A Note on the CR Cohomology of Levi-Flat Minimal Orbits in Complex Flag Manifolds

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Summary. - We prove a relation between the $\bar{\partial}_M$ cohomology of a minimal orbit $M$ of a real form $G_0$ of a complex semisimple Lie group $G$ in a flag manifold $G/Q$ and the Dolbeault cohomology of the Matsuki dual open orbit $X$ of the complexification $K$ of a maximal compact subgroup $K_0$ of $G_0$, under the assumption that $M$ is Levi-flat.

1. Introduction

Many authors have studied the $\bar{\partial}_M$ cohomology of CR manifolds (see e.g. [6, 7, 12] and references therein). In particular, since Andreotti and Fredricks [2] proved that every real analytic CR manifold $M$ can be embedded in a complex manifold $X$, it is natural to try to find relations between the $\bar{\partial}_M$ cohomology of $M$ and the Dolbeault cohomology of $X$.

In this paper we examine this problem for a specific class of homogeneous CR manifolds, namely minimal orbits in complex flag manifolds that are Levi-flat.

Given a (generalized) flag manifold $Y = G/Q$, with $G$ a complex semisimple Lie group and $Q$ a parabolic subgroup of $G$, a real form

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$G_0$ of $G$ acts on $Y$ with finitely many orbits. Among these there is exactly one orbit that is compact, the minimal orbit $M = G_0 \cdot o$. Let $K_0$ be a maximal compact subgroup of $G_0$, and $K$ its complexification. Then $X = K \cdot o$ is an open dense complex submanifold of $G/Q$ and contains $M$ as an embedded submanifold. It is known that $M$ is a deformation retract of $X$, so $H^p(M, \mathbb{C}) = H^p(X, \mathbb{C})$ ([4], [8]).

Let $E$ be a $K$-homogeneous complex vector bundle on $X$ and $E|_M$ its restriction to $M$. Under the additional assumption that $M$ is Levi-flat we prove that the restriction map from the Dolbeault cohomology $H^{p,q}(X, E)$ to the $\bar{\partial}_M$ cohomology $H^{p,q}(M, E|_M)$ is continuous, injective and has a dense range. More precisely we show that

$$H^{p,q}(M, E|_M) = \mathcal{O}_M(M) \otimes_{\mathcal{O}_X(X)} H^{p,q}(X, E)$$

where $\mathcal{O}_M(M)$ (resp. $\mathcal{O}_X(X)$) is the space of CR (resp. holomorphic) functions on $M$ (resp. $X$), and that the restriction map from $\mathcal{O}_X(X)$ to $\mathcal{O}_M(M)$ is injective, continuous and has a dense range.

2. Preliminaries on minimal orbits in complex flag manifolds

Let $G$ be a complex connected semisimple Lie group, with Lie algebra $g$, and let $Q$ be a parabolic subgroup of $G$, with Lie algebra $q$. Then $Q$ is the normalizer of $q$, $Q = N_G(q)$ and is connected. The coset space $Y = G/Q$ is a compact complex manifold, called a flag manifold (it is a complex smooth projective variety). It is not restrictive to assume that $q$ does not contain any simple ideal of $g$.

Let $\sigma$ be an anti-holomorphic involution of $G$; we will also denote by $\sigma$ its differential at the identity and we will write $\hat{x} = \sigma(x)$. A real form of $G$ is an open subgroup $G_0$ of $G^\sigma$. It is a Lie subgroup and its Lie algebra $g_0$ satisfies $g_0 = g^\sigma$ and $g = \mathbb{C} \otimes g_0$.

Let $K_0$ be a maximal compact subgroup of $G_0$, and $\theta$ the corresponding Cartan involution: $K_0 = G_0^\theta$. Still denoting by $\theta$ the complexification of $\theta$, there is exactly one open subgroup $K$ of $G^\theta$ such that $K \cap G_0 = K_0$. Let $k$ and $k_0$ be the corresponding Lie subalgebras.

