On Hyperbolic $\pi$-Orbifolds with Arbitrary many Singular Components

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Summary. - We construct a family of $(n + 1)$-component links $\mathcal{L}_n$ which are closures of rational 3-string braids $(\sigma_1^{-1/2} \sigma_2^2)^n$, and show that for $n \geq 3$ they arise as singular sets of hyperbolic $\pi$-orbifolds. Moreover, their 2-fold branched coverings are described by Dehn surgeries.

1. Introduction

The concept of a hyperelliptic involution came originally from the theory of Riemann surfaces. Let $S_g$ be a Riemann surface of genus $g$, $g > 1$. An involution $\tau \in \text{Iso}^+(S_g)$ is said to be hyperelliptic if the quotient space $S_g/\langle \tau \rangle$ is homeomorphic to the 2-dimensional sphere $S^2$. A Riemann surface is said to be hyperelliptic if it admits a hyperelliptic involution, i.e. if it can be obtained as a 2-fold branched covering of $S^2$. For properties of hyperelliptic Riemann surfaces see [4].

This concept can be generalized to higher dimensions in the natural way. Let $M$ be an $n$-dimensional manifold. Suppose that there exists an involution $\tau : M \to M$ such that the quotient space $M/\langle \tau \rangle$ is homeomorphic to the $n$-dimensional sphere $S^n$. Then, $\tau$ is said to be a hyperelliptic involution and $M$ is said to be a hyperelliptic

(*) Supported by the grant NSh-8526.2006.1, the grant of RFBR, and the grant of Siberian Branch of RAN.
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Keywords: Hyperbolic 3-Manifolds, Hyperelliptic Involution, $\pi$-Orbifold.
AMS Subject Classification: Primary: 57M25.
manifold. If M admits a geometric structure then we assume in the
definition that \( \tau \) is an isometry.

Three-dimensional hyperelliptic manifolds are objects of a special
interest because of the relation with knot theory. If M is a
3-dimensional hyperelliptic manifold, with a hyperelliptic involution
\( \tau \), then M is the 2-fold branched covering of \( S^3 \) branched over some
link (in particular, a knot) \( L \). The covering is given by the action of
\( \tau \) and each point of \( L \) has branching index 2. According to the ter-
minology of orbifold theory (see [16, 19]), this situation means that
M is the 2-fold covering of a \( \pi \)-orbifold \( O = S^3(L) \) with underling
set \( S^3 \) and singular set \( L \) with singular angle \( \pi \) at each point of \( L \).

It is known that in the 3-dimensional case there are eight model
geometries: \( E^3, \mathbb{H}^3, S^3, \mathbb{H}^2 \times \mathbb{E}^1, S^2 \times \mathbb{E}^1, Sol, Nil, \) and \( PSL(2, \mathbb{R}) \)
[16, 19]. It was shown in [8] that for each of these geometries there
exist hyperelliptic manifolds (with \( \tau \) be an isometry).

Examples of hyperbolic 3-manifolds of small volume admitting
one, two, or three hyperelliptic involutions can be found in [11];
we note that the maximal number of non-conjugate hyperelliptic
involutions of a hyperbolic manifold is nine, see [12], [6].

Let \( M \) be a hyperbolic hyperelliptic 3-manifold with hyperelliptic
involution \( \tau \). Then, the quotient \( \pi \)-orbifold \( M/\langle \tau \rangle = S^3(L) \) is also
hyperbolic.

A link \( L \) in \( S^3 \) is said to be hyperbolic if the complement \( S^3 \setminus L \)
is a hyperbolic manifold. We will say that \( L \) is \( \pi \)-hyperbolic if the
\( \pi \)-orbifold \( O = S^3(L) \) is hyperbolic. Obviously, hyperbolicity of a
link does not imply \( \pi \)-hyperbolicity of it (for example, hyperbolic
2-bridge links are not \( \pi \)-hyperbolic).

Most of known examples of \( \pi \)-hyperbolic links have few com-
ponents. Among them are knots \( 8_{18} \) and \( 9_{49} \), 2-component link
\( 10^2_{138} \), knots and 3-component links arising as closed 3-string braids
\( (\sigma_1 \sigma_2^{-1})^n, n \geq 4 \) (here we use standard notations for knots and links
according to [15] and for braids according to [1]). Discussions of the
2-fold branched coverings of these knots and links can be found in
[9, 10, 11].

