Petri Maps for Very Ample Line Bundles

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Summary. - Here we construct several smooth curves $C$ and several very ample and special line bundles $L$ on $C$ such that the multiplication map $H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \to H^0(C, \omega_C)$ (i.e. the Petri map) is surjective.

1. Introduction

Let $C$ be a smooth and connected projective curve and $L \in \text{Pic}(C)$ such that $h^0(C, L) \geq 2$ and $h^1(C, L) \geq 2$. A key role in the classical theory of special divisors on $C$ is played by the Petri map, i.e. by the multiplication map $\mu_L : H^0(C, L) \otimes H^0(C, \omega_C \otimes L^*) \to H^0(C, \omega_C)$ ([2] or [3], Ch. II). The best situation occurs when $\mu_L$ is injective, but this is a rather restrictive assumption for the pair $(C, L)$, because it implies $h^0(C, L) \cdot h^0(C, \omega_C \otimes L^*) \leq h^0(C, \omega_C)$, i.e. $h^0(X, L) \cdot (h^0(X, L) + p_a(C) - 1 - \deg(L)) \leq p_a(C)$. This condition is not satisfied when $L$ induces an embedding of $C$ in $\mathbb{P}^n$ as a curve of high genus with respect to its degree and in particular it is never satisfied for many pairs $(C, L)$ which arise quite often in the Castelnuovo theory of curves in $\mathbb{P}^n$ ([3], Ch. III). The next best thing would be the surjectivity of $\mu_L$, because it would give the key...

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information \( \dim(\ker(\mu_L)) \). Notice that the surjectivity of \( \mu_L \) is an open condition for a flat family of pairs \((C, L)\) for which \( h^0(C, L) \) is constant. Following [1] we use the following classical definition.

**Definition 1.1.** Fix an integer \( k > 0 \) and curve \( C \subset \mathbb{P}^n \) (i.e. a locally closed equidimensional one-dimensional subscheme of \( \mathbb{P}^n \)). \( C \) is said to be \( k \)-normal if the restriction map \( H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \rightarrow H^0(C, \mathcal{O}_C(k)) \) is surjective. \( C \) is said to be strongly \( k \)-normal if it is \( t \)-normal for every integer \( t \) such that \( 1 \leq t \leq k \).

The definition of \( \mu_L \) makes sense even if \( C \) is singular and we will get for free the surjectivity of the Petri map for pairs \((C, L)\) with \( C \) Gorenstein (but perhaps with multiple components) and \( C \) any member of a suitable linear system on a certain degree \( 2n - 2 \) linearly normal \( K_3 \) surface \( S \subset \mathbb{P}^n \).

**Theorem 1.2.** Fix integers \( d, g, n, x \) such that \( n \geq 3 \) and \( 0 \leq d - n < g < d^2/(4n - 4) - (n - 1)/4 \). Set \( r := \lfloor (d - \sqrt{d^2 - (4n - 4)g/(2n - 2)}) \rfloor \), \( d_0 := d - (2n - 2)r \) and \( g_0 := (n - 1)r^2 - dr + g \); notice that \( r \geq 1 \) and \( 0 \leq g_0 \leq d - n \). Assume \( 1 \leq 3x \leq r \). Then there exists a smooth linearly normal \( K_3 \) surface \( S \subset \mathbb{P}^n \) with degree \( 2n - 2 \) and the following properties. Set \( H := \mathcal{O}_S(1) \). There is a smooth and connected curve \( C_0 \subset S \) such that \( \deg(C_0) = d_0 \), \( p_a(C_0) = g_0 \), \( h^0(S, \mathcal{O}_S(H - C_0)) = h^0(S, \mathcal{O}_S(C_0 - H)) = 0 \). \( \text{Pic}(S) \) is freely generated by \( H \) and \( C_0 \) and a general element of \( |C_0 + rH| \) is smooth. Fix any \( C \in |C_0 + rH| \). \( C \) is strictly \( r \)-normal. \( C \) is \((r + 1)\)-normal if and only if \( C_0 \) is linearly normal, i.e. if and only if \( n + g_0 = d_0 + h^1(C_0, \mathcal{O}_{C_0}(1)) \). Set \( L := \mathcal{O}_C(x) \). Then \( \mu_L \) is surjective.

