. Further, the results show that the energy variation induced within the bunch can take an unacceptably large value when shorter electron bunches are used. Chambers with circular section present higher oscillations compared to the ones with higher elliptical aspect ratios $a/b$. Moreover we showed that the electromagnetic fields in case of chambers with higher aspect ratios is confined in the center of the chamber, thus allowing eventually to open it on the long axis side without modifying the fields configuration.

Using an elliptical vacuum chamber of aluminum an elliptical aspect ratio $a/b > 3$, the amplitude and the number of oscillations can be reduced.
2.4. UNDULATOR VACUUM CHAMBER SELECTION

(a) $W_2(x, x_1, z = 0)/W_2(x = 0, x_1 = 0, z = 0)$ vs $x = x_1$.

(b) $W_2(y, y_1, z = 0)/W_2(y = 0, y_1 = 0, z = 0)$ vs $x = x_1$.

(c) $\max_x |W_2(x, x_1)|/\max_x |W_2(x = \epsilon, x_1 = \epsilon)|$ vs $x = x_1, \epsilon << 1 \mu m$.

(d) $\max_y |W_2(y, y_1)|/\max_y |W_2(y = \epsilon, y_1 = \epsilon)|$ vs $y = y_1, \epsilon << 1 \mu m$.

Figure 2.3: Growth of the longitudinal wake functions and non linearity with the offset of the tranverse wake functions.
CHAPTER 2. WAKEFIELD EFFECTS IN THE UNDULATOR

Figure 2.4: Longitudinal impedance vs $\log_{10}k$ ($k$ in m$^{-1}$) obtained using the AC conductivity model for aluminium (a) and copper (b), semi-axis $b = 3$ mm and with several aspect ratio values $a/b$.

Figure 2.5: Dependence of longitudinal wake functions on the longitudinal displacement behind the leading charge. Semi-minor axis $b = 3$ mm, AC conductivity models for aluminum (a) and copper (b) and several aspect ratio values $a/b$. 
2.4. UNDULATOR VACUUM CHAMBER SELECTION

Figure 2.6: Dependence of tranverse wake functions on the longitudinal displacement behind the leading charge in a small region around the vacuum chamber axis. Semi-minor axis $b = 3$ mm and several values of the aspect ratio $a/b$ are considered. AC conductivity model for aluminium is used.

(a) $\partial W_y/\partial y_1$, $V/pC/m^2$, the leading charge off axis and the trailing charge on-axis.

(b) $\partial W_x/\partial x_1$, $V/pC/m^2$, the leading charge off axis and the trailing charge on-axis.

(c) $\partial W_y/\partial y$, $V/pC/m^2$, the trailing charge off axis and the leading charge on-axis.

(d) $\partial W_x/\partial x$, $V/pC/m^2$, the trailing charge off axis and the leading charge on-axis.

(e) $\partial W_y/\partial y_1 + \partial W_y/\partial y$, $V/pC/m^2$.

(f) $\partial W_x/\partial x_1 + \partial W_x/\partial x$, $V/pC/m^2$, the leading and the trailing charges the leading and the trailing charges off-axis.
Figure 2.7: Energy variation per unit length induced within flat top bunches with different lengths and charge of 1 nC. Semi-minor axis $b = 3$ mm, AC conductivity model and several aspect ratio values $a/b$ are used.
2.4. UNDULATOR VACUUM CHAMBER SELECTION

Figure 2.8: Transversal kick per unit length and off-axis displacement, along the short axis direction, induced within flat top bunches with different lengths and charge 1 nC at 1.2 GeV. Semi-minor axis $b = 3$ mm, AC conductivity model and several aspect ratio values $a/b$ are used.
Figure 2.9: Longitudinal wake function intensity color map at $z = 0$. Leading charge on axis
Chapter 3

One dimensional free electron laser theory

In this chapter we focus the attention on the FEL Theory concerning the single pass amplifier configuration, following the approach described in [23]. We consider a one dimensional model with an electron beam consisting of \( N \) electrons travelling through a wiggler having a magnetostatic periodic field. In order to properly describe properly the evolution of the system, it is essential to consider the self-consistent coupling of the particle dynamics with the field dynamics. The particle dynamics can be described by the relativistic Newton-Lorentz equations, while the field dynamics is described by the Maxwell equations having, as source term, the electron transverse current.

3.1 Electron equations

We write the wiggler magnetostatic field as

\[
B_w = \nabla \times A_w, \tag{3.1}
\]

where \( A_w \) is the magnetic vector potential and the electromagnetic co-propagating field \((E, B)\) as

\[
E = -\frac{1}{c} \frac{\partial A}{\partial t}, \tag{3.2}
\]

\[
B = \nabla \times A. \tag{3.3}
\]

Where \( A \) is the magnetic vector potential related to the e.m. fields \( E, B \). Momentum and energy evolutions are ruled, for each electron, by the Newton-
Lorentz equations:
\[
\frac{d(\gamma m v)}{dt} = e \left[ E + \frac{v}{c} \times (B_w + B) \right] \tag{3.4}
\]
\[
\frac{d(\gamma mc^2)}{dt} = eE \cdot v \perp \tag{3.5}
\]
By Eq. (3.5), the energy exchange depends only on the transversal velocity of the electrons, since we consider \(\{E, B\}\) as a plane wave that propagates on the directions of the wiggler axis.

For simplicity, we consider the case of the elicoidal wiggler with a vector potential:
\[
\mathbf{A}_w(z) = \frac{\tilde{a}_w}{\sqrt{2}} \left( \hat{e} e^{-ik_w z} + \hat{e}^* e^{ik_w z} \right) \tag{3.6}
\]
where \(\hat{e} = \hat{x} + i\hat{y}\) is a unit vector.

The vector potential of the co-propagating field can be written as:
\[
\mathbf{A}(z, t) = -\frac{i}{\sqrt{2}} \left( \hat{e} \tilde{a}(z, t) e^{i(k_0 z - \omega_0 t)} - \hat{e}^* \tilde{a}^*(z, t) e^{-i(k_0 z - \omega_0 t)} \right) \tag{3.7}
\]
where \(\tilde{a}(z, t)\) is a complex quantity, and the total magnetic vector potential is
\[
\mathbf{A}_{\text{tot}}(z, t) = \mathbf{A}_w(z) + \mathbf{A}(z, t) \tag{3.8}
\]
Further, the following wiggler parameter \(a_w\) and the normalized vector potential amplitude \(a\) are introduced:
\[
a_w(z) = \frac{\tilde{a}_w}{\sqrt{2}} \left( \hat{e} e^{-ik_w z} + \text{c.c.} \right) \quad ; \quad a_w = e \frac{mc^2}{e} \tilde{a}_w \tag{3.9}
\]
\[
a(z, t) = -\frac{i}{\sqrt{2}} \left( \hat{e} a(z, t) e^{i(k_0 z - \omega_0 t)} - \text{c.c.} \right) \quad ; \quad a = e \frac{mc^2}{e} \tilde{a} \tag{3.10}
\]
Substituting into Eq. (3.4) yields:
\[
\frac{d}{dt}(\gamma m v \perp) = e \left[ E + \frac{v}{c} \times (B_w + B) \right] = e \left[ -\frac{1}{c} \frac{\partial \mathbf{A}_{\text{tot}}}{\partial t} + \frac{v}{c} \times \nabla \times (\mathbf{A}_w + \mathbf{A}) \right] \\
- \frac{e}{c} \left[ \frac{\partial \mathbf{A}_{\text{tot}}}{\partial t} + \frac{\partial z}{\partial t} \frac{\partial \mathbf{A}_{\text{tot}}}{\partial z} + \frac{\partial z}{\partial t} \frac{\partial \mathbf{A}_{\text{tot}}}{\partial z} \right] = -\frac{e}{c} \frac{d \mathbf{A}_{\text{tot}}}{dt} \tag{3.11}
\]
Using the adimensional quantity \(a_{\text{tot}} = \frac{e A_{\text{tot}}}{mc^2}\), one obtains:
\[
\frac{d}{dt}(\gamma \beta \perp) = -\frac{d}{dt} a_{\text{tot}} \tag{3.12}
\]
3.1. ELECTRON EQUATIONS

In case of perfect on axis injection ($\beta_\perp(t, z = 0) = 0$), it is:

$$\beta_\perp = -\frac{a_{tot}}{\gamma}$$  \hspace{1cm} (3.13)

Eq.(3.5) describes the energy exchange between the electrons and the field, and can be manipulated as follows:

$$\frac{d\gamma}{dt} = \frac{e}{mc^2} E \cdot v_\perp = \frac{e}{mc^2} \left( -\frac{1}{c} \frac{\partial A_{tot}}{\partial t} \right) \cdot (c\beta_\perp) = \frac{\partial a_{tot}}{\gamma} \cdot a_{tot} = \frac{1}{2\gamma} \frac{\partial |a_{tot}|^2}{\partial t}$$  \hspace{1cm} (3.14)

$$|a_{tot}|^2 = a_w^2 - ia_w \left( a(z, t)e^{i(k_0z-\omega_0t+k_wz)} - c.c. \right) + |a(z, t)|^2$$  \hspace{1cm} (3.15)

where:

$$a_w \cdot a^* = \frac{a_w}{2} \left[ \left( \hat{\theta}e^{-ik_{w}z} + \hat{\epsilon}^*e^{ik_{w}z} \right) \left( \hat{\theta}^*a^*e^{-i(k_0z-\omega_0t)} - \hat{\epsilon}a^*e^{i(k_0z-\omega_0t)} \right) \right]$$  \hspace{1cm} (3.16)

Introducing the phase $\theta$, defined by:

$$\theta = (k_0 + k_w)z - \omega_0 t$$  \hspace{1cm} (3.17)

eq 3.15 can be rewritten as:

$$|a_{tot}|^2 = a_w^2 - ia_w \left( a(z, t)e^{i\theta} - c.c. \right) + |a(z, t)|^2$$  \hspace{1cm} (3.18)

Differentiating Eq. (3.17) with respect to the time, the phase of the ponderomotive field is found as:

$$v_p = \frac{\omega_0}{k_0 + k_w}$$  \hspace{1cm} (3.19)

Using the paraxial approximation $z = v_{||}t$, $\frac{d}{dt} \approx v_{||} \frac{d}{dz} \approx c \frac{d}{dz}$, from Eq.(3.14) the following electron energy equation is obtained:

$$\frac{d\gamma}{dz} = -\frac{k_0a_w}{2\gamma} \left( a(z, t)e^{i\theta} + c.c. \right)$$  \hspace{1cm} (3.20)

where we have assumed $|\partial_t a(z, t)| << |\omega_0 a(z, t)|$ and neglected the term $\partial_t |a(z, t)|^2$. An equation for the electrons phase is obtained by differentiating Eq. (3.17) on $z$:

$$\frac{d\theta}{dz} = k_0 + k_w - \omega_0 \frac{1}{c\beta_{||}} = k_w + k_0 \left( 1 - \frac{1}{\beta_{||}} \right)$$  \hspace{1cm} (3.21)

The resonance condition is found solving the equation $\frac{d\theta}{dz} = 0$, thus obtaining:

$$k_w = -k_0 \left( 1 - \frac{1}{\beta_{||}} \right)$$  \hspace{1cm} (3.22)
which is solved as follows, in terms of the resonant wavelength:

\[ \lambda_0 = \lambda_w \frac{1 - \beta_\parallel}{\beta_\parallel} \quad (3.23) \]

We rewrite \( \beta_\parallel \) in terms of \( \gamma = (1 - \beta^2)^{-1/2} \) and \( a_{\text{tot}} \) in Eq. (3.13) as:

\[ \beta_\parallel^2 = 1 - \beta_\parallel^2 = 1 - \frac{a_{\text{tot}}^2}{\gamma^2} = 1 - \frac{1 + a_{\text{tot}}^2}{\gamma^2} \quad (3.24) \]

In the relativistic limit \( \beta_\parallel \to 1 \), it follows \( \frac{1 + a_{\text{tot}}^2}{\gamma^2} \ll 1 \), and expanding to the first order \( \beta_\parallel \approx 1 - \frac{1 + a_{\text{tot}}^2}{2 \gamma^2} \), the resonance condition can be written as:

\[ \lambda_0 \approx \lambda_w \left( \frac{1 - 1 + \frac{1 + a_{\text{tot}}^2}{2 \gamma^2}}{1 - \frac{1 + a_{\text{tot}}^2}{2 \gamma^2}} \right) \approx \lambda_w \frac{1 + a_{\text{tot}}^2}{2 \gamma^2} \quad (3.25) \]

When \( a_{\text{tot}} \approx a_w \) one obtains:

\[ \lambda_0 \approx \lambda_w \frac{1 + a_w^2}{2 \gamma^2}. \quad (3.26) \]

Considering the relativistic limit \( \frac{1}{\beta_\parallel} \approx 1 + \frac{1 + a_{\text{tot}}^2}{2 \gamma^2} \), and substituting into Eq. (3.21), yields the phase equation:

\[ \frac{d\theta}{dz} = k_w + k_0 \left( 1 - \frac{1}{\beta_\parallel} \right) = k_w - k_0 \frac{1 + a_{\text{tot}}^2}{2 \gamma^2} \quad (3.27) \]

Considering the resonance condition in \( \gamma \), Eq. (3.25) is written as:

\[ \gamma_0^2 \approx \frac{1 + a_w^2 k_0}{k_w} \quad (3.28) \]

Substituting Eq. (3.28) into Eq. (3.27), and assuming \( a_{\text{tot}} \approx a_w \), yields the phase equation:

\[ \frac{d\theta}{dz} = k_w \left( 1 - \frac{\gamma_0^2}{\gamma^2} \right) \quad (3.29) \]

### 3.2 Field equation

Let us decompose the density current \( J \) in the sum of transverse and longitudinal components, \( J = J_\perp + J_\parallel \), where \( \nabla \times J_\perp = 0 \) and \( \nabla \cdot J_\perp = 0 \). Adopting the gauge of Coulomb [34], the equation for the magnetic vector potential can be written as

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A = -\frac{4\pi}{c} J_\perp. \quad (3.30)
\]
3.2. FIELD EQUATION

For a set of $N$ electrons having transverse velocity $v_{\perp j}$ and position $x_j(t)$, $j = 1..N$ the current in Eq. (3.30) can be written as

$$
J_{\perp} = e \sum_{j=1}^{N} v_{\perp j} \delta(x - x_j(t))
$$

(3.31)

Substituting (3.7) and (3.31) into 3.30 yields:

$$
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \left( -\frac{i}{\sqrt{2}} \left( \dot{\gamma}a(z, t) \frac{mc^2}{e} e^{i(k_0 z - \omega_0 t) - c.c.} \right) \right) =

- \frac{4\pi}{c} e \sum_{j=1}^{N} c\beta_{\perp} \delta(x - x_j(t))
$$

(3.32)

Substituting $a_{tot}$ (Eq. (3.13)) and solving on $\dot{\gamma}$, one obtains:

$$
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) a(z, t) e^{i(k_0 z - \omega_0 t)} =

\frac{i4\pi e^2}{me^2 n_{\perp}} \sum_{j=1}^{N} a_{\omega} e^{-ik_w z} - ia(z, t) e^{i(k_0 z - \omega_0 t)} \gamma_j \delta(z - z_j(t))
$$

(3.33)

where $n_{\perp}$ denotes the transverse electron density. We derive the left hand side of the expression in Eq.(3.33). Noting that $\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) e^{i(k_0 z - \omega_0 t)} = 0$, we get:

$$
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) e^{i(k_0 z - \omega_0 t)} =

e^{i(k_0 z - \omega_0 t)} \left( \frac{\partial^2 a(z, t)}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 a(z, t)}{\partial t^2} + 2ik_0 \frac{\partial a(z, t)}{\partial z} + \frac{2i\omega_0}{c^2} \frac{\partial a(z, t)}{\partial t} \right)
$$

(3.34)

The following Slow Variating Envelope Approximation (SVEA) is introduced:

$$
\left| \frac{\partial a}{\partial z} \right| \ll |k_0 a|
$$

(3.35)

$$
\left| \frac{\partial a}{\partial t} \right| \ll |\omega_0 a|
$$

(3.36)

which allows to neglect the second derivatives on $z$ and $t$ of Eq. (3.34), thus yielding:

$$
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) a(z, t) e^{i(k_0 z - \omega_0 t)} \approx 2ik_0 e^{i(k_0 z - \omega_0 t)} \left( \frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) a(z, t)
$$

(3.37)
CHAPTER 3. 1D FEL THEORY

Since the complex variable $a$ is slowly varying on the scale of a radiation wavelength, it can be driven only by a transverse current averaged on a volume having longitudinal dimension $\lambda_0$ consisting of several wavelengths $\lambda_0$.