In the present paper, we construct explicit examples of \( \pi \)-hy-
perbolic links with an arbitrary number \( n \) of components, for any posi-
tive integer \( n \). We will present quite simple examples of such a type.
Moreover we describe the 3-manifolds that are the 2-fold branched coverings of the links under consideration.

2. $\pi$-hyperbolic links

To define a family of links we start with the notion of a rational 3-string braid.

Let $\sigma_1$ and $\sigma_2$ be standard generators of the braid group $B_3$ on 3 strings. Elements of $B_3$ are of the form $\omega = \sigma_{i_1}^{p_1} \cdots \sigma_{i_k}^{p_k}$, where $i_1, \ldots, i_k$ are equal to 1 or 2, and $p_1, \ldots, p_k$ are integers. To construct a geometric braid corresponding to $\omega$, with each multiplier $\sigma_{i_j}^{p_j}$ we associate $|p_j|$ half-twists on strings $i_j$ and $i_j + 1$ in the direction depending of sign of $p_j$. In other words, we are putting $p_j$–tangle with strings $i_j$ and $i_j + 1$ as incoming arcs.

We generalize this construction in the following way (see also [7]).

Let $p_j$ and $q_j$ be coprime integers. By $\sigma_{i_j}^{p_j/q_j}$ we denote the geometrical object called a rational braid, which is obtained by putting the rational $p_j/q_j$–tangle with strings $i_j$ and $i_j + 1$ as incoming arcs. The product of two rational braids is defined similarly to the product of usual braids. Thus, an expression $\omega = \sigma_{i_1}^{p_1/q_1} \cdots \sigma_{i_k}^{p_k/q_k}$, with $i_1, \ldots, i_k$ equal to 1 or 2, and $p_j$ and $q_j$ be coprime for each $j = 1, \ldots, k$, defines a rational braid obtained by putting rational tangles in respect to each multiplier.

Consider a rational 3-string braid $\sigma_1^{-1/2} \sigma_2^2$ pictured in Figure 1.

Denote by $\mathcal{L}_n$, $n \geq 1$, the closure of the rational 3-string braid $(\sigma_1^{-1/2} \sigma_2^2)^n$ (see Figure 2, where the 4-component link $\mathcal{L}_3$ is pictured). Obviously, $\mathcal{L}_n$ has $(n+1)$ components.

**Theorem 2.1.** For any integer $n \geq 3$ the $(n+1)$-component link $\mathcal{L}_n$ is $\pi$-hyperbolic.
Proof. Let $\mathcal{O}_n = S^3(\mathcal{L}_n)$ be the $\pi$-orbifold with singular set $\mathcal{L}_n$. By the definition $\mathcal{L}_n$ has a cyclic symmetry $\rho$ of order $n$ which permutes blocks $\sigma_1^{-1/2}\sigma_2^2$. The symmetry $\rho$ induces a cyclic symmetry of order $n$ of the orbifold $\mathcal{O}_n$; we denote this symmetry also by $\rho$. The singular set of the quotient orbifold $\mathcal{O}'_n = \mathcal{O}_n/\langle \rho \rangle$ is the 3-component link $\mathcal{R}$ presented in the left part of Figure 3, i.e. $\mathcal{O}'_n = S^3(\mathcal{R})$. One of its components is the image of the axis of $\rho$ and has singularity index $n$. Two other components are images of $\mathcal{L}_n$ and have singularity index 2.

Using Reidemeister moves one can redraw $\mathcal{R}$ as in the right part of Figure 3, and then as in the left part of Figure 4.

Let $\mathcal{O}''_n$ be the 2-fold covering of $\mathcal{O}'_n$, branched over one component of $\mathcal{R}$ having singularity index 2. The singular set of $\mathcal{O}''_n$ is the 2-component link $\mathcal{Q}$ presented in the right part of Figure 4, i.e. $\mathcal{O}''_n = S^3(\mathcal{Q})$. One its component, say $\mathcal{Q}_1$, has singularity index $n$, and other, say $\mathcal{Q}_2$, has singularity index 2.

