Classical and non-Classical Positive Solutions of a Prescribed Curvature Equation with Singularities

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Summary. - We investigate the existence of positive solutions of a prescribed curvature equation with a nonlinearity having one or two singularities. Our approach relies on the method of lower- and upper-solutions, truncation arguments and energy estimates.

1. Introduction

In this paper, we are interested in the existence of positive solutions of the curvature problem

\[-(\varphi(u'))' = \lambda f(t, u), \quad u(0) = 0, \quad u(1) = 0,\]

(1)

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where

$$\varphi(v) = \frac{v}{\sqrt{1 + v^2}}.$$  

Problem (1) is the one-dimensional counterpart of the elliptic Dirichlet problem

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right) = \lambda f(x, u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \quad (2)$$

The existence of positive solutions of problems (1) and (2) has been discussed in the last two decades by several authors (see [1]–[5],[8]–[22]) in connection with various qualitative assumptions on $f$.

In our recent paper [2], two types of solutions have been considered for problem (1) and referred to as classical or non-classical, respectively.

A **classical** solution of (1) is a function $u : [0, 1] \to \mathbb{R}$, with $u \in W^{2,1}(0, 1)$, which satisfies the equation in (1) a.e. in $[0, 1]$ and the boundary conditions $u(0) = u(1) = 0$. A **non-classical** solution of (1) is a function $u : [0, 1] \to \mathbb{R}$ such that $u \in W^{2,1}_{\text{loc}}(0, 1)$, $|u'(0)| = +\infty$ or $|u'(1)| = +\infty$ and $u' \in C^0([0, 1], [-\infty, +\infty])$, which satisfies the equation in (1) a.e. in $[0, 1]$ and the boundary conditions $u(0) = u(1) = 0$. Such solutions are said to be **positive** if $u(t) > 0$ on $]0, 1[$ and $u'(0) > 0 > u'(1)$.

In [2] existence and multiplicity of positive solutions of (1) have been established under various types of assumptions on the behaviour of the function $f$ at zero and at infinity. In the present work we discuss cases where $f$ exhibits singularities at zero or at some point $R > 0$; our model nonlinearities are $u^{-p}$, $(R - u)^{-q}$ and $u^{-p}(R - u)^{-q}$, with $p, q > 0$.

Unlike the fact that a large amount of work has been done for a class of quasilinear elliptic equations in the presence of a singularity at zero, few results have been obtained for the curvature problem (cf. [5]). Singularities on the right have been considered only recently for semilinear elliptic problems; to the best of our knowledge they have never been considered before for the curvature problem.

This paper is organized as follows. Section 2 deals with a nonlinearity that is allowed to be singular at the origin. Within this setting, we prove the existence of at least one positive solution for
small values of the parameter $\lambda > 0$. In Section 3 we analyze the nature of the solutions for small values of $\lambda > 0$. Namely, we prove that if $f$ is non-singular or has a weak singularity at the origin (see assumption (7)) and $\lambda$ is small enough, any small positive solution is classical. On the other hand, if the singularity at the origin is too strong (see assumption (8)), then any positive solution is non-classical. In Section 4, we work out the case of a singularity on the right, while Section 5 deals with possibly two singularities. In this last case, under a quite general assumption, we show that the presence of the second singularity gives rise to a second positive solution. Finally, we present a non-existence result for large values of $\lambda$ in Section 6, while in the Appendix we show some numerical illustrations of our results.

Throughout this paper the following conditions will be considered. Let $f : [0, 1] \times I \to \mathbb{R}$, with $I$ an interval, be a given function.

We say that $f$ is a $L^1$-Carathéodory function if, for a.e. $t \in [0, 1]$, $f(t, \cdot) : I \to \mathbb{R}$ is continuous; for every $u \in I$, $f(\cdot, u) : [0, 1] \to \mathbb{R}$ is measurable; for every compact set $K \subset I$ there is $h \in L^1(0, 1)$ such that, for a.e. $t \in [0, 1]$ and every $u \in K$, $|f(t, u)| \leq h(t)$.

We say that $f$ is locally $L^1$-Lipschitz with respect to the second variable if, for every compact set $K \subset I$, there is $\ell \in L^1(0, 1)$ such that, for every $u_1, u_2 \in K$ and a.e. $t \in [0, 1],$

$$|f(t, u_1) - f(t, u_2)| \leq \ell(t) |u_1 - u_2|.$$

2. The singularity at the origin

**Theorem 2.1.** Let $f : [0, 1] \times [0, R] \to [0, +\infty[$, with $R \in [0, \infty]$, be a $L^1$-Carathéodory and locally $L^1$-Lipschitz function. Assume

(h$_1$) $\liminf_{u \to 0} f(t, u) > 0$, uniformly a.e. on $[0, 1]$, i.e. there exist $\eta > 0$ and $\delta > 0$ such that, for a.e. $t \in [0, 1]$ and for every $u \in [0, \delta]$, we have $f(t, u) \geq \eta$.

Then there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0]$, problem (1) has at least one positive solution.