The groups $G_0$ and $K$ act on $Y$ via left multiplication. There is exactly one closed $G_0$-orbit $M$ ([1, 19]) and it is contained in
the unique open $K$ orbit $X$ ([3, 14]), we denote by $j: M \to X$ the inclusion. The open orbit $X$ is dense in $Y$ and is the dual orbit of $M$, in the terminology of [14].

The manifolds $M$ and $X$ do not depend on the choice of $G$ and $G_0$, but only on $g_0$ and $q$ ([1, 14]). So there is no loss of generality assuming that $G$ is simply connected, that $G_0$ is connected and that $M$ and $X$ are the orbits through the point $o = eQ$. We will write $M = M(g_0, q)$.

The isotropy subgroups at $o$ of the actions of $G_0$ and $K$ are $G_+ = G_0 \cap Q$ and $L = K \cap Q$, with Lie algebras $g_+ = g_0 \cap q$ and $l = \mathfrak{k} \cap q$. Since $M$ is compact, the action on $M$ of maximal compact subgroup $K_0$ is transitive: $M = K_0/K_+$, where $K_+ = K_0 \cap Q = G_0 \cap L$ and $\text{Lie}(K_+) = \mathfrak{k}_+ = \mathfrak{k}_0 \cap \mathfrak{q} = g_0 \cap l$.

In the language of [1] the pair $(g_0, q)$ is an effective parabolic minimal CR algebra and $M$ is the associated minimal orbit. On $M$ there is a natural CR structure induced by the inclusion into $X$.

We recall that $M$ is totally real if the partial complex structure is trivial. We give a more complete characterization of totally real minimal orbits:

**Theorem 2.1.** The following are equivalent:

1. $M$ is totally real.
2. $\bar{q} = q$.
3. $l = \mathfrak{k}_+^\mathbb{C}$.
4. $X$ is a Stein manifold.
5. $X$ is a smooth affine algebraic variety defined over $\mathbb{R}$ and $M$ is the set of its real points.

**Proof.** (5) $\Rightarrow$ (4) because every closed complex submanifold of a complex vector space is Stein.

(4) $\Rightarrow$ (3) Since $X$ is Stein, its covering $\tilde{X} = K/L^0$ is also Stein ([17]). Furthermore $K$ is a linear algebraic group that is the complexification of a maximal compact subgroup $K_0$; the result then follows from Theorem 3 of [15] and Remark 2 thereafter.

(3) $\Rightarrow$ (5) If $l$ is the complexification of $\mathfrak{k}_+$, then $L$ is the complexification of $K_+$. Hence $X = K/L$ is the complexification of
\[ M = \mathbf{K}_0/\mathbf{K}_+ \] in the sense of [9], and (5) follows from Theorem 3 of the same paper.

(1) \(\iff\) (2) is easy, and proved in [1].

(5) \(\Rightarrow\) (1) is obvious.

(2) \(\Rightarrow\) (3) We have that \(\bar{\mathfrak{k}} = \mathfrak{k}\), thus \(\mathfrak{k} \cap \mathfrak{q} = (\mathfrak{k} \cap \mathfrak{q} \cap \mathfrak{g}_0)^{\mathbb{C}} = \mathfrak{k}_+^{\mathbb{C}}\).

We denote by \(\mathcal{O}_N\) the sheaf of smooth CR functions on a CR manifold \(N\). If \(N\) is complex or real, then \(\mathcal{O}_N\) is the usual sheaf of holomorphic or smooth (complex valued) functions. For every open set \(U \subset N\), the space \(\mathcal{O}_N(U)\) is a Fréchet space (with the topology of uniform convergence of all derivatives on compact sets).

**Corollary 2.2.** If \(M\) is totally real then \(j^*(\mathcal{O}_X(X))\) is dense in \(\mathcal{O}_M(M) = C^\infty(M)\).

**Proof.** Let \(X \subset \mathbb{C}^N\) be an embedding as in (5) of Theorem 2.1, so that \(M = X \cap \mathbb{R}^N\). The restrictions of complex polynomials in \(\mathbb{C}^N\) are contained in \(\mathcal{O}_X(X)\) and dense in \(\mathcal{O}_M(M)\) (see e.g. [18]).

