Resolvent Estimates for Magnetic Schrödinger Operators and Their Applications to Related Evolution Equations

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Abstract. In this article we survey some basic results for the magnetic Schrödinger operator with external potential which has a strong singularity. The following topics are treated under suitable decay conditions on the magnetic field and external potential: Selfadjointness of the operator, Growth estimates of generalized eigenfunctions, Principle of limiting absorption, Uniform resolvent estimates, and Smoothing properties for corresponding evolution equations.

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1. Introduction and results

We consider the magnetic Schrödinger equation

\[ -\sum_{j=1}^{n} \{\partial_j + ib_j(x)\}^2 u + c(x)u - \kappa^2 u = f(x), \quad x \in \mathbb{R}^n, \]

where \(\partial_j = \partial/\partial x_j\) \((j = 1, \cdots, n)\), \(i = \sqrt{-1}\), \(\kappa \in \Pi_{\pm} = \{\kappa \in \mathbb{C}; \pm \Re \kappa > 0, \Im \kappa > 0\}\), \(b_j(x)\) are real valued \(C^1\)-functions of \(x \in \mathbb{R}^n\), \(c(x)\) is a real valued continuous function of \(x \in \mathbb{R}^n\\setminus\{0\}\) and \(f(x) \in L^2(\mathbb{R}^n)\). \(b(x) = (b_1(x), \cdots, b_n(x))\) represents a magnetic potential. Thus the magnetic field is defined by its rotation \(\nabla \times b(x)\). The external potential \(c(x)\) may have a singularity like \(O(|x|^{-2})\) at \(x = 0\).

Notation: Let \(a \cdot b\) and \(a \times b\) respectively denote the inner product and exterior product of \(a, b \in \mathbb{R}^n\). More generally, we put

\[ \nabla \cdot v(x) = \partial_1 v_1(x) + \cdots + \partial_n v_n(x), \quad \nabla \times v(x) = (\partial_j v_k(x) - \partial_k v_j(x))_{1 \leq j < k \leq n} \]

for \(\nabla = (\partial_1, \cdots, \partial_n)\) and \(v(x) = (v_1(x), \cdots, v_n(x))\). We also put \(\nabla_b = \nabla + ib(x)\), \(\Delta_b = \nabla_b \cdot \nabla_b\), \(r = |x|\), \(\tilde{x} = x/r\) and \(\partial_r = \tilde{x} \cdot \nabla\). The inner product and
norm of $L^2$ are defined by
\[(f, g) = \int f(x)\overline{g(x)} \, dx \quad \text{and} \quad \|f\| = \sqrt{(f, f)}.\]

Here we specify by $\int dx$ the integration over $\mathbb{R}^n$. For function $\xi = \xi(r) > 0$ let $L^2_\xi = L^2_\xi(\mathbb{R}^n)$ be the weighted $L^2$-space with norm
\[\|f\|_\xi = \left\{ \int \xi(r)|f(x)|^2 \, dx \right\}^{1/2} < \infty.\]

Moreover, for $0 < s < t$ we put $B_{s,t} = \{x; s < |x| < t\}$, $B_t = \{x; |x| < t\}$, $B'_t = \mathbb{R}^n \backslash B_t$ and $S_t = \{x; |x| = t\}$.

Throughout this paper we assume the existence of $c_\infty(x) \in L^\infty$ such that
\[(A1) \quad c(x) - c_\infty(x) \geq \frac{\beta}{r^2} \quad \text{with} \quad \beta > -\frac{(n-2)^2}{4}.\]

With this condition we define the operator $L$ acting in $L^2$ as follows:
\[
\begin{cases}
Lu = -\Delta_b u + c(x)u & \text{for } u \in \mathcal{D}(L), \\
\mathcal{D}(L) = \{u \in L^2 \cap H^2_{\text{loc}}(\mathbb{R}^n \backslash \{0\}); (-\Delta_b + c)u, r^{-1}u \in L^2\}. 
\end{cases}
\]

Here $H^j = H^j(\mathbb{R}^n) (j = 1, 2, \cdots)$ is the usual Sobolev space on $\mathbb{R}^n$ and $H^2_{\text{loc}}(\Omega)$ is the $H^2$-space on each compact set of the domain $\Omega$.

**Theorem 1.** (i) If $u \in \mathcal{D}(L)$, then we have $\nabla_b u \in [L^2]^n$.

(ii) $L$ gives a lower semibounded selfadjoint operator in $L^2$.

(iii) If $c(x) \to 0$ as $|x| \to \infty$, then the essential spectrum $\sigma_e(L)$ of $L$ is included in the half line $[0, \infty)$.

A proof is given in Mochizuki [12, Theorems 1.1 and 1.3] under a stronger restriction on the singularity of $c(x)$ (the so called Stummel conditions are required there). On the other hand, in Kalf-Schmincke-Walter-Wist [9, Theorem 3] is treated the case with $b(x) \equiv 0$ and $c(x)$ having a strong singularity like $O(r^{-2})$. In both works (ii) is obtained, based on (i), as a Friedrichs extension of lower semibounded symmetric operators ([5]).

Theorem 1 shows that $\kappa^2$ with $\kappa \in \Pi_\pm$ is in the resolvent set of $L$. Thus, equation (1) has a unique solution $u = R(\kappa^2)f \in L^2$, where $R(\kappa^2) = (L - \kappa^2)^{-1}$ is the resolvent of $L$.

In order to study the essential spectrum of $L$ we add the following conditions.
(A2) There exist constants $R_0 > 0$ and $C_0 > 0$ such that

$$\max \left\{ |\nabla \times b(x)|, |c(x) + \frac{(n-1)(n-3)}{4r^2}| \right\} \leq C_0 \mu(r), \quad r = |x| > R_0,$$

where $\mu = \mu(r)$ is a smooth, positive $L^1$-function of $r \in \mathbb{R}_+ = (0, \infty)$.

(A3) The unique continuation property holds for $-\Delta b + c(x)$.

**Theorem 2.** Assume (A1) and (A2) with $\mu(r)$ also satisfying

$$\mu(r) = o(r^{-1}) \quad \text{as} \quad r \to \infty. \quad (3)$$

Let $\lambda > 0$ and let $u \in H^2_{loc}(\mathbb{R}^n \setminus \{0\})$ solve the homogeneous equation

$$-\Delta_b u + c(x)u - \lambda u = 0. \quad (4)$$

If the support of $u$ is not compact, then

$$\liminf_{t \to \infty} \int_{S_t} \left| \tilde{x} \cdot \nabla b u + \frac{n-1}{2r} u - i\kappa u \right|^2 dS \neq 0,$$

where $\kappa = \sqrt{\lambda}$ or $\kappa = -\sqrt{\lambda}$.

**Theorem 3.** Assume (A1), (A3) and (A2) with $\mu(r)$ satisfying (3) and also

$$\int_r^\infty \mu(s) ds \geq r\mu(r) \quad \text{for} \quad r \geq R_0. \quad (5)$$

Then the resolvent $R(\kappa^2)$ is continuously extended to $\Pi_{\pm} \cup (0, \infty)$ as an operator from $L^2_{\mu^{-1}}$ to $L^2_{\mu}$. Thus, the positive spectrum of $L$ is absolutely continuous with respect to the Lebesgue measure.

Theorem 2 gives a real generalization of the Rellich growth estimates for the Laplace operator in exterior domain ([15]). A similar result

$$\lim_{t \to \infty} t^\epsilon \int_{S_t} \left| \tilde{x} \cdot \nabla b u + \frac{n-1}{2r} u - i\kappa u \right|^2 dS = \infty \quad (\forall \epsilon > 0)$$

has been obtained in Ikebe-Uchiyama [7]. In this case we only use condition (3) and it is not necessary to assume $\mu \in L^1(\mathbb{R}_+)$. To show Theorem 2 we employ the methods developed in Jäger-Rejto [8] and Mochizuki [13] for non-magnetic Schrödinger operators with oscillating long range potentials. Theorem 3 is then a direct result of Theorem 2 (see e.g., Eidus [3], Mochizuki [12]).