**Proof.** Step 1 – The modified problem. Let $R \in [0, R]$. For each
\[ n \in \mathbb{N}, \ n > 1, \ \text{define} \]
\[ f_n(t, u) = \begin{cases} 
  f(t, \bar{R}/n) & \text{if } u \leq \bar{R}/n, \\
  f(t, u) & \text{if } \bar{R}/n < u \leq \bar{R}, \\
  f(t, \bar{R}) & \text{if } \bar{R} < u,
\end{cases} \]  
(3)

and consider the modified problem
\[ -(\varphi_n(u'))' = \lambda f_n(t, u), \quad u(0) = 0, \quad u(1) = 0. \]  
(5)

**Step 2 – Construction of an upper solution \( \beta \) of (5):** for any \( r \in ]0, \bar{R}[ \), there exist \( \lambda_0 > 0, \ n_0 \in \mathbb{N} \) and \( \beta \in W^{2,1}(0,1) \) such that \( 0 < \beta(t) \leq r \) in \([0,1]\) and for any \( \lambda \in ]0, \lambda_0[ \) and any \( n \geq n_0 \), \( \beta \) is an upper solution of (5). Let us fix \( 0 < \hat{r} < r < \bar{R} \). From the \( L^1 \)-Carathéodory conditions, there exists \( h \in L^1(0,1) \) such that \( |f(t,u)| \leq h(t) \) for a.e. \( t \in [0,1] \) and every \( u \in [\hat{r}, r] \). Let then \( H \in W^{2,1}(0,1) \) be the solution of

\[ -H'' = h(t), \quad H(0) = 0, \quad H(1) = 0 \]

and take \( \beta = \hat{r} + \kappa H \), where \( \kappa > 0 \) is small enough so that

\[ \beta = \hat{r} + \kappa H < r. \]

We then compute for \( \lambda > 0 \) small enough and \( n \in \mathbb{N} \) large enough

\[ -(\varphi_n(u'))' = \lambda f_n(t, u), \quad u(0) = 0, \quad u(1) = 0. \]

**Step 3 – Construction of a lower solution \( \alpha \leq \beta \) of (5):** for any \( \lambda \in ]0, \lambda_0[ \), there exist \( n_1 \in \mathbb{N} \) and \( \alpha \in W^{2,1}(0,1) \) such that \( 0 < \alpha(t) \leq \beta(t) \) in \([0,1]\) and, for any \( n \geq n_1 \), \( \alpha \) is a lower solution of (5). From assumption \( (h_1) \), we can find \( r_0 \in ]0, \hat{r}[ \) so that for all large \( n \), a.e. \( t \in [0,1] \) and every \( u \in ]0, r_0[ \)

\[ f_n(t, u) \geq r_0 \frac{\pi^2}{\lambda}. \]
The function $\alpha(t) = r_0 \sin(\pi t) \leq \beta(t)$ is such that

$$-\alpha''(t) = \pi^2 \alpha(t) \leq \pi^2 r_0 \leq \lambda f_n(t, \alpha(t)) \leq \lambda \frac{f_n(t, \alpha(t))}{\varphi'_n(\alpha'(t))}.$$

**Step 4 – Existence of a solution $u_n$ of the modified problem (5).** Since $\alpha$ and $\beta$ are lower and upper solutions of (5) for all large $n$, with $\alpha(t) \leq \beta(t)$ on $[0, 1]$, and the equation in (5) can be written as

$$-u'' = \frac{f_n(t, u)}{\varphi'_n(u')}.$$

where the right-hand side is bounded by a $L^1$-function, a standard result (see [6, Theorem II-4.6]) yields the existence of a solution $u_n$ of (5) satisfying

$$\alpha(t) \leq u_n(t) \leq \beta(t) \quad \text{in } [0, 1]. \quad (6)$$

**Step 5 – Existence of a solution $u$ of (1).** Let $a \in ]0, 1/2[$. From (6) and Step 3 we know that, for all $t \in [a, 1-a]$, we have $\alpha(a) \leq u_n(t) \leq r$. Hence using the $L^1$-Carathéodory conditions, there exists $h \in L^1(0, 1)$ such that for $n$ large enough and a.e. $t \in [a, 1-a]$, we have

$$0 \leq f_n(t, u_n(t)) = f(t, u_n(t)) \leq h(t).$$

Also from the concavity of $u_n$, we deduce that, for all $t \in [a, 1-a]$,

$$\frac{r}{a} \geq u_n'(a) \geq u'_n(t) \geq u'_n(1-a) \geq -\frac{r}{a}.$$

Hence for $n$ large enough and all $t \in [a, 1-a]$, we get

$$\varphi'_n(u'_n(t)) = \varphi'(u'_n(t)) \geq \varphi'(\frac{r}{a}).$$

It follows that

$$0 \leq -u''_n(t) = \lambda \frac{f(t, u_n(t))}{\varphi'(u'_n(t))} \leq \lambda \frac{h(t)}{\varphi'(\frac{r}{a})}.$$