Now we construct a 2-fold covering of $\mathcal{O}''_n$ branched over $\mathcal{Q}_2$ as
follows. Using Reidemeister moves one can redraw $Q$ as in the left part of Figure 5, and then as in the right part of Figure 5.

Let us denote by $O_n'''$ the 2-fold covering of $O_n''$ branched over $Q_2$. The singular set of $O_n'''$ is the 2-component link $P$ presented in Figure 6, i.e. $O_n''' = S^3(P)$. Both its component have singularity index $n$.

Using Reidemeister moves $P$ can be redrawn as in the left part of Figure 7, and then as in the right part of Figure 7. Comparing Figure 7 with the standard picture for a 2-bridge link (see, for example [3, p. 195], one can conclude that $P$ is the 2-bridge link corresponding to the rational parameter $40/9 = 4 + 1/2 + 1/4$.

Thus $O_n'''$ is the orbifold with the singular set the 2-bridge 40/9-link and the singularity index $n$ on both components. The hyperbolicity of orbifolds $\alpha/\beta(n)$ with singular set a 2-bridge knot or link $\alpha/\beta$ and singularity index $n$ is described in [2, Example A.0.2, p. 174] and in [5]. In particular, $\alpha/\beta(n)$ is hyperbolic if $\alpha > 5$, $|\beta| > 1$, and $n \geq 3$. Therefore, the orbifold $O_n'''$ is hyperbolic if $n \geq 3$. Since by the construction $O_n'''$ is commensurable with $O_n$, the $\pi$-orbifold $O_n$
is also hyperbolic, and the link $L_n$ is $\pi$-hyperbolic for $n \geq 3$.

Geometrical invariants of manifolds and orbifolds from the proof can be found by using a computer program SnapPea [17]. Thus, one can see that $\text{vol}(S^3 \setminus \mathcal{R}) = 7.70691\ldots$ and $\text{vol}(S^3 \setminus \mathcal{P}) = 8.51908\ldots$. Moreover, for initial values of $n$ the following table of volumes holds:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\text{vol}(S^3 \setminus L_n)$</th>
<th>$\text{vol} \mathcal{O}_n$</th>
<th>$\text{vol} \mathcal{O}_n'$</th>
<th>$\text{vol} \mathcal{O}_n''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>16.59112\ldots</td>
<td>2.56897\ldots</td>
<td>0.85632\ldots</td>
<td>3.42529\ldots</td>
</tr>
<tr>
<td>4</td>
<td>25.76187\ldots</td>
<td>5.60143\ldots</td>
<td>1.40036\ldots</td>
<td>5.60143\ldots</td>
</tr>
<tr>
<td>5</td>
<td>34.42142\ldots</td>
<td>8.32706\ldots</td>
<td>1.66541\ldots</td>
<td>6.66165\ldots</td>
</tr>
</tbody>
</table>

3. 2-fold branched coverings of links

In this section we will describe 3-manifolds $M_n$ that are 2-fold coverings of $S^3$ branched over links $L_n$. 
In [18] there was introduced a family of closed orientable 3-manifolds Takahashi manifolds obtained by Dehn surgery with rational coefficients \( p_k/q_k \) and \( r_k/s_k \), \( k = 1, \ldots, n \), on \( S^3 \), along the 2n-component link \( T_n \) (see Figure 8) which is a closed chain of 2n unknotted components. These manifolds have been studied and generalized in [7, 13].

A Takahashi manifold is said to be periodic when the surgery coefficients have the same cyclic symmetry of order \( n \) as the 2n-component link \( T_n \), i.e. the coefficients are \( p_k/q_k = p/q \) and \( r_k/s_k = r/s \) alternately, for \( k = 1, \ldots, n \). Let us denote such Takahashi manifold by \( M_n(p/q; r/s) \). By [7, 18] the manifold \( M_n(p/q; r/s) \) is a 2-fold branched covering of \( S^3 \) branched over the link that is the closure of a rational 3-string braid \( (\sigma_1^{p/q} \sigma_2^{r/s})^n \). By the definition, if \( p/q = -1/2 \) and \( r/s = 2/1 \) then we get the link \( L_n \) from the previous section. Therefore, the following description of 2-fold branched coverings of \( L_n \) holds.

