An Orlicz Extension of Some New Sequence Spaces

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SUMMARY. - The aim of this note is to introduce and study a new concept of lacunary σ-convergence with respect to an Orlicz function and examine some properties of the resulting sequence spaces.

1. Introduction and Background

Let \( w \) denote the set of all real and complex sequences \( x = (x_k) \). By \( l_\infty \) and \( c \), we denote the Banach spaces of bounded and convergent sequences \( x = (x_k) \) normed by \( ||x|| = \sup_n |x_n| \), respectively. A linear functional \( L \) on \( l_\infty \) is said to be a Banach limit [1] if it has the following properties:

1. \( L(x) \geq 0 \) if \( x \geq 0 \) (i.e. \( x_n \geq 0 \) for all \( n \)),
2. \( L(e) = 1 \) where \( e = (1,1,\ldots) \),
3. \( L(Dx) = L(x) \), where the shift operator \( D \) is defined by \( D(x_n) = x_{n+1} \).

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Let $B$ be the set of all Banach limits on $l_\infty$. A sequence $x$ is said to be almost convergent to a number $L$ if $L(x) = L$ for all $L \in B$. Lorentz [8] has shown that

$$\hat{c} = \{ x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n \}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + x_{n+2} + \cdots + x_{n+m}}{m + 1}.$$

Shaefer [14] defines the $\sigma$-convergence as follows: Let $\sigma$ be a mapping of the set of positive integers into itself. A continuous linear functional $\phi$ on $l_\infty$ is said to be an invariant mean or a $\sigma$-mean if and only if

1. $\phi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_n \geq 0$ for all $n$;
2. $\phi(e) = 1$ where $e = (1,1,1,\ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in l_\infty$.

Let $V_\sigma$ denote the set of bounded sequences which have unique $\sigma$-mean. If $x \in V_\sigma$ and $\phi(x) = l$, then we write $l = V_\sigma - \lim x$. In case $\sigma$ is the translation mapping $n \to n + 1$, $\sigma$-mean reduces to the unique Banach limit and $V_\sigma$ reduces to $\hat{c}$. We denote by $V_\sigma$ the space of $\sigma$-convergent sequences. It is known that $x \in V_\sigma$ if and only if

$$\frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)}$$

has a limit as $m \to \infty$, uniformly in $n$.

By a lacunary $\theta = (k_r); r = 0, 1, 2, \ldots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by $q_r$.

Recently Das and Mishra [3] have introduced the space $AC_\theta$ of lacunary almost convergent sequences as follows:

$$AC_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{n_r} \sum_{k \in I_r} (x_{k+n} - L) = 0, \text{ for some } L \text{ uniformly in } n \right\}.$$
Note that in the special case where $\theta = 2^r$, we have $AC_\theta = \hat{c}$.

Quite recently, concept of lacunary $\sigma$-convergent was introduced and studied by Savas [13] which is a generalization of the idea of lacunary almost convergence due to Das and Mishra [3]. If $x \in V^\theta_\sigma$ denotes the set of all lacunary $\sigma$-convergent sequences, then Savas [13] defined

$$V^\theta_\sigma = \left\{ x = (x_k) : \lim_{r} t_{r,n}(x) = L, \text{ uniformly in } n \text{ for some } L \right\}$$

where

$$t_{r,n}(x) = \frac{1}{\nu_r} \sum_{k \in I_r} x_{\sigma^k(n)}.$$ 

Note that for $\sigma(n) = n + 1$, the space $V^\theta_\sigma$ is the same as $AC_\theta$. We write $V^\theta_\sigma = V^\theta_{\sigma_0}$ whenever $L = 0$.

Recall in [6] that an Orlicz function $M : [0, \infty) \to [0, \infty)$ is continuous, convex, non decreasing function defined by $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called the modulus function which is defined and characterized by Ruckle [11].

Lindeastrau and Tzafriri [7] used the concept of Orlicz function to construct the following sequence space:

$$L_M = \left\{ x \in w : \sum_{k} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}.$$ 

The space $L_M$ with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space.