3. Levi-flat orbits and the fundamental reduction

In this paper we consider Levi-flat minimal orbits. They are orbits \(M = M(g_0, \mathfrak{q})\), where \(\mathfrak{q}' = \mathfrak{q} + \mathfrak{q}\) is a subalgebra (necessarily parabolic) of \(\mathfrak{g}\). Let \(Q' = N_G(\mathfrak{q}')\), \(Y' = G/Q'\), \(G'_+ = G \cap Q'\), \(M' = M(g_0, \mathfrak{q}') = G_0/G'_+\), \(K'_+ = K_0 \cap Q'\), \(L' = K \cap Q'\) and \(X' = K/L'\).

From Theorem 2.1 we have that \(M'\) is totally real and \(X'\) is Stein. The inclusion \(Q \rightarrow Q'\) induces a fibration

\[ \pi: Y = G/Q \longrightarrow G/Q' = Y' \]  

with complex fiber \(F \simeq Q'/Q\). This fibration is classically referred to as the Levi foliation, and is a special case of the fundamental reduction of [1]. In fact Levi-flat minimal orbits are characterized by the property that the fibers of the fundamental reduction are totally complex.

We identify \(F\) with \(\pi^{-1}(eQ)\).

**Lemma 3.1.** \(\pi^{-1}(M') = M\), \(\pi^{-1}(X') = X\) and \(F\) is a compact connected complex flag manifold.
Proof. First we observe that $F$ is connected because $Q'$ is connected. Let $F'$ be the fiber of the restriction $\pi|_M:M \to M'$. Then $F'$ is totally complex and CR generic in $F$, hence an open subset of $F$. Proposition 7.3 and Theorem 7.4 of [1] show that there exists a connected real semisimple Lie group $G''_0$ acting on $F'$ with an open orbit, and that the Lie algebra of the isotropy is a $t$-subalgebra (i.e. contains a maximal triangular subalgebra) of $q''_0$. Hence a maximal compact subgroup $K''_0$ of $G''_0$ has an open orbit, which is also closed. Since $F'$ is open in $F$ and $F$ is connected, $K''_0$ is transitive on $F$, and $F'=F$, proving the first two statements.

Furthermore the isotropy subgroup $G''_+$ for the action of $G''_0$ on $F$ and the homogeneous complex structure are exactly those of a totally complex minimal orbit, hence by [1, §10] $F$ is a complex flag manifold.

The total space $M$ is locally isomorphic to an open subset of $M' \times F$, hence to $U \times \mathbb{R}^k$, where $U$ is open in $\mathbb{C}^n$, for some integers $n$ and $k$.

For a Levi flat CR manifold $N$ and a nonnegative integer $p$, let $\Omega^p_N$ be the sheaf of $p$-forms that are CR (see [7] for precise definitions). They are $\mathcal{O}_N$-modules and $\Omega^0_N \cong \mathcal{O}_N$.

Let $\mathcal{A}^{p,q}_N$ be the sheaf of (complex valued) smooth $(p,q)$-forms on $N$, $\partial_N$ the tangential Cauchy-Riemann operator and $\mathcal{Z}^{p,q}_N$, (resp. $\mathcal{B}^{p,q}_N$) the sheaf of closed (resp. exact) $(p,q)$-forms. As usual we denote by $H^{p,q}(N) = \mathcal{Z}^{p,q}_N(N)/\mathcal{B}^{p,q}_N(N)$ the cohomology groups of the $\partial_M$ complex on smooth forms. The Poincaré lemma is valid for $\partial_N$ (see [11]), thus the complex:

$$0 \to \Omega^p_N \to \mathcal{A}^{p,0}_N \partial_N \to \cdots \partial_N \mathcal{A}^{p,q}_N \partial_N \to \cdots$$

is a fine resolution of $\Omega^p_N$. This implies that $H^{p,q}(N) \cong H^q(\Omega^p_N)$.

