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Manifold Spines and Hyperbolicity Equations

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Dedicated to the memory of our dear friend Marco Reni

Summary. - *We give a combinatorial representation of compact connected orientable* 3*-dimensional manifolds with boundary and their special spines by a class of graphs with extrastructure which are strictly related to o-graphs defined and studied in [3] and [4]. Then we describe a simple algorithm for constructing the boundary of these manifolds by using a list of* 6*-tuples of non-negative integers. Finally we discuss some combinatorial methods for determining the hyperbolicity equations. Examples of hyperbolic* 3*-manifolds of low complexity illustrate in particular cases the constructions and algorithms presented in the paper.*

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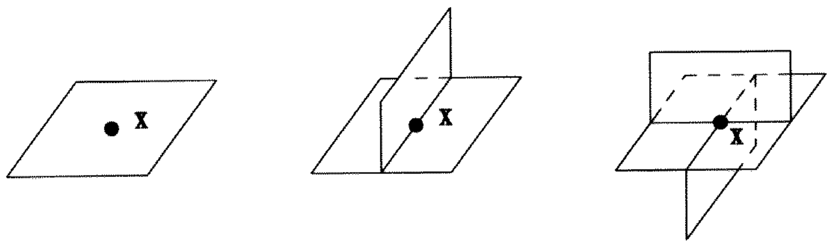
1. Introduction

A compact (orientable) 3-manifold M with boundary can be constructed in the following way: take a finite collection of disjoint 3-simplexes in the standard 3-space; identify their faces pairwise by (orientation-reversing) isometries; remove regular neighbourhoods of the vertices (the images of the vertices of the 3-simplexes) from the resulting quotient space. In other words, M is obtained by gluing together 3-simplexes with truncated vertices. So M is also called a *gluing manifold*. In this paper we describe a combinatorial representation of gluing 3-manifolds and their special spines via certain graphs (strictly related to o-graphs [3], [4]) encoded by 5-tuples of non-negative integers. This allows to study compact 3-manifolds (and determine their topological invariants) by a computer. Then we give an algorithm for constructing the boundary of M directly from the graph of the corresponding gluing. The algorithm produces a list of 6-tuples of non-negative integers which completely encodes a triangulation of the boundary. Finally, we describe some procedures to determine the hyperbolicity equations of the gluing manifold M from the boundary triangulation mentioned above. Examples of hyperbolic 3-manifolds of low complexity illustrate our constructions and algorithms in special cases.

2. Special spines

Throughout the paper, 3-*manifold* means compact, connected, orientable PL 3-manifold. Let M^3 be a 3-manifold with non-empty boundary. A compact connected 2-dimensional subpolyhedron $P \subset \mathrm{Int}(M)$ is said to be a *spine* of M if M collapses to P or, equivalently, if the open manifold $M \setminus P$ is homeomorphic to $\partial M \times [0, 1)$. Of course, M is a regular neighbourhood of P in the sense of [22] and [30]. By a *spine* of a closed 3-manifold M we mean a spine of the 3-manifold with boundary obtained from M by removing an open 3-ball. Two spines of a 3-manifold M differ by a 3-deformation (for more details, see for example [8]), and much information about M can be derived from any member of this 3-deformation class; in particular, all the homotopy (homology) invariants of M. Unfortunately, many different 3-manifolds can admit the same spine. For this, Casler introduced in [5] a special class of 2-dimensional polyhedra, and proved that any 3-manifold M collapses to some polyhedron of that class, called a *special spine* of M. Moreover, he proved that a special spine uniquely determines the 3-manifold. Subsequently, the theory of special spines was developed by Matveev in a series of papers [12], [11], [13], [14] and [15] (see also [4], [19], [9] and [10]). Here many classical representations of 3-manifolds as Heegaard diagrams, surgery presentations and triangulations were described in terms of special spines. We recall now basic definitions and results of the theory of special spines (for more details, see the quoted papers). A compact 2-dimensional polyhedron P is called *simple* if every point in P has a link homeomorphic to either a circle, a circle with a diameter, or a circle with three radii. The tipical regular neighbourhoods of the points of P are shown in Figure 1.



(a) (b) (c)

Figure 1: Typical neighbourhoods in special spines.

Any point of P having a neighbourhood of type 1.c is called a *vertex* of P, and the set of such vertices is denoted by $V(P)$. The *singular set* of P, written $S(P)$, is formed by the points of P having neighbourhoods of type 1.b and 1.c (i.e. $S(P)$ is the union of $V(P)$ with the set of points lying on triple lines of P). A neighbourhood of type 1.b (resp. 1.c) terminates with two (resp. four) *triods*, i.e. cones over three points. A simple polyhedron P admits naturally a stratified structure as $V(P) \subset S(P) \subset P$, where any connected component of $P \setminus S(P)$ is an open 2-manifold. A simple polyhedron P is called *special* (or, *standard*) if it contains at least one vertex, the connected components of $S(P) \setminus V(P)$ are open arcs, and the connected components of $P \setminus S(P)$ are open 2-cells. Examples of special polyhedra are given by fake surfaces obtained by pasting 2-cells along pairwise disjoint 2-sided closed simple curves drawn on a closed surface. A spine of a compact 3-manifold is said to be *special* (or, *standard*) if it is a special polyhedron. For example, the Bing house with two rooms is a special spine of a closed 3-cell. An important advantage of special spines with respect to general ones is that a compact 3-manifold can be uniquely recovered from a special spine of it. More precisely, there is the following basic result in the theory of special spines for compact 3-manifolds [5] (extended to general case in [11]).

THEOREM 2.1. *(Existence) Any compact connected* 3*-manifold possesses a special spine (with at least two vertices).*

(Unicity) Let P_i be a special spine of a compact connected 3*-manifold M_i, for any $i = 1, 2$. Then any homeomorphism from P_1 onto P_2 extends to a homeomorphism from M_1 onto M_2. In other words, two compact connected* 3*-manifolds with homeomorphic special spines are homeomorphic.*

Theorem 2.1 says that special spines give a combinatorial representation of compact 3-manifolds. However, a 3-manifold may have different special spines. Any two special spines representing the same 3-manifold are proved to be joined by a finite sequence of elementary moves. We briefly describe these moves following [12] and [13] (compare also with [19]). The elementary move T_1 consists in altering a regular neighbourhood of a vertex of a special polyhedron P as indicated in Figure 2.a. The elementary move T_2 changes a regular neighbourhood of some edge in P as shown in Figure 2.b.

The following result about the representation of 3-manifolds via special spines was independently proved in [12] and [19].

THEOREM 2.2. *(Equivalence) Two special spines (with at least two vertices) represent homeomorphic* 3*-manifolds if and only if one can be transformed into the other by a finite sequence of elementary moves of type T_1 and T_2, and their inverses.*

Theorem 2.2 can be stated in a different (but equivalent) form by substituting the moves of type T_i with the so-called *Matveev-Piergallini move* (briefly, MP-*move*) shown in Figure 3 (the polyhedron is left unchanged outside the considered neighbourhood).

