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Proofs of the Results in "Boundary-Value Problems for Weakly Nonlinear Ordinary Differential Equations"

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Abstract. *We provide the proofs of the results, announced in [1], concerning the existence (and non-existence) of solutions to Dirichlet and periodic boundary value problems associated to second order differential equations with asymmetric nonlinearities.*

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1. Introduction

In our paper [1], published in 1976, we have announced results on the Dirichlet and periodic boundary value problems for the equation

$$-x''(t) = g(x(t)) - f(t) \tag{1}$$

on $[0, \pi]$ (though we could also study Neumann boundary conditions). Here, $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $f \in L^1[0, \pi]$. We were interested in understanding the set of integrable functions f for which this problem has a solution. In the following, we refer to (1_D) for the Dirichlet problem, and to (1_p) for the π-periodic problem associated with (1).

We were interested in cases where $g(y)/y$ has limits at plus and minus infinity (possibly infinite). In particular, we discussed degenerate cases where both limits are finite and the limit problem has a non-trivial solution satisfying the boundary conditions, or in cases where one or both limits is plus infinity. Most of the other cases were largely covered in earlier work. A second motivation was to try to understand what might be true for analogous nonlinear elliptic boundary value problems. The idea was that there seemed to be no chance of a good result for the elliptic problem unless the results for the ordinary differential equation case were similar for the two different types of boundary conditions. Note that in two dimensions one thinks of a thin annulus as a perturbation of a circle (at least intuitively).

At the end of that paper, we wrote that

> "as the full proofs are long and complicated, the author does not at present plan to publish them, but will reconsider this if he has sufficient evidence of interest in them."

After so many years, in which those handwritten unpublished proofs have been used and cited by several mathematicians, we think it could be of some interest to make them more easily available. This is why, mostly motivated by the interest shown in all these years on the arguments introduced in [1], which have been quoted by many authors and further developed in several interesting papers, we decided to finally publish them in the present form.

It will be useful to recall the main results in [1]. Throughout the paper, we assume that the limits

$$\mu = \lim_{y \to +\infty} \frac{g(y)}{y}, \qquad \nu = \lim_{y \to -\infty} \frac{g(y)}{y}$$

exist (possibly $\pm\infty$). Let

$$X_D = \{u \in C^1[0,\pi] : u(0) = u(\pi) = 0,\ u'' \in L^1[0,\pi]\}$$

and define $H_D : X_D \to L^1[0,\pi]$ by $(H_D x)(t) = x''(t) + g(x(t))$. Note that X_D is a Banach space under the norm $\|x\| + \|x''\|$, where $\|x\|$ denotes the L^1-norm of x. Let R_D denote the range of H_D. Similar definitions are given for X_p, H_p and R_p in the case of the periodic boundary conditions on $[0,\pi]$.

In Section 1 of [1], we considered the case when μ, ν are finite and positive, and we stated three theorems. We assume that the limits

$$I^{\pm} = \lim_{y \to \pm\infty} (g(y) - \mu y^+ + \nu y^-)$$

exist, where $y^+ = \max\{y, 0\}$, $y^- = \max\{-y, 0\}$, and the limits are allowed to be infinite. Let ϕ_β denote the solution of $-x''(t) = \mu(x(t))^+ - \nu(x(t))^-$ for which $\phi_\beta(0) = 0$ and $\phi'_\beta(0) = \beta$. We will also write ϕ_+ for ϕ_{+1} and ϕ_- for ϕ_{-1}.

We first consider the Dirichlet problem, for which we have the following two theorems. We assume $\mu \neq \nu$, and that there exist positive integers k and l such that $|k - l| \leq 1$ and $k\mu^{-1/2} + l\nu^{-1/2} = 1$.

THEOREM 1.1. *Suppose that $k = l$.*

(i) *If* $I^+ \neq I^-$ *and at least one of* I^+ *and* I^- *is infinite,* H_D *is proper and onto.*

(ii) *If* I^+ *and* I^- *are both finite, let*

$$T = \left\{ f \in L^1[0,\pi] : \int_0^\pi f_{+1}(t)\,\mathrm{d}t \int_0^\pi f_{-1}(t)\,\mathrm{d}t > 0 \right\},$$

where $f_{\pm 1}(t) = f(t)\phi_{\pm 1}(t) - I^+(\phi_{\pm 1}(t))^+ + I^-(\phi_{\pm 1}(t))^-$. *Then,* T *is nonempty and* $T \subseteq R_D$. *Moreover,* R_D *contains a relatively closed unbounded proper subset of* $L^1[0,\pi]\backslash\bar{T}$.

THEOREM 1.2. *Suppose that* $k - l = 1$ *and* $\mu > \nu$.

(i) *If* $I^+ \neq I^-$ *and either* $I^+ = -\infty$ *or* $I^- = \infty$, *then* H_D *is proper and onto.*

(ii) *If* $I^+ \neq I^-$ *and either* $I^+ = \infty$ *or* $I^- = -\infty$, *then* H_D *is proper,* R_D *is closed, and* $R_D \neq L^1[0,\pi]$.

(iii) *If* I^- *and* I^+ *are both finite, let*

$$T = \left\{ f \in L^1[0,\pi] : \int_0^\pi f_{+1}(t)\,\mathrm{d}t > 0 \right\}.$$

Then, T *is non-empty and* $T \subseteq R_D$. *Moreover,* R_D *contains a relatively closed unbounded proper subset of* $L^1[0,\pi]\backslash\bar{T}$.

Concerning the periodic boundary value problem, we stated the following theorem, where we assumed that $\mu \neq \nu$ and that there is a positive integer k such that $k\mu^{-1/2} + k\nu^{-1/2} = 1$.

THEOREM 1.3. (i) *If at least one of* I^- *and* I^+ *is infinite and* $I^- \neq I^+$, *then* H_p *is proper and onto.*

(ii) *If both* I^- *and* I^+ *are finite, let*

$$F(\theta) = \int_0^\pi f(t)\phi_{+1}(t+\theta)\,\mathrm{d}t - 2k(\mu^{-1}I^+ - \nu^{-1}I^-)\,,$$
$$T = \left\{ f \in L^1[0,\pi] : F(\theta) \neq 0 \textit{ for every } \theta \in [0,\pi] \right\}\,,$$

and

$$S = \left\{ f \in L^1[0,\pi] : (F(\theta))^2 + (F'(\theta))^2 > 0 \textit{ for every } \theta \in [0,\pi] \right\}\,.$$

Then, T *is non-empty,* $T \subseteq R_p$, *and* $(R_p \cap S)\backslash T$ *is a relatively closed non-empty proper subset of* $S\backslash T$.

In Section 2 of [1] we stated two further theorems. We consider there the case where $\mu = \infty$ and $\nu = i^2$, where i is a positive integer for (1_D) and a non-negative integer for (1_p). We assume that $I^- = \lim_{y\to-\infty} g_i(y)$ exists (where $g_i(y) = g(y) - i^2 y$) and that there exists an $M > 0$ such that $g(y) + M(y-x) + M \geq g(x)$ if $y \geq x \geq M$. If $I^- = -\infty$, we also assume that for every $\varepsilon > 0$ there exist $N_1, N_2 > 0$ such that $g_i(y) \leq (1-\varepsilon)g_i(x)$ if $y \leq x \leq -N_1$ and $g_i(y) \geq (1+\varepsilon)g_i(x)$ if $N_2 x \leq y \leq x \leq -N_1$.

We first consider the Dirichlet problem. If $I^- = -\infty$ and $i > 0$, define

$$P^l = \liminf_{\alpha\to\infty} \frac{T(\alpha)}{S(\alpha)}\,, \quad P^u = \limsup_{\alpha\to\infty} \frac{T(\alpha)}{S(\alpha)}\,,$$

where

$$S(\alpha) = -\frac{1}{i\alpha}\int_0^{\pi/i} g_i\Big(-\frac{\alpha}{i}\sin it\Big)\sin it\,\mathrm{d}t\,,$$

and $T(\alpha)$ is the first positive zero of the solution of $-x''(t) = g(x(t))$, $x(0) = 0$, $x'(0) = \alpha$.

Theorem 1.4. *(i) If $i = 1$ and I^- is finite, then*

$$\Big\{f \in L^1[0,\pi] : \int_0^\pi f(t)\sin t\,\mathrm{d}t > 2I^-\Big\} \subseteq R_D\,.$$

Moreover, if $g(y) - y > I^-$ for all y, equality holds.

(ii) Suppose that

(a) $i > 1$ and $I^- > -\infty$, or

(b) $i = 1$ and $I^- = \infty$, or

(c) $i = 1$, $I^- = -\infty$ and $P^u < \frac{1}{2}$, or

(d) $i > 1$, $I^- = -\infty$ and either $P^l > i(i-1)^{-1}$, or $P^u < i(i+1)^{-1}$.

Then H_D is proper, R_D is closed, and $R_D \neq L^1[0,\pi]$.

(iii) Suppose that

(a) $i = 1$, $I^- = -\infty$ and either $P^l > 1$ or $\frac{1}{2} < P^l \leq P^u < 1$, or

(b) $i > 1$, $I^- = -\infty$ and either $i(i+1)^{-1} < P^l \leq P^u < 1$, or $1 < P^l \leq P^u < i(i-1)^{-1}$.

Then, H_D is proper and onto.

Concerning the periodic problem, we stated the following

THEOREM 1.5. *(i) If $i = 0$, (1_p) is solvable if $\int_0^\pi f(t)\,\mathrm{d}t > \pi I^-$. This condition is also necessary if I^- is finite and $g(y) > I^-$ for all y.*

(ii) Suppose that $i \geq 1$ and

(a) $I^- > -\infty$, or

(b) $I^- = -\infty$ and either $P^u < 1$ or $P^l > 1$.

Then, H_p is proper and onto.

This paper is organized as follows. In Section 2, we prove some technical lemmas on the distance between zeros of solutions of the differential equations.

In Section 3, we prove a simple abstract result and, in Section 4, we prove the results in Section 1 of [1], i.e., Theorems 1.1, 1.2 and 1.3 above.

In Section 5, we prove the results in Section 2 of [1], i.e., Theorems 1.4 and 1.5 above.

2. Technical Estimates

Subsection (i)

Suppose that z is a solution of

$$-x''(t) = g(x(t)) - f(t) \tag{2}$$

on $[0, \pi]$. Assume that $yg(y) \geq 0$ for y large. Thus we can write $g = g_1 + g_2$ where g_1 and g_2 are continuous, $yg_1(y) \geq 0$ for all y, $|g_2(y)| \leq K$ for all y and g_2 has compact support. Let

$$G(y) = \int_0^y g_1(u)\,\mathrm{d}u$$

and

$$E(t) = \frac{1}{2}(z'(t))^2 + G(z(t))\,.$$

By adding a constant to g and f (and with a little more care), we may assume that $\int_0^\infty g_2(u)\,\mathrm{d}u = 0$ (and thus $G(y) = \int_0^y g(u)\,\mathrm{d}u$ if y is large). This will be convenient later (in Subsection (iii)).

LEMMA 2.1. *(i) There exist K_1, K_2, depending only on $\|f\|_1$ (and K), such that $E(t_1) \leq K_2(E(t_2) + K_1)$ for $t_1, t_2 \in [0, \pi]$.*

(ii) There exists K_3 depending only on $\|f\|_1$ such that

$$|E(t_1) - E(t_2)| \leq K_3\sqrt{E(t_3) + K_1} \qquad \text{for} \qquad t_1, t_2, t_3 \in [0, \pi]\,.$$

Proof. Let us first prove (i).

$$\begin{aligned} |E'(t)| &\le |f(t)+g_2(z(t))||z'(t)| \\ &\le (|f(t)|+K)((z'(t))^2+1) \\ &\le (|f(t)|+K)+(|f(t)|+K)E(t)\,. \end{aligned} \tag{3}$$

Hence (i) follows from Gronwall's inequality.
Let us now prove (ii).

$$|E(t_1)-E(t_2)| \le \int_{t_1}^{t_2} |E'(t)|\,\mathrm{d}t \le \int_{t_1}^{t_2} (|f(t)|+K)|z'(t)|\,\mathrm{d}t\,.$$

The result follows from this, since $|z'(t)| \le \sqrt{2E(t)} \le K_3\sqrt{E(t_3)+K_2}$ by part (i). □

This lemma shows that, if $E(t)$ is large at one point, then $E(t)$ is large throughout the interval and, *in this case*, the variation of $E(t)$ across the interval is of smaller order than $E(t)$.