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) a(z,t) = \frac{1}{2k_0 m c^2} \frac{4\pi e^2 n_\perp}{\lambda_0} \cdot \left[ a_w \sum_j e^{-i\theta_j} \frac{1}{\gamma_j} - ia \sum_j \frac{1}{\gamma_j} \right]$$  (3.38)

By introducing the plasma frequency $\omega_p^2 = \frac{4\pi e^2 n}{m}$, the electron density $n = n_\perp / l_\parallel$, and evaluating the mean quantities $\langle ... \rangle$ over a volume with many electrons, the above equation can be written in the following form:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) a(z,t) = k_0 \frac{1}{2} \left( \frac{\omega_p}{\omega_0} \right)^2 \left[ a_w \langle e^{-i\theta} \rangle - ia \langle \frac{1}{\gamma} \rangle \right]$$  (3.39)

3.3 Vlasov-Maxwell approach

We now consider the electrons as a continuous distribution $\psi(z, t, \gamma)$. Thus the number of electrons on a section is estimated as:

$$n_\perp \int_\gamma \psi(z, t, \gamma) \, d\gamma.$$  (3.40)

In the sequel we will consider, instead of an elicoidal undulator, a planar one. Although the derivation of the electron motion for the latter undulator is more complicated, the equation obtained are formally the same [24]. Further we will manipulate the FEL equations to obtain the Vlasov-Maxwell system of equations which will be the starting point to study the impact of the initial conditions on the FEL.

In the paraxial approximation $\frac{d}{dz} \approx \frac{1}{c} \frac{\partial}{\partial t} + \frac{1}{v_\parallel} \frac{\partial}{\partial t}$, where $v_\parallel$ is the velocity of the macroscopic distribution, Eqs. (3.29), (3.20) and (3.39) can be written as the following system:

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_\parallel} \frac{\partial}{\partial t}\right) \theta = k_w \left( 1 - \frac{\gamma_0^2}{\gamma^2} \right)$$  (3.41)

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_\parallel} \frac{\partial}{\partial t}\right) \gamma = -\frac{k_0 a_w [JJ]}{2\gamma} (a(z, t)e^{i\theta} + c.c.)$$  (3.42)

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) a(z, t) = \frac{k_0}{2} \left( \frac{\omega_p}{\omega_0} \right)^2 [JJ] a_w e^{-i\theta} \int_\gamma \frac{\psi(z, t, \gamma)}{\Gamma} \, d\gamma$$  (3.43)
3.3. VLASOV-MAXWELL APPROACH

where $\Gamma$ represents the set of all possible Lorentz factors. In the latter system of equations we have introduced the constant factors needed for a planar undulator, the continuous distribution, neglected the last term in right hand side of Eq. (3.39) and $[JJ] = J_0(\xi) - J_1(\xi)$.

Further we will introduce the electric field envelope $A$, that is related to the electric field by $E(z,t) = \Re(A(z,t)e^{ik_0z-\omega_0t})$. Switching to the International System, we can write:

\[
\left(\frac{\partial}{\partial z} + \frac{1}{v_{||}} \frac{\partial}{\partial t}\right) \theta = k_w \left(1 - \frac{\gamma_0^2}{\gamma^2}\right) \quad (3.44)
\]

\[
\left(\frac{\partial}{\partial z} + \frac{1}{v_{||}} \frac{\partial}{\partial t}\right) \gamma = \frac{ea_w[JJ]k_w}{\sqrt{2\gamma_0^2 mc^2}} (A(t,z) e^{i\theta} + \text{c.c.}) \quad (3.45)
\]

\[
\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) A(t,z) = \frac{ea_w[JJ]}{2\sqrt{2} \epsilon_0} \sqrt{\frac{2}{\gamma^2 \gamma_0^2 mc^2}} e^{-i\theta} \int_{\gamma} \frac{\psi(z,t,\gamma)}{\gamma} d\gamma \quad (3.46)
\]

Finally, we perform the change of coordinates $(z,t) \rightarrow (Z,\theta)$, with $Z = k_w z$, and $\theta = (k_0 + k_w)z - \omega_0 t$. Thus Eqs.(3.44), (3.45 and (3.46) become:

\[
\frac{\partial}{\partial Z} \theta = 1 - \frac{\gamma_0^2}{\gamma^2} \quad (3.47)
\]

\[
\frac{\partial}{\partial \gamma} \gamma = -\frac{D_2}{\gamma} (A(\theta,Z) e^{i\theta} + \text{c.c.}) \quad (3.48)
\]

\[
\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial \theta}\right) A(\theta,Z) = D_1 e^{-i\theta} \int_{\Gamma} \frac{\psi(Z,\theta,\gamma)}{\gamma} d\gamma \quad (3.49)
\]

where $D_1 = \frac{ea_w[JJ]}{2\sqrt{2} \epsilon_0}$ and $D_2 = \frac{ea_w[JJ]}{\sqrt{2\gamma_0^2 mc^2}}$.

In addition to the system of Eqs. (3.47), (3.48) and (3.49), the evolution of the distribution of charge is ruled by the Vlasov equation

\[
\left(\frac{\partial}{\partial Z} + \frac{\partial \theta}{\partial Z} \frac{\partial}{\partial \theta} + \frac{\partial \gamma}{\partial Z} \frac{\partial}{\partial \gamma}\right) \psi(Z,\theta,\gamma) = 0 \quad (3.50)
\]

We expand the Eq. (3.47) to the first order in $\gamma$ in the neighborhood of $\gamma_0$ obtaining

\[
\frac{\partial}{\partial Z} \theta = 2\frac{\gamma - \gamma_0}{\gamma_0} = p \quad (3.51)
\]

where $p = 2\frac{\gamma - \gamma_0}{\gamma_0}$ is a measure of the relative displacement from the resonant energy $\gamma_0$. Furthermore, we assume $\gamma \approx \gamma_0$ in Eqs. (3.48) and (3.49).
Substituting into Eq. (3.50) yields:

\[
\left( \frac{\partial}{\partial Z} + p \frac{\partial}{\partial \theta} - 2 \frac{D_2}{\gamma_0} \left( A(\theta, Z)e^{i\theta} + A^*(\theta, Z)e^{-i\theta} \right) \frac{\partial}{\partial p} \right) \psi(Z, \theta, \gamma) = 0 \quad (3.52)
\]

The latter equation together with the Maxwell Equation in (3.49)

\[
\left( \frac{\partial}{\partial Z} + \frac{\partial}{\partial \theta} \right) A(\theta, Z) = \frac{D_1}{\gamma_0} e^{-i\theta} \int_p \psi(Z, \theta, p) dp \quad (3.53)
\]

will be the starting point for the development of our evaluation concerning the impact on the FEL process of the initial condition on the electron bunch.
Chapter 4

Green functions for a Free Electron Laser with initial energy chirp and curvature

In this chapter we derive the Green functions that allow to evaluate the electric field envelope along the undulator, in terms of the initial conditions. In particular, we will first consider a system with a planar undulator and an electron beam having a linear energy chirp, and find an exact expression for the seeded FEL Green functions. Subsequently we will consider an electron beam having both a linear energy chirp and a quadratic curvature and find an approximated expression that gives a good accuracy for the exponential growth regime. As for the previous case, an exact Green function will be derived for the initial conditions, with the quadratic curvature. The Green function will give the expression for the fields in both the lethargy and the exponential growth regime. Finally we will consider the harmonic generation scheme, consisting of two undulators separated by a dispersive section. In the first short undulator the FEL is operated in the seeded amplifier configuration, and has the purpose of inducing an energy modulation on the electrons. In the dispersive section the energy modulation is changed into an electron density modulation, which is the source for the FEL process in the long undulator, called radiator. In order to study this configuration, we will evaluate a Green function for the case of an initial modulation density on the electrons.
 CHAPTER 4. GREEN FUNCTIONS WITH PARABOLIC ENERGY PROFILE

4.1 Vlasov-Maxwell Analysis for an Initial Value Problem

We have to solve the following system of Eqs. (3.52) and (3.53):

\[
\begin{align*}
\frac{\partial}{\partial Z} \psi + p \frac{\partial}{\partial \theta} \psi - 2 \frac{D_2}{\gamma_0^2} (A(\theta, Z)e^{i\theta} + A^*(\theta, Z)e^{-i\theta}) \frac{\partial}{\partial p} \psi &= 0 \\
\frac{\partial}{\partial Z} A(\theta, Z) + \frac{\partial}{\partial \theta} A(\theta, Z) &= \frac{D_1}{\gamma_0} e^{-i\theta} \int_p \psi(Z, \theta, p) dp 
\end{align*}
\]

(4.1)

As a first step we first solve the first equation of the system. To this aim, we write the electron distribution function as

\[
\psi(Z, \theta, p) = \psi_0(\theta - pZ, p) + \psi_1(Z, \theta, p) 
\]

(4.2)

where \( \psi_0 \) does not have a modulation on the wavelength scale, but can represent the energy profile of the electron beam, and \( \psi_1 \) is a small perturbation of \( \psi_0 \).

In the most general case that we will consider, we assume that the electron beam energy profile has both a linear chirp and a quadratic curvature. This leads to the following expression for \( \gamma \):

\[
\gamma = \gamma_0 + \frac{\partial \gamma}{\partial t} \bigg|_{t=0, z=0} t + \frac{1}{2} \frac{\partial^2 \gamma}{\partial t^2} \bigg|_{t=0, z=0} t^2 
\]

(4.3)

Thus, in the negligible uncorrelated energy spread limit, we will assume that

\[
\psi_0 = \delta(p + \mu \theta_0 + \nu \theta_0^2/2) 
\]

(4.4)

where \( \theta_0 = \theta - pZ \), and

\[
\mu = \frac{2}{\gamma_0 \omega_0} \frac{\partial \gamma}{\partial t} \bigg|_{t=0, z=0} \\
\nu = -\frac{2}{\gamma_0 \omega_0^2} \frac{\partial^2 \gamma}{\partial t^2} \bigg|_{t=0, z=0} 
\]

(4.5) \hspace{1cm} (4.6)

are the linear chirp parameter and the quadratic curvature parameter, respectively. We substitute Eq. (4.4) into the first equation in (4.1), obtaining:

\[
\frac{\partial}{\partial Z} \psi_1 + p \frac{\partial}{\partial \theta} \psi_1 = 2 \frac{D_2}{\gamma_0^2} (A(\theta, Z)e^{i\theta} + A^*(\theta, Z)e^{-i\theta}) \frac{\partial}{\partial p} \psi_0 = 0 
\]

(4.7)
where we have neglected $\frac{\partial}{\partial p} \psi_1$ with respect to $\frac{\partial}{\partial p} \psi_0$. Eq. (4.7) yields as solution:

$$\psi_1 = 2D_2^2 e^{i\theta_0} \frac{\partial}{\partial p} \psi_0 \int_0^Z e^{ip(Z'-Z)} A(\theta - p(Z - Z'), Z')dZ' + e^{i\theta_0} F_b(\theta_0, p)$$  (4.8)

where we have introduced the function $F_b$ to take into account eventual non-zero initial conditions on the electron beam energy or density modulation on the wavelength scale. In particular, we will assume $F_b \propto \delta(p + \mu\theta_0 + \nu\theta_0^2/2)$

Substituting the solution into the second equation of the system (4.1), assuming $\mu Z << 1$ and $\nu\theta Z << 1$ and taking into account the arrival times of the single electrons, we obtain [7]:

$$\frac{\partial}{\partial Z} A(\theta, Z) + \frac{\partial}{\partial \theta} A(\theta, Z) = \frac{D_1}{\gamma_0} \sum_j e^{-i\theta_j + i(\mu\theta_j Z + \nu(\theta_0^2/2)Z)} \delta(\theta - \theta_j)$$

$$+ i(2\rho)^3 \int_0^Z (Z - Z')e^{i(Z-Z')(\mu\theta + \nu\theta^2/2)} A(\theta + (Z - Z')(\mu\theta + \nu\theta^2/2), Z')dZ'$$

$$+ \frac{D_1}{\gamma_0} e^{iZ(\mu\theta + \nu\theta^2/2)} F_b(\theta + Z(\mu\theta + \nu\theta^2/2))$$  (4.9)

where $\theta_j = -\omega_0 t_j$ and $(2\rho)^3 = 2D_1 D_2/\gamma_0^3$ with $\rho$ being the Pierce parameter [30].

The first term in the right hand side of Eq. (4.9) represents the electron shot noise at the undulator entrance, and is responsible for the Self Amplified Spontaneous Emission (SASE). Since we are interested only in the seeded process and in the initial density modulation $F_b$, at this stage we drop the SASE source, and the expression is simplified as:

$$\frac{\partial}{\partial Z} A(\theta, Z) + \frac{\partial}{\partial \theta} A(\theta, Z) = \frac{D_1}{\gamma_0} e^{iZ(\mu\theta + \nu\theta^2/2)} F_b(\theta)$$

$$+ i(2\rho)^3 \int_0^Z (Z - Z')e^{i(Z-Z')(\mu\theta + \nu\theta^2/2)} A(\theta, Z')dZ'$$  (4.10)

Introducing the Laplace transform

$$f(\theta, s) = \int_0^\infty dZe^{-sZ} A(\theta, Z),$$  (4.11)
Eq. (4.10) is then rewritten in the Laplace complex frequency domain as

\[ s f(\theta, s) - A(\theta, 0) + \frac{\partial f(\theta, s)}{\partial \theta} = \frac{D_1}{\gamma_0} \frac{F_b(\theta)}{s - i(\mu \theta + \nu \theta^2/2)} f(\theta, s) \]

(4.12)

Which can be written in the form:

\[ \frac{\partial f(\theta, s)}{\partial \theta} + \left( s - \frac{i(2\rho)^3}{F_b(\theta)} \right) f(\theta, s) = A(\theta, 0) + \frac{D_1}{\gamma_0} \frac{F_b(\theta)}{s - i(\mu \theta + \nu \theta^2/2)^2} \]

(4.13)

The solution of Eq. (4.13) is:

\[ f(\theta, s) = \int_{-\infty}^{\theta} e^{\int_{\theta'}^{\theta} \left( s - \frac{i(2\rho)^3}{[s - i(\mu \theta' + \nu \theta'^2/2)]^2} \right) d\theta''} \left( A(\theta', 0) + \frac{D_1}{\gamma_0} \frac{F_b(\theta)}{s - i(\mu \theta' + \nu \theta'^2/2)} \right) \]

(4.14)

The integral at the exponential factor can be solved exactly as:

\[ \int_{\theta'}^{\theta} \frac{i(2\rho)^3}{[s - i(\mu \theta'' + \nu \theta''^2/2)]^2} d\theta'' = \frac{\mu + \theta'' \nu}{i \theta'' + \mu \theta'' + \nu \theta''^2/2} \]

\[ \frac{\left[ \frac{2 \nu \tan \left( \frac{\mu \theta'' + \nu \theta''^2}{\sqrt{2 \theta'' + \nu \theta''^2}} \right) \right]}{\mu^2 - 2i \nu s} \]

(4.15)

In order to evaluate the FEL electric field envelope along the undulator, an inverse Laplace transform of \( f(\theta, s) \) in (4.14) has to be performed. This gives:

\[ A(\theta, Z) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} f(\theta, s) e^{sZ} \ ds \]