Next, we shall show uniform resolvent estimates for $\kappa \in \Pi_{\pm}$. To this aim we restrict ourselves to the case $n \geq 3$ and replace (A2) by some smallness conditions.
Theorem 4. (i) Let \( n \geq 3 \). Assume (A1) and

\[
\text{(A4)} \quad \max\{|\nabla \times b(x)|, |c(x)|| \leq \epsilon_0 r^{-2} \quad \text{in} \quad \mathbb{R}^n,
\]

where \( 0 < \epsilon_0 < 1/4\sqrt{2} \) \((n = 3)\) or \( < \sqrt{(n-1)(n-3)}/8 \) \((n \geq 4)\). Then we have for each \( \kappa \in \Pi_\pm \),

\[
\int \frac{1}{r^2} |u|^2 \, dx \leq C_1 \int r^2 |f|^2 \, dx \quad \text{with}
\]

\[
C_1 = \frac{8}{1 - 32\epsilon_0^2} \quad (n = 3) \quad \text{or} \quad \frac{8}{(n-1)(n-3) - 8\epsilon_0^2} \quad (n \geq 4).
\]

(ii) Let \( n \geq 3 \). Assume (A1) and

\[
\text{(A5)} \quad \max\{|\nabla \times b(x)|, |c(x)|| \leq \epsilon_0 \min\{\mu(r), r^{-2}\} \quad \text{in} \quad \mathbb{R}^n,
\]

where \( \mu(r) \) is a smooth, positive \( L^1 \)-function of \( r \in \mathbb{R}_+ \) satisfying also

\[
\mu'(r) \leq 0 \quad \text{in} \quad \mathbb{R}_+.
\]

Then we have for each \( \kappa \in \Pi_\pm \),

\[
\int \left\{ \mu(|\nabla u|^2 + |\kappa u|^2) - \mu \frac{n-1}{2r} |u|^2 \right\} \, dx \leq C_2 \int \max\{\mu^{-1}, r^2\} |f(x)|^2 \, dx
\]

with \( C_2 = 4\|\mu\|_{L^1} \left( 5 + 4\epsilon_0^2 C_1 \right) \).

Remark 1 The functions \((1 + r)^{-1-\delta}\) and \((1 + r)^{-1} \log(e + r)^{-1-\delta}\) \((0 < \delta \leq 1)\) are typical examples of \( \mu(r) \). As is easily seen, all the conditions (3), (5) and (6) are verified by these functions.

As a corollary of Theorem 4, we are able to obtain space-time weighted estimates (smoothing properties, cf., Kato [10]) for the Schrödinger evolution equation

\[
i \frac{\partial u}{\partial t} - Lu = 0, \quad u(0) = f \in L^2,
\]

and for the relativistic Schrödinger evolution equation

\[
i \frac{\partial u}{\partial t} - \sqrt{L + m^2} u = 0, \quad u(0) = f \in L^2
\]

with \( m \geq 0 \). Note that the smoothing effects for (8) give those for the Klein-Gordon \((m > 0)\) or the wave equation \((m = 0)\) in the energy space.
Theorem 5. (i) Under the conditions of Theorem 4 (i), we have
\[
\left|\int_{0}^{\pm\infty} \left| r^{-1} \int_{0}^{t} e^{-i(t-\tau)L} h(\tau) d\tau \right|^2 dt \right| \leq C_1 \left| \int_{0}^{\pm\infty} \| r h(t) \|^2 dt \right|
\]
for \( h(t) \) satisfying \( r^{-1} h(t) \in L^2(\mathbb{R} \times \mathbb{R}^n) \), and
\[
\left|\int_{0}^{\pm\infty} \| r^{-1} e^{-iL} f \|_2^2 dt \right| \leq 2 \sqrt{C_1} \| f \|^2 \quad \text{for} \quad f \in L^2.
\]

(ii) Under the conditions of Theorem 4 (ii), we have
\[
\left|\int_{0}^{\pm\infty} \| \min\{r^{1/2}, r^{-1}\} e^{-it\sqrt{L+m^2}} f \|_2^2 dt \right| \leq 4 \sqrt{\max\{C_1, C_2\}} \| f \|^2.
\]

Theorems 4 and 5 summarize the main results of the recent work of Mochizuki [14]. Theorem 4 (i) generalizes the corresponding results of Kato-Yajima [11], where the operator in question is restricted to the Laplace operator in \( \mathbb{R}^n \) \( (n \geq 3) \). The Fourier transformation method employed there is not applicable in our case. We are based on the partial integration method, and the proof of Theorems 2, 3 and 4 are all reduced to one quadratic identity given in Proposition 1 of §3. Another important tools are modifications of the Hardy inequality (Lemma 3 of §2 and Lemma 9 of §5).

Results similar to Theorem 5 have been studied by many authors in connection with local smoothing properties (see, e.g., Yajima [16], Cuccagna-Schirmer [1], D’Ancona-Fanelli [2], Erdogan-Goldberg-Schlag [4] and Georgiev-Stefanov-Tarulli [6]). Note that these works are restricted to the case where the vector potential \( b(x) \) itself is required to be small and to decay sufficiently fast (the smallness of \( b(x) \) is not required in [4]). On the other hand, no such a requirement is in our case, and the smallness is required on \( \nabla \times b(x) \). To remove it seems difficult without any decay conditions on \( b(x) \).

Theorems 1 to 5 will be proved in the following §2 to §6, respectively. Finally, we give an extension of Theorem 4 (i) and add some remarks in §7. Theorem 6 there asserts that the smallness of \( c(x) \) is not essential for uniform resolvent estimates.

2. Proof of Theorem 1

In this section, we give a proof of Theorem 1 following Kalf et al [7, Theorem 3] (cf., also Mochizuki [12, Theorems 1.1 and 1.3]).

For \( \alpha \in \mathbb{R} \) and \( u \in H^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) we have
\[
\left| \tilde{x} \cdot \nabla u \right|^2 = \left| \tilde{x} \cdot \nabla u + \frac{\alpha}{r} u - \frac{\alpha}{r} u \right|^2
\]
\[
\int_{B_{\epsilon,t}} |\tilde{x} \cdot \nabla b u + \frac{\alpha}{r} u|^2 \, dx = \int_{B_{\epsilon,t}} \tilde{x} \cdot \nabla b u + \frac{\alpha}{r} u \bigg| \bigg|^2 \, dx \\
- \left( \int_{S_t} - \int_{S_{\epsilon}} \right) \frac{\alpha}{r} |u|^2 \, dS + \int_{B_{\epsilon,t}} \frac{(n-2)\alpha - \alpha^2}{r^2} |u|^2 \, dx.
\]

**Lemma 1.** We have

\[
\int_{B_{\epsilon,t}} |\tilde{x} \cdot \nabla b u + \frac{\alpha}{r} u|^2 \, dx = \int_{B_{\epsilon,t}} \tilde{x} \cdot \nabla b u + \frac{\alpha}{r} u \bigg| \bigg|^2 \, dx \\
- \left( \int_{S_t} - \int_{S_{\epsilon}} \right) \frac{\alpha}{r} |u|^2 \, dS + \int_{B_{\epsilon,t}} \frac{(n-2)\alpha - \alpha^2}{r^2} |u|^2 \, dx.
\]