From Arzelà-Ascoli Theorem, we infer that a subsequence $(u_n)_n$ converges in $C^1([a, 1-a])$ to a function $u \in C^1([a, 1-a])$. By the $C^1_{\text{loc}}$-estimates previously obtained, $u$ satisfies

$$-u'' = \lambda \frac{f(t, u)}{\varphi'(u')}.$$
Using a diagonalization argument, \( u \) can be extended onto \([0,1]\), so that \( u \in W^{2,1}_{\text{loc}}(0,1) \) satisfies the equation in (1) in \([0,1]\). Moreover, as \( u \) is positive and concave, there exist
\[
\lim_{t \to 0} u(t) \in [0, +\infty[ \quad \text{and} \quad \lim_{t \to 1} u(t) \in [0, +\infty[.
\]

**Step 6** – \( \lim_{t \to 0} u(t) = 0 \) or \( \lim_{t \to 1} u'(t) = +\infty \). Notice that \( u_n \) is a solution of the Cauchy problem
\[
-u'' = \lambda \frac{f_n(t,u)}{\varphi'_n(u')}, \quad u(\frac{1}{2}) = u_n(\frac{1}{2}), \quad u'(\frac{1}{2}) = u'_n(\frac{1}{2}),
\]
and \( u \) is a solution of the limit problem
\[
-u'' = \lambda \frac{f(t,u)}{\varphi'(u')}, \quad u(\frac{1}{2}) = \lim_{n \to \infty} u_n(\frac{1}{2}), \quad u'(\frac{1}{2}) = \lim_{n \to \infty} u'_n(\frac{1}{2}).
\]
If it were \( \lim_{t \to 0} u(t) > 0 \) and \( \lim_{t \to 0} u'(t) \in \mathbb{R} \), by continuity with respect to parameters and initial conditions, which follows from the \( L^1 \)-Carathéodory and locally \( L^1 \)-Lipschitz conditions, we should get
\[
\lim_{t \to 0} u(t) = \lim_{n \to \infty} u_n(0) = 0,
\]
which is a contradiction.

**Step 7** – \( \lim_{t \to 0} u(t) = 0 \) or \( \lim_{t \to 1} u'(t) = -\infty \). The argument is similar to the previous one.

**Conclusion** – If \( \lim_{t \to 0} u(t) = \lim_{t \to 1} u(t) = 0 \), then the continuous extension of \( u \) on \([0,1]\) is a solution of (1), which may be classical or non-classical. Otherwise, the extension of \( u \) obtained by setting \( u(0) = u(1) = 0 \) is a non-classical solution of (1).

**Remark 2.2.** From Step 2 in the proof of Theorem 2.1, we see that, for each \( \lambda > 0 \) small enough, a solution \( u_\lambda \) of (1) exists such that \( \|u_\lambda\|_{\infty} \to 0 \) as \( \lambda \to 0 \).

**Remark 2.3.** Suppose that, in addition to the assumptions of Theorem 2.1, the following condition holds
\[
\lim_{u \to R} f(t,u) = 0, \quad \text{uniformly a.e. on } [0,1].
\]
Then, for any \( \lambda \in ]0, +\infty[ \), problem (1) has at least one positive solution.

To prove this claim we only need to verify that an upper solution \( \beta \) of (5) can be constructed for any given \( \lambda > 0 \). We fix \( \lambda \) and, in case \( R \neq +\infty \), we extend \( f \) by setting \( f(t, u) = 0 \) for a.e. \( t \in [0, 1] \) and all \( u \geq R \). We define the upper solution \( \beta \) by setting 

\[
\beta(t) = M + \varepsilon t(1 - t),
\]

where \( M \in ]0, R[ \) and \( \varepsilon \in ]0, \frac{1}{2}[ \) are such that \( ]M, M + \varepsilon[ \subset ]0, R[ \) and \( f(t, u) \leq \frac{\varepsilon}{\lambda} \) for a.e. \( t \in [0, 1] \) and all \( u \in [M, M + \varepsilon] \). We have then

\[
-\beta''(t) = 2\varepsilon \geq (1 + \varepsilon^2)^{3/2} \geq \lambda(1 + \beta'(t)^2)^{3/2} f(t, \beta(t))
\]
a.e. in \([0, 1]\).

3. Classical and non-classical solutions

**Theorem 3.1.** Assume \( f : [0, 1] \times ]0, R[ \to ]0, +\infty[ \), with \( R \in ]0, +\infty[ \), is a \( L^1 \)-Carathéodory function and \( v \) is a positive solution of (1) for some \( \lambda > 0 \). Assume further there exists a function \( g : ]0, R[ \to ]0, +\infty[ \) having a bounded antiderivative such that, for a.e. \( t \in [0, 1] \) and all \( u \in ]0, \|v\|_{\infty} [ \),

\[
f(t, u) \leq g(u).
\]

Then, if \( \lambda > 0 \) is small enough, \( v \) is a classical solution.