(4.16)

For convenience, we will refer to the variable \( \xi = \theta - \theta' \), and the equations are rewritten using the following notations:

\[ \hat{z} = 2\rho Z, \]
\[ \hat{s} = \rho \theta, \]
\[ \hat{\xi} = \rho \xi, \]
\[ \hat{\alpha} = -\mu/(2\rho^2), \]
\[ \hat{\beta} = \nu/(2\rho^3), \]
\[ \hat{p} = s/(2\rho) \]

(4.17)
4.1. VLAVOS-MAXWELL FOR AN INITIAL VALUE PROBLEM

Substituting (4.14) into Eq. (4.16) and using the notation in Eq. (4.17), one obtains:

\[
A(\hat{s}, \hat{z}) = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\hat{p}}{\pi i} \frac{e^{-\frac{\hat{p}(\hat{s} - 2\hat{\xi}) + 4i\hat{\beta}}{\alpha - \beta(\hat{s} - \hat{\xi})}}}{2\hat{p} + i(\hat{s} - \xi)(2\alpha - \beta(\hat{s} - \xi))}
\]

where the field envelope is expressed as the sum of two contributions: the convolution of the seeded FEL Green function with the seed laser \(A\) at the undulator entrance, and the convolution of the Green function with the initial density modulation \(F_b\) at the undulator entrance:

\[
A(\hat{s}, \hat{z}) = \int_{\sigma-i\infty}^{\sigma+i\infty} G_s(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}, \hat{\beta}) A(\hat{s} - \hat{\xi}, 0) + G_b(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}, \hat{\beta}) F_b(\hat{s} - \hat{\xi}, 0) d\hat{\xi}
\]

where:

\[
G_s = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\hat{p}}{\pi i} \frac{e^{-\frac{\hat{p}(\hat{s} - 2\hat{\xi}) + 4i\hat{\beta}}{\alpha - \beta(\hat{s} - \hat{\xi})}}}{2\hat{p} + i(\hat{s} - \xi)(2\alpha - \beta(\hat{s} - \xi))}
\]

and

\[
G_b = \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{d\hat{p}}{\pi i \rho_0} \frac{D_1 e^{-\frac{\hat{p}(\hat{s} - 2\hat{\xi}) + 4i\hat{\beta}}{\alpha - \beta(\hat{s} - \hat{\xi})}}}{2\hat{p} + i(\hat{s} - \xi)(2\alpha - \beta(\hat{s} - \xi))}
\]

Note that, due to the energy chirp and to the energy curvature in the electron bunch, the Green functions depend separately on \(\xi\) and \(\hat{s}\), which are the FEL amplifier time variables, thus losing the translational invariance property. The Green functions have the translational invariance property only in the limit case \(\hat{\alpha} = \hat{\beta} = 0\). For this reason, the expression in Eq. (4.19) is not a convolution, strictly speaking.
CHAPTER 4. GREEN FUNCTIONS WITH PARABOLIC ENERGY PROFILE

4.2 Exact Green function for a linear energy chirped seeded Free Electron Laser

In this section we work out a solution for the seeded FEL Green function $G_s$ in the case of a linear energy chirp on the electrons but without energy curvature $\hat{\beta} = 0$, and discuss its properties. The derivation and the results of this study are published in [32].

We have to evaluate:

$$G_{s,\text{lin}}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \lim_{\hat{\beta} \to 0} G_s = \frac{1}{\pi i} \int_{\sigma - i \infty}^{\sigma + i \infty} d\hat{p} \ e^{\hat{\beta}(\hat{z} - 2\hat{\xi})^2 \frac{2\hat{\xi}}{(\hat{p} + i\hat{\alpha})(\hat{p} + i\alpha(\hat{s} - \hat{\xi}))}} \tag{4.22}$$

An approximation of the Green function in Eq. (4.22) was found for a SASE FEL process in the case of a linear energy chirped electron beam [5], using a saddle point approximation method.

Here we present an exact expression for the integral in (4.22), obtained by exploiting the residual theorem in conjunction with a series expansion of the integrand function. At first, we introduce an auxiliary integral that fulfills the conditions of the Jordan’s Lemma and is related to the original integral. Subsequently, since the integrand function presents two essential singularities, we evaluate the two residual values by expanding it in Laurent series in the neighborhood of each singularity.

To this aim, let us first introduce the auxiliary integral

$$I_{\text{aux}} = \frac{1}{\pi i} \int_{\sigma - i \infty}^{\sigma + i \infty} d\hat{p} \ e^{\hat{\beta}(\hat{z} - 2\hat{\xi})^2 \frac{2\hat{\xi}}{(\hat{p} + i\hat{\alpha})(\hat{p} + i\alpha(\hat{s} - \hat{\xi}))}} \tag{4.23}$$

which is related to

$$I = \frac{1}{\pi i} \int_{\sigma - i \infty}^{\sigma + i \infty} d\hat{p} \ e^{\hat{\beta}(\hat{z} - 2\hat{\xi})^2 \frac{2\hat{\xi}}{(\hat{p} + i\hat{\alpha})(\hat{p} + i\alpha(\hat{s} - \hat{\xi}))}}, \tag{4.24}$$

by the equation

$$I = I_{\text{aux}} + 2\delta(\hat{z} - 2\hat{\xi}) \tag{4.25}$$

First of all, we observe that the integrand function in (4.23) has two singularities in $\hat{p}_1 = -i\hat{\alpha}\hat{s}$ and $\hat{p}_2 = i\hat{\alpha}(\hat{s} - \hat{\xi})$.

Furthermore, by the residual theorem and the Jordan’s lemma, it results $I_{\text{aux}} = 0$ for $\hat{z} - 2\hat{\xi} < 0$, while for $\hat{z} - 2\hat{\xi} > 0$ the quantity $I_{\text{aux}}/2$ is equal to the sum of the residuals of the integrand function in the singularities $\hat{p}_1$ and
\[ \hat{p}_2. \] On the other hand, such residuals coincide with those of the integrand function in (4.22). Therefore, taking \( \sigma > 0 \), we can write:

\[
I_{aux} = \begin{cases} 
2 \sum_{j=1}^{2} \text{Res}[e^{(\hat{p} - \hat{p}_j)\xi + \frac{2i\xi}{(\hat{p} + i\alpha)(\hat{p} + i\alpha(\xi - \xi)^{-1})}}, \hat{p} = \hat{p}_j] & \text{for } \hat{\xi} < \hat{z}/2 \\
0 & \text{for } \hat{\xi} > \hat{z}/2
\end{cases}
\]

Therefore, for \( \hat{\xi} < \hat{z}/2 \) it results:

\[
I_{aux} = 2(a_{-1}\big|_{\hat{p}_1} + a_{-1}\big|_{\hat{p}_2}) \tag{4.27}
\]

where \( a_{-1}\big|_{\hat{p}_j} \) is the multiplicative coefficient of the term \((\hat{p} - \hat{p}_j)^{-1}\) in the Laurent series expansion of the integrand function in a neighbourhood of the singularity \( \hat{p}_j \).

In order to find the coefficients \( a_{-1}\big|_{\hat{p}_1} \) and \( a_{-1}\big|_{\hat{p}_2} \), the integrand function in (4.22) can be expressed as:

\[
e^{(\hat{p} - \hat{p}_j)\xi + \frac{2i\xi}{(\hat{p} + i\alpha)(\hat{p} + i\alpha(\xi - \xi)^{-1})}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \left( \hat{p}(\hat{\xi} - 2\hat{\xi}) + \frac{2i\xi}{(\hat{p} + i\alpha\hat{s})(\hat{p} + i\alpha(\hat{s} - \hat{\xi}))} \right)^n
\]

\[
= \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} \hat{p}_j^k (\hat{\xi} - 2\hat{\xi})^k \frac{(2i\xi)^{n-k}}{(\hat{p} - \hat{p}_1)^{n-k}(\hat{p} - \hat{p}_2)^{n-k}}
\]

where the known expansion formula of the power of a binomial term has been used. To calculate \( a_{-1}\big|_{\hat{p}_1} \), in (4.28) we introduce the variable change \( \hat{p} = x + \hat{p}_1 \), and to calculate \( a_{-1}\big|_{\hat{p}_2} \), we set \( \hat{p} = x + \hat{p}_2 \), obtaining respectively, for the right hand side of (4.28):

\[
\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (x + \hat{p}_1)^k (\hat{\xi} - 2\hat{\xi})^k \frac{(2i\xi)^{n-k}}{x^{n-k}(x + (\hat{p}_1 - \hat{p}_2))^{n-k}} \tag{4.29}
\]

\[
\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (x + \hat{p}_2)^k (\hat{\xi} - 2\hat{\xi})^k \frac{(2i\xi)^{n-k}}{x^{n-k}(x + (\hat{p}_2 - \hat{p}_1))^{n-k}} \tag{4.30}
\]

Eqs. (4.29) and (4.30) can be written in the form

\[
\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{(\hat{\xi} - 2\hat{\xi})^k (2i\xi)^{n-k}}{(n-k)!k!} \frac{1}{x^{n-k}(x + \hat{p}_j - \hat{p}_{3-j})^{n-k}} \tag{4.31}
\]
setting \( j = 1 \) and \( j = 2 \), respectively. In order to calculate the coefficient for \( x^{-1} \), we consider the Taylor series expansion of the function \( \frac{(x + \hat{p}_j)^k}{(x + \hat{p}_j - \hat{p}_{3-j})^{n-k}} \) in (4.31), in the neighborhood of \( x = 0 \):

\[
\frac{(x + \hat{p}_j)^k}{(x + \hat{p}_j - \hat{p}_{3-j})^{n-k}} = \sum_{w=0}^{+\infty} \frac{1}{w!} \frac{d^w}{dx^w} \left. \frac{(x + \hat{p}_j)^k}{(x + \hat{p}_j - \hat{p}_{3-j})^{n-k}} \right|_{x=0} x^w
\]

(4.32)

and we calculate the multiplicative coefficient for \( x^w = x^{n-k-1} \). Note that, since \( w \geq 0 \), we have \( k \leq n - 1 \). After some algebraic manipulations, the coefficient for \( x^w \) results to be given by:

\[
H(n, k, \hat{p}_j, \hat{p}_{3-j}) = \frac{1}{(n - k - 1)!} \frac{d^{(n-k-1)}}{dx^{(n-k-1)}} \left. \frac{(x + \hat{p}_j)^k}{(x + \hat{p}_j - \hat{p}_{3-j})^{n-k}} \right|_{x=0} = \frac{\min[n-k-1,k]}{g!(n-k-1-g)!} \frac{(-1)^{n-k-1-g} k!(2n-2k-g-2)!}{(n-k-1)!(k-g)!} \hat{p}_j^{k-g}(\hat{p}_j - \hat{p}_{3-j})^{-2n+2k+1+g}.
\]

(4.33)

The coefficient \( a_{-1}|_{\hat{p}_j} \) is then given by:

\[
a_{-1}|_{\hat{p}_j} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!k!} (2i\hat{\xi})^{n-k}(\hat{\xi} - 2\hat{\xi})^k H(n, k, \hat{p}_j, \hat{p}_{3-j})
\]

(4.34)

Substituting (4.34) into (4.27) yields:

\[
I_{aux} = 2 \sum_{n=0}^{+\infty} \sum_{k=0}^{n-1} \frac{(2i\hat{\xi})^{n-k}(\hat{\xi} - 2\hat{\xi})^k}{(n-k)!k!} \left( H(n, k, \hat{p}_1, \hat{p}_2) + H(n, k, \hat{p}_2, \hat{p}_1) \right)
\]

(4.35)

To simplify (4.35), we observe that both \( \hat{p}_1 \) and \( \hat{p}_2 \) are proportional to \(-i\hat{\alpha}\).

Setting \( \hat{p}_1 = \frac{\hat{p}_1}{i\hat{\alpha}} = \hat{s} \) and \( \hat{p}_2 = \frac{\hat{p}_2}{i\hat{\alpha}} = \hat{s} - \hat{\xi} \), we rewrite \( I_{aux} \) as

\[
I_{aux} = 2 \sum_{n=0}^{+\infty} \sum_{k=0}^{n-1} \frac{(2i\hat{\xi})^{n-k}(\hat{\xi} - 2\hat{\xi})^k(-i\hat{\alpha})^{3k-2n+1}}{(n-k)!k!} \left( H(n, k, \hat{s}, \hat{p}_2) + H(n, k, \hat{s}, \hat{p}_1) \right)
\]

(4.36)

which using (4.33) becomes:

\[
I_{aux} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n-1} \frac{(\hat{\xi} - 2\hat{\xi})^k2^{n-k+1}n^{n-2k+1}\hat{\alpha}^{3k-2n+1}\hat{\xi}^{n-k}}{(n-k)!(n-k-1)!} \sum_{g=0}^{\min[n-k-1,k]} \frac{(-1)^g(2n-2k-g-2)!}{g!(n-k-1-g)!(k-g)!} \left( \hat{p}_1^{k-g}(\hat{p}_1 - \hat{p}_2)^{-2n+2k+1+g} + \hat{p}_2^{k-g}(\hat{p}_2 - \hat{p}_1)^{-2n+2k+1+g} \right)
\]

(4.37)
4.2. LINEAR ENERGY CHIRPED EXACT GREEN FUNCTION

Since $\tilde{p}_1 = \tilde{s}$ and $\tilde{p}_2 = \tilde{s} - \tilde{\xi}$, (4.37) can be written as:

$$
I_{aux} = \sum_{n=0}^{+\infty} \sum_{k=0}^{n-1} \frac{(\hat{z} - 2\hat{\xi})^{n-k} 2^{n-k+1} \alpha^{3k-n+1} \hat{\xi}^{k+1-n}}{(n-k)!(n-k-1)!} \times \\
\sum_{g=0}^{\min\{n-k-1,k\}} \frac{\hat{\xi}^{2g}}{g!(n-k-1-g)!(k-g)!} (\hat{z} - 2\hat{\xi})^{n-k-2} \alpha^{3k-n+1} \hat{\xi}^{k+1-n} (4.38)
$$

Introducing the index changes:

$$
n = 3h + t + 2 ; \quad k = 2h + t + 1 (4.39)
$$

and recalling that $0 \leq k \leq n - 1$, we obtain the inequalities

$$
h \geq 0 ; \quad t \geq -2h - 1. (4.40)
$$

Thus, we can write:

$$
I_{aux} = \sum_{h=0}^{+\infty} \sum_{t=-2h-1}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t+2h+t+1} \alpha^{2h} \hat{\xi}^{2h-t-1} \hat{\alpha}^{h-t-1}}{h!(h+1)!} \times \\
\sum_{g=0}^{\min\{h, 1+2h+t\}} \frac{(2h-g)! \xi^{1+2h+t-g}((\hat{z} - 2\hat{\xi})^{1+2h+t-g})}{g!(h-g)!(2h+t+1-g)!} (4.41)
$$

We now show that the multiplicative coefficients of the terms $\hat{\alpha}^r$ with $r < 0$ are zero. To this aim let us indicate by $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ the integrand in Eq. (4.24), that is:

$$
f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = e^{\hat{p}(\hat{z}-2\hat{\xi}) + \hat{s}(\hat{p} + \hat{\alpha}(\hat{s}-\hat{\xi}))}. (4.42)
$$