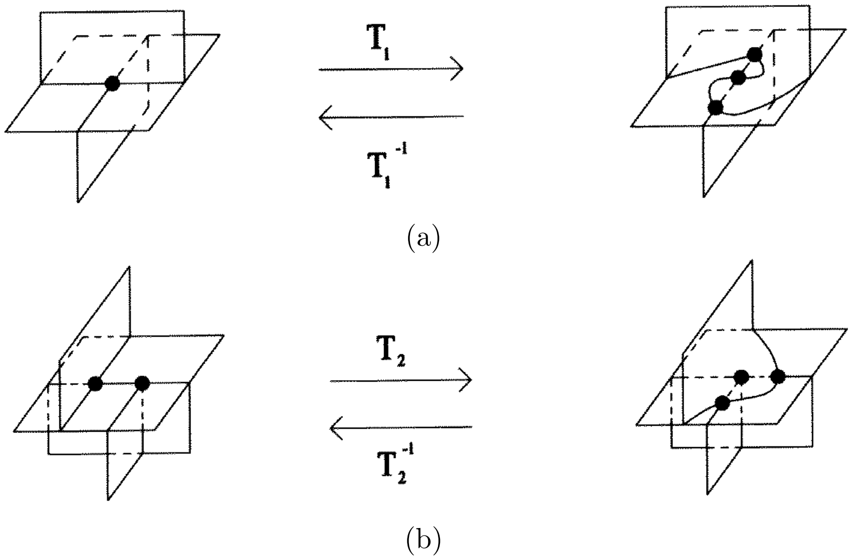


Figure 2: The moves of type T_1 and T_2.

We observe that the property of being special is not hereditary, i.e. a subpolyhedron of a special polyhedron is not in general special. So it was introduced in [13] (see also [14] and [15]) the class of almost special polyhedra, which is in fact hereditary. A compact 2-dimensional polyhedron P is said to be *almost special* if it embeds in some special polyhedron, i.e. the link of any point of P can be embedded into the circle with three radii. A *vertex* of P is a point whose link is homeomorphic to the circle with three radii. A spine of a compact 3-manifold is called *almost special* if it is an almost special polyhedron. For a compact 3-manifold M, a topological invariant, called the complexity of M, was defined in [13] and [14] by using the notion of almost special spine. More precisely, the *complexity* of M, written $c(M)$, is the smallest integer k ($k \geq 0$) such that M possesses an almost special spine with k vertices.

The following result (see [13] and [14]) illustrates two important properties of the complexity.

THEOREM 2.3. *(Finiteness). For any integer $k \geq 0$, there exists only a finite number of distinct closed irreducible connected 3-manifolds of complexity k.*

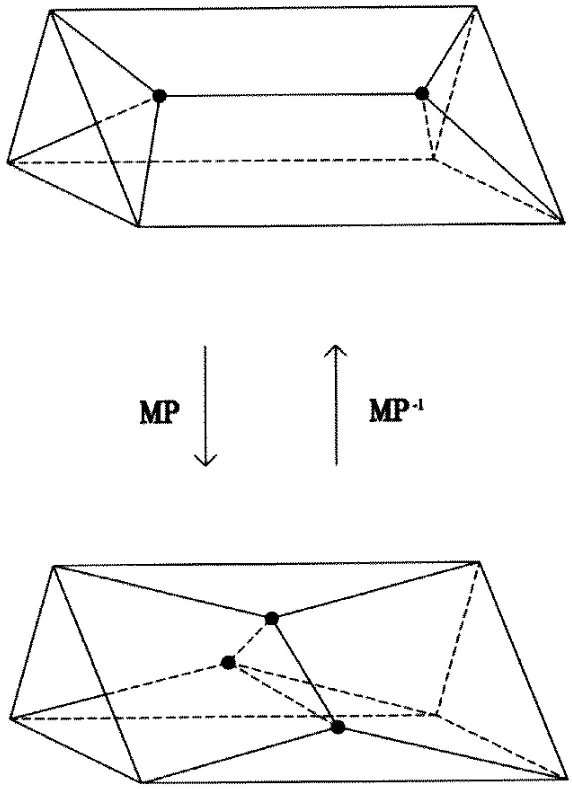


Figure 3: The Matveev-Piergallini move.

(Additivity). The complexity of a connected sum $M \# M'$ of two compact connected 3-manifolds M and M' equals the sum of their complexities, that is

$$c(M \# M') = c(M) + c(M').$$

In particular, it was proved in [14] and [15] that the number $n(k)$ of closed orientable irreducible connected 3-manifolds of complexity $k \leq 6$ is given by the following table:

k	0	1	2	3	4	5	6
$n(k)$	3	2	4	7	14	31	74

Moreover, all the closed orientable 3-manifolds of complexity $k \leq 8$ are *graph manifolds* in the sense of [28] (so they are not hyperbolic). However, there exist closed orientable hyperbolic 3-manifolds of complexity 9 (see [14], [15], and [16]). Among them, we find the smallest known closed hyperbolic 3-manifold with respect to the volume (which is $0.94272\ldots$). This manifold was independently obtained by Matveev and Fomenko [16] and by Weeks [29] (it can be constructed by closing the torus boundary of the hyperbolic 3-manifold described in Section 5).

3. o-Graphs

Benedetti and Petronio described in [3] a nice representation of compact connected 3-manifolds with non-empty boundary by means of certain planar graphs with some extra structures, called *o-graphs* (closed 3-manifolds are included in this representation by removing an open 3-cell). Such graphs translate essentially the combinatorial representation of bordered 3-manifolds via special spines in terms of graphic tools. So these representations of 3-manifolds are in fact equivalent. We present now a short informal outline of the subject, and refer to [3] for more detailed definitions and results. Let Γ be a finite connected planar quadrivalent graph with some marked vertices and simple normal crossings. We call *edges* of Γ the locally embedded segments with marked endpoints, and suppose that the edges cover Γ. Then Γ is said to be an *o-graph* if it has an under-over specification (as in the usual projection of a link) at each marked vertex, and an edge-colouring with colour set $\mathbb{Z}_3$. Such graphs are related with (oriented) special polyhedra in a natural way. To any oriented special polyhedron P we can associate a suitable o-graph $\Gamma = \Gamma(P)$ representing P (by an invertible construction) as follows. The graph $\Gamma = \Gamma(P)$ coincides with the singular set $S(P)$ of P (as cellular 1-complex), and the marked vertices of Γ are precisely the vertices of P (those having regular neighbourhoods of type 1.c in Figure 1). The choice of an embedding from a regular neighbourhood of each vertex of P into the standard 3-space induces the under-over specification at each marked vertex of Γ. Now suppose that x and y are marked vertices of Γ joined by an edge, and denote by $N(x)$ and $N(y)$ regular neighbourhoods of x and y in P, respectively. As illustrated in Figure 1.c, these neighbourhoods have exactly four terminal triods eachone. We have to match carefully a precise triod in the boundary of $N(x)$ to one in the boundary of $N(y)$: those intersecting the edge mentioned above (which connects x and y). Enumerating the three branches of each triod by $\mathbb{Z}_3$, the edge-colouring of Γ describes how to drill the triod of $N(x)$ before gluing it to the triod of $N(y)$. Thus the compact 3-manifold M, uniquely defined by thickening the special polyhedron P, can be completely represented by the o-graph $\Gamma = \Gamma(P)$. However, M may have different o-graphs. Any two o-graphs representing the same manifold are proved to be joined by a finite sequence of elementary moves [3]. The elementary moves of type R arise naturally from the well-known moves of Reidemeister on the planar projections of links, and they are illustrated in Figure 4a. The elementary move of type C, shown in Figure 4b, takes in account all the possible choices of embeddings from a regular neighbourhood of a vertex of P into the standard 3-space. The elementary move of type MP translates for o-graphs an oriented version of the MP-move on oriented polyhedra (see Figure 4c). Here we use the convention that colours on outer edges of these local pictures are allowed and they must be summed up, modulo 3. The following is the main result proved in [3].

Theorem 3.1. *(Existence). Any compact connected* 3*-manifold with non-empty boundary can be completely represented by an o-graph (with at least two marked vertices).*

(Equivalence). Two o-graphs with at least two marked vertices (regarded up to isotopies of the plane) represent homeomorphic 3*-manifolds if and only if one can be transformed into the other by a finite sequence of elementary moves of type R, C, MP, and their inverses.*

Further developments in the o-graph calculus can be found in two recent papers of Theis (see [26] and [25]). Here the author defines many local transformations of o-graphs which give a graph-theoretical descriptions of various topological constructions of 3-manifolds as puncturing, connected sums, adjoining a handle, closing a boundary component, products, and mapping tori.

