Kakutani’s Splitting Procedure in Higher Dimension

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Dedicated to the memory of Fabio Rossi

Summary. - In this paper we will generalize to higher dimension the splitting procedure introduced by Kakutani for $[0,1]$. This method will provide a sequence of nodes belonging to $[0,1]^d$ which is uniformly distributed. The advantage of this approach is that it is intrinsically $d$-dimensional.

1. Introduction

A partition $\pi$ of $I = [0,1]$ is a finite covering of $I$ by a family of intervals $[t_{i-1}, t_i]$, with $1 \leq i \leq k$ and $t_{i-1} < t_i$, with pairwise disjoint interiors. In 1976 Kakutani introduced the very interesting notion of uniformly distributed sequence of partitions of the interval $[0,1]$.

Definition 1.1. If $\pi$ is any partition of $[0,1]$, and $\alpha \in \]0,1[$, its Kakutani’s $\alpha$-refinement $\alpha \pi$ is obtained by splitting all the intervals of $\pi$ having maximal length in two parts, having lengths (left and right) proportional to $\alpha$ and $\beta = 1 - \alpha$, respectively.

Kakutani’s sequence of partitions $\{\kappa_n\}$ is obtained by successive $\alpha$-refinements of the trivial partition $\omega = \{[0,1]\}$. For example, if $\alpha < \beta$, $\kappa_1 = \{[0,\alpha], [\alpha,1]\}$, $\kappa_2 = \{[0,\alpha], [\alpha,\alpha + \alpha \beta],[\alpha + \alpha \beta,1]\}$, and so on.

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**Definition 1.2.** Given a sequence of partitions \( \{\pi_n\} \), with 
\[
\pi_n = \{[t^n_{i-1}, t^n_i], 1 \leq i \leq k(n)\},
\]
we say that it is uniformly distributed, if for any continuous function \( f \) on \([0,1]\) we have 
\[
\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t^n_i) = \int_0^1 f(t) \, dt.
\]

We denote, as usual, by \( \delta_t \) the Dirac measure concentrated in \( t \).

**Remark 1.3.** It follows from the definition that uniform distribution of the sequence \( \{\pi_n\} \) is equivalent to the weak convergence of the sequence of measures 
\[
\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{t^n_i}
\]
to the Lebesgue measure \( \lambda \) on \([0,1]\).

**Remark 1.4.** It is obvious that the uniform distribution of the sequence of partitions \( \{\pi_n\} \) is equivalent to each of the following two conditions:

1. For any choice of points \( \tau_i \in [t^n_{i-1}, t^n_i] \) we have 
\[
\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(\tau^n_i) = \int_0^1 f(t) \, dt,
\]
for any continuous function \( f \) on \([0,1]\).

2. For any choice of points \( \tau_i \in [t^n_{i-1}, t^n_i] \) the sequence of measures 
\[
\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{\tau^n_i}
\]
converges weakly to the Lebesgue measure \( \lambda \) on \([0,1]\).

The following beautiful theorem is the main result of [5]:
Theorem 1.5. For any $\alpha \in ]0,1[$ the sequence of partitions $\{\kappa_n\}$ is uniformly distributed.

This result got a considerable attention in the late seventies, when other authors provided different proofs of Kakutani’s theorem and also proved its stochastic version [8]. The paper [1] extended the notion to compact metric spaces, and put in connection to a question raised by De Bruijn and Post, which has been addressed also in [7].

The aim of this paper is to extend Kakutani’s splitting procedure to higher dimension.

It is convenient to introduce for later convenience the useful standard notation for the so called “$\alpha$-dyadic” intervals. Let $I(\alpha) = [0, \alpha]$ and $I(\beta) = [\alpha, 1]$. If $I(\gamma_1 \ldots \gamma_m) = [a, b]$ (with $\gamma_k \in \{\alpha, \beta\}$ for $1 \leq k \leq m$), then

$I(\gamma_1 \ldots \gamma_m \alpha) = [a, a + \alpha(b-a)]$

and

$I(\gamma_1 \ldots \gamma_m \beta) = [a + \alpha(b-a), b].$

Naturally $\lambda(I(\gamma_1 \ldots \gamma_m)) = \gamma_1 \ldots \gamma_m = \alpha^p \beta^q$, where $p + q = m$ and $p$ is the number of occurrences of $\alpha$ among the $\gamma_k$’s, while $q$ is the number of the occurrences of $\beta$.

2. Splitting the $d$-dimensional cube

By $I^d = [0,1]^d$ we denote the unit cube of $\mathbb{R}^d$. By a cartesian $d$-rectangle (or simply a rectangle) contained in $I^d$ we always mean a set of the type $R = \prod_{j=1}^d [a_j, b_j]$. We denote by $v_i = (a_1, \ldots, a_d)$ the left endpoint of $R$.