Assuming that $\hat{s}, \hat{\xi}$ and $\hat{z}$ are assigned, the integration problem posed above can be reduced to that of calculating, for any real $\hat{\alpha}$, the sum of the residual values of $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ in the two singularities, given by

$$
\text{Res}(\hat{p}_1) + \text{Res}(\hat{p}_2) = \frac{1}{2\pi i} \oint_C f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) d\hat{p} (4.43)
$$

where the integration line $C$ includes, as inner points, the singularities $\hat{p}_1$ and $\hat{p}_2$ as well as the origin of the complex plane $\hat{p}$. Therefore there exists a
neighbourhood $U$ of $\tilde{\alpha} = 0$ where, for any $\hat{p} \in C$, the function $f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$ is bounded and has continuous derivatives of any order with respect to $\hat{\alpha}$. Therefore it can be expanded in Taylor series:

$$f(\hat{p}, \hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\partial^n f}{\partial \hat{\alpha}^n} \right)_{\hat{\alpha}=0} \hat{\alpha}^n$$

where all of the derivatives depend on $\hat{p}$ ($\hat{s}$, $\hat{\xi}$ and $\hat{z}$ are assigned). Substituting (4.44) into (4.43) and integrating yields

$$\text{Res}(\hat{p}_1) + \text{Res}(\hat{p}_2) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \oint_C \left( \frac{\partial^n f}{\partial \hat{\alpha}^n} \right)_{\hat{\alpha}=0} \, d\hat{p} \right] \hat{\alpha}^n$$

The right hand side of Eq. (4.45) does not involve terms $\hat{\alpha}^n$ with $n < 0$. Therefore, recalling (4.27), the same holds for the series expression in Eq. (4.41). Taking this into account, (4.41) can be written as:

$$I_{aux} = \sum_{h=0}^{+\infty} \sum_{t=0}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t}\Phi^{1+h-t-1} \hat{\alpha}^t \hat{\xi}^{1-h} \hat{z}^t}{h!(h+1)!} \times$$

$$\sum_{g=0}^{h} \frac{(2h - g)!\hat{\xi}^g((-1)^g\hat{\xi}^{1+2h+t-g} - (\hat{s} - \hat{\xi})^{1+2h+t-g})}{g!(h-g)!(2h + t + 1 - g)!}$$

The latter expression can be further simplified into the following:

$$I_{aux}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \sum_{t=0}^{+\infty} \hat{\alpha}^t \sum_{h=0}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t}\Phi^{1+h-t-1} \hat{\alpha}^t \hat{\xi}^{1-h} \hat{z}^t}{h!(1+h)!} \sum_{w=0}^{t} \frac{(-1)^w \hat{\xi}^w \hat{z}^{1+h-t-w}(h+t-w)!}{w!(t-w)!(1+2h+t-w)!}$$

The Green function is finally expressed as:

$$G_{s,lin}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) = \delta(\hat{\xi} - \hat{z}/2) + I_{aux}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha})$$

which corresponds to the final expression:

$$G_{s,lin}(\hat{s}, \hat{\xi}, \hat{z}, \hat{\alpha}) =$$

$$\sum_{t=0}^{+\infty} \hat{\alpha}^t \sum_{h=0}^{+\infty} \frac{(\hat{z} - 2\hat{\xi})^{1+2h+t}\Phi^{1+h-t-1} \hat{\alpha}^t \hat{\xi}^{1-h} \hat{z}^t}{h!(1+h)!} \sum_{w=0}^{t} \frac{(-1)^w \hat{\xi}^w \hat{z}^{1+h-t-w}(h+t-w)!}{w!(t-w)!(1+2h+t-w)!} + \delta(\hat{\xi} - \hat{z}/2)$$
In order to check the expression in Eq. (4.48), we performed a numerical inverse Laplace transform of Eq. (4.22). This has been done choosing a circular path in the complex plane, including the singularities to evaluate the line integral. The algorithm to perform the integration has to take into account some numerical issues. The path in the complex plane should not be too close to the singularities \( \hat{p}_1 = -i\hat{\alpha}\hat{s} \) and \( \hat{p}_2 = i\hat{\alpha}(\hat{\xi} - \hat{s}) \), and also the term \( e^{\hat{p}(\hat{z} - 2\hat{\xi})} \) should not be too large, compared to the result. Since the evaluation of the Green function for a single value of \( \hat{s} \) requires the evaluation of the path integral for different values of \( \hat{\xi} \) from 0 to \( \hat{z}/2 \), the path integral should be chosen carefully for each value of \( \hat{\xi} \), in order to have a reliable result. Further the numerical integration can not evaluate the solution for \( \hat{\xi} \) close to \( \hat{z}/2 \), because it cannot represent a Dirac delta. The comparison proved that the expression given in Eq. (4.48) is correct, but also showed the advantage of having a series expansion instead of using a numerical integration.

### 4.2.1 Comparison between the proposed series expression for the Green function and the saddle point approximation

In this section we give some amplitude and phase plots of the Green function, calculated by using Eq. (4.48). Such results are compared with those obtained via saddle point approximation.

At first, we consider a short 19 periods undulator, such as that employed in the FERMI@Elettra project[1]. The normalized length of the undulator is \( \hat{z} = 1.87 \) and the linear chirp parameter is set to \( \hat{\alpha} = 0.04 \). Figure 4.1 shows the Green functions obtained at the beam positions \( \hat{s}_0 = 0 \), \( \hat{s}_1 = 15 \) and \( \hat{s}_2 = -15 \). The system is almost in the lethargy regime, and it can be noted that the approximated and exact expressions of the Green functions are different in both amplitude and phase. Furthermore, the exact formula in (4.48) shows the presence of a Dirac delta pulse at \( \hat{\xi} = \hat{z}/2 \), which represents the laser seed moving at the velocity of light through the undulator. This allows a correct characterization of the group velocity of the FEL pulse in the lethargy regime, as shown in Fig. 1 in [31]. As a second test case, we consider a 190 periods undulator. The normalized length of the undulator is \( \hat{z} = 18.7 \), and the linear chirp parameter is set to \( \hat{\alpha} = 0.01 \). In this case the system is not in the lethargy regime, and the asymptotic and the exact representations of the Green function are in very good agreement, as is shown
in Figure 4.2. In particular, the phase plots of the Green function differ only for \( \xi \) coordinates where the gain is low. This result allows to use also the asymptotic expression of the Green function in the exponential gain regime, obtaining good estimates of both the amplitude and the phase of the FEL pulse.

### 4.3 Saddle point approximated Green function for a seeded Free Electron Laser with energy chirp and curvature

As done by other authors in the case of the linear chirped FEL Green function, a closed analytical form for the Green function can be estimated by a saddle point approximation. The saddle point \( \hat{p}_s \) is found as the solution of

\[
\frac{d \hat{F}(\hat{p})}{d \hat{p}} \bigg|_{\hat{p}=\hat{p}_s} = 0
\]  

(4.49)
4.3. SADDLE POINT APPROXIMATED GREEN FUNCTION

Figure 4.2: Green function plots for a 190 periods undulator. Exact formula (Bold thickness), asymptotic approximation (Normal thickness). \( s = 0 \) (solid), \( s = 15 \) (dashed), \( s = -15 \) (dash-dotted). The Green function amplitudes for different \( s \) coordinates are superposed.

where \( \hat{F} \) is the exponent in the integrand of Eq. (4.20), which is purely imaginary when \( \hat{p} \) is on the imaginary axis, and is also called fasor in literature. The expression given by:

\[
\hat{F} = \hat{p}(\hat{z} - 2\hat{\xi}) + 4i\hat{\beta} \frac{\arctan\left(\frac{\hat{\alpha} - \hat{\beta}}{\sqrt{2\hat{\beta} - \hat{\alpha}^2}}\right) - \arctan\left(\frac{\hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi})}{\sqrt{2\hat{\beta} - \hat{\alpha}^2}}\right)}{(2i\hat{\beta} - \hat{\alpha}^2)^{3/2}} - \frac{4i\hat{\beta}}{2\hat{\beta} - 2\hat{\alpha} + 2i\hat{\beta}} + \frac{\hat{\beta}(\hat{s} - \hat{\xi})}{(\hat{s} - \hat{\xi})(2\hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi})) - 2i\hat{\beta}} \frac{\hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi})}{\hat{\alpha}^2 - 2i\hat{\beta}} \frac{\hat{\alpha} - \hat{\beta}}{(\hat{s} - \hat{\xi}) (2\hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi})) - 2i\hat{\beta}} \frac{\hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi})}{\hat{\alpha}^2 - 2i\hat{\beta}}
\]

At the saddle point, the phasor has the highest real part, therefore, for long undulators, the growth mode with the highest real part dominates the other ones. The Green function is then approximated as

\[
g\left(\hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}\right) \approx \frac{2 \exp \left[\hat{F} \left(\hat{p}_n, \hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}\right)\right]}{2\pi \hat{F}''\left(\hat{p}_n, \hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}\right)}^{1/2}. \tag{4.50}
\]
Since the expression in Eq. (4.50) is difficult to handle, it is expanded in power series on $\mu$ and $\nu$, and the phasor becomes

$$
\tilde{F} = \hat{F}(\tilde{z}-2\xi) + \sum_{m=0}^{M} \left[ \sum_{n=0}^{m} \frac{i^{m+1}(m+1)! \left( \tilde{z}^{2m-n+1} - (\tilde{s} - \hat{\xi})^{2m-n+1} \right) (-\hat{\alpha})^n \hat{\beta}^{m-n}}{n!(m-n)!(2m-n+1)!} \hat{p}^{2+m} \right]^{(4.51)}
$$

To estimate $\hat{p}$, we introduce the slippage off resonance condition $\hat{\zeta} = \tilde{z} - 6\xi$, and the mode laser temporal duration related variable $\hat{\eta} = \tilde{s} - \hat{\xi}$. Then, an order analysis is performed, fixing $\hat{\zeta} \sim \varepsilon$, $\hat{\eta} \sim 1$, $\hat{\alpha} \sim \varepsilon^2$, $\hat{\beta} \sim \varepsilon^3$ and $\tilde{z} \sim \varepsilon^{-1}$. The saddle point $\hat{p}_S \approx p_{s0} + p_{s1}\varepsilon + p_{s2}\varepsilon^2 + p_{s3}\varepsilon^3 + p_{s4}\varepsilon^4$ is obtained solving the equation $d\tilde{F}(\hat{p})/d\hat{p} = 0$ order-by-order in $\varepsilon$, and for this purpose we choose $M = 5$ in Eq. (4.51). The approximated saddle point is found as

$$
\hat{p}_S = i^{18} \left[ \frac{i^{3}(\tilde{z}-6\xi)}{2\varepsilon} + \frac{\hat{\alpha}}{2}(\tilde{s} - 2\delta) + \frac{i\hat{\beta}}{2} \left( \frac{\tilde{z}-12\xi}{105} + \frac{\hat{\alpha}^2}{\varepsilon} \right) + \sqrt{\frac{\hat{\alpha}^2}{\varepsilon}} \right] + \frac{\hat{\alpha}^2}{216} \frac{\tilde{s} - 2\delta}{270} - \frac{\hat{\alpha}^4}{6} - \frac{\hat{\beta}^2}{216} \left( \frac{\tilde{z}-6\xi}{6} - \frac{(\tilde{s} - \hat{\xi})^2}{z} \right)
$$

Substituting it back to Eq. (4.51) and to the second derivative of the phasor, it yields to the following expressions for the Green function

$$
\tilde{F}(\hat{p}_S) = + \frac{i^{3/2}(\tilde{z}-6\xi)(\tilde{s}+6\xi)}{512} + \frac{i^{3}(\tilde{z}-2\xi)(\tilde{s} - 2\delta)}{45} + \frac{\hat{\alpha}^2}{18} \left( \frac{\tilde{z} - 12\xi}{105} + \frac{\hat{\alpha}^2}{\varepsilon} \right) + \frac{\hat{\alpha}^4}{135} \left( \frac{\tilde{s} - 2\delta}{6} + \frac{\hat{\beta}^2}{18} \right)
$$

$$
\tilde{F}(\hat{p}_S)^{(4.53)}
$$
and

\begin{align}
\hat{F}^{n}(\hat{p}_{s}) &= -\frac{i}{2} (\hat{s}^{2} - 2\hat{s} \hat{\xi} - 36\hat{\xi}^{2}) + i \frac{5\hat{\xi}^{2}}{108} \left( \frac{\hat{\alpha}^{2}}{\hat{z}} - \frac{\hat{\alpha}\hat{\beta}}{\hat{z}} + \frac{\hat{\beta}^{2}}{12} \right) \\
&\quad + \frac{i\hat{\xi}^{3}\hat{\beta}(\hat{s} - \hat{\xi})}{108} \left( \hat{s} \hat{\beta} - 2\hat{\alpha} \right) + \frac{i\hat{\xi}^{3}(\hat{s} - \hat{\xi})^{2}}{108} \left(\hat{s}^{2} - \hat{s} \hat{\beta} - 36\hat{\xi}^{2} \right)
\end{align}

Terms with large denominators give small contributions to the Green function and are cut in the following equations:

\begin{align}
\hat{F}^{n}(\hat{p}_{s}) &= i \frac{\hat{s}}{2} - \frac{i\hat{\xi}^{2}}{18} \left( \hat{s} - \hat{\xi} \right) \left( \hat{s} - \hat{\xi} + 12\hat{s} \right) \\
&\quad + \frac{i\hat{\xi}^{3}{\hat{s} - \hat{\xi}}^{2}}{432} \left( \frac{\hat{\alpha}^{2}}{\hat{z}} - \frac{\hat{\alpha}\hat{\beta}}{\hat{z}} + \frac{\hat{\beta}^{2}}{45} \right) \\
\hat{F}^{n}(\hat{p}_{s}) &= i \frac{\hat{s}}{2} - \frac{i\hat{\xi}^{2}}{18} \left( \hat{s} - \hat{\xi} \right) \left( \hat{s} - \hat{\xi} + 12\hat{s} \right)
\end{align}

The expressions in Eqs. (4.55) and (4.56) give a sufficiently good accuracy, and will be used for further calculations.

It is interesting to find a Green function that gives a closed form for the FEL radiation along the undulator when the initial seed is Gaussian. To this aim we rewrite Eq. (4.50) as

\begin{align}
g \left( \hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta} \right) \approx \sqrt{\frac{2}{\pi}} e^{\hat{F}(\hat{p}_{s}, \hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}) - \frac{i}{2} \ln[\hat{F}^{n}(\hat{p}_{s}, \hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta})]}
\end{align}

The term \( \frac{i\hat{\xi}^{3}{\hat{s} - \hat{\xi}}^{2}}{432} \left( \frac{\hat{\alpha}^{2}}{\hat{z}} - \frac{\hat{\alpha}\hat{\beta}}{\hat{z}} + \frac{\hat{\beta}^{2}}{45} \right) \) in Eq. (4.53) has been expanded in power series up to the second order from the stationary phase point of the unchirped Green function \( \hat{\xi} = \hat{z}/6 \). Further the terms with powers of \( \hat{\alpha} \) and \( \hat{\beta} \) larger than the second one have large denominators and their contribution to the Green function can be neglected. From Eq. (4.50), the expression of the
Green function, which is primitiviable when convoluted with a Gaussian seed laser, is as follows:

\[
g(\hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}) \approx \frac{i}{\sqrt{\pi \hat{z}}} \exp \left\{ i \frac{1}{2} \hat{z} - \frac{\hat{z} - 6 \hat{\xi}}{2 \hat{z}} - i \frac{\hat{\alpha}}{6} (\hat{z} - 6 \hat{\xi})^2 + i \frac{\hat{\beta}}{2} \hat{z} - 2 \hat{\xi} (\hat{\xi} - 2 \hat{s}) + \frac{i \hat{\beta} \hat{s}}{216} (6 \hat{s} + \hat{z} - 12 \hat{\xi}) \right. \\
+ \frac{i \hat{\alpha} \hat{s} \hat{z}}{36} \left[ \frac{18 \hat{\xi}^2 - 6 \hat{\xi}}{18} + (\hat{s} - \hat{\xi})(\hat{\xi} - 6 \hat{\xi} + 12 \hat{s}) \right] + \frac{i (6 \hat{s} - \hat{z})(\hat{z} - 6 \hat{\xi}) \hat{\beta}}{216} (6 \hat{s} + \hat{z} - 12 \hat{\xi}) \\
- \frac{1}{432} \left[ \frac{\hat{\alpha} \hat{\xi}^2}{\hat{z}} - \frac{\hat{\alpha} \hat{\beta} \hat{s}}{6} + \frac{\hat{\beta} \hat{s} \hat{z}^2}{135} \right] \left( 1 - i \frac{\hat{\alpha} \hat{\xi}}{\hat{z}} \right) + \frac{i \hat{\beta} \hat{s} \hat{z}^2 (\hat{\xi} - 6 \hat{\xi})}{432} \left[ 2 \hat{\alpha} \hat{\beta} \hat{s} - i \frac{\hat{\alpha} \hat{\beta} \hat{s} \hat{z}}{6} \left( 2 \hat{\alpha} \hat{\xi} - \hat{\beta} \hat{s} \hat{z} \right) \right] \\
+ \frac{i \hat{\beta} \hat{s} (6 \hat{\xi} - 6 \hat{\xi})}{432} \left[ \left( \frac{\hat{\alpha} \hat{\xi}^2}{\hat{z}} - \frac{\hat{\alpha} \hat{\beta} \hat{s}}{6} + \frac{\hat{\beta} \hat{s} \hat{z}^2}{45} \right) \left( 1 - i \frac{\hat{\alpha} \hat{\xi}}{\hat{z}} \right) - \frac{\hat{\beta} \hat{s} (\hat{\xi} - 6 \hat{\xi})}{6} \left( 1 + i \frac{\hat{\alpha} \hat{\xi}}{\hat{z}} \right) \right] \right\} 
\]

(4.58)

### 4.3.1 Properties of the Green function

The advantage of having an analytical formula expressed as an exponential over a series expansion is that properties like bandwidth, frequency shift, and frequency chirp are easier to determine, since they depend on the derivatives of the Green function.