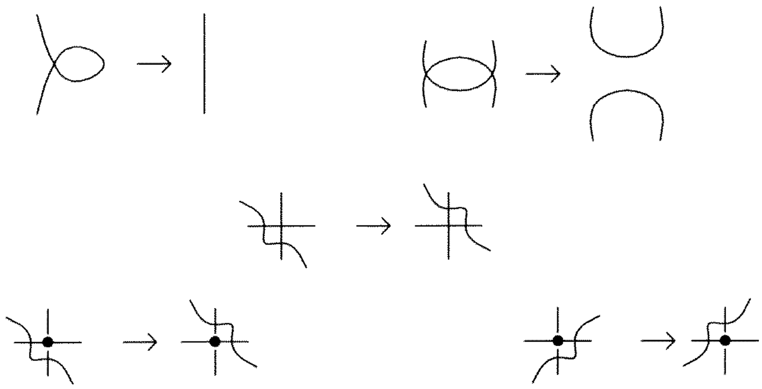


Figure 4a: The elementary moves of type R.

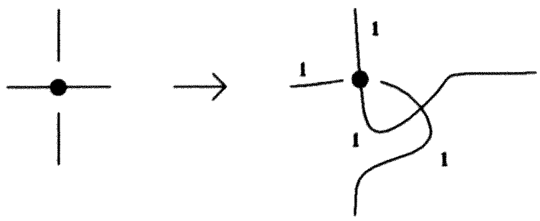


Figure 4b: The elementary move of type C.

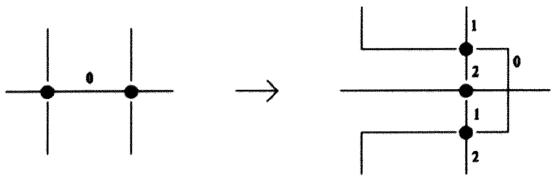


Figure 4c: The elementary move of type MP.

4. Gluing manifolds

The face identification procedure is a very standard method for constructing compact 3-manifolds. Any compact 3-manifold with boundary can be constructed by gluing together (truncated) tetrahedra along their faces. More precisely, take a finite family $\mathcal{F}$ of disjoint 3-simplexes in the Euclidean 3-space E^3, and identify their faces pairwise via a collection ϕ of orientation-reversing isometries of E^3. We call ϕ a *side pairing* for $\mathcal{F}$. Of course, not every side pairing for $\mathcal{F}$ yields an orientable closed 3-manifold. However, the resulting quotient space $Q = \mathcal{F}/\phi$ is a closed orientable pseudo-manifold in the sense of [7], and we call it the *gluing space* of the pair $(\mathcal{F}, \phi)$. The only troublesome points in Q are the vertices (the images of the vertices of the 3-simplexes of $\mathcal{F}$). They have regular neighbourhoods that are cones over closed surfaces. By [24] the gluing space Q is a closed 3-manifold if and only if its Euler characteristic vanishes. If we remove regular neighbourhoods of the vertices from Q, then we obtain a compact orientable 3-manifold $M = M(\mathcal{F}, \phi)$. In other words, M is constructed by gluing together tetrahedra with truncated vertices (see for example Figure 5), and we call it the *gluing 3-manifold* with non-empty boundary, defined by the pair $(\mathcal{F}, \phi)$. We present now a combinatorial description of gluing (pseudo)manifolds and their special spines by certain graphs, which are strictly related with the o-graphs discussed in the previous section. Our graphs can be easily encoded by 5-tuples of non-negative integers. This permits to handle (and modify) them by using a computer program. The goal is to obtain simplified o-graphs (with respect to the number of marked vertices) which may represent either the same gluing (pseudo)manifolds or other spaces corresponding to specified topological constructions.

Let Δ^3 be a standard 3-simplex in the Euclidean 3-space. We colour its vertices by $\mathbb{Z}_4$ in the following way: fix an edge and label its vertices by 0 and 2; then label the vertices of the opposite edge by 1 and 3, according to the right-hand rule (see Figure 5).

We label each face of Δ^3 by the number of its opposite vertex, and the barycentres of the edges in Δ^3 by the elements of $\mathbb{Z}_3$, as indicated in Figure 5. In this way, the barycentres of any two opposite edges of Δ^3 have the same label. Now we consider Δ^3 as a simplicial complex, and take the 2-skeleton of its dual cellular decomposition. So we obtain a polyhedron $N = N(x)$ which is homeomorphic to a regular neighbourhood of a vertex x of a special spine (see Figure 1c) (here x is the barycentre of Δ^3). The polyhedron N intersects each face of Δ^3 in a triod (i.e. a space homeomorphic to a picture T). The endpoints of each triod in N are precisely the barycentres of the edges in Δ^3 (coloured by $\mathbb{Z}_3$). Moreover, we require that the bottom point of any triod T takes the label 0, and that the endpoints of its branches are numbered counter-clockwise with respect to an outer observer

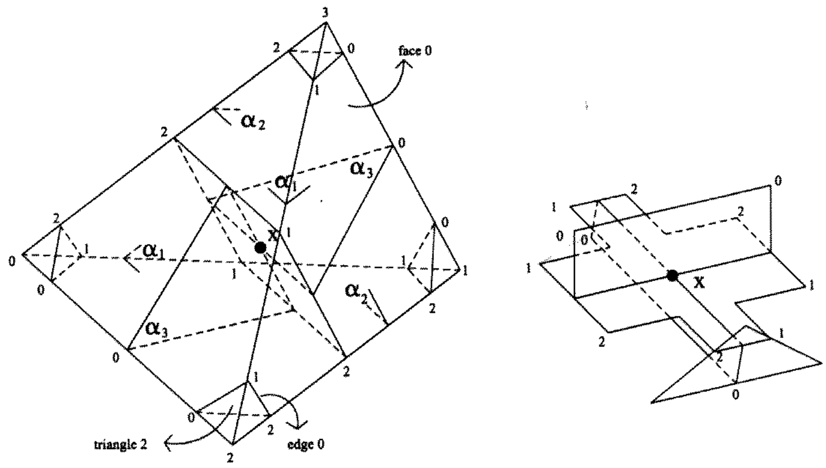


Figure 5: The polyhedron $N = N(x)$ (of type 1.c) embedded in Δ^3 and the truncated 3-simplex.

(see Figure 5: to simplify the picture we use inner points of the edges instead of the barycentres; of course, the resulting embedded polyhedron is again homeomorphic to N). These endpoints have the same labels of the barycentres of the edges of the corresponding face in Δ^3. Let us denote by T_i the triod T lying in the face i of Δ^3, for any $i \in \mathbb{Z}_4$. Suppose now to have a side pairing ϕ for a finite family $\mathcal{F}$ of disjoint tetrahedra, i.e. a partition of their faces into pairs. Then we label the parts of any 3-simplex of $\mathcal{F}$ as done for Δ^3. The identification of two faces labeled by i and j via an orientation-reversing isometry of ϕ yields a gluing of the triods T_i and T_j. To preserve the orientation of the quotient space $Q = \mathcal{F}/\phi$, we have to consider only three possible gluings of T_i with T_j; eachone of them can be represented by a transposition of $\mathbb{Z}_3$ which fixes one of the endpoints of a triod. Let $p_0 = (1\ 2)$, $p_1 = (0\ 2)$ and $p_2 = (0\ 1)$ be the transpositions which fix the endpoint of a triod labeled by 0,1, and 2, respectively. Of course, we can identify the transposition p_i with the colour $i \in \mathbb{Z}_3$ (we can always require in addition that any gluing of two faces with different parity is realized by a permutation p_i, for $i \neq 0$). Under gluing of tetrahedra of $\mathcal{F}$ via ϕ, the polyhedra of type N combine together to form a special polyhedron $P = P(\mathcal{F}, \phi)$.