A partition of $I^d$ will always mean in this paper a finite collection of rectangles $\{R_i, 1 \leq i \leq k\}$ as defined above, with disjoint interiors and which cover $I^d$.

The following definition is the natural extension of Kakutani’s one-dimensional splitting procedure.

Definition 2.1. Fix $\alpha \in ]0, 1[$. If $\pi = \{R_i, 1 \leq i \leq k\}$ is any partition of $[0,1]^d$, its Kakutani’s $\alpha$-refinement $\alpha \pi$ is obtained by splitting all the rectangles of $\pi$ having maximal $d$-dimensional measure $\lambda_d$ in
two rectangles, dividing in two segments the longest side such that
the lower and upper part have length proportional to $\alpha$ and $\beta = 1 - \alpha$,
respectively. If the rectangle $R$ has several sides with the same length,
we split the side with the smallest coordinate index $j$.

We define now the generalized Kakutani sequence of partitions
$\{\kappa_n^d\}$ of $I^d$ as the successive $\alpha$-refinements of the trivial partition
$\omega = \{I^d\}$.

The definition of uniformly distributed sequence of partitions extends
naturally to higher dimension.

**Definition 2.2.** Given a sequence of partitions $\{\pi_n\}$, with $\pi_n = \{R^n_i, 1 \leq i \leq k(n)\}$, we say that it is uniformly distributed if for any
continuous function $f$ on $I^d$, we have

$$\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(v^n_i) = \int_{I^d} f(t) \, dt.$$ 

As in the previous section, it is possible to allow, in the above expression, other choices of the points $\sigma^n_i \in R^n_i$ and to express uniform
distribution as the weak convergence of

$$\frac{1}{k(n)} \sum_{i=1}^{k(n)} \delta_{\sigma^n_i},$$

for any choice of $\sigma_i \in R^n_i$, to the $d$-dimensional Lebesgue measure $\lambda_d$
on $[0, 1]$.

Our aim is to prove that the $d$-dimensional Kakutani’s sequence
of partitions $\{\kappa_n^d\}$ is uniformly distributed. This will be obtained introducing
a convenient notation and proving two preparatory lemmas.

Let us begin with the following notation. By $R(\alpha)$ and $R(\beta)$
we denote the rectangles $[0, \alpha] \times [0, 1]^{d-1}$ and $[\beta, 1] \times [0, 1]^{d-1}$, respectively. If $R(\gamma_1, \ldots, \gamma_m) = \prod_{i=1}^{d} [a_i, b_i]$ (with $\gamma_k \in \{\alpha, \beta\}$ for
$1 \leq k \leq m$), then we define

$$R(\gamma_1, \ldots, \gamma_m \alpha) = \prod_{i=1}^{j-1} [a_i, b_i] \times [a_j, a_j + \alpha (b_j - a_j)] \times \prod_{i=j+1}^{d} [a_i, b_i]$$
and

\[ R(\gamma_1, \ldots, \gamma_m, \beta) = \prod_{i=1}^{j-1} [a_i, b_i] \times [a_j + \alpha(b_j - a_j), b_j] \times \prod_{i=j+1}^{d} [a_i, b_i], \]

if

\[ b_j - a_j > b_k - a_k \]

for all \( 1 \leq k < j \) and

\[ b_j - a_j \geq b_h - a_h \]

for all \( j \leq h \leq d \).

**Lemma 2.3.** The diameter of the Kakutani partition \( \kappa_n^d \) tends to zero, when \( n \) tends to infinity.

**Proof.** As in the one-dimensional case, every rectangle of \( \kappa_n^d \) is eventually subdivided in two parts, therefore given any \( m \in \mathbb{N} \) there exists \( n_0 \) such that for \( n \geq n_0 \) every \( R_n^i \) in \( \kappa_n^d \) results from at least \( md \) splittings. This implies that each side of \( R_n^i \) has length at most \( L^m \), where \( L = \max\{\alpha, \beta\} < 1 \), and therefore its diameter is smaller than \( L^m \sqrt{d} \).

We have to introduce now in this context a notion which is widely used in the theory of uniformly distributed sequences of points (compare for instance Chapter 3 of [6] or Chapter 1 of [2]).

**Definition 2.4.** We say that a class of functions \( \mathcal{F} \) is determining for the uniform convergence of partitions whenever, for a given sequence of partitions \( \{\pi_n\} (\pi_n = \{R_n^i, 1 \leq i \leq k(n)\}) \), from

\[ \lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(v_n^i) = \int_0^1 f(t) \, dt \]

for any \( f \in \mathcal{F} \), it follows that \( \{\pi_n\} \) is uniformly distributed.

