A Note on Non-homogeneous Hyperbolic Operators with Low Regularity Coefficients

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Abstract. In this paper we obtain an energy estimate for a complete strictly hyperbolic operator with second order coefficients satisfying a log-Zygmund-continuity condition with respect to \( t \), uniformly with respect to \( x \), and a log-Lipschitz-continuity condition with respect to \( x \), uniformly with respect to \( t \).

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1. Introduction

Let us consider the second order operator

\[
P = \partial_t^2 - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t)\partial_{x_j})
\]

and suppose that \( P \) is strictly hyperbolic, i.e. there exist two positive constants \( \lambda_0 \leq \Lambda_0 \) such that

\[
\lambda_0 |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t) \xi_i \xi_j \leq \Lambda_0 |\xi|^2
\]

for all \( t \in \mathbb{R} \) and all \( \xi \in \mathbb{R}^n \).

It is well-known (see e.g. [5] and [8]) that, if the coefficients \( a_{ij} \) are Lipschitz-continuous, then the following energy estimate holds for the operator \( P \): for all \( s \in \mathbb{R} \), there exists \( C_s > 0 \) such that

\[
\sup_{t \in [0,T]} (\|u(t,\cdot)\|_{H^{s+1}} + \|\partial_t u(t,\cdot)\|_{H^s}) \leq

\leq C_s \left( \|u(0,\cdot)\|_{H^{s+1}} + \|\partial_t u(0,\cdot)\|_{H^s} + \int_0^T \|Pu(t,\cdot)\|_{H^s} \, dt \right)
\]
for every function $u \in C^2([0, T]; H^\infty(\mathbb{R}^n))$.

In particular, the previous energy estimate implies that the Cauchy problem for (1) is well-posed in the space $H^\infty$, with no loss of derivatives.

On the contrary, if the coefficients $a_{ij}$ are not Lipschitz-continuous, then (3) is no more true in general, as it is shown by an example given by Colombini, De Giorgi and Spagnolo in the paper [2]. Nevertheless, under suitable weaker regularity assumptions on the coefficients, one can recover the $H^\infty$-well-posedness again, but this time from an energy estimate with loss of derivatives.

A first result of this type was obtained in the quoted paper [2]. The authors supposed that there was a constant $C > 0$ such that, for all $\varepsilon \in [0, T]$,

$$
\int_0^{T-\varepsilon} |a_{ij}(t + \varepsilon) - a_{ij}(t)| \, dt \leq C \varepsilon \log \left( 1 + \frac{1}{\varepsilon} \right).
$$

(4)

The Fourier transform with respect to $x$ of the equation, together with the new “approximate energy technique” (i.e. the approximation of the coefficients is different in different zones of the phase space), enabled them to obtain the following energy estimate: there exist strictly positive constants $K$ (independent of $s$) and $C_s$ such that

$$
\sup_{t \in [0, T]} (\|u(t, \cdot)\|_{H^{s+1}} + \|\partial_t u(t, \cdot)\|_{H^{s-1}}) \leq
$$

$$
\leq C_s \left( \|u(0, \cdot)\|_{H^{s+1}} + \|\partial_t u(0, \cdot)\|_{H^{s}} + \int_0^T \|P u(t, \cdot)\|_{H^{s}} \, dt \right)
$$

(5)

for all $u \in C^2([0, T]; H^\infty(\mathbb{R}^n))$.

Considering again the case that the coefficients of $P$ depend only on the time variable, in the recent paper [7] (see also [9]) Tarama has weakened the regularity hypothesis further, supposing a log-Zygmund type integral condition, i.e. that there exists a constant $C > 0$ such that, for all $\varepsilon \in [0, T/2]$,

$$
\int_{\varepsilon}^{T-\varepsilon} |a_{ij}(t + \varepsilon) + a_{ij}(t - \varepsilon) - 2a_{ij}(t)| \, dt \leq C \varepsilon \log \left( 1 + \frac{1}{\varepsilon} \right).
$$

(6)

Nevertheless, he has been able to prove the well-posedness to the Cauchy problem for (1) in the space $H^\infty$: the improvement with respect to [2] was obtained introducing a new type of approximate energy, which involves the second derivatives of the approximating coefficients.

Much more difficulties arise if the operator $P$ has coefficients depending both on the time variable $t$ and on the space variables $x$. This case was considered by Colombini and Lerner in the paper [4]. They supposed a pointwise log-Lipschitz regularity condition, i.e. that there exists $C > 0$ such that, for all
Because the coefficients of the operator $P$ depend also on the space variables, here the Littlewood-Paley dyadic decomposition with respect to $x$ takes the place of the Fourier trasform, and it is, togheter with the approximate energy technique, the key tool to obtain the energy estimate: for all fixed $\theta \in [0, 1/4]$, there exist $\beta, C > 0$ and $T^* \in [0, T]$ such that

$$\|u(t, \cdot)\|_{H^{-\theta+1}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta-1}} \leq C \left( \|u(0, \cdot)\|_{H^{-\theta+1}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|P u(t, \cdot)\|_{H^{-\theta-1}} \, dt \right)$$

for all $u \in C^2([0, T^*]; H^\infty(\mathbb{R}^n))$.

In this case, the loss of derivatives gets worse with the increasing of time.

In a recent paper ([3]), Colombini and Del Santo considered the case of one space variable (i.e. $n = 1$) and studied again the case of the coefficient $a$ depending both on $t$ and $x$, but under a special regularity condition: they mixed condition (6) togheter with (7). In particular, they supposed $a$ to be log-Zygmund-continuous with respect to $t$, uniformly with respect to $x$, and log-Lipschitz-continuous with respect to $x$, uniformly with respect to $t$. The dyadic decomposition technique and the Tarama’s approximate energy enabled them to obtain an estimate similar to (8).

The reason why they focused on the case $n = 1$ is that the case of several space variables needs some different and new ideas in the definition of the microlocal energy: this point still remains as an open problem.

In the present note, we will consider the case of the non-homogeneous operator

$$Lu = \partial^2_t u - \partial_x (a(t, x) \partial_x u) + b_0(t, x) \partial_t u + b_1(t, x) \partial_x u + c(t, x) u,$$

where the coefficient $a$ satisfy the same regularity assumptions as in [3]. We will also suppose that $b_0, b_1 \in L^\infty(\mathbb{R}_t; \mathcal{C}^\omega(\mathbb{R}_x)), \omega > 0$, and $c$ bounded on $\mathbb{R}_t \times \mathbb{R}_x$. We will apply the Littlewood-Paley decomposition and the Tarama’s approximate energy again to obtain an energy estimate with a loss of derivatives that depends on $t$, as in (8).

One can find the estimate of the second order coefficient $a$ in the paper [3], however, for reader’s convenience, we will give here all the details.
2. Main result

Let \( a : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a function such that, for positive constants \( \lambda_0 \leq \Lambda_0 \) and \( C_0 \), one has, for all \( (t, x) \in \mathbb{R}^2 \) and all \( \tau > 0, y > 0 \),

\[
\lambda_0 \leq a(t, x) \leq \Lambda_0 \quad (10)
\]

\[
\sup_{(t, x)} |a(t + \tau, x) + a(t - \tau, x) - 2a(t, x)| \leq C_0 \tau \log \left( \frac{1}{\tau} + 1 \right) \quad (11)
\]

\[
\sup_{(t, x)} |a(t, x + y) - a(t, x)| \leq C_0 y \log \left( \frac{1}{y} + 1 \right) \quad (12)
\]

Moreover, let

\[
b_0, b_1 \in L^\infty(\mathbb{R}_t; C^\omega(\mathbb{R}_x)) \quad (13)
\]

where \( \omega > 0 \), and

\[
c \in L^\infty(\mathbb{R}_t \times \mathbb{R}_x) \quad (14)
\]

**Theorem 2.1.** Let us consider, on the whole space \( \mathbb{R}^2 \), the operator

\[
Lu = \partial^2_t u - \partial_x (a(t, x) \partial_x u) + b_0(t, x) \partial_t u + b_1(t, x) \partial_x u + c(t, x) u \quad (15)
\]

where the coefficients \( a, b_0, b_1 \) and \( c \) satisfy the hypothesis (10)-(14).

