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Web Functions: Survey of Results and Perspectives

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Summary. - *We recall some of our previous results on web functions, we give some new numerical results concerning a simple model and we state some open problems.*

1. Introduction

Let Ω be an open bounded convex domain of $\mathbb{R}^n$ $(n \geq 2)$, let $f\colon \mathbb{R}^+ \to \overline{\mathbb{R}}$ be a lower semicontinuous (l.s.c.) function and consider the functional J defined by

$$J(u) = \int_\Omega [f(|\nabla u|) - u] \, dx \ .$$

For some concrete models leading to this kind of functionals we refer to the references in [4].

We study the following problem of existence of minima,

$$\min_{u \in W_0^{1,1}(\Omega)} J(u) \ . \tag{1}$$

Since we make no convexity assumption on f, the minimum in (1) may not exist. In such case, the standard procedure [5] is to modify the functional J by relaxation and to try to recover informations

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concerning problem (1) from the properties of the relaxed functional, of its minimum and of its minimizing sequences. Recently [3, 4, 6], we have been trying to proceed in a different fashion, namely by maintaining the functional J and by modifying the space where to seek the minimum: we considered the subspace $\mathcal{K} \subset W_0^{1,1}(\Omega)$ of web functions (functions depending only on the distance from the boundary $\partial\Omega$ of the set Ω) and we proved in [3, 6], under mild assumptions on f, that the minimum problem

$$\min_{u\in\mathcal{K}} J(u) \tag{2}$$

always admits a unique solution $\overline{u}$. Clearly, in order to justify such approach, one should then verify that $\overline{u}$ describes in some sense the minimization problem (1); to this end, we gave in [4] some estimates of the relative error

$$\mathcal{E} = \frac{\min_{u\in\mathcal{K}} J(u)}{\inf_{u\in W_0^{1,1}(\Omega)} J(u)} \tag{3}$$

for several meaningful models. In order to have $\mathcal{E}$ well defined, the denominator in (3) must be different from 0; by means of a suitable normalization, in [4] we showed that it is always possible to reduce to functions f satisfying $f(0) = 0$ so that $J(0) = 0$. Then, either $u \equiv 0$ (which is a web function!) solves (1) or $\inf_{W_0^{1,1}(\Omega)} J < 0$ so that $\mathcal{E}$ is well defined. In the latter case, since $\mathcal{K} \subset W_0^{1,1}(\Omega)$ one has $\mathcal{E} \in [0,1]$ and $\mathcal{E}$ represents the relative error of the approximation: the closer $\mathcal{E}$ is to 1, the better the approximation is. The results we obtained show that our web function approach is promising, that is, $\mathcal{E}$ is close to 1 in many concrete examples: even in the cases where problem (1) admits a solution (for instance when f is convex) this approach gives important information about it.

In this paper we first recall some results from [3, 4, 6], then we give some new numerical results concerning the case where $f(s) = s^2/2$; finally, we state three open problems which seem of particular interest to us: the first one is a problem in functional analysis, the second is a problem in optimal design suggested by Buttazzo [1], the third is a possible alternative definition of web function.

2. Some results, old and new

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded open convex set and let W_Ω denote its inradius, namely the supremum of the radii of the open balls contained in Ω. The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure of a set $A \subset \mathbb{R}^n$ will be denoted respectively by $\mathcal{L}(A)$ and $\mathcal{H}(A)$. Assume that

$$\begin{gathered} f \not\equiv +\infty \quad \text{is a l.s.c. function s.t.} \\ \exists M > \frac{\mathcal{L}(\Omega)}{\mathcal{H}(\partial\Omega)}\,,\ \exists b \in \mathbb{R}\,,\ f(s) \geq Ms - b \quad \forall s \geq 0\ . \end{gathered} \tag{4}$$

We denote by f^* the polar function of f and by f^{**} the bipolar function of f, see [5]; let

$$\sigma = \max\{s \geq 0;\ f^{**}(s) = \min f^{**}\} \tag{5}$$

and define the *normalized non-decreasing bipolar function* f_{**} of f by

$$f_{**}(s) = \begin{cases} 0 & \text{if } 0 \leq s \leq \sigma \\ f^{**}(s) - f^{**}(\sigma) & \text{if } s \geq \sigma\ . \end{cases}$$

Finally, denote by f_* the polar function of f_{**}. In [4, Proposition 1] we showed that any function f satisfying (4) may be normalized without altering the minimization problem and therefore we can always reduce to the case where $f^{**}(0) = 0$ and f^{**} is non-decreasing. Moreover, if f satisfies (4), a necessary condition for $\inf_{u \in W_0^{1,1}(\Omega)} J(u) < 0$ is that $f_*(W_\Omega) > 0$, see [4, Proposition 2].

