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On Self-Injectivity and p-Injectivity

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Summary. - *A generalization of injectivity is studied and several properties are developed. Von Neumann regular rings are characterized. Sufficient conditions are given for a ring to admit a strongly regular classical left quotient ring. A nice characterization of strongly regular rings is given. Special direct summands of left self-injective regular and left continuous regular rings are considered.*

1. Introduction

Since several years, injectivity, p-injectivity and their generalizations have drawn the attention of numerous authors (cfr. for example [2, 4, 5, 8, 10, 20, 22, 24, 41] and [11]-[15]). Here we consider modules satisfying a condition ★ (see (2.1)). Such modules contain their complement submodules as direct summands. Semi-prime rings satisfying ★ are also studied. Self.injective regular rings are characterized using condition ★. Strongly regular rings are characterized in terms of certain annihilators. In the left continuous regular ring, the sum of all reduced ideals is a direct summand.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z, Y will stand respectively for the Jacobson radical, the left singular ideal and the right singular ideal of A. An ideal of A will always mean a two-sided ideal of A. Of course, J,

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Z, Y are all ideals of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. For any left A-module M, $Z(M) = \{y \in M : l(y)$ is an essential left ideal of $A\}$ is the left singular submodule of M. ${}_AM$ is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = (0)$). Thus $Z = Z({}_AA)$ and $Y = Z(A_A)$. A is called semi-primitive or semi-simple (resp. left non-singular or right non-singular) if $J = (0)$ (resp. $Z = 0$ or $Y = (0)$).

Following C. Faith [6], we will write that A is VNR if it is a von Neumann regular ring. It is well-known that A is VNR if and only if every left (right) A-module is flat (M. Harada (1956); M. Auslander (1957)). Also, A is VNR if and only if every left (right) A-module is p-injective (cfr. [2, 4, 10, 11, 14, 24, 25, 28]). Note that the Harada-Auslander characterization may be weakened as follows: A is VNR if and only if every cyclic singular left A-module is flat (see [29], Theorem 5) (cf. G. O. Michler's comment in MR 80i-16021). Flatness and p-injectivity are distinct concepts.

2. On Self-Injectivity and p-Injectivity

Recall that a left A-module, M, is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to one of A into M ([6, p.122], [18, p.577], [23, p.340], [28]).

A is called a left p-injective ring if ${}_AA$ is p-injective (p-injectivity is similarly defined on the right side). Indeed, the study of flat modules over non-VNR rings has motivated various authors to consider p-injective modules over rings which are not necessarily VNR (cfr. the bibliography). K. R. Goodearl's classic [7] has motivated extensive research in the area of VNR rings and associated rings. According to a theorem of P. Menal - P. Vamos [6, p.108], any arbitrary ring may be embedded in a FP-injective ring (and hence in a p-injective ring). This has given an impetus to the study of p-injective rings (cfr. [6, Theorem 6.4], [9, 11, 16, 17]). In 1974, we introduced the concept of p-injective modules [28] to study VNR rings, self-injective rings and associated rings. This is later generalized to YJ-injectivity ([18, p.578], [24, 34, 41]), also called GP-injectivity by other authors [4, 12, 15]. Recall that a left A-module M is YJ-injective if, for any $a \in A, a \neq 0$, there exists a positive integer n (depending on a) such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to one of A into M [34]. A is called a left YJ-injective ring if ${}_AA$ is YJ-injective. YJ-injectivity is similarly defined on the right side. Note that A is left YJ-injective if and only if, for any $a \in A, a \neq 0$, there exists a positive integer n such that a^nA is a non-zero right annihilator [34, Lemma 3] (cfr. also [15, 24, 41]).

We here consider the following generalization of injectivity.

Definition 2.1. *We say that a left A-module M satisfies ★ if, for any left submodule N containing a non-zero complement left submodule of M, every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$.*

We will write that A satisfies ★ if ${}_AA$ satisfies ★. It is clear that simple left A-modules and uniform left A-modules satisfy ★.