Then, for all fixed

\[
\theta \in \left[ 0, \min \left\{ \frac{1}{2}, \frac{\omega}{1 + \log 2} \right\} \right],
\]

there exist \( \beta^* > 0 \), a time \( T \in \mathbb{R} \) and a constant \( C > 0 \) such that

\[
\sup_{[0, T]} (\|u(t, \cdot)\|_{H^{1-\theta - \beta^*}} + \|\partial_t u(t, \cdot)\|_{H^{-\theta - \beta^*}}) \leq C \left( \|u(0, \cdot)\|_{H^{1-\theta}} + \|\partial_t u(0, \cdot)\|_{H^{-\theta}} + \int_0^T \|Lu(t, \cdot)\|_{H^{1-\theta - \beta^*}} \, dt \right) \quad (16)
\]

for all \( u \in C^2([0, T]; H^\infty(\mathbb{R}_x)) \).

3. Proof of Theorem 2.1

3.1. Approximation of the coefficient \( a(t, x) \)

Let \( \rho \in C^\infty_0(\mathbb{R}) \) be an even function such that:

1. \( 0 \leq \rho \leq 1 \)
2. \( \text{supp } \rho \subset [-1, 1] \)
3. \( \int \rho(s) \, ds = 1 \)
4. \( |\rho'(s)| \leq 2; \)
for all $0 < \varepsilon \leq 1$, we set $\rho_\varepsilon(s) = \frac{1}{\varepsilon} \rho\left(\frac{s}{\varepsilon}\right)$.

Then, for all $0 < \varepsilon \leq 1$, we define

$$a_\varepsilon(t, x) := \int_{\mathbb{R}_t \times \mathbb{R}_x} \rho_\varepsilon(t - s) \rho_\varepsilon(x - y) a(s, y) \, dsdy.$$  \hspace{1cm} (17)

**Lemma 3.1.** The following inequalities hold true:

1. for all $\varepsilon \in [0, 1]$, for all $(t, x) \in \mathbb{R}^2$, one has

$$\lambda_0 \leq a_\varepsilon(t, x) \leq \Lambda_0;$$  \hspace{1cm} (18)

2. for all $\varepsilon \in [0, 1]$, one has

$$\sup_{(t, x)} |a_\varepsilon(t, x) - a(t, x)| \leq \frac{3}{2} C_0 \varepsilon \log \left(\frac{1}{\varepsilon} + 1\right);$$  \hspace{1cm} (19)

3. for all $\sigma \in [0, 1]$, a constant $c_\sigma > 0$ exists such that, for all $\varepsilon \in [0, 1]$,

$$\sup_{(t, x)} |\partial_\sigma a_\varepsilon(t, x)| \leq c_\sigma (\Lambda_0 + C_0) \varepsilon^{\sigma - 1};$$  \hspace{1cm} (20)

4. for all $\varepsilon \in [0, 1]$, one has

$$\sup_{(t, x)} |\partial_t a_\varepsilon(t, x)| \leq C_0 \|\rho'\|_{L^1} \log \left(\frac{1}{\varepsilon} + 1\right);$$  \hspace{1cm} (21)

$$\sup_{(t, x)} |\partial_t^2 a_\varepsilon(t, x)| \leq \frac{C_0}{2} \|\rho''\|_{L^1} \frac{1}{\varepsilon} \log \left(\frac{1}{\varepsilon} + 1\right);$$  \hspace{1cm} (22)

$$\sup_{(t, x)} |\partial_t \partial_\sigma a_\varepsilon(t, x)| \leq C_0 \|\rho'\|_{L^1}^2 \frac{1}{\varepsilon} \log \left(\frac{1}{\varepsilon} + 1\right).$$  \hspace{1cm} (23)

**Proof.** Inequalities in (18) immediately follow from the fact that $|\rho| \leq 1$.

Relation (19), instead, follows from (11), after one has observed that

$$a_\varepsilon(t, x) - a(t, x) = \frac{1}{2} \int_{\mathbb{R}_x} \rho_\varepsilon(s) \int_{\mathbb{R}_y} \rho_\varepsilon(x - y)(a(t + s, y) + a(t - s, y) - 2a(t, y)) \, dyds,$$

where we have used the fact that $\rho$ is an even function.

Moreover, one has

$$\partial_t^2 a_\varepsilon(t, x) = \frac{1}{2} \int \rho''_\varepsilon(s) \int \rho_\varepsilon(x - y)(a(t + s, y) + a(t - s, y) - 2a(t, y)) \, dyds,$$

from which one can deduce (22).
Inequalities (21) and (23) derive from (12) in a very similar way. Finally, relation (20) is a consequence of the fact that (10) and (11) imply that for all \( \sigma \in [0,1] \), a constant \( c'_{\sigma} > 0 \) exists such that, for all \( \tau > 0 \), one has
\[
\sup_{(t,x)} |a(t + \tau, x) - a(t, x)| \leq c'_{\sigma} (A_0 + C_0) \tau^\sigma .
\]
(24)

3.2. Littlewood-Paley decomposition

We collect here some well-known facts about dyadic decomposition, referring to [1], [4] and [6] for the details.

Let \( \varphi_0 \in C_0^\infty(\mathbb{R}_\xi) \) be an even function, decreasing on \([0, +\infty[\), such that \( 0 \leq \varphi_0 \leq 1 \) and
\[
\varphi_0(\xi) = 1 \text{ if } |\xi| \leq 1 , \quad \varphi_0(\xi) = 0 \text{ if } |\xi| \geq 2.
\]

We set \( \varphi(\xi) = \varphi_0(\xi) - \varphi_0(2\xi) \) and, for \( \nu \in \mathbb{N} \setminus \{0\} \), \( \varphi_\nu(\xi) := \varphi(2^{-\nu} \xi) \).

For a tempered distribution \( u \in H^{-\infty}(\mathbb{R}) \), we define
\[
u(x) := \varphi_\nu(D_x)u(x) = \frac{1}{2\pi} \int e^{ix\xi} \varphi_\nu(\xi) \hat{u}(\xi) d\xi = \frac{1}{2\pi} \int \varphi_\nu(y) u(x - y) dy;
\]
for all \( \nu \), \( u_\nu \) is an entire analytic function belonging to \( L^2 \).

Moreover, for all \( s \in \mathbb{R} \) there exists a constant \( C_s > 0 \) such that
\[
1 \frac{1}{C_s} \sum_{\nu=0}^{+\infty} 2^{2\nu s} \|u_\nu\|^2_{L^2} \leq \|u\|^2_{H^s} \leq C_s \sum_{\nu=0}^{+\infty} 2^{2\nu s} \|u_\nu\|^2_{L^2}
\]
(25)
and the following inequalities (called “Bernstein’s inequalities”) hold:
\[
\|\partial_x u_\nu\|_{L^2} \leq 2^{\nu+1} \|u_\nu\|_{L^2} \quad \text{for all } \nu \geq 0
\]
(26)
\[
\|u_\nu\|_{L^2} \leq 2^{-(\nu-1)} \|\partial_x u_\nu\|_{L^2} \quad \text{for all } \nu \geq 1.
\]
(27)

We end this subsection quoting a result which will be useful in the following; for its proof, see [4]. We denote with \([A,B]\) the commutator between two linear operators \( A \) and \( B \) and with \( L(L^2) \) the space of bounded linear operators from \( L^2 \) to \( L^2 \).

**Lemma 3.2.** 1. There exist \( C > 0, \nu_0 \in \mathbb{N} \) such that, for all \( a \in L^\infty(\mathbb{R}) \) satisfying
\[
\sup_{x \in \mathbb{R}} |a(x + y) - a(x)| \leq C_0 y \log \left( 1 + \frac{1}{y} \right)
\]
for all \( y > 0 \), one has, for all \( \nu \geq \nu_0 \),
\[
\| [\varphi_\nu(D_x), a(x)] \|_{L(L^2)} \leq C (\|a\|_{L^\infty} + C_0) 2^{-\nu} \nu.
\]
(28)
2. There exist $C > 0$, $\nu_0 \in \mathbb{N}$ such that, for all $b \in C^\infty(\mathbb{R})$ and all $\nu \geq \nu_0$, one has
\[
\| [\varphi_\nu(D_x),b(x)] \|_{C^\omega(L^2)} \leq C \|b\|_{C^\omega} 2^{-\nu}.
\] (29)

3.3. Approximate and total energy

Let $T_0 > 0$ and $u \in C^2([0,T_0];H^\infty(\mathbb{R}))$. If we set $u_\nu(t,x) = \varphi_\nu(D_x)u(t,x)$, we obtain
\[
(Lu)_\nu = \partial_t^2 u_\nu - \partial_x(a(t,x)\partial_x u_\nu) - \partial_x([\varphi_\nu(D_x),a]\partial_x u) + b_0(t,x)\partial_t u + b_1(t,x)\partial_x u + [\varphi_\nu(D_x),b_0]\partial_t u + [\varphi_\nu(D_x),b_1]\partial_x u + c(t,x)u + [\varphi_\nu(D_x),c]u.
\] (30)

Now, we introduce the approximate energy of $u_\nu$ (see [3] and [7]), setting
\[
ev_{\nu,\epsilon}(t) := \int_\mathbb{R} \left( \frac{1}{\sqrt{a_\epsilon}} \left| \partial_t u_\nu + \frac{\partial_x \sqrt{a_\epsilon}}{2\sqrt{a_\epsilon}} u_\nu \right|^2 + \sqrt{a_\epsilon} |\partial_x u_\nu|^2 + |u_\nu|^2 \right) dx,
\] (31)
and, taken $\theta$ as in the hypothesis of Theorem 2.1, we define the total energy of $u$: \[
E(t) := \sum_{\nu=0}^{+\infty} e^{-2\beta(\nu+1)t} 2^{-2\nu \theta} e_{\nu,2^{-\nu}}(t),
\] (32)
where $\beta > 0$ will be fixed later on.