An Orlicz function $M$ is said to satisfy $\Delta_2$-condition of all values of $u$, if there exists a constant $K > 0$ such that

$$M(2u) \leq KM(u), \text{ for all } u \geq 0.$$ 

The $\Delta_2$-condition is equivalent to

$$M(lu) \leq KlM(u)$$
for all values of $u$ and $l \geq 1$.

In the present paper, we introduce and study some properties of the following three sequence spaces that are defined using the Orlicz function.

Let $M$ be an Orlicz function then

$$V_{\sigma_0}^\theta(M) = \left\{ x = (x_k) : \lim_{r \to \infty} M \left( \frac{|t_{r,n}(x)|}{\rho} \right) = 0 \text{ uniformly in } n \text{ for some } \rho > 0 \right\},$$

$$V_{\sigma}^\theta(M) = \left\{ x = (x_k) : \lim_{r \to \infty} M \left( \frac{|t_{r,n}(x)-l|}{\rho} \right) = 0 \text{ uniformly in } n \text{ for some } \rho > 0 \right\},$$

and

$$V_{\sigma_\infty}^\theta(M) = \left\{ x = (x_k) : \sup_{r,n} M \left( \frac{|t_{r,n}(x)|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

If $t_{r,n}(x)$ is replaced by $x$, then we have the following sequence spaces:

$$c_0(M) = \left\{ x = (x_k) : \lim_{k} M \left( \frac{|x_k|}{\rho} \right) = 0 \text{ for some } \rho > 0 \right\},$$

$$c(M) = \left\{ x = (x_k) : \lim_{k} M \left( \frac{|x_k-l|}{\rho} \right) = 0 \text{ for some } l > 0 \text{ and } \rho > 0 \right\},$$

and

$$l_\infty(M) = \left\{ x = (x_k) : \sup_{k} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

It is easy to see that $V_{\sigma_0}^\theta$, $V_{\sigma}^\theta$, and $V_{\sigma_\infty}^\theta$ are linear spaces over the complex field. With consider of the above sequence spaces we now present the following theorem.
2. Main Result

**Theorem 2.1.** The linear spaces $V_0^\theta(M)$, $V_\sigma^\theta(M)$ and $V_\sigma^\infty(M)$ are Banach spaces with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sup_{r,n} M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \leq 1 \right\}.$$

*Proof.* It is clear that the spaces are normed spaces with the above norm. We shall only establish that $V_\sigma^\infty(M)$ is a Banach space. The others can be established in a manner similar to $V_\sigma^\infty(M)$. Let $(x^i_k)_k$ be a Cauchy sequence in $V_\sigma^\infty(M)$. Let $s, x_0 > \epsilon$ be fixed such that $M\left(\frac{s x_0}{2}\right) \geq 1$. Then for each $\epsilon x_0 s > 0$ there exists a positive integer $N$ such that for all $i, j \geq N$

$$||x^i - x^j|| \leq \frac{\epsilon}{x_0 s}.$$

The definition of norm above implies that for all $i, j \geq N$

$$\sup_{r,n} M \left( \frac{|t_{r,n}(x^i - x^j)|}{||x^i - x^j||} \right) \leq 1,$$

since $||x^i - x^j||$ is positive so we can substitute $\rho$ for $||x^i - x^j||$. Thus

$$M\left( \frac{|t_{r,n}(x^i - x^j)|}{||x^i - x^j||} \right) \leq 1 \text{ for all } r, n \geq 0 \text{ and for all } i, j \geq N.$$

Since $M\left(\frac{s x_0}{2}\right) \geq 1$ we have

$$M\left( \frac{|t_{r,n}(x^i - x^j)|}{||x^i - x^j||} \right) \leq M\left( \frac{s x_0}{2}\right) \text{ for all } r, n.$$  

This implies that

$$|t_{r,n}(x^i - x^j)| \leq \frac{x_0 s}{2} \frac{\epsilon}{x_0 s} = \frac{\epsilon}{2} \text{ for all } r, n.$$