Proposition 2.2. *Let M be a left A-module satisfying ★. Then any complement left submodule of M is a direct summand of M.*

Proof. Let C be a non-zero complement left submodule of M; I a relative complement of C in M such that $E = C \oplus I$ is an essential submodule of ${}_AM$. If $p : E \longrightarrow C$ is the natural projection, the set of submodules U of M containing E such that p extends to a left A-homomorphism of U into C has, by Zorn's Lemma, a maximal member L. Let $g : L \longrightarrow C$ be the extension of p to L. If $j : C \longrightarrow M$ is the inclusion map, then $j \circ g : L \longrightarrow M$ and by hypothesis, $j \circ g$ extends to an endomorphism f of ${}_AM$. Suppose that $f(M) \nsubseteq C$. Since C is a relative complement of I in M, then $(f(M)+C) \cap I \neq (0)$. If $d \in (f(M) + C) \cap I$, $d \neq 0$, $d = f(m) + c$, $m \in M$, $c \in C$, and $F = \{v \in M : f(v) \in E\}$ is therefore a submodule of M which strictly contains L (because $m \in F$, since $f(m) = d - c \in E$, and $m \notin L$). Now define $t : F \longrightarrow E$ by $t(v) = f(v)$ for all $v \in F$. Then $p \circ t : F \longrightarrow C$ extends p to F, which contradicts the maximality of L. Therefore $f(M) \subseteq C$ which yields $f(M) = C$. Now $C \cap \ker f = (0)$ and if $b \in M$, $b = f(b) + (b - f(b)) \in C + \ker f$ which leads to $M = C \oplus \ker f$. □

If A is a left self-injective regular ring, then for any essentially finitely generated left A-module M, $M = Z(M) \oplus N$, where N is a non-singular injective left A-module [39, Corollary 10]. This motivates the study of non-singular injective modules.

PROPOSITION 2.3. *Let A be a semi-prime ring satisfying $\bigstar$. If M, N are non-singular injective left A-modules, then there exists a central idempotent $e \in A$ such that ${}_AeM$ is isomorphic to a submodule of ${}_AeN$ and ${}_A(1-e)N$ is isomorphic to a submodule of ${}_A(1-e)M$.*

Proof. Let

$$E = \left\{ (Q,P) : \begin{array}{l} \text{are respectively left submodules of } M \text{ and } N, \\ {}_AQ \text{ is isomorphic to } {}_AP \end{array} \right\}.$$

The set S of all the families $\{(Q_i, P_i)\}$ of elements of E such that $\{Q_i\}$ and $\{P_i\}$ are independent families of submodules of M and N, respectively, has a maximal member $\{(Q_i, P_i)\}_{i \in I_0}$ (cfr. the proof of [30, Lemma 2]). If M_o, N_o are the injective hulls of $\bigoplus_{i \in I_0} Q_i$, $\bigoplus_{i \in I_0} P_i$ respectively in M, N, then $M = M_0 \oplus M_1$ and $N = N_0 \oplus N_1$. Since $Q_i \approx P_i$, with $i \in I_0$, then $M_0 \approx N_0$. Write $T = \{a \in A : aM_1 = 0\}$. Then T is an ideal of A which is a complement left ideal of A (in as much as M_1 is non-singular and for any element c in an essential extension of ${}_AT$ in ${}_AA$, $Lc \subseteq T$ for some essential left ideal L of A). By (2.2), ${}_AT$ is a direct summand of ${}_AA$. If $T = Ae$, $e = e^2 \in A$, then e is central in A (because A is semi-prime). It follows that $eM = eMo \approx eN_0 \subseteq eN$. Now suppose that $(1-e)N_1 \neq 0$. If $b \in (1-e)N_1$, $b \neq 0$ since ${}_AAb$ is non-singular, then $l(b)$ is again a direct summand of ${}_AA$ by (2.2) which yields ${}_AAb \approx {}_AAu$, $u = u^2 \in A$, and $eu = 0$. Since $u \neq 0$, then $u \in T$ (otherwise, $u = ue = eu = 0$). Therefore $uy \neq 0$ for some $y \in M_1$ and ${}_AAuy$ is again projective (being a cyclic non-singular left A-module). Looking at the epimorphism $Au \longrightarrow Auy$, we conclude that Auy is isomorphic to a submodule of $Ab \subseteq N_1$, which contradicts the maximality of $\{(Q_i, P_i)\}_{i \in I_0}$ in S. Therefore $(1-e)N_1 = 0$ and hence

$$(1-e)N = (1-e)N_0 \approx (1-e)M_0 \subseteq (1-e)M.$$

□

COROLLARY 2.4. *If A is a prime ring satisfying $\bigstar$, then for non-singular injective left A-modules, M, N, either M is isomorphic to a submodule of N or N is isomorphic to a submodule of M.*

Well-known examples of self-injective rings are quasi-Frobenius rings, pseudo-Frobenius rings and the maximal quotient rings of non-singular rings.