**Remark 3.3.** From (25) and Bernstein’s inequalities (26)-(27), it’s easy to see that there exist two positive constants $C_\theta$ and $C'_\theta$ such that
\[
E(0) \leq C_\theta (\|\partial_t u(0,\cdot)\|_{H^{-\theta}} + \|u(0,\cdot)\|_{H^{1-\theta}})
\]
\[
E(t) \geq C'_\theta (\|\partial_t u(t,\cdot)\|_{H^{-\theta-\beta^*}} + \|u(t,\cdot)\|_{H^{1-\theta-\beta^*}})
\]
where we have set $\beta^* = \beta(\log 2)^{-1}$.

First, we derive $e_{\nu,\epsilon}$, defined by (31), with respect to the time variable, and
we obtain

\[
\frac{d}{dt} \psi_{\nu, \varepsilon}(t) = \int \frac{2}{\sqrt{a_\varepsilon}} \Re \left( \partial_t^2 u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_k \sqrt{a_\varepsilon}}{2 \sqrt{a_\varepsilon}} u_\nu \right) \right) dx \\
+ \int \frac{2}{\sqrt{a_\varepsilon}} \Re \left( R_\nu u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_k \sqrt{a_\varepsilon}}{2 \sqrt{a_\varepsilon}} u_\nu \right) \right) dx \\
+ \int \partial_k \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 dx \\
+ \int 2 \sqrt{a_\varepsilon} \Re (\partial_x u_\nu \cdot \partial_x \partial_t u_\nu) dx \\
+ \int 2 \Re (u_\nu \cdot \partial_t u_\nu) dx ,
\]

where \( R_\nu v = \partial_t \left( \frac{\partial_k \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}} \right) v - \left( \frac{\partial_k \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}} \right)^2 v \). Now, we can put in the previous relation the value of \( \partial_t^2 u_\nu \), given by (30).

Integrating by parts and taking advantage of the spectral localisation of \( u_\nu \), we have

\[
\int \frac{2}{\sqrt{a_\varepsilon}} \Re \left( \partial_x (a \partial_x u_\nu) \cdot \left( \partial_t u_\nu + \frac{\partial_k \sqrt{a_\varepsilon}}{2 \sqrt{a_\varepsilon}} u_\nu \right) \right) dx =
\]

\[
= \int \frac{2 \partial_x \sqrt{a_\varepsilon}}{a_\varepsilon} a \Re \left( \partial_x u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_k \sqrt{a_\varepsilon}}{2 \sqrt{a_\varepsilon}} u_\nu \right) \right) dx \\
- \int \frac{\partial_k \sqrt{a_\varepsilon}}{a_\varepsilon} \frac{a}{\sqrt{a_\varepsilon}} |\partial_x u_\nu|^2 dx \\
- \int \frac{2}{\sqrt{a_\varepsilon}} \Re (\partial_x u_\nu \cdot \partial_x \partial_t u_\nu) dx \\
- \int \frac{a}{\sqrt{a_\varepsilon}} \partial_x \left( \frac{\partial_k \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}} \right) \Re (\partial_x u_\nu \cdot \partial_x u_\nu) dx \\
\]

taken care of the fact that \( b_0, b_1 \) and \( c \) are real-valued, finally we obtain the
complete expression for the time derivative of the approximate energy:

\[
\frac{d}{dt} e_{\nu,\varepsilon}(t) = \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( (Lu)_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
+ \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( R_{\nu} u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
+ \int \partial_t \sqrt{a_{\varepsilon}} \left( 1 - \frac{a}{a_{\varepsilon}} \right) |\partial_x u_\nu|^2 dx \\
+ \int 2 \left( \sqrt{a_{\varepsilon}} - \frac{a}{a_{\varepsilon}} \right) \Re(\partial_x u_\nu \cdot \partial_x \partial_t u_\nu) dx \\
+ \int 2 \partial_x \sqrt{a_{\varepsilon}} a \Re \left( \partial_x u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{a}{\sqrt{a_{\varepsilon}}} \partial_x \left( \frac{\partial_t \sqrt{a_{\varepsilon}}}{\sqrt{a_{\varepsilon}}} \right) \Re(\partial_x u_\nu \cdot \nu_\nu) dx \\
+ \int 2 \Re(u_\nu \cdot \partial_t u_\nu) dx \\
+ \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( (\partial_x ([\varphi_\nu(D_x), a] \partial_x u)) \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} b_0(t, x) \Re \left( \partial_t u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( ([\varphi_\nu(D_x), b_0] \partial_x u) \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} b_1(t, x) \Re \left( \partial_x u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( ([\varphi_\nu(D_x), b_1] \partial_x u) \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} c(t, x) \Re \left( u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx \\
- \int \frac{2}{\sqrt{a_{\varepsilon}}} \Re \left( ([\varphi_\nu(D_x), c] u) \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{\varepsilon}}}{2\sqrt{a_{\varepsilon}}} u_\nu \right) \right) dx.
\]
3.4. Estimate for the approximate energy

We want to obtain an energy estimate; so, let us start to control each term of (33).
Through the rest of the proof, we will denote with $C$, $C'$, $C''$ and $\hat{C}$ constants depending only on $\lambda_0$, $\Lambda_0$, $C_0$ and on the norms of the coefficients of the lower order terms of the operator $L$ in their respective functional spaces, and which are allowed to vary from line to line.

3.4.1. Terms with $a$ and $a_\varepsilon$

Thanks to relations (10), (20) with $\sigma = 1/2$, (22) and Bernstein’s inequalities, we deduce that there exists $C > 0$, depending only on $\lambda_0$, $\Lambda_0$ and $C_0$, such that, for all $\nu \in \mathbb{N}$,

$$\left| \int \frac{2}{\sqrt{a_\varepsilon}} \Re \left( R_\varepsilon u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_\varepsilon u_\nu}{2\sqrt{a_\varepsilon}} \right) \right) \right| \leq C \frac{1}{\varepsilon} \log \left( \frac{1}{\varepsilon} + 1 \right) 2^{-\nu} e_{\nu, \varepsilon}(t) .$$

In the same way, from (10), (19) and (20), we have

$$\left| \int \partial_x \sqrt{a_\varepsilon} \left( 1 - \frac{a}{a_\varepsilon} \right) |\partial_x u_\nu|^2 \right| \leq C \log \left( \frac{1}{\varepsilon} + 1 \right) e_{\nu, \varepsilon}(t) ,$$

for a constant $C$ depending again only on $\lambda_0$, $\Lambda_0$ and $C_0$.