Subsection (ii)

In this case, we assume that

$$|g(y)-\mu y| \le \tau y + M\,, \tag{4}$$

for all $y \ge 0$, where $\mu > 0$, $\tau < \mu$. Suppose that z is a solution of (2) on $[t_0, t_1]$ (where $0 \le t_0 \le \pi$, $t_0 \le t_1 \le 2\pi$ and f is extended to $[0, 2\pi]$ by periodicity) and assume that $z(t_0) = 0$, $z(t_1) = 0$ (or $t_1 = 2\pi$), $z(t) \ge 0$ on $[t_0, t_1]$ and $z'(t_0) = \alpha$, where α is large and positive.

In the following, we will only prove our results for the case $t_0 = 0$, but we will usually state the final results for general t_0. Now $E(0) = \frac{1}{2}\alpha^2$. Thus, by Lemma 2.1(ii), $|E(t) - \frac{1}{2}\alpha^2| \le K_4\alpha$ on $[0, t_1]$ if α is large. Since

$$0 < \mu - \tau \le \liminf_{y\to\infty} \frac{2G(y)}{y^2} \le \limsup_{y\to\infty} \frac{2G(y)}{y^2} \le \mu + \tau\,,$$

it follows that $z(t) \le (\mu-\tau)^{-1}(E(t))^{1/2} \le K_5\alpha$ on $[0, t_1]$, if α is large. Let $w(t) = \alpha^{-1}z(t)$. Then $w(0) = 0$, $w'(0) = 1$ and, on $[0, t_1]$, $w''(t) - \mu w(t) = h(t)$, where $h(t) = \alpha^{-1}[g(z(t)) - \mu z(t) - f(t)]$. Then,

$$|h(t)| \le \alpha^{-1}[\tau|z(t)| + M + |f(t)|] \le \tau K_5 + \alpha^{-1}(M + |f(t)|)$$

(by our inequality for $z(t)$). Thus h is small in L^1 if τ is small and α is large. Hence, by continuous dependence, $w(t)$ is near $\mu^{-1/2}\sin\sqrt{\mu}\,t$ in the C^1-norm on $[0, t_1]$ if τ is small and α is large. (This is easily proved by using the Green's function for the initial-value problem for $w''(t) + \mu w(t)$.) Thus we have proved:

Lemma 2.2. *Given $\epsilon > 0$, there is a $\tau > 0$ such that, if $|g(y) - \mu y| \le \tau y + M$ for $y \ge 0$, $\|f\|_1 \le M$ and $\alpha \ge T(= T(\epsilon, M))$, then $\|\alpha^{-1} z(t) - \phi_+(t - t_0)\|'_\infty \le \epsilon$ (where $\phi_+(t) = \mu^{-1/2} \sin \sqrt{\mu}\, t$ and $\|\cdot\|'_\infty$ is the usual C^1-norm restricted to $[t_0, t_1]$).*

Let $\tilde{t} = t_0 + \mu^{-1/2}\pi$ and *assume that* $\tilde{t} < 2\pi$. By Lemma 2.2, either $z(t)$ has its first zero in $(0, 2\pi]$ close to but less than or equal to $\tilde{t}$, or $z(t) \ge 0$ on $[0, \tilde{t} + a]$ (where $a > 0$). (Note that we have reverted to assuming that $t_0 = 0$.) Lemma 2.2 then implies that $\alpha^{-1} z(\tilde{t}+a)$ is close to $\phi_+(\tilde{t}+a)$. Since $z(\tilde{t}+a) \ge 0$ and $\phi_+(\tilde{t} + a) = \mu^{-1/2} \sin \sqrt{\mu}\, a < 0$, this is only possible if a is small. Thus, if $\tilde{t} < 2\pi$, $z(t)$ must have its first zero in $(0, t_1]$ near $\tilde{t}$, i.e., t_1 is near $\tilde{t}$. We now want a more precise estimate for t_1. Let

$$Z(t) = z(t)\phi'_+(t) - z'(t)\phi_+(t)\,.$$

Since z is a solution of (2),

$$Z'(t) = [g(z(t)) - \mu z(t) - f(t)]\phi'_+(t)\,.$$

By integrating from 0 to t_1, we get

$$-z'(t_1)\phi_+(t_1) = \int_0^{t_1} [g(z(t)) - \mu z(t) - f(t)]\phi'_+(t)\,\mathrm{d}t\,. \tag{5}$$

Since t_1 is near $\tilde{t}$, $\phi_+(t_1) = -(t_1 - \tilde{t})(1 + \omega(t_1 - \tilde{t}))$, where $\omega(r) \to 0$ as $r \to 0$. Moreover, by our earlier estimate for $E(t)$, $|\frac{1}{2}(z'(t_1))^2 - \frac{1}{2}\alpha^2| \le K_4\alpha$. Hence, $-\alpha - 2K_4 < z'(t_1) < -\alpha + 2K_4$ if α is large (remember that $z'(t_1) \le 0$). Thus $z'(t_1) = -\alpha(1 + \alpha^{-1}s(\alpha))$ where $|s(\alpha)| \le 2K_4$. So, equation (5) becomes

$$\alpha(t_1 - \tilde{t})(1 + \omega(t_1 - \tilde{t}))(1 + \alpha^{-1}s(\alpha)) = -\int_{t_0}^{t_1} [g(z(t)) - \mu z(t) - f(t)]\phi'_+(t - t_0)\,\mathrm{d}t \tag{6}$$

(where we have given the formula for the general case). By integrating $Z'(t)$ from t_0 to $\tilde{t}$, we also have

$$-z(\tilde{t}) = \int_{t_0}^{\tilde{t}} [g(z(t)) - \mu z(t) - f(t)]\phi'_+(t - t_0)\,\mathrm{d}t \tag{7}$$

(since $\phi'_+(\tilde{t}) = -1$). Equations (6) and (7) will now be used to obtain our estimates for t_1.

Lemma 2.3. *Suppose that $\epsilon > 0$ and $g(y) - \mu y \ge S$ if $y \ge y_0$. There exists $\tau(= \tau(\epsilon, S)) > 0$ such that, if $|g(y) - \mu y| \le \tau y + M$ for $y \ge 0$, $\|f\|_1 \le M$ and $\alpha \ge T(= T(\epsilon, S, M))$, then*

$$t_1 - \tilde{t} \le \alpha^{-1}\left[\int_{\tilde{t}}^{t_1} (f(t) - S)\phi_+(t - t_0)\,\mathrm{d}t + \epsilon\right].$$

Proof. If τ is small and α is large, t_1 is near $\tilde{t}$ and $\alpha^{-1}z(t)$ is near $\phi_+(t)$ on $[0, t_1]$. Hence, $z(t) \geq y_0$ on $[0, t_1]$ except possibly near the endpoints. We can make the part of $[0, t_1]$ where this inequality fails arbitrarily small and, on this part of the interval, $g(z(t)) - \mu z(t)$ is bounded (because $0 \leq z(t) \leq y_0$).

Case (i) $t_1 \leq \tilde{t}$. Thus $\phi_+1(t) \geq 0$ on $[0, t_1]$ and the right hand side of equation (6) becomes (for $t_0 = 0$)

$$\int_0^{t_1} f(t)\phi_+(t)\,\mathrm{d}t - \int_A [g(z(t)) - \mu z(t)]\phi_+(t)\,\mathrm{d}t - \int_B (g(z(t)) - \mu z(t))\phi_+(t)\,\mathrm{d}t \leq \int_0^{t_1} f(t)\phi_+(t)\,\mathrm{d}t - S\int_A \phi_+(t)\,\mathrm{d}t + m(B)K_6$$

where $A = \{t \in [0, t_1] : z(t) \geq y_0\}$, $B = [0, t_1]\backslash A$ and $K_6 = \sup\{|g(y) - \mu y| : 0 \leq y \leq y_0\}$. Since $m(B)$ and $t_1 - \tilde{t}$ are small if τ is small and α is large (and thus $\int_A \phi_+(t)\,\mathrm{d}t$ is near $\int_0^{t_1} \phi_+(t)\,\mathrm{d}t$), the result follows from this inequality and equation (6).

Case (ii) $t_1 > \tilde{t}$. In this case, there is an additional difficulty because $\phi_+(t) < 0$ on $[\tilde{t}, t_1]$. However, if we have a bound for $z(t)$ on $[\tilde{t}, t_1]$, we can estimate the integral from $\tilde{t}$ to t_1 (in equation (6)) by the same argument as we used for the integral over B in case (i) and the proof can be completed as in case (i). Since $\phi_+(t) \geq 0$ on $[0, \tilde{t}]$, we can use equation (7) and, by estimating the right hand side of (7) by a similar argument to that in case (i) (noting that $z(t) \geq 0$ on $[0, \tilde{t}]$), we find that $z(\tilde{t}) \leq K_7$. Since t_1 is near $\tilde{t}$, $\alpha^{-1}z'(t)$ is near $\phi'_+(t)$ and $\phi'_+(\tilde{t}) < 0$, we see that $z'(t) \leq 0$ on $[\tilde{t}, t_1]$. Hence $0 \leq z(t) \leq K_7$ on $[\tilde{t}, t_1]$. So, by our comments above, the proof can be completed as before. $\square$

Remark. By the lemma, τ and T can be chosen to work simultaneously for the sequence of function $\{g(y) + \frac{1}{n}y\}_{n \geq n_0}$ provided that $g(y) - \mu y \geq S$ for $y \geq y_0$, $y^{-1}g(y) \to 0$ as $y \to \infty$, and n_0 is sufficiently large.

LEMMA 2.4. *Suppose that* $\epsilon > 0$ *and* $g(y) - \mu y \leq S$ *if* $y \geq y_0$. *There exists* $\tau (= \tau(\epsilon, S))$ *such that, if* $|g(y) - \mu y| \leq \tau y + M$ *for* $y \geq 0$, $\|f\|_1 \leq M$ *and* $\alpha \geq T (= T(\epsilon, S, M))$, *then*

$$t_1 - \tilde{t} \geq \alpha^{-1}\left[\int_{t_0}^{t_1} (f(t) - S)\phi_+(t - t_0)\,\mathrm{d}t - \epsilon\right].$$

Proof. Case (i) $t_1 \leq \tilde{t}$. This is similar to the proof of case (i) of Lemma 2.3.

Case (ii) $t_1 > \tilde{t}$. As in case (ii) of Lemma 2.3, it suffices to show that $R := \int_{\tilde{t}}^{t_1} (g(z(t)) - \mu z(t))\phi_+(t)\,\mathrm{d}t$ is small. Let $K_8 = \int_{t_0}^{t_1} (f(t) - S)\phi_+(t)\,\mathrm{d}t + 1$. If $t_1 - \tilde{t} \geq \alpha^{-1}K_8$, the result is trivial. If not, $|\phi_+(t)| \leq [t - \tilde{t}| \leq \alpha^{-1}K_8$ on

$[\tilde{t}, t_1]$. Since $|g(z(t)) - \mu z(t)| \leq \tau z(t) + M \leq \tau K_5 \alpha + M$, it follows that if $t_1 - \tilde{t} \leq \alpha^{-1} K_8$, then

$$|R| \leq K_8(\tau K_5 + \alpha^{-1} M)\,,$$

and thus R is small if τ is small and α is large. Hence the result follows. □

A remark similar to the one after Lemma 2.3 is true (except that we consider $\{g(y) - \frac{1}{n}y\}_{n \geq n_0}$).