The Green function contains a time independent contribution due to the undulator parameters and a time dependent contribution coming from the chirp and curvature on the electron bunch energy, thus it depends on both variables \( \hat{\xi} \) and \( \hat{s} \) separately. The properties of the amplitude and phase of the Green function strongly affect the characteristic of the FEL radiation along the undulator. Furthermore, the ratio between the Green function temporal duration and the seed laser pulse temporal duration plays an important role as well. When the temporal duration of the seed laser is longer than that of the Green function, FEL radiation properties can be evinced directly by the Green function. In particular, the central frequency shift and the frequency chirp of the FEL depend on the electron bunch (time dependent) structure. Nevertheless, the study of the standing alone Green function can give some insight on the characteristics of the FEL radiation independently of the seed temporal duration and shape.

The effects of a linear chirp and quadratic curvature on the real and imaginary part of the exponential factor in Eq. (4.58) are studied in the following.

We give the expressions for the real \( R \) and imaginary \( I \) part of the exponential factor in Eq. (4.58), that are useful to discuss the properties of the Green
4.3. SADDLE POINT APPROXIMATED GREEN FUNCTION

The position of the peak of the imaginary part which gives the stationary function,

\[ R = \frac{-1}{2} + \frac{3\xi}{\bar{z}} + \frac{3\sqrt{3}(\bar{z} - 2\xi)(\bar{z} + 6\xi)}{8\bar{z}} - \alpha^2 \left( \bar{z}\sqrt{3\bar{z}} - 1 \right) + \frac{\beta}{144} \frac{\bar{z}^2 \bar{z}^2}{\bar{z}} \left\{ \left[ \frac{1}{\bar{z}} \right] \right\} \]

and

\[ I = \frac{3(\bar{z} - 2\xi)(\bar{z} + 6\xi)}{8\bar{z}} + \alpha \left( \bar{z} - 2\xi \right)(-2\bar{s} + \bar{\xi}) \]

Both Eqs. (4.59) and (4.60) are quadratic functions of \( \bar{\xi} \). The position of the peak of the real part, which gives the maximum amplitude of the Green function, is given by:

\[ \bar{\xi}_{Rpp} = \left( \frac{1}{3\sqrt{3}} \right) \left\{ \frac{3\xi}{6} \left[ \frac{-\bar{z}^2(-1 + \sqrt{3})\xi^2}{216(2 + \sqrt{3})} - \frac{\bar{z}^3(4 - 6\sqrt{3} + 3\sqrt{3})\alpha\beta}{648(2 + \sqrt{3})} + \frac{\bar{z}^3[20\sqrt{3} - 180\xi + 3(29 + \sqrt{3})\xi^2\beta^2]}{116640(2 + \sqrt{3})} \right] \right\} \]

The position of the peak of the imaginary part which gives the stationary phase point, is given by:

\[ \bar{\xi}_{Ipp} = \left( \frac{\xi}{6} \right) \left\{ 1 + \frac{12\bar{z} + \bar{\xi}}{9} + \frac{\bar{z}(-\sqrt{3} + \bar{\xi})}{216}\bar{\xi} + \frac{\bar{z}^2(36\bar{z}^2 + 6\bar{\xi})}{54\beta} + \frac{\bar{z}^2[6\bar{\xi} - 6\bar{s}]}{648\alpha\beta} - \frac{\bar{z}^3[9\sqrt{3} + 60\sqrt{3}\xi]}{38880} \right\} \]

Both the linear and the quadratic terms have small influence on the position of the amplitude peak of the Green function in Eq. (4.61), since \( \alpha \) and \( \beta \) only provides second order terms contributions.

The linear chirp and the curvature affect much more the phase of the Green function. The position of the stationary point of the phase is given by Eq. (4.62), and it is plotted in Fig. 4.3 as a function of the chirp parameters.

To estimate the central frequency shift of the FEL radiation along the undulator, we consider the derivatives of \( I \) on \( \bar{s} \) and \( \bar{\xi} \). Derivatives on \( \bar{\xi} \) will
Figure 4.3: Position of the stationary phase point in $\hat{s}$ units with $\hat{z} = 25$ and $\hat{s} = \hat{z}/6$.

give information on the shift in case of a seed much shorter than the Green function, while derivatives on $\hat{s}$ will give information when the seed is longer than the Green function. The expressions for the derivatives are as follows:

$$
\frac{\partial I}{\partial \xi} = \frac{3}{2} - \frac{9\hat{\xi}}{\hat{z}} + \frac{\hat{\alpha}(4\hat{s} + \hat{z} - 4\hat{\xi})}{2} + \frac{\hat{\beta}\left(\hat{z}^2 + 48\hat{s}\hat{\xi} - 4\hat{z}\hat{\xi} - 12\hat{s}^2 - 10\hat{s}\hat{z}\right)}{12}
+ \frac{\hat{\alpha}^2\hat{z}(\hat{z} - \sqrt{3})}{144} - \frac{\hat{\alpha}\hat{\beta}\hat{z}^2(6\hat{s} + \hat{z} - 12\hat{\xi})}{432} + \frac{\hat{z}^3\hat{\beta}^2}{144} \left(\frac{\hat{s} - \hat{\xi}}{\hat{z}\sqrt{3}} + \frac{\sqrt{3} + 9\hat{z} - 60\hat{\xi}}{180}\right) 
$$

(4.63)

$$
\frac{\partial I}{\partial \hat{s}} = -\hat{\alpha}(\hat{z} - 2\hat{\xi}) + \hat{\beta} \left[\hat{s}(\hat{z} - 2\hat{\xi}) + \frac{\hat{z}^2}{36} - \frac{5\hat{z}\hat{\xi}}{6} + 2\hat{\xi}^2\right] + \frac{\hat{z}^2\hat{\alpha}\hat{\beta}\left(\sqrt{3} - 6\hat{\xi}\right)}{432}
+ \frac{\hat{z}^2\hat{\beta}^2}{144} \left[\frac{1}{\sqrt{3}} \left(\hat{\xi} - \hat{s} - \frac{\hat{z}}{12}\right) + \frac{\hat{s}\hat{z}}{3} - \frac{\hat{z}^2}{36}\right].
$$

(4.64)
4.3. SADDLE POINT APPROXIMATED GREEN FUNCTION

For the frequency chirp in the FEL radiation, we consider the second derivatives of the phase on \( \hat{s} \) and \( \hat{\xi} \), obtaining the expressions:

\[
\frac{\partial^2 I}{\partial \hat{\xi}^2} = -\frac{9}{\hat{z}} - 2\alpha + \left(4\hat{s} - \frac{\hat{z}}{3}\right)\beta + \frac{\hat{z}^2 \hat{\alpha} \hat{\beta}}{36} - \frac{\sqrt{3} + \hat{z}}{432}\beta^2 \hat{z}^2, \tag{4.65}
\]

and

\[
\frac{\partial^2 I}{\partial \hat{s}^2} = \hat{\beta}(\hat{z} - 2\hat{\xi}) + \frac{\hat{z} - \sqrt{3}}{432}\hat{z}^2 \hat{\beta}^2. \tag{4.66}
\]

To evaluate the shift and the chirp of the FEL central frequency, we have to calculate, respectively, the first and the second derivative in \( \hat{s} \) of the phase of Eq. (4.18). Actually, in the case of a seed laser much shorter than the Green function (i.e. approaching to the delta Dirac function), the FEL radiation is given straightforwardly by the Green function. Thus deriving the convoluted signal with respect to \( \hat{s} \) is like to deriving the Green function with respect to \( \hat{\xi} \).

Thus, to evaluate the shift of the FEL central frequency in this case, we consider the first order derivative of the phase on \( \hat{\xi} \) as in Eq. (4.63). We evaluate it in the correspondence of the maximum amplitude of the Green function (i.e. for \( \hat{\xi} = \hat{\xi}_{Rpp} \) as expressed in Eq. (4.61)) choosing a delta function seed. Expanding to the first order in \( \hat{\alpha} \) and \( \hat{\beta} \), we obtain:

\[
\frac{\partial I}{\partial \hat{\xi}} \bigg|_{\hat{s}=\hat{s}_{Rpp}, \hat{\xi}=\hat{\xi}_{Rpp}} \approx -\frac{\sqrt{3}}{\hat{z}} + \frac{\hat{z}}{2} \hat{\alpha} - \frac{\hat{z}^2}{36} \beta + \frac{6 - \sqrt{3} \hat{z}}{54} \beta \tag{4.67}
\]

The second derivative of the phase on \( \hat{\xi} \) in Eq.(4.65) gives the radiation frequency chirp in case of a short seed:

\[
\frac{\partial^2 I}{\partial \hat{\xi}^2} \bigg|_{\hat{s}=\hat{s}_{Rpp}} \approx -\frac{9}{\hat{z}} - 2\alpha + \left(\frac{4}{3\sqrt{3}} + \frac{\hat{z}}{3}\right)\beta \tag{4.68}
\]

In this configuration, Eq. (4.68) shows that both the linear chirp and the quadratic curvature have influence on the frequency chirp, but the main effect is due to the linear component. The first term is the intrinsic frequency chirp due to the FEL interaction [7]; the second term is inherited from the electron bunch energy chirp.

On the contrary, when the seed laser is longer than the Green function,
the former is almost constant in the convolution of Eq. (4.18), and the FEL radiation phase is mostly given by the phase of the Green function amplitude peak. Thus, the first and second derivatives on \( \hat{s} \) of the FEL radiation phase are approximately given by the corresponding derivatives on \( \hat{s} \) of the Green function phase at its peak amplitude. The first order derivative of the phase on \( \hat{s} \) in Eq. (4.64), evaluated in correspondence of the maximum amplitude of the Green function and expanded to the first order of \( \hat{\alpha} \) and \( \hat{\beta} \), is given by:

\[
\frac{\partial I}{\partial \hat{s}} \bigg|_{\hat{s} = \hat{s}_{\text{typ}}} \approx \frac{\sqrt{3} - 3\hat{z}}{9} 2\hat{\alpha} - \frac{\sqrt{3}(12\hat{s} + \hat{z})}{54} - 3\hat{z}(12\hat{s} - \hat{z}) - \frac{4}{\hat{\beta}}.
\]

Considering the order analysis between \( \hat{\alpha} \), \( \hat{\beta} \) and \( \hat{z} \), Eq. (4.69) shows that even if the quadratic curvature has influence on the central frequency shift, the main effect comes from the linear chirp. The second order derivative of the Green function phase on \( \hat{s} \) is calculated in (4.66) and its value at the peak amplitude, considering only the leading terms, is given by:

\[
\frac{\partial^2 I}{\partial \hat{s}^2} \bigg|_{\hat{s} = \hat{s}_{\text{typ}}} \approx \frac{3z - \sqrt{3}}{9} 2\hat{\beta} - 2z^2 z - \sqrt{3} \hat{\beta}^2.
\]

It is worthwhile to emphasize that, regarding to the FEL radiation frequency chirp, the energy curvature of the electrons becomes the main source of the frequency chirp in case of long seed according to Eq. (4.66).

Similarly to [7], we have calculated the temporal duration and the bandwidth of the Green function; obtaining:

\[
\sigma^2_{t,\hat{\alpha},\hat{\beta}} = \frac{k_wz}{9\sqrt{3}\rho\omega_0^2} \frac{1}{1 - 2\eta},
\]

\[
\sigma^2_{\omega,\hat{\alpha},\hat{\beta}} = \frac{3\sqrt{3}\rho\omega_0^2}{k_wz} \left( 1 + \frac{k^2 + 2k\eta + 4\eta^2}{1 - 2\eta} \right),
\]

where

\[
\eta = \frac{k_w^3z\hat{\beta}\rho^3 \left[ -36\hat{\alpha} + \hat{\beta} (\sqrt{3} + 6k_wz\rho) \right]}{2916},
\]

\[
\kappa = \frac{2}{9} k_wz\hat{\alpha} - \frac{10}{27} k^2wz^2 \hat{\beta} \rho^2 + \frac{k^3w^3z\beta^2\rho^3}{486\sqrt{3}} - \frac{4}{9} k_wz\hat{\beta} \rho^2 (k_0z - \omega_0t).
\]
4.4. EXACT GREEN FUNCTION WITH PARABOLIC PROFILE

Eq. (4.71) reveals that the rms temporal duration is affected, through the parameter \( \eta \), only by the energy curvature in the electron bunch, and this is a second order effect. The Green function bandwidth in Eq. (4.72) can be further simplified neglecting \( 2\eta \) compared to 1 and neglecting also third order quantities in \( \hat{\alpha} \) and \( \hat{\beta} \), obtaining the following approximated expression:

\[
\sigma_{\omega, \hat{\alpha}, \hat{\beta}}^2 = \frac{3\sqrt{3}\rho\omega_0^2}{k_w z} (1 + \kappa + \kappa^2).
\] (4.75)

The FEL pulse travels with a group velocity [7] of \( v_g = \omega_0 / (k_0 + 2k_w/3) \), and we have \( k_0 z - \omega_0 t = -2k_w z/3 \). The complete expression of the group velocity as a function of \( \hat{\alpha} \) and \( \hat{\beta} \) can be well approximated by the above expression, since the terms in \( \hat{\alpha} \) and in \( \hat{\beta} \) are at least of the second order. So, the expression of \( \kappa \) in Eq. (4.74) can be simplified as

\[
\kappa = \frac{2}{9} k_w z \hat{\alpha} \rho - \frac{2}{27} k_w^2 z^2 \hat{\beta} \rho^2 + \frac{k_w^3 z^3 \hat{\beta}^2 \rho^3}{48\sqrt{3}}.
\] (4.76)

4.4 Exact Green function for a seeded Free Electron Laser with energy chirp and curvature

We expand the exponential factor of the integrand of Eq. (4.20) into the following power series.

\[
\hat{p}(\hat{s} - 2\hat{\xi}) + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\beta^{2+m}} \frac{i^{m+1}(m+1)!}{(2m-n+1)^{2m-n-1}} \frac{(-\hat{\alpha})^n \hat{\beta}^{m-n}}{(2m-n+1)!}.
\] (4.77)