Let $E_N$ be a homogeneous CR vector bundle on $N$ (i.e. a complex vector bundle with transition functions that are CR) with fiber $E$, and let $\mathcal{E}_N$ be the sheaf of its CR sections.

We denote by $E^p_N$, the bundle of CR, $E_N$-valued, $p$-forms, with associated sheaf of CR sections $\mathcal{E}^p_N = \Omega^p_N \otimes_{\mathcal{O}_N} \mathcal{E}_N$.

Let $\mathcal{A}^{p,q}_{E_N} = \mathcal{A}^{p,q}_N \otimes_{\mathcal{O}_N} \mathcal{E}_N$, denote by $\partial_{E_N}$ the tangential Cauchy-Riemann operator on $E_N$ and let $\mathcal{Z}^{p,q}_{E_N}$ (resp. $\mathcal{B}^{p,q}_{E_N}$) be the sheaf
of $\partial E_N$-closed (resp. exact) smooth forms with values in $E_N$.

Then $H^{p,q}(N, E_N) = Z^{p,q}_{N,E_N}(N)/B^{p,q}_{N,E_N}(N)$, but we also have:

$$H^{p,q}(N, E_N) = H^q(E_N^p) = H^{0,q}(N, E^p_N) = Z^{0,q}_{N,E^p_N}(N)/B^{0,q}_{N,E^p_N}(N).$$

For any open set $U \subset N$, the spaces $\mathcal{A}^{p,q}_{N,E_N}(U)$ and $Z^{p,q}_{N,E_N}(U)$ are Fréchet spaces with the topology of uniform convergence of all derivatives on compact sets. If $B^{p,q}_{N,E_N}(N)$ is closed in $Z^{p,q}_{N,E_N}(N)$, then $H^{p,q}(N, E_N)$, with the quotient topology, is also a Fréchet space.

4. Statements and proofs

Let $E_F$ be a $L'$-homogeneous holomorphic vector bundle on $F$. The $L'$ action induces a natural $L'$ action on $\mathcal{A}^{p,q}_{F,E_F}$, hence on $H^{p,q}(F, E_F)$, because the action of $L'$ preserves closed and exact forms. Since $F$ is a compact complex manifold, $H^{p,q}(F, E_F)$ is finite dimensional and we can construct the $K$-homogeneous holomorphic vector bundle on $X'$:

$$H^{p,q}_{X'}(F, E_F) = K \times_{L'} H^{p,q}(F, E_F).$$

In a similar way we define a $K_0$-homogeneous complex vector bundle on $M'$:

$$H^{p,q}_M(F, E_F) = K_0 \times_{K_1} H^{p,q}(F, E_F).$$

The following theorem has been proved by Le Potier ([13], see also [5]):

**Theorem 4.1.** Let $X$, $X'$ and $F$ be as above, $E_X$ a $K$-homogeneous holomorphic vector bundle on $X$ and $E_X|_F$ its restriction to $F$. Then there exists a spectral sequence $pE^{s,t}_r$, converging to $H^{p,q}(X, E_X)$, with

$$pE^{s,t}_2 = \bigoplus_i H^{i,s-i}(X', H^{p-i,t+i}_X(F, E_X|_F)).$$

For $p = 0$ the spectral sequence collapses at $r = 2$ and, recalling that $X'$ is a Stein manifold, we obtain:

$$H^{0,q}(X, E_X) = H^{0,0}(X', H^{0,q}_X(F, E_X|_F)).$$

Recalling that $H^{p,q}(X, E_X) = H^{0,q}(X, E^p_X)$ we finally obtain:
Proposition 4.2. Let \( X, X' \) and \( F \) be as above, \( E_X \) a \( K \)-homogeneous holomorphic vector bundle on \( X \) and \( E_X|_F \) its restriction to \( F \). Then:

\[
H^{p,q}(X, E_X) = H^{0,0}(X', H^{0,q}_{X'}(F, E_X^p|_F))
\]
as Fréchet spaces.