In particular, the vertex set of P is formed by the images of the barycentres of the tetrahedra in $\mathcal{F}$. We can represent the regular neighbourhood of a vertex x of P by a square whose vertices, labeled by $\mathbb{Z}_4$, bijectively correspond to the triods T_0, T_1, T_2 and T_3. This square also represents a 3-simplex of $\mathcal{F}$; in fact, the vertices of the square correspond to the faces of the represented 3-simplex. Let now x and y be vertices (not necessarily different) of P which are joined by an edge in P. This means that a triod $T_i \subset N(x)$ must be glued with a triod $T_j \subset N(y)$ maintaining fixed a branch labeled k. In our graphic representation, we have to join the vertices i and j of the squares, representing $N(x)$ and $N(y)$, by an edge coloured $k \in \mathbb{Z}_3$ (the colour k corresponds to the transposition p_k). Therefore, we have constructed a cubic graph $G = G(\mathcal{F}, \phi)$, embedded in the Euclidean 3-space, formed by squares and coloured edges (between them). We call it (and its planar projection with normal under-overcrossings as in the usual sense of links) the *graph of the gluing*. The graph G induces immediately an o-graph $\Gamma = \Gamma(\mathcal{F}, \phi)$ in the sense of [3] which represents the special polyhedron $P = P(\mathcal{F}, \phi)$. It suffices to substitute any square with a marked vertex, and to define the under-over specification at the vertex by assuming that the diagonal $0 - 2$ overcrosses the diagonal $1 - 3$.

Proposition 4.1. *Let $\mathcal{F}$ be a finite family of disjoint 3-simplexes in the Euclidean 3-space, and ϕ a side pairing for $\mathcal{F}$ formed by orientation-reversing isometries. Let Q be the closed orientable pseudomanifold defined by $(\mathcal{F}, \phi)$ as a quotient space, and M the compact orientable 3-manifold with non-empty boundary obtained by gluing the truncated tetrahedra of $\mathcal{F}$ via ϕ. Then Q and M are completely represented by the graph $G = G(\mathcal{F}, \phi)$ of the gluing. Furthermore, G defines an o-graph $\Gamma = \Gamma(\mathcal{F}, \phi)$ which represents a special spine of M.*

Enumerating all the 3-simplexes of $\mathcal{F}$, we can algebraically describe the corresponding gluing spaces and graphs by 5-tuples of non-negative integers $(n_1, f_1, n_2, f_2, p_j)$. This means that the face f_1 of the 3-simplex n_1 of $\mathcal{F}$ is glued to the face f_2 of the 3-simplex n_2 by the transposition p_j, for $j \in \mathbb{Z}_3$. In this way, the graph of the gluing and the corresponding o-graph can be easily handled by a computer.

We illustrate the combinatorial constructions described above by using a special spine, given by Matveev and Fomenko in [16], and depicted in Figure 6 (we colour the two vertices of the spine by 1 and 2). The graph of the gluing and the corresponding o-graph are shown in Figure 7. They represent a compact orientable 3-manifold with torus boundary having complexity 2, and a special spine of it. Of course, the gluing and the corresponding graphs can be completely described by the 5-tuples of integers: $(1,0,2,3,1)$, $(1,1,2,2,2)$, $(1,2,2,1,1)$, $(1,3,2,0,2)$.

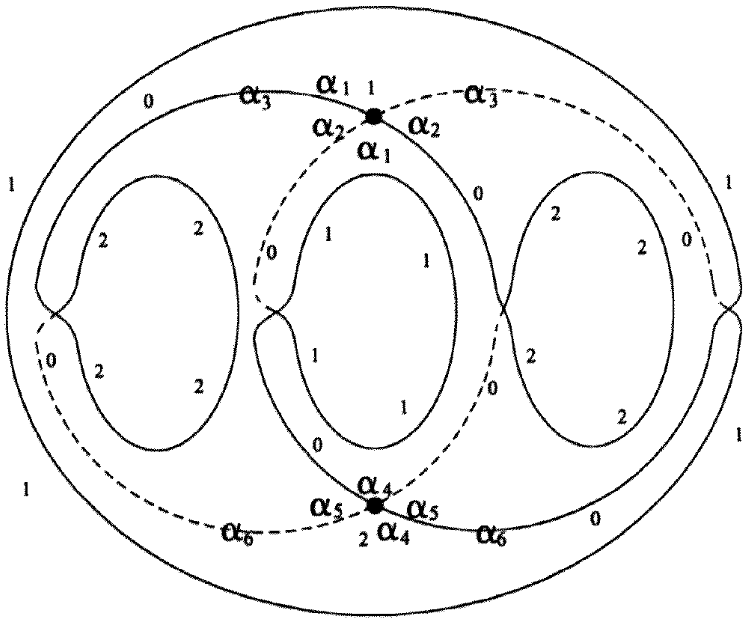


Figure 6: A special spine of the Matveev-Fomenko 3-manifold with torus boundary.

5. The construction of the boundary

Let M be a compact orientable connected 3-manifold (with non-empty boundary) obtained by gluing together 3-simplexes with truncated vertices. We describe now a simple numeric algorithm for

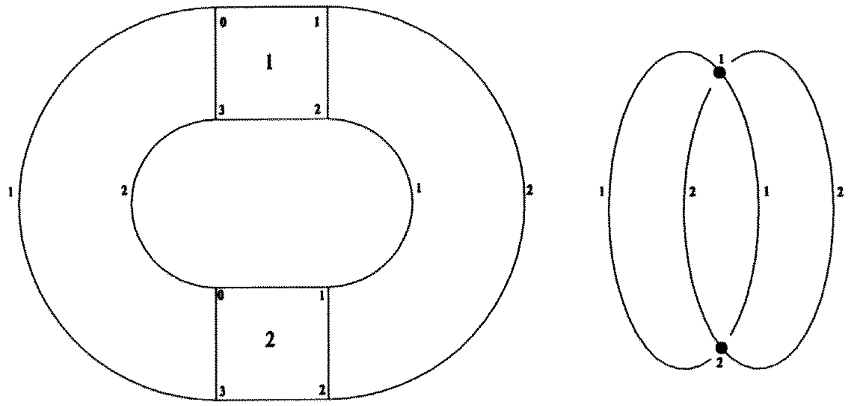


Figure 7: The graph of the gluing which produces the Matveev-Fomenko manifold, and the corresponding o-graph.

constructing the boundary of M. The algorithm produces a list of 6-tuples of non-negative integers which can be read off directly from the graph of the gluing. Let us denote by $\tilde{\Delta}^3$ the truncated tetrahedron obtained from Δ^3 by removing regular neighbourhoods of the vertices. There are four triangles in the boundary of $\tilde{\Delta}^3$ which correspond to the removed vertices of Δ^3 (see Figure 5). We label every triangle with the number of the corresponding removed vertex. The vertices of these triangles lie on the edges of the 3-simplex Δ^3, and are labeled like the barycentres of the corresponding edges, as indicated in Figure 5. Finally, every edge of these triangles is labeled like its opposite vertex (in the triangle). We can construct the following table which works for every truncated tetrahedron $\tilde{\Delta}^3$. Its meaning is the following: in every face of the tetrahedron Δ^3 there are exactly three edges which belong to different triangles lying on the boundary of the truncated tetrahedron $\tilde{\Delta}^3$. These edges are always labeled by different elements of $\mathbb{Z}_3$. For example, on face 0 of Δ^3 there are: edge 0 which belongs to triangle 2 of $\tilde{\Delta}^3$; edge 1 which belongs to triangle 1 of $\tilde{\Delta}^3$; and edge 2 which belongs to triangle 3 of $\tilde{\Delta}^3$ (see Figure 5 and Table 1).

line	face of Δ^3	edge	triangle of $\tilde{\Delta}^3$
1	0	0	2
2	0	1	1
3	0	2	3
4	1	0	3
5	1	1	0
6	1	2	2
7	2	0	0
8	2	1	3
9	2	2	1
10	3	0	1
11	3	1	2
12	3	2	0

Table 1.