Consider the set of web functions relative to Ω

$$\mathcal{K} = \{u \in W_0^{1,1}(\Omega);\ u(x) = u(d(x,\partial\Omega))\ \forall x \in \Omega\}\ ,$$

where $d(\cdot,\partial\Omega)$ denotes the distance function from the boundary. We also consider the one-parameter family of subsets of Ω defined by

$$\Omega_t = \{x \in \Omega;\ d(x,\partial\Omega) > t\} \qquad \forall t \in [0, W_\Omega]$$

and their boundaries $\partial\Omega_t$. In our setting a major role is played by the functions

$$\nu(t)=\frac{\mathcal{L}(\Omega_t)}{\mathcal{H}(\partial\Omega_t)}\ ,\qquad \alpha(t)=\mathcal{H}(\partial\Omega_t)\qquad t\in[0,W_\Omega]\ .$$

Giving a full generalization of [6, Theorem 1], the following result has been proved in [3]:

THEOREM 2.1. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded convex set and assume that f satisfies* (4)*. Then, the function*

$$\overline{u}(x)=\int_0^{d(x,\partial\Omega)}f_*'(\nu(t))\,dt$$

is the unique solution of (2)*.*

In [4] we obtained the following lower bound for $\mathcal{E}$:

THEOREM 2.2. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded convex set and let $R=(\frac{\mathcal{L}(\Omega)}{\omega_n})^{1/n}$; assume that f satisfies* (4) *and $f_*(\frac{R}{n})>0$. Let*

$$\mathcal{E}_1=\frac{\int_0^{W_\Omega}\alpha(t)f_*(\nu(t))dt}{n\omega_n\int_0^R t^{n-1}f_*(\frac{t}{n})dt}\qquad \mathcal{E}_2=\frac{W_\Omega\int_0^{W_\Omega}\alpha(t)f_*(\nu(t))dt}{f_*(W_\Omega)\int_0^{W_\Omega}\mathcal{L}(\Omega_t)\,dt}$$

then

$$\mathcal{E}\geq\max\{\mathcal{E}_1,\mathcal{E}_2\}. \tag{6}$$

Both $\mathcal{E}_1$ and $\mathcal{E}_2$ have interest in the estimate (6) depending on how thin is the domain Ω: we quote the following example taken from [4]:

EXAMPLE 2.3. Let $f(s)=s^2/2$ and let $\Omega=(0,1)\times(0,2W_\Omega)$, $0<W_\Omega\leq 1/2$. We have that $\mathcal{E}_2$ is monotone decreasing on $[0,1/2]$, $\lim_{W_\Omega\to 0}\mathcal{E}_2(W_\Omega)=2/3$ and $\mathcal{E}_2(1/2)=3/8$. On the other hand, $\mathcal{E}_1$ is monotone increasing on $[0,1/2]$, approaches 0 as W_Ω tends to 0 and $\mathcal{E}_1(1/2)=\pi/4$.

The explicit computation of $\mathcal{E}_1$ and $\mathcal{E}_2$ gives that $\mathcal{E}_2>\mathcal{E}_1$ for

$$0<W_\Omega<\frac{3\pi-\sqrt{9\pi^2-12\pi}}{4\pi}\approx 0.181$$

. $\square$

From now on, we essentially deal with the simple case where $f(s) = s^2/2$ and $n = 2$; in such case, also a lower bound for $\mathcal{E}$ involving optimal Sobolev constants is available, see [4, Theorem 5]. However, in this case numerical analysis is helpful and gives well-approximated values of $\mathcal{E}$: in what follows we list some new results obtained numerically.