Moreover, again from (10) and (19) and Bernstein’s inequalities, we obtain

$$\left| \int 2 \left( \sqrt{a_\varepsilon} - \frac{a}{\sqrt{a_\varepsilon}} \right) \Re (\partial_x u_\nu \cdot \partial_x \partial_t u_\nu) \right| \leq C \varepsilon \log \left( \frac{1}{\varepsilon} + 1 \right) \| \partial_x u_\nu \|_{L^2} \| \partial_x \partial_t u_\nu \|_{L^2} \leq C \varepsilon \log \left( \frac{1}{\varepsilon} + 1 \right) 2^{\nu+1} \| \partial_x u_\nu \|_{L^2} \| \partial_t u_\nu \|_{L^2} ;$$

but we have

$$\| \partial_t u_\nu \|_{L^2} \leq \left\| \partial_t u_\nu + \frac{\partial_\varepsilon u_\nu}{2\sqrt{a_\varepsilon}} \right\|_{L^2} + \left\| \frac{\partial_t \sqrt{a_\varepsilon} u_\nu}{2\sqrt{a_\varepsilon}} \right\|_{L^2}$$

and

$$\left\| \frac{\partial_t \sqrt{a_\varepsilon} u_\nu}{2\sqrt{a_\varepsilon}} \right\|_{L^2} \leq C \varepsilon^{-1/2} \| u_\nu \|_{L^2}$$

$$\left\| \frac{\partial_t \sqrt{a_\varepsilon} u_\nu}{2\sqrt{a_\varepsilon}} \right\|_{L^2} \leq C \varepsilon^{-1/2} 2^{-\nu} \| u_\nu \|_{L^2} \quad (\nu \geq 1) ,$$
that give us the following:

$$\left| \int 2 \left( \sqrt{a_\varepsilon} - \frac{a}{\sqrt{a_\varepsilon}} \right) \Re(\partial_x u_\nu \cdot \partial_x \overline{u_\nu}) \, dx \right| \leq C(\varepsilon 2^\nu + 1) \log \left( \frac{1}{\varepsilon} + 1 \right) e_{\nu, \varepsilon}(t).$$

In a very similar way, from (21) one has

$$\left| \int \frac{\partial_x \sqrt{a_\varepsilon}}{a_\varepsilon} \Re \left( \partial_x u_\nu \cdot \left( \frac{\partial_t \overline{u_\nu}}{\sqrt{a_\varepsilon}} + \frac{\partial_x \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right) \right) \, dx \right| \leq C \log \left( \frac{1}{\varepsilon} + 1 \right) e_{\nu, \varepsilon}(t);$$

moreover, from (20) with $\sigma = 1/2$, (21) and (23) we deduce

$$\left| \int \frac{\partial_x \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}} \partial_x \left( \frac{\partial_t \sqrt{a_\varepsilon}}{\sqrt{a_\varepsilon}} \right) \Re(\partial_x u_\nu \cdot \overline{u_\nu}) \, dx \right| \leq C \frac{1}{\varepsilon} \log \left( \frac{1}{\varepsilon} + 1 \right) 2^{-\nu} e_{\nu, \varepsilon}(t).$$

Finally, we have

$$\left| \int 2 \Re(u_\nu \cdot \overline{\partial_t u_\nu}) \, dx \right| \leq C \varepsilon^{-1/2} 2^{-\nu} e_{\nu, \varepsilon}(t).$$

3.4.2. Terms with $b_0$, $b_1$ and $c$

Thanks to the hypothesis (13)-(14), one has that there exist suitable constants, depending only on $\lambda_0$, $\Lambda_0$, $C_0$ and on the norms of $b_0$ and $b_1$ in the space
\[ L^\infty(\mathbb{R}; C^\omega(\mathbb{R}_x)) \] and of \( c \) in \( L^\infty(\mathbb{R}_t \times \mathbb{R}_x) \), such that

\[
\left| \int \frac{2}{\sqrt{a_\varepsilon}} b_0(t, x) \Re \left( \partial_t u_\nu \cdot \left( \partial_x u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right) \right) \, dx \right| \leq C \| \partial_t u_\nu \|_{L^2} \left\| \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right\|_{L^2}
\]

\[
\leq (C + C' 2^{-\nu} \varepsilon^{-1/2} e_{\nu,\varepsilon}(t)) ;
\]

\[
\left| \int \frac{2}{\sqrt{a_\varepsilon}} b_1(t, x) \Re \left( \partial_x u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right) \right) \, dx \right| \leq C \int \sqrt{a_\varepsilon} |\partial_x u_\nu|^2 + \frac{1}{\sqrt{a_\varepsilon}} \left| \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right|^2 \, dx
\]

\[
\leq 2 e_{\nu,\varepsilon}(t) ;
\]

\[
\left| \int \frac{2}{\sqrt{a_\varepsilon}} c(t, x) \Re \left( u_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right) \right) \, dx \right| \leq C \int |u_\nu| \frac{1}{\sqrt{a_\varepsilon}} \left| \partial_t u_\nu + \frac{\partial_t \sqrt{a_\varepsilon}}{2\sqrt{a_\varepsilon}} u_\nu \right| \, dx
\]

\[
\leq 2 C e_{\nu,\varepsilon}(t) ;
\]

where we have dealt with \( \| \partial_t u_\nu \|_{L^2} \) as before.

Now, we join the approximation parameter \( \varepsilon \) with the dual variable \( \xi \), setting

\[
\varepsilon = 2^{-\nu} ;
\]
so, from (33) and the previous inequalities, we obtain

\[
\frac{d}{dt} e_{\nu, 2^{-\nu}}(t) \leq \tilde{C} (\nu + 1) e_{\nu, 2^{-\nu}}
\]

(34)

\[
+ \int \frac{2}{\sqrt{a_2^{-\nu}}} \Re \left( (Lu)_\nu \cdot \left( \partial_t u_\nu + \frac{\partial_l \sqrt{a_2^{-\nu}} u_\nu}{2\sqrt{a_2^{-\nu}}} \right) \right) \, dx
\]

\[
+ \int \frac{2}{\sqrt{a_2^{-\nu}}} \Re \left( (\partial_2 (\varphi_\nu(D_x), a) |\partial_2 u) \cdot \left( \partial_t u_\nu + \frac{\partial_l \sqrt{a_2^{-\nu}} u_\nu}{2\sqrt{a_2^{-\nu}}} \right) \right) \, dx
\]

\[
- \int \frac{2}{\sqrt{a_2^{-\nu}}} \Re \left( (\varphi_\nu(D_x), b_0) |\partial_2 u) \cdot \left( \partial_t u_\nu + \frac{\partial_l \sqrt{a_2^{-\nu}} u_\nu}{2\sqrt{a_2^{-\nu}}} \right) \right) \, dx
\]

\[
- \int \frac{2}{\sqrt{a_2^{-\nu}}} \Re \left( (\varphi_\nu(D_x), b_1) |\partial_2 u) \cdot \left( \partial_t u_\nu + \frac{\partial_l \sqrt{a_2^{-\nu}} u_\nu}{2\sqrt{a_2^{-\nu}}} \right) \right) \, dx
\]

for a suitable constant \( \tilde{C} \), which depends only on \( \lambda_0, \Lambda_0, C_0 \) and on the norms of the coefficients of the operator \( L \) in their respective functional spaces.

### 3.5. Estimates for the commutator terms

Now, we have to deal with the commutator terms. As we will see, it’s useful to consider immediately the sum over \( \nu \in \mathbb{N} \).

First, we report an elementary lemma (see also [3]), which we will use very often in the next calculations.

**Lemma 3.4.** There exist two continuous, decreasing functions \( \alpha_1, \alpha_2 : [0, 1[ \to ]0, +\infty[ \) such that \( \lim_{c \to 0^+} \alpha_j(c) = +\infty \) for \( j = 1, 2 \) and such that, for all \( \delta \in ]0, 1[ \) and all \( n \geq 1 \), the following inequalities hold:

\[
\sum_{j=1}^{n} e^{\delta_j} j^{-1/2} \leq \alpha_1(\delta) e^{\delta n} n^{-1/2},
\]

\[
\sum_{j=n}^{+\infty} e^{-\delta_j} j^{1/2} \leq \alpha_2(\delta) e^{-\delta n} n^{1/2}.
\]

Before going on, we take \( \beta > 0 \) and \( T \in ]0, T_0[ \) such that

\[
\beta T = \frac{\theta}{2} \log 2.
\]

**Remark 3.5.** Notice that, thanks to the hypothesis of Theorem 2.1, this condition implies that

\[
\beta T \leq \frac{\omega - \theta}{2}.
\]
Moreover, for all $t \in [0,T]$, we have:

\[
\begin{align*}
\beta t + \frac{\theta}{2} \log 2 & \geq \frac{\theta}{2} \log 2 > 0 \\
\beta t + \frac{\theta}{2} \log 2 & \leq \theta \log 2 \leq \frac{1}{2} \log 2 < 1 \\
(1 - \theta) \log 2 - \beta t & \geq \left(1 - \frac{3}{2} \frac{\theta}{2}\right) \log 2 \geq \left(1 - \frac{3}{4}\right) \log 2 > 0 \\
(1 - \theta) \log 2 - \beta t & \leq (1 - \theta) \log 2 \leq \log 2 < 1.
\end{align*}
\]

Finally, we set (with the same notations used in the subsection 3.2)

\[
\psi_{\mu} = \varphi_{\mu-1} + \varphi_{\mu} + \varphi_{\mu+1} \quad (\varphi_{-1} \equiv 0).
\]

As $\psi_{\mu} \equiv 1$ on the support of $\varphi_{\mu}$, we can write

\[
\partial_x u_{\mu} = \varphi_{\mu}(D_x) \partial_x u = \Psi_{\mu}(\varphi_{\mu}(D_x) \partial_x u) = \Psi_{\mu} \partial_x u_{\mu},
\]

where $\Psi_{\mu}$ is the operator related to $\psi_{\mu}$. So, given a generic function $f(t,x)$, one has

\[
[\varphi_{\mu}(D_x), f] \partial_x u = \left[\varphi_{\mu}(D_x), f\right] \sum_{\mu \geq 0} \partial_x u_{\mu} = \sum_{\mu \geq 0} \left([\varphi_{\mu}(D_x), f]\Psi_{\mu}\right) \partial_x u_{\mu}. \tag{35}
\]

After these preliminary remarks, we now can go on with commutators’ estimates.