Recall that A is left continuous (in the sense of Y. Utumi) if every left ideal of A which is isomorphic to a complement left ideal is a direct summand of ${}_AA$. In [32], left continuous rings are generalized as follows: A is a left GQ-injective ring if, for any left ideal C of A which is isomorphic to a complement left ideal of A, every left A-homomorphism of C into A extends to an endomorphism of ${}_AA$.

THEOREM 2.5. *The following conditions are equivalent:*

1. *A is a left self-injective regular ring;*
2. *A is a left non-singular left p-injective ring satisfying ★;*
3. *A is a left non-singular left GQ-injective ring satisfying ★.*

Proof. Evidently (1) implies (2) and (3).

Now assume (2). Since A is left p-injective, then every left ideal which is isomorphic to a direct summand of ${}_AA$ is itself a direct summand of ${}_AA$. Since A satisfies ★, then every complement left ideal of A is a direct summand of ${}_AA$ by (2.2). A is therefore a left non-singular left continuous ring which is then VNR by a well-known result of Y. Utumi. Then any non-zero left ideal I of A contains a non-zero idempotent. Consequently, every left A-homomorphism of I into A extends to an endomorphism of ${}_AA$. A is therefore left self-injective and (2) implies (1).

Assume (3). Since A is left GQ-injective, then $J = Z$ and A/J is VNR [32, Proposition 1]. Since A is left non-singular, then A is VNR. Then for any non-zero left ideal I of A (which contains a non-zero idempotent), every left A-homomorphism of I into A extends to an endomorphism of ${}_AA$ and hence (3) implies (1). □

As before, write A is ELT (resp.MELT) if each essential (resp. maximal essential, if it exists) left ideal of A is an ideal of A.

COROLLARY 2.6. *If A is a semi-prime ELT left GQ-injective ring satisfying ★, then A is a left and right self-injective regular, left and right V-ring of bounded index.*

Proof. If we suppose that $Z \neq (0)$, then exists $z \in Z, z \neq 0$ such that $z^2 = 0$ [32, Lemma 7]. Since $I = l(z)$ is an ideal of A, then $(Az)^2 = AzAz \subseteq IAz \subseteq Iz = (0)$ which contradicts the semi-primeness of A. Therefore $Z = (0)$ and A is left self-injective regular by (2.5)(3). The corollary follows from [31, Lemma 1.1]. □

COROLLARY 2.7. *A is simple Artinian if and only if A is a prime ELT left GQ-injective ring satisfying* $\bigstar$.

Rings whose simple modules are either injective or projective and various generalizations are studied since several years (cfr. for example, [2, 5, 12, 13, 15, 20]). Such rings need not be semi-prime as shown by the following example.

EXAMPLE 2.8. *If A denotes the 2×2 upper triangular matrix ring over a field, then A is an Artinian, hereditary ring whose simple one-sided modules are either injective or projective but is not a semi-prime ring (indeed, the Jacobson radical J of A is non-zero with $J^2 = (0)$). Also, all singular one-sided modules are injective while all non-singular one-sided modules are projective.*

For a left A-module M, if N is a submodule of M,

$$Cl_M(N) = \{y \in M : Ly \subseteq N \text{ for some essential left ideal } L \subseteq A\}$$

is the closure of N in M. $Cl_M(0) = Z(M)$ is the singular submodule of M.

PROPOSITION 2.9. *Let A be a semi-prime ring whose simple right modules are either YJ-injective or projective. If M is a homomorphic image of a left A-module satisfying $\bigstar$, then $Z(M)$ is a direct summand of ${}_AM$.*

Proof. Let Q be a left A-module satisfying $\bigstar$, $g : Q \longrightarrow M$ an epimorphism of left A-modules. By (2.2), every complement left submodule of Q is a direct summand of Q. Since A is a semi-prime ring whose simple right modules are either YJ-injective or projective, then $Z = O$ [38, Proposition 1]. Since g is an epimorphism, $g^{-1}(Z(M)) = Cl_Q(\ker g)$, then by [27, Theorem 4], $g^{-1}(Z(M))$ is a complement left submodule of Q. Therefore $Q = g^{-1}(Z(M)) \oplus N$. It follows that $M = g(Q) = Z(M) \oplus g(N)$, where $g(N) \approx N$. □

A well-known theorem of I. Kaplansky asserts that a commutative ring is VNR if and only if it is a V-ring. In the non-commutative case, the work of O. E. Villamayor has motivated many papers on generalizations of V-rings and VNR rings (cfr. the bibliography of [18]).