The transposition p_j, for any $j \in \mathbb{Z}_3$, acts on the edges of the triangles (contained in the boundary of $\tilde{\Delta}^3$) as described by the following table:

edge	p_0	p_1	p_2
0	0	2	1
1	2	1	0
2	1	0	2

Table 2.

The boundary of the manifold M is of course constructed by gluing together the edges of the triangles lying on the boundaries of the truncated tetrahedra. This gluing can be completely described by 6-tuples $(n_1, t_1, l_1, n_2, t_2, l_2)$ of non-negative integers: the edge l_1 of the triangle t_1 of the tetrahedron n_1 must be glued to the edge l_2 of the triangle t_2 of the tetrahedron n_2. If n is the number of (truncated) tetrahedra, then the boundary ∂M of M is completely represented by $6n$ 6-tuples of integers. This permits to construct the triangulation of ∂M by a computer program. In fact, we give an algorithm for getting the 6-tuples $(n_1, t_1, l_1, n_2, t_2, l_2)$ from the 5-tuples $(n_1, f_1, n_2, f_2, p_j)$, $j \in \mathbb{Z}_3$, which encode the graph of the corresponding gluing (see Section 4). It can be obtained by the following steps: for every $l_1 \in \mathbb{Z}_3$

1) the triangle t_1 is given by Table 1 at the line $3f_1 + l_1 + 1$;

2) the edge l_2 of the triangle t_2 is obtained from Table 2; it corresponds to the edge l_1 and the transposition p_j;

3) the triangle t_2 is given by Table 1 at the line $3f_2 + l_2 + 1$.

For example, the boundary of the Matveev-Fomenko manifold, described in Section 4, can be completely encoded by twelve 6-tuples of integers. These 6-tuples are directly deduced from the four 5-tuples which encode the graph of the corresponding gluing (use the algorithm described above):

$$(1,0,2,3,1) \begin{cases} (1,2,0,2,0,2) \\ (1,1,1,2,2,1) \\ (1,3,2,2,1,0) \end{cases}$$

$$(1,1,2,2,2) \begin{cases} (1,3,0,2,3,1) \\ (1,0,1,2,0,0) \\ (1,2,2,2,1,2) \end{cases}$$

$$(1,2,2,1,1) \begin{cases} (1,0,0,2,2,2) \\ (1,3,1,2,0,1) \\ (1,1,2,2,3,0) \end{cases}$$

$$(1,3,2,0,2) \begin{cases} (1,1,0,2,1,1) \\ (1,2,1,2,2,0) \\ (1,0,2,2,3,2). \end{cases}$$

Gluing the edges of the triangles (lying on the boundaries of the truncated tetrahedra labeled by $n_1 = 1$ and $n_2 = 2$) according to the previous list of 6-tuples yields immediately the triangulation of a torus (see Figure 8). This is the boundary of the Matveev-Fomenko manifold represented by the special spine and by the o-graph depicted in Figures 6 and 7, respectively.

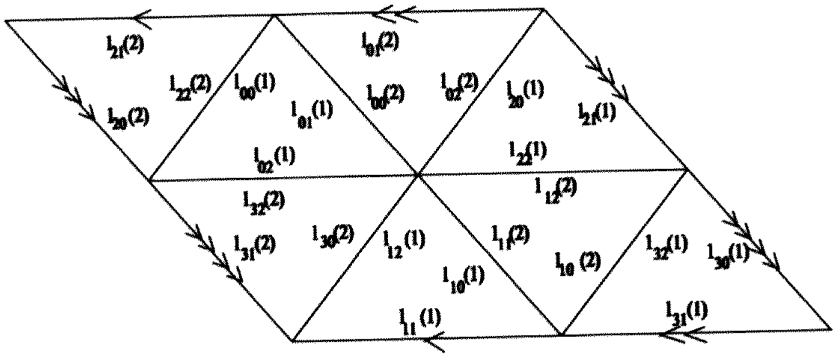


Figure 8: The boundary of the Matveev-Fomenko manifold.

6. Hyperbolicity equations

Let M be a compact connected oriented 3-manifold with non-empty boundary obtained by taking n standard tetrahedra, by gluing in pair the faces of them, and by removing the vertices. So M is triangulated by truncated tetrahedra, i.e. tetrahedra in which we remove the open star of vertices in the second barycentric subdivision. We investigate when M can be endowed with a hyperbolic structure (i.e. a Riemannian metric with constant negative curvature) and when this structure is complete. We summarize well-known results about hyperbolic geometry and topology of 3-manifolds; for more details see, for example, [27], [2], [1], [18] and [20]. As a consequence of Margulis' lemma, a hyperbolic structure on $\mathrm{Int}(M)$ can be constructed if ∂M consists of tori; therefore, we will assume this hypothesis for M. A well-known approach for constructing this structure is to endow each tetrahedron with a hyperbolic structure and try to extend it to $\mathrm{int}(M)$. Each tetrahedron can be realized as an ideal tetrahedron in the hyperbolic 3-space $\mathbb{H}^3 =]0, \infty[\times \mathbb{C}$ with its vertices at infinity. Since any two ideal triangles are isometric, ideal tetrahedra can be glued via isometries of their faces. Furthermore, the natural hyperbolic structure defined in the interior of any tetrahedron naturally extends to the interior of its faces. We recall that the dihedral angles at opposite edges of an ideal tetrahedron are always equal and the congruent class of an ideal tetrahedron is completely determined by these angles, α, β and γ, say. The intersection of this ideal tetrahedron with a horosphere centred at a vertex is a Euclidean triangle with angles α, β and γ, and the similarity class of the triangle completely determines the ideal tetrahedron. Every similarity class of triangles has a representative with vertices 0, 1, and z, where $\mathrm{Im}(z) > 0$. In fact, take a Euclidean triangle in the complex plane $\mathbb{C}$ with vertices v, u and t, according to the positive orientation of the boundary of the triangle. Then consider the orientation-preserving similarity of the plane which maps v to 0, u to 1 and t to the complex number $z_1 = z(v) = \dfrac{t-v}{u-v}$, where $\mathrm{Im}(z_1) > 0$ (see Figure 9).

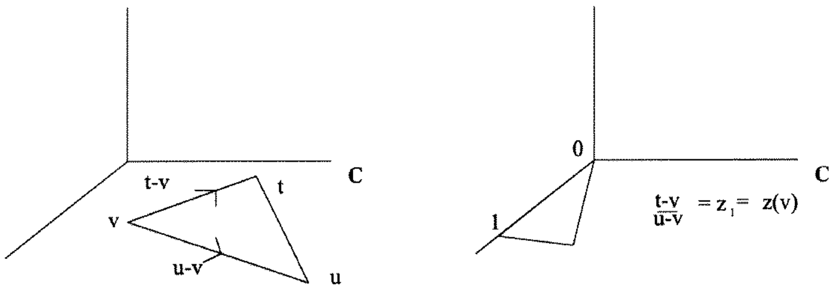


Figure 9: The orientation-preserving similarity which maps v to 0, u to 1 and t to $z_1 = z(v)$.

