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On $\Gamma$-convergence in Anisotropic Orlicz-Sobolev Spaces

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SUMMARY. - In this paper we consider $\Gamma$-convergence for a class of lower semicontinuous functionals defined on Orlicz-Sobolev spaces. Particularly we prove compactness results for these type of functionals. Moreover, we compare $\Gamma$-convergence and convergence of minima.

Introduction

In the study of variational problems in applied mathematics the concept of variational convergence called $\Gamma$-convergence has come to be a very important tool. One reason is its compactness properties for general classes of functionals and topologies. In addition almost all other variational convergences follow as consequences of the $\Gamma$-convergence. For an introduction to the theory we refer to Dal Maso [9].

In this paper we study $\Gamma$-convergence for a class of lower semicontinuous functionals defined on the Orlicz-Sobolev class $W^1_{LG}(\Omega)$ defined below. There are many advantages of such a development. The analysis in Orlicz-Sobolev spaces uses properties like convexity and growth ($\Delta_2$-property) in such a way that one can obtain variational solutions to larger classes of nonlinear problems than in usual Sobolev spaces, see e.g. [5].

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The paper is organized as follows: In Section 1 we give some preliminary results on $\Gamma$-convergence and on Orlicz-Sobolev spaces. The main results are presented in Section 2. In particular we prove a $\Gamma$-compactness result (Theorem 2.2) for functionals defined on $W^1L^G(\Omega)$. The framework uses the localization method as presented in [9]. We also compare $\Gamma$-convergence and convergence of minima (Theorem 2.3 and Theorem 2.4). Section 3 is devoted to the proof of Theorem 2.2 and contains in particular an Orlicz-space version of the fundamental estimate. In Section 4, finally, we give some concluding remarks.

1. Preliminary results

Let $X$ be a topological space and let $\mathcal{N}(x)$ denote the set of all open neighborhoods of $x \in X$. Further, let $\{F_h\}$ be a sequence of functions from $X$ into $\overline{\mathbb{R}}$.

**Definition 1.1.** The $\Gamma$-lower and $\Gamma$-upper limits of the sequence $\{F_h\}$ are the functions from $X$ into $\overline{\mathbb{R}}$ defined by

$$F^\prime(x) = \Gamma - \lim \inf_{h \to \infty} F_h(x) = \sup_{\omega \in \mathcal{N}(x)} \lim \inf_{h \to \infty} \inf_{z \in \omega} F_h(z)$$

and

$$F^{\prime\prime}(x) = \Gamma - \lim \sup_{h \to \infty} F_h(x) = \sup_{\omega \in \mathcal{N}(x)} \lim \sup_{h \to \infty} \inf_{z \in \omega} F_h(z),$$

respectively. If these two limits coincide, i.e. if there exists a unique function $F : X \to \overline{\mathbb{R}}$ such that

$$F = \Gamma - \lim \inf_{h \to \infty} F_h(x) = \Gamma - \lim \sup_{h \to \infty} F_h(x),$$

we say that the sequence $\{F_h\}$ $\Gamma$-converges to $F$.

**Remark 1.2.** By the definition it is obvious that $\{F_h\}$ $\Gamma$-converges to $F$ if and only if

$$\Gamma - \lim \sup_{h \to \infty} F_h \leq F \leq \Gamma - \lim \inf_{h \to \infty} F_h.$$
This means that \( \Gamma \)-convergence and lower semicontinuity are closely related concepts. We have the following sequential characterization of \( \Gamma \)-convergence, see [9, Proposition 8.1]:

**Theorem 1.3.** Let \( X \) be a separable metric space and let \( \{ F_h \} \) be a sequence of functionals from \( X \) into \( \mathbb{R} \). Then

(i) for every \( x \in X \) and for every sequence \( \{ x_h \} \) converging to \( x \),

\[ F'(x) \leq \liminf_{h \to \infty} F_h(x_h); \]

(ii) for every \( x \in X \) there exists a sequence \( \{ x_h \} \) converging to \( x \) such that

\[ F'(x) = \liminf_{h \to \infty} F_h(x_h); \]