**Lemma 2.** (i) Let \( r^{-1} u \in L^2 \). Then we have

\[
\lim_{\epsilon \to 0} \int_{S_{\epsilon}} r^{-1} |u|^2 \, dS = 0, \quad \lim_{\rho \to \infty} \int_{S_{\rho}} r^{-1} |u|^2 \, dS = 0.
\]

(ii) Let \( u \in L^2 \). Then there exist sequences \( \epsilon_k \to 0, \, t_k \to \infty (k \to \infty) \) such that

\[
\partial_r \int_{S_1} r^n |u(r\omega)|^2 \, dS_{\omega} \bigg|_{r=\epsilon_k} \geq 0, \quad \partial_r \int_{S_1} r^n |u(r\omega)|^2 \, dS_{\omega} \bigg|_{r=t_k} \leq 0.
\]

**Proof** (i) is obvious since we have

\[
\int r^{-2} |u|^2 \, dx = \int_0^\infty r^{-1} \int_{S_r} r^{-1} |u|^2 \, dS \, dr < \infty.
\]

(ii) is also verified from the inequality

\[
\int |u|^2 \, dx = \int_0^\infty r^{-1} \int_{S_1} r^n |u(r\omega)|^2 \, dS_{\omega} \, dr < \infty.
\]

**Proof of Theorem 1** (i) By means of the Gauss formula we have for \( u \in D(L) \),

\[
\text{Re} \int_{B_{\epsilon,t}} (-\Delta_b u + cu) \overline{\varphi} \, dx = \int_{B_{\epsilon,t}} (|\nabla_b u|^2 + c|u|^2) \, dx - \text{Re} \left( \int_{S_t} - \int_{S_{\epsilon}} \right) (\tilde{x} \cdot \nabla u) \overline{\varphi} \, dS.
\]

Combine this identity and Lemma 1. Then noting

\[
-\text{Re} \int_{S_r} (\tilde{x} \cdot \nabla u) \overline{\varphi} \, dS = -\frac{1}{2r} \partial_r \int_{S_1} r^n |u(r\omega)|^2 \, dS_{\omega} + \frac{n}{2r} \int_{S_r} |u|^2 \, dS,
\]

we obtain

\[
\text{Re} \int_{B_{\epsilon,t}} (-\Delta_b u + cu) \overline{\varphi} \, dx = \int_{B_{\epsilon,t}} \left| \frac{\tilde{x} \cdot \nabla b u + \alpha}{r} u \right|^2 \, dx
\]
\[
+ \int_{B_{\cdot t}} \left( \frac{(n-2)\alpha - \alpha^2}{r^2} + c \right) |u|^2 \, dx
+ \left( \int_{S_{\cdot r}} - \int_{S_{\cdot t}} \right) \frac{n-2\alpha}{2r} |u|^2 \, dS
- \frac{1}{2} \left[ \partial_r \int_{S_{\cdot r}} r^n |u(r\omega)|^2 \, dS_{\omega} \right]_e.
\]

Put \( \alpha = n/2 \) in this equation. Then since \( r^{-1} u, |c|^{1/2} u \in L^2 \), the first inequality of Lemma 2 (ii) shows
\[
\int_{B_{\cdot r}} \left| \nabla_b u + \frac{n}{2r} u \right|^2 \, dx < \infty.
\]

Going back to Lemma 1 with \( \alpha = n/2 \) and using the first inequality of Lemma 2 (i), we conclude
\[
\int_{B_{\cdot r}} |\nabla_b u|^2 \, dS < \infty. \tag{11}
\]

On the other hand, since (10) reduces to
\[
\text{Re} \int_{B_{\cdot r}} (-\Delta_b u + cu) \, dx \geq \int_{B_{\cdot r}} \left( |\nabla_b u|^2 + c|u|^2 \right) \, dx
- \frac{1}{2} \left[ r^{-1} \partial_r \int_{S_{\cdot r}} r^n |u(r\omega)|^2 \, dS_{\omega} \right]_e - \frac{n}{2} \int_{S_{\cdot r}} r^{-1} |u|^2 \, dS,
\]
the second inequality of Lemma 2 (ii) implies
\[
\int_{B_{\cdot r}} |\nabla_b u|^2 \, dS < \infty. \tag{12}
\]

(11) and (12) prove the assertion (i). \( \square \)

Let \( H^1_b \) be the completion of \( C_0^\infty = C_0^\infty(\mathbb{R}^n) \) with respect to the norm
\[
||u||_{H^1_b} = \left[ \int \left( |\nabla_b u|^2 + |u|^2 \right) \, dx \right]^{1/2} < \infty. \tag{13}
\]

A modified Hardy inequality is given by

**Lemma 3.** Let \( u \in H^1_b \). Then we have
\[
\int \frac{(n-2)^2}{4r^2} |u|^2 \, dx \leq \int |\hat{x} \cdot \nabla_b u|^2 \, dx.
\]
Choose \( \alpha = \frac{n-2}{2} \) in Lemma 1. Then letting \( t \to \infty \), we have

\[
\int_{B'_r} |\tilde{x} \cdot \nabla_b u|^2 dx \geq \int_{S_r} \frac{n-2}{2r} |u|^2 dS + \int_{B'_r} \frac{(n-2)^2}{4r^2} |u|^2 dx.
\]

By assumption, we can let \( \epsilon \to 0 \) to conclude the desired inequality.

Proof of Theorem 1 (ii) Let \( u, v \in \mathcal{D}(L) \). Then with the help of (i), especially noting

\[
\liminf_{\epsilon \to 0} \int_{S_r} |(\tilde{x} \cdot \nabla_b u)\tilde{v}| dS = \liminf_{t \to \infty} \int_{S_t} |(\tilde{x} \cdot \nabla_b u)\tilde{v}| dS = 0,
\]

we easily see that

\[
(Lu, v) = \int \{ \nabla_b u \cdot \nabla_b v + cu\tilde{v} \} dx.
\]

Since \( \mathcal{D}(L) \) is dense in \( L^2 \), this shows that \( L \) is a symmetric operator. Moreover, by means of (A1) and Lemma 3,

\[
(Lu, u) \geq \left( \frac{(n-2)^2}{4} + \beta \right) \| r^{-1}u \|^2 - C_\infty \| u \|^2, \quad C_\infty = \max |c_\infty(x)|,
\]

which proves the lower semi boundedness of \( L \).

To show that \( L \) coincides with the Friedrichs extension of the differential operator \(-\Delta_b + c(x)\) in \( C_0^\infty(\mathbb{R}^n \setminus \{0\}) \), let \( \{u_k\} \subset C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) satisfying

\[
s - \lim_{k \to \infty} u_k = u \quad \text{in} \quad L^2,
\]

\[
\lim_{j, k \to \infty} (-\Delta_b + c)(u_j - u_k, u_j - u_k) = 0.
\]

It then follows from (14) that \( \{r^{-1}u_k\} \) forms a Cauchy sequence. Thus, \( r^{-1}u \in L^2 \).

This implies that \( \mathcal{D}(L) \) coincides with the domain of the Friedrichs extension.

Proof of Theorem 1 (iii) Let \( L_1 = L - c_\infty(x) \) with domain \( \mathcal{D}(L_1) = \mathcal{D}(L) \).