Applying [38, Propositions 2 and 9], we get

Remark 2.10. *If A is a MELT ring whose simple left modules are YJ-injective, then $J = Z = Y = (0)$.*

QUESTION 1. Are the rings in (2.10) fully left idempotent?

Remark 2.11. *If A contains a non-singular maximal left ideal, then A is left non-singular.*

Proof. Let M be a maximal ideal of A such that $Z(M) = (0)$. If ${}_AM$ is essential in ${}_AA$, then $M \cap Z = Z(M) = (0)$ implies that $Z = (0)$. If ${}_AM$ is a direct summand of ${}_AA$, suppose that $Z \neq O$. Since $M \cap Z = (0)$, then $A = M \oplus Z$. Now Z cannot contain a non-zero idempotent which implies that $Z = (0)$, a contradiction! Therefore $Z = (0)$ in any case. □

Note that the analogous result holds for reduced rings. Indeed, if A contains a reduced maximal left ideal, then A is reduced [37, Lemma 2].

Lemma 2.12. *Let A be a ring whose simple left modules are either p-injective or projective. Then the centre of A is VNR.*

Proof. Let C denote the centre of A. For any $c \in C$, set $L = Ac + l(c)$. Let K be a complement left ideal of Asuch that $L \oplus K$ is an essential left ideal of A. Then $Kc = cK \subseteq L \cap K = (0)$ which implies that $K \subseteq l(c)$, whence $K = K \cap l(c) \subseteq K \cap L = (0)$. Therefore L is an essential left ideal of A. Now suppose that $L \neq A$. Let M be a maximal left ideal of A containing L. Then ${}_AA/M$ must be p-injective. Define $g : Ac \longrightarrow A/M$ by $g(ac) = a + M$ for all $a \in A$. Since ${}_AA/M$ is p-injective, there exists $y \in A$ such that $1 + M = g(c) = cy + M$. Now $cy = yc \in M$ implies that $1 \in M$, which contradicts $M \neq A$. We have shown that $A = L = Ac + l(c)$. Then $c = bc^2$, $b \in A$ and therefore $c = cbc$. Now set $d = c^2b^3$. Then

$$cdc = (cbc)bcbc = (cbc)bc = c$$

and

$$c^2b = bc^2 = c.$$

For every $u \in A$,

$$bc^2u = cu = uc = ubc^2 = c^2ub$$

and hence

$$b^3c^2u = c^2ub^3.$$

Now

$$du = c^2b^3u = b^3c^2u = c^2ub^3 = uc^2b^3 = ud$$

which proves that $d \in C$. C is therefore a VNR ring. □

THEOREM 2.13. *The following conditions are equivalent for a ring A with centre C :*

1. *A is VNR;*
2. *every simple left A-module is either p-injective or projective and for each maximal ideal N of C, A/AN is VNR.*

Proof. (1) implies (2) evidently. (2) implies (1) by [1, Theorem 3] and (2.12). □

The next result is motivated by recurrent questions of V. A. Hiremath in private communications concerning classical quotient rings (which are not necessarily semi-simple, Artinian). See, for very nice results of Hiremath, consult the bibliography of R. Wisbauer [23].

PROPOSITION 2.14. *Let A be an ELT left p.p. ring whose complement left ideals are ideals of A. Then A admits a classical left quotient ring Q which is strongly regular.*