LEMMA 2.5. *Suppose that $g(y) - \mu y \to I^+$ (where I^+ is finite) as $y \to +\infty$. Then, given $\epsilon > 0$, there is a $T > 0$ such that*

$$\left| t_1 - \tilde{t} - \alpha^{-1} \int_{t_0}^{t_1} (f(t) - I^+)\phi_+(t - t_0)\,\mathrm{d}t \right| \leq \alpha^{-1}\epsilon$$

if $\|f\|_1 \leq M$ and $\alpha \geq T (= T(\epsilon, M))$.

Proof. This follows from Lemmas 2.3 and 2.4. □

This is Lemma 1 of [1].

Remark. Lemma 2.5 remains true if we replace $\int_{t_0}^{t_1}$ by $\int_{\tilde{t}_0}^{\tilde{t}_1}$ where $\tilde{t}_0 \to t_0$ as $\alpha \to +\infty$ and $\tilde{t}_1 \to t_1$ as $\alpha \to +\infty$. This follows if we can show that the change to the integral is small. To see this, note that

$$\left| \int_{t_1}^{\tilde{t}_1} (I^+ - f(t))\phi_+(t - t_0)\,\mathrm{d}t \right| \leq \sup\{|\phi_+(t - t_0)| : t_1 \leq t \leq \tilde{t}_1\}\tilde{K} \leq \epsilon\tilde{K}$$

(since $\phi_+(\tilde{t}) = 0$ and $t_1, \tilde{t}_1$ are near $\tilde{t}$ if α is large). Similar comments apply to our other lemmas (including Lemma 2.6 later). Moreover, we could replace $\phi_+(t - t_0)$ by $\phi_+(t - \tilde{t}_0)$.

Finally, for this subsection, we need a more precise estimate for $t_1 - \tilde{t}$ when $g(y) - \mu y = r(t) + \frac{1}{n}y$, where $y^{-1}r(y) \to 0$ as $y \to \infty$, $r(y) \to \infty$ as $y \to \infty$ and we assume that, for every $\epsilon > 0$, there exist $N_1, N_2 > 0$ such that $|r(y)| \geq (1 - \epsilon)|r(x)|$ if $y \geq x \geq N_1$ and $|r(y)| \leq (1 + \epsilon)|r(x)|$ if $N_1 \leq x \leq y \leq N_2 x$. Let

$$S_\mu(\alpha) = \alpha^{-1} \int_0^{\mu^{-1/2}\pi} r(\alpha\phi_+(t))\phi_+(t)\,\mathrm{d}t\,.$$

LEMMA 2.6. *Given $\epsilon > 0$, there exists $n_0 > 0$ such that, if $\|f\|_1 \leq M$, $n \geq n_0$ and $\alpha \geq T (= T(\epsilon, M))$, then*

$$t_1 - \tilde{t} \leq -(1 - \epsilon)S_\mu(\alpha)\,.$$

(Here we allow n to be ∞.)

Proof. As before, it suffices to estimate L, the right hand side of equation (6). Now

$$L \leq -\int_{t_0}^{t_1} r(z(t))\phi_+(t)\,\mathrm{d}t - \frac{1}{n}\int_0^{t_1} z(t)\phi_+(t)\,\mathrm{d}t + M\,.$$

Since $\tilde{t}$ is near t_1 and $\alpha^{-1}z(t)$ is near $\phi_+(t)$, the second term is non-positive. Since obviously the first term tends to $-\infty$ as $\alpha \to \infty$, the result will follow if we show that the ratio of the first term and $\alpha S_\mu(\alpha)$ tends to 1 as $n \to \infty$ and $\alpha \to \infty$. We merely sketch the easy but tedious proof of this. One shows (using our regularity of growth assumptions on r) that the contribution to both integrals (remember that $\alpha S_\mu(\alpha)$ is defined by an integral) from near the endpoints is relatively small (compared with the integrals over the central portion) and that, over the most of the interval,

$$(1-\epsilon')r(\alpha\phi_+(t)) \leq r(z(t)) \leq (1+\epsilon')r(\alpha\phi_+(t))\,.$$

(Remember that $\alpha^{-1}z(t)$ is near $\phi_+(t)$. Here we assume the other regularity of growth assumption on r.) Hence the integrals over "most" of the interval are asymptotically the same and so the result follows. □

Remarks 1. If we considered the equality $g(y) - \mu y = r(y) - \frac{1}{n}y$, we would obtain a similar result except that the inequality becomes $t_1 - \tilde{t} \geq -(1+\epsilon)S_\mu(\alpha)$. By combining this result with Lemma 2.6, we see that, if $g(y) - \mu y = r(y)$, then $(t_1 - \tilde{t})/S_\mu(\alpha) \to -1$ as $\alpha \to \infty$ uniformly in f for $\|f\|_1 \leq M$.

2. Similar results hold if $r(y) \to -\infty$ as $y \to \infty$.

3. The argument in the proof of Lemma 2.6 can be used to show that, if $M > 0$,

$$\frac{S_\mu(\alpha+t)}{S_\mu(\alpha)} \to 1 \quad \text{as} \quad \alpha \to \infty \quad \text{uniformly in } t \text{ for } |t| \leq M\,.$$

Subsection (iii)

In this subsection, we consider the case where $y^{-1}g(y) \to \infty$ as $y \to \infty$. We define t_1 as before and $T(\alpha)$ as in [1]. (Although g is only continuous, $T(\alpha)$ is well defined if α is large.)

We also assume that there exists an $M_1 > 0$ such that $g(y) + M_1(y-x) + M_1 \geq g(x)$ if $y \geq x \geq 0$. (This is equivalent to the assumption in [1].) Much of the work below is true without this assumption. Suppose that $z'(0) = \alpha$, where α is large, and $\|f\|_1 \leq M$ (where z is a solution of equation (2) with $z(0) = 0$). By Lemma 2.1(ii), there is a $K_1 > 0$ such that

$$\frac{1}{2}(\alpha - K_1)^2 < E(t) < \frac{1}{2}(\alpha + K_1)^2 \tag{8}$$

on $[0, t_1]$. Thus, if $|z(t)| \leq C$, $|z'(t)| \sim \alpha$. In particular, $z'(t) \neq 0$, and the time for a solution to move from zero to C is asymptotic to $\frac{C}{\alpha}$. We first prove that $t_1 \to 0$ as $\alpha \to \infty$. If $n > 0$, there is a $K(n)$ such that

$$g(y) - n^2 y \geq K(n) \quad \text{for} \quad y \geq 0 \quad (\text{since } y^{-1}g(y) \to \infty \text{ as } y \to \infty)\,.$$

Let $\phi(t) = n^{-1} \sin nt$ and $W(t) = z\phi' - \phi z'$. By differentiating and using equation (2), we find that $W'(t) \geq -|f(t)| - |K(n)|$ and hence $W(t) \geq K_2$ on $[0, t_1]$. If $t_1 \geq t_2 := n^{-1}\pi$, we deduce by putting $t = t_2$ that $z(t_2) \leq K_2$. Since $z(t)$ must be large when $z'(t) = 0$ (by equation (8)), $z(t) > K_2$ for $t \sim \frac{2K_2}{\alpha}$ and since $z'(t) \neq 0$ if $z(t) \leq K_2$ (see earlier), $z'(t_2) < 0$. Hence $z'(t) \sim -\alpha$ on $[t_2, t_1]$ (by equation (8), cp earlier). Thus $t_1 \leq t_2 + \frac{2K_2}{\alpha}$ if α is large. Since n was arbitrary, it follows that $t_1 \to 0$ as $\alpha \to \infty$.

We want more precise estimates for t_1. First note that since $G(y) \to \infty$ as $y \to \infty$ and $G'(y) > 0$ if y is large, $G^{-1}(z)$ is well defined and increasing if z is large. Let $x_{\max} = \sup\{z(t) : 0 \leq t \leq t_1\}$. Since $G(x_{\max}) \geq \frac{1}{4}\alpha^2$ if α is large (by equation (8)), $x_{\max} \geq G^{-1}(\frac{1}{4}\alpha^2)$. Also by equation (8), $|z'(t)| \leq 2\alpha$ on $[0, t_1]$. Since

$$t_1 \geq \frac{2x_{\max}}{\sup\{|z'(t)| : 0 \leq t \leq t_1\}}\,,$$

we see that

$$t_1 \geq \alpha^{-1} G^{-1}\left(\frac{1}{4}\alpha^2\right). \tag{9}$$

We need more precise estimates for t_1. The idea is to compare $z(t)$ with solutions x_1, x_2 satisfying

$$-x_1''(t) = g(x_1(t)), \quad x_1(0) = 0, \quad x_1'(0) = \alpha - 2K_1\,,$$

and

$$-x_2''(t) = g(x_2(t)), \quad x_2(0) = 0, \quad x_2'(0) = \alpha + 2K_1\,.$$

Let $E_1(t), E_2(t)$ denote the functions obtained when, in the definition of $E(t), z(t)$ is replaced by $x_1(t)$ and $x_2(t)$ respectively. By equation (8), we see that $E_1(t) < E(t) < E_2(t)$ on $[0, t_1]$.

It follows that $z(t) \geq x_1(t)$ as long as $z'(s) \geq 0$ on $[0, t]$. This follows because otherwise there is v in $(0, t]$ such that $z(v) = x_1(v)$ and $x_1'(v) \geq z(v)$. (Remember that $z(t) > x_1(t)$ for small non-zero t.) Thus $E_1(v) \geq E(v)$, which contradicts the result of the previous paragraph. Similarly $z(t) \leq x_2(t)$ as long as $x_2'(s) \geq 0$ on $[0, t]$. We need extra estimates. To do this we use our assumptions on the regularity of the growth of g. Using this and by making a simple calculation, we see that, as long as $x_1(t) \leq z(t)$,

$$(z - x_1)''(t) \leq M_1(z - x_1)(t) + |f(t)| + M_1\,. \tag{10}$$

By the usual comparison argument, it follows that $(z - x_1)(t) \leq W(t)$ where $W''(t) = M_1 W(t) + |f(t)| + M_1$, $W(0) = 0$, $W'(0) = (z - x_1)'(0) = 2K_1$. By substituting this estimate back in (10) and integrating, we get that $(z-x_1)'(t) \leq K_3$ as long as $z(t) \geq x_1(t) \geq 0$ i.e.,

$$z'(t) - x_1'(t) \leq K_3 \tag{11}$$

on the same interval. Similarly,

$$x_2'(t) - z'(t) \leq K_3$$

as long as $z(t) \leq x_2(t)$. Let t_2 be the first point where $z'(t) = 0$, t_3 be the first point where $x_1'(t) = 0$ and t_4 be the first point where $x_1'(t) = -K_3$. If $t_2 \geq t_4$, we see from what we have already proved that $x_1(t) \leq z(t)$ on $[0, t_2]$. Thus, by equation (11), $z'(t_4) + K_3 \leq K_3$, i.e. $z'(t_4) \leq 0$ and thus $t_2 \leq t_4$. *Hence, we always have that* $t_2 \leq t_4$. By a similar argument, $t_5 \leq t_6$ where t_5 is the first positive zero of $x_2'(t)$ and t_6 is the smallest t where $z'(t) = -K_3$. Note that $t_3 = \frac{1}{2}T(\alpha - 2K_1)$ and $t_5 = \frac{1}{2}T(\alpha + 2K_1)$. We let t_7 be the largest t in $[0, t_1]$ for which $z'(t) = -K_3$. Obviously $t_7 \geq t_6$. We now want to estimate $t_7 - t_2$, $t_4 - t_3$ and $\frac{t_5}{t_3}$. We need two sublemmas whose proofs we defer till later.