(iii) for every \( x \in X \) and for every sequence \( \{ x_h \} \) converging to \( x \),

\[ F''(x) \leq \limsup_{h \to \infty} F_h(x_h); \]

(iv) for every \( x \in X \) there exists a sequence \( \{ x_h \} \) converging to \( x \) such that

\[ F''(x) = \limsup_{h \to \infty} F_h(x_h). \]

Consequently \( \{ F_h \} \) \( \Gamma \)-converges to a function \( F \in X \) if and only if

(v) for every \( x \in X \) and for every sequence \( \{ x_h \} \) converging to \( x \),

\[ F(x) \leq \liminf_{h \to \infty} F_h(x_h) \]

and

(vi) for every \( x \in X \) there exists a sequence \( \{ x_h \} \) converging to \( x \) such that

\[ F(x) = \lim_{h \to \infty} F_h(x_h) \]

Moreover, \( \Gamma \)-convergence enjoys the following compactness property, see [9], Theorem 8.5:
THEOREM 1.4. Let $X$ be a separable metric space. Then every sequence $\{F_n\}$ of functionals from $X$ into $\mathbb{R}$ has a $\Gamma$-convergent subsequence.

We recall that a Young function $A : [0, \infty) \to [0, \infty]$ is a function of the form

$$A(t) = \int_0^t a(x)\,dx$$

where the function $a : [0, \infty) \to [0, \infty]$ is increasing, left continuous and not identically zero and not identically infinity on the interval $(0, \infty)$.

The Orlicz space $L_A(\Omega)$ is the set of measurable functions $f$ on $\Omega$ such that $\|f\|_{A,\Omega} < \infty$, where

$$\|f\|_{A,\Omega} = \inf \left\{ \theta > 0 : \int_{\Omega} A \left( \frac{|f(x)|}{\theta} \right) \,dx \leq 1 \right\}$$

(the Luxemburg norm on $L_A(\Omega)$)

A $G$-function $G : \mathbb{R}^m \to [0, \infty]$ is a function with the following properties:

(i) $G(0) = 0$;
(ii) $\lim_{|x|\to\infty} G(x) = \infty$, $\left[ x \in \mathbb{R}^m : |x| = \left( \sum_{i=1}^m x_i^2 \right)^{1/2} \right]$;
(iii) $G$ is convex
(iv) $G$ is symmetric i.e. $G(-x) = G(x)$, $x \in \mathbb{R}^m$;
(v) the set $G^{-1}(\infty) = \{ x \in \mathbb{R}^m : G(x) = \infty \}$ is separated from $0$;
(vi) $G$ is lower semi-continuous.

Additionally we will assume that $G$ is monotonically increasing in each variable separately, that $G$ and $G^*$ (the convex polar) satisfies $\Delta_2$ condition (this will guarantee that the separability and reflexivity of function spaces defined below, see [6]). The vector valued Orlicz-space $L_G(\Omega)$ is defined as follows:

Let $G$ be a $G$-function and let $\Omega$ be a domain in $\mathbb{R}^n$, let $u_1, u_2, \ldots, u_m$ be real valued measurable functions defined on $\Omega$ and let $u =$
(u_1, u_2, \ldots, u_m) be a vector valued function. Then, u is said to belong to \( L_G(\Omega) \) if there exists a \( \lambda > 0 \) such that
\[
\int_\Omega G(\lambda u(x)) < \infty.
\]

The space \( L_G(\Omega) \) is equipped with a norm corresponding to the Luxemburg norm given by
\[
\|u\|_{G, \Omega} = \inf \left\{ \theta > 0 : \int_\Omega G \left( \frac{u}{\theta} \right) \, dx \leq 1 \right\}.
\]

There should not be any ambiguity for the same notations \( L_A(\Omega) \) and \( L_G(\Omega) \) used for Young function and \( G \)-function, respectively.

For a \( G \)-function \( G \), the complementary function \( G^*_+ \) is defined by
\[
G^*_+(u) = \sup_{v \geq 0} (u \cdot v - G(v)),
\]
where \( u \cdot v = \sum_{i=1}^m u_i v_i \).