The other two choices of the starting vertex produce the complex numbers

$$z_2 = z(t) = \frac{u-t}{v-t} \quad \text{and} \quad z_3 = z(u) = \frac{v-u}{t-u}.$$

The complex numbers z_1, z_2 and z_3 are called the *vertex invariants* of the triangle. They depend only on the orientation-preserving similarity class of the triangle and satisfy the following equations:

1) $z_1 z_2 z_3 = -1$; and

2) $1 - z_1 + z_1 z_2 = 0$.

Consequently, z_1 determines z_2 and z_3. Setting $z_1 = z$, we have $z_2 = \dfrac{z-1}{z}$ and $z_3 = \dfrac{1}{1-z}$. Therefore the complex number z, where

$\operatorname{Im} z > 0$, completely determines the orientation-preserving similarity class of the triangle of vertices v, u and t. The number z, associated to the triangle by a choice of a starting vertex v, is called the *modulus* of the triangle with respect to v. Now we can describe the parametrization of an ideal tetrahedron in $\mathbb{H}^3$. If T is an ideal tetrahedron in $\mathbb{H}^3$ and an edge E of T is fixed, then we can associate to T a complex number z, with $\operatorname{Im} z > 0$, in the following way: realize T in the half-space model in such a way that one of the endpoints of the preferred edge E is at infinity; consider the Euclidean triangle obtained by intersecting T with a suitably high horizontal plane; let z be the modulus of the triangle with respect to the vertex lying on the preferred edge. The six choices of a preferred edge produce the numbers $z_1 = z$, $z_2 = \dfrac{z-1}{z}$ and $z_3 = \dfrac{1}{1-z}$ (each being obtained twice), and opposite edges of T have the same number. The complex numbers z_1, z_2, z_3 are called the *edge invariants* of T, and their arguments are equal to the dihedral angles of T (see Figure 10).

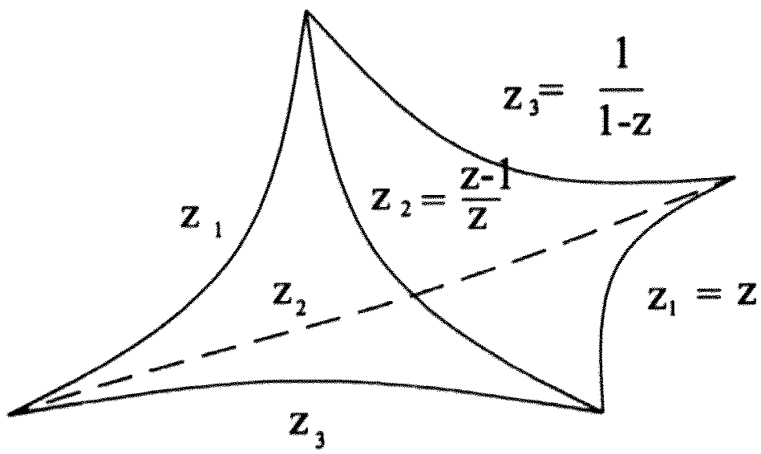


Figure 10: The edge invariants of an ideal tetrahedron T in $\mathbb{H}^3$.

We return to the manifold M obtained by gluing n tetrahedra $\Delta_1^3, \Delta_2^3, \ldots, \Delta_n^3$; each of them, realized as an ideal tetrahedron T_i in $\mathbb{H}^3$, is parametrized by a complex number $z(i)$, for $i = 1, 2, \ldots, n$, or equivalently, by the complex numbers $z_1(i)$, $z_2(i)$ and $z_3(i)$. If

$z_{j_1}(i_1), \ldots, z_{j_r}(i_r)$, $j_1, \ldots, j_r \in \{1,2,3\}$, are the parameters along the various edges of the tetrahedra projecting onto an edge e of M, then the conditions for extending the hyperbolic structure (already defined on the interior of any tetrahedron and on the interior of its triangular faces) to e are given by the following:

THEOREM 6.1. *Let M be the manifold defined above. The hyperbolic structure defined by $z(i)$ on the tetrahedra T_i extends to the edge e if and only if*

1) $z_{j_1}(i_1) \cdot z_{j_2}(i_2) \cdots z_{j_r}(i_r) = 1$; *and*

2) $arg\, z_{j_1}(i_1) + arg\, z_{j_2}(i_2) + \cdots + arg\, z_{j_r}(i_r) = 2\pi$.

A lemma of [2] implies that condition 2) is a consequence of 1). So it suffices to require that the product of the complex parameters, corresponding to the dihedral angles incident to the edge e, is equal to 1. This condition must be applied for all the edges of M, which are exactly n (use the fact that $\chi(\partial M) = 0$). Therefore, we obtain a system of n equations in the unknowns $z(1), z(2), \ldots, z(n)$ (as $z_2(i)$ and $z_3(i)$ can be expressed in terms of $z_1(i) = z(i)$, for any $i = 1, 2, \ldots, n$). These equations are called "consistency (or compatibility) equations ".
Now we investigate when the hyperbolic structure defined on M is complete. Let $L_1, \ldots, L_k$ be the links of the k removed vertices of M. They are constructed by gluing in pairs the edges of the Euclidean triangles obtained intersecting the tetrahedra with horospheres centered at the vertices. By hypothesis, these links are homeomorphic to tori. The hyperbolic structure on $\mathrm{Int}(M)$ implies that the similarity structure globalizes to toric links. For the completeness we have the following result (see for example [17] and [18]):

THEOREM 6.2. *The hyperbolic structure of the tetrahedra extends to a global complete hyperbolic structure on the interior of M if and only if on each torus of ∂M the above-defined decomposition into similarity triangles is compatible with a global Euclidean structure.*

To translate this fact into equations, we recall that the similarity structure on L_j induces a coniugacy class of homomorphisms η : $\Pi_1(L_j) \to \mathrm{Aff}(\mathbb{C})$, called the *holonomy* of the structure. Then L_j

is complete if and only if the holonomy maps $\Pi_1(L_j)$ isomorphically onto a freely acting discrete group of Euclidean isometries of $\mathbb{C}$, i.e. on a lattice group of translations of $\mathbb{C}$. Now, every element of $\mathrm{Aff}(\mathbb{C})$ is of the form $\varphi(z) = az + b$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. Moreover, φ is a Euclidean translation if and only if $a = 1$ (a is also called the *dilation component* of φ). Since the derivative of φ is $\varphi'(z) = a$, it follows that φ is a Euclidean translation if and only if $\varphi'(z) = 1$. Taking a pair of simplicial generators m_j and l_j of $\Pi_1(L_j)$, we require that the derivative of the holonomy of each generator equals 1, i.e. $\eta'(m_j) = \eta'(l_j) = 1$. The derivative of the holonomy of a simplicial loop can be computed as the product of all moduli found on one of the sides (left or right) of the loop. So we obtain two equations in the unknowns $z(1), \ldots, z(n)$ for each L_j, $j = 1, 2, \ldots, k$. Therefore, the structure is complete if and only if $z(1), \ldots, z(n)$ satisfy $2k$ further equations, called "completeness equations ". As a consequence of Mostow's rigidity theorem, if there exists a hyperbolic complete structure on M, then this structure is unique.
Let us return to the Matveev-Fomenko manifold with torus boundary obtained by gluing two tetrahedra, according to Figure 11. We realize the tetrahedra as ideal tetrahedra in $\mathbb{H}^3$, and parametrize them by the modules z and w, with $\mathrm{Im}\, z > 0$ and $\mathrm{Im}\, w > 0$.