Sublemma 1. *$yg(y) \geq \frac{1}{2}G(y)$ if y is large.*

Sublemma 2. *$\frac{T(\alpha+u)}{T(\alpha)} \to 1$ as $\alpha \to \infty$ uniformly in u for $|u| \leq C_1$ (for each $C_1 > 0$).*

We estimate $t_7 - t_2$. First note that $g(y) \geq -K$ on $[0, \infty)$ (where K was defined in Subsection (i)). Hence, if $0 \leq t_8 \leq t_9 \leq t_1$, $z'(t_9) - z'(t_8) \leq \int_{t_8}^{t_9} (K + f(t))\,\mathrm{d}t \leq K_4$ (since z is a solution of equation (2) and $t_1 \leq 2\pi$). Since $z'(t_2) = 0$ and $z'(t_7) = -K_3$, a tedious but easy argument shows that we can deduce that $|z'(t)| \leq |K_3| + |K_4|$ on $[t_2, t_7]$. Hence, by equation (8), $G(z(t)) \geq \frac{1}{4}\alpha^2$ on $[t_2, t_7]$ if α is large. Since this implies that $z(t)$ is large on $[t_2, t_7]$, $z(t)g(z(t)) \geq \frac{1}{2}G(z(t)) \geq \frac{1}{8}\alpha^2$ on $[t_2, t_7]$. However, by equation (8), $G(z(t)) \leq \alpha^2$ if α is large and hence $z(t) \leq G^{-1}(\alpha^2)$. Thus

$$g(z(t)) \geq \frac{\alpha^2}{8G^{-1}(\alpha^2)}\,.$$

Now

$$\begin{aligned} -K_3 = -z'(t_7) + z'(t_2) &= \int_{t_2}^{t_7} [g(z(t)) - f(t)]\,\mathrm{d}t \\ &\geq (t_7 - t_2)\frac{\alpha^2}{8G^{-1}(\alpha^2)} - M \end{aligned}$$

(by the previous line). Thus

$$(t_7 - t_2) \le \frac{G^{-1}(\alpha^2)K_5}{\alpha^2}\,.$$

Since $y^{-1}g(y) \to \infty$ as $y \to \infty$, $y^{-2}G(y) \to \infty$ as $y \to \infty$ and thus $G^{-1}(\alpha^2) = O(\alpha)$. Thus we see that $t_7 - t_2 \le \alpha^{-1}K_6$ if α is large. By the derivation of equation (9), we see that, for any $C > 0$, $\alpha t_2 \ge C$ if α is large enough (and similar results hold for t_3 and t_5). Thus $t_7 - t_2 = o(\inf(t_2, t_3, t_5))$ as $\alpha \to \infty$ (uniformly in f for $\|f\|_1 \le M$). Since $t_5 \le t_6 \le t_7$ (see earlier), it follows that $t_2 \ge (1-\epsilon)t_5$ if α is large. Since we can establish by similar arguments that $t_4 - t_3 = o(\inf(t_2, t_3))$, we find that $t_2 \le (1+\epsilon)t_3$ (since $t_2 \le t_4$). However, by Sublemma 2, $\frac{t_3}{t_5} \to 1$ as $\alpha \to \infty$. Hence we have that $\frac{t_2}{t_3} \to 1$ as $\alpha \to \infty$ (uniformly in f for $\|f\|_1 \le M$). Also, by Sublemma 2, $\frac{t_3}{\frac{1}{2}T(\alpha)} \to 1$ as $\alpha \to \infty$. Hence we eventually find that $\frac{t_2}{\frac{1}{2}T(\alpha)} \to 1$ as $\alpha \to \infty$ (uniformly in f for $\|f\|_1 \le M$).

Thus we have estimated the time for $z(t)$ to reach the point where $z'(t) = 0$. (In fact, there may be more than one such point but any such point lies between t_2 and t_7 and hence the difference between the first and last such point is $o(T(\alpha))$.) Since we can use a similar argument to estimate the time for $z(t)$ to move from $z'(t) = 0$ back to zero, we have established the following lemma. (Where, as usual, we have included a t_0.)

Lemma 2.7 (Lemma 2 of [1]). *Given $\epsilon, M > 0$ there is a $T > 0$ such that $|t_1 - t_0 - T(\alpha)| \le \epsilon T(\alpha)$ if $\alpha \ge T$ and $\|f\|_1 \le M$.*

In fact, we have yet to establish Sublemmas 1 and 2.

Proof of Sublemma 1. Since $g(x) \le g(y) + M_1(y-x) + M_1$ if $0 \le x \le y$, we find by integrating that, if y is large,

$$\begin{aligned} G(y) = \int_0^y g(x)\,\mathrm{d}x &\le g(y)y + \frac{1}{2}M_1y^2 + M_1y \\ &\le g(y)y + \frac{1}{2}G(y) \qquad \text{if } y \text{ is large} \end{aligned}$$

(since $y^{-2}G(y) \to \infty$ as $y \to \infty$). Hence the result follows. □

Proof of Sublemma 2. It obviously suffices to prove the result for $0 \le u \le C_1$. The proof that $\limsup_{\alpha\to\infty} \frac{T(\alpha+u)}{T(\alpha)} \le 1$ is similar to the proof that $t_2 \ge (1-\epsilon)t_5$. (Let x_α denote the solution of $-x''(t) = g(x(t))$ for which $x_\alpha(0) = 0$, $x'_\alpha(0) = \alpha$. By using the constancy of $\frac{1}{2}(x'_\alpha(t))^2 + \bar{G}(x_\alpha(t))$ in time (where $\bar{G}(y) = \int_0^y g(u)\,\mathrm{d}u$), one shows (cp earlier) that $x_{\alpha+u}(t) \ge x_\alpha(t)$ as long as $x'_{\alpha+u}(t) \geq 0$ and then the proof follows closely the proof that $t_2 \geq (1-\epsilon)t_5$ and hence we omit the details).

We must be a little more careful to prove that $\liminf_{\alpha\to\infty} \frac{T(\alpha+u)}{T(\alpha)} \geq 1$. It suffices to consider the case where $T(\alpha+u) \leq T(\alpha)$.

If t_{10} denotes the point where $x'_{\alpha+u}(t) = 0$, an earlier argument (cp the derivation of equation (11)) implies that $x_{\alpha+u}(t_{10}) - x_\alpha(t_{10}) \leq K_6$. Since the maximum of $x_{\alpha+u}$ is greater than the maximum of x_α (because the first is $\bar{G}^{-1}(\frac{1}{2}(\alpha+u)^2)$ and the second is $\bar{G}^{-1}(\frac{1}{2}\alpha^2)$), it follows that, at t_{10}, x_α is within K_6 of its maximum value. (Remember $x_{\alpha+u}$ achieves its maximum at t_{10}.)

Define t_{12} to be the point where $x'_\alpha(t) = 0$ and t_{11} to be the point in $[t_{10}, t_{12}]$ where $x'_\alpha(t_{11}) = \frac{1}{2}\alpha$ (Set $t_{11} = t_{10}$ if no such point exists.) On $[t_{10}, t_{11}]$, $x'_\alpha(t) \geq \frac{1}{2}\alpha$ and $x_\alpha(t_{11}) - x_\alpha(t_{10}) \leq x_\alpha(t_{12}) - x_\alpha(t_{10}) \leq K_6$. Hence $t_{11} - t_{10} \leq 2\alpha^{-1}K_6$. On $[t_{11}, t_{12}]$, $0 \leq x'_\alpha(t) \leq \frac{1}{2}\alpha$ and hence, since $\frac{1}{2}(x'_\alpha(t))^2 + \bar{G}(x_\alpha(t)) = \frac{1}{2}\alpha^2$, $\bar{G}(x_\alpha(t)) \geq \frac{1}{4}\alpha^2$, i.e. $G(x_\alpha(t)) \geq \frac{1}{4}\alpha^2$ (since $\bar{G}(y) = G(y)$ if y is large). Thus, since $yg(y) \geq \frac{1}{2}G(y)$ for y large, $g(x_\alpha(t)) \geq \frac{1}{8}x_m^{-1}\alpha^2$ on $[t_{11}, t_{12}]$ (where x_m is the maximum of $x_\alpha(t)$). Since $-x''_\alpha(t) = g(x_\alpha(t)) \geq \frac{1}{8}x_m^{-1}\alpha^2$ on $[t_{11}, t_{12}]$, we see by the two integrations that $x_\alpha(t) \leq x_m - \frac{1}{16}(t-t_{12})^2 x_m^{-1}\alpha^2$ on $[t_{11}, t_{12}]$. Since $x_m - x_\alpha(t_{11}) \leq K_6$, it follows that

$$t_{12} - t_{11} \leq 4\alpha^{-1}(x_m)^{\frac{1}{2}} K_6^{\frac{1}{2}} .$$

However, $t_{12} \geq \frac{1}{2}\alpha^{-1}x_m$ (cp the derivation of equation (9)). Thus $t_{12} - t_{11}$ is $o(t_{12})$. Similarly $t_{11} - t_{10}$ is $o(t_{12})$. Thus $t_{12} - t_{10}$ is $o(t_{12})$ i.e. $\frac{t_{12}}{t_{10}} \to 1$ as $\alpha \to \alpha$. This is the required result. (A glance through the proof shows that it is uniform in u for $0 \leq u \leq C_1$.) □

Remarks. 1. In applications, it often happens that $yg(y) \geq 0$ for all large y. In this case, it follows easily from Lemma 2.1(ii) that there is $\bar{K} > 0$ depending only on k and $\|f\|_1$ such that

$$\Big| |z'(t_3)| - |z'(t_4)| \Big| \leq \bar{K} ,$$

whenever $t_3, t_4 \in [0, \pi]$, $z(t_3) = 0$ and $z(t_4) = 0$. Thus, if we try to apply the results of this section between each successive pair of zeros, the corresponding α's differ only by $\bar{K}$ in modulus. By checking each of our results (using some of the remarks after them) we see that this change of α has only a higher order effect on our estimates (i.e. it can be incorporated in the ϵ in each case). Thus the change of α does not matter very much.

2. Obviously we could also obtain analogues of our results for the case when $z(t_0) = z(t_1) = 0$, $z(t) < 0$ on $[t_0, t_1]$.

Appendix to Section 2

It turns out that we need one more lemma.

LEMMA 2.8. *Suppose that $\epsilon > 0$. Then there exist δ, $\alpha_0 > 0$ such that*

$$\frac{T(\alpha+u)}{T(\alpha)} \le 1+\epsilon \quad \text{if} \quad \alpha \ge \alpha_0 \quad \text{and} \quad |u| \le \delta|\alpha|\,.$$

Proof. It obviously suffices to assume that $u > 0$. We use the notation in the proof of Lemma 2.7. We need only consider the case where $T(\alpha+u) > T(\alpha)$. In this case, arguing as in the proof of Lemma 2.7, we see that

$$(x_{\alpha+u}(t) - x_\alpha(t))'' \le M_1(x_{\alpha+u}(t) - x_\alpha(t)) + M_1 \tag{12}$$

for $0 \le t \le \frac{1}{2}T(\alpha)$ and thus that

$$x_{\alpha+u}(t) - x_\alpha(t) \le W(t) \le K_1 ut + K_2$$

(by explicitly calculating W). By substituting this back in (12) and integrating we get that

$$x'_{\alpha+u}\left(\frac{1}{2}T(\alpha)\right) = x'_{\alpha+u}\left(\frac{1}{2}T(\alpha)\right) - x'_\alpha\left(\frac{1}{2}T(\alpha)\right) \le K_3u + K_4\,.$$

Thus $x'_{\alpha+u}(t) \le K_3u + K_4$ for $\frac{1}{2}T(\alpha) \le t \le \frac{1}{2}T(\alpha+u)$. Hence, if δ is small, $x'_{\alpha+u}(t) \le \frac{1}{2}\alpha$ for t in the same range. Since $\frac{1}{2}x'_\alpha(t)^2 + \bar{G}(x_\alpha(t)) = \frac{1}{2}(\alpha+u)^2$ it follows that $G(x_{\alpha+u}(t)) \ge \frac{1}{4}(\alpha+u)^2$ for t in the same range. Since $g(y) \ge \frac{1}{2}y^{-1}G(y)$ for y large, it follows that $g(x_{\alpha+u}(t)) \ge \frac{1}{8}(\bar{x}_m)^{-1}\alpha^2$ for t as above (where $\bar{x}_m = G^{-1}((\alpha+u)^2)$). Since $-x''_{\alpha+u}(t) = g(x_{\alpha+u}(t))$, it follows by integrating this equation from $\frac{1}{2}T(\alpha)$ to $\frac{1}{2}T(\alpha+u)$ that

$$K_3u + K_4 \ge \frac{1}{2}(T(\alpha+u) - T(\alpha))\frac{1}{8}(\bar{x}_m)^{-1}\alpha^2,$$

i.e.

$$T(\alpha+u) - T(\alpha) \le 16\alpha^{-2}\bar{x}_m(\delta K_3\alpha + K_4) \qquad (\text{since } u \le \delta\alpha)\,.$$

However (cp equation (9)), $\frac{1}{2}T(\alpha+u) \ge (\alpha+u)^{-1}\bar{x}_m$. Thus, if δ is small and α is large, $T(\alpha+u) - T(\alpha) \le \epsilon' T(\alpha+u)$. The result follows easily from this. $\square$

Some of the above ideas can be used to prove that there exist $K_5, K_6 > 0$ such that $\alpha^{-1}K_5G^{-1}\left(\frac{1}{2}\alpha^2\right) \le T(\alpha) \le \alpha^{-1}K_6G^{-1}\left(\frac{1}{2}\alpha^2\right)$ for α large and that $T(\alpha) \sim \frac{2G^{-1}\left(\frac{1}{2}\alpha^2\right)}{\alpha}$ if g grows sufficiently rapidly. These are useful estimates for $T(\alpha)$.