Let \( G \) be a \( G \)-function of \((n+1)\) variables. The anisotropic Orlicz-Sobolev space, denoted by \( W^1 L_G(\Omega) \), is defined to be the space of weakly differentiable functions \( u \) for which
\[
(u, Du) = (u, D_1 u, D_2 u, \ldots, D_n u)
\]
belongs to \( L_G(\Omega) \). A norm for the space \( W^1 L_G(\Omega) \) is given by
\[
\|u\| = \|(u, Du)\|_{G, \Omega}.
\]

For further details regarding Orlicz-Sobolev spaces we refer to the monographs [1] and [6].

Given two functions \( A \) and \( B \), the notation \( A \ll B \) means that for every \( \lambda > 0 \)
\[
\lim_{t \to \infty} \frac{A(t)}{B(\lambda t)} = 0.
\]

Let us recall the following imbedding result (see [4]).

**Theorem 1.5.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with the cone property, let \( f \) be a continuous non-negative function on \([0, \infty)\) and let \( G \) be a \( G \)-function of \((n+1)\) variables on \([0, \infty)\) such that
\[
G^*_+(0, f(s), f(s), \ldots, f(s)) \leq s.
\]
Furthermore, let $A$ be a Young function given by
\[
A^{-1}(|t|) = \frac{1}{\eta} \int_{0}^{t} \frac{f^{-1}(s)}{s^{1/n}} ds
\]
for some constant $\eta > 0$. If $B$ is a Young function such that $B \ll A$, then $W^{1}L_{G}(\Omega)$ is compactly imbedded in $L_{B}(\Omega)$.

2. The main results

Let the function $G$ be defined as above and let us define $G_{0}$ and $B$ as
\[
G_{0}(\xi_{1}, \xi_{2}, \ldots, \xi_{n}) = G(0, \xi_{1}, \xi_{2}, \ldots, \xi_{n})
\]
and
\[
B(u) = G(u, u, \ldots, u),
\]
respectively, where we assume that $B$ satisfies all the hypotheses of Theorem 1.3 above. We have the following compactness result:

**Theorem 2.1.** Suppose that $G$ satisfies the $\Delta_{2}$-condition. Then every sequence of functionals $F_{h} : L_{B}(\Omega) \to \mathbb{R}$ has a $\Gamma(L_{B})$-convergent subsequence.

**Proof.** Since $G$ satisfies the $\Delta_{2}$-condition, $L_{B}(\Omega)$ is separable, see e.g. Kučer et. al. [6], and thus the result follows from the compactness Theorem 1.2 above.

Let us now define the space $\mathcal{M} = \mathcal{M}(c, \beta)$ of Caratheodory functions $f : \Omega \times \mathbb{R}^{n} \to [0, +\infty)$ satisfying the conditions:

1. $f(x, \xi)$ is convex in $\xi$.
2. $G_{0}(\xi_{1}, \ldots, \xi_{n}) \leq f(x, \xi) \leq c(1 + G_{0}(\xi_{1}, \ldots, \xi_{n})).$
3. $G$ satisfies the $\Delta_{2}$-condition with constant $\beta$.

Let us also define the class $\mathcal{F}(\mathcal{M})$ of functionals $F : L_{B}(\Omega) \times \mathcal{A}(\Omega) \to [0, +\infty)$ given by
\[
F(u, A) = \int_{A} f(x, Du(x)) dx,
\]
for \( f \in \mathcal{M} \) and \( A \in \mathcal{A}(\Omega) \), where \( \mathcal{A}(\Omega) \) denotes the family of all open subsets of \( \Omega \). We extend in the usual way the functionals to \( +\infty \) on \( L_B(\Omega) \setminus W^1L_C(\Omega) \).

The main objective is now to establish a result which says that the \( \Gamma \)-limit of a sequence

\[
F_h(u, A) = \int_A f_h(x, Du(x))dx,
\]

in \( \mathcal{F}(\mathcal{M}) \) has an integral representation

\[
F_0(u, A) = \int_A \varphi(x, Du(x))dx,
\]

where also \( \varphi \in \mathcal{M} \).