### 3.5.1. Term with $[\varphi_{\nu}(D_x), a]$

Due to Bernstein’s inequalities, we have

\[
\left\| \partial_t \left( \partial_t u_{\nu} + \frac{\partial_t \sqrt{a_{2-\nu}}}{2 \sqrt{a_{2-\nu}}} u_{\nu} \right) \right\|_{L^2} \leq C \ 2^\nu \ (e_{\nu,2-\nu}(t))^{1/2}.
\]

So, using (35) and the fact that $a_{\nu}$ is real-valued, one has

\[
\begin{align*}
& \left| \int \frac{2}{\sqrt{a_{2-\nu}}} \Re \left( \partial_x \left([\varphi_{\nu}(D_x), a] \partial_x u_{\nu} \right) \cdot \left( \partial_t u_{\nu} + \frac{\partial_t \sqrt{a_{2-\nu}}}{2 \sqrt{a_{2-\nu}}} u_{\nu} \right) \right) dx \right| \\
& \leq C \sum_{\mu} \left\| [([\varphi_{\nu}(D_x), a] \Psi_{\mu}) \partial_x u_{\mu}] \right\|_{L^2} 2^\nu (e_{\nu,2-\nu}(t))^{1/2} \\
& \leq C \sum_{\mu} \left\| [\varphi_{\nu}(D_x), a] \Psi_{\mu} \right\|_{L^2} (e_{\nu,2-\nu}(t))^{1/2} 2^\nu (e_{\nu,2-\nu}(t))^{1/2},
\end{align*}
\]
with the constant $C$ which depends only on $\lambda_0$, $\Lambda_0$ and $C_0$.

Then,

$$\sum_{\nu \geq 0} e^{-2\beta(\nu+1)t_2 - 2\nu \theta} \int \frac{2}{\sqrt{a_2 - \nu}} \Re \left( \partial_x (|\varphi_\nu(D_x), a| \partial_x u) \left( \frac{\partial_t u_\nu + \partial_t \sqrt{a_2 - \nu}}{2\sqrt{a_2 - \nu}} u_\nu \right) \right) \, dx$$

$$\leq C \sum_{\nu, \mu} k_{\nu\mu} (\nu + 1)^{1/2} e^{-\beta(\nu+1)t_2 - \nu \theta} (\nu_2 - \nu \nu)^{1/2} (\mu + 1)^{1/2} e^{-\beta(\mu+1)t_2 - \mu \theta} (\nu_2 - \mu \nu)^{1/2}$$

where we have set

$$k_{\nu\mu} = e^{-(\nu-\mu) \beta t_2 - (\nu-\mu) \theta 2^\nu (\nu + 1)^{-1/2} (\mu + 1)^{-1/2} ||[\varphi_\nu(D_x), a]|_{L(\mathbb{C})}}.$$  \hspace{1cm} (36)

Observe that, if $|\nu - \mu| \geq 3$, then $\varphi_\nu \psi_\mu \equiv 0$, so $[\varphi_\nu(D_x), a]|_{\Psi_\mu} = [\varphi_\nu(D_x), a]|_{\Psi_\mu}$. Therefore, from lemma 3.2, in particular from (28), we deduce that

$$\| [\varphi_\nu(D_x), a(t, x)]_{\Psi_\mu} \|_{L(\mathbb{C})} \leq \begin{cases} C2^{-\nu}(\nu + 1) & \text{if } |\nu - \mu| \leq 2, \\ C2^{-\max\{\nu, \mu\}}(\max\{\nu, \mu\} + 1) & \text{if } |\nu - \mu| \geq 3, \end{cases}$$

where the constant $C$ depends only on $\Lambda_0$ and $C_0$.

Now our aim is to apply Schur’s lemma, so to estimate the quantity

$$\sup_\mu \sum_\nu |k_{\nu\mu}| + \sup_\nu \sum_\mu |k_{\nu\mu}|.$$  \hspace{1cm} (37)

To do this, we will use lemma 3.4 and the inequalities stated in remark 3.5.

1. Fix $\mu \leq 2$.

$$\sum_{\nu \geq 0} |k_{\nu\mu}| \leq C e^{(\mu+1)\beta t_2} 2^{(\mu+1)\theta}(\mu + 1)^{-1/2} \sum_{\nu} e^{-(\nu+1)\beta t_2 - (\nu+1)\theta}(\nu + 1)^{1/2}$$

$$= C e^{(\mu+1)\beta t_2} 2^{(\mu+1)\theta}(\mu + 1)^{-1/2} \sum_{\nu} e^{-(\nu+1)(\beta t + \theta \log 2)}(\nu + 1)^{1/2}$$

$$\leq C e^{3\beta t} 2^{\theta \alpha_2 (\beta t + \theta \log 2)}$$

$$\leq C 2^{2\theta} \alpha_2 (\theta \log 2).$$
2. Now, take $\mu \geq 3$ and first consider
\[
\sum_{\nu=0}^{\mu-3} |k_{\nu\mu}| \leq 
\]
\[
\leq C e^{(\mu+1)\beta t} 2^{-(-\nu+1)(1-\theta)} (\mu + 1)^{1/2} \sum_{\nu=0}^{\mu-3} e^{-(\nu+1)\beta t} 2^{(\nu+1)(1-\theta)} (\nu + 1)^{-1/2}
\]
\[
\leq C e^{(\mu+1)\beta t} 2^{-(-\nu+1)(1-\theta)} (\mu + 1)^{1/2} \sum_{\nu=0}^{\mu-3} e^{(\nu+1)(-\beta t + (1-\theta) \log 2)} (\nu + 1)^{-1/2}
\]
\[
\leq C e^{(\mu+1)\beta t} 2^{-(-\nu+1)(1-\theta)} (\mu + 1)^{1/2} \alpha_1 (\beta t + (1-\theta) \log 2) \cdot e^{(-\beta t + (1-\theta) \log 2)(\mu+1) (\mu - 2)^{-1/2}}
\]
\[
\leq C 2^{\frac{\theta}{2}} \alpha_1 \left( \left(1 - \frac{3}{2} \theta \right) \log 2 \right).
\]

For the second part of the sum, one has
\[
\sum_{\nu=\mu}^{+\infty} |k_{\nu\mu}| \leq 
\]
\[
\leq C e^{(\mu+1)\beta t} 2^{(\mu+1)\theta} (\mu + 1)^{-1/2} \sum_{\nu=\mu-2}^{+\infty} e^{-(\nu+1)\beta t} 2^{-(-\nu+1)\theta} (\nu + 1)^{1/2}
\]
\[
\leq C e^{(\mu+1)\beta t} 2^{(\mu+1)\theta} (\mu + 1)^{-1/2} \alpha_2 (\beta t + \theta \log 2) \cdot e^{(-\beta t + \theta \log 2)(\mu-1) (\mu - 1)^{-1/2}}
\]
\[
\leq C 2^{\frac{\theta}{2}} \alpha_2 (\theta \log 2).
\]