In particular $|t_{1,n}(x^i - x^j)| = |x^i_{\sigma(n)} - x^j_{\sigma(n)}| \to 0$ as $i, j \to \infty$ for each fixed $n$. Hence $(x^i)$ is a Cauchy sequence in the complex plane.
Therefore for each $\epsilon (0 < \epsilon < 1)$, there exists a positive integer $N$ such that $|t_{r,n}(x^i - x^j)| < \epsilon$, for all $i, j \geq N$ and for all $r, n$. Using the continuity of $M$ and taking the limit as $j \to \infty$ we have that

$$\sup_{r \geq N} M \left( \frac{|t_{r,n}(x_k^i - x_k^j)|}{\rho} \right) \leq 1.$$ 

Taking the infimum over such $\rho$’s we get the following for all $n$

$$\inf \left\{ \rho > 0 : \sup_{r \geq N} M \left( \frac{|t_{r,n}(x_k^i - x_k^j)|}{\rho} \right) \leq 1 \right\} < \epsilon,$$

for all $i \geq N$. Since $x^i \in V^\theta_{\infty}(M)$ and $M$ is an Orlicz function, it follows that $x \in V^\theta_{\infty}(M)$. This completes the proof.

\[\square\]

**Theorem 2.2.** Let $M$ be an Orlicz function that satisfies the $\Delta_2$-condition then

- $c_0(M) \subset V^\theta_0(M)$,
- $c(M) \subset V^\theta(M),$

and

- $l_\infty(M) \subset V^\theta_{\sigma_\infty}(M).$

The verification of this theorem is routine and thus omitted.

It is quite natural to expect that the spaces $V^\theta_{\sigma_0}(M)$, $V^\theta_{\sigma}(M)$ and $V^\theta_{\sigma_\infty}(M)$ can be extended to $V^\theta_{\sigma_0}(M, p)$, $V^\theta_{\sigma}(M, p)$ and $V^\theta_{\sigma_\infty}(M, p)$ in manner similar to the extension of $c$, $c_0$, and $l_\infty$ to $c(p)$, $c_0(p)$, and $l_\infty(p)$ respectively (see, Simons [15] and Moddox [9]).

In this section of this paper, we study the spaces $V^\theta_{\sigma_0}(M, p)$, $V^\theta_{\sigma}(M, p)$ and $V^\theta_{\sigma_\infty}(M, p)$ which are defined below:

Let $M$ be Orlicz function, $p = (p_r)$ be any sequence of positive real numbers.

- $V^\theta_{\sigma_0}(M, p) = \left\{ x = (x_k) : \lim_{r \to \infty} M \left( \frac{|t_{r,n}(x)|}{\rho} \right)^{p_r} = 0 \right\}$, uniformly in $n$ for some $\rho > 0$,

- $V^\theta_{\sigma}(M, p) = \left\{ x = (x_k) : \lim_{r \to \infty} M \left( \frac{|t_{r,n}(x) - L|}{\rho} \right)^{p_r} = 0 \right\}$, uniformly in $n$ for some $L, \rho > 0$. 

and

\[ V^\theta_{\sigma_{\infty}}(M, p) = \left\{ x = (x_k) : \sup_{r, n} \left( M \left( \frac{|x_{r,n}(x)|}{\rho} \right) \right)^{p_r} < \infty \right\} . \]

If \( p = p_r \) is a constant sequence, i.e., \( p_r = p \) for all \( r \), then we write

\[ V^\theta_{\sigma_0}(M, p) = V^\theta_{\sigma_0}(M), \quad V^\theta_{\sigma}(M, p) = V^\theta_{\sigma}(M) \quad \text{and} \quad V^\theta_{\sigma_{\infty}}(M, p) = V^\theta_{\sigma_{\infty}}(M). \]