The main result of this paper is the following compactness result:

**Theorem 2.2.** For every sequence \( \{F_h\} \) in \( \mathcal{F}(\mathcal{M}) \) there exists a subsequence \( \{F_{h_k}\} \) and a functional \( F_0 \in \mathcal{F}(\mathcal{M}) \) such that \( F_{h_k}(\cdot, A) \to \Gamma(L_B) \)-converges to \( F_0 \) for every \( A \in \mathcal{A}(\Omega) \).

**Remark 2.3.** \( F_0(u, \cdot) \) is the restriction of a Borel measure to \( \mathcal{A}(\Omega) \) and moreover, the local property of the \( \Gamma \)-limit shows that in the integral representation (2.1) the function \( \varphi \in \mathcal{M} \) is independent of \( A \).

**Remark 2.4.** By the definition of \( \Gamma \)-convergence it easily follows that

(i) \( F_0 \) is lower semicontinuous.

(ii) If \( H \) is continuous, then

\[
F_0 + H = \Gamma(L_B) - \lim_{h \to 0} F_h + H.
\]

Theorem 2.2 will be proven in the next section. We end this section by giving examples of the relationship between \( \Gamma \)-convergence and convergence of minima. Let \( F_h \) and \( F \) belong to \( \mathcal{F}(\mathcal{M}) \) and let \( H : L_B(\Omega) \to \mathbb{R} \) be a continuous functional with the property that there exist some constants \( c > 0 \) and \( b \in \mathbb{R} \) such that
\[ H(u) \geq c \int_\Omega B(u(x))dx - b \]  

(2)

for all \( u \in L_B(\Omega) \). Let us put

\[ m_h = \inf_{u \in W^{1,L_G}(\Omega)} \{ F_h(u) + H(u) \} \]  

(3)

and

\[ m = \inf_{u \in W^{1,L_G}(\Omega)} \{ F(u) + H(u) \} \]  

(4)

**Theorem 2.5.** If \( \{ F_h \} \) \( \Gamma \)-converges to \( F \) in \( L_B(\Omega) \) then \( m_h \) converges to \( m \).

*Proof.* We recall that for any topological vector space \( X \) it holds that

\[ \min_{x \in X} F(x) = \liminf_{x \in X} F_h(x) \]  

(5)

whenever \( \{ F_h \} \) is a \( X \)-equi-coercive sequence of functionals which \( \Gamma \)(\( X \))-converges to \( F \) (see e.g. [9] Theorem 7.8). The minima in (3) and (4) can be taken over \( L_B(\Omega) \) instead of \( W^{1,L_G}(\Omega) \). Moreover, by Remark 3, \( \{ F_h + H \} \) \( \Gamma \)-converges to \( \{ F + H \} \) in \( L_B(\Omega) \). It holds that

\[ F_h + H \geq k_1 \Psi - k_2 \]

for some positive constants \( k_1 \) and \( k_2 \), where

\[ \Psi(u) = \begin{cases} \int_\Omega G(u(x), Du(x)) dx & \text{if } u \in W^{1,L_G}(\Omega) \\ +\infty & \text{otherwise} \end{cases} \]

This follows from the fact that

\[
G(u(x), Du(x)) = G\left( \frac{1}{2} (2u(x)) + \frac{1}{2} \nabla 0 \right) + \frac{1}{2} (2Du(x))
\]

\[
\leq G\left( 2 \frac{1}{2} (u(x), 0, \ldots, 0) \right) + \frac{1}{2} G(2(0, Du(x)))
\]

\[
\leq \beta(G((u(x), 0, \ldots, 0)) + G((0, Du(x))))
\]

\[
\leq c \left( G((u(x), \ldots, u(x))) + G(Du(x)) \right)
\]
Moreover, we observe that $\Psi(u) \leq 1$ if $\|u\| \leq 1$ (by the definition of the Luxemburg norm) and that $\|u\| \leq \Psi(u)$ if $1 < \|u\|$ (use that by the definition of the Luxemburg norm and by convexity $1 < \Psi(\frac{t}{\theta}) \leq \theta^{-1} \Psi(u)$ for all $1 < \theta < \|u\|$). Thus, the set $\{u : \Psi(u) \leq t\}$ is bounded in $W^1 L_G(\Omega)$ for all $t > 0$. Moreover, by the imbedding result Theorem 1.5, it holds that $\{u : \Psi(u) \leq t\}$ is compact in $L_B(\Omega)$ which implies that the sequence $\{F_h + H\}$ is equi-coercive in $L_B(\Omega)$. Consequently we obtain that $m_h \rightarrow m$ by replacing $X$ by $L_H(\Omega)$, $F_h$ by $F_h + H$ and $F$ by $F + H$ in (5). \[\square\]