We work over an algebraically closed field \( K \) such that \( \text{char}(K) = 0 \). Our proofs use an existence theorem for curves in certain \( K3 \) surfaces proved only in characteristic zero ([4], [5]).

**2. Proof of Theorem 1.2**

All the results, except the surjectivity of \( \mu_L \) were proved in [1]. More precisely, the results were proved in [1], Prop. 2.2 and proof of Th. 1.4, but only stated when \( C \) is smooth; however, no difference is for the case in which \( C \) is an arbitrary element of \(|C_0 + rH|\).
key tool for the quoted results in [1] was [4], Th. 4.6. Let \( \eta_t : H^0(S, \mathcal{O}_S(tH)) \otimes H^0(S, \mathcal{O}_S(C_0 + (r-t)H)) \rightarrow H^0(S, \mathcal{O}_S(C_0 + rH)) \) denote the multiplication map. Since \( C \) is strictly \( r \)-normal, the restriction map \( \rho_t : H^0(S, \mathcal{O}_S(t)) \rightarrow H^0(C, \mathcal{O}_C(t)) \) is surjective for every integer \( t \) such that \( 1 \leq t \leq r \). Since \( h^0(S, \mathcal{O}_S((t-r)H-C_0)) = 0 \) if \( t \leq r \), we get the bijectivity of \( \rho_t \) for \( 1 \leq t \leq r \). Since \( \omega_S \cong \mathcal{O}_S \), we have \( \omega_C \cong \mathcal{O}_C(C) \) (adjunction formula). Hence for any integer \( z \) we have the exact sequence

\[
0 \rightarrow \mathcal{O}_S(C_0 + (z-r)H) \rightarrow \mathcal{O}_S(C_0 + zH) \rightarrow \omega_C((z-r)) \rightarrow 0 \tag{1}
\]

Since \( S \) is arithmetically Cohen-Macaulay (or use Kodaira vanishing and duality), we have \( h^1(S, \mathcal{O}_S(yH)) = 0 \) for every integer \( y \). Hence the restriction map \( \tau_t : H^0(S, \mathcal{O}_S(C_0 + zH)) \rightarrow H^0(C, \omega_C((z-r))) \) is bijective for every integer \( z < r \) and surjective with one-dimensional kernel for \( z = r \). Hence to show that \( \mu_L \) is surjective, it is sufficient to show that \( \eta_t \) is surjective. We fix a general linear subspace \( W \) of \( H^0(S, \mathcal{O}_S(xH)) \) such that \( \dim(W) = 3 \). Since \( \mathcal{O}_S(H) \) is very ample, \( W \) spans \( \mathcal{O}_S(xH) \). Hence for all integers \( y \) the vector space \( W \) induces the following Koszul complex type exact sequence

\[
0 \rightarrow \mathcal{O}_S(C_0 + (r-3x+y)H) \rightarrow \mathcal{O}_S(C_0 + (r-2x+y)H) \rightarrow \mathcal{O}_S(C_0 + (r+y)H) \rightarrow 0 \tag{2}
\]

Set \( A_y : = \ker(\alpha_y) = \ker(\beta_y) \). Since \( W \subseteq H^0(S, \mathcal{O}_S(xH)) \) and \( \rho_x \) and \( \tau_r \) are surjective, to prove the surjectivity of \( \mu_L \) it is sufficient to prove \( h^1(S, A_0) = 0 \). Since \( 3x \leq r \), then \( h^2(S, \mathcal{O}_S(C_0 + (r-3x)H)) = h^0(S, \mathcal{O}_S(-C_0 + (3x-r)H)) = 0 \). Hence we would get \( h^1(S, A_0) = 0 \) if \( h^1(S, \mathcal{O}_S(C_0 + (r-2x)H)) = 0 \). By [1], proof of part (iiiib) of Proposition 2.2, the linear system \( |C_0| \) has no base point. Since \( C_0 \) is irreducible and \( H \) ample, we get \( h^1(S, \mathcal{O}_S(C_0 + (r-2x)H)) = 0 \) by Kodaira vanishing if either \( 2x < r \) or \( 2x = r \) and \( C_0 \cdot C_0 > 0 \), concluding the proof.

References

[2] Ph. Griffiths, E. Arbarello, M. Cornalba and J. Harris,  


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