Without loss of generality we can assume \( c_\infty(x) \to 0 \) as \( r \to \infty \). Then since

\[
(L_1 u, u) \geq C(\beta) \| \nabla_b u \|^2, \quad C(\beta) = 1 \quad (n = 2), = 1 + \frac{4\beta}{(n-2)^2} \quad (n \geq 3),
\]

we easily see that the multiplication operator \( c_\infty(x) \) is \( L_1 \)-compact. This implies \( \sigma_e(L) = \sigma_e(L_1) \). \( L_1 \) being positive, we conclude (iii).
3. Proof of Theorem 2

First we prepare a quadratic identity for solutions $u$ of equation (1). We put $v = e^{-i\kappa r (n-1)/2}e^{\sigma(r)} u$, $g = e^{-i\kappa r (n-1)/2}e^{\sigma(r)} f$ and rewrite (1) as follows:

$$-\nabla_b \cdot \nabla_b v + \left(-2i\kappa + \frac{n-1}{r} + 2\sigma'\right) \tilde{x} \cdot \nabla_b v
$$

$$+ \left(c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa \sigma'\right) v = g. \quad (15)$$

Let $\phi = \phi(r) = e^{-2i\kappa r - n + 1}\varphi(r)$, where $\varphi(r)$ is a smooth, nonnegative function of $r > 0$. We multiply by $\phi \tilde{x} \cdot \nabla_b v$ on both sides of (15) to obtain

$$-\text{Re} \nabla \cdot \left( (\phi \nabla_b v) \tilde{x} \cdot \nabla_b v \right) + \frac{1}{2} \nabla \cdot (\phi \tilde{x} |\nabla_b v|^2) - \left( \frac{\phi'}{2} + \phi \frac{n-1}{2r} \right) |\nabla_b v|^2$$

$$-\text{Re} \phi \left( \tilde{x} \times \nabla_b v \right) \cdot \left( \nabla \times i\theta \right) v + \phi \left( 2i\kappa r + \frac{n-1}{r} + 2\sigma' \right) |\tilde{x} \cdot \nabla_b v|^2$$

$$+ \text{Re} \phi \left( c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa \sigma' \right) \tilde{x} \cdot \nabla_b v = \text{Re} \left\{ \phi g \tilde{x} \cdot \nabla_b v \right\}.$$

Integrate this over $B_{R,t}$. Then noting

$$\nabla_b v = e^{-i\kappa r (n-1)/2} \left\{ \nabla_b (e^{\sigma} u) + \tilde{x} \left( \frac{n-1}{2r} - i\kappa \right) (e^{\sigma} u) \right\},$$

$$\phi' (r) = \phi(r) \left( -2i\kappa - \frac{n-1}{r} + \frac{\varphi'}{\varphi} \right),$$

we obtain

**Proposition 1.** Let $u \in H^2_{\text{gc}}(\mathbb{R}^n \setminus \{0\})$ solves (1). Put $u_\sigma = e^{\sigma} u$, $f_\sigma = e^{\sigma} f$ and

$$\theta_\sigma = \theta_\sigma(x, \kappa) = \nabla_b u_\sigma + \tilde{x} \left( \frac{n-1}{2r} - i\kappa \right) u_\sigma.$$

Then

$$\left[ \int_{S_t} - \int_{S_{R,t}} \right] \phi \left( -|\tilde{x} \cdot \theta_\sigma|^2 + \frac{1}{2} |\theta_\sigma|^2 \right) dS + \int_{B_{R,t}} \phi \left( \frac{\varphi'}{\varphi} - \frac{1}{r} \right) |\tilde{x} \cdot \theta_b|^2$$

$$+ \left( \text{Im} \kappa - \frac{\varphi'}{2\varphi} + \frac{1}{r} \right) |\theta_\sigma|^2 + 2\sigma' |\tilde{x} \cdot \theta_\sigma|^2 + \text{Re} J_\sigma(x, \kappa)$$

$$= \phi \left( c + \frac{(n-1)(n-3)}{4r^2} + \sigma'' - \sigma'^2 + 2i\kappa \sigma' \right) \tilde{x} \cdot \nabla_b v.$$
\[+ \text{Re} \left[ (\sigma'' - \sigma'') u_{\sigma} \theta_{\sigma} \right] \right\] dx = \text{Re} \int_{B_{R,t}} \varphi f_{\sigma} \bar{x} \cdot \theta_{\sigma} \, dx,\]

where

\[J_{\sigma}(x, \kappa) = -(\bar{x} \times \theta_{\sigma}) \cdot (\nabla \times ib) u_{\sigma} + \left( c + \frac{(n-1)(n-3)}{4r^2} \right) u_{\sigma} \bar{x} \cdot \theta_{\sigma}.\]

In this section we prove Theorem 2 based on this identity.

**Lemma 4.** Let \( u \) be a solution of the homogeneous equation (4). Then for each \( \lambda > 0 \) and \( r > 0 \) we have

\[
\text{Im} \left[ \int_{S_r} (\bar{x} \cdot \nabla_b u_{\sigma}) \bar{\eta}_{\sigma} \, dS \right] = 0,
\]

(16)

\[
\int_{S_r} \left\{ |\bar{x} \cdot \nabla_b u_{\sigma} + \frac{n-1}{2r} u_{\sigma}|^2 + \lambda |u_{\sigma}|^2 \right\} dS = \int_{S_r} |\bar{x} \cdot \theta_{\sigma}|^2 dS,
\]

(17)

where \( \theta_{\sigma} = \theta_{\sigma}(x, \pm \sqrt{\lambda}) \).

Proof. We multiply by \( \bar{\eta} \) on both sides of (4) and integrate by parts over \( B_r \). Then the imaginary part gives

\[-\text{Im} \int_{S_r} (\bar{x} \cdot \nabla_b u) \bar{\eta} dS = 0.\]

\( \sigma(r) \) being real, this implies (16). (17) is obvious from (16). \( \square \)

The following lemma is a direct consequence of (A2) and (17).

**Lemma 5.** Let \( u \) solve (4). Then there exists \( C_3 > 0 \) independent of \( \sigma(r) \) such that

\[
\int_{S_r} |J_{\sigma}(x, \kappa)| dS \leq C_3 \mu(r) \int_{S_r} |\theta_{\sigma}|^2 dS \quad \text{for} \quad r > R_0.
\]

Proof of Theorem 2. We define \( F(r), F_{\sigma,\tau}(r) \) as follows:

\[
F(r) = \frac{1}{2} \int_{S_r} \{2|\bar{x} \cdot \theta|^2 - |\theta|^2 \} \, dS,
\]

\[
F_{\sigma,\tau} = \frac{1}{2} \int_{S_r} \{2|\bar{x} \cdot \theta_{\sigma}|^2 - |\theta_{\sigma}|^2 + (\sigma'^2 - \tau)|u_{\sigma}|^2 \} \, dS,
\]

where \( \theta \) means \( \theta_{\sigma} \) with \( \sigma \equiv 0 \) and \( \tau = \tau(r) > 0 \) is another weight function.

It follows from Proposition 1 with \( \sigma \equiv 0, \varphi \equiv 0, \kappa^2 = \lambda > 0 \) and \( f \equiv 0 \) that

\[
F(t) - F(R) = \int_{B_{R,t}} \left\{ \frac{1}{r} (|\theta|^2 - |\bar{x} \cdot \theta|^2) + \text{Re} J(x, \pm \sqrt{\lambda}) \right\} \, dx,
\]
where $J$ means $J_\sigma$ with $\sigma \equiv 0$. We choose $R_1 \geq R_0$ such that $\frac{1}{r} - C_3 \mu(r) \geq 0$ for $r \geq R_1$. Then differentiating both sides and using Lemma 5, we obtain

$$\frac{d}{dt} F(t) \geq -2C_3 \mu(t) F(t) \quad \text{for} \quad t \geq R_1.$$ 

Assume here that there exists a sequence $r_k \to \infty$ such that $F(r_k) > 0$. Then we can choose $r_k \geq R_1$ to obtain

$$F(t) F(r_k) \geq \exp \left\{ -2C_3 \int_{r_k}^t \mu dr \right\}.$$ 

Since $\mu(r) \in L^1(\mathbb{R}_+)$, this proves the uniform positivity near infinity of $F(t)$.