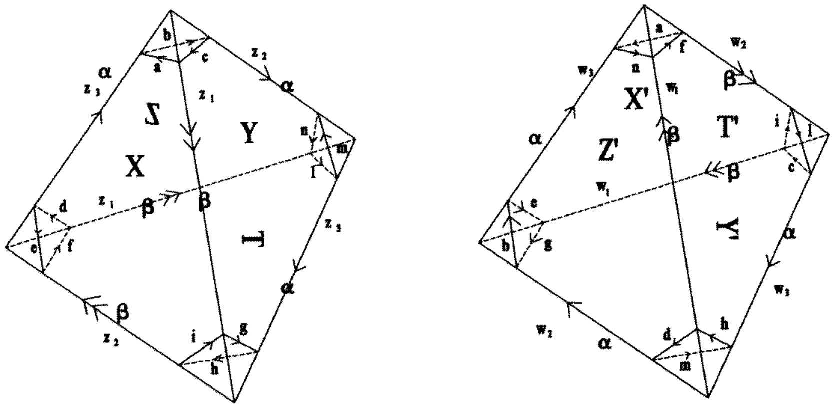


Figure 11: The side pairing of the Matveev-Fomenko manifold.

Let L be the link of the unique removed vertex, shown in Figure

12. Then L intersects the edges α and β in two points A and B. The gluing consistency equations for the two edges can be read directly on L, and they are

$$(i) \qquad w_3z_3w_2z_3w_3z_2 = 1$$
$$(ii) \qquad w_1z_1w_2z_1w_1z_2 = 1$$

or equivalently

$$(i) \qquad z_2z_3^2w_2w_3^2 = 1$$
$$(ii) \qquad z_1^2z_2w_1^2w_2 = 1.$$

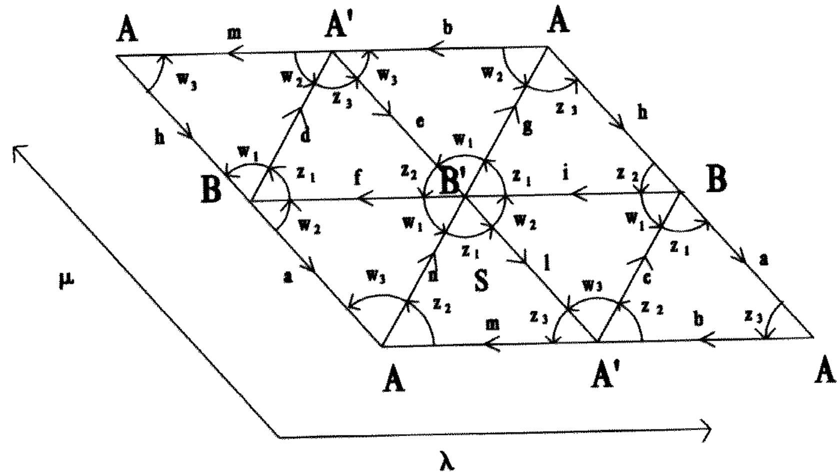


Figure 12: The link L of the removed vertex.

Since $(z_1z_2z_3w_1w_2w_3)^2 = 1$, conditions (i) and (ii) are equivalent. Thus we can consider only one of the consistency equations, for example (i). Setting $z_2 = \dfrac{z-1}{z}$, $z_3 = \dfrac{1}{1-z}$, $w_2 = \dfrac{w-1}{w}$ and $w_3 = \dfrac{1}{1-w}$, equation (i) becomes

$$(*) \qquad z(z-1)w(w-1) = 1$$

which has the solutions

$$z = \frac{1 \pm \sqrt{1 + 4(w(w-1))^{-1}}}{2}.$$

For each value of w there is a unique solution for z, with $\operatorname{Im} z > 0$, provided that $1 + 4(w(w-1))^{-1} < 0$. Following [20], the solutions w belong to the set

$$\{\operatorname{Im} w > 0\} \setminus \left\{ \operatorname{Re} w = \frac{1}{2}; \operatorname{Im} w \geq \frac{\sqrt{15}}{2} \right\}.$$

Now we compute the derivative of the holonomy of the similarity structure on L as a product of ratios. Furthermore, the ratio of any two vectors in the same triangle can be computed in terms of the vertex invariants. In fact, let λ be the element of $\Pi_1(L)$ represented by the base of the parallelogram in Figure 12. We assign the value 1 to the edge m of the triangle S, developing the triangulation of L onto $\mathbb{C}$ along λ until we meet another copy of S (see Figure 13).
Then we can express the encountered edges to go from m to m' in terms of the vertex invariants, i.e. we have

$$\frac{1}{l} = -z_3, \quad \frac{l}{c} = -w_3, \quad \frac{c}{b} = -z_2, \quad \frac{b}{a'} = -z_3, \quad \frac{a'}{n'} = -w_3, \quad \frac{n'}{m'} = -z_2.$$

So we obtain

$$\frac{1}{m'} = \frac{1}{l}\frac{l}{c}\frac{c}{b}\frac{b}{a'}\frac{a'}{n'}\frac{n'}{m'} = z_2^2 z_3^2 w_3^2 = \frac{w_3^2}{z_1^2} = \frac{1}{(1-w)^2 z^2}.$$

The value m' is $\eta'(\lambda)$, hence

$$\eta'(\lambda) = z^2(1-w)^2.$$

Let μ be the element of $\Pi_1(L)$ represented by the left side of the parallelogram in Figure 12. From Figure 13, assigning the value 1 to the edge a yields

$$\eta'(\mu) = z_1 z_2 w_1 w_2 w_3^2 = \frac{w_3}{z_3} = \frac{1-z}{1-w}.$$

We obtain $\eta'(\mu) = 1$ if and only if $z = w$, and $\eta'(\lambda) = 1$ if and only if $z^2(1-z)^2 = 1$, i.e. $z(1-z) = 1$ or $z(z-1) = 1$. The first equation has the solution $z = \dfrac{1 \pm \sqrt{3}i}{2}$; the second equation has the solution $z = \dfrac{1 \pm \sqrt{5}}{2}$. Hence, the unique solution with $\operatorname{Im} z > 0$ and

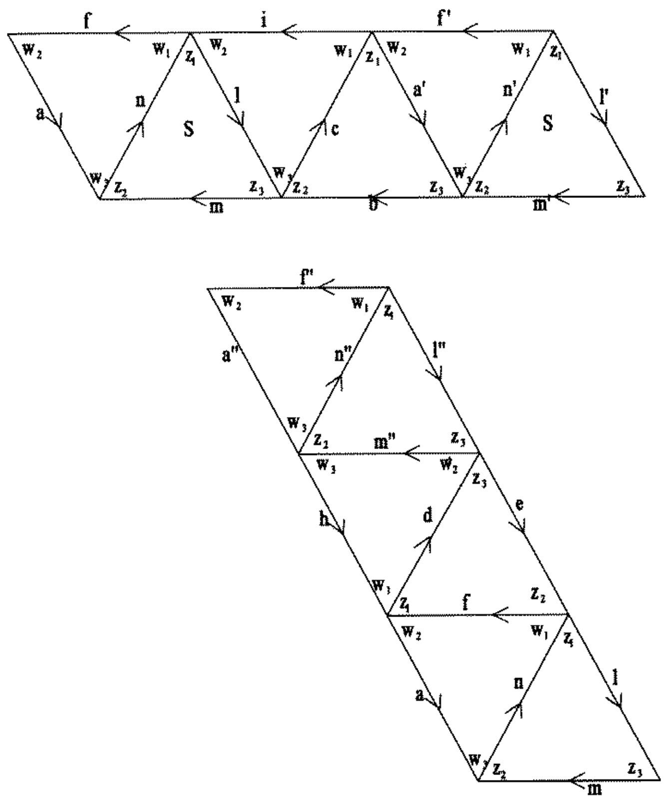


Figure 13: Developing the triangulation of L along λ, $\mu \in \Pi_1(L)$.

