On some Semilinear Periodic Parabolic Problems

T. Godoy and U. Kaufmann

Summary. - Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. We study existence and nonexistence of positive solutions for some semilinear Dirichlet periodic parabolic problems of the form $Lu = h(x, t, u)$ in $\Omega \times \mathbb{R}$ for a class of Caratheodory functions $h : \Omega \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ such that $h(., 0) = 0$ and $\lim_{\xi \to 0^+} \xi^{-1} h(., \xi) = 0$ or $\pm \infty$. All results remain true for the corresponding elliptic problems.

1. Introduction

Let $\Omega$ be a $C^{2+\theta}$ bounded domain in $\mathbb{R}^N$, $\theta \in (0, 1)$, $N \geq 2$. For $T > 0$ and $1 \leq p \leq \infty$, let $L^p_T$ be the Banach space of $T$-periodic functions $f$ on $\Omega \times \mathbb{R}$ (i.e. satisfying $f(x, t) = f(x, t + T)$ a.e. $(x, t) \in \Omega \times \mathbb{R}$) such that $f|_{\Omega \times (0, T)} \in L^p(\Omega \times (0, T))$, equipped with the norm $\|f\|_{L^p_T} := \|f|_{\Omega \times (0, T)}\|_{L^p(\Omega \times (0, T))}$. Let $C_T$ be the space of continuous and $T$-periodic functions on $\Omega \times \mathbb{R}$ provided with the $L^\infty$ norm, and let $C^{1+\theta, (1+\theta)/2}_T$ be the space of $T$-periodic functions belonging to $C^{1+\theta, (1+\theta)/2}(\Omega \times \mathbb{R})$.

Let $\{a_{ij}\}$, $\{b_j\}$, $1 \leq i, j \leq N$, be two families of $T$-periodic functions satisfying $a_{ij} \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$, $a_{ij} = a_{ji}$ and $b_j \in L^p_T$, and

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Authors’ addresses:
Tomás Godoy and Uriel Kaufmann, Facultad de Matemática, Astronomía y Física, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina; E-mail: godoy@mate.uncor.edu, kaufmann@mate.uncor.edu
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assume that
\[ \sum a_{ij} (x,t) \xi_i \xi_j \geq \alpha_0 |\xi|^2 \]
for some \( \alpha_0 > 0 \) and all \( (x,t) \in \Omega \times \mathbb{R} \), \( \xi \in \mathbb{R}^N \). Let \( A \) be the \( N \times N \) matrix whose \( i,j \) entry is \( a_{ij} \), let \( b = (b_1, ..., b_N) \), let \( 0 \leq c_0 \in L_\infty^0 \) and let \( L \) be the parabolic operator given by
\[ Lu = u_t - \text{div} (A \nabla u) + \langle b, \nabla u \rangle + c_0 u \]

Let \( W = \{ u \in L^2 ((0, T), H^1_0 (\Omega)) : u_t \in L^2 ((0, T), H^{-1} (\Omega)) \} \). For \( h \in L_T^2 \), we say that \( u \) is a (weak) solution of the periodic problem
\[ \begin{aligned}
L u &= h, & \quad & \text{in } \Omega \times \mathbb{R} \\
u &= 0, & \quad & \text{on } \partial \Omega \times \mathbb{R} \\
u &= T \text{-periodic, if } u \text{ is } T \text{-periodic, } u_{|\Omega \times (0,T)} \in W \end{aligned} \]  
if \( u \) is \( T \text{-periodic, } u_{|\Omega \times (0,T)} \in W \) and
\[ \int_{\Omega \times (0,T)} \left[ -u \frac{\partial g}{\partial t} + (A \nabla u, \nabla g) + \langle b, \nabla u \rangle g + c_0 ug \right] = \int_{\Omega \times (0,T)} hg \]
for all \( g \in C_0^\infty (\Omega \times (0,T)) \). For \( u \in W \), the inequality \( Lu \geq h \) (respectively \( \leq \)) in \( \Omega \times \mathbb{R} \) will be understood in the analogous weak sense.