Next assume the contrary that $F(r) \leq 0$ for $r \geq R_2 (\geq R_1)$, and $u$ does not have compact support. In this case we put $\varphi = r$, $\kappa = \lambda$ and $f \equiv 0$ in Proposition 1, and subtract the identity

$$\frac{1}{2} \left( \int_{S_t} - \int_{S_r} \right) r (\sigma'^2 - \tau) |u_\sigma|^2 - \frac{1}{2} \int_{B_{R,t}} r \left\{ \text{Re} \left[ (\sigma'^2 - \tau) u_\sigma \bar{x} \cdot \theta_\sigma \right] \right. 
+ \left. \left( \frac{1}{r} \sigma'^2 + \sigma'' \sigma' - \frac{1}{r} \tau - \frac{1}{2} r' \right) |u_\sigma|^2 \right\} dx = 0.$$ 

Then it follows that

$$\frac{d}{dt} [t F_{\sigma, \tau}(t)] = \int_{S_t} r \left\{ \frac{1}{2r} |\theta_\sigma|^2 + \text{Re} J_\sigma(r) + 2\sigma' \bar{x} \cdot \nabla_b u_\sigma + \frac{n-1}{2r} |u_\sigma|^2 \right\} dx. \quad (17)$$

We choose here

$$\sigma(r) = \frac{m}{1 - \epsilon} r^{1-\epsilon}, \quad \tau(r) = r^{-2\epsilon} \log r$$

with $m \geq 1$ and $1/3 < \epsilon < 1/2$. Then noting (17) and assumption $\mu(r) = o(r^{-1})$, we can show (as for the details, see e.g. Mochizuki [13]) that there exists $R_3 \geq R_2$ such that for any $m \geq 1$,

$$\frac{d}{dt} [t F_{\sigma, \tau}(t)] \geq \int_{S_t} r \left( \frac{1}{2r} - o(r^{-1}) \right) |\theta_\sigma|^2 dS \geq 0 \quad \text{in} \quad t \geq R_3.$$
Moreover, by assumption there exists $R_4 \geq R_3$ such that $\int_{S_{r_4}} |u_\sigma|^2 dS > 0$. Thus, we can choose $m$ large to satisfy $F_{\sigma, \tau}(R_4) > 0$. Combining these properties, we conclude that $F_{\sigma, \tau}(t) > 0$ for $t \geq R_4$. Note here that

$$F_{\sigma, \tau}(t) = e^{2\sigma} \left\{ F(r) + \sigma' \frac{d}{dr} \int_{S(r)} |u|^2 dS + (2\sigma'^2 - \tau) \int_{S(r)} |u|^2 dS \right\}.$$ 

$F(r) \leq 0$ near infinity by assumption, and the third term of the right becomes nonpositive when $r$ goes large. Hence,

$$\frac{d}{dt} \int_{S(t)} |u|^2 dS > 0$$

for $r$ large enough.

The desired conclusion thus holds.

4. Proof of Theorem 3

We put

$$\varphi_1(r) = \left( \int_r^\infty \mu(s) ds \right)^{-1}.$$ 

Then as is easily seen

$$\varphi'_1(r) = \mu(r)\varphi_1(r)^2,$$ 

and $\mu\varphi_1$ and hence $\varphi' = \mu\varphi^2$ is not in $L^1(\mathbb{R}_+)$. Moreover, it follows from (5) that

$$\frac{\varphi'_1(s)}{\varphi_1(s)} ds = \mu(r)\varphi_1(r) \leq \frac{1}{r} \text{ for } r \geq R_0.$$ 

We shall show that for any $0 < a < b < \infty$, the resolvent $R(\kappa^2) \in B(L^2_{\mu^{-1}}, L^2_{\mu})$ restricted in $\kappa \in K_\pm = \{ \kappa; a \leq \pm \text{Re}\kappa \leq b, 0 < \text{Im}\kappa \leq 1 \}$ is continuously extended to $K_\pm \cup [a, b]$.

The proof is based on Theorem 2 and the following two lemmas.

**Lemma 6.** Let $u = R(\kappa^2)f$ with $\kappa \in K_\pm$ and $f \in L^2_{\mu^{-1}}$. Then there exists $C = C(K_\pm) > 0$ such that $u = R(\kappa^2)f$ satisfies

$$\|\theta\|^2_{\varphi_1^{-1}, B_{R_4}} \leq C \left\{ \|u\|^2_{\mu} + \|f\|^2_{\mu^{-1}} \right\}, \quad R \geq R_4.$$ 

**Proof** We choose $R \geq R_4$ and $t > R + 1$ in Proposition 1, and put $\sigma = 0$ and $\varphi = \chi\varphi_1$ there, where $\chi = \chi(r)$ is a smooth function such that $\chi(r) = 0$ ($r \leq R$) and $\chi(r) = 1$ ($r \geq R + 1$). Then

$$\int_{S_t} \varphi_1 \left( |\hat{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right) dS = \int_{B_{R_4}} \varphi_1 \left( \varphi'_1 \right)^2 \theta^2$$
\[ + \left( \frac{1}{r} - \frac{\varphi_1'}{\varphi_1} \right) (|\theta|^2 - |\hat{x} \cdot \theta|^2) + \frac{1}{2} \text{Re} \left[ J(x, \kappa) - f \hat{x} \cdot \theta \right] \right) dx \\
+ \int_{B_{R,R+1}} \chi' \varphi_1 \left( |\hat{x} \cdot \theta|^2 - \frac{1}{2} |\theta|^2 \right) dx. \quad (21) \]

It follows from (A2) and (19) that
\[ \varphi_1 |J(x, \kappa)| \leq C_0 \varphi_1 \mu |u \hat{x} \cdot \theta| \leq C_0 \varphi_1^{1/2} |\hat{x} \cdot \theta|^{1/2} |u|. \]

Moreover, by the ellipticity of equation (1), we see
\[ \int_{B_{R,R+1}} |\theta|^2 dx \leq C \left\{ \|u\|_{\mu}^2 + \|f\|_{\mu}^{2-1} \right\}. \]

Thus, noting (20), applying the Schwarz inequality and letting \( t \to \infty \) in (21), we obtain the assertion of the lemma.

**Lemma 7.** Let \( \kappa \in K_{\pm} \) and \( f \in L^2_{\mu-1} \). Then there exists \( R_5 \geq R_4 \) such that \( u = R(\kappa^2 f) \) satisfies for \( R \geq R_5 \),
\[ a \|u\|^2_{\mu,B_R} \leq C \varphi_1(R)^{-1} \left\{ \|\hat{x} \cdot \theta\|^2_{\varphi_1^{-1},B_R} + \|u\|^2_{\mu} + \|f\|^2_{\mu-1} \right\}. \]

**Proof** By the Gauss formula
\[ \text{Im} \int_{B_r} f \dd \bar{u} dx = -\text{Im} \int_{S_r} (\hat{x} \cdot \nabla u) \dd \bar{u} dS - \text{Im} \kappa^2 \int_{B_r} |u|^2 dx. \]

It then follows that
\[ \text{Im} \kappa^2 \int_{B_r} |u|^2 dx + \text{Re} \kappa \int_{S_r} |u|^2 dS = -\text{Im} \left[ \int_{S_r} (\hat{x} \cdot \theta) \dd \bar{u} dS + \int_{B_r} f \dd \bar{u} dx \right]. \]