3. Abstract Results

Assume that $H : X \to Y$ is positive homogeneous (i.e. $H(\lambda x) = \lambda H(x)$ if $\lambda \geq 0$), where X and Y are Banach spaces. Let $R(H)$ denote the range of H.

LEMMA 3.1. *(i) If $G : X \to Y$ satisfies $\|G(x)\| \leq K$ for $x \in X$ and $z \notin \overline{R(H)}$, then $tz \notin R(H+G)$ if t is sufficiently large and positive. (Thus, if $R(H)$ is not dense in Y, $R(H+G) \neq Y$.)*

(ii) If $v \in R(H)$, then, for each $t > 0$, there is a $u(t) \in R(H+G)$ such that $\|u(t) - tv\| \leq K$.

Proof. (i) If $z \notin \overline{R(H)}$, there is an $a > 0$ such that $\|H(x) - z\| \geq a$ for all x in H. If $ta > K$, $tz \notin R(H+G)$. This follows because, if $H(x) + G(x) = tz$, then $ta \leq \|H(x) - tz\| = \|G(x)\| \leq K$.

(ii) Since $v \in R(H)$, $v = H(u)$ where $u \in X$. Set $u(t) = H(tu) + G(tu) \in R(H+G)$. Then $\|u(t) - tv\| = \|H(tu) + G(tu) + tH(u)\| = \|G(tu)\| \leq K$. $\square$

The following lemma is well-known and we omit the proof. Assume that $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and let $x(t,a)$ denote the solution of $x'(t) = f(t, x(t))$ for which $x(0,a) = a$.

LEMMA 3.2. *If K is a compact connected subset of $\mathbb{R}^n$ and if there is a $K_1 > 0$ such that $\|x(t,a)\| \leq K_1$ when $a \in K$ and $t \in [0,\pi]$, then $\{x(\pi,a) : a \in K\}$ is also compact and connected.*

This immediately justifies shooting arguments when we do not have uniqueness (but have a bound of the type in the lemma).

4. Proof of the First Three Stated Theorems

In this section, we prove Theorems 1.1, 1.2 and 1.3.

4.1. The Dirichlet Problem

Define $\bar{H}_D : X_D \to L^1[0,\pi]$ by $\bar{H}_D(x)(t) = x''(t) + \mu(x(t))^+ - \nu(x(t))^-$,

$$\bar{f}_\pm(t) = f(t)\phi_\pm(t) \quad \text{and} \quad T_1 = \left\{ f \in L^1[0,\pi] : \int_0^\pi \bar{f}_+ \int_0^\pi \bar{f}_- < 0 \right\}.$$

Note that the corresponding formula for H_D in [1] had an incorrect sign and that it is easy to prove that

$$L^1[0,\pi] \backslash \overline{T}_1 = \left\{ f \in L^1[0,\pi] : \int_0^\pi \bar{f}_+ \int_0^\pi \bar{f}_- > 0 \right\}.$$

Moreover, a similar result holds for $L^1[0,\pi]\backslash\bar{T}$.

If $\bar{H}_D(u) = f$ and $\phi_+(\pi) = 0$, we see that, integrating by parts (and noting that $\bar{H}_D(\phi_+) = 0$),

$$(\mu - \nu)\left[\int_A u\phi_+ - \int_B u\phi_+\right] = \int_0^\pi f\phi_+\,, \tag{13}$$

where $A = \{x \in [0,\pi] : u(x) > 0, \phi_+(x) < 0\}$ and $B = \{x \in [0,\pi] : u(x) < 0, \phi_+(x) > 0\}$. Moreover, a similar result holds if ϕ_+ is replaced by ϕ_- (when $\phi_-(\pi) = 0$).

LEMMA 4.1. *Let $f(t) = c\chi_{[a,b]}$, where $c < 0$ and $\chi_{[a,b]}$ is the characteristic function of $[a,b]$ and assume that z is a solution of $-x''(t) = \mu(x(t))^+ - \nu(x(t)) - f(t)$ on $[t_0,t_1]$, where $z(t_0) = 0$, $t_0 \le a < b \le t_1$ and $z(t) \ne 0$ on $(t_0,a]$ if $t_0 < a$.*

(i) If $z'(t_0) \le 0$, then $z(t) \ne 0$ on $[t_0,d]$ where d is the infimum of t_1 and $t_0 + \nu^{-\frac{1}{2}}\pi$.

(ii) If $z'(t_0) > 0$ and $t_0 + \mu^{-\frac{1}{2}}\pi \le t_1$, then z has a zero t_2 in $(t_0, t_0 + \mu^{-\frac{1}{2}})$. Moreover $z'(t_2) \ne 0$ and the next zero t_3 of z (if one exists in $[t_0,t_1]$) satisfies $t_3 - t_2 \le \nu^{-\frac{1}{2}}\pi$.

Proof. This follows easily from a Wronskian argument (cp the proof of Proposition 1(i) in [2]. □

LEMMA 4.2. *Under the assumption of Theorem 1.1, there exist f_1, f_2 in T_1 such that $f_1 \notin R(\bar{H}_D)$ and $f_2 \in R(\bar{H}_D)$.*

Proof. We first find f_1. Choose x_0 in $(0,\pi)$ so that $\pi - x_0 < \pi \inf\{\mu^{-1/2}, \nu^{-1/2}\}$ and hence, by the easily constructed explicit formulae for ϕ_+ and ϕ_-, $\phi_+(t) < 0$ and $\phi_-(t) > 0$ on $[x_0,\pi)$. Hence, if we define $f_1(t) = c\chi_{[x_0,\pi]}$ where $c < 0$, $f_1 \in T_1$. Moreover, if we let ϕ_α denote the solution of $-x''(t) = \mu(x(t))^+ - \nu(x(t))^- - f_1(t)$ for which $x(0) = 0$, $x'(0) = \alpha$, we see that, if $\alpha \ge 0$, then $\phi_\alpha(t) = \alpha\phi_+(t)$ for $t \le x_0$. Hence we apply Lemma 4.1 to deduce that $\phi_\alpha(\pi) < 0$ if $\alpha \ge 0$. (Set $t_0 = x_0$ and $t_0 = \pi - \nu^{-\frac{1}{2}}\pi$ if $\alpha > 0$.) Similarly $\phi_\alpha(\pi) < 0$ if $\alpha < 0$. Thus $\phi_\alpha(\pi) \ne 0$ for all α and thus $f_1 \notin R(\bar{H}_D)$.

Now $\phi_+(t) < 0$ and $\phi_-(t) > 0$ on $[x_0,\pi)$. Choose u smooth with support in $[x_0,\pi)$ such that u takes both positive and negative values. Let $f_2 = \bar{H}_D(u)$. Hence $f_2 \in R(\bar{H}_D)$. Since the support of u is contained in $[x_0,\pi]$ and ϕ_+ is negative on this interval, equation (13) implies that $\int_0^\pi f_2\phi_+ < 0$ (where for simplicity we are assuming that $\mu > \nu$). Similarly, $\int_0^\pi f_2\phi_- > 0$. Hence we have constructed the required f_2. □

Proof of Theorem 1.1. (i) Let ϕ_α denote a solution of equation (2) for which $\phi_\alpha(0) = 0$, $\phi'_\alpha(0) = \alpha$. If α is large we see by applying Lemma 2.2 between successive zeros (and by using Remark 1, after Lemma 2.7), that $\alpha^{-1}\phi_\alpha(t)$ is near $\phi_+(t)$ on $[0,\pi]$ in the C^1 norm. We only prove part (i) when $I^+ = \infty$ and I^- is finite. The other cases are similar. Lemma 2.3 implies that for any $C > 0$ the distance between successive zeros of z (between which z is positive) (where $z = \phi_\alpha$) is less then $\frac{\pi}{\sqrt{\mu}} - \alpha^{-1}C$ if α is large (because we can take S arbitrarily large since $I^+ = \infty$).

On the other hand, Lemma 2.5 implies that the distance between successive zeros of z (between which z is negative) is $\frac{\pi}{\sqrt{\nu}} + W(\alpha)$, where $|W(\alpha)| \le \frac{K}{\alpha}$ (since I^- is finite). Thus, choosing C large, we see that the $2k^{\text{th}}$ positive zero of z is less than $\frac{k\pi}{\sqrt{\mu}} + \frac{k\pi}{\sqrt{\nu}}$ i.e. less than π. However, this zero is near π since $\alpha^{-1}z(t)$ is near $\phi_+(t)$. Now, near π, $\alpha^{-1}z'(t)$ is near $\phi_+'(t)$ which is positive (and not near zero). Thus it follows that $\phi_\alpha(\pi) > 0$ for α large. Similarly, $\phi_\alpha(\pi) < 0$ for α large negative. Hence, by a shooting argument, there is an α with $\phi_\alpha(\pi) = 0$. Hence $f \in R(H_D)$ as required. Moreover, since our estimates hold uniformly in f for $f \in L^1[0,\pi]$, the above argument shows that, if $M > 0$, then there is an α_0 such that $\phi_\alpha(\pi) \neq 0$ if ϕ_α is a solution of (2) with $\|f\|_1 \le M$ and $|\alpha| \ge \alpha_0$. Hence, if $H_D(z) = f$ where $\|f\|_1 \le M$, then $|z'(0)| \le \alpha_0$. Hence, by Lemma 2.1, we have a bound for $E(t)$ uniformly in f for $\|f\|_1 \le M$. Hence we have a bound for z in C^1-norm uniformly in f for $\|f\|_1 \le M$. Properness follows easily from this. (Note that we have in fact proved that H_D^{-1} maps bounded sets of $L^1[0,\pi]$ to bounded sets in X_D.)

(ii) We define ϕ_α as in part (i) and note as there that $\alpha^{-1}z(t)$ is near $\phi_+(t)$ if α is large and positive. Thus the zeros of $z(t)$ are near those of $\phi_+(t)$. Hence in applying Lemma 2.5, we can replace t_0 and t_1 by the corresponding zeros of ϕ_+. (Here we are using the remark after Lemma 2.5.) Hence we find that, if α is large, then the $2k^{\text{th}}$ zero of $z(t)$ is at $\pi + \alpha^{-1}\int_0^\pi f_{+1}(t)\,\mathrm{d}t + o(\alpha^{-1})$ (where the $o(\alpha^{-1})$ term satisfies this condition uniformly in f for $\|f\|_1 \le M$). Hence, as in part (i), we find that if $\int_0^\pi f_{+1}(t)\,\mathrm{d}t \neq 0$, then $\phi_\alpha(\pi) \neq 0$ if α is large and $\operatorname{sgn}\phi_\alpha(\pi) = -\operatorname{sgn}\int_0^\pi f_{+1}(t)\,\mathrm{d}t$. Similarly, if $\int_0^\pi f_{-1}(t)\,\mathrm{d}t \neq 0$, then $\phi_\alpha(\pi) \neq 0$ if α is large and negative and $\operatorname{sgn}\phi_\alpha(\pi) = \operatorname{sgn}\int_0^\pi f_{-1}(t)\,\mathrm{d}t$. Thus $T \subseteq R(H_D)$, by a shooting argument. Since $|\int_0^\pi f_{+1}(t)\,\mathrm{d}t|$ and $|\int_0^\pi f_{-1}(t)\,\mathrm{d}t|$ have positive lower bounds on compact subsets of $L^1[0,\pi]\backslash\bar{T}$ and since our estimate for the $2k^{\text{th}}$ zero of $\phi_\alpha(t)$ holds uniformly on bounded subsets of $L^1[0,\pi]$, it follows easily that, for a compact subset of H of $L^1[0,\pi]\backslash\bar{T}$, there is an $\alpha_0 > 0$ such that $\phi_\alpha(\pi) \neq 0$ if $f \in H$ and $|\alpha| \ge \alpha_0$. As in part (i), it follows that there is a $K > 0$ such that $\|z\|_\infty \le K$ if $H_D(z) \in H$. A simple compactness argument now ensures that $R(H_D) \cap H$ is closed and thus $R(H_D) \cap (L^1[0,\pi]\backslash\bar{T})$ is closed.