For \( 1 \leq r \leq \infty \) let \( W^{2,1}_r (\Omega \times (t_0, t_1)) \) be the Sobolev space of the functions \( u \in L^r (\Omega \times (t_0, t_1)), u = u(x_1, ..., x_N, t), u_t, u_{x_i}, \text{ and } u_{x_i x_j} \) belong to \( L^r \Omega \times (t_0, t_1)) \) for \( 1 \leq i, j \leq N \), and let \( W^{2,1}_{r,T} \) be the space of \( T \text{-periodic functions such that } u_{|\Omega \times (0,T)} \in W^{2,1}_r (\Omega \times (0,T)) \). For \( f \in L_T^r, r > 1 \), we say that \( u \) is a strong solution of (1) if \( u \in W^{2,1}_{r,T} \) and the equation holds a.e. in the pointwise sense.

Let \( f, g : \Omega \times \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R} \) be two Carathéodory functions, i.e. \((x,t) \rightarrow f(x,t,\xi) \) is measurable for all \( \xi \geq 0 \) and \( \xi \rightarrow f(x,t,\xi) \) is continuous in \([0, \infty) \) a.e. \((x,t) \in \Omega \times \mathbb{R} \), and the same for \( g \). Assume that \( f(\cdot, \xi) \) and \( g(\cdot, \xi) \) belong to \( L_T^r, r > (N + 2)/2 \), for all \( \xi \geq 0 \) and that both are \( T \text{-periodic in } t \). Let

H1. There exist \( c_f, p_1, p_2, \xi, \bar{\xi} > 0 \) and \( 0 \leq a \in L_T^\infty \) such that
\[ c_f \xi^{p_1} \leq f(x,t,\xi) \text{ for all } \xi \in (0, \xi_0] \text{ a.e. } (x,t) \in \Omega \times \mathbb{R}, \quad (2) \]
\[ f(x,t,\xi) \leq a(x,t) \xi^{p_2} \text{ for all } \xi \in (\xi_0, \infty) \text{ a.e. } (x,t) \in \Omega \times \mathbb{R} \quad (3) \]
H2. There exist \( b, c, q_1, q_2, \xi, \eta > 0 \) with \( q_1 > p_2 \) if \( p_2 \geq 1 \) such that
\[
c g \xi^{q_1} \leq g (x, t, \xi) \quad \text{for all } \xi \in [\xi, \infty) \ a.e. (x, t) \in \Omega \times \mathbb{R},
\]
\[
g (x, t, \xi) \leq b \xi^{q_2} \quad \text{for all } \xi \in (0, \xi] \ a.e. (x, t) \in \Omega \times \mathbb{R}
\]

Our aim in this paper is to study existence and nonexistence of positive solutions for semilinear periodic parabolic problems of the form
\[
\begin{aligned}
Lu &= \lambda f (x, t, u) - g (x, t, u) \quad \text{in } \Omega \times \mathbb{R} \\
u &= 0 \quad \text{on } \partial \Omega \times \mathbb{R} \\
u &= \text{T-periodic}
\end{aligned}
\]

where \( \lambda > 0 \) is a real parameter and \( f, g \) satisfy conditions H1 and H2. Let us mention that as a consequence of our proofs the results remain true for the corresponding elliptic problems. For applications we refer to [2], [16].

In order to describe our results and relate them to others in the literature, let us take as an example of the above situation the problem
\[
\begin{aligned}
Lu &= \lambda a (x, t) h (u) u^p - b (x, t) u^q := H (x, t, u) \quad \text{in } \Omega \times \mathbb{R} \\
u &= 0 \quad \text{on } \partial \Omega \times \mathbb{R} \\
u &= \text{T-periodic}
\end{aligned}
\]

where \( 0 < a_0 \leq a \in L^\infty_T, 0 < b_0 \leq b \in L^\infty_T, p, q > 0 \) and \( h : [0, \infty) \rightarrow \mathbb{R} \) is a continuous function such that \( h (0) \geq 0 \) with \( h' (0) > 0 \) if \( h (0) = 0 \), and sup_{\xi > 0} h (\xi) < \infty.