By \( \chi_C \) we will denote the characteristic function of \( C \).

**Lemma 2.5.** Assume \( \{C_n\} \) is a sequence of finite partitions of \( I^d \) whose elements \( C_n^i, 1 \leq i \leq k(n) \), are rectangles and \( \text{diam} \, C_n \) tends to zero. Suppose moreover that for each \( C_j^m \) we have

\[ \lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{C_j^m}(v_n^i) = \lambda_d(C_j^m), \]

(1)
where \( v^n_i \) is the left endpoint of \( C^n_i \). Then the family \( F \) of the characteristic functions of the \( C^n_i \)'s is determining.

**Proof.** It is well known that the family of the characteristic functions of all the rectangles \( R = \prod_{j=1}^{d}[a_j, b_j] \) is determining. So let \( R \subset I^d \) be a (non degenerate) rectangle and denote by \( B \) the unit ball of \( \mathbb{R}^d \). Fix \( \varepsilon \in [0, 1] \) and let us denote by \( R_\varepsilon = (\cup_{z \in R}(z + \varepsilon B)) \cap I^d \).

Let \( n_0 \in \mathbb{N} \) be such that for \( n \geq n_0 \), \( \text{diam } C_n < \varepsilon \). For such an \( n \), let \( C_n(R) \) be the collection of all the sets in \( C_n \) intersecting \( R \), and let us denote by \( C_R \) their union. Then we have \( R \subset C_R \subset R_\varepsilon \) and therefore

\[
\lambda_d(R) \leq \lambda_d(C_R) \leq \lambda(R_\varepsilon) \leq \lambda_d(R) + c\varepsilon ,
\]

where \( c \) is an appropriate constant.

The same inclusions imply that, for arbitrarily small \( \varepsilon \),

\[
\limsup_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v^n_i) \leq \lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{C_R}(v^n_i) = \lambda_d(C_R) \leq \liminf_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v^n_i) + c\varepsilon \leq \limsup_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v^n_i) + c\varepsilon .
\]

The equality in the first line follows from (1). It follows now from (2) and (3) that

\[
\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_R(v^n_i) = \lambda_d(R) ,
\]

for any rectangle \( R \subset I^d \), and the conclusion follows.

We are now in position to prove the main result of this paper.

**Theorem 2.6.** The sequence of partitions \( \{\kappa_n^d\} \) introduced in Definition 2.1 is uniformly distributed.

**Proof.** We apply the previous lemma to the sequence of partitions \( \{\kappa_n^d\} \). Since by Lemma 2.3 its diameter tends to zero, we only have
to prove that given any \( s \in \mathbb{N} \) and any rectangle \( R = R_j^{s} \) belonging to \( \kappa_d^s \), we have that

\[
\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{R}(v_i^n) = \lambda_d(R) .
\]

But from the previous discussion we know that \( R_j^{s} = R(\gamma_1 \ldots \gamma_m) \) for appropriate values \( \gamma_k \in \{\alpha, \beta\} \). On the other hand there is a one to one correspondence between the rectangles \( R(\gamma_1 \ldots \gamma_m) \) showing up in the partitions \( \kappa_d^s \) and the intervals \( I(\gamma_1 \ldots \gamma_m) \) appearing in the one-dimensional partitions \( \kappa_n \). Since

\[
\lambda_d(R(\gamma_1 \ldots \gamma_m)) = \gamma_1 \ldots \gamma_m = \lambda(I(\gamma_1 \ldots \gamma_m)),
\]

the rectangle \( R(\gamma_1 \ldots \gamma_m) \) is split into \( R(\gamma_1 \ldots \gamma_m \alpha) \) and \( R(\gamma_1 \ldots \gamma_m \beta) \) exactly when the interval \( I(\gamma_1 \ldots \gamma_m) \) undergoes the same procedure. Now Kakutani’s theorem says that \( I = I(\gamma_1 \ldots \gamma_m) \) is subdivided the right number of times, so that

\[
\lim_{n \to \infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{I}(t_i^n) = \lambda(I),
\]

and therefore the analogous identity (4) holds for \( R = R(\gamma_1 \ldots \gamma_m) \).

3. Conclusions

The interest of this result is that it is intrinsically \( d \)-dimensional and this may be useful in applications to integration in higher dimension, where it is important (and not very easy) to find good sets of nodes.

Given \( \kappa_n^d \), the centers of gravity of the rectangles \( R_i^n \) seem to be a convenient choice of nodes.

In a subsequent paper we will compare our results, and other intrinsically multidimensional methods we are developing, with methods which are based on the subdivision of the one-dimensional factors of \( I^d \) as proposed in [3] and [4].
References


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