**Theorem 2.6.** Assume that all hypotheses are satisfied as in Theorem 2.5 except that 2 is replaced by the assumption that there exists a bounded set $U$ in $W^1 L_G(\Omega)$ such that

$$\inf_{u \in W^1 L_G(\Omega)} \{F_h(u) + H(u)\} = \inf_{u \in U} \{F_h(u) + H(u)\}$$

for all $h$. Then, if $\{F_h\}$ $\Gamma$-converges to $F$ in $L_B(\Omega)$ it holds that $m_h$ converges to $m$.

**Proof.** We recall that for any topological vector space $X$ it holds that

$$\min_{x \in X} F(x) = \lim_{x \in X} \inf F_h(x)$$

whenever $\{F_h\}$ $\Gamma(X)$-converges to $F$ and there exists a compact set $K$ such that

$$\inf_{x \in X} \{F_h(u)\} = \inf_{x \in K} \{F_h(u)\}$$

for all $h$ (see [9], Theorem 7.4.). Minimizing over $X = L_B(\Omega)$, and $K = \overline{U}$ and replacing $F_h$ by $F_h + H$ and $F$ by $F + H$ in the desired result.

\[\square\]

3. Some results related to Theorem 2.2 and its proof

The proof of Theorem 2.2 will be divided into a number of lemmas. Inspired by the pedagogical presentation in Dal Maso [9] we will establish the result by using localization and by proving that functionals $F \in \mathcal{F}(\mathcal{M})$ satisfies the fundamental estimate in Orlicz-Sobolev spaces. A necessary condition for the integral representation (2.1) is
that $F_0(u, \cdot)$ is a measure. For this purpose we introduce increasing set functions:

**Definition 3.1.** A set function $\sigma : \mathcal{A}(\Omega) \to [0, +\infty]$ is called

(i) an increasing set function if $\sigma(\emptyset) = 0$ and $\sigma(A_1) \leq \sigma(A_2)$ for $A_1 \subset A_2$.

(ii) subadditive if

$$\sigma(A_1 \cup A_2) \leq \sigma(A_1) + \sigma(A_2),$$

for all $A_1, A_2 \in \mathcal{A}(\Omega)$.

(iii) superadditive if

$$\sigma(A_1 \cup A_2) \geq \sigma(A_1) + \sigma(A_2),$$

for all $A_1, A_2 \in \mathcal{A}(\Omega)$ with $A_1 \cap A_2 = \emptyset$.

(iv) inner regular if

$$\sigma(A) = \sup\{\sigma(B) : B \in \mathcal{A}(\Omega), B \subseteq A\},$$

for all $A \in \mathcal{A}(\Omega)$.

**Lemma 3.2.** Let $\sigma : \mathcal{A}(\Omega) \to [0, +\infty]$ be an increasing set function. The following statements are equivalent:

(1) $\sigma$ is is the restriction to $\mathcal{A}(\Omega)$ of a Borel measure on $\Omega$;

(2) $\sigma$ is subadditive, superadditive and inner regular;

(3) the set function

$$\nu(E) = \inf\{\sigma(A) : A \in \mathcal{A}(\Omega), E \subseteq A\}$$

is a Borel measure on $\Omega$.

**Proof.** See e.g. [9, Theorem 14.23].
We will use the properties of increasing set functions to obtain the integral representation of the $\Gamma$-limit $F_0$. We begin with

**Lemma 3.3.** Let $\{F_h\}$ be a sequence of functionals in $\mathcal{F}(\mathcal{M})$. Suppose that for every $u \in W^1L_G(\Omega)$

$$\sigma'(A) = \Gamma(L_B) - \liminf_h F_h(u, A)$$

and

$$\sigma''(A) = \Gamma(L_B) - \limsup_h F_h(u, A)$$

define inner regular increasing set functions. Then there exists a subsequence $\{F_{h_k}(u, A)\}$ which $\Gamma(L_B)$-converges for all $u \in W^1L_G(\Omega)$ and $A \in \mathcal{A}(\Omega)$.