When \( p = 1 < q \) and \( h \equiv 1 \), (7) becomes the well-known logistic equation that has been widely studied in recent years. A necessary and sufficient condition for the existence of positive solutions is \( \lambda > \lambda_1 (a) \), where \( \lambda_1 (a) \) is the (unique) positive principal eigenvalue of the linear problem with weight
\[
\begin{aligned}
Lu &= \lambda a (x, t) u \quad \text{in } \Omega \times \mathbb{R} \\
u &= 0 \quad \text{on } \partial \Omega \times \mathbb{R} \\
u &= \text{T-periodic}
\end{aligned}
\]

(see e.g. [14], [17] and the references therein for the elliptic problem and [12], [10] for the periodic parabolic case). However, if for instance
If $0 < p, q < 1$ and $h \equiv 1$, in the elliptic case it is also known that there exists some $\Lambda \geq 0$ such that (7) has a positive solution for all $\lambda > \Lambda$ and that there is no positive solution if $0 < \lambda < \Lambda$. In fact, it was proved under additional smoothness assumptions on $a$ and $b$ that if $p \leq q$ then $\Lambda = 0$, and that if $p > q$ then $\Lambda > 0$ (see e.g. [21], [4], [15] and its references). For the periodic parabolic problem, recently the authors have found existence of positive solutions for all $\lambda$ large enough in [11]. We note however that there it is asked that either $q > 1 - 1/(N + 2)$ or $b$ satisfies a quite strong assumption.

Finally, to our knowledge no results are known for (7) when $1 < p < q$, even if $h \equiv 1$, while this elliptic problem has been studied for example in [20], [13] for $a = b = h \equiv 1$ and $(N + 2) / (N - 2) < p < q$, and recently in [6] for a quasilinear equation that includes the case $1 < p < (N + 2) / (N - 2)$, $q < 2N / (N - 2)$, $a \equiv 1$ and $b \geq 0$ satisfying some additional conditions. Theorem 3.1 shows that similar existence results as the ones quoted above are still valid in this situation, and that there exists a lower estimate for $\Lambda$ and a positive solution for $\lambda = \Lambda$ in this case.

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2. Preliminaries

We start collecting some known facts about periodic parabolic problems with weight.

 Remark 2.1. i) Let $a \in L^r_T$, $r > (N + 2)/2$, and let

$$P(a) := \int_0^T \operatorname{esssup}_{x \in \Omega} a(x,t) \, dt.$$ 

Then $P(a) > 0$ is necessary and sufficient for the existence of a (unique) positive principal eigenvalue $\lambda_1(a)$ for problem (8) (cf. [8], Theorem 3.6). We note that the case $P(a) = +\infty$ is allowed (cf. [8], p. 218).

ii) Let $0 \leq \lambda < \lambda_1(a)$ if $\lambda_1(a)$ exists or $\lambda \geq 0$ if $\lambda_1(a)$ does not exist. Then $(L - \lambda a)^{-1} : L^r_T \to C_T (r > (N + 2)/2)$ is a well defined compact and positive operator (cf. [9], Lemma 2.9). In particular, if $Lu \geq \lambda au$ (respectively $\leq$) then $\lambda \leq \lambda_1(a)$ if $\lambda_1(a)$ exists (respectively $\lambda \geq \lambda_1(a)$).

iii) The following comparison principle holds: if $a_1, a_2 \in L^r_T$, $P(a_1) > 0$ and $a_1 \leq a_2$ in $\Omega \times \mathbb{R}$, then $\lambda_1(a_1) \geq \lambda_1(a_2)$ and, if in addition $a_1 < a_2$ in a set of positive measure, then $\lambda_1(a_1) > \lambda_1(a_2)$ (cf. [8], Remark 3.7).

The following remark compiles necessary information of some singular periodic parabolic problems.