Proof. Given $a, c \in A$, c being a non-zero-divisor, let K be a complement left ideal of A such that $L = Ac \oplus K$ is an essential left ideal of A. Since K is an ideal of A, then $Kc \subseteq K \cap Ac = (0)$ which implies that $K = (0)$ (c being a non-zero-divisor). Then $L = Ac$ is an essential left ideal which, by hypothesis, is an ideal of A. Now $ca \in L$ yields $ca = dc$ for some $d \in A$. We have just shown that A satisfies the left Ore Condition which is equivalent to A having a classical left quotient ring Q. Since $Z = (0)$ and every complement left ideal of A is an ideal of A, then A is a reduced ring [37, Lemma 3]. By [35, Theorem 2], every element a of A is of the form $a = ce$, where c is a non-zero-divisor and e is a central idempotent. Now given $q \in Q$, $q = b^{-1}a$ with $b, a \in A$, b being a non-zero-divisor. If $a = ce$ as above, then

$$\begin{aligned} q &= b^{-1}a \\ &= b^{-1}ce \\ &= b^{-1}cebb^{-1}c^{-1}c \\ &= b^{-1}ceb(b^{-1}c^{-1})c \\ &= (b^{-1}ce)b(db^{-1})c \end{aligned}$$

for some $d \in A$. Since e is a central idempotent,

$$q = (b^{-1}ce)bd(b^{-1}ce) = q(bd)q,$$

which proves that Q is VNR. By [33, Proposition 1.5], Q is a reduced ring and hence Q is strongly regular. □

Recall that if every ideal of A is a complement left ideal of A, then every ideal of A is generated by a central idempotent [36, Proposition 2] (consequently, A is biregular). We also know that A is strongly regular if and only if A is a reduced ring whose finitely generated right ideals are principal complement right ideals of A [31, Theorem 2.6].

QUESTION 2. Is A strongly regular if A is a reduced ring whose finitely generated right ideals are complement right ideals?

We proceed to give a new characterization of strongly regular rings.

LEMMA 2.15. *Let T be a non-zero ideal of A which contains no non-zero nilpotent left ideal of A. If e is an idempotent in T such that Ae is an ideal of A, then e is central in A.*

Proof. Since Ae is an ideal of A, $Ae = AeA$ and $eA \subseteq Ae$. Then $eA(1-e) \subseteq Ae(1-e) = 0$ implies that $ea = eae$ for every $a \in A$. Now $A = eA \oplus (1-e)A$ and for any $u \in (1-e)A$, $b \in A$, $b = ec + (1-e)d$, with $c, d \in A$, whence $bu = ecu + (1-e)du$ and since $eA \subseteq Ae$, $ec = we$ for some $w \in A$. Therefore

$$bu = weu + (1-e)du = (1-e)du \in (1-e)A$$

which shows that $(1-e)A$ is also an ideal of A. Then $((1-e)Ae)^2 = (0)$ implies that $(1-e)Ae = (0)$ (since T contains no non-zero nilpotent left ideal of A). Now $ae = eae$ for each $a \in A$ which proves that e is central in A. □

Theorem 2.16. *The following conditions are equivalent:*

1. *A is strongly regular;*
2. *for every $b \in A$, $Ab + r(AbA)$ is an ideal of A which is a complement left ideal of A.*

Proof. Assume (1). For any $b \in A$, $Ab = AbA$ is generated by a central idempotent. If $Ab = Ae$, $e = e^2$ being central, then $r(AbA) = (1-e)A = A(1-e)$ and $Ab + r(AbA) = Ae + A(1-e) = A$. Therefore (1) implies (2).

Assume (2). For every $b \in A$, $T = Ab + r(AbA)$ is an ideal of A which implies that $AbA \subseteq T$, whence $T = AbA + r(AbA)$ is a complement left ideal of A. Suppose there exists $d \in A$ such that $(AdA)^2 = (0)$. Then $r(AdA)$ is an essential left ideal of A. But $r(AdA) = AdA + r(AdA)$ is a complement left ideal of A by hypothesis. Therefore $r(AdA) = A$ which yields $d = 0$. We have shown that A must be a semi-prime ring. For any $c \in A$, let $C = AcA$. Since A is semi-prime, then $C \cap r(C) = O$. Set $L = C \oplus r(C)$. If ${}_AK$ is a relative complement of ${}_AL$ in ${}_AA$, then $E = L \oplus K$ is an essential left ideal of A. Now $LK \subseteq L \cap K = (0)$ implies that $K \subseteq r(L) \subseteq r(C)$, whence $K = K \cap r(C) \subseteq K \cap L = (0)$. Therefore L is an essential left ideal of A. But $L = Ac + r(AcA)$ is a complement left ideal of A by hypothesis. Therefore $L = A$ which proves that $C = Au$, $u = u^2 \in A$. Since A is semi-prime, u is central in A by (2.15). We have proved that A is a biregular ring. Now for every $b \in A$,