A statement analogous to the last proposition holds for \( M \):

Proposition 4.3. Let \( M, M' \) and \( F \) be as above, \( E_M \) a \( K_0 \)-homogeneous CR vector bundle on \( M \) and \( E_M|_F \) its restriction to \( F \). Then:

\[
H^{p,q}(M, E_M) = H^{0,0}(M', H^{0,q}_{M'}(F, E_M^p|_F))
\]
as Fréchet spaces.

Proof. Fix \( p, q \), let \( Z_{M'} = \pi_*(Z^0_{M,E_M^p}), B_{M'} = \pi_*(B^0_{M,E_M^p}) \) and \( \mathcal{H}_{M'} \) be the sheaf of sections of \( H^{0,q}_{M}(F, E_M^p|_F) \) We already know that \( H^{p,q}(M, E_M) \simeq Z_{M'}(M')/B_{M'}(M') \).

We now define a map \( \phi: Z_{M'} \to \mathcal{H}_{M'} \) as follows.

Let \( U \subset M' \), \( x \in U \), \( x = gK'_{K_0} \), \( g \in K_0 \) and \( \xi \in Z_{M'}(U) \). Let \( \xi_g = (g^{-1} \cdot \xi)|_F \). This is a closed \( E_M^p \)-valued \((0,q)\)-form on \( F \), that determines a cohomology class \([\xi_g]\) in \( H^{0,q}(F, E_M^p|_F) \). Then the class of \((g, [\xi_g])\) in \( H^{0,q}_{M}(F, E_M^p|_F) \) does not depend on the particular choice of \( g \), but only on \( x \), hence it defines a section \( s_\xi = \phi(\xi) \) of \( \mathcal{H}_{M'} \) on \( U \).

The sheaves \( Z_{M'}, B_{M'}, \mathcal{H}_{M'}, \ker \phi \) are \( \mathcal{O}_{M'} \)-modules and \( \phi \) is a morphism of \( \mathcal{O}_{M'} \)-modules. Since \( \mathcal{O}_{M'} \) is fine, to prove that \( \phi(M') \) is continuous, surjective and that its kernel is exactly \( B_{M'}(M') \) it suffices to check that this is true locally, and this reduces to a straightforward verification.

We prove now the main theorem of this paper:

Theorem 4.4. Let \( M \) and \( X \) be as above, \( E_X \) a \( K \)-homogeneous holomorphic vector bundle over \( X \). Then:

\[
H^{p,q}(M, E_X|_M) \simeq \mathcal{O}_M(M) \otimes_{\mathcal{O}_X(X)} H^{p,q}(X, E_X).
\]
Proof. Define $M'$ and $X'$ as above, fix integers $p, q \geq 0$ and let $\mathbf{H}_{X'} = \mathbf{H}^{0,q}_{X'}(F, E^p_{X'}|_F)$, $\mathbf{H}_{M'} = \mathbf{H}^{0,q}_{M'}(F, E^p_{X'}|_F) = \mathbf{H}_{X'}|_{M'}$. By Propositions 4.2 and 4.3, we have that $H^{p,q}(M, E_X|_M) = \Gamma(M', \mathbf{H}_{M'})$ and $H^{p,q}(X, E_X) = \Gamma(X', \mathbf{H}_{X'})$.

Since $\dim_{\mathbb{R}} M' = \dim_{\mathbb{C}} (X')$, a global section of $\mathbf{H}_{X'}$ that is zero on $M'$ must be zero on $X'$, i.e. the restriction map $\Gamma(X', \mathbf{H}_{X'}) \rightarrow \Gamma(M', \mathbf{H}_{M'})$ is injective.

On the other hand, $X'$ is Stein, thus $\mathbf{H}_{X'}$ is generated at every point by its global sections. Together with the fact that $\mathbf{H}_{M'} = \mathbf{H}_{X'}|_{M'}$ this implies that

$$\Gamma(M', \mathbf{H}_{M'}) = \mathcal{O}_{M'}(M') \otimes_{\mathcal{O}_{(X')}} \Gamma(X', \mathbf{H}_{X'}),$$

where in the right hand side we implicitly identify global holomorphic sections on $X'$ with their restrictions to $M'$.