3. Fix now $\nu \geq 0$; we have
\[
\sum_{\mu=0}^{\nu+2} |k_{\nu\mu}| \leq 
\]
\[
\leq C e^{-(\nu+1)\beta t} 2^{-(-\nu+1)\theta} (\nu + 1)^{1/2} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)\beta t} 2^{(\mu+1)\theta} (\mu + 1)^{-1/2}
\]
\[
\leq C e^{-(\nu+1)\beta t} 2^{-(-\nu+1)\theta} (\nu + 1)^{1/2} \alpha_1 (\beta t + \theta \log 2) \cdot e^{(\beta t + \theta \log 2)(\nu+3) (\nu + 3)^{-1/2}}
\]
\[
\leq C 2^{\frac{\theta}{2}} \alpha_1 (\theta \log 2).
\]
For the second part of the series, the following inequality holds:

\[ \sum_{\mu = \nu + 3}^{+\infty} |k_{\nu\mu}| \leq C e^{-\nu (1-\theta)} e^{(\nu+1)\frac{1}{2}} \sum_{\mu = \nu + 3}^{+\infty} e^{(\mu+1)\beta t} e^{-\nu (1-\theta)} (\mu + 1)^{\frac{1}{2}} \leq C e^{-\nu (1-\theta)} e^{\alpha_2 (\beta t + (1 - \theta) \log 2)} \times e^{(\mu+1)(\nu+4)} (\nu+4)^{\frac{1}{2}} \]

\[ \leq C^2 e^{-\nu (1-\theta)} (1 - \frac{3}{2} \theta) \log 2 \times (\nu+4)^{\frac{1}{2}} \]

\[ \leq C e^{-\nu (1-\theta)} e^{\alpha_2 (\beta t + (1 - \theta) \log 2)(\nu+4)} (\nu+4)^{\frac{1}{2}} \]

In conclusion, there exists a positive function \( \Pi \), with \( \lim_{\theta \to 0} \Pi(\theta) = +\infty \), such that

\[ \sup_{\mu} \sum_{\nu} |k_{\nu\mu}| \leq C \Pi(\theta) \]

and so

\[ \sum_{\nu \geq 0} e^{-2\beta (\nu+1) t} e^{-2\nu \theta} \int \frac{2}{\sqrt{a_2 - \nu}} \Re \left( \partial_t ([\varphi_\nu(D_x), b_0(t,x)] \partial_t u) \right) \left( \frac{\partial_t u_\nu}{2\sqrt{a_2 - \nu}} \right) dx \leq C \Pi(\theta) \sum_{\nu \geq 0} (\nu+1) e^{-2\beta (\nu+1) t} e^{-2\nu \theta} e_{\nu,2-\nu}(t) \]

### 3.5.2. Terms with \([\varphi_\nu(D_x), b_0]\) and \([\varphi_\nu(D_x), b_1]\)

Now, let us consider

\[ \left| \int \frac{2}{\sqrt{a_2 - \nu}} \Re \left( ([\varphi_\nu(D_x), b_0(t,x)] \partial_t u) \left( \frac{\partial_t u_\nu}{2\sqrt{a_2 - \nu}} \right) \right) dx \right| \leq 2 \left\| [\varphi_\nu(D_x), b_0(t,x)] \partial_t u \right\|_{L^2} \left\| \frac{1}{\sqrt{a_2 - \nu}} \partial_t u_\nu \right\|_{L^2} \leq 2 \left\| [\varphi_\nu(D_x), b_0(t,x)] \partial_t u \right\|_{L^2} (e_{\nu,2-\nu}(t))^{1/2} \]

Thanks to relation (35), we have

\[ \left\| [\varphi_\nu(D_x), b_0(t,x)] \partial_t u \right\|_{L^2} = \left\| [\varphi_\nu(D_x), b_0(t,x)] \sum_{\mu \geq 0} \Psi_\mu \partial_t u_\mu \right\|_{L^2} \leq \sum_{\mu \geq 0} \left\| [\varphi_\nu(D_x), b_0(t,x)] \Psi_\mu \right\|_{L^2} \left\| \partial_t u_\mu \right\|_{L^2} \]
As we have done before, we have, for constants depending only on \( \lambda_0, \Lambda_0 \) and \( C_0 \),
\[
\| \partial_t u_\mu \|_{L^2} \leq \left( C e^{-\mu} \left\| \frac{\partial_t \sqrt{a_2 - \mu} u_\mu}{\sqrt{a_2 - \mu}} \right\|_{L^2} \right)^{1/2}
\leq \left( C e^{-\mu} + C' \sqrt{a_2 - \mu} \| \partial_x u_\mu \|_{L^2} \right)^{1/2}
\leq (C + C' e^{-\mu})^{1/2} \left( e^{-\mu} \right)^{1/2}
\leq C \left( e^{-\mu} \right)^{1/2}.
\]
Due to lemma 3.2, we obtain
\[
\| [\varphi_\nu(D_x), b_0(t,x)] \Psi_{\mu} \|_{L^2(L^2)} \leq \begin{cases} C 2^{-\nu \omega} & \text{if } |\nu - \mu| \leq 2 \\ C 2^{\max(\mu, \nu) \omega} & \text{if } |\nu - \mu| \geq 3 \end{cases}
\]
where \( C \) is a constant depending only on \( \| b_0 \|_{L^\infty(\mathbb{R}^+_t; C^0(\mathbb{R}))} \).

As a matter of fact, the kernel of operator \( [\varphi_\nu(D_x), b_0] \) is
\[
n(x,y) = \hat{\varphi}(2^\nu (x-y)) 2^\mu (b_0(t,x) - b_0(t,y)) ;
\]
so, to evaluate its norm we apply Schur’s lemma and, thanks to the fact that \( b_0 \) is \( \omega \)-holder, we get the desired estimate.

Therefore,
\[
\left| \sum_{\nu \geq 0} e^{-2\beta(t+1)} 2^{-2\nu} \int \frac{2}{\sqrt{a_2 - \nu}} \Re \left( [\varphi_\nu(D_x), b_0]\partial_t u \left( \frac{\partial_x u_\nu}{\sqrt{a_2 - \nu}} \right) \right) dx \right| \leq \sum_{\nu, \mu \geq 0} e^{-\beta(t+1)} 2^{-\nu \theta} \left( e^{-\beta(t+1)} \right)^{1/2} e^{-\beta(t+1)} 2^{-\mu \theta} \left( e^{-\mu} \right)^{1/2} l_{\nu \mu} ,
\]
where we have defined
\[
l_{\nu \mu} := e^{-\nu \theta} 2^{-\nu \theta} \left( e^{-\nu} \right)^{\theta} \left( e^{-\mu} \right)^{\theta} \left[ [\varphi_\nu(D_x), b_0(t,x)] \Psi_{\mu} \right]_{L^2(L^2)} . \tag{38}
\]
As made before, we are going to estimate \( l_{\nu \mu} \) applying Schur’s lemma.

1. Let us fix \( \mu \leq 2 \). Then
\[
\left| \sum_{\nu \geq 0} l_{\nu \mu} \right| \leq C e^{\beta(t+1)} 2^{(\mu+1)\theta} \sum_{\nu \geq 0} e^{-\nu \theta} 2^{-\nu \theta} 2^{-\nu \omega} \leq C e^{3\beta t} 2^{3\theta} \sum_{\nu \geq 0} e^{-\nu \theta} (\beta t + \theta \log 2) (\nu + 1)^{1/2} \leq C e^{3\beta t} 2^{3\theta} \alpha_2 (\beta t + \theta \log 2) \leq C 2^{3\theta} \alpha_2 (\theta \log 2) .
\]
2. Now, take $\mu \geq 3$ and consider first
\[
\sum_{\nu = 0}^{\mu - 3} |l_{\nu\mu}| \leq C e^{\mu \beta t} 2^\mu \theta 2^{-\mu \omega} \sum_{\nu = 0}^{\mu - 3} e^{-\nu \beta t} 2^{-\nu \theta}
\]
\[
\leq C e^{\mu (\beta t - (\omega - \theta) \log 2)} (\mu - 2)
\]
\[
\leq C e^{-\mu (\omega - \frac{2}{7}) \log 2} (\mu - 2)
\]
\[
\leq C M(\omega, \theta),
\]
where $M(\omega, \theta)$ is the maximum of the function $z \mapsto e^{-\gamma z} (z - 2)$, with $\gamma = (\omega - \frac{\theta}{2}) \log 2$.