$$A = Ab \oplus r(AbA) = AbA \oplus r(AbA)$$

and if $r(AbA) = Aw$, where w is a central idempotent, then

$$A = Ab \oplus Aw = A(1-w) \oplus Aw.$$

Then $b = b(1-w) + bw$ and $bw = wb \in Ab \cap Aw = (0)$, whence $b = b(1-w)$ which yields $Ab \subseteq A(1-w)$. Since $A = Ab \oplus Aw$,

$$A(1-w) = Ab \oplus (Aw \cap A(1-w)) = Ab$$

and therefore Ab is generated by a central idempotent. Thus (2) implies (1). □

Applying [36, Proposition 2] to (2.16), we get:

Corollary 2.17. *A is a finite direct sum of division rings if and only if every ideal of A is a complement left ideal and for every $b \in A$, $Ab + r(AbA)$ is an ideal of A.*

(2.15) also yields the next remark

Remark 2.18. *If e is an idempotent in A and Ae is a minimal left ideal of A which is an ideal of A, then e is central in A.*

Remark 2.19. *If M is an injective maximal left ideal of A, $M = Ae$, $e = e^2 \in A$ and $A(1-e)$ is an ideal of A, then A is a left self-injective ring.*

A condition for non-singularity.

Proposition 2.20. *Let A be a MELT ring such that for any maximal essential left ideal M of A, A/M_A is flat. Then $Z = (0)$.*

Proof. Suppose that $Z \neq (0)$. By [32, Lemma 7], there exists $z \in Z$, $z \neq 0$ such that $z^2 = 0$. Let M be a maximal left ideal of A containing $l(z)$. Then M is an essential left ideal of A and M is an ideal of A by hypothesis. Therefore A/M_A is flat. Since $z \in l(z) \subseteq M$, $z = dz$ for some $d \in M$ [3, p.458]. Therefore $1 - d \in l(z) \subseteq M$ and since $d \in M$, $1 \in M$ which contradicts $M \neq A$. We have proved that $Z = (0)$. □

Note that the ring considered in (2.20) needs not be semi-prime (cfr. (2.8)).

Finally, we consider the reduced ideals in a ring.

Proposition 2.21. *Let A be a semi-prime left YJ-injective ring. Then T, the sum of all reduced ideals of A, is the unique maximal strongly regular ideal of A and T is a left annihilator.*

Proof. Suppose that $l(r(T))$ is not a reduced ideal of A. Then there exists $w \in l(r(T))$, $w \neq 0$ such that $w^2 = 0$. Now $T = TA$ and if $Tw = O, TAw = O$ and $Aw^2 \subseteq l(r(T)) \cdot r(T) = (0)$ which contradicts A semi-prime. Therefore $Tw \neq O$ and hence there exists a reduced ideal R of A such that $Rw \neq (0)$. This implies that $R \cap Aw \neq (0)$. Let $z \in R \cap Aw$, $z \neq 0$. Then by [34, Lemma 5], $z = zvz$ for some $v \in R$ and $Az = Ae$, $e = vz$ being an idempotent. Now $z = bw$, $b \in A$ and

$$vbwe = vze = e^2 = e \neq 0$$

which implies that $we \neq 0$. But $(we)^2 = wvbwwe = 0$, which contradicts R being a reduced ideal of A. We have shown that $l(r(T))$ must be a reduced ideal of A. It follows that $T = l(r(T))$ is the unique maximal reduced ideal of A. By [34, Lemma 5], T is the unique maximal strongly regular ideal of A and $T = l(r(T))$ is a left annihilator. □

Corollary 2.22. *If A is a left self-injective regular ring and T is the sum of all reduced ideals of A, then $A = T \oplus Q$, where T is a left and right self-injective strongly regular ring and Q is a left self-injective regular ring such that every non-zero ideal of Q contains a non-zero nilpotent element.*

Note that if Q is a left continuous regular ring such that every non-zero ideal of Q contains a non-zero nilpotent element, then Q is left self-injective [19, Theorem 3]. Consequently, the next decomposition follows.

Corollary 2.23. *If A is a left continuous regular ring and T is the sum of all reduced ideals of A, then $A = T \oplus Q$, where T is a left and right continuous strongly regular ring and Q is a left self-injective regular ring such that every non-zero ideal of Q contains a non-zero nilpotent element.*

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