**Proof.** Let \( G \) be an antiderivative of \( g \) and define

\[
E(t) = 1 - \frac{1}{\sqrt{1 + \beta'(t)^2}} + \lambda G(v(t)).
\]

We have

\[
E'(t) = v'(t)(\sqrt{1 + \beta'(t)^2})' + \lambda G'(v(t))
\geq v'(t)(\sqrt{1 + \beta'(t)^2})' + \lambda f(t, v(t)) = 0,
\]

for a.e. \( t \in [0, 1] \) such that \( v'(t) \geq 0 \). From the concavity of \( v \), we know there exists \( t_0 \in ]0, 1[ \) such that \( v'(t) \geq 0 \) on \([0, t_0]\) and \( v'(t_0) = 0 \). Assume by contradiction \( v'(0) = +\infty \). Since \( \lim_{t \to 0} v(t) = v_0 \geq 0 \), we have

\[
\lambda G(R) \geq \lambda G(v(t_0)) = E(t_0) \geq \lim_{t \to 0} E(t) = 1 + \lambda G(v_0) \geq 1
\]
which is impossible for small values of $\lambda$.

A similar argument shows that $v'(1) \in \mathbb{R}$. \hfill \Box

**Remark 3.2.** Theorem 3.1 applies in particular if

$$g(u) = \frac{K}{u^p},$$

where $K > 0$ and $p \in ]0, 1[$.

**Theorem 3.3.** Assume $f : [0, 1] \times ]0, +\infty[ \rightarrow [0, +\infty[$, with $R \in [0, +\infty[$, is a $L^1$-Carathéodory function and $v$ is a positive solution of (1) for some $\lambda > 0$.

Assume further there exist $\varepsilon > 0$ and a function $h : ]0, \varepsilon[ \rightarrow [0, +\infty[$ having an unbounded antiderivative such that, for a.e. $t \in [0, 1]$ and all $u \in ]0, \varepsilon[$,

$$f(t, u) \geq h(u). \quad (8)$$

Then $v$ is a non-classical solution. More precisely, we have

$$\lim_{t \to 0} v(t) > 0 \quad \text{and} \quad \lim_{t \to 1} v(t) > 0.$$

**Proof.** Assume, by contradiction, that $v$ is a positive solution of (1) satisfying $\lim_{t \to 0} v(t) = 0$. Define

$$E(t) = 1 - \frac{1}{\sqrt{1 + v'^2(t)}} + \lambda H(v(t)),$$

where $H$ is an antiderivative of $h$. We have

$$E'(t) = v'(t)[(\varphi(v'(t)))' + \lambda h(v(t))]$$

$$\leq v'(t)[(\varphi(v'(t)))' + \lambda f(t, v(t))] = 0,$$

for a.e. $t \in [0, 1]$ such that $v'(t) \geq 0$ and $v(t) \leq \varepsilon$. Let $t_0 \in ]0, 1[$ be such that $v'(t) > 0$ and $v(t) \leq \varepsilon$ on $[0, t_0]$. Hence, we get the contradiction

$$E(t_0) \leq \lim_{t \to 0} E(t) = 1 + \lambda \lim_{t \to 0} H(v(t)) = -\infty.$$

A similar argument shows that assuming that $\lim_{t \to 1} v(t) = 0$ leads to a contradiction as well. \hfill \Box

**Remark 3.4.** Theorem 3.1 applies in particular if

$$h(u) = \frac{K}{u},$$

where $K > 0$. 
4. Singularity on the right

**Theorem 4.1.** Let \( f : [0, 1] \times [0, R] \rightarrow [0, +\infty] \), with \( R \in ]0, +\infty[ \), be a \( L^1 \)-Carathéodory and locally \( L^1 \)-Lipschitz function. Assume

\[
\lim_{u \to R} f(t, u)(R - u)^p = +\infty, \quad \text{uniformly a.e. on } [0, 1].
\]

Then there exists \( \lambda_0 > 0 \) such that for each \( \lambda \in ]0, \lambda_0[ \) problem \( (1) \) has at least one positive solution.

**Proof.** Step 1 – The modified problem. For each \( n \in \mathbb{N} \), \( n > 1 \), we define

\[
f_n(t, u) = \begin{cases} f(t, 0) & \text{if } u \leq 0, \\ f(t, u) & \text{if } 0 < u \leq \frac{n-1}{n}R, \\ f(t, \frac{n-1}{n}R) & \text{if } \frac{n-1}{n}R < u,
\end{cases}
\]

and consider the modified problem

\[
-(\varphi_n'(u'))' = \lambda f_n(t, u), \quad u(0) = 0, \quad u(1) = 0,
\]

(9)

where \( \varphi_n \) is defined from (4).

Step 2 – Construction of an upper solution \( \beta \) of (9); for any \( r \in ]0, R[ \), there exist \( \lambda_0 > 0 \), \( n_0 \in \mathbb{N} \) and \( \beta \in W^{2,1}(0, 1) \) such that \( 0 < \beta(t) \leq r \) in \( [0, 1] \) and, for any \( \lambda \in ]0, \lambda_0[ \) and any \( n \geq n_0 \), \( \beta \) is an upper solution of (9). To prove this claim, we repeat the argument used in Step 2 of the proof of Theorem 2.1.