**Proof.** Consider the countable family $\mathcal{R}$ of all finite unions of open rectangles of $\Omega$ with rational vertices. For every fixed sequence $\{F_h\}$ we can use a diagonal procedure and Theorem 2.1 to extract a subsequence $\{F_{h_k}(u, R)\}$ which $\Gamma(L_B)$-converges for all $R \in \mathcal{R}$ and $u \in W^1L_G(\Omega)$. Now, let $A \in \mathcal{A}(\Omega)$ and $u \in W^1L_G(\Omega)$. By hypothesis $\sigma'(A)$ and $\sigma''(A)$ define inner regular increasing set functions. This gives

$$\Gamma(L_B) - \liminf_h F_{h_k}(u, A) = \sigma'(A) =$$

$$= \sup\{\sigma'(B) : B \in \mathcal{A}(\Omega), B \subset A\}$$

$$= \sup\{\sigma'(R) : R \in \mathcal{R}(\Omega), R \subset A\}$$

$$= \sup\{\sigma''(R) : R \in \mathcal{R}(\Omega), R \subset A\}$$

$$= \sup\{\sigma''(B) : B \in \mathcal{A}(\Omega), B \subset A\}$$

$$= \sigma''(A) = \Gamma(L_B) - \limsup_h F_{h_k}(u, A).$$

We proceed by proving a fundamental estimate in $L_B$ which will guarantee that the $\Gamma$-limits define inner regular increasing set functions.

**Definition 3.4.** We say that $F$ satisfies the $L_B$-fundamental estimate if for every $A$, $A'$ and $B$ in $\mathcal{A}(\Omega)$ with $A' \subset A$ and $\alpha > 0$ there exists $M_\alpha > 0$ such that for all $u$, $v \in W^1L_G(\Omega)$ there exists a cut-off function $\psi$ between $A'$ and $A$ such that
\[ F(\psi u + (1 - \psi) v, A' \cup B) \leq (1 + \alpha)(F(u, A) + F(v, B)) + \\
+ M_\alpha \int_{(A \cap B) \setminus A'} B(u - v) \, dx + \alpha. \]

Moreover, we say that the class \( \mathcal{F}(\mathcal{M}) \) satisfies the \( L_H \)-fundamental estimate uniformly if every functional \( F \in \mathcal{F}(\mathcal{M}) \) satisfies the fundamental estimate and the constant \( M_\alpha > 0 \) can be chosen uniformly on \( \mathcal{F}(\mathcal{M}) \).

**Remark 3.5.** Let \( A, A' \in \mathcal{A}(\Omega) \) with \( A' \subset \subset A \). We say that \( \psi \) is a cut-off function between \( A' \) and \( A \) if \( \psi \) is smooth with compact support in \( A, 0 \leq \psi \leq 1 \) and \( \psi \equiv 1 \) on \( A' \).