So, consider the functional

$$J(u) = \int_\Omega \left(\frac{|\nabla u|^2}{2} - u \right) dx \ .$$

To the minimization problem of J is associated the Euler equation

$$\begin{cases} -\Delta u &= 1 \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega \ ; \end{cases} \tag{7}$$

the unique solution of (7) is precisely the minimum of the functional J over the space $W_0^{1,1}(\Omega)$ (in fact the solution is smooth). Therefore, by multiplying the equation in (7) by $\overline{u}$ and by integrating by parts we obtain

$$f(s) = \frac{s^2}{2} \ , \ \overline{u} \text{ minimizes } J \quad \Longrightarrow \quad J(\overline{u}) = -\frac{1}{2} \int_\Omega \overline{u} \ . \tag{8}$$

On the other hand, by (25) in [4] we have

$$I_{\mathcal{K}} = \min_{u \in \mathcal{K}} J(u) = -\frac{1}{2} \int_0^{W_\Omega} \frac{\mathcal{L}^2(\Omega_t)}{\mathcal{H}(\partial\Omega_t)} \, dt \tag{9}$$

so that (3) becomes

$$\mathcal{E} = \frac{\int_0^{W_\Omega} \frac{\mathcal{L}^2(\Omega_t)}{\mathcal{H}(\partial\Omega_t)} \, dt}{\int_\Omega \overline{u}} \tag{10}$$

RECTANGLES. Consider the case where $\Omega = (0,\ell) \times (0,1)$ with $\ell \in (0,1]$ and denote by $\mathcal{E}(\ell)$ the corresponding error (10). By separating variables we find that the unique solution $\overline{u}$ of (7) is given by

$$\overline{u}(x,y) = \frac{\ell x - x^2}{2} +$$
$$-\frac{4\ell^2}{\pi^3} \sum_{k=0}^{\infty} \frac{\sin[(2k+1)\pi x]}{(2k+1)^3 (e^{(2k+1)\pi/\ell}+1)} \left\{ e^{(2k+1)\pi y/\ell} + e^{(2k+1)\pi(1-y)/\ell} \right\} :$$

therefore, by (8) we have

$$J(\overline{u}) = -\frac{1}{2}\int_\Omega \overline{u} = -\frac{\ell^3}{24} + \frac{8\ell^4}{\pi^5} \sum_{k=0}^{\infty} \frac{e^{(2k+1)\pi/\ell}-1}{e^{(2k+1)\pi/\ell}+1} \frac{1}{(2k+1)^5} \ . \tag{11}$$

In order to evaluate the minimum of J over $\mathcal{K}$, note that in this case we have $W_\Omega = \frac{\ell}{2}$ and

$$\alpha(t) = 2(1+\ell-4t) \ , \qquad \nu(t) = \frac{(\ell-2t)(1-2t)}{2(1+\ell-4t)} \ ;$$

then, from (9) we get

$$\begin{aligned} I_\mathcal{K} = I_\mathcal{K}(\ell) &= -\frac{1}{4}\int_0^{\ell/2} \frac{(\ell-2t)^2(1-2t)^2}{1+\ell-4t}\, dt \\ &= \frac{1}{256}(1-\ell)^4 \log\frac{1-\ell}{1+\ell} + \frac{\ell}{128}(\ell^2-4\ell+1) \ . \end{aligned} \tag{12}$$

From (11) and (12) we deduce the explicit value of $\mathcal{E}$:

$$\mathcal{E}(\ell) = \frac{\frac{1}{256}(1-\ell)^4 \log\frac{1-\ell}{1+\ell} + \frac{\ell}{128}(\ell^2-4\ell+1)}{-\frac{\ell^3}{24} + \frac{8\ell^4}{\pi^5}\sum_{k=0}^{\infty} \frac{\exp[(2k+1)\pi/\ell]-1}{\exp[(2k+1)\pi/\ell]+1} \frac{1}{(2k+1)^5}} \ .$$

A numerical computation performed with MATHEMATICA® gives the following picture representing the function $\mathcal{E} = \mathcal{E}(\ell)$ for $\ell \in (0,1]$:

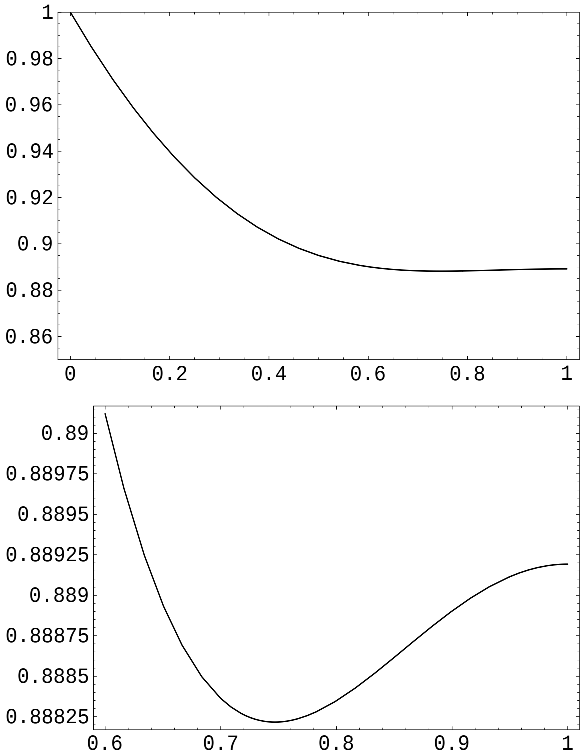

Figure 1: $\mathcal{E} = \mathcal{E}(\ell)$

These pictures reveal a new striking and unexpected phenomenon: the map $\ell \mapsto \mathcal{E}(\ell)$ is *not decreasing* on $(0, 1]$, contrary to what the intuition suggests. In the second picture of Figure 1 we show in more detail the behavior of $\mathcal{E}$ in a neighborhood of its minimum which is achieved for $\ell \approx 0.75$.

ELLIPSES. Let $0 < b < 1$ and let

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2;\ x^2 + \frac{y^2}{b^2} < 1 \right\} .$$

Then, the unique solution $\overline{u}$ of (7) is given by

$$\overline{u}(x, y) = \frac{b^2 - b^2 x^2 - y^2}{2(1 + b^2)} ;$$

hence, if B_1 denotes the unit ball in $\mathbb{R}^2$, by (8) we infer

$$J(\overline{u}) = -\frac{1}{2}\int_{\Omega} \overline{u} = -\frac{b^3}{4(1+b^2)}\int_{B_1}(1-x^2-y^2)dxdy = -\frac{\pi b^3}{8(1+b^2)}. \tag{13}$$

Let

$$\theta_t = \arcsin\sqrt{\max\left[\frac{t^2-b^4}{b^2(1-b^2)}, 0\right]} \ ;$$

in [4] we proved that

$$\alpha(t) = 4\int_{\theta_t}^{\pi/2}\left[\sqrt{\sin^2\theta + b^2\cos^2\theta} - \frac{tb}{\sin^2\theta + b^2\cos^2\theta}\right] d\theta$$

and

$$\mathcal{L}(\Omega_t) = 2\int_{\theta_t}^{\pi/2}\left[b - \frac{tb^2}{(\sin^2\theta + b^2\cos^2\theta)^{3/2}} + -t\sqrt{\sin^2\theta + b^2\cos^2\theta} + \frac{t^2 b}{\sin^2\theta + b^2\cos^2\theta}\right] d\theta \ .$$

A numerical computation performed with MATHEMATICA® gives the following picture representing the function $\mathcal{E} = \mathcal{E}(b)$ for $b \in (0, 1]$:

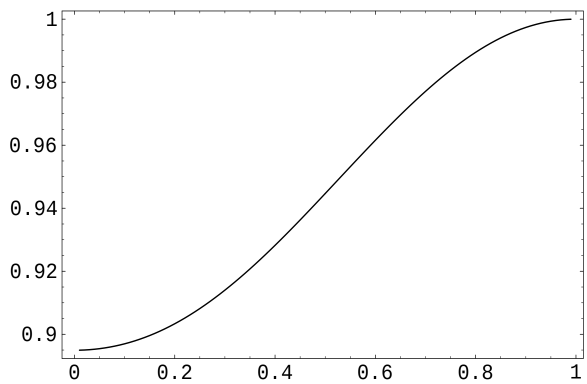

Figure 2: $\mathcal{E} = \mathcal{E}(b)$

This picture shows that for ellipses the function $\mathcal{E} = \mathcal{E}(b)$ is indeed increasing as the intuition suggests. Note that the graph is not symmetric with respect to its middle point $b = 0.5$.