Step 3 – Construction of a lower solution \( \alpha \) of (9); there exist \( n_1 \in \mathbb{N} \) and \( \alpha \in W^{2,1}(0, 1) \) such that for any \( \lambda \in ]0, \lambda_0[ \) and any \( n \geq n_1 \), \( \alpha \) is a lower solution of (9), \( 0 < \alpha(t) < R \) in \( ]0, 1[ \) and \( \max(\alpha - \beta) > 0 \).

Let \( \lambda > 0 \) be fixed. We first choose \( \bar{r} \) such that

\[
r < \bar{r} < R, \quad (R - \bar{r})^{p-1} < \frac{\lambda}{8R^2},
\]

and for a.e. \( t \in [0, 1] \) and all \( u \in [\bar{r}, R[ \)

\[
f(t, u) \geq \frac{1}{(R - u)^p} \geq \frac{1}{(R - \bar{r})^p}.
\]
Next we write

$$M_0 = \frac{\lambda}{2(R - \bar{r})^p},$$

$$k_0 = \frac{\bar{r}}{M_0} = \frac{2\bar{r}(R - \bar{r})^p}{\lambda} < \frac{1}{8},$$

$$\rho_0 = 1 - \sqrt{1 - 8k_0} = \frac{2k_0}{1 + \sqrt{1 - 8k_0}}.$$ 

The function $\alpha$ defined by

$$\alpha(t) = \begin{cases} 
\bar{r} + 2M_0 \rho_0 (t - \frac{1}{2} + \rho_0) & \text{if } 0 \leq t < \frac{1}{2} - \rho_0, \\
\bar{r} + M_0 (t - \frac{1}{2} + \rho_0)(\frac{1}{2} + \rho_0 - t) & \text{if } \frac{1}{2} - \rho_0 \leq t < \frac{1}{2} + \rho_0, \\
\bar{r} + 2M_0 \rho_0 (\frac{1}{2} + \rho_0 - t) & \text{if } \frac{1}{2} + \rho_0 \leq t \leq 1,
\end{cases}$$

is such that

$$\max (\alpha - \beta) > 0 \quad \text{and} \quad \max \alpha = \alpha(\frac{1}{2}) = \bar{r} + M_0 \rho_0^2 < R.$$

To prove the last inequality, we compute

$$M_0 \rho_0^2 < 4M_0 k_0^2 = 4 \frac{\bar{r}^2}{M_0} \leq \frac{8R^2 (R - \bar{r})^p}{\lambda} \leq R - \bar{r}.$$ 

Further, we check that $\alpha$ is a lower solution of (9) for $n$ large enough:

(i) $\alpha(0) = \alpha(1) = \bar{r} + 2M_0 \rho_0 (-\frac{1}{2} + \rho_0) = 0,$

(ii) $-\alpha''(t) = 2M_0 = \frac{\lambda}{(R - \bar{r})^p} \leq \lambda f(t, \alpha(t)) \leq \lambda f_n(t, \alpha(t)) \varphi'_n(\alpha'(t)),$

if $\frac{1}{2} - \rho_0 \leq t < \frac{1}{2} + \rho_0,$

(iii) $-\alpha''(t) = 0 \leq \lambda f(t, \alpha(t)) \leq \lambda f_n(t, \alpha(t)) \varphi'_n(\alpha'(t))$, if $0 \leq t < \frac{1}{2} - \rho_0$ or $\frac{1}{2} + \rho_0 \leq t \leq 1.$

**Step 4 – Existence of a solution $u_n$ of (9) for $n$ large enough.** Notice that (9) can be written as

$$-u'' = \lambda \frac{f_n(t, u)}{\varphi'_n(u')} , \quad u(0) = 0, \ u(1) = 0,$$

where the right-hand side of the equation is bounded by a $L^1$–function. Using the lower and upper solutions obtained in Step 2 and
Step 3, and applying Theorem 4.1 in [7], we obtain a solution \( u_n \) of (9) and points \( t_n' \) and \( t_n'' \) \([0, 1]\) such that

either \( u_n(t_n') > \beta(t_n') \) or \( u_n(t_n') = \beta(t_n') \) and \( u_n'(t_n') = \beta'(t_n') \)

and

either \( u_n(t_n'') < \alpha(t_n'') \) or \( u_n(t_n'') = \alpha(t_n'') \) and \( u_n'(t_n'') = \alpha'(t_n'') \).

(10)

**Step 5 – The functions \( u_n \) are bounded away from \( R \).** We choose now \( \hat{r} \) such that

\[
\bar{r} < \hat{r} < R, \quad (R - \hat{r})^{p-1} < \frac{\lambda}{32R^2}, \quad (R - \hat{r})^p < \frac{\lambda}{32(2M_0\rho_0 - \bar{r})}.
\]

Let us prove that \( u_n(t) \leq \hat{r} + (R - \hat{r})/(2\hat{r}) \). Assume by contradiction there exists \( s_n \) such that \( \max u_n = u_n(s_n) > \hat{r} + (R - \hat{r})/(2\hat{r}) \). Define then \( s_n' < s_n < s_n'' \) such that \( u_n(s_n') = u_n(s_n'') = \hat{r} \).