**Proposition 3.1.** For any \( n \geq 1 \) the two-fold covering of \( S^3 \) branched over \( L_n \) is the periodic Takahashi manifold \( M_n = M_n(-1/2; 2/1) \).

In virtue [13, 18] the fundamental group of \( M_n(p/q; r/s) \) has the following presentation:

\[
\langle x_1, \ldots, x_n, y_1, \ldots, y_n \mid y_i^{-p} = x_i^{s} x_{i-1}^{-s}, \quad x_i^{-r} = y_i^{q} y_{i+1}^{-q}, \quad i = 1, \ldots, n \rangle,
\]

where all indices are taken by mod \( n \). Hence the following cyclic
presentation holds:

$$\pi_1(M_n(-1/2; 2/1)) = \langle x_1, \ldots, x_n \mid w(x_i, x_{i+1}, x_{i+2}) = 1, \quad i = 1, \ldots, n \rangle.$$  

with the defining word $w(x_i, x_{i+1}, x_{i+2}) = x_i^2(x_ix_i^{-1})^2(x_ix_i^{-1})^2$.

4. Covering diagram

To complete the discussion of links $L_n$ and manifolds $M_n$ let us describe a covering diagram in which they are involved.

Before formulating the main result of this section we have to talk about the types of $n$-fold cyclic branched coverings of links we want to consider. Obviously, a knot has an unique $n$-fold cyclic branched covering. Let $L = K_1 \cup K_2$ be a link in the 3-sphere with two components. Denote by $\pi_1(S^3 \setminus L)$ the fundamental group of the link complement and by $m_1$ and $m_2$ meridians of the components $K_1 \cup K_2$ of the link, oriented in an arbitrary way. The homology group $H_1(S^3 \setminus L)$ of the link complement is isomorphic to $\mathbb{Z}^2$ and generated by the homology classes of the meridians. Each surjection $\psi : \pi_1(S^3 \setminus L) \to H_1(S^3 \setminus L) \to \mathbb{Z}_n$ onto the cyclic groups $\mathbb{Z}_n$ of order $n$ defines a cyclic $n$-fold branched covering $M = M(\psi)$ of $S^3$ branched over $L$. According to [14] we call $M$ a strictly–cyclic $n$-fold covering of $L$ if the corresponding surjection $\psi$ maps (the homotopy class of) meridians $m_1$ and $m_2$ of $L$ to the same generator of the cyclic group $\mathbb{Z}_n$. Note that strictly–cyclic coverings are also called uniform coverings in [20].

Let us denote by $M'_n$ the strictly–cyclic $n$-fold covering of $S^3$ branched over the 2-component 2-bridge link 40/9. Remark that $M'_n$ is a generalized periodic Takahashi manifold in the sense of [13].

**Theorem 4.1.** For the above described manifolds and orbifolds the following diagram of coverings holds:
where singular sets $L_n$, $R$, $Q$, and $P$ of orbifolds $O_n$, $O_n'$, $O_n''$, and $O_n'''$ are presented in Figures 2, 3, 4, and 7, respectively.

Proof. By the proof of Theorem 2.1 and by Proposition 3.1 we already have the following sequences of coverings:

$$M_n \xrightarrow{2} O_n \xrightarrow{n} O_n'$$

and

$$M_n' \xrightarrow{n} O_n''' \xrightarrow{2} O_n'' \xrightarrow{2} O_n'.$$

Let us denote by $\Gamma_n'$ the group of the orbifold $O_n'$, i.e. $O_n' = \mathbb{H}^3 / \Gamma_n'$. Let $\alpha$, $\beta$, and $\gamma$ be generators of $\Gamma_n'$ corresponding to generators of $\pi_1(S^3 \setminus R)$ pictured in Figure 9.