POLYGONS. Let Ω^m be the regular polygon with m ($m \geq 3$) sides circumscribed to $B \subset \mathbb{R}^2$ (the unit ball). The constants and functions relative to Ω^m are given by

$$W_{\Omega^m} = 1 \quad \mathcal{L}(\Omega_t^m) = m(1-t)^2 \tan\frac{\pi}{m} \quad \mathcal{H}(\partial\Omega_t^m) = 2m(1-t)\tan\frac{\pi}{m}.$$

Therefore, (9) yields

$$I_{\mathcal{K}} = -\frac{m}{4}\tan\frac{\pi}{m}\int_0^1 (1-t)^3\,dt = -\frac{m}{16}\tan\frac{\pi}{m}\ . \tag{14}$$

Now Ω^m can be decomposed into $2m$ equivalent right triangles with catheti of lengths 1 and $\tan\frac{\pi}{m}$. Let T_m denote one these triangles, let B_m be its basis (half of one side of Ω^m) and let A_m be the union of its two other sides. Consider the following mixed problem

$$\begin{cases} -\Delta v &= 1 \quad \text{in} \quad T_m \\ v &= 0 \quad \text{on} \quad B_m \\ \dfrac{\partial v}{\partial n} &= 0 \quad \text{on} \quad A_m \end{cases}$$

and denote by v_m its unique solution; by a symmetry argument, the restriction to T_m of the unique solution $\overline{u}_m$ of (7) on Ω^m coincides with v_m. Therefore, by (8) we get

$$J(\overline{u}_m) = -\frac{1}{2}\int_{\Omega^m}\overline{u}_m = -m\int_{T_m} v_m\ ;$$

this, together with (3) and (14), gives

$$\mathcal{E}(m) = \frac{\tan(\frac{\pi}{m})}{16\int_{T_m} v_m}\ .$$

A numerical computation performed with MatLab® gives the following approximated values for the function $E = 1000 \cdot \mathcal{E}(m)$:

m	3	4	5	6	7	8	9	10	20
$E \approx$	834	889	921	942	956	966	972	976	990

Also in this case the map $\mathcal{E} = \mathcal{E}(m)$ is monotonic and, in particular, the equilateral triangle is the regular polygon having the least $\mathcal{E}$.

3. Three open problems

3.1. A problem from functional analysis

As noticed in [6, Remark 6], if $f_{**} \in C^1(\mathbb{R}^+)$ then the (unique) solution $\overline{u}$ of (2) satisfies the following generalized Euler equation

$$\int_\Omega \left(\operatorname{div}\Big[f'_{**}(|\nabla\overline{u}|)\frac{\nabla\overline{u}}{|\nabla\overline{u}|}\Big]+1\right)\phi = 0 \qquad \forall\phi\in\mathcal{K}.$$

For simplicity, we restrict again our attention to the case where $f(s) = s^2/2$ so that we may set the problem in the Hilbert space $H^1_0(\Omega)$ and the previous weak Euler equation becomes

$$\int_\Omega \nabla\overline{u}\nabla\phi = \int_\Omega \phi \qquad \forall\phi\in\mathcal{K}\cap H^1_0(\Omega).$$

In terms of functional analysis, this means that $\overline{u}$ satisfies the Euler equation projected onto the subspace $\mathcal{K}$ (which has empty interior): then one should find out which is the behavior of $\overline{u}$ in the orthogonal complement $\mathcal{K}^\perp$. In particular, it could be of some interest to evaluate

$$\sup_{\|\phi\|=1}\int_\Omega[\nabla\overline{u}\nabla\phi-\phi] \ :$$

this is also an estimate of the error made when approximating (1) with (2). Further, is this supremum attained? Is it attained by a function $\phi\in\mathcal{K}^\perp$?