Since \( \text{Im} \kappa^2 \) and \( \text{Re} \kappa \) have the same sign, and since \( \pm \text{Re} \kappa \geq a > 0 \), multiplying by \( \mu(r) \) and integrating over \( (R, \infty) \) with respect to \( r \), we obtain
\[ a \|u\|^2_{\mu,B_R} \leq \int_{B_R} \mu |\hat{x} \cdot \theta||u| dx + \varphi_1(R)^{-1} \int |f||u| dx. \]

The inequality of the lemma then follows from the relation \( \mu^{1/2} = \varphi_1^{-1} \varphi_1^{1/2} \) and the Schwarz inequality. \( \square \)

**Proof of Theorem 3** Let \( \{\kappa_k, f_k\} \subset K_{\pm} \times L^2_{\mu-1} \) converge to \( \{\kappa_0, f_0\} \) as \( k \to \infty \). Since the other case is easier, we assume that \( \pm \kappa_0 = \sqrt{\lambda} \in [a, b] \). Let \( u_k = R(\kappa_k^2 f_k) \). Then since \( \varphi_1(R)^{-1} \to 0 \) as \( R \to \infty \), the Rellich compactness criterion, Lemmas 6 and 7 show that \( \{u_k\} \) is compact in \( L^2_{\mu} \) if it is bounded.
in the same space. Moreover, Lemma 6 shows that every accumulation point \( u_0 \in L^2_\mu \) satisfies the inequality
\[
\left\| \tilde{x} \cdot \nabla_b u_0 + \left( \frac{n-1}{2r} - i\kappa_0 \right) u_0 \right\|_{\varphi'_1} < \infty.
\]
The boundedness of \( \{u_k\} \) is proved by contradiction. In fact, assume that there exists a subsequence, which we also write \( \{u_k\} \), such that \( \|u_k\|_\mu \to \infty \) as \( k \to \infty \). Put \( v_k = u_k / \|u_k\|_\mu \). Then as it is explained above, \( \{\kappa_k, v_k\} \) has a convergent subsequence, and if we denote the limit by \( \{\kappa_0, v_0\} \), then it satisfies the homogeneous equation (4) with \( \lambda = \kappa_0^2 \) and also
\[
\|v_0\|_\mu = 1, \quad \left\| \tilde{x} \cdot \nabla_b v_0 + \left( \frac{n-1}{2r} - i\kappa_0 \right) v_0 \right\|_{\varphi'_1} < \infty. \tag{22}
\]
The second inequality implies
\[
\liminf_{r \to \infty} \int_{S_r} \left| \tilde{x} \cdot \nabla_b v_0 + \left( \frac{n-1}{2r} - i\kappa_0 \right) v_0 \right|^2 dS = 0
\]
since \( \varphi'_1(r) \notin L^1([R_5, \infty)) \). Comparing this with Theorem 2, we see that \( v_0 \) has a compact support in \( x \in \mathbb{R}^n \). Hence, \( v_0 \equiv 0 \) by the unique continuation property for solutions to (4). But this contradicts to the first equation of (22).

We have shown that the sequence \( \{u_k\} \) is precompact in \( L^2_\mu \) and satisfies inequality (22). But if we apply Theorem 2 once more, then \( \{u_k\} \) itself is shown to converge.

The proof of Theorem 3 is thus completed. \( \square \)

5. Proof of Theorem 4

In this section we shall prove Theorem 4 by a series of lemmas for the solution \( u = R(\kappa^2)f \) of (1) (the proof of this and next sections are essentially the same as in Mochizuki [14]).

**Lemma 8.** Let \( \varphi = \varphi(r) \) be a positive increasing function of \( r > 0 \) satisfying
\[
\frac{\varphi'(r)}{\varphi(r)} \leq \frac{1}{r}. \tag{23}
\]
Then we have
\[
\int \varphi \left( \text{Im} \kappa + \frac{\varphi'}{2\varphi} \right) \left\{ |\theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right\} dx
\leq \int \varphi \left( |f| + \max \{|\nabla \times b|, |c|\} |u| \right) |\theta| dx.
\]
Proof In the identity of Proposition 1 we put \( \sigma \equiv 0 \). Then letting \( R \to 0 \) and \( t \to \infty \), we obtain

\[
\int \varphi \left\{ \left( \frac{1}{r} - \frac{\varphi'}{\varphi} \right) ((\theta)^2 - |\tilde{x} \cdot \theta|^2) + \left( \text{Im} \kappa + \frac{\varphi'}{2\varphi} \right) |\theta|^2 + \text{Re} J(x, \kappa) \right\} dx
= \text{Re} \int \varphi \tilde{x} \cdot \theta dx.
\]  

(24)

Here

\[
\varphi \left| J(x, \kappa) \right| - \frac{(n-1)(n-3)}{4r^2} \left| \tilde{x} \cdot \theta \right| \leq \varphi \max \{ |\nabla \times b|, |c| \} |u||\theta|
\]

and

\[
\text{Re} \int \varphi \frac{(n-1)(n-3)}{4r^2} \left| u \right|^2 dx = \int \varphi \left( \text{Im} \kappa - \frac{\varphi'}{2\varphi} + \frac{1}{r} \right) \frac{(n-1)(n-3)}{4r^2} |u|^2 dx.
\]

Substitute these relations to (24). Then assumption (23) on \( \varphi \) and the Schwarz inequality show the inequality of the lemma.

\[ \Box \]

**Lemma 9.** We have

\[
\int \frac{1}{4r^2} |u|^2 dx \leq \int |\tilde{x} \cdot \theta|^2 dx.
\]

Proof We begin with the identity

\[
\left| \nabla_b u \right|^2 = \left| \nabla_b u + \tilde{x} \frac{\alpha}{r} u \right|^2 - \nabla \cdot \left( \tilde{x} \frac{\alpha}{r} |u|^2 \right) + \frac{(n-2)\alpha - \alpha^2}{r^2} |u|^2
\]

which is similar to (9). Multiply by \( \xi = \xi(r) > 0 \) on both sides. Then

\[
\xi |\tilde{x} \cdot \nabla_b u|^2 = |\tilde{x} \cdot \nabla_b (\sqrt{\xi} u) + \frac{\alpha}{r} \sqrt{\xi} u|^2 - \nabla \cdot \left[ \tilde{x} \left( \frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) \sqrt{\xi} u^2 \right]
\]

\[
+ \left\{ \frac{n-1}{r} \left( \frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) + \left( \frac{\alpha}{r} + \frac{\xi'}{2\xi} \right)' - 2 \frac{\alpha}{r} \left( \frac{\alpha}{r} + \frac{\xi'}{2\xi} \right) + \left( \frac{\alpha}{r} + \frac{\xi'}{2\xi} \right)^2 \right\} |\sqrt{\xi} u|^2.
\]

Integrating this over \( B_{\epsilon, t} \), we have

\[
\int_{B_{\epsilon, t}} \xi |\tilde{x} \cdot \nabla_b u|^2 dx = \int_{B_{\epsilon, t}} \left| \tilde{x} \cdot \nabla_b (\sqrt{\xi} u) + \frac{\alpha}{r} \sqrt{\xi} u \right|^2 dx
\]

\[
- \left[ \int_{S_t} - \int_{S_t} \right] \left( \frac{\xi'}{2\xi} + \frac{\alpha}{r} \right) |\sqrt{\xi} u|^2 dS
\]

\[
+ \int_{B_{\epsilon, t}} \left\{ \frac{(n-2)\alpha - \alpha^2}{r^2} + \frac{(n-1)|\xi'|}{2r\xi} + \frac{2\xi''\xi - \xi'^2}{4\xi^2} \right\} |\sqrt{\xi} u|^2 dx.
\]
We here replace \( u \) by \( v = e^{-i\kappa r(n-1)/2}u \) and choose \( \xi = r^{-n+1}e^{-2i\kappa r} \) and \( \alpha = \frac{n-2}{2} \). Then, since
\[
\xi |v|^2 = |u|^2, \quad \xi \langle \tilde{x} \cdot \nabla_b v \rangle = |\tilde{x} \cdot \nabla_b |^2
\]
and
\[
\frac{(n-2)\alpha - \alpha^2}{r^2} + \frac{(n-1)\xi'}{2r\xi} + \frac{2\xi''\xi - \xi'^2}{4\xi^2} = \frac{1}{4r^2} + (\text{Im} \kappa)^2,
\]
letting \( \epsilon \to 0 \) and \( t \to \infty \), we obtain the inequality of the lemma.