The expression in Eq. (4.77) is found expanding first the expression on the l.h.s. of Eq. 4.15 on the variables \( \mu \) and \( \nu \), performing the integral over \( \theta'' \) and, changing the variables using the rules specified in (4.17). In detail:

\[
\frac{i(2\rho)^3}{(s - i(\mu \theta + \nu \theta^2/2))^2} = \sum_{n=0}^{+\infty} \sum_{M=0}^{+\infty} \frac{(n + M + 1)!}{M! n!} \frac{\lambda^{n+1+2M}}{2^{M} s^{n+M+2}} \lambda^n \lambda^M
\] (4.78)
Substituting \( M = m - n \) one obtains:

\[
\frac{i(2\rho)^3}{(s - i(\mu\theta + \nu\theta^2/2))^2} = \sum_{m=n}^{+\infty} \sum_{n=0}^{+\infty} \frac{(m + 1)!i(2\rho)^3 i^{m+1} \theta^{m+2(m-n)}}{(m-n)! n!} \frac{\mu^n \nu^{m-n}}{2^{m-n} s^{m+2}}
\]

(4.79)

Since \( m \geq n \) we can rewrite the sums using as first counter \( m \) which is related to the exponent of \( s \)

\[
\frac{i(2\rho)^3}{(s - i(\mu\theta + \nu\theta^2/2))^2} = \sum_{m=0}^{+\infty} \sum_{n=0}^{m} \frac{(m + 1)!i(2\rho)^3 i^{m+1} \theta^{m-n}}{(m-n)! n!} \frac{\mu^n \nu^{m-n}}{2^{m-n} s^{m+2}}
\]

(4.80)

Finally, performing the integral over \( \theta \) and using the substitutions of Eq. (4.17), one obtains the expression in (4.77)

We introduce then the auxiliary integral

\[
I_{aux} = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{1}{\hat{p}^2 + m} \sum_{m=0}^{+\infty} \frac{i^{m+1} (m+1)!}{\hat{s}^{m-n+1} - (\hat{s} - \hat{\xi})^{2m-n+1}} (-\hat{\alpha})^n \hat{\beta}^{m-n} \frac{d\hat{p}}{\pi i}
\]

(4.81)

which is related to \( I \) by the equation

\[
\mathcal{G} = I_{aux} + 2\delta(\hat{s} - 2\hat{\xi})
\]

(4.82)

We further manipulate the expression in Eq. 4.81 to obtain a Laurent series expansion in the variable \( \hat{p} \). First of all, by the definition of the exponential we have

\[
\sum_{A=0}^{+\infty} \frac{1}{A!} \hat{p}(\hat{s} - 2\hat{\xi}) + \sum_{A=0}^{+\infty} \sum_{J=0}^{A} \frac{A!}{J!(A-J)!} \hat{p}^{A-J} (\hat{s} - 2\hat{\xi})^{A-J} \left( \sum_{m=0}^{+\infty} \frac{T(m)}{\hat{p}^{2+m}} \right)^J
\]

(4.83)

Using the power of the binomial, we obtain

\[
\sum_{A=0}^{+\infty} \frac{1}{A!} \sum_{J=0}^{A} \frac{A!}{J!(A-J)!} \hat{p}^{A-J} (\hat{s} - 2\hat{\xi})^{A-J} \left( \sum_{m=0}^{+\infty} \frac{T(m)}{\hat{p}^{2+m}} \right)^J
\]

(4.84)

where

\[
T(m) = \sum_{n=0}^{m} \frac{i^{m+1} (m+1)!}{(2m-n+1)2^{m-n-1}n!(m-n)!} \frac{(-\hat{\alpha})^n \hat{\beta}^{m-n}}{2^{m-n} s^{m+2}}
\]

(4.85)
The expression in Eq. (4.84) can be also rewritten as

\[
\sum_{A=0}^{\infty} \sum_{J=0}^{A} (\hat{z} - 2\hat{\xi})^{A-J} p^{A-3J} \left( \sum_{m=0}^{\infty} \frac{T(m)}{p^m} \right)^J
\]  

(4.86)

We define \( A \) so that:

\[
\sum_{X=0}^{+\infty} \frac{A(J,X)}{p^X} = \left( \sum_{m=0}^{+\infty} \frac{T(m)}{p^m} \right)^J
\]  

(4.87)

where \( A \) does not depend on \( p \), in this way \( p^X A \) is the coefficient of the \( X \)-th negative power of \( \left( \sum_{m=0}^{+\infty} \frac{T(m)}{p^m} \right)^J \). The residual is evaluated with the coefficient of the first negative power of \( p \)

\[
I_{aux} = \sum_{A=0}^{\infty} \sum_{J=0}^{A} \frac{(\hat{z} - 2\hat{\xi})^{A-J}}{J!(A-J)!} A(J, A - 3J + 1) = \sum_{X=0}^{\infty} \sum_{J=1}^{\infty} \frac{(\hat{z} - 2\hat{\xi})^{X+2J-1}}{J!(X + 2J - 1)!} A(J, X)
\]  

(4.88)

with \( A \) yet to be defined explicitly as

\[
A(J, X) = \sum_{w_0=0}^{\infty} \sum_{w_1=0}^{\infty} \sum_{w_2=0}^{\infty} \cdots \delta_k(J, \sum_{i=0}^{\infty} w_i) \delta_k(X, \sum_{i=0}^{\infty} iw_i) \prod_{i=0}^{\infty} \frac{T(i)^{w_i}}{w_i!}
\]  

(4.89)

where \( \delta^k \) denotes the Kronecker delta. Substituting into the Eq. (4.82) the expressions in (4.88) and (4.89) we obtain the expression for the green function

\[
G = \delta(\hat{\xi} - \hat{\alpha}/2) + \sum_{X=0}^{\infty} \sum_{J=1}^{\infty} \frac{(\hat{z} - 2\hat{\xi})^{X+2J-1}}{J!(X + 2J - 1)!} \sum_{w_0=0}^{\infty} \sum_{w_1=0}^{\infty} \cdots \delta_k^{w_0}(X, \sum_{i=0}^{\infty} iw_i) \prod_{i=0}^{\infty} \frac{T(i)^{w_i}}{w_i!}
\]  

(4.90)

We note that the expression in Eq. (4.85) is a polynomial in \( \hat{\alpha} \) and \( \hat{\beta} \) where the degree of each term is \( m \). Since the chirp parameters \( \hat{\alpha} \) and \( \hat{\beta} \) are in practical cases small numbers, the contribution for the higher orders is negligible. This allows to truncate the expression in Eq. (4.90) with great accuracy, considering only the first \( w_i \) terms in the sums. Furthermore, the Kronecker delta limits the range of the sums so that \( w_k \) can assume integer values from 1 to the minimum between \( J \) and \( X/k \). The expression in Eq. (4.90) has been validated performing a numerical inverse Laplace transform of Eq. (4.20). As specified for the linear energy chirped Green function, the numerical inverse Laplace transform has to be carried on carefully to obtain reliable results.
4.5 Green function for a Free Electron Laser with density modulation as source, energy chirp and curvature

The exponential factor of the integrand of Eq. (4.21) is the same of the one in Eq. (4.20) and we will use the same series expansion as in (4.77) further we expand the factor \( \frac{D_1}{\rho \gamma_0 2 \hat{p} + i(\hat{s} - \hat{\xi})(2 \hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi}))} \) as:

\[
D_1 \frac{1}{\rho \gamma_0 2 \hat{p} + i(\hat{s} - \hat{\xi})(2 \hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi}))} = \frac{D_1}{\rho \gamma_0} \sum_{H=0}^{\infty} \frac{i^{2H+1}(\hat{s} - \hat{\xi})^H(2 \hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi}))^H}{p^{1+H}} \tag{4.91}
\]

Further we notice that the expression in Eq. (4.21) satisfies the conditions of the Jordan’s Lemma, therefore we don’t need to introduce an auxiliary integral as done for the seeded FEL Green function. As it has been done for the seeded Green function, we expand the exponential factor in power series obtaining Eq. (4.86) that has to be multiplied by the expression found in Eq. (4.91).

\[
G_b = \sum_{H=0}^{\infty} \sum_{J=0}^{\infty} \sum_{W_l=0}^{\infty} \frac{W_k \sum_{h=1}^{\infty} hW_h + 2J + H + 1}{(1+H+2J+\sum_{h=1}^{\infty} hW_h)!} (\hat{\xi} - 2\hat{\xi})^{\sum_{h=1}^{\infty} hW_h} \times \frac{T(l)}{W_l!} R(H) \delta_{J,\sum_{h=0}^{\infty} W_h} \tag{4.92}
\]

where \( T(m) \) is defined in Eq. (4.85) and \( R(H) \) is defined as

\[
R(H) = \frac{D_1}{\rho \gamma_0} \sum_{H=0}^{\infty} \frac{i^{2H+1}(\hat{s} - \hat{\xi})^H(2 \hat{\alpha} - \hat{\beta}(\hat{s} - \hat{\xi}))^H}{p^{1+H}} \tag{4.93}
\]

The expression in Eq. (4.92) has been validated performing a numerical inverse Laplace transform of Eq. (4.21).
Chapter 5

Impact of chirp and curvature on the FEL radiation

In this chapter we evaluate the radiation envelope along the undulator using the integral representations derived in Chapter 4. The results are obtained by the convolution of the Green functions with the source terms, the seed laser envelope at the undulator entrance Eq. (4.20) and the bunching Eq. (4.21), convolutions can be evaluated numerically for arbitrary source terms, in principle. However, in the interesting case a seed with gaussian envelope, using the saddle point approximated expressions for the Green function with both linear chirp and curvature, an analytical closed form can be determined. This is particularly useful since properties like the frequency shift and the frequency chirp depend on the derivative of the phase of the envelope, that are easy to evaluate with the closed form, while with the series expansions can be evaluated only numerically. We will discuss the impact on the FEL radiation properties showing both the results obtained with the approximated Green functions and the more accurate exact series expansions. Further we will propose a method to obtain ultra-short FEL pulses using both a frequency chirp on the seed laser and the curvature on the energy profile of the electron bunch. Finally we will evaluate the effect of a curvature on the electrons in the cascade harmonic generation configuration.
5.1 Coherent Seed Laser Pulse

The integral representation in Eq. (4.20) allows us to evaluate numerically the radiation along the undulator with arbitrary kinds of seed lasers at the undulator entrance. Of particular interest is the case of seed lasers having a Gaussian profile. Further, if the seed laser is a fundamental Gaussian mode with eventually a linear frequency chirp on it, the expression of the FEL can be determined analytically, using the approximated Green function.

We assume the electric field of the seed laser to be

\[ E_s(t, z) = E_0 e^{i(k_0 z - \omega_0 t)} e^{-(iB_l + A_l)(t-z/c)^2} e^{-iB_c (t-z/c)}. \]  

(5.1)

Notice that, a time jitter on the seed laser can be considered by substituting \( t \) with \( t - t_j \).

At the undulator entrance the field can be written as

\[ E_s(t, z=0) = E_0 e^{-i\omega_0 (t-t_j)} e^{-(iB_l + A_l)(t-t_j)^2} e^{-iB_c (t-t_j)}. \]  

(5.2)

Using the notation previously introduced, \( E = A e^{i(\theta-Z)} \), and neglecting the \( e^{i\omega_0 t_j} \) phase constant, one obtains:

\[ A(\hat{s}, 0) = E_0 e^{-Q(\hat{s} - \hat{s}_j)^2 + iQ_c(\hat{s} - \hat{s}_j) - iQ_l(\hat{s} - \hat{s}_j)^2} \]  

(5.3)

where \( Q = \frac{A_l}{\rho^2 \omega_0^2}, Q_c = \frac{B_c}{\rho \omega_0}, Q_l = \frac{B_l}{\rho^2 \omega_0^2}, \) and \( s_j = -\rho \omega_0 t_j \).

5.1.1 Seed Laser Pulse Duration

In a high-gain Self Amplified Spontaneous Emission (SASE) FEL, the FEL Green function temporal duration defines the temporal coherent length. Yet, in a seeded FEL, the FEL process starts with a coherent seed. Hence, the Green function temporal duration would be shorter than the initial coherent seed laser pulse duration to start with. Eventually, the Green function temporal duration can be longer than the initial seed laser pulse duration. Therefore, it is interesting to study different cases where the initial seed laser pulse temporal duration is short, or long, compared to the Green function temporal duration evaluated at the exit of the undulator.

According to Eq. (5.3), the seed laser squared temporal rms duration is \( \sigma_{t,seed}^2 = 1/(2Q_p^2 \omega_0^2) \). On the other hand, the Green function squared temporal duration is \( \sigma_{t,GF}^2 = \hat{\tau}/(18\sqrt{3} \rho^2 \omega_0^2) \). In the following, we show that the
linear energy chirp and the energy curvature along the electron bunch have different effects on the FEL radiation, depending on the ratio between the Green function temporal duration and the seed laser pulse temporal duration. We introduce the parameter

$$K = \frac{\sigma_{t,GF}^2}{\sigma_{t,seed}^2} = \frac{Q \tilde{\chi}}{9\sqrt{3}}$$

(5.4)

to characterize the above mentioned different cases. When the seed laser is short, the FEL depends mostly on the Green function itself. This is in some sense similar to a SASE FEL, which starts from shot noise. For the other extreme, when the seed laser is long, for an electron bunch having energy chirp and energy curvature, different parts of the electron bunch have different phases, and thus interference plays an important role. This is an unique feature of a seeded FEL.

Furthermore, the electron bunch is considered to be much longer than the seed laser, since the Green function has been derived using the coasting beam model.

## 5.2 Time-Frequency FEL characterization

In order to characterize the longitudinal properties of the FEL pulse jointly in both the time and the frequency domain, we introduce the Wigner distribution function,

$$W(z, t, \omega) \equiv \int_{-\infty}^{\infty} \tilde{E}(z, \omega - \frac{\Omega}{2}) \tilde{E}^*(z, \omega + \frac{\Omega}{2}) e^{i\Omega t} d\Omega,$$

(5.5)

where * denotes the complex conjugate and

$$\tilde{E}(z, \omega) \equiv \int_{-\infty}^{\infty} E(z, t) e^{i\omega t} dt,$$

(5.6)

is the Fourier transform of the field without worrying about the normalization. The Wigner function in Eq. (5.5) is useful to evaluate the expectation values and moments. In this regard, we will indicate with $< \hat{F} >$ any quantity evaluated as

$$< \hat{F} > = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt d\omega W(t, \omega, z) \hat{F}}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dt d\omega W(t, \omega, z)}.$$

(5.7)
For the FEL radiation, the Wigner function of $t$ and $\omega$ is evaluated numerically, while for the Gaussian seed laser an analytical expression can be derived. Figures 5.1(a), 5.1(b) and 5.1(c) show the Wigner function plots for the Gaussian laser seed of Eq. (5.2) for different values of the parameters that characterize the frequency chirp.