 Remark 2.2. Let $0 < \alpha < 1/(N + 2)$, $0 < \beta < 1$, and consider the problem

$$\begin{cases}
Lv = -v^{-\alpha} + \lambda v^\beta & \text{in } \Omega \times \mathbb{R} \\
v = 0 & \text{on } \partial \Omega \times \mathbb{R} \\
v \text{ is } T\text{-periodic}
\end{cases}$$

Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ (9) has a positive strong solution $v \in W^{2,1}_{r,T}$ for some $r > N + 2$. Moreover, $v \in C^{1+\theta/(1+\theta)}_{r,T}$ and $\frac{\partial v}{\partial n} < 0$ on $\partial \Omega \times \mathbb{R}$, where $\nu$ denotes the outward normal unit vector (cf. [7], Theorems 3.1 and 3.3).
3. The theorem

Let
\[ P^0 := \text{interior of the positive cone of } C_T^{1+\theta,(1+\theta)/2} \]

**Theorem 3.1.**

i) Let \( f, g \) satisfying H1 and H2. Then there exists \( \Lambda \geq 0 \) such that (6) has a (strictly) positive solution \( u = u_\lambda \in L^\infty_T \) for all \( \lambda > \Lambda \), and if \( 0 < \lambda < \Lambda \) then there is no positive solution for (6). Moreover, \( u_\lambda \) can be chosen such that
\[
\lim_{\lambda \to \infty} \| u_\lambda \|_{L^\infty_T} = \infty \tag{10}
\]
Assume in addition that (3) and (4) hold for all \( \xi > 0 \).

ii) If \( p_2 = 1 \), then
\[
\Lambda \geq \lambda_1 (a) \tag{11}
\]

iii) If \( p_2 > 1 \), then there exists a positive solution \( u_\Lambda \in L^\infty_T \) for \( \lambda = \Lambda \) and
\[
\Lambda \geq \lambda_1 (a)^{(q_1-p_2)/(q_1-1)} \left( c_g / \| a \|_{L^\infty_T} \right)^{(p_2-1)/(q_1-1)} \tag{12}
\]

iv) If \( 0 < q_1 < p_2 < 1 \), then
\[
\Lambda \geq c_g^{(1-p_2)/(1-q_1)} / \left( \| a \|_{L^\infty_T} \| L^{-1} \|_{L^\infty_T}^{(p_2-q_1)/(1-q_1)} \right) := \tilde{\Lambda} \tag{13}
\]
Also, either if \( f(., \xi), g(., \xi) \in L^r_T \) for some \( r > N+2 \) and all \( \xi \geq 0 \) or if in addition (2) and (5) hold for all \( \xi > 0 \), then \( u_\lambda \in W^{2,1}_{r,T} \cap P^0 \) whenever such \( u_\lambda \) exists.

**Proof.** In order to prove (i) we start constructing a subsolution for (6). Let \( \alpha, \beta, \lambda_0 \) be as in Remark 2.2, and let \( v = v_{\lambda^*} \in W^{2,1}_{r,T} \cap P^0 \) be a solution of (9) corresponding to some \( \lambda^* > \lambda_0 \). Let \( \xi > 0 \) be given by H1 and H2 (clearly we may assume that both \( \xi \) coincide). Choose \( k = \xi / \| v \|_{\infty} \) and \( \varepsilon = \varepsilon (k) > 0 \) such that \( b_\xi q_2 + \lambda^* k^{1-\beta} \xi^{\beta} \leq k^{1+\alpha} \xi^{-\alpha} \) for all \( 0 < \xi \leq \varepsilon \). Define \( v_k := kv \) and \( h(x, t, \xi) := \lambda f(x, t, \xi) - g(x, t, \xi) \),
and pick \( \lambda \geq (\lambda^* k^{1-\beta} \xi^\beta + b \xi^{q_2}) / (c_f \varepsilon^{p_1}) \). Since \( v_k \leq \xi \), from (9), (2) and (5) we have that

\[
Lv_k = -k^{1+\alpha} v_k^{-\alpha} + \lambda^* k^{1-\beta} v_k^\beta \\
\leq -b \xi^{q_2} \chi_{0<v_k<\varepsilon} + \lambda^* k^{1-\beta} \xi^\beta \chi_{\varepsilon<v_k} \\
\leq (\lambda c_f \xi^{p_1} - b v_k^{q_2}) \chi_{0<v_k<\varepsilon} + (\lambda c_f \varepsilon^{p_1} - b \xi^{q_2}) \chi_{\varepsilon<v_k} \\
\leq h(x, t, v_k)
\]

and therefore \( v_k \) is a subsolution of (6).