**Lemma 3.6.** The class \( \mathcal{F}(\mathcal{M}) \) satisfies the \( L_H \)-fundamental estimate uniformly.

**Proof.** Let \( F \in \mathcal{F}(\mathcal{M}) \) and let \( A, A' \) and \( B \) in \( \mathcal{A}(\Omega) \) with \( A' \subset \subset A \). Define
\[
\delta = \text{dist}(A', \partial A)
\]
and take \( 0 < \eta < \delta \) and \( 0 < r < \delta - \eta \). Let \( \psi \) be a cut-off function between
\[
\{x \in A : \text{dist}(x, A') < r\} \text{ and } \{x \in A : \text{dist}(x, A') < r + \eta\},
\]
with \( |D\psi| \leq 2/\eta \). Define the sets
\[
B^\eta_1 = \{x \in B : r < \text{dist}(x, A') < r + \eta\},
\]
\[
I_1 = \{x \in B : \text{dist}(x, A') \geq r + \eta\}
\]
and
\[
I_2 = \{x \in A' \cup B : \text{dist}(x, A') \leq r\}.
\]
For \( u, v \in W^1L_G(\Omega) \) a repeated use of the convexity and the \( \Delta_2\)-
property of $G$ yield
\[
F(\psi u + (1 - \psi)v, A' \cup B) \\
= \int_{A' \cup B} f(x, \psi Du + (1 - \psi)Dv + (u - v)D\psi)dx \\
= \int_{1_1} f(x, Dv)dx + \int_{1_2} f(x, Du)dx + \int_{B^2} f(x, \psi Du + (1 - \psi)Dv + (u - v)D\psi)dx \\
\leq F(u, A) + F(v, B) + c \int_{B^2} (1 + G_0(\psi Du + (1 - \psi)Dv + (u - v)D\psi))dx \\
\leq F(u, A) + F(v, B) + c \int_{B^2} (1 + \beta G_0(\frac{1}{2}(\psi Du + (1 - \psi)Dv)) + \frac{1}{2}(u - v)D\psi))dx \\
\leq F(u, A) + F(v, B) + \\
c \int_{B^2} (1 + \frac{\beta \psi}{2} G_0(Du) + \frac{\beta (1 - \psi)}{2} G_0(Dv))dx + \\
\int_{B^2} \frac{\beta \psi}{2} G_0((u - v)D\psi/|D\psi|)dx \\
\leq F(u, A) + F(v, B) + \frac{c \beta}{2} \int_{B^2} (1 + G_0(Du) + G_0(Dv))dx \\
+ \frac{c \beta}{2} \int_{(A \cap B) \setminus A'} G_0((u - v)D\psi/|D\psi|)dx \\
\leq F(u, A) + F(v, B) + \frac{c \beta}{2} \int_{B^2} (1 + G_0(Du) + G_0(Dv))dx \\
+ \frac{c \beta}{2} \int_{(A \cap B) \setminus A'} B(u - v)dx
\]
where $\kappa = 1 - \frac{\log \eta}{\log 2}$. Now define
\[
\mu(U) = \frac{c \beta}{2} \int_U (1 + G_0(Du) + G_0(Dv))dx.
\]
By the structure conditions
\[\mu(A \cap B) \leq \frac{c \beta}{2}(m(A \cap B) + F(u, A) + F(v, B)).\]
Moreover, for every \(N = 1, 2, \ldots\),
\[\mu(A \cap B) \geq \sum_{k=1}^{N} \mu(\{x \in B : \delta \frac{k-1}{N} < \text{dist}(x, A') < \delta \frac{k}{N}\}).\]
Consequently, for every \(N = 1, 2, \ldots\) there exists \(k \in \{1, \ldots, N\}\) such that
\[\mu(\{x \in B : \delta \frac{k-1}{N} < \text{dist}(x, A') < \delta \frac{k}{N}\}) \leq \frac{c \beta}{2N}(m(A \cap B) + F(u, A) + F(v, B)).\]
Hence, for fixed \(\alpha > 0\), by choosing
\[N \geq \frac{1}{\alpha} \max\{\frac{c \beta}{2}m(A \cap B), \frac{c \beta}{2}\}, \quad \eta = \delta \frac{k-1}{N} \quad \text{and} \quad r = \delta \frac{k}{N},\]
we obtain
\[M_\alpha = \frac{c \beta \alpha}{2},\]
which depends only on \(A, A', B, c\) and \(\beta\) and can thus be chosen uniformly in the class \(\mathcal{F}(\mathcal{M})\).

In the next two lemmas we apply the fundamental estimate to show that the \(\Gamma\)-limits satisfies the measure properties subadditivity and inner regularity.

**Lemma 3.7.** Let \(\{F_h\}\) be a sequence in \(\mathcal{F}(\mathcal{M})\) which satisfies the \(L_B\)-fundamental estimate as \(h \to \infty\). Then
\[F'(u, A' \cup B) \leq F'(u, A) + F''(u, B)\]
and
\[F''(u, A' \cup B) \leq F''(u, A) + F'''(u, B),\]
for all \(u \in W^{1}L_{C}(\Omega)\) and \(A, A'\) and \(B\) in \(\mathcal{A}(\Omega)\) with \(A' \subset \subset A\).
Proof. By Theorem 1.1 there exists two sequences \( \{u_h\} \) and \( \{v_h\} \) converging to \( u \) strongly in \( L_B(\Omega) \) such that

\[
F'(u, A) = \liminf_h F_h(u_h, A)
\]

and

\[
F''(u, B) = \limsup_h F_h(v_h, B).
\]