Figure 5.1: Seed laser Wigner function plots
5.2. TIME-FREQUENCY FEL CHARACTERIZATION

5.2.1 FEL expression

The expression in Eq. (5.3), together with the saddle point approximated Green function of Eq. (4.58), allows us to obtain a closed form expression for the FEL as the following:

\[
A(\hat{s}, \hat{z}) = E_0 \frac{i^{1/6}}{2\sqrt{\hat{z}}} e^{i\hat{a}/\hat{a}^2} \left[ \text{erf}i \left( \frac{b + a\hat{z}}{2\sqrt{a}} \right) - \text{erf}i \left( \frac{b}{2\sqrt{a}} \right) \right],
\]

(5.8)

where \text{erf}i is the complex error function, and the functions \(a(\hat{s}, \hat{z}, \hat{\alpha}, \hat{\beta})\), \(b(\hat{s}, \hat{z}, \hat{\alpha}, \hat{\beta})\), and \(c(\hat{s}, \hat{z}, \hat{\alpha}, \hat{\beta})\) are the following:

\[
a = -Q - iQ_1 - \frac{9i\hat{s}}{\hat{z}} - i\hat{\alpha} + i\frac{12\hat{s} - \hat{z}}{6}\hat{\beta} + i\frac{i\hat{s}^2\hat{\alpha}\hat{\beta}}{36} - \frac{i\hat{s}}{432}\hat{z}^2 \left( 1 + i\frac{\hat{s}}{\hat{z}} \right) \beta^2,
\]

(5.9)

\[
b = -iQ_c + 2(Q + iQ_1)(\hat{s} - \hat{s}_j) + 3\hat{s} + \frac{3}{2}\hat{z} + \frac{4\hat{s} + \hat{z}}{2}i\hat{\alpha} - \frac{1}{12}i\hat{\beta}(6\hat{s}\hat{z} + (6\hat{s} - \hat{z})(2\hat{s} + \hat{z}))
\]

\[
+ \frac{i\hat{s}}{72} \left( i\hat{s}^3 - 1 \right) \hat{\alpha}^2 - \frac{6\hat{s} + \hat{z}}{216}\hat{z}^2 \hat{\alpha}\hat{\beta} + \frac{i\hat{s}^2\hat{z}^2}{216}\beta^2 \left( \hat{s} + \hat{z} + 1 + \frac{6\hat{z}}{60} \right),
\]

(5.10)

\[
c = iQ_c(\hat{s} - \hat{s}_j) - (Q + iQ_1)(\hat{s} - \hat{s}_j)^2 - \frac{1}{2} + \frac{3\hat{s}^2}{4}\hat{z} - i\hat{s}\hat{z}\hat{\alpha} + \frac{i\hat{z}}{324}(162\hat{s}^2 + 9\hat{s}\hat{z} - 2\hat{z}^2) \hat{\beta}
\]

\[
+ \frac{i\hat{s}}{216}\hat{z}^2 \left( \hat{s} - \frac{1 - i\hat{s}}{12} \right) \hat{\alpha}\hat{\beta} + \frac{i\hat{s}^2\hat{z}^2}{2592} \left( \frac{1 - i\hat{s}}{45} \right) \left( \frac{5\hat{s}^2}{45} - 6\hat{z}^2 \right) - \hat{s}\hat{z} \left( 1 + \frac{1}{\hat{z}} \right) \hat{\beta}^2
\]

(5.11)

Regarding to the properties of the FEL light, such as the central frequency shift and the frequency chirp, it is useful to study the phase of the envelope of the radiation. For this purpose, Eq. (5.8) can be seen as the product of three functions: the exponential part \(e^{c - b^2/(4a)}\), the part involving the complex error functions, and the factor \(E_0 i^{1/6}/(2\sqrt{a}\sqrt{\hat{z}})\). Figure 5.2 shows an example for the phase of each term. The dependance on \(\hat{s}\) is given mostly by the exponential part, and the \text{erf}i part is almost constant where the amplitude of the envelope is large. For this reason the expression for the phase of the exponential part is given in equation (5.13):

\[
RE \equiv \Re \left( c - \frac{b^2}{4a} \right) = H + \frac{DF^2 - DG^2 - 2EFG}{D^2 + E^2}
\]

(5.12)

\[
IM \equiv \Im \left( c - \frac{b^2}{4a} \right) = I + \frac{EG^2 - EF^2 - 2DFG}{D^2 + E^2}
\]

(5.13)
where:

\[
D = -4Q_l \frac{18}{\zeta} - 4\hat{\alpha} + \left(8\hat{s} - \frac{2\hat{\zeta}}{3}\right)\hat{\beta} + \frac{1}{18} \hat{\zeta}^2 \hat{\alpha} \hat{\beta} - \hat{\zeta}^2 \hat{\beta}^2 \frac{\sqrt{3} + \hat{\zeta}}{216} \tag{5.14}
\]

\[
E = -4Q - \frac{18\sqrt{3}}{\zeta} - \frac{\hat{\zeta}^2 \hat{\alpha} \hat{\beta}}{6\sqrt{3}} + \hat{\zeta}^2 \hat{\beta}^2 \frac{1 + \sqrt{3} \hat{\zeta}}{216} \tag{5.15}
\]

\[
F = \frac{3\sqrt{3}}{2} + 2Q(\hat{s} - \hat{s}_j) + \frac{3}{\zeta} + \hat{\zeta} \hat{\alpha} \frac{1 - \sqrt{3} \hat{\zeta}}{144}
\]

\[
+ \hat{\zeta}^2 \hat{\alpha} \frac{6\hat{s} + \hat{\zeta}}{144\sqrt{3}} - \frac{\hat{\zeta}^2 \hat{\beta}^2}{432} \left(\hat{s} + \frac{\hat{\zeta}}{60}\right) \tag{5.16}
\]

\[
G = \frac{3}{2} - Q_c + 2Q_l(\hat{s} - \hat{s}_j) + \frac{4\hat{s} + \hat{\zeta}}{2} \hat{\alpha} + \hat{\zeta} \hat{\alpha}^2 \hat{\zeta} - \sqrt{3}
\]

\[
+ \frac{\hat{\zeta}^2 - 12\hat{s}^2 - 10\hat{s} \hat{\zeta}}{12} - \hat{\zeta}^2 \hat{\alpha} \frac{6\hat{s} + \hat{\zeta}}{144\sqrt{3}} - \hat{\zeta}^2 \hat{\beta}^2 \frac{\hat{s} + \frac{\hat{\zeta}}{60}}{432} \left(\hat{s} + \frac{1 + 3\sqrt{3} \hat{\zeta}}{60}\right) \tag{5.17}
\]

\[
H = Q_c(\hat{s} - \hat{s}_j) - Q_l(\hat{s} - \hat{s}_j)^2 + \frac{3\hat{\zeta}}{8} - \hat{s} \hat{\zeta} \hat{\alpha} + \frac{\hat{\zeta}}{324}(162\hat{s}^2 + 9\hat{s} \hat{\zeta} - 2\hat{\zeta}^2) \hat{\beta}
\]

\[
- \frac{\hat{\zeta}^4 \hat{\alpha} \hat{\beta}}{10368} + \frac{\sqrt{3} \hat{\zeta}^2}{432} \left(\hat{s} - \frac{\hat{\zeta} - 3\sqrt{3} \hat{s} \hat{\zeta}}{24}\right) \hat{\beta}
\]

\[
+ \frac{\hat{\beta}^2 \hat{\zeta}^2}{5184} \left(\frac{\sqrt{3} - \hat{\zeta}}{45} (4\hat{\zeta}^2 - 270\hat{s}^2)\right) - \hat{s} \hat{\zeta} \left(\sqrt{3} + \hat{\zeta}\right) \tag{5.18}
\]

\[
I = -Q(\hat{s} - \hat{s}_j)^2 - \frac{1}{2} + \frac{3\sqrt{3}}{8} \hat{\zeta} - \left(\hat{s} - \frac{\hat{\zeta}}{12} + \frac{\hat{\zeta}^2}{4\sqrt{3}}\right) \frac{\hat{\zeta}^2 \hat{\alpha} \hat{\beta}}{432}
\]

\[
+ \left(\frac{\sqrt{3} - 1}{45} \frac{4\hat{\zeta}^2 - 270\hat{s}^2}{432}\right) + \hat{s} \hat{\zeta} \left(1 + \sqrt{3} \hat{\zeta}\right) \frac{\hat{\zeta}^2 \hat{\beta}^2}{5184}. \tag{5.19}
\]

For an unchirped electron beam, Eq. (5.12) has the maximum at

\[
\hat{s}_{peak} = \hat{s}_j + \hat{\zeta} + \frac{1}{3\sqrt{3}} + \frac{3Q_l(\sqrt{3} + Q_c \hat{\zeta}) - Q \left(3 + \sqrt{3} Q_c \hat{\zeta}\right)}{6(6\sqrt{3} Q + Q^2 \hat{\zeta} + Q_l \hat{\zeta})}, \tag{5.20}
\]

we calculate the central frequency shift and frequency chirp with the first and the second derivatives on \(\hat{s}\) of the function in Eq. (5.13), and evaluate the at \(\hat{s} = \hat{s}_{peak}\). For a seed laser with \(Q_c = 0, Q_l = 0\, and \, \hat{s}_j = 0\, and using
5.2. TIME-FREQUENCY FEL CHARACTERIZATION

Figure 5.2: Comparison among the different contributions to the phase of Eq. (5.8) (blue dashed curve): phase of $e^{c-b^2/(4a)}$ as in Eq. (5.13) (yellow curve), phase of $E_0 t^{1/6}/(2\sqrt{a}\sqrt{z})$ (green line), and phase of $erfi[(b+a\hat{z})/(2\sqrt{a})] - erfi[b/(2\sqrt{a})]$ (red line). As a reference, the amplitude of Eq. (5.8) in arbitrary units is shown as the black curve. Phases are expressed in radians.

$K$ as defined in Eq. (5.4), we have:

\[
\frac{\partial IM}{\partial \hat{s}} = \left( \frac{24K}{2+3K} - 2\sqrt{3} \hat{z} + \frac{6 + 12K - 3\sqrt{3}(2+3K)\hat{z}}{1+3K(1+K)} \right) \frac{\hat{\alpha}}{12\sqrt{3}}
\]

\[
- \frac{3\sqrt{3}K}{2\hat{z}+3K\hat{z}} + \frac{\hat{\beta}}{72} \left( -\frac{10}{(2+3K)^2} + \frac{1+4\sqrt{3}\hat{z}}{2+3K} + 2(4 + \hat{z}^2) \right)
\]

\[
\frac{\partial^2 IM}{\partial \hat{s}^2} = - \frac{27K^2}{\hat{z}+3K\hat{z}+3K^2\hat{z}^2} + \frac{6K + 27K^2 + 36K^3 + 18K^4}{(1+3K(1+K))^2} \frac{\hat{\beta}}{\hat{\alpha}}
\]

\[
+ \left( 4 - \frac{14}{2+3K} \right) \frac{\hat{\beta}}{2\sqrt{3}} + \frac{2\sqrt{3} + (4+3K)\hat{z}}{6(1+3K(1+K))} \hat{\beta} - \frac{2K(1+2K)}{(1+3K(1+K))^2\sqrt{3}} \hat{\beta}
\]

\[
- \frac{2\sqrt{3}(1+6K)+3(2+3K)\hat{z}}{6(2+3K)(1+3K(1+K))^2} K^2 \hat{\beta} + \frac{1}{(1+3K(1+K))^3/9} \hat{\alpha}^2.
\]

These expressions are useful to estimate the central frequency shift and the frequency chirp of the FEL pulse at the end of the undulator.
5.3 Effects of the initial conditions on the FEL radiation

In this section we will consider different properties of the FEL radiation and discuss the impact of the initial conditions on the energy profile of the electrons.

5.3.1 FEL envelope distortions

Different behaviors of the FEL radiation, characterized by the value of $K$ introduced in Eq. (5.4) are considered in this section. Further, we show both the results obtained with the saddle point approximated Green function and with the exact series expansion. Figure 5.3 shows the amplitude of the FEL envelope of Eq. (5.8) for different values of $K$ and different values of the chirp and curvature parameters $\dot{\alpha}$ and $\dot{\beta}$. The seed laser parameters $Q_c$ and $Q_l$ are set to 0, and similarly the jitter $s_j$. Figures obtained with the saddle point approximated closed form for the FEL pulse are also compared with the envelope profiles obtained using the exact series expansion expression of Eq.(4.90). With our notation, a point moving at the velocity of the bunch is, at $\hat{s}$ coordinate equal to a constant, while the seed laser is moving at the speed of light and is centered at $\hat{s} = \hat{z}/2$. For $\dot{\alpha} = 0 = \dot{\beta}$, the FEL group velocity sets the FEL pulse center around $\hat{s} = \hat{z}/6$ [7].

Figure 5.3(a) illustrates the case with a very short seed laser. The peak position of the Green function amplitude is almost independent of the chirp and the curvature, so the FEL envelope has its maximum amplitude at nearly the same location ($\hat{s} \approx \hat{z}/6$) as for the case of $\dot{\alpha} = 0 = \dot{\beta}$. With a longer seed laser as in Fig. 5.3(b) and in Fig. 5.3(c), $\dot{\alpha}$ and $\dot{\beta}$ start to play a more important role, affecting both the group velocity and the gain of the FEL. In particular, Fig. 5.3(c) shows that the amplitude peak of the FEL is shifted from the coordinate $\hat{s} \approx \hat{z}/6$ for large values of $\dot{\alpha}$, which indicates that the group velocity is different due to the chirp and curvature parameters. This is due to the interference among the different phases of the Green function calculated at different $\hat{s}$ values.

Finally, Fig. 5.3(d) represents the case involving a very long seed laser. In this case, when the energy chirp and the energy curvature along the electron bunch are large, we obtain a strong interference among different parts of the electron bunch, thus leading to a heavy change of the FEL radiation
5.3. EFFECTS OF THE INITIAL CONDITIONS ON THE FEL

shape. In fact, Fig. 5.4(a) shows the amplitude and the phase of the Green function with the parameters of the blue curve in Fig. 5.3(d) for two different values of \( \hat{s} \). The black curve refers to the seed laser amplitude. Although in the case of the red curve, the maximum amplitude of the Green function matches the maximum amplitude of the seed laser, the amplitude of the radiation is lower compared to that of the blue curve case, as shown in Fig. 5.4(b). The reason is that the contributions given in the blue curve case are summed up more coherently compared to those of the red curve case, as it can be seen from the phase of the Green functions (dashed blue and red curves). This can also be seen from Fig. 5.4(b), which represents the quantity \( \int_0^x d\xi A(\hat{s} - \hat{\xi}, 0) g(\hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta}) \) in the complex plane, with \( x \) varying from 0 to \( \hat{z}/2 \).

5.3.2 Frequency shift and chirp of the FEL radiation

Figure 5.5 shows the frequency chirps estimated with the Eq. (5.22) multiplied by \(-\omega_0^2\rho^2\). For a short seed laser, Fig. 5.5(a) shows a linear behavior with respect to the both \( \hat{\alpha} \) and \( \hat{\beta} \) parameters. The effect of \( \hat{\alpha} \) on the linear frequency chirp is larger for shorter seeds. For a very short seed, the FEL can acquire a large intrinsic chirp, i.e. the FEL radiation is chirped even in case of an energy unchirped electron bunch at undulator entrance. Equation (5.22) can be simplified into the following expressions for very short seeds:

\[
\frac{\partial^2 IM}{\partial \hat{s}^2} = -\frac{9}{\hat{z}} + \frac{9}{K\hat{z}} + 2\hat{\alpha} + \left(2 - \frac{7}{3K} + \frac{\hat{z}\sqrt{3}}{6K}\right)\frac{\hat{\beta}}{\sqrt{3}} \quad (K \to \infty). \tag{5.23}
\]

So, even for \( \hat{\alpha} = 0 \), there is an intrinsic chirp equal to \(-9/\hat{z}\) existing along the FEL pulse. For the seed laser temporal duration similar to the temporal coherent duration determined by the Green function at the undulator exit, Eq. (5.22) can be simplified as

\[
\frac{\partial^2 IM}{\partial \hat{s}^2} = -\frac{27(2 + 5K)}{49\hat{z}} + \frac{3\hat{\alpha}}{343}(165 + 38K) + \frac{144K - 263}{21609}\hat{z}\hat{\alpha}^2 + \frac{302 - 141K}{1029}\hat{z}\hat{\beta} + \frac{1331K - 316}{1715\sqrt{3}}\hat{\beta} \quad (K \to 1). \tag{5.24}
\]

In this case the intrinsic chirp developed on the FEL is smaller, and approaches the value \(-27/(7\hat{z})\) for \( K \to 1 \), when \( \hat{\alpha} = 0 \). The effect on the
Figure 5.3: FEL Envelope amplitude of different $\hat{\alpha}$ and $\hat{\beta}$ values and for a fixed normalized undulator length. Envelope amplitude is expressed in arbitrary units.

longer seeds is represented in Fig. 5.5(b). Equation (5.22) for a long seed
5.3. EFFECTS OF THE INITIAL CONDITIONS ON THE FEL

(a) Seed laser amplitude (black), Green function for $\hat{s} = -0.83$: amplitude (blue solid) and phase (blue dashed), Green function for $\hat{s} = 1.85$: amplitude (red solid) and phase (red dashed). Amplitudes of the seed laser and the Green function are scaled. Phases are expressed in radians.