The theorem follows from the observation that $\mathcal{O}_M(M) \simeq \mathcal{O}_{M'}(M')$ and $\mathcal{O}_X(X) \simeq \mathcal{O}_{X'}(X')$ because the fiber $F$ of $\pi$ is a compact connected complex manifold. \qed

This, together with Corollary 2.2, implies the following.

**Corollary 4.5.** With the same assumptions, the inclusion $j: M \rightarrow X$ induces a map:

$$j^*: H^{p,q}(X, E_X) \rightarrow H^{p,q}(M, E_X|_M)$$

that is continuous, injective and has a dense range. \qed

5. An example

Let $G = \text{SL}(4, \mathbb{C})$, and $Q$ be the parabolic subgroup of upper triangular matrices. We consider the real form $G_0 = \text{SU}(1,3)$, identified with the group of linear transformations of $\mathbb{C}^4$ that leave invariant the Hermitian form associated to the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $G/Q$ is the set of complete flags $\{\ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4\}$ and $M = G_0 \cdot eQ$ is the submanifold $\{\ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4 | \ell^3 = (\ell^1)^\perp\}$. 
Let $Q'$ be the set of block upper triangular matrices of the form

$$Q' = \left\{ g = \begin{pmatrix} ** & * \\ 0 & ** \\ 0 & 0 & * \end{pmatrix} \mid g \in G \right\},$$

so that $M'$ is the totally real manifold $\{ \ell^1 \subset \ell^3 \subset \mathbb{C}^4 \mid \ell^3 = (\ell^1)^\perp \}$ and $M$ fibers over $M'$ with typical fiber $F$ isomorphic to $\mathbb{CP}^1$. The fibration is given by

$$(\ell^1, \ell^2, \ell^3) \mapsto (\ell^1, \ell^3). \quad (*)$$

Choose $K$ to be the stabilizer in $G$ of the subspaces $V = \text{Span}(e_1 - e_4)$ and $W = \text{Span}(e_1 + e_4, e_2, e_3)$ so that $K$ is isomorphic to $S(GL(1, \mathbb{C}) \times GL(3, \mathbb{C}))$ and $K_0$ to $S(U(1) \times U(3))$. Then $X$ is the set of flags $\{ \ell^1 \subset \ell^2 \subset \ell^3 \subset \mathbb{C}^4 \}$ in a generic position with respect to the subspaces $V$ and $W$, and $X'$ is the set of flags $\{ \ell^1 \subset \ell^3 \subset \mathbb{C}^4 \}$ in a generic position with respect to $V$ and $W$. The map from $X$ to $X'$ given by $(*)$ is a fibration with typical fiber isomorphic to $\mathbb{CP}^1$ and $X'$ is a Stein manifold.

Finally let $E = X \times \mathbb{C}$ be the trivial line bundle. According to Propositions 4.2 and 4.3 the cohomology of $M$ and $X$ is given by

$$H^{p,q}(X) = H^{p,q}(X, E) = H^{0,0}(X', H_X^{0,q}(F, E^p|_F)),\quad H^{p,q}(M) = H^{p,q}(M, E|_M) = H^{0,0}(M', H_M^{0,q}(F, E^p|_F)).$$

Recalling that $H^{p,q}(F) \simeq \mathbb{C}$ if $p = q = 0$ or $p = q = 1$ and 0 otherwise, we obtain:

$$\begin{cases} H^{p,q}(X) \simeq O_X(X) \simeq O_X'(X') & \text{if } p = q = 0 \text{ or } p = q = 1; \\ H^{p,q}(X) = 0 & \text{otherwise}; \end{cases}$$

and analogously:

$$\begin{cases} H^{p,q}(M) \simeq O_M(M) \simeq O_{M'}(M') & \text{if } p = q = 0 \text{ or } p = q = 1; \\ H^{p,q}(M) = 0 & \text{otherwise}; \end{cases}$$

and it is clear that

$$H^{p,q}(M) = O_M(M) \otimes O_X(X) H^{p,q}(X).$$
References


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