**Claim 1:** \( \max u_n \leq 2M_0\rho_0 \). Since \( u_n \) is concave and (10) holds, we compute for \( t \geq t_n'' \)

\[
u_n(t) \leq u_n(t_n') \frac{t}{t_n'} \leq \alpha(t_n') \frac{t}{t_n'} \leq \alpha'(0) t \leq \alpha'(0) = 2M_0\rho_0.
\]

In a similar way, for \( t \leq t_n'' \) we get

\[
u_n(t) \leq -\alpha'(1)(1 - t) \leq -\alpha'(1) = 2M_0\rho_0,
\]

and the claim follows.

**Claim 2:** \( s_n - s_n' < \frac{1}{4} \). Notice that for any \( t \in [s_n', s_n] \) one has

\[-(\varphi_n(u_n'(t)))' = \lambda f_n(t, u_n(t)) \geq \frac{\lambda}{(R - \hat{r})^p}.
\]

It follows that

\[
u_n'(t) \geq \varphi_n(u_n'(t)) \geq \frac{\lambda}{(R - \hat{r})^p} (s_n - t)
\]

and

\[
2M_0\rho_0 - \hat{r} \geq u_n(s_n) - \hat{r} \geq \frac{\lambda}{2(R - \hat{r})^p} (s_n - s_n')^2.
\]

The claim follows then since

\[
(s_n - s_n')^2 \leq \frac{2}{\lambda}(2M_0\rho_0 - \hat{r})(R - \hat{r})^p < \frac{1}{16}.
\]

**Claim 3:** \( s_n' - s_n < \frac{1}{4} \). This follows from the argument in Claim 2.
Claim 4: \( s_n' < \frac{1}{4} \). Define the energy

\[
E_n(t) = \Phi_n(u_n'(t)) + \frac{\lambda}{(R - \hat{r})^{p-1}} u_n(t),
\]

where

\[
\Phi_n(v) = \int_0^{\varphi_n(v)} \varphi_n^{-1}(s) \, ds = v \varphi_n(v) - \int_0^v \varphi_n(s) \, ds.
\]

Since for \( t \in [s_n', \hat{s}_n] \)

\[
E_n'(t) = u_n'(t) \left[ (\varphi_n(u_n'(t)))' + \frac{\lambda}{(R - \hat{r})^{p-1}} \right] \leq u_n'(t) \left[ (\varphi_n(u_n'(t)))' + \lambda f_n(t, u_n(t)) \right] = 0,
\]

we can write

\[
E_n(s_n') = \Phi_n(u_n'(s_n')) + \frac{\lambda}{(R - \hat{r})^{p-1}} \hat{r} \geq E_n(\hat{s}_n) = \frac{\lambda}{(R - \hat{r})^{p-1}} u_n(\hat{s}_n).
\]

It follows that

\[
u_n^2(s_n') \geq u_n'(s_n') \varphi_n(u_n'(s_n')) \\
\geq \Phi_n(u_n'(t)) \\
\geq \frac{\lambda}{(R - \hat{r})^{p-1}} (u_n(\hat{s}_n) - \hat{r}) \\
\geq \frac{\lambda}{2(R - \hat{r})^{p-1}}.
\]

As

\[
\hat{r} = u_n(s_n') = \int_0^{s_n'} u_n'(s) \, ds \geq s_n' u_n'(s_n'),
\]

the claim follows

\[
s_n'^2 \leq \frac{\hat{r}^2}{u_n^2(s_n')} \leq \frac{2\hat{r}^2}{\lambda} (R - \hat{r})^{p-1} \leq \frac{2R^2}{\lambda} (R - \hat{r})^{p-1} < \frac{1}{16}.
\]

Claim 5: \( 1 - s_n'^2 < \frac{1}{4} \). To prove this claim, we repeat the argument in Claim 4.
Conclusion – We come now to a contradiction since the previous claims imply that \( 1 = s' + (s_n' - s' + (s''_n - s_n') + (1 - s''_n) < \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \).

**Step 6 – Existence of a solution of (1).** From Step 5, we know that

\[
u_n(t) \leq \hat{R} = \hat{r} + \left(\frac{\hat{R} - \hat{r}}{2}\right) < R.
\]

Hence for \( n \) large enough and a.e. \( t \in [0, 1] \), \( f(t, u_n(t)) = f_n(t, u_n(t)) \). Let \( a \in [0, 1/2] \). The concavity of \( u_n \) implies that, for all \( t \in [a, 1 - a] \),

\[
\frac{\hat{R}}{a} \geq u'_n(a) \geq u'_n(t) \geq u'_n(1 - a) \geq -\frac{\hat{R}}{a},
\]

so that for \( n \) large enough and all \( t \in [a, 1 - a] \) \( \varphi'(u'_n(t)) = \varphi'_n(u'_n(t)) \).