Proof of Theorem 4 (i) We choose \( \varphi = r \) in Lemma 8. Then noting (A4), we have for any \( 0 < \epsilon \leq 1 \),
\[
\frac{1}{2} \int \left\{ (1 - \epsilon)|\theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right\} dx \leq \frac{1}{\epsilon} \int \left( r^2 |f|^2 + c_0 r^{-2} |u|^2 \right) dx.
\]
Combining this and Lemma 9 leads us to
\[
-\epsilon^2 + \frac{(n-2)^2 \epsilon - 8\epsilon_0}{8\epsilon} \int \frac{1}{r^2} |u|^2 dx \leq \frac{1}{\epsilon} \int r^2 |f|^2 dx.
\]
Thus, we conclude the inequality of (ii) by choosing \( \epsilon = \min\{\sqrt{8\epsilon_0}, 1\} \). 

LEMMA 10. Assume \( c(x) \geq -\frac{(n-2)^2}{4r^2} \). Then for \( \mu \) satisfying (6) we have
\[
\frac{1}{2} \int \left\{ \mu \text{Im} \frac{1}{r} |u|^2 - \mu \frac{n-1}{r} |u|^2 + \mu \left( |\nabla_b v|^2 + |\kappa u|^2 \right) \right\} dx \leq \frac{1}{2} \int \mu \left( |\theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right) dx + \|\mu\|_{L^1} \int |f(x)||\text{i}k\mu| dx.
\]

Proof We multiply by \(-\text{i}k\mu\) on both sides of (1) and integrate the real part over \( B_r \) to obtain
\[
\frac{1}{2} \int_{S_r} \left\{ -|\nabla_b u - \text{i}\kappa u|^2 + |\nabla_b u|^2 + |\kappa|^2 |u|^2 \right\} dS
\]
\[
+ \text{Im} \kappa \int_{B_r} (|\nabla_b u|^2 + |\kappa|^2 |u|^2) dx = -\text{Re} \int_{B_r} \text{i}k \mu u dx.
\]
Multiply \( \mu(r) \) on both sides and integrate over \( (0, \infty) \). Then noting
\[
\mu|\nabla_b u - \text{i}k\tilde{x}u|^2 = -\nabla \cdot \{ \tilde{x} \mu \frac{n-1}{2r} |u|^2 \} + \mu \left( |\theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right)
\]
we obtain
\[ \frac{1}{2} \int \left\{ \mu \im \frac{n-1}{r} |u|^2 - \mu \frac{n-1}{2r} |u|^2 + \mu \left( |\nabla b u|^2 + |\kappa u|^2 \right) \right\} \, dx \]
\[ + \im \kappa \int_0^\infty \mu dr \int_{B_r} \left( \frac{n-1}{r} |u|^2 + \left| |\nabla b u|^2 + |\kappa u|^2 \right| \right) \, dx \]
\[ = \frac{1}{2} \int \mu \left( |\theta|^2 + \frac{(n-1)(n-3)}{4r^2} |u|^2 \right) \, dx + \re \int_0^\infty \mu dr \int_{B_r} f(x)|\kappa u| \, dx. \]
Note here \( c(x) \geq -\frac{(n-2)^2}{4r^2} \). Then Lemma 1 with \( \alpha = \frac{n-2}{2} \) and \( \epsilon = 0 \) shows
\[ \im \kappa \int_0^\infty \mu dr \int_{B_r} \left( |\nabla b u|^2 + c(x)|u|^2 \right) \, dx \]
\[ \geq -\im \kappa \int_0^\infty \mu dr \int_{S_r} \frac{n-2}{2r} |u|^2 dS = -\int \mu \im \kappa \frac{n-2}{2r} |u|^2 dx, \]
and the desired inequality holds. \( \square \)

**Proof of Theorem 4 (ii)** We combine Lemmas 10 and 8 with \( \varphi(r) = \int_0^r \mu(\sigma) d\sigma \).

It is obvious that this \( \varphi \) satisfies (23). Then since \( \varphi(r) \leq \| \mu \|_{L^1} \), it follows that
\[ \frac{1}{2} \int \left\{ -\mu \frac{n-1}{2r} |u|^2 + \mu \left( |\nabla b u|^2 + |\kappa u|^2 \right) \right\} \, dx \]
\[ \leq 4 \| \mu \|^2_{L^1} \int \mu^{-1} \left( |f|^2 + \max\{|\nabla \times b|, |c|\}|u|^2 \right) \, dx + \| \mu \|_{L^1} \int |f| |\kappa u| \, dx. \]

Thus, noting
\[ \| \mu \|_{L^1} \int |f| |\kappa u| \, dx \leq \| \mu \|^2_{L^1} \int \mu^{-1} |f|^2 \, dx + \frac{1}{4} \int \mu |\kappa u|^2 \, dx, \]
we conclude
\[ \int \left\{ \mu (|\nabla b u|^2 + |\kappa u|^2) - \mu \frac{n-1}{2r} |u|^2 \right\} \, dx \]
\[ \leq 4 \| \mu \|^2_{L^1} \int \mu^{-1} \left( |f|^2 + 4 \max\{|\nabla \times b|, |c|\}|u|^2 \right) \, dx. \] \( (25) \)

The use of (A5) and the inequality of (i) implies that
\[ \int \mu^{-1} \max\{|\nabla \times b|, |c|\}|u|^2 \, dx \leq c_0^2 C_1 \int r^2 |f|^2 \, dx. \]

Thus, substituting this in (25) gives the desired inequality. \( \square \)
6. Proof of Theorem 5

The following proposition summarizes abstract results which allows us to employ the resolvent estimate for a selfadjoint operator to a space-time weighted estimate for the associated evolution equation. As for the proof see, e.g., Mochizuki [14].

Let $\Lambda$ be a selfadjoint operator in the Hilbert space $H$, and for $z \in \mathbb{C} \setminus \mathbb{R}$ let $R(z)$ be the resolvent of $\Lambda$. Suppose that $A$ is a densely defined, closed operator from $H$ to another Hilbert space $H_1$.

**Proposition 2.** Assume that there exists $C > 0$ such that

$$
\sup_{z \in \mathbb{R}} \| R(z) A^* f \|_{H_1} < \sqrt{C} \| f \|_{H_1}
$$

(26)

for $f \in D(A^*)$. Then we have

$$
\left| \int_0^{\pm \infty} \left\| \int_0^t \Lambda e^{-i(t-\tau)} A^* h(\tau) d\tau \right\|^2_{H_1} dt \right| \leq C \left| \int_0^{\pm \infty} \| h(t) \|^2_{H_1} dt \right|,
$$

(27)

$$
\sup_{t \in \mathbb{R}} \left| \int_0^t \Lambda e^{i\tau} A^* h(\tau) d\tau \right|^2_{H_1} \leq 2 \sqrt{C} \left| \int_0^{\pm \infty} \| h(t) \|^2_{H_1} dt \right|
$$

(28)

for each $h(t) \in L^2(\mathbb{R}; D(A^*))$, and

$$
\left| \int_0^{\pm \infty} \| A e^{-it\Lambda} f \|^2_{H_1} dt \right| \leq 2 \sqrt{C} \| f \|^2_{H_1}
$$

(29)

for each $f \in H$.