On the other side, if \( p_2 \geq 1 \) we have \( p_2 < q_1 \) and so recalling (3) and (4) we see that for some constant \( K > 0 \) it holds that

\[
h(x, t, K) \leq \lambda \|a\|_\infty K^{p_2} - c_g K^{q_1} \leq 0 \leq L(K)
\]

Hence, \( K \) is a supersolution of (6). Suppose now \( 0 < p_2 < 1 \), and fix \( 0 < \delta < \lambda_1 (1) \) and \( K > \max \left( \bar{\xi}, (\lambda \|a\|_\infty / \delta)^{1/(1-p_2)} \right) \), where \( \bar{\xi} \) is given by H1. From Remark 2.1 (ii) there exists \( 0 \leq w \in L^\infty \) solution of the Dirichlet periodic problem \( Lw = \delta (w + K) \) in \( \Omega \times \mathbb{R} \). Moreover,

\[
h(x, t, w + K) \leq \lambda a(x, t) (w + K)^{p_2} \leq \lambda \|a\|_\infty (w + K) / K^{1-p_2} \\
\leq \delta (w + K) \leq L(w + K)
\]

and thus \( w + K \) is a supersolution of (6). Hence, in any case we can apply [5], Theorem 1, to obtain a solution \( 0 < u \leq L^\infty \) of (6).

Let \( \Lambda := \inf \{ \lambda > 0 : \text{there exists } 0 < u_\lambda \in L^\infty \text{ solution of (6)} \} < \infty \). Let \( \lambda > \Lambda \) and let \( \lambda > \bar{\lambda} > \Lambda \) such that there exists \( u_{\bar{\lambda}} \in L^\infty \) solution of (6) for \( \lambda = \bar{\lambda} \). Clearly \( u_{\bar{\lambda}} \) is a subsolution of (6). Moreover, as above we can choose a supersolution \( w \geq \|u_{\bar{\lambda}}\|_\infty \) and then again Theorem 1 in [5] gives a solution of (6).

Let us prove (10). Let \( \lambda_j \) be an increasing sequence such that \( \lambda_j \to \infty \), and let \( u_{\lambda_j} \) be the corresponding positive solutions of (6). An inspection of the above part of the proof shows that we can choose \( u_{\lambda_j} \) such that \( \lambda_j \to \lambda_{\lambda_j} \) is increasing and so there exists \( \lim_{j \to \infty} \|u_{\lambda_j}\|_\infty := l \leq \infty \). Suppose \( l < \infty \), and let \( 0 < u_\infty := \lim_{j \to \infty} u_{\lambda_j} \). Dividing (6) by \( \lambda_j \) and going to the limit we find that \( f(x, t, u_\infty) = 0 \), which is not possible. Therefore, part (i) of the theorem is proved.
Assume now that (3) and (4) hold for all \( \xi > 0 \), and let \( \lambda > \Lambda \), \( 0 < u \in L^\infty_T \) be the solution found above. If \( p_2 = 1 \), (3) and (4) imply \( Lu \leq \lambda au \) and hence Remark 2.1 (ii) gives \( \lambda \geq \lambda_1 (a) \) and thus (11) follows (note that since (3) holds for all \( \xi > 0 \), (2) and (3) say that \( a \) is not identically zero, i.e. \( P(a) > 0 \) and so \( \lambda_1 (a) \) exists).