Of course, before answering to these questions one should characterize the space $\mathcal{K}^\perp$. Let us just mention that in the particular case where $\Omega = B_1$ (the unit ball) the space of web functions $\mathcal{K}$ coincides with the subset $R\subset H^1_0$ of radially symmetric functions and its orthogonal complement can be characterized as follows:

LEMMA 3.1. *The orthogonal complement $R^\perp$ of R in $H^1_0(B_1)$ is given by*

$$R^\perp = \left\{\psi\in H^1_0(B_1);\ \int_{\partial B_t}\psi\,d\mathcal{H} = 0 \ \textit{for a.e.}\ t\in(0,1)\right\} . \qquad (15)$$

Proof. Denote by R' the r.h.s. in (15). We have to show that $\psi \in H_0^1(B_1)$ satisfies $\int_{B_1} \nabla u \cdot \nabla \psi = 0$ for all $u \in R$ if and only if $\psi \in R'$. By a density argument, it is enough to prove that

$$\int_{B_1} \nabla u \cdot \nabla \psi \, dx = 0 \qquad \forall u \in R \cap C_c^\infty(B_1) \tag{16}$$

if and only if $\psi \in R'$. If u belongs to $C_c^\infty(B_1)$, we have $\int_{B_1} \nabla u \cdot \nabla \psi = -\int_{B_1} \Delta u \, \psi$. Moreover, $u \in R$ if and only if $\Delta u \in R$, hence (16) is equivalent to

$$\int_{B_1} \phi(|x|)\psi(x) \, dx = 0 \qquad \forall \phi \in C_c^\infty([0,1)) \ . \tag{17}$$

From the coarea formula we deduce that (17) is equivalent to

$$\int_{B_1} \phi(|x|)\psi(x)dx = \int_0^1 \phi(t) \left[\int_{\partial B_t} \psi(y) d\mathcal{H}(y) \right] dt \quad \forall \phi \in C_c^\infty([0,1)).$$

Then (17) holds if and only if the term in brackets vanishes for a.e. $t \in [0,1]$, that is if and only if $\psi \in R'$. □

3.2. A problem from shape optimization

Assume that $n = 2$ and let $f(s) = s^2/2$; then we know that the unique solution of (1) is also the unique solution of (7). Of course, the ratio defined in (10) depends on the domain Ω, $\mathcal{E} = \mathcal{E}(\Omega)$ and it is not difficult to verify that it is invariant under rescaling, namely $\mathcal{E}(k\Omega) = \mathcal{E}(\Omega)$ for all $k > 0$ and all convex domain $\Omega \ni 0$: in fact, both the numerator and the denominator in (10) are homogeneous of degree 4 under rescaling. Then, in order to have a universal lower bound for $\mathcal{E}(\Omega)$ (independent of Ω) it would be interesting to solve the following problem from optimal design

$$\min_{\Omega \in \mathcal{C}} \mathcal{E}(\Omega)$$

where

$$\mathcal{C} = \{\Omega \subset \mathbb{R}^2; \ \Omega \text{ convex}, \ \mathcal{L}(\Omega) = 1\}.$$

This problem was suggested to us by Buttazzo [1] in occasion of the FGI2000 Conference on Optimization held in Montpellier. The questions are both to evaluate the infimum of $\mathcal{E}(\Omega)$ and to establish if it is attained. We performed some of the numerical experiments described in the previous section in order to have some feeling about this problem.

More generally, the problem may be set in higher dimensions n. Even more general, for any f satisfying (4) does there exist an optimal domain Ω minimizing $\mathcal{E}(\Omega)$? How is it related to the function f? Note that for general f one cannot expect invariance under rescaling.

3.3. A different definition of web functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex set whose barycenter is the origin O and for all $t \in (0,1]$ let

$$\Omega^t = \{x \in \Omega;\ \exists y \in \Omega,\ x = ty\}\ ;$$

these sets are none other than the transformed of Ω by the homothety of ratio t and centered at O. Let

$$\mathcal{K}_* = \{u \in W_0^{1,1}(\Omega);\ u = \text{constant on } \partial\Omega^t\ \forall t \in (0,1]\}\ ,$$

then, $\mathcal{K}_*$ is a closed subset of $W_0^{1,1}(\Omega)$; we have $\mathcal{K}_* = \mathcal{K}$ in some cases (e.g., when Ω is a ball, a regular polyhedra) but in general $\mathcal{K}_* \neq \mathcal{K}$. Therefore, instead of (2) one could consider the problem

$$\min_{u \in \mathcal{K}_*} J(u)\ . \tag{18}$$

The natural question is: which one among (2) and (18) better approximates (1)? Let us mention that if Ω is an ellipsoid and $f(s) = s^2/2$, then the unique solution of (1) belongs to $\mathcal{K}_*$; moreover, the functions in $\mathcal{K}_*$ have regular level lines and perhaps the minimum over $\mathcal{K}_*$ (if it exists) is a smooth function. On the other hand, an argument in favour of $\mathcal{K}$ are the results in [2, 7] where is given a class of functions f for which (1) admits a solution which is a linear function of $d(x, \partial\Omega)$ (i.e. a web function).