(b) $\int_0^\pi d\xi \hat{A} (\hat{s} - \hat{\xi}, 0) g (\hat{s}, \hat{z}, \hat{\xi}, \hat{\alpha}, \hat{\beta})$ in the complex plane, with $\hat{s} = -0.83$ (blue) and $\hat{s} = 1.85$ (red). To better compare the amplitude of the signals, the phase of each curve has been shifted by a constant.

Figure 5.4: Interference for the long seed laser case.

The laser can be simplified as

$$\frac{\partial^2 IM}{\partial \hat{s}^2} = 6K \hat{\alpha} + \frac{\hat{z} \hat{\alpha}^2}{9} (1 - 9K) + \frac{\hat{z} \hat{\beta}}{6} (4 - 9K) + \frac{K - 2}{4\sqrt{3}} \hat{\beta} \quad (K \to 0), \quad (5.25)$$

For $K = 0.001$, $\hat{\alpha}$ gives a small effect on the frequency chirp. The main contribution due to $\hat{\alpha}$ in Eq. (5.25) is the quadratic term, thus the chirp has the same sign for both positive and negative $\hat{\alpha}$; the effect of $\hat{\beta}$ on the frequency chirp is a linear one. Furthermore, the effect of $\hat{\beta}$ is larger compared to the effect of $\hat{\alpha}$, considering the order analysis introduced in the derivation of the saddle point approximated expression. With $K = 0.01$, $\hat{\alpha}$ and $\hat{\alpha}^2$ give comparable contributions to the chirp in Eq. (5.25). However, for a positive $\hat{\alpha}$, these two contributions are summed, while for a negative $\hat{\alpha}$ they tend to cancel with each other, thus yielding an asymmetrical behavior as shown in Fig. 5.5(b). The effect of $\hat{\beta}$ is linear, and considering the order analysis it is larger than the effect of $\hat{\alpha}$.
5.3.3 Velocity of the centroid of the FEL pulse

Another interesting property is the velocity of the FEL pulse travelling through an undulator in the seeded FEL process. As shown in Fig. 5.3, the envelope shape of the FEL pulse can change through the undulator. For this reason it is difficult to evaluate the group velocity of the FEL pulse. As a consequence, we introduce the centroid velocity \( v_c \) [35] which is the velocity of the centroid of the pulse:

\[
v_c(z) = \frac{\langle z \rangle}{\langle t \rangle} = \frac{z \int_{-\infty}^{+\infty} E(t, z) E^*(t, z) \, dt}{\int_{-\infty}^{+\infty} t E(t, z) E^*(t, z) \, dt}
\] (5.26)

To study the centroid velocity of the FEL pulse it is necessary to use the Green functions as exact series expansions. In fact, the approximated formulas have reliable results only in the exponential growth regime. In figure 5.6 we show the relative difference between the centroid velocity \( v_c \) and the velocity of light in the vacuum \( c \), for the two different regimes of lethargy and exponential growth. Note, in particular, that in the lethargy regime \( v_c \) is close to the velocity of light, while in the exponential growth regime it approaches \( v_{th} = \ldots \)
5.3. EFFECTS OF THE INITIAL CONDITIONS ON THE FEL

Figure 5.6: \((v_c - c)/c\) as a function of \(z\). The solid thick line represents the centrovelocity of the FEL pulse. For comparison, the velocity of light is dashed, the theoretical velocity in exponential growth \(v_{th}\) is dotted and the bunch velocity \(v_b\) is dash-dotted.

\[ v_b + (c - v_b)/3, \text{ where } v_b = \frac{\omega_0}{k_0 + k_w}. \]

Further, in figure 5.7 we show the effect of a linear chirp on the centrovelocity, using the Green function of Eq. 4.48. Note that the centrovelocity can exceed the velocity of light. This is due to the fact that not all the initial seed signal is amplified in the same way. When the head of the pulse is amplified more than the tail, we can find cases with centrovelocity greater than the velocity of light.

5.3.4 Impact of chirp and curvature on the cascade configuration

Figure 5.8 represents the schematic of an HG FEL: in the modulator a laser seed imprints an energy modulation on the electrons; in the dispersive section the energy modulation is converted to density modulation, so that the electron current distribution functions contains a frequency content at higher harmonics. However, due to the energy chirp and curvature, the electron bunching at the radiator entrance presents a quadratic phase that can have a strong impact on the FEL radiation.

To study the evolution of the FEL pulse along the radiator, we need to
determine an expression for the bunching at the radiator entrance. To this aim we consider an electron bunch with both energy linear chirp and curvature having a Gaussian energy distribution on a fixed phase coordinate, characterized by the rms Lorentz factor $\sigma_\gamma$. Through the first modulator the laser seed induces an energy modulation $\Delta \gamma$ on the Lorentz factor. The amplitude of the bunching factor for the $n$-th harmonic has been evaluated analytically in [4], for a given strength of the dispersive section $d\theta/d\gamma$ as

$$|B_n| = e^{-\frac{1}{2}n^2\sigma_\gamma^2 \left(\frac{d\theta}{d\gamma}\right)^2} J_n(n\Delta \gamma \frac{d\theta}{d\gamma})$$

(5.27)

with

$$\frac{d\theta}{d\gamma} = \frac{2\pi N_w}{\gamma_0} + \frac{k_0 + k_w}{\gamma_0} R_{56},$$

(5.28)

which includes the phase advance both in the modulator and the dispersion section [25], with $N_w$ being the number of periods in the modulator and $R_{56}$
characterizing the dispersion strength in the "chicane". In addition to the amplitude of the density modulation, we determine the phase accumulated through the modulator and the dispersive section due to the overall energy chirp and curvature on the electron bunch. For a short undulator, we can evaluate with good approximation the phase from the linearized pendulum equation \( \frac{d\theta}{dz} = k_w \frac{2}{\gamma_0^2} \). For a \( N_w \) periods modulator, this yields a contribution \( \Delta \theta_M \) to the bunching phase:

\[
\Delta \theta_M(\theta) = 4\pi \frac{\gamma(\theta) - \gamma_0}{\gamma_0} N_w. \tag{5.29}
\]

Similarly, the dispersive section gives its own contribution to the phase \( \Delta \theta_C \). For a given \( R_{56} \) value it can be calculated as

\[
\Delta \theta_C(\theta) = (k_0 + k_w) \frac{\gamma(\theta) - \gamma_0 R_{56}}{\gamma_0}. \tag{5.30}
\]

At the \( n \)-th harmonic, the bunching factor will be finally evaluated as

\[
B_n(\theta) = |B_n| e^{i[n(\Delta \theta_M(\theta) + \Delta \theta_C(\theta))]} \tag{5.31}
\]

When convoluted with a constant bunching at the undulator entrance, Eq. (4.92) gives an electric field envelope that grows linearly during the start-up and then evolves in the exponential growth regime, as shown in Figures 5.9 and 5.10. We consider now an HG FEL with the set of parameters employed in the FERMI@Elettra project. The modulator is 19 periods long, and it is tuned at a wavelength of 240 nm, the \( R_{56} \) parameter characterizing the dispersive section is 30 \( \mu m \) and the radiator is 400 periods long and is tuned on the 10-th harmonic of the modulator wavelength. We consider two possible initial distribution functions for the electrons at the modulator entrance, that have been generated with LiTrack [33] and an ideal flat-energy electron bunch. Fitting the particle distribution with a second order polynomial, we calculate the linear chirp and the quadratic curvature parameters needed for our calculations, which are reported in Table 5.1. The r.m.s. uncorrelated energy spread for electron distributions is set to 150 keV. The laser seed at the undulator entrance is supposed to be constant in amplitude, and longer than the electron bunch, and induces a modulation of 1.5 MeV.

Figures 5.11, 5.12 and 5.13 show the Wigner function plot for the three different bunches considered. The ideal flat bunch yields an unchirped FEL pulse with the shortest bandwidth. In the other cases, instead, the FEL
pulse presents a larger bandwidth and a strong frequency chirp. In fact, the curvature on the electron energies yields a parabolic behavior of the phase of the bunching at the radiator entrance, which gives a frequency chirped radiation. As shown in Fig. 5.13, the pulse generated by the bunch B2 presents a larger frequency chirp and a shorter temporal length compared to the pulse of B1 in Fig. 5.12 due to the larger curvature on the energies of the electrons.

5.4 Ultra short FEL pulses using a frequency chirp on the seed in single pass amplifier configuration

In this section we consider the effect on the FEL originated from a central frequency shift and from a frequency chirp on the Gaussian seed laser, by
5.4. ACHIEVING ULTRA-SHORT FEL PULSES

Figure 5.10: Electric field envelope peak growth as function of the undulator periods with log scale.

Figure 5.11: Wigner function plot for the FB case.

means of the $Q_c$ and $Q_l$ parameters introduced in Eq. (5.3). A laser frequency shift $Q_c$ degrades the gain of the FEL, especially in the case of a long seed laser, due to its small bandwidth. On the contrary, a short seed laser has a broad spectrum which can support the FEL start up even with a large central frequency shift. A frequency chirp on the seed laser gives it a phase curvature. With reference to Fig. 5.4(a), the addition of a phase curvature to the seed laser can lead to a larger FEL power for a particular slice with a certain $\hat{s}$ coordinate, while other slices far from that $\hat{s}$ value will result in a smaller amplitude. In this way, the FEL pulse can be shorter than the seeded laser pulse. In particular, the quantities $Q_c$ and $Q_l$ can be chosen in
such a way as to compensate the Green function phase curvature, when the seed laser peak amplitude matches the Green function peak. The values that satisfy this condition are:

\[
Q_c = \frac{\sqrt{3}}{z} - \frac{z\alpha}{2} - \frac{z - \sqrt{3}z\alpha^2}{144} - \frac{12 - 2\sqrt{3}z - 3z^2}{108}\beta \\
- \frac{z^2\alpha\beta}{216\sqrt{3}} + \frac{z^3\left(17\sqrt{3} + 3z\right)\beta^2}{77760} 
\]

\[
Q_l = -\frac{9}{2z} - \alpha + \frac{z^2\alpha\beta}{72} + \frac{4\sqrt{3} + 3z}{18}\beta - \frac{z^2\left(\sqrt{3} + z\right)}{864}\beta^2. 
\]

In this way, the phase of the convolution product is constant when the peak amplitude of the seed laser matches the peak of the Green function. In this
condition, as long as \( \hat{s} \) is far from the compensated slice, the phase of the Green function for different \( \hat{s} \) is not compensated, thus leading to a smaller FEL gain. Figure 5.14(a) shows the superposition of the phases for the seed laser and for the Green function. Figure 5.14(b) shows the comparison between the FEL envelope amplitude obtained for the unchirped seed laser and a chirp seed laser with \( Q_c \) and \( Q_l \) in Eqs. (5.32) and (5.33). In the latter case the FEL pulse is shorter compared to the seed laser. This result can be very useful for applications requiring a very short FEL pulse without constrains on the bandwidth.

Figure 5.14: Frequency chirp and central frequency shift on the seed laser with \( K = 0.01, \hat{z} = 10, \hat{\alpha} = 0.05, \) and \( \hat{\beta} = -10^{-3}. \)
Chapter 6

Conclusion

In this work we described the effects of the resistive wall wakefields in an elliptical cross-section undulator vacuum chamber, and derived tools to study the impact of a parabolic energy profile on a seeded FEL process and on Harmonic Generation (HG) cascade configuration.

The first part of the study concerns on the analytical derivation of the longitudinal and transversal wakefields in an elliptical cross-section vacuum chamber[17]. The analytical formula allows to evaluate the resistive wall coupling impedance for an arbitrary elliptical aspect-ratio beam pipe, choosing the position of the leading charge on the cross section and characterizing the metal wall through the finite conductivity and the electron relaxation time. This new formula gives reliable results at high frequencies, using the AC conductivity model, while the analytical results obtained in the literature using the wall impedance, give reliable results only for low frequencies. The wakefields are then calculated numerically inverting the Fourier domain expressions, and accurate results are obtained both in the short range and in the long range. From the wake functions behavior, we pointed out that the copper chambers present higher oscillations compared to the aluminum chambers. This is due to the higher electron relaxation time of the copper. Higher electron relaxation time gives strong oscillations in the short range for both the longitudinal and the transverse wakes. This is particularly important for the medium and the short bunch configurations, while there is almost no effect in case of long bunches. The results show that the energy variation induced within the bunch, for a fixed bunch charge, is larger for short electron bunches, and can take an unacceptably large value. Moreover the electromagnetical field in case of chambers with large aspect ratios is
CHAPTER 6. CONCLUSION

confined in the center of the chamber, thus allowing eventually to open it on the long axis side without modifying the field configuration.

In the second part of our study, we considered the FEL theory within the one dimensional Vlasov-Maxwell framework to evaluate the impact, on the radiated light along the undulator, produced by the parabolic energy profile of the electron beam. Initial conditions have been introduced to take into account of both the parabolic energy profile and an initial electron density modulation at the radiation wavelength. This led to an integral representation in the Laplace domain, where the FEL is expressed as a convolution of the source terms with a time-dependent Green function. The Green functions are derived performing an analytical inverse Laplace transform. We first obtained a Green function for the FEL in seeded amplifier configuration with both energy chirp and curvature, using a saddle point approximation method [26]. This Green function allows to predict the FEL radiation with good approximation in the exponential growth regime. We also evaluated analytically properties of the Green function such as the temporal duration and the bandwidth, and pointed out that the linear energy chirp and the curvature have a strong effect on the Green function phase. Further, since the Green function can be convoluted analytically with a Gaussian seed laser with eventually a linear frequency chirp on it, it is useful to study effects such as the central frequency shift and the frequency chirp on the radiation. This study is reported in [27] and shows that the impact of the energy profile depends on the ratio between the temporal length of the Green function and the temporal length of the seed laser. In particular, we pointed out that the quadratic curvature gives a frequency chirp on the FEL light and that it has a stronger impact for the long seed cases. Conversely, a linear energy chirp on the electrons produces a central frequency shift in the long seed cases and a frequency chirp in the short seed cases. In the latter cases its effect is stronger compared to the curvature effect. This effect depends on the Green function phase, that determines in which way different amplified parts of the seed are summed up along the undulator.

Further, we derived the exact expressions of the Green functions for the seeded amplifier configuration, both in the case of a linear energy chirp on the electrons [32] and in the case where both linear energy chirp and quadratic curvature are present. The exact solution is given in the form of a series expansion and allows to obtain more accurate results compared to the saddle point approximated solution. Exact solutions present a Dirac delta pulse that represents the seed laser travelling through the undulator chamber at the
velocity of light. This allows to characterize correctly the envelope evolution for both the lethargy regime and the exponential growth regime, and to correctly evaluate the velocity of the centroid of the FEL pulse. Finally, exact formulas can accept larger values for the linear chirp and quadratic curvature parameters, compared to the saddle point approximated formula.

We also derived a Green function for an initial electron density modulation. Such Green function is useful to evaluate the impact of the energy profile in the harmonic generation cascade configuration. In our work we provide formulas to evaluate the initial density modulation at the radiator entrance having an electron bunch with parabolic energy profile at the modulator entrance, further we show that an initial energy curvature is responsible of a frequency chirp on the FEL light in the cascade configuration and of a bandwidth growth. Finally we proposed a method to obtain ultra-short FEL pulses using a linear frequency chirp on the seed laser and a suitable parabolic energy profile for the electron bunch.

The Green functions have been derived in the hypothesis of negligible uncorrelated energy spread. It will be interesting, as an extension of this work to consider the use of an energy electron distribution which is not in the form of a Dirac delta function. The use of a Gaussian energy distribution will give a more realistic electron phase space evolution and even more accurate results for the FEL envelope along the undulator.
Bibliography


[33] K. Bane and P. Emma, PAC 2005, 4266-4268, Knoxville