For the second part of the sum, we have
\[
\sum_{\nu = \mu - 2}^{\nu + 2} |l_{\nu\mu}| \leq C e^{(\mu + 1) \beta t} 2(\mu + 1) \theta \sum_{\nu = \mu - 2}^{\nu + 2} e^{-(\nu + 1) \beta t} 2^{-\nu \omega} \frac{(\mu + 1)^{1/2}}{(\nu + 1)^{1/2}}
\]
\[
\leq C e^{(\mu + 1) \beta t} 2(\mu + 1) \theta (\mu - 1)^{-1/2} \alpha_2 (\beta t + \theta \log 2)
\]
\[
\leq C e^{2(\beta t + \theta \log 2)} \alpha_2 (\theta \log 2)
\]
\[
\leq C 2^{\frac{3}{2} \theta} \alpha_2 (\theta \log 2).
\]

3. Fix now $\nu$. Initially, we have
\[
\sum_{\mu = 0}^{\nu + 2} |l_{\nu\mu}| \leq C e^{-(\nu + 1) \beta t} 2^{-(\nu + 1) \theta} 2^{-\nu \omega} \sum_{\mu = 0}^{\nu + 2} e(\mu + 1) \beta t + \theta \log 2) \frac{(\mu + 1)^{1/2}}{(\nu + 1)^{1/2}}
\]
\[
\leq C e^{-(\nu + 1) \beta t} 2^{-(\nu + 1) \theta} 2^{-\nu \omega} (\nu + 3)^{1/2} \alpha_1 (\beta t + \theta \log 2)
\]
\[
\cdot e^{(\nu + 3) \beta t} 2^{(\nu + 3) \theta} (\nu + 3)^{-1/2}
\]
\[
\leq C e^{2(\beta t + \theta \log 2)} \alpha_1 (\theta \log 2)
\]
\[
\leq C 2^{3 \theta} \alpha_1 (\theta \log 2).
\]

Moreover,
\[
\sum_{\mu = \nu + 1}^{\nu + 3} |l_{\nu\mu}| \leq C e^{-\nu \beta t} 2^{-\nu \theta} \sum_{\mu = \nu + 3}^{\nu + 3} e^{-\mu (\beta t - (\omega - \theta) \log 2)} \frac{\mu^{1/2}}{\mu^{1/2}}
\]
\[
\leq C e^{-\nu \beta t} 2^{-\nu \theta} (\nu + 3)^{-1/2} \alpha_2 ((\omega - \theta) \log 2 - \beta t)
\]
\[
\cdot e^{(\nu + 3) \beta t} 2^{-\nu (\nu + 3) (\omega - \theta)} (\nu + 3)^{1/2}
\]
\[
\leq C e^{3 \beta t} 2^{3 \theta} 2^{-\nu (\nu + 3) \omega} \alpha_2 \left( \left( \omega - \frac{3}{2} \theta \right) \log 2 \right)
\]
\[
\leq C 2^{\frac{3}{2} \theta} \alpha_2 \left( \left( \omega - \frac{3}{2} \theta \right) \log 2 \right).
\]
From all these inequalities, thanks to Schur’s lemma, one has that there exists a constant $\overline{M}(\omega, \theta)$, depending only on $\|b_0\|_{L^\infty(\mathbb{R}, C^r(\mathbb{R}))}$ and on the parameter $\theta$ (which we have fixed at the beginning of the calculations), such that

$$\sup_{\mu} \sum_{\nu} |l_{\nu\mu}| + \sup_{\mu} \sum_{\nu} |l_{\nu\mu}| \leq C \overline{M}(\omega, \theta);$$

from this relation, finally we get

$$\sum_{\nu \geq 0} e^{-2\beta(t+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2-\nu}^2}} \Re \left( [\varphi_\nu(D_x), b_0] \partial_x u \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{2-\nu}}}{2 \sqrt{a_{2-\nu}}} u_\nu \right) \right) dx \leq$$

$$\leq C \overline{M}(\omega, \theta) \sum_{\nu=0}^{+\infty} (\nu + 1) e^{-2\beta(t+1)} t^2 \beta 2^{-2\nu\theta} e_{\nu, 2^{-\nu}}(t).$$

With regard to the term with the commutator $[\varphi_\nu(D_x), b_1(t, x)]$, notice that, as we made before, one has, for constants depending only on $\lambda_0$, $A_0$ and $C_0$,

$$\left| \text{sup}_{\mu} \sum_{\nu \geq 0} \| [\varphi_\nu(D_x), b_1] \Psi_\mu \|_{L^2} \| \sqrt{a_{2-\nu}} \partial_x u_\mu \|_{L^2} \right| \leq C \sum_{\nu \geq 0} \| [\varphi_\nu(D_x), b_1] \Psi_\mu \|_{L^2} \left( e_{\nu, 2^{-\nu}} \right)^{1/2} \left( e_{\nu, 2^{-\nu}} \right)^{1/2}.$$

Therefore,

$$\sum_{\nu \geq 0} e^{-2\beta(t+1)} 2^{-2\nu\theta} \int \frac{2}{\sqrt{a_{2-\nu}^2}} \Re \left( [\varphi_\nu(D_x), b_1] \partial_x u \left( \partial_t u_\nu + \frac{\partial_t \sqrt{a_{2-\nu}}}{2 \sqrt{a_{2-\nu}}} u_\nu \right) \right) dx \leq C \sum_{\nu, \mu \geq 0} e^{-\beta(t+1)} 2^{-\nu\theta} (e_{\nu, 2^{-\nu}})^{1/2} e^{-\beta(t+1)} 2^{-\mu\theta} (e_{\mu, 2^{-\mu}})^{1/2} h_{\nu\mu},$$

where we have set again

$$h_{\nu\mu} = e^{-\nu\theta} 2^{-\nu\theta} \| [\varphi_\nu(D_x), b_1] \Psi_\mu \|_{L^2}.$$
where we have defined we obtain
\[
\sum_{\nu \geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu \theta} \int \frac{2}{\sqrt{a_{2}}-r} \Re \left( [\varphi_{r}(D_{x}), b_{1}] \partial_{x} u \left( \partial_{t} u_{\nu} + \frac{\partial_{t} \sqrt{a_{2}}-v}{2 \sqrt{a_{2}}-v} u_{\nu} \right) \right) dx \leq 
\]
\[
\leq C \left( \omega, \theta \right) \sum_{\nu=0}^{+\infty} (\nu + 1) e^{-2\beta t(\nu+1)} 2^{-2\nu \theta} e_{\nu,2-v}(t).
\]

3.5.3. Term with \([\varphi_{r}(D_{x}), c]\)

Finally, we have to deal with the commutator \([\varphi_{r}(D_{x}), c(t,x)]\).

First, observe that there exist constants, depending only on \(\lambda_0, \Lambda_0\) and \(C_0\) as usual, such that one has

\[
\left| \int \frac{2}{\sqrt{a_{2}}-r} \Re \left( [\varphi_{r}(D_{x}), c] \partial_{x} u \left( \partial_{t} u_{\nu} + \frac{\partial_{t} \sqrt{a_{2}}-v}{2 \sqrt{a_{2}}-v} u_{\nu} \right) \right) dx \right| \leq 
\]
\[
\leq C \| [\varphi_{r}(D_{x}), c] u \|_{L^{2}} \left\| \frac{1}{\sqrt{a_{2}}-v} \left( \partial_{t} u_{\nu} + \frac{\partial_{t} \sqrt{a_{2}}-v}{2 \sqrt{a_{2}}-v} u_{\nu} \right) \right\|_{L^{2}} 
\]
\[
\leq C \sum_{\mu \geq 0} \| [\varphi_{r}(D_{x}), c] \Psi_{\mu} \|_{L^{2}} \| u_{\mu} \|_{L^{2}} \left( e_{\nu,2-v}(t) \right)^{1/2} 
\]
\[
\leq 2 C \sum_{\mu \geq 0} \| [\varphi_{r}(D_{x}), c] \Psi_{\mu} \|_{L^{2}} 2^{-\mu} \| D_{x} u_{\mu} \|_{L^{2}} \left( e_{\nu,2-v}(t) \right)^{1/2} 
\]
\[
\leq 2 C \sum_{\mu \geq 0} \| [\varphi_{r}(D_{x}), c] \Psi_{\mu} \|_{L^{2}} 2^{-\mu} \left( e_{\nu,2-v}(t) \right)^{1/2} \left( e_{\nu,2-v}(t) \right)^{1/2} .
\]

Thereby, we get the estimate

\[
\sum_{\nu \geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu \theta} \int \frac{2}{\sqrt{a_{2}}-r} \Re \left( [\varphi_{r}(D_{x}), c] \partial_{x} u \left( \partial_{t} u_{\nu} + \frac{\partial_{t} \sqrt{a_{2}}-v}{2 \sqrt{a_{2}}-v} u_{\nu} \right) \right) dx \leq 
\]
\[
\leq 2 C \sum_{\nu,\mu \geq 0} e^{-\beta t(\nu+1)} 2^{-\nu \theta} \left( e_{\nu,2-v}(t) \right)^{1/2} e^{-\beta t(\mu+1)} 2^{-\mu \theta} \left( e_{\mu,2-v}(t) \right)^{1/2} m_{\nu,\mu},
\]

where we have defined

\[
m_{\nu,\mu} = e^{-\nu \theta} 2^{-\nu \theta} \left( [\varphi_{r}(D_{x}), c] \Psi_{\mu} \right) ||_{L^{2}} .
\]

The kernel of the operator \([\varphi_{r}(D_{x}), c]\) is

\[
n'(x,y) = \tilde{\psi}(2^{\nu}(x-y)) 2^{\nu} (c(t,y) - c(t,x)) ;
\]
so, remembering that $c$ is bounded over $\mathbb{R} \times \mathbb{R}$, from Schur’s lemma one gets

$$\| \varphi_{\nu}(D_x), c \|_{L(L^2)} \leq C \quad \forall \nu \geq 0,$$

where the constant $C$ depends only on $\| c \|_{L^\infty(\mathbb{R} \times \mathbb{R})}$.