The approach for proving the existence of a solution of (18) and finding an explicit form of it should be the following. Let $\rho(x) \doteq \inf\{\lambda > 0;\ x \in \lambda\Omega\}$ be the gauge function of the convex set Ω, and define $\delta(x) \doteq 1 - \rho(x)$. The function δ, which plays the role of the distance from the boundary when dealing with $\mathcal{K}$, is concave on $\overline{\Omega}$ and vanishes on $\partial\Omega$. The set $\mathcal{K}^*$ is none other than the set of all functions $u \in W_0^{1,1}(\Omega)$ depending only on δ, that is $u(x) = \phi(\delta(x))$ for some $\phi\colon [0,1] \to \mathbb{R}$, with $\phi(0) = 0$; then $\nabla u(x) = \phi'(\delta(x))\nabla\delta(x)$ and from the coarea formula we get

$$\begin{aligned} J(u) &= \int_\Omega \Big[f\Big(|\phi'(\delta(x))|\,|\nabla\delta(x)|\Big) - \phi(\delta(x))\Big]\, dx \\ &= \int_0^1 dt \int_{\{\delta=t\}} \left(\frac{f(|\phi'(t)|\,|\nabla\delta(y)|)}{|\nabla\delta(y)|} - \frac{\phi(t)}{|\nabla\delta(y)|}\right) d\mathcal{H}(y) \qquad (19) \\ &= \int_0^1 [\tilde{f}(t, |\phi'(t)|) - \lambda(t)\phi(t)]\, dt =: I(\phi) \end{aligned}$$

where

$$\tilde{f}(t,s) = \int_{\{\delta=t\}} \frac{f(s|\nabla\delta(y)|)}{|\nabla\delta(y)|}\, d\mathcal{H}(y), \quad \lambda(t) = \int_{\{\delta=t\}} \frac{1}{|\nabla\delta(y)|}\, d\mathcal{H}(y).$$

Let $\tilde{f}_*$ denote the polar function of the normalized non-decreasing bipolar function $\tilde{f}^{**}$ w.r.t. s; under suitable assumptions on $\tilde{f}$, any minimizer ϕ of I should satisfy the Euler–Lagrange differential inclusion

$$\phi'(t) \in \partial_s \tilde{f}_*(t, \Lambda(t)) \quad \text{for a.e. } t \in [0,1]$$

where $\partial_s \tilde{f}_*(t,\Lambda(t))$ denotes the (partial w.r.t. s) subdifferential of $\tilde{f}_*$ evaluated at $s = \Lambda(t)$ and

$$\Lambda(t) \doteq \int_t^1 \lambda(s)\, ds = \mathcal{L}(\{\delta > t\}) = \mathcal{L}(\Omega^t)\ .$$

If $\tilde{f}_*(t,s)$ is differentiable w.r.t. s in $(t,\Lambda(t))$ for a.e. $t \in [0,1]$ we get the explicit form of u:

$$u(x) = \int_0^{\delta(x)} \frac{\partial \tilde{f}_*}{\partial s}(t, \Lambda(t))\, dt\ .$$

Let us perform the explicit computations in the case of the ellipse $\Omega = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2/b^2 < 1\}$ and $f(s) = s^2/2$ (in this case,

$f = f^* = f^{**} = f_{**} = f_*)$. We have

$$\delta(x,y) = 1 - \sqrt{x^2 + \frac{y^2}{b^2}}, \qquad \tilde{f}(t,s) = \pi \frac{1+b^2}{2b}(1-t)s^2,$$

$$\Lambda(t) = \pi b(1-t)^2,$$

hence

$$u(x,y) = \frac{b^2}{1+b^2} \int_0^{\delta(x,y)} (1-t)\, dt = \frac{b^2 - b^2x^2 - y^2}{2(1+b^2)} \ ,$$

which is also the minimizer of J over $W_0^{1,1}(\Omega)$.

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