Using the Wirtinger algorithm [3] one can see that $\Gamma_n'$ has the
following presentation:

\[
\langle \alpha, \beta, \gamma \mid \alpha^n = 1, \beta^2 = 1, \gamma^2 = 1, \beta \alpha \gamma = \alpha \gamma \beta, \\
\alpha^{-1} \gamma \beta^{-1} \alpha^{-1} \gamma^{-1} \beta^{-1} \alpha^{-1} \gamma^{-1} \beta \gamma \cdot \\
\cdot \beta^{-1} \alpha \beta \gamma^{-1} \beta^{-1} \alpha \gamma^{-1} \beta \gamma^{-1} \alpha \beta \gamma^{-1} = 1 \rangle.
\]

Consider a group

\[H_n = \mathbb{Z}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a \mid a^n = 1 \rangle \oplus \langle b \mid b^2 = 1 \rangle \oplus \langle c \mid c^2 = 1 \rangle\]

and define an epimorphism \(\varphi_n : \Gamma' \rightarrow H_n\) by setting \(\varphi_n(\alpha) = a, \varphi_n(\beta) = b, \varphi_n(\gamma) = c\). Let \(\Gamma_n, \Gamma''_n, \Gamma'''_n, G_n,\) and \(G'_n\) be such groups that \(O_n = \mathbb{H}^3 / \Gamma_n, O''_n = \mathbb{H}^3 / \Gamma''_n, O'''_n = \mathbb{H}^3 / \Gamma'''_n, M_n = \mathbb{H}^3 / G_n,\) and \(M'_n = \mathbb{H}^3 / G'_n\).

For the covering \(O''_n \rightarrow O'_n\) a lift of the loop \(\beta\) is a trivial loop, lifts \(\tilde{\alpha}\) and \(\tilde{\gamma}\) of \(\alpha\) and \(\gamma\) are loops about components of the singular set \(Q\) of \(O''_n\) generating subgroups \(\mathbb{Z}_n\) and \(\mathbb{Z}_2\), respectively. Thus, \(\Gamma'_n = \varphi^{-1}_n(\langle a \mid a^n = 1 \rangle \oplus \langle c | c^2 = 1 \rangle)\). For the covering \(O'''_n \rightarrow O''_n\) a lift of the loop \(\tilde{\gamma}\) is a trivial loop, a lift \(\tilde{\alpha}\) of the loop \(\alpha\) is a loop about the singular set \(P\) of \(O'''_n\) generating subgroup \(\mathbb{Z}_n\). Thus, \(\Gamma'''_n = \varphi^{-1}_n(\langle a | a^n = 1 \rangle)\). For the covering \(M'_n \rightarrow O'''_n\) the preimage of the loop \(\tilde{\alpha}\) is a trivial loop. Thus, \(G'_n = \text{Ker}(\varphi_n)\).

For the covering \(O_n \rightarrow O'_n\) a lift of the loop \(\alpha\) is a trivial loop, lifts \(\tilde{\beta}\) and \(\tilde{\gamma}\) of loops \(\beta\) and \(\gamma\) are loops about components of the singular set \(L_n\) of \(O_n\) generating subgroups \(\mathbb{Z}_2\) and \(\mathbb{Z}_2\). Thus, \(\Gamma_n = \varphi^{-1}_n(\langle b | b^2 = 1 \rangle \oplus \langle c | c^2 = 1 \rangle)\). For the group \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle b | b^2 = 1 \rangle \oplus \langle c | c^2 = 1 \rangle\) we denote \(d = b + c\) and consider a group \(\mathbb{Z}_2 = \langle d | d^2 = 1 \rangle\). For the covering \(M_n \rightarrow O_n\) loops \(\tilde{\beta}\) and \(\tilde{\gamma}\) lift to trivial loops. Thus, \(G_n = \varphi^{-1}_n(\langle d | d^2 = 1 \rangle)\).

Therefore we get the following diagram of subgroups (where \(A \xrightarrow{m} B\) denotes that \(A\) is a subgroup of \(B\) of index \(m\))
\[ \Gamma_n = \varphi_n^{-1}( (b) \oplus (c)) \]
\[ \Gamma_n' = \varphi_n^{-1}( (a) \oplus (c)) \]

that implies the diagram of coverings. \[\square\]

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Received November 2, 2006.