If we let \( \sigma(n) = n + 1 \), the spaces \( V^\theta_{\sigma_0}(M, p) \), \( V^\theta_{\sigma}(M, p) \) and \( V^\theta_{\sigma_{\infty}}(M, p) \) reduce to the following sequence spaces:

\[ \hat{V}_0(M, p) = \left\{ x = (x_k) : \lim_{r \to \infty} \left( M \left( \frac{|d_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0 \right\}, \]

\[ \hat{V}(M, p) = \left\{ x = (x_k) : \lim_{r \to \infty} \left( M \left( \frac{|d_{r,n}(x) - Le|}{\rho} \right) \right)^{p_r} = 0 \right\}, \]

and

\[ \hat{V}_{\infty}(M, p) = \left\{ x = (x_k) : \sup_{r, n} \left( M \left( \frac{|d_{r,n}(x)|}{\rho} \right) \right)^{p_r} < \infty \right\} \]

where

\[ d_{r,n}(x) = \frac{1}{h_r} \sum_{k \in I_r} x_{k+n}, \]

We now consider the following theorem:

**Theorem 2.3.** The linear spaces \( V^\theta_{\sigma_0}(M, p) \), \( V^\theta_{\sigma}(M, p) \) and \( V^\theta_{\sigma_{\infty}}(M, p) \) are paranormed spaces with

\[ g(x) = \inf \left\{ \frac{\rho}{\rho^H} : \left\{ \sup_{r} \left( M \left( \frac{|x_{r,n}(x)|}{\rho} \right) \right)^{p_r} \leq 1, n = 1, 2 \right\} \right\}, \]

where \( H = \max\{1, \sup_{r} p_r\}. \)

**Proof.** This can be proved by using the techniques similar to those used in Theorem 1 and Theorem 2 in [10]. \( \square \)
Theorem 2.4. Let $p_k$ and $q_k$ be two sequences of real numbers such that $0 < p_k \leq q_k < \infty$ for each $k$. Then

$$V^\theta_{\sigma_0}(M, p) \subseteq V^\theta_{\sigma_0}(M, q).$$

Proof. Let $x \in V^\theta_{\sigma_0}(M, p)$. Then there exists some $\rho > 0$ such that

$$\lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0,$$

uniformly in $n$. This implies that

$$M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \leq 1$$

for sufficiently large $n$. Since $M$ is non-decreasing, we obtain the following

$$\lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right)^{q_r} \leq \lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$$

uniformly in $n$ that is $x \in V^\theta_{\sigma_0}(M, q)$. This completes the proof.

Theorem 2.5. (1) Let $0 < \inf p_k \leq p_k \leq 1$. Then

$$V^\theta_{\sigma_0}(M, p) \subseteq V^\theta_{\sigma_0}(M).$$

(2) Let $0 < p_k \leq \sup_k p_k \leq \infty$. Then

$$V^\theta_{\sigma_0}(M) \subseteq V^\theta_{\sigma_0}(M, p).$$

Proof. (1) Let $x \in V^\theta_{\sigma_0}(M, p)$, that is $\lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$, uniformly in $n$. Since $0 < \inf p_k \leq p_k \leq 1$ we have

$$\lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right) \leq \lim_{r \to \infty} \left( M \left( \frac{|t_{r,n}(x)|}{\rho} \right) \right)^{p_r} = 0$$

uniformly in $n$ and hence $x \in V^\theta_{\sigma_0}(M)$.

(2) Let $p_k \geq 1$ for each $k$ and $\sup_k p_k < \infty$. Let $x \in V^\theta_{\sigma_0}(M)$, then for each $\epsilon (0 < \epsilon < 1)$ there exists a positive integer $N$ such that
\[ M\left(\frac{|f_{r,n}(x)|}{\rho}\right) \leq \epsilon \text{ for all } r \geq N \text{ and for all } n. \text{ Since } 0 < p_k \leq \sup p_k < \infty, \text{ we have for all } n. \]

\[ \lim_{r \to \infty} \left( M\left(\frac{|f_{r,n}(x)|}{\rho}\right)\right)^{p_r} \leq \lim_{r \to \infty} \left( M\left(\frac{|f_{r,n}(x)|}{\rho}\right)\right) \leq \epsilon < 1. \]

Therefore \( x \in V_{o_0}^{\theta}(M, p) \). This completes the proof. \( \square \)

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**References**


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