Let f_1 be that constructed in Lemma 4.2. Remembering that $H_D = \bar{H}_D +$ a bounded map, Lemma 3.1 implies that $tf_1 \notin R(H_D)$ if t is large. On the other hand, it is easy to check that $tf_1 \notin L^1[0,\pi]\backslash\bar{T}$ if t is large (since $f_1 \in T_1$). Finally Lemma 3.1 implies that, for a suitable $K > 0$, there is a $u(t)$ in $R(H_D)$

with $\|u(t)-tf_2\| \le K$. It follows by a simple calculation that $u(t) \in L^1[0,\pi]\backslash\bar{T}$ if t is large. This proves Theorem 1.1. (T is non-empty because it contains step functions with suitable small support.) □

Proof of Theorem 1.2. The proofs of parts (i) and (iii) of this theorem are essentially the same as the proof of Theorem 1.1, so we merely point out the differences. Note that $\phi_+(\pi) = 0$, $\phi'_+(\pi) < 0$ and $\phi_-(\pi) < 0$ (by our assumptions on k, ℓ, μ, ν). f_1 can be constructed as before while we let $f_2 = \bar{H}_D(u)$ where u is as before. The rest of the proof is as before.

(ii) The proof that $R(H_D)$ is closed and H_D is proper is similar to the proof that $R(H_D) \cap (L^1[0,\pi]\backslash\bar{T})$ is relatively closed in $L^1[0,\pi]\backslash\bar{T}$ in Theorem 1.1(iii).

It remains to prove that $R(H_D) \ne L^1[0,\pi]$. To see this, choose $x_0 < \pi$ so that $|\pi - x_0| \le \frac{\pi}{4\sqrt{\nu'}}$, $\phi'_+(t) < 0$, $\phi_-(t) < 0$ and $\phi'_-(t) \ne 0$ on $[x_0, \pi)$ and define $f_n = Cn\chi_{[x_0, x_0+1/n]}$ where $C < 0$. If $R(H_D) = L^1[0,\pi]$, then there is a z_n in X_D with $H_D(z_n) = f_n$. Since the f_n are uniformly bounded in $L^1[0,\pi]$ we can argue as in the proof of Theorem 1.1 and get uniform estimates for the zeros of the solutions $\phi_{\alpha,n}$ of $-x''(t) = g(x(t)) - f_n(t)$, $x(0) = 0$, $x'(0) = \alpha$. As in Theorem 1.1 , we find that there is an $\alpha_0 > 0$ such that $\phi_{\alpha,n}(\pi) \ne 0$ if $|\alpha| \ge \alpha_0$ for all n. Thus $|z'_n(0)| \le \alpha_0$ and hence, by Lemma 2.1, $E_n(t) = \frac{1}{2}(z'_n(t))^2 + G(z_n(t))$ is uniformly (*in* n) bounded on $[0,\pi]$. It follows easily that $\|z_n\|'_\infty$ is uniformly bounded.

Thus, by the differential equation satisfied by z_n, z''_n is uniformly bounded on $[0,\pi]\backslash[x_0, x_0+1/n]$. Hence, by a simple compactness argument, a subsequence of the z_n converges in $C[0,\pi]$ to z where z is C^2 except at x_0, $-z''(t) = g(z(t))$ for $t \ne x_0$ and $z'(x_0^+) - z'(x_0^-) = C$ (where $z'(x_0^\pm)$ are the right and left limits of z' at x_0). The last equality follows because

$$-z'_n\left(x_0 + \frac{1}{n}\right) + z'_n(x_0) = \int_{x_0}^{x_0+\frac{1}{n}} [g(z_n)) - f_n(t)]\,\mathrm{d}t \to -C \text{ as } n \to \infty\,.$$

We now prove that, if C is large negative, then $|z'(0)|$ must be large. More precisely, we prove that, given $K > 0$ there is a $K_1 > 0$ such that $|z'(0)| \ge K$ if $C \le -K_1$. To see this, note that if $|z'(0)| \le K$, then by Lemma 2.1 (applied to z), we see that $|z(x_0)| + |z'(x_0^-)| \le K_2$. Hence, if C is large negative, $|z(x_0)| \le K_2$ and $z'(x_0^+)$ is large negative. Hence there is a $x_1 > x_0$ such that $z'(x_1) < 0$ and $-K_2 \le z(x_1) \le 0$ if C is large negative. (If $z'(x_0) > 0$, $z(t)$ must be zero after a short time.) Now $E(t) = \frac{1}{2}(z'(t))^2 + G(z(t)) = \frac{1}{2}(z'(x_0^+))^2 + G(z(x_0))$ for $t > x_0$. Since $z'(x_0^+)$ is large negative if $|z'(x_0)| \le K$ and C is large negative, $E(t)$ is large if $|z'(0)| \le K$ and C is large negative.

We prove that, *in this case*, $z(t) < 0$ on $[x_1, \pi]$. This follows because otherwise there exist t_2, t_3 such that $x_1 < t_2 < t_3 \le \pi$ such that $z'(t_2) = 0$, $z(t_3) = 0$, $z(t) < 0$ on (x_1, t_3). Since $z''(t) = g(z(t))$ on $[x_1, \pi]$ and $(z'(t_3))^2 =$

$2E(t_3) = 2E(x_0^+)$ (and thus $z'(t_3)$ is large negative), we can use our earlier estimates to estimate $t_3 - t_2$. We find that $t_3 - t_2$ is near $\frac{\pi}{2\sqrt{\nu}}$ if C is large negative. This is impossible since $\pi - x_0 \leq \frac{\pi}{4\sqrt{\nu}}$.

Hence, since $z(\pi) = 0$, it follows that $|z'(0)| \geq K$ *if* C *is large negative.* In this case, we can use our earlier estimates to study z on $[0, x_0]$. If K is large and $z'(0) \geq K$, we see (since $[z'(0)]^{-1}z(t)$ is near $\phi_+(t)$ on $[0, x_0]$) that $z(x_0) > 0, -z'(x_0^-) > 0$ and both are large (and thus $E(x_0^-)$ is large). We compare z with w, where $-w''(t) = g(w(t))$, $w = z$ for t near zero. Now $w(x_0) = z(x_0)$, $z'(x_0^+) < z'(x_0^-) = w'(x_0) < 0$. It follows easily using the first integral for z and w on $[x_0^+, \pi]$ that $z(t) < w(t)$ on $[x_0^+, \pi]$ as long as both are positive. Since our earlier estimates for the distance between zeros easily imply that w has a zero on $[x_0, \pi]$) (if K is large) it follows that z has a zero in (x_0, π), say at t_4. Now $z'(t_4)^3 = 2E(x_0^+)$ which is large hence, if t_5 is the next zero of z, $t_5 - t_4 \approx \frac{\pi}{\sqrt{\nu}}$. Hence $t_5 > \pi$, i.e. $z(\pi) \neq 0$, which is impossible since $z(\pi) = 0$. Thus $z'(0) \leq -K$. In this case, $[z'(0)]^{-1}z(x_0)$ is near $\phi_-(x_0)$ and so $z(x_0)$ is large negative. Hence $E(x_0^+)$ is large. There are now two cases.

Case (i) $z'(x_0^+) \leq 0$. We can then show that $z(t) < 0$ on $[x_0, \pi]$ by a similar argument to that at the end of the previous paragraph. Thus, in this case $z(\pi) \neq 0$.

Case (ii) $z'(x_0^+) > 0$. In this case, we compare z with w where w is as defined before. Now $z(x_0) = w(x_0)$ and $0 < z'(x_0^+) < w(x_0^+)$. Hence, by arguing as before, $z(t) < w(t)$ as long as $w(t) < 0$. But by using our earlier estimates for the distance between zeros, we see that $w(t) < 0$ on $[x_0, \pi]$ if K is large. (Remember that $w'(0) \leq -K$ and that $(w'(0))^{-1}w(t)$ is near $\phi_-(t)$.) It follows that $z(\pi) < 0$ and so we once again have a contradiction. Thus the assumption that H_D is onto leads to a contradiction. Hence H_D is not onto. □

Remarks. 1. Our methods can be applied if we replace $-y''$ by rather more general self-adjoint second order differential operator and also if we allow some dependence of s in g.

2. It would be of interest to study $R(H_D)$ further in the cases where H_D is not onto. $R(H_D)$ is not well understood in these cases. (Some additional results are known when μ and ν are close to the same eigenvalue of $-y''$.)

4.2. The Periodic Boundary-Value Problem

Define $\bar{H}_p : X_p \to L^1[0, \pi]$ by $\bar{H}_p(x)(t) = x''(t) + \mu(x(t))^+ - \nu(x(t))^-$, $\bar{F}(\theta) = \int_0^\pi f(t)\phi_+(t + \theta)\,\mathrm{d}t$, and

$$T_2 = \{f \in L^1[0, \pi] : \bar{F}(\theta) \text{ has only simple zeros, and } \bar{F}(\theta) \text{ has at least one such zero}\}.$$

Since $\bar{F}'(\theta) = -\int_0^\pi f(t)\phi_+'(t+\theta)\,\mathrm{d}t$ (by the dominated convergence theorem), T_2 is open. T_2 is non-empty because it contains suitable step functions with small support. It is also easy to see that $\bar{F}$ is C^1. Note that T, T_2, $R(\bar{H}_D)$, $R(H_p)$ are all translation invariant (i.e., if g belongs to one of them and is extended to be periodic, then all translates of g also belong to the set) and that, if f is constant, then $\bar{F}(\theta) \neq 0$ for all θ.

Finally, if $z \in X_p$ and $H_p(z) = f$, then

$$0 = \bar{E}(\pi) - \bar{E}(0) = \int_0^\pi f(t)z'(t)\,\mathrm{d}t\,. \tag{14}$$

This will be useful later.

Lemma 4.3. *Suppose that the assumptions of Theorem 1.3 hold. Then*

(i) there is an f in T_2 with $f \notin R(\bar{H}_p)$, and

(ii) $T_2 \cap R(\bar{H}_p) \neq \emptyset$.