Again, we are going to estimate the kernel $m_{\nu \mu}$ to apply Schur’s lemma.

1. First, we take $\mu \leq 2$ and we have

\[
\sum_{\nu \geq 0} |m_{\nu \mu}| \leq Ce^{(\mu+1)\beta t} 2^{(\mu+1)(1-\theta)} 2^{-\mu} \sum_{\nu \geq 0} e^{-(\nu+1)\beta t} 2^{-(\nu+1)\theta} \left( \nu + 1 \right)^{1/2} \\
\leq Ce^{3\beta t} 2^{3\theta} \sum_{\nu \geq 0} e^{-\left(\nu+1\right)\left(\beta t + \theta \log 2\right)} \left( \nu + 1 \right)^{1/2} \\
\leq Ce^{3\beta t} 2^{3\theta} \alpha_2(\beta t + \theta \log 2) \\
\leq C 2^{2\theta} \alpha_2(\theta \log 2).
\]

2. Now, we fix $\mu \geq 3$ and we consider the first part of the series:

\[
\sum_{\nu \geq 0} |m_{\nu \mu}| \leq \sum_{\nu = 0}^{\mu-3} |m_{\nu \mu}| \\
\leq Ce^{(\mu+1)\beta t} 2^{-(\mu+1)(1-\theta)} 2^{-\mu} \sum_{\nu = 0}^{\mu-3} e^{-(\nu+1)\beta t} 2^{(\nu+1)(1-\theta)} 2^{-(\nu+1)\theta} \left( \nu + 1 \right)^{1/2} \\
\leq Ce^{(\mu+1)\beta t} 2^{-(\mu+1)(1-\theta)} \left( \mu - 2 \right)^{1/2} \sum_{\nu = 0}^{\mu-3} e^{(\nu+1)((1-\theta) \log 2 - \beta t)} \left( \nu + 1 \right)^{-1/2} \\
\leq Ce^{(\mu+1)\beta t} 2^{-(\mu+1)(1-\theta)} \left( \mu - 2 \right)^{1/2} \alpha_1 \left( (1 - \theta) \log 2 - \beta t \right) \\
\leq Ce^{3\beta t} 2^{3\theta} \alpha_1 \left( 1 - \frac{3\theta}{2} \right) \log 2 \\
\leq C 2^{2\theta} \alpha_1 \left( 1 - \frac{3\theta}{2} \right) \log 2.
\]

For the second part, one has:

\[
\sum_{\nu = \mu - 2}^{+\infty} |m_{\nu \mu}| \\
\leq Ce^{(\mu+1)\beta t} 2^{(\mu+1)(1-\theta)} 2^{-\mu} \sum_{\nu = \mu - 2}^{+\infty} e^{-(\nu+1)\beta t} 2^{-(\nu+1)\theta} \left( \nu + 1 \right)^{1/2} \\
\leq Ce^{(\mu+1)\beta t} 2^{(\mu+1)(1-\theta)} 2^{-\mu} \alpha_2(\beta t + \theta \log 2) e^{-(\mu-1)\beta t} 2^{-(\mu-1)\theta} \\
\leq C 2^{3\theta} \alpha_2(\theta \log 2).
\]
3. Now, we fix $\nu \geq 0$. Initially, we consider 
\[
\sum_{\mu=0}^{\nu+2} |m_{\nu \mu}| \leq C e^{-(\nu+1)^{\beta t}} 2^{-2(\nu+1)\theta} \sum_{\mu=0}^{\nu+2} e^{(\mu+1)(\beta t + \theta \log 2)} 2^{-\mu} \frac{1}{\mu+1}^{1/2} \frac{1}{(\mu+1)^{1/2}} 
\leq C e^{-(\nu+1)^{\beta t}} 2^{-2(\nu+1)\theta} (\nu + 3)^{1/2} \alpha_1 (\beta t + \theta \log 2) \cdot e^{(\nu+3)^{\beta t}} g^{(\nu+3)\theta} (\nu + 3)^{-1/2} 
\leq C e^{2(\beta t + \theta \log 2)} \alpha_1 (\theta \log 2) 
\leq C 2^{\beta t} \alpha_1 (\theta \log 2) .
\]

The second part of the series, instead, can be treated as follow:
\[
\sum_{\mu=\nu+3}^{\infty} |m_{\nu \mu}| \leq C e^{-(\nu+1)^{\beta t}} 2^{(\nu+1)(1-\theta)} \sum_{\mu=\nu+3}^{\infty} e^{(\mu+1)^{\beta t}} 2^{-\mu} (\nu+1)^{(1-\theta)} 2^{-\nu} \frac{1}{\mu+1}^{1/2} \frac{1}{(\mu+1)^{1/2}} 
\leq C e^{-(\nu+1)^{\beta t}} 2^{(\nu+1)(1-\theta)} (\nu + 4)^{-1/2} \sum_{\mu=\nu+3}^{\infty} e^{-(\mu+1)(\theta \log 2) - \beta t} (\mu + 1)^{1/2} 
\leq C e^{-(\nu+1)^{\beta t}} 2^{(\nu+1)(1-\theta)} \alpha_2 ((1-\theta) \log 2 - \beta t) e^{(\nu+4)^{\beta t}} 2^{-(\nu+4)(1-\theta)} 
\leq C 2^{\beta t} \alpha_2 \left(1 - \frac{3}{2} \theta\right) \log 2 .
\]

Finally, we obtain:
\[
\left| \sum_{\nu \geq 0} e^{-2\beta t(\nu+1)} 2^{-2\nu t} \int \frac{2}{\sqrt{\alpha_2 - \nu}} \Re \left( [\varphi, (D_x), e] \partial \nu u + \frac{\partial \sqrt{a_{2-\nu}}}{\partial \nu} u_{\nu} \right) dx \right| 
\leq C \Pi(\theta) \sum_{\nu \geq 0} (\nu + 1) e^{-2\beta t(\nu+1)} 2^{-2\nu t} e^{(\nu+1)(1-\theta)} ,
\]
where the function $\Pi$ is the same used in the estimate of the term $[\varphi, (D_x), a]$.

3.6. End of the proof of theorem 2.1

Now we can complete the proof of theorem 2.1.

First, remembering the definition of total energy given by (32), we have that there exists a constant $C > 0$, depending only on $\theta$, such that 
\[
\left| \sum_{\nu \geq 0} e^{-\beta t(\nu+1)} 2^{-2\nu t} \int \frac{2}{\sqrt{\alpha_2}} \Re \left( (Lu)_{\nu} \cdot \left( \partial \nu u_{\nu} + \frac{\partial \sqrt{a_{2-\nu}}}{\partial \nu} u_{\nu} \right) \right) dx \right| 
\leq C (E(t))^{1/2} \|Lu\|_{H^{\theta - \beta t}} .
\]
Therefore, if we set $\tilde{\Pi}(\omega, \theta) = \max \{\tilde{M}(\omega, \theta), \Pi(\theta)\}$, from relation (34) and from the estimates proved in the previous subsection, we have that, for suitable constants, depending only on $\lambda_0$, $\Lambda_0$, $C_0$ and on the norms of the coefficients of operator $L$ in their respective functional spaces, the following inequality holds:

$$\frac{d}{dt} E(t) \leq \left( C + C' \tilde{\Pi}(\omega, \theta) - 2\beta \right) \sum_{\nu=0}^{+\infty} (\nu + 1) e^{-2\beta(\nu+1)t} 2^{-2\nu} e_{\nu,2^{-\nu}}(t)$$

$$+ \ C'' (E(t))^{1/2} \|Lu\|_{H^{-s-\beta^*}}.$$ 

Now, let us fix $\beta$ large enough, such that $C + C' \tilde{\Pi}(\omega, \theta) - 2\beta \leq 0$: we can always do this, on condition that we take $T$ small enough. With this choice, we have

$$\frac{d}{dt} E(t) \leq C'' (E(t))^{1/2} \|Lu\|_{H^{-s-\beta^*}};$$

now, the thesis of the theorem follows from Gronwall’s lemma and remark 3.3.

References


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