It follows that

\[
0 \leq -u''_n(t) = \lambda \frac{f(t, u_n(t))}{\varphi'(u'_n(t))} \leq \lambda \frac{h(t)}{\varphi'(\hat{R}_a)}
\]

where \( h \in L^1(0, 1) \) is such that, for a.e. \( t \in [0, 1] \) and every \( u \in \hat{R} \), \( f(t, u) \leq h(t) \). From Arzelà-Ascoli Theorem, a subsequence of \((u_n)\) converges in \( C^1([a, 1 - a]) \) to a function \( u \in C^1([a, 1 - a]) \) which satisfies

\[
-u'' = \lambda \frac{f(t, u)}{\varphi'(u')}
\]

Using a diagonalization argument, \( u \) can be extended onto \([0, 1]\). We observe further that, as \( \max u_n \geq \min \beta \), the same holds true for \( u \) and hence it is non-trivial. As \( u \) is concave and positive, we have

\[
\lim_{t \to 0} u(t) \in [0, +\infty[ \quad \text{and} \quad \lim_{t \to 1} u(t) \in [0, +\infty[.
\]

Arguing as in Step 6 and Step 7 of the proof of Theorem 2.1, we obtain

\[
\lim_{t \to 0} u(t) = 0 \quad \text{or} \quad \lim_{t \to 0} u'(t) = +\infty
\]

and

\[
\lim_{t \to 1} u(t) = 0 \quad \text{or} \quad \lim_{t \to 1} u'(t) = -\infty.
\]

Hence, we conclude as in the proof of Theorem 2.1. \( \Box \)
Remark 4.2. Suppose that, in addition to the assumptions of Theorem 4.1, the following condition holds
\[ \lim_{u \to 0} \frac{f(t, u)}{u} = 0, \quad \text{uniformly a.e. on } [0, 1]. \]
Then, for any \( \lambda \in ]0, +\infty[ \), problem (1) has at least one positive solution.

To prove this claim we only need to verify that an upper solution \( \beta \) of (5) can be constructed for any given \( \lambda > 0 \). We fix \( \lambda \) and define the upper solution \( \beta \) by setting 
\[ \beta(t) = \varepsilon t(1 - t), \quad \text{where } \varepsilon \in ]0, \frac{1}{2}[, \]
such that \( f(t, u) \leq \frac{u}{\lambda} \) for a.e. \( t \in [0, 1] \) and all \( u \in ]0, \varepsilon[ \). We have then
\[ -\beta''(t) = 2\varepsilon \geq (1 + \varepsilon^2)^{3/2} \geq \lambda(1 + \beta'(t)^2)^{3/2} f(t, \beta(t)) \]
a.e. in \([0, 1]\).

5. Two singularities

Theorem 5.1. Let \( f : [0, 1] \times ]0, R[ \to ]0, +\infty[ \), with \( R \in ]0, +\infty[ \), be a \( L^1 \)-Carathéodory and locally \( L^1 \)-Lipschitz function. Assume that \( (h_1) \) and \( (h_2) \) hold. Then there exists \( \lambda_0 > 0 \) such that for each \( \lambda \in ]0, \lambda_0[ \) problem (1) has at least two positive solutions.

Proof. The modified problem. For each \( n \in \mathbb{N}, n > 1 \), we define
\[ f_n(t, u) = \begin{cases} f(t, R/n) & \text{if } u \leq R/n, \\ f(t, u) & \text{if } R/n < u \leq \frac{n-1}{n}R, \\ f(t, \frac{n-1}{n}R) & \text{if } \frac{n-1}{n}R < u, \end{cases} \]
and consider the modified problem
\[ -(\varphi_n(u'))' = \lambda f_n(t, u), \quad u(0) = 0, \quad u(1) = 0. \quad (11) \]
where \( \varphi_n \) is defined from (4).

From Step 2 and Step 3 in the proof of Theorem 4.1 there exists \( \lambda_0 > 0 \) such that, for all \( \lambda \in ]0, \lambda_0[ \), there are upper solutions \( \beta_1, \beta_2 \) and a lower solution \( \alpha_2 \) of (11) such that, for all large \( n \), \( \min(\beta_2 - \beta_1) > 0 \), \( \min \beta_1 > 0 \), and \( \max(\alpha_2 - \beta_2) > 0 \).
From Step 3 in the proof of Theorem 2.1 there exists a lower solution $\alpha_1$ of (11) such that, for all large $n$, $\alpha_1(t) > 0$ on $[0,1]$ and $\max \alpha_1 < \min \beta_1$.

Accordingly, for each $\lambda \in [0,\lambda_0]$ there are sequences $(u_n)_n$ and $(v_n)_n$ of solutions of (11) satisfying, for some $R' < R$, $\alpha_1(t) \leq u_n(t) \leq \beta_1(t)$ on $[0,1]$ and $\min \beta_2 \leq \max v_n \leq R'$. Arguing as in Step 5 of the proof of Theorem 2.1 and as in Step 6 of the proof of Theorem 4.1 we obtain positive solutions $u$ and $v$ of (1) as limits of subsequences of $(u_n)_n$ and $(v_n)_n$, respectively. Since $\alpha_1(t) \leq u(t) \leq \beta_1(t)$ on $[0,1]$ and $\min \beta_2 \leq \max v_n$, we have $u \neq v$. 