Proof. Without loss of generality, we may assume that $\mu > \nu$.
(i) Choose f to be $\chi_{[a,b]}$ where $b-a$ is small. It is easy to show that $f \in T_2$ if $b-a$ is small. Suppose by way of contradiction that there is a z in X_D with $H_p(z) = f$. By equation (14), $\int_0^\pi \chi_{[a,b]} z' = 0$, i.e., $z(b) = z(a)$. Now, for $t \notin [a,b]$, $-z''(t) = \mu(z(t))^+ - \nu(z(t))^-$. Since $\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} \leq 1$, it follows that, if $|b-a|$ is small, $z(t)$ has a zero on $[0,\pi]\backslash[a,b]$. Hence, by translating f, we may assume without loss of generality that $z(0) = 0$ and $0 < a < b < \pi$. We assume that $z'(0) > 0$. (The other cases are similar.) Let $\tau = z'(0)$. Hence $z(t) = \tau\phi_+(t)$ for $t \in [0,\pi]\backslash[a,b]$. (Remember that z and ϕ_+ are periodic.) Since $b-a$ is small and $z(b) = z(a)$ (and thus $\phi_+(b) = \phi_+(a)$), b and a must be both near $\tilde{u}$ where $\phi_+'(\tilde{u}) = 0$ and so $\phi_+(t) \neq 0$ on $[a,b]$. Now if $t_1 < a < b < t_2$ are the zeros of ϕ_+ adjacent to $[a,b]$, we see that $-z''(t_1) = \tau\phi_+'(t_1) \neq 0$, $z'(t_2) = \tau\phi_+'(t_2)$, and $t_2 - t_1 = \frac{\pi}{\sqrt{\mu}}\left(\frac{\pi}{\sqrt{\nu}}\right)$ if ϕ_+ is positive (negative) on (t_1, t_2). (Note that, by our remarks above, t_1 and t_2 are adjacent zeros of ϕ_+.) We can then use Lemma 4.1 to obtain a contradiction. (This is easy but tedious. One uses Lemma 4.1 to show that $z(t)$ has no zeros on (t_1, t_2) and that one cannot end up with the correct spacing of the zeros.) We obtain a similar contradiction if we assume that $z'(0) \leq 0$ and hence $f \notin R(\bar{H}_p)$.

(ii) We choose u so that $\bar{H}_p(u) \in T_2$. This will give the required result. We choose u_1 to be a smooth function taking both positive and negative values with support in an interval $[a,b]$ (where $b-a$ is small). By using the argument to derive equation (11), we see that

$$\begin{aligned}\bar{F}_1(\theta) &= \int_0^{\pi} f_1(t)\phi_+(t+\phi)\,\mathrm{d}t \qquad (15)\\ &= (\mu-\nu)\Big[\int_{A(\theta)} u_1(t)\phi_+(t+\theta)\,\mathrm{d}t - \int_{B(\theta)} u_1(t)\phi_+(t+\theta)\,\mathrm{d}t\Big]\end{aligned}$$

where $A(\theta) = \{x \in [a,b] : u_1(t) > 0, \phi_+(t+\theta) < 0\}$, $B(\theta) = \{x \in [a,b] : u_1(t) < 0, \phi_+(t+\theta) > 0\}$ and $f_1 = \bar{H}_p(u_1)$. It follows easily from this formula that $\bar{F}_1(\theta)$ changes sign and that $\bar{F}_1(\theta)$ can only vanish if $\phi_+(t+\theta)$ has a zero t' in $[a,b]$. Suppose in addition that we choose u_1 so that $u_1(t) > 0$ on $(a,\bar{a})$ (where $\bar{a} = \frac{a+b}{2}$) and $u_1(t-\bar{a})\phi_+(t-\bar{a})$ is odd.

Now by considering the integrals from a to $\bar{a}$ and $\bar{a}$ to b separately it is tedious but easy to show that $\bar{F}_1(\theta)$ can only vanish if $\phi_+(\bar{a}+\theta) = 0$ and that, in this case $\bar{F}_1(\theta)$ is decreasing in θ near such a θ if $\phi'_+(\bar{a}+\theta) < 0$ and $\bar{F}_1(\theta)$ is increasing in θ near such a θ if $\phi'_+(\bar{a}+\theta) > 0$. (This is a tedious but elementary comparison of integrals.) Now we can evaluate $\bar{F}'_1(\theta)$ by explicit calculation of the derivative from the definition (using the right hand side of (15))). We find that $\bar{F}'_1(\theta) = 0$ if $\phi_+(\bar{a}+\theta) = 0$ and $\phi'_+(\bar{a}+\theta) < 0$ and

$$\bar{F}'_1(\theta) = (\mu-\nu)\Big[\int_a^{\bar{a}} u_1(t)\phi'_+(t+\theta)\,\mathrm{d}t - \int_{\bar{a}}^{b} u_1(t)\phi'_+(t+\theta)\,\mathrm{d}t\Big] > 0$$

if $\phi'_+(\bar{a}+\theta) > 0$ and $\phi_+(\bar{a}+\theta) = 0$. Choose u_2 to be a non-zero, non-negative function with support in $[c,d]$ where $[c,d]$ is close to $[a,b]$ and does not intersect it. (Thus $\phi'_+(t+\theta) < 0$ on $[c,d]$ whenever $\phi'(\bar{a}+\theta) < 0$ and $\phi(\bar{a}+\theta) = 0$.) By an argument used to derive equation (9), we see that

$$\bar{F}_2(\theta) = \int_0^{\pi} f_2(t)\phi_+(t+\theta)\,\mathrm{d}t = (\mu-\nu)\int_{A_3(\theta)} u_2(t)\phi(t+\theta)\,\mathrm{d}t\,,$$

where $f_2 = H_p(u_2)$, $A_3(\theta) = \{t \in [0,\pi] : \phi_+(t+\theta) < 0\}$. It follows easily that

$$\bar{F}'_2(\theta) = (\mu-\nu)\int_{A_3(\theta)} u_2(t)\phi'(t+\theta)\,\mathrm{d}t\,.$$

It follows from the choice of c and d that $\bar{F}'_2(\theta) < 0$ if θ is such that $\phi(\bar{a}+\theta) = 0$ and $\phi'(\bar{a}+\theta) < 0$. If we finally define u to be $u_1 + \epsilon u_2$ where ϵ is small and positive and let $f_3 = H(u_1+\epsilon u_2) = H(u_1) + \epsilon H(u_2)$ (since u_1 and u_2 have disjoint support), it is easy to check that f_3 has the required properties. (Remember that $\bar{F}(\theta_1) = \bar{F}_1(\theta) + \epsilon\bar{F}_2(\theta)$ and that $\bar{F}'_1(\theta) \le 0$ and $\bar{F}'_2 < 0$ in a neighborhood of a point θ_0 where $\phi_+(\bar{a}+\theta_0) = 0$ and $\phi'_+(\bar{a}+\theta_0) < 0$.) □

Proof of Theorem 1.3. We only prove (ii). The proof of (i) is similar to the proof that $T \subseteq R_p$ and is omitted. (The proof that, under the assumptions of (i), H_p is proper is similar to part of the proof below.) Suppose that $f \in L^1[0,\pi]$ such that $F(\theta) > 0$ for all $\theta \in [0,\pi]$. Thus $F(\theta) \geq k > 0$ on $[0,\pi]$. Consider the problem

$$-x''(t) = g_n(x(t)) - f(t) \tag{16}$$

on $[0,\pi]$, $x(0) = x(\pi)$, $x'(0) = x'(\pi)$, where $g_n(y) = g(y) - \frac{1}{n}y^+$. The results in Section 6 of [2] imply that if n is large this equation has a solution u_n. If we have a bound for $\|u_n\|_\infty$ we can pass to the limit by standard arguments and find that $f \in R(H_p)$.

Thus it suffices to prove a contradiction if we assume that $\{\|u_n\|_\infty\}_{n=1}^\infty$ is not bounded. We may assume that $\|u_n\|_\infty \to \infty$ as $n \to \infty$ since we can achieve this by passing to a subsequence. By Lemma 2.1, $E_n(t) = \frac{1}{2}(u_n'(t))^2 + G_n(u_n(t)) \to \infty$ as $n \to \infty$ uniformly in t (where $G_n(y) = G(y) - \frac{1}{2n}(y^+)^2$). Thus u_n has only simple zeros. We first show that u_n must have a zero. If not, $u_n(t) > 0$ for all t (or $u_n(t) < 0$ for all t).

We obtain a contradiction in the first case. (The other is similar.) If $u_n(t)$ achieves its minimum at t_n, then $G_n(u_n(t_n)) = E_n(t_n)$ which is large and hence $u_n(t_n)$ is large. Thus $\inf\{u_n(t) : t \in [0,\pi]\} \to \infty$ as $n \to \infty$. But by integrating,

$$\int_0^\pi g_n(u_n(t))\,\mathrm{d}t = \int_0^\pi f(t)\,\mathrm{d}t$$

which is impossible if $u_n(t)$ is large positive for all t (since $g_n(y) \geq \left(\frac{1}{2}\mu - \frac{1}{n}\right)y$ for y large). Thus u_n must have a zero at $\theta_n \in [0,\pi]$. Since u_n is periodic, we can choose θ_n so $u_n'(\theta_n) > 0$. Thus $v_n = u_n(t+\theta_n)$ is a solution of (16) when f is replaced by $f(t+\theta_n)$ and $v_n(0) = 0$. We can use the results of Section 2 to estimate the $2k^{\text{th}}$ zero of v_n (cp the proof of Theorem 1.1). We find that, if $\alpha_n = v_n'(0)$, then the $2k^{\text{th}}$ zero of v_n (which is the only zero near π) is at t_n where

$$\begin{aligned} t_n &\geq \pi + \alpha_n^{-1}\left[\int_0^\pi (f(t+\theta_n)\phi_+(t) + I^+(\phi_+(t))^+ + I^-(\phi_+(t))^-)\,\mathrm{d}t\right] + o(\alpha_n^{-1}) \\ &= \pi + \alpha_n^{-1}F(-\theta_n) + o(\alpha_n^{-1})\,. \end{aligned}$$

Since $\alpha_n \to \infty$ as $n \to \infty$ and $F(-\theta_n) \geq k > 0$, it follows that $t_n > \pi$ and hence $u_n(\pi) \neq 0$ (since $\alpha_n^{-1}v_n(t)$ is near $\phi_+(t)$).

Hence $u_n(0) \neq u_n(\pi)$ and so we have a contradiction. Thus $f \in R(H_p)$. If $F(\theta) < 0$ for all θ, we use a similar argument except that we define $g_n(y) = g(y) + \frac{1}{n}y^+$.

We now prove that $R(H_p) \cap S$ is closed in S. As before, it suffices to prove that, if $H_p(u_n) = f_n$, where $f_n \in S$, $f_n \to f$ in $L^1[0,\pi]$ as $n \to \infty$ (where $f \in S$), then $\|u_n\|_\infty$ is bounded. *If not*, we can argue as in the previous paragraph and find solutions v_n of $H_p(v_n) = f_n(t+\theta_n)$ for which $v_n(0) = 0$, $v_n'(0) > 0$ and $\|v_n\|_\infty \to \infty$ as $n \to \infty$. By choosing a subsequence we may assume that $\theta_n \to \theta_0$ as $n \to \infty$.

Now, as in the previous paragraph, we find that the $2k^{\text{th}}$ zero of v_n is at t_n, where $t_n = \pi + \alpha_n^{-1} F_n(-\theta_n) + o(\alpha_n^{-1})$. (Here $\alpha_n = v_n'(0)$ and $F_n(\theta) = \int_0^\pi f_n(t)\phi_+(t+\theta)\,\mathrm{d}t$.) Since $F_n(\theta) \to F(\theta)$ uniformly in θ, we obtain a contradiction by the same argument as in the previous paragraph unless $F(-\theta_n) \to 0$ as $n \to \infty$. Thus $F(-\theta_0) = 0$. But, by equation (14), $\int_0^\pi f_n(t)u_n'(t)\,\mathrm{d}t = 0$, i.e. $\int_0^\pi f_n(t+\theta_n)v_n'(t)\,\mathrm{d}t = 0$. Since $\alpha_n^{-1}v_n'(t) \to \phi_+'(t)$ uniformly in t as $n \to \infty$ (cp Lemma 2.2), it follows that, in the limit,

$$\int_0^\pi f(t+\theta_0)\phi_+'(t)\,\mathrm{d}t = 0$$

i.e.

$$F'(-\theta_0) = \int_0^\pi f(t)\phi_+'(t-\theta_0)\,\mathrm{d}t = 0\,.$$

Thus $F(-\theta_0) = F'(-\theta_0) = 0$, which contradicts our assumption that $f \in S$. Hence the result follows.

The remainder of the proof is similar to the proof of Theorem 1.1(iii) (using Lemma 4.3 instead of Lemma 4.2). □

Our methods can be used to obtain results for more general periodic boundary-value problems but it is unclear whether our methods give as strong results for these more general problems. (Note that equation (14) does not always hold for more general problems.)