**Proof of Theorem 5 (i)** Set $\Lambda = L$, $H = H_1 = L^2$ and $A = r^{-1}$ (multiplication operator). Then $A^* = A$ and $R(z) = R(z)$, and if we let $z = \kappa^2$, then it follows from Theorem 4 (i) that

$$
\| A R(z) A^* f \| = \| r^{-1} R(z) A^* f \| \leq \sqrt{C_1} \| r A^* f \| = \sqrt{C_1} \| f \|.
$$

Thus, the estimates (27) and (29) can be written as

$$
\left| \int_0^{\pm \infty} \| r^{-1} \int_0^t e^{-i(t-\tau)L} h(\tau) d\tau \|^2_{H_1} dt \right| \leq C_1 \left| \int_0^{\pm \infty} \| r h(t) \|^2_{H} dt \right|,
$$

$$
\left| \int_0^{\pm \infty} \| r^{-1} e^{-itL} f \|^2_{H_1} dt \right| \leq 2 \sqrt{C_1} \| f \|^2.
$$

These are what to be proved. \qed
To show Theorem 5 (ii) we consider the Klein-Gordon equation
\[ i \partial_t u = \Lambda u, \quad u(t) = \{w(t), \partial_t w(t)\}, \quad \Lambda = \begin{pmatrix} 0 & i \\ -i(L + m^2) & 0 \end{pmatrix} \]
in the energy space \( \mathcal{H} = H^1_b \times L^2 \), where \( H^1_b \) is the completion of \( C_0^{\infty}(\Omega) \) in the norm
\[ \|f_1\|_{H^1_b}^2 = \frac{1}{2} \int \{ |\nabla_b f_1|^2 + (c(x) + m^2)|f_1|^2 \} dx. \]
Then \( \Lambda \) with domain
\[ D(\Lambda) = \{ f_1 \in H^1_b; \Delta_b f_1 \in L^2 \} \times \{ f_2 \in H^1_b \cap L^2 \} \]
forms a selfadjoint operator in \( \mathcal{H} \), and its resolvent is given by
\[ \mathcal{R}(z) = (L + m^2 - z^2)^{-1} \begin{pmatrix} z \\ -i(L + m^2) \end{pmatrix}. \]
Let \( A : \mathcal{H} \to H_1 = L^2 \) be defined by
\[ Af = \min\{ \sqrt{\mu(r)}, r^{-1} \} \sqrt{L + m^2} f_1 \text{ for } f = \{ f_1, f_2 \} \in \mathcal{H}. \]
Then the adjoint operator \( A^* \) is given by
\[ A^* g = \left\{ \sqrt{L + m^2}^{-1} \min\{ \sqrt{\mu(r)}, r^{-1} \} g, 0 \right\} \text{ for } g \in L^2. \]

**Proof of Theorem 5 (ii)** By definition
\[ AR(z)A^* g = \min\{ \sqrt{\mu(r)}, r^{-1} \} z(L + m^2 - z^2)^{-1} \min\{ \sqrt{\mu(r)}, r^{-1} \} g \quad (30) \]
for \( g \in D(A^*) \). Then since
\[
\int \left| \min\{ \sqrt{\mu}, r^{-1} \} z(L + m^2 - z^2 - 2)^{-1} f \right|^2 dx \leq m^2 \int r^{-2}(L + m^2 - z^2)^{-1} f^2 dx \\
+ \int \mu | - m^2 + z^2 ||L(L + m^2 - z^2)^{-1} f|^2 dx,
\]
using Theorem 4, we obtain
\[ \|AR(z)A^* g\| \leq \sqrt{m^2C_1 + C_2}\|g\|. \]
We return to Proposition 3 with this inequality. Then (29) shows that
\[
\left| \int_0^{\pm\infty} \| Ae^{-it\Lambda} f \|^2 dt \right| = \left| \int_0^{\pm\infty} \| \min\{ \sqrt{\mu(r)}, r^{-1} \} \sqrt{L + m^2 w(t)} \|^2 dy \right|
\]
\[
\leq 2\sqrt{m^2C_1 + C_2}\|f\|_H^2.
\]

Since
\[
w(t) = \cos(t\sqrt{L + m^2})f_1 + \sqrt{L + m^2}^{-1}\sin(t\sqrt{L + m^2})f_2,
\]
choosing \(f = \{\sqrt{L + m^2}^{-1}g, 0\}\) and \(f = \{0, g\}\) for \(g \in L^2\), we obtain
\[
\left|\int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\cos(t\sqrt{L + m^2})g\|_2^2 dy\right| \leq \sqrt{m^2C_1 + C_2}\|g\|_2^2
\]
and
\[
\left|\int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\sin(t\sqrt{L + m^2})g\|_2^2 dy\right| \leq \sqrt{m^2C_1 + C_2}\|g\|_2^2,
\]
respectively. These inequalities imply assertion (ii).

\section*{7. An extension of Theorem 4 (i) and final remarks}

As is proved in Theorems 3 and 4, the resolvent \(R(\kappa^2)\) of \(L\) satisfies the following properties under \((A_1)\) and \((A_4)\).

(a) As an operator from \(L^2_{\xi-1}\) to \(L^2_{\xi}\), where \(\xi = (1 + r)^{-2}\), \(R(\kappa^2)\) is continuously extended to \(\kappa \in C_+: \{\kappa \in C; \text{Im}\kappa \geq 0\}\).

(b) \(R(\kappa^2) \in \mathcal{B}(L^2_{\xi-1}, L^2_{\xi})\), \(\kappa \in C_+\), is a compact operator.

(c) \((1 + |\kappa|)\|R(\kappa^2)\|_{\mathcal{B}(L^2_{\xi-1}, L^2_{\xi})}\) is uniformly bounded in \(\kappa \in C_+\).

Now, let us consider a perturbation of \(L\):
\[
L_2 = L + c_2(x),
\]
where \(c_2(x)\) is a real valued \(L^\infty\)-function satisfying
\[
(A6) \quad |c_2(x)| \leq C(1 + r)^{-2} \quad \text{for some} \quad C > 0,
\]
and the unique continuation property holds for \(-\Delta_\delta + c(x) + c_2(x)\).

\((A7)\) \(L_2\) has at most a finite number of negative eigenvalues, and \(\kappa^2 = 0\) is neither an eigenvalue nor a resonance of \(L_2\).

Under these conditions Theorem 4 (i) is easily extended to the operator \(L_2\).

\textbf{Theorem 6.} Let \(-\lambda_0\) be the largest negative eigenvalue of \(L_2\). Choose \(\delta > 0\) to satisfy \(\delta^2 < \lambda_0\), and let \(C_{\kappa, \delta} = \{\kappa \in C; 0 < \text{Im}\kappa \leq \delta\}\). Then there exists \(C > 0\) such that
\[
\int (1 + r)^{-2}|R_2(\kappa^2)f|^2 dx \leq C \int (1 + r)^2|f|^2 dx
\]
for each \(\kappa \in C_{\kappa, \delta}\) and \(f\) satisfying \((1 + r)f \in L^2\).
Finally, we summarize related problems not proved in this article.

1. The finiteness of negative eigenvalues should be ascertained under condition (A6).

2. It is not known, whether the essential spectrum $\sigma_e(L)$ of $L$ fills the non-negative real line or not.

3. The inhomogeneous smoothing property corresponding to the first estimate of Theorem 5 (i) is not obtained here for the relativistic Schrödinger equation (9).

4. What happens when the smallness of the magnetic field $\nabla \times b(x)$ like (A4) is not required?

References


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