6. Non-existence for large values of $\lambda$

**Theorem 6.1.** Let $f : [0,1] \times [0,R[ \to [0,\infty[$, with $R \in ]0,\infty]$, be a $L^1$-Carathéodory function. Assume that for some $a \in L^1(0,1)$, with $a(t) \geq 0$ a.e. in $[0,1]$ and $a(t) > 0$ on a set of positive measure, we have

$$ \frac{f(t,u)}{u} \geq a(t), \quad \text{for a.e. } t \in [0,1] \text{ and for every } u \in ]0,R[. $$

Let $\Lambda_1$ be the principal eigenvalue of the problem

$$ -u'' = \Lambda a(t)u, \quad u(0) = 0, \quad u(1) = 0. $$

Then for each $\lambda > \Lambda_1$ problem (1) has no positive solution.

**Proof.** Let us prove that the existence of a positive solution $v$ of problem (1), for some $\lambda > \Lambda_1$, yields the existence of a positive solution $u \in W^{2,1}(0,1)$ of

$$ -u'' = \lambda a(t)u, \quad u(0) = 0, \quad u(1) = 0, $$

which is impossible as $\lambda \neq \Lambda_1$.

**Step 1 – Construction of an upper solution $\beta$ of (13).** Let $\beta$ be the continuous extension on $[0,1]$ of the restriction to $]0,1[$ of $v$. Since

$$ -\beta''(t) = \lambda(1 + (\beta'(t))^2)^{3/2} f(t,\beta(t)) \geq \lambda a(t)\beta(t), \quad \text{a.e. in } [0,1], $$

$\beta(0) \geq 0$ and $\beta(1) \geq 0$, we have that $\beta$ is an upper solution of (13) according to Definition II-4.1 in [6].
Figure 1: Phase plane for $f(u) = 1/\sqrt{u}$ and $\lambda = 1/2$.

**Step 2 – Construction of a lower solution $\alpha$ of (13).** Let $\alpha$ be an eigenfunction of (12) corresponding to $\Lambda_1$ such that $0 < \alpha(t) \leq \beta(t)$ on $[0,1]$. Then $\alpha$ is a lower solution of (13), as

$$-\alpha''(t) = \Lambda_1 a(t) \alpha(t) \leq \lambda a(t) \alpha(t).$$

**Conclusion** – As $\alpha(t) \leq \beta(t)$ for all $t \in [0,1]$, Theorem II-4.6 in [6] guarantees that (13) has a positive solution $u \in W^{2,1}(0,1)$. This yields the contradiction.

**A. Numerical illustrations**

In this appendix, we present some numerics on autonomous model examples. The computations have been performed by using the MATLAB built-in function *ode45*. On the phase-plane portraits, the bold part of an orbit corresponds to a time interval of length $1$.

In Figure 1, we depict a phase-plane example for a weak singularity in zero. One can see that the time to travel to zero from the
maximum is increasing. On the left of the bold orbit, the time required to reach zero (in the future or in the past) is less than $1/2$ while it is larger than $1/2$ for the orbits on the right.

The same phenomenon is depicted for a strong singularity in Figure 2. Observe that in this case, as proved in Theorem 3.3, no orbit reaches zero.

Figure 2: Phase plane for $f(u) = 1/u^2$ and $\lambda = 1/2$.

Figure 3: Small classical solution for $f(u) = 1/\sqrt{u}$ and $\lambda = 1/2$. 
Figure 4: Non-classical solution for $f(u) = 1/u^2$ and $\lambda = 1/2$.

We depict in Figure 3 and Figure 4 the corresponding solutions. In the case of the strong force, the solutions display jumps at the endpoints of the interval.

Figure 5: Phase plane for $f(u) = \frac{1}{\sqrt{u(5-u)^2}}$ and $\lambda = 1/2$. Zoom on the small orbits.
Figure 6: Phase plane for $f(u) = \frac{1}{\sqrt{u(5-u)^2}}$ and $\lambda = 1/2$. Zoom on the large orbits.

The case of two singularities is illustrated by Figure 5 and Figure 6. The time to reach zero is increasing from left to right for small orbits while, for large orbits, the derivative blows up before reaching zero.

Figure 7: Small classical solution for $f(u) = \frac{1}{\sqrt{u(5-u)^2}}$ and $\lambda = 1/2$. 
Hence, as proved in Theorem 5.1, we obtain a small classical solution (see Figure 7) and a large solution which turns here to be non-classical (see Figure 8).

Figure 8: Large non-classical solution for $f(u) = \frac{1}{\sqrt{u(5-u)}}$ and $\lambda = 1/2$.

Figure 9: Phase plane for $f(u) = -\frac{1}{u \ln u}$ and $\lambda = 1/2$. 
Finally, Figure 9 concerns a case which is not totally covered by our results. Indeed, Theorem 2.1 ensures the existence of a small solution for the model nonlinearity $f(u) = -\frac{1}{u \ln u}$; this is non-classical by Theorem 3.3 (see Figure 10). Since the singularity of $f$ at $u = 1$ is not strong, Theorem 5.1 does not apply. However, the numerical computation suggests the existence of a second large non-classical solution (see Figure 11). This example may motivate a further analysis.
References


[14] M. Marzocchi, Multiple solutions of quasilinear equations involving


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