If we now apply the \( L_B \)-fundamental estimate as \( h \to \infty \) to the functions \( u_h \) and \( v_h \) with fixed \( \alpha > 0 \), there exist \( M_\alpha \) and \( h_\alpha \) such that for all \( h > h_\alpha \) there exists a sequence of functions

\[
w_h = \psi_h u_h + (1 - \psi_h)v_h,
\]

where \( \psi_h \) are cut-off functions between \( A' \) and \( A \) such that

\[
F_h(w_h, A' \cup B) \leq (1 + \alpha)(F(u_h, A) + F(v_h, B)) + M_\alpha \int_{(A \cap B) \setminus A'} B(u_h - v_h) \, dx + \alpha,
\]

Now \( w_h \to u \) in \( L_B(\Omega) \). Moreover, since convergence in \( L_B(\Omega) \) implies \( B \)-mean convergence, see e.g. Kufner et. al. [6], p. 157, it follows that

\[
\int_{(A \cap B) \setminus A'} B(u_h - v_h) \, dx \to 0.
\]

Consequently,

\[
F'(u, A' \cup B) \leq \liminf_h F_h(w_h, A' \cup B)
\]

\[
\leq (1 + \alpha)(\liminf_h F_h(u_h, A) + \liminf_h F_h(v_h, B)) + \alpha
\]

\[
= (1 + \alpha)(F'(u, A) + F''(v, B)) + \alpha.
\]

Since \( \alpha \) can be chosen arbitrarily the first inequality follows. The second inequality is proved the same way.

The last lemma concerns inner regularity of the \( \Gamma \)-limits.
LEMMA 3.8. Let \( \{ F_h \} \), \( F' \) and \( F'' \) be defined as in Lemma 3.4. Let 
\( u \in W^1 L_G(\Omega) \). If \( F'(u, \cdot) \) and \( F''(u, \cdot) \) are increasing set functions and if 
\[
F''(u, A) \leq \tilde{C} \int_A (1 + G_0(Du)) dx,
\]
for all \( A \in \mathcal{A}(\Omega) \), then \( F'(u, \cdot) \) and \( F''(u, \cdot) \) are inner regular and moreover \( F''(u, \cdot) \) is subadditive.

Proof. Since \( \{ F_h \} \) satisfies the \( L_B \)-fundamental estimate the proof follows along the line of Proposition 11.6 in [2], by taking Lemma 3.4 into account.

Proof of Theorem 2.2. We extend as above the functionals to \( +\infty \) on 
\( L_B(\Omega) \setminus W^1 L_G(\Omega) \). By Lemma 3.3 \( \{ F_h \} \) satisfies the \( L_B \)-fundamental estimate. Therefore, by Lemma 3.5, the \( \Gamma \)-lower and \( \Gamma \)-upper limits define inner regular increasing set functions. Compactness thus follows from Lemma 3.2 and the measure properties again follows from Lemma 3.5 if we take Lemma 3.1 into account.

4. Some final comments and concluding remarks

Theorem 2.2 opens the possibility to find representations of the \( \Gamma \)-limit for large classes of interesting problems. In particular in the periodic case, i.e. when \( f_h \) is of the form 
\[
f_h(x, \xi) = f(hx, \xi),
\]
it is possible, with the obvious modifications, to apply classical homogenization methods analogous to those presented in for instance Dal Maso [9]. Moreover, for the case when \( f_h \) is of the form 
\[
f_h(x, \xi) = f(x, hx, \ldots, h^m x, \xi),
\]
one can mimic the reiterated homogenization techniques presented in [3] and obtain homogenization results. Similar compactness and homogenization results are clearly also obtainable for corresponding nonlinear parabolic operators by combining the compactness result in this paper with the G-convergence and multi-scale convergence methods described in e.g. [11, 10, 7, 8, 12, 13]. These interesting questions will be discussed in a forthcoming paper.
REFERENCES


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