5. Proof of the Last Two Theorems

We now prove the results of Section 2 in [1]. We do not need to introduce any additional techniques.

Proof of Theorem 1.4. The necessity in Theorem 1.4(i) follows easily if we note that, if u is a solution of (2), then

$$0 = \int_0^\pi [-u''(t) - u(t)]\sin t\,\mathrm{d}t = \int_0^\pi [g(u) - u(t) - f(t)]\sin t\,\mathrm{d}t\,.$$

The sufficiency in Theorem 1.4(i) and Theorem 1.4(iii) now follows by shooting arguments as in Section 4 (using the estimates of Section 2 for the distance between successive zeros). We illustrate by considering Theorem 1.4(iii) when (b) holds and $i(i+1)^{-1} < P^l \le P^u < 1$. As before, let z_α denote a solution of eq. (2), with $z_\alpha(0) = 0$, $z_\alpha'(0) = \alpha$. If α is large positive, then the first zero of z_α is at $(1+\epsilon(\alpha))T(\alpha)$ where $\epsilon(\alpha) \to 0$ as $\alpha \to \infty$ (by Lemma 2.7). By

Lemma 2.6, the second zero is at $(1+\epsilon_1(\alpha))T(\alpha)+\frac{\pi}{i}-(1+\epsilon_2(\alpha))S(\alpha)$. ($S$ was defined in the Introduction.) By continuing we find that the $(2i-1)^{\text{th}}$ zero of z_α is less than π, the $2i^{\text{th}}$ zero is at $\pi+i(1+\epsilon_1(\alpha))T(\alpha)-i(1+\epsilon_2(\alpha))S(\alpha)$ and the $(2i+1)^{\text{th}}$ zero is at $\pi+(i+1)(1+\epsilon_1(\alpha))T(\alpha)-i(1+\epsilon_2(\alpha))S(\alpha)$. Since $P^u<1$, it follows that the $2i^{\text{th}}$ zero is less than π and, since $P^l>i(i+1)^{-1}$, the $(2i+1)^{\text{th}}$ zero is greater than π (*for α large*). Moreover, since $z'_\alpha(0)>0$, $z'_\alpha(t)>0$ at the $2i^{\text{th}}$ zero of z_α. (Remember that the zeros are simple.) Hence $z_\alpha(\pi)>0$ if α is large. If α is large negative, we see by similar arguments that the $2i^{\text{th}}$ zero of z_α is less than π and the $(2i+1)^{\text{th}}$ zero is greater than π (it is near $\frac{i+1}{i}\pi$) and hence $z_\alpha(\pi)>0$ if α is large negative. Hence we obtain the existence of a solution by a shooting argument. We obtain properness as before as the above argument shows that $z_\alpha(\pi)\neq 0$ if $|\alpha|\geq\alpha_0$ where α_0 depends only on $\|f\|_1$. The properness (and hence that $R(H_D)$ is closed) in part (ii) follows by the same argument.

It remains to prove that $R(H_D)\neq L^1[0,\pi]$ under the assumptions of Theorem 1.4(ii). That this is true under 4(ii)(b) follows by a similar argument to that in the proof of necessity in Theorem 1.4(i). The proof in the remaining cases are similar and we illustrate by proving that H_D is not onto under 4(iii)(d) when $P^u<i(i+1)^{-1}$. Set $a=\pi-\frac{\pi}{4i}$. If H_D were onto, then by noting that $Cn\chi_{[a,a+\frac{1}{n}]}\in R(H_D)$ and by using a similar limit argument to that in the proof of Theorem 1.2(ii), there exists $z\in C[0,\pi]$ such that z is C^2 for $t\neq a$, $z(0)=z(\pi)=0$, $-z''(t)=g(z(t))$ for $t\neq a$ and $z'(a^+)-z'(a^-)=C$. As in that proof, we find that, if C is large positive, then $|z'(0)|$ is large. We prove that, if $\beta=z'(0)>0$, then $z(\pi)\neq 0$ and so we have a contradiction. (We can obtain a contradiction by a similar argument if $\beta<0$.) Since $z'(0)$ is large positive, we can use our earlier estimates to find the zeros of z for $t<a$. We find that the $(2i-1)^{\text{th}}$ zero of z is at $t(\beta)=\frac{i-1}{i}\pi+i(1+\epsilon_1(\beta))T(\beta)-(i-1)(1+\epsilon_2(\beta))S(\beta)$ and that, on $[t(\beta),a]$, $\beta^{-1}z(t)$ is near $\phi_-(t-t(\beta))$. Hence $z(a)>0$. (Remember that $t(\beta)$ is near $\frac{i-1}{i}\pi$ and $a-t(\beta)$ is near $\frac{3\pi}{4i}$.) By comparing z with a solution w of $-w''(t)=g(w(t))$ which agrees with z for $t<a$, we see that $z(t)>w(t)$ as long as $z(t)\leq 0$. (Remember that $z'(a^+)>w'(a)$.) Hence the $2i^{\text{th}}$ zero of z is less than the $2i^{\text{th}}$ zero of w (which is at $\pi+[i(1+\epsilon_1(\beta))T(\beta)-(1+\epsilon_2(\beta))S(\beta)]<\pi$ since $P^u<1$). If we prove that the $(2i+1)^{\text{th}}$ zero of z is less than π, then, since the $(2i+2)^{\text{th}}$ zero is approximately $\frac{\pi}{i}$ away, it will follow that $z(\pi)\neq 0$. *There are two cases.*

1. *If C is $o(\beta)$*, then the change of $E(t)$ across $t=a$ is $o(\beta^2)$ and thus $|z'(t_{2i})-z'(0)|$ is $o(\beta)$ (where t_{2i} is the $2i^{\text{th}}$ zero of z). Hence, by Lemma 2.8, $t_{2i+1}-t_{2i}\leq(1+\epsilon_3(\beta))(\bar{t}_{2i+1}-\bar{t}_{2i})\leq(1+\epsilon'_1(\beta))T(\beta)$ (where t_k and $\bar{t}_k$ are the k^{th} zeros of z and w respectively). Since $t_{2i}\leq\bar{t}_{2i}$, we see that $t_{2i+1}\leq\pi+(i+1)(1+\epsilon'_1(\beta))T(\beta)+i(1+\epsilon_2(\beta))S(\beta)$. (Here we have used our earlier estimate for the $2i^{\text{th}}$ zero of w.) Since $P^u<i(i+1)^{-1}$, it follows that $t_{2i+1}<\pi$, as required.

2. *If* $C \geq \mu\beta$, then on $[a, t_{2i}]$, $\frac{1}{2}z'(t)^2 + \bar{G}(z(t)) = \frac{1}{2}(\beta + C)^2$ and $\frac{1}{2}(w'(t))^2 + \bar{G}(w(t)) = \frac{1}{2}\beta^2$. Since $z(t) > w(t)$ and $\bar{G}(y) \geq \bar{G}(x) - M$ if $y \leq x < 0$ (since $g(u) < 0$ if u is large negative), we see that $z'(t) - w'(t) \geq \frac{2\beta C - M}{2\beta + C}$. Hence if β is large and $C \geq \mu\beta$ then $z'(t) - w'(t) \geq \frac{1}{2}\tau\beta$ on $[a, t_{2i}]$ where $\tau = \inf\{1, \mu\}$. Hence, by integrating, $w(t_{2i}) \leq -\frac{1}{2}\tau\beta(t_{2i} - a)$. Thus, either $t_{2i} \leq \pi - \frac{\pi}{8i}$ or $w(t_{2i}) \leq \gamma\beta$ (where $\gamma = \frac{\pi}{16i}\tau$). Remember that $a = \pi - \frac{\pi}{4i}$. Since $|w'(t)| \leq \beta$, if follows that either $t_{2i} \leq \left(1 - \frac{1}{8i}\right)\pi$ or $\bar{t}_{2i} - t_{2i} \geq \gamma$. Since $\bar{t}_{2i} < \pi$, we see that always $t_{2i} \leq \pi - \gamma_1$, where $\gamma_1 = \inf\{\gamma, \frac{\pi}{8i}\}$. If β is large, $t_{2i+1} - t_{2i}$ is small (since $T(r) \to 0$ as $r \to \infty$). Hence, if β is large, $t_{2i+1} < \pi$, as required. □

We now consider the periodic boundary-value problem. Note that if a solution of the periodic boundary-value problem has only simple zeros, then it has an *even number* of zeros in $(0, \pi)$.

Proof of Theorem 1.5. We first prove part (ii). We only consider the case where $I^- = -\infty$ and $P^l > 1$. The other cases are similar. In this case, we note that, by the results in Section 6 of [2] the equation

$$-x''(t) = g_n(x(t)) - f(t), \quad x(0) = x(\pi), \quad x'(0) = x'(\pi)$$

has a solution u_n (where $g_n(y) = g(y) + \frac{1}{n}y^-$). If we establish a bound for $\|u_n\|_\infty$, we can pass to the limit by a standard argument.

If not, we can argue as in the proof of Theorem 1.3 and find that $E_n(t)$ is large for all t in $[0, \pi]$ and u_n has a zero θ_n, with $u_n'(\theta_n) \to \infty$. As there, we may, by replacing u_n by $u_n(t + \theta)$ and f by $f(t + \theta_n)$ assume that $u_n(0) = 0$ and $u_n'(0) \to \infty$ as $n \to \infty$ (since $E_n(t) \to \infty$ as $n \to \infty$).

Since $P^l > 1$, a similar argument to that in the proof of Theorem 1.4 shows that the $2i^{\text{th}}$ zero of u_n is larger than π but near π. (Remember that $\{f(t + \theta_n)\}$ is bounded in $L^1[0, \pi]$.) Since the $(2i - 2)^{\text{th}}$ zero is near $\frac{i-1}{i}\pi$ and the $2(i+1)^{\text{th}}$ zero is near $\frac{i+1}{i}\pi$ and since u_n must have an even number of zeros in $(0, \pi]$ if it satisfies the boundary conditions (u_n has only simple zeros since $E_n(t) \to \infty$ as $n \to \infty$), we have a contradiction. Thus $\|u_n\|_\infty$ is bounded and hence $f \in R(H_p)$. A similar argument establishes properness. (For this, we replace g_n by g.)

The proof of necessity in Theorem 1.5(i) is similar to the proof of necessity in Theorem 1.4(i). The proof of sufficiency is similar to the proof in the previous paragraph and we only point out the differences. First, we may, by adding a constant to g and f, assume that $yg(y) \geq 0$ for $|y|$ large (and thus Lemma 2.1 applies). Secondly, we need an analogue of our results of Subsection (ii) of Section 2 for the distance between zeros when $\mu = 0$. As easy modification of the proof of Lemma 2.2 shows that $\alpha^{-1}z(t)$ is near t on $(0, \pi]$ and thus, if α is large, $z(t)$ has *no zeros* on $[0, \pi]$. This replaces our estimates for the distance between the zeros in Subsection (ii) of Section 2. Using this estimate, the argument proceeds as above. There is one slight difference when I^- is finite. If $u_n(t) < 0$ for all t and $\{\|u_n\|_\infty\}$ is not bounded, then as in the proof of Theorem 1.3, $u_n(t) \to -\infty$ as $n \to \infty$ uniformly in t.

Now

$$\int_0^\pi f(t)\,\mathrm{d}t = \int_0^\pi g_n(u_u(t))\,\mathrm{d}t \le \int_0^\pi g(u_n(t)) \to \pi I^-$$

as $n \to \infty$. Hence we have a contradiction. □

If $i = 0$ or $I^- > -\infty$, we can delete our extra regularity of growth assumption on g for y large positive in the above two theorems. If $I^- = \infty$ in Theorem 1.5(i), H_p is proper. It seems possible that our methods can be used for more general equations, though there are difficulties in getting analogues of the estimates for the distance between zeros in Subsection (iii) of Section 2. It would be of interest to understand the structure of $R(H_D)$ under the assumptions of Theorem 1.4(ii). Finally, the estimates for $T(\alpha)$ mentioned at the end of the Appendix to Section 2 are of use in verifying the assumptions of Theorem 1.4 and 1.5.

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