We prove (iii). Let \( \Lambda < \lambda_j \) be a decreasing sequence such that \( \lambda_j \to \Lambda \) and let \( u_{\lambda_j} \) be the positive solutions of (6) for \( \lambda = \lambda_j \). As before, we can choose \( u_{\lambda_j} \) such that \( j \to u_{\lambda_j} \) is decreasing and so \( \| u_{\lambda_j} \|_\infty \leq c \) for some \( c > 0 \) not depending on \( j \). Moreover, since \( \| h(., u_{\lambda_j}) \| \leq \max_{0 \leq \xi \leq \| u_{\lambda_j} \|_\infty} |h(., \xi)| \), the assumptions on \( f \) and \( g \) give that \( \| h(x, t, u_{\lambda_j}) \|_{L^r_T} \leq c \) with \( c \) not depending on \( j \) \((r > (N + 2)/2)\). Thus, from the compactness of \( L^{-1} : L^r_T \to C_T \) (cf. Remark 2.1 (ii)) we get some \( 0 \leq u_\Lambda \in L^\infty_T \) solution of (6) for \( \lambda = \Lambda \). In order to show that \( u_\Lambda \) is not identically zero it suffices to prove that \( \lim_{j \to \infty} \| u_{\lambda_j} \|_\infty \neq 0 \). Now, suppose \( \lim_{j \to \infty} \| u_{\lambda_j} \|_\infty = 0 \), and let \( v_j := u_{\lambda_j}/\| u_{\lambda_j} \|_\infty \). Recalling (3) and (4) we get

\[
0 < v_j \leq L^{-1} \left( \lambda_j a v_j u_{\lambda_j}^{p_2 - 1} - c_{p_2} v_j u_{\lambda_j}^{q_1 - 1} \right)
\]

and thus going to the limit the continuity of \( L^{-1} \) implies \( v_j \to 0 \) which is not possible, and so the first assertion of (iii) is proved.

Let \( k = (c_g/\lambda \| a \|_\infty)^{1/(q_1 - p_2)} \). Since \( 1 < p_2 < q_1 \) we have

\[
L(ku) \leq \lambda ak^{1-p_2} (ku)^{p_2} - c_{p_2} k^{1-q_1} (ku)^{q_1}
\]

\[
\leq \lambda ak^{1-p_2} (ku) \chi_{(0 < ku \leq 1)} + (\lambda ak^{1-p_2} - c_{p_2} k^{1-q_1}) (ku)^{p_2} \chi_{(ku > 1)}
\]

\[
\leq \lambda ak^{1-p_2} (ku) \chi_{(0 < ku \leq 1)}
\]

and so from the last statements in Remark 2.1 (ii) we get \( \lambda \geq \lambda_1 (ak^{1-p_2}) = \lambda_1 (a) k^{p_2 - 1} \) which in turn implies (12).

In order to prove (13) we proceed by contradiction. Suppose there exists a positive solution \( u \) for \( \lambda = \bar{\Lambda} \). Choose \( k := (\bar{\Lambda} \| a \| \| L^{-1} \|)^{-1/(1-p_2)} \). Recalling (3), (4) and that \( 0 < p_2 < 1 \) a computation shows that \( \| ku \|_\infty \leq 1 \). Taking into account this and
that $q_1 < p_2$ we find

$$L(ku) \leq \tilde{\Lambda}k^{1-p_2}(ku)^{p_2} - c_g k^{1-q_1} (ku)^{q_1}$$

$$\leq (\|a\| \|L^{-1}\|)^{-1} a (ku)^{p_2} - \|L^{-1}\|^{-1} (ku)^{q_1}$$

Contradiction.

To end the proof, note that any of the last assumptions imply $h(x,t,u) \in L^r_T$ for some $r > N + 2$. Since the operator $L^{-1} : L^r_T \to W^{2,1}_{r,T}$ is continuous (see e.g. [19], Section 4) it follows that $u \in W^{2,1}_{r,T}$, and from the Sobolev imbedding theorems (e.g. [18], Lemma 3.3, p. 80) and the strong maximum principle (e.g. [2], Theorem 13.5) we get that $u \in P^o$. \qed

References


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