$\operatorname{Im} w > 0$ is $z = w = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$. So M is complete if and only if both the tetrahedra are regular (compare with [16]).

Now we explain a slightly different method to parametrize ideal tetrahedra and to produce hyperbolicity equations for a compact oriented 3-manifold M with non-empty boundary (consisting of tori) whose interior has a fixed truncated triangulation. Let $\tilde{\Delta}^3_r$ be the r-th truncated 3-simplex of M, obtained from the 3-simplex Δ^3_r of vertices V_0, V_1, V_2 and V_3. We denote by F_i, $i = 0, 1, 2, 3$, the boundary triangle of $\tilde{\Delta}^3_r$ corresponding to the vertex V_i of Δ^3_r and by P_{ij}, $j = 0, 1, 2$, the vertices of F_i in $\tilde{\Delta}^3_r$. Let $\mathbb{H}^3 =]0, \infty[\times \mathbb{C}$ be the hyperbolic 3-space and let α_0 be the map which realizes Δ^3_r as an ideal tetrahedron of $\mathbb{H}^3$ such that $\alpha_0(V_0) = (\infty, 0)$. Let $\pi_1 : \mathbb{H}^3 \to]0, \infty[$ and $\pi_2 : \mathbb{H}^3 \to \mathbb{C}$ denote the canonical projections and τ_j, $j = 0, 1, 2$, the isometry of $\mathbb{H}^3$ defined by

$$\tau_j : (x, z) \to (x, z - \pi_2 \circ \alpha_0(P_{0j})).$$

Let J be the isometry of $\mathbb{H}^3$ defined by

$$J(x, z) = \left(\frac{x}{x^2 + z\overline{z}}, \frac{\overline{z}}{x^2 + z\overline{z}} \right).$$

The composite map $\alpha_{j+1} = J \circ \tau_j \circ \alpha_0$, $j = 0, 1, 2$, realizes Δ^3_r as an ideal tetrahedron of $\mathbb{H}^3$ such that $\alpha_{j+1}(V_{j+1}) = (\infty, 0)$ (see Figure 14).

We set

$$\begin{aligned} l_{i0} &= \pi_2 \circ \alpha_i(P_{i2}) - \pi_2 \circ \alpha_i(P_{i1}) \\ l_{i1} &= \pi_2 \circ \alpha_i(P_{i0}) - \pi_2 \circ \alpha_i(P_{i2}) \\ l_{i2} &= \pi_2 \circ \alpha_i(P_{i1}) - \pi_2 \circ \alpha_i(P_{i0}) \end{aligned}$$

for any $i = 0, 1, 2, 3$. Since

$$\frac{l_{01}}{l_{00}} = \frac{l_{11}}{l_{10}} = \frac{l_{21}}{l_{20}} = \frac{l_{31}}{l_{30}}$$

and

$$\frac{l_{02}}{l_{00}} = \frac{l_{12}}{l_{10}} = \frac{l_{22}}{l_{20}} = \frac{l_{32}}{l_{30}},$$

we set $x_{2r-1} = \dfrac{l_{i1}}{l_{i0}}$ and $x_{2r} = \dfrac{l_{i2}}{l_{i0}}$ (hence $\dfrac{l_{i2}}{l_{i1}} = x^{-1}_{2r-1} x_{2r}$). In this way we have two variables, x_{2r-1} and x_{2r}, for every truncated tetrahedron $\tilde{\Delta}^3_r$ of M. Since $l_{i1} + l_{i2} + l_{i0} = 0$, we get the equation $x_{2r-1} + x_{2r} + 1 = 0$ which we call a *hyperbolicity equation of the first type*.

For every truncated tetrahedron $\tilde{\Delta}^3_r$ table 3 holds ($i = 0, 1, 2, 3$).

Each connected component of the boundary of M is represented by a 2-dimensional polyhedron, obtained by gluing the edges of the triangles of the truncated tetrahedra. This polyhedron can be completely described by 6-tuples of integers (see Section 4). Starting from an arbitrary edge l_{ih} of a triangle F_i of a truncated tetrahedron $\tilde{\Delta}^3_r$, we associate to it the value 1. The remaining two edges of F_i are

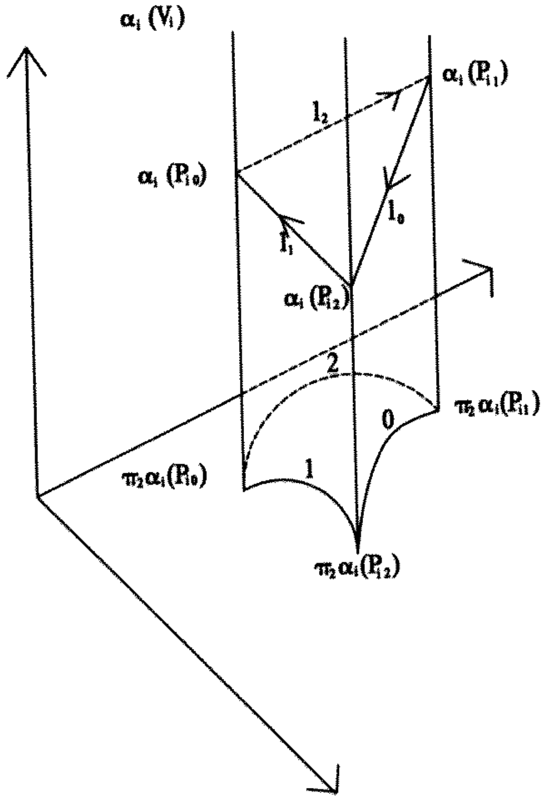


Figure 14: The realization of Δ^3_r as an ideal tetrahedron of $\mathbb{H}^3$.

represented by expressions depending on x_{2r-1} and x_{2r}, according to Table 3. If a triangle F'_j of a truncated tetrahedron $\tilde{\Delta}^3_s$ is glued to F_i of $\tilde{\Delta}^3_r$ along an edge having the value a, then we associate the value $-a$ to the edge of F'_j which is attached to F_i. Starting from this edge we can determine the expressions for the other two edges of F'_j according to Table 3 (use s instead of r). In this way we can express all the edges in terms of the variables x_{2l-1}, x_{2l}, for any $l = 1, 2, \ldots, n$ (where n is the number of tetrahedra). By this procedure, some edges (in particular the edges of the boundary of the polygon, but not only them) can be achieved in two different ways. So they can be obtained by two different expressions. Requir-

$\frac{l_{ih}}{l_{ik}}$	$h=0$	$h=1$	$h=2$
$k=0$	1	x_{2r-1}	x_{2r}
$k=1$	x_{2r-1}^{-1}	1	$x_{2r-1}^{-1}x_{2r}$
$k=2$	x_{2r}^{-1}	$x_{2r-1}x_{2r}^{-1}$	1

Table 3.

ing that the expressions representing the same edge are equal yields the *hyperbolicity equations of the second type*.

Let us explain this method for the Matveev-Fomenko manifold. We denote by $l_{ij}(r), r=1,2$, the edge number j of triangle i of tetrahedron r. The edges are glued in pairs according to Figure 8. The variables are x_1, x_2, x_3, x_4 and the equations of the first type are

$$x_1+x_2+1=0 \quad \text{and} \quad x_3+x_4+1=0.$$

To get the equations of the second type, we set arbitrarily $l_{00}(1)$=1, and hence $l_{01}(1)=x_1$ and $l_{02}(1)=x_2$, according to Table 3. Since $l_{00}(1)$, $l_{01}(1)$ and $l_{02}(1)$ are glued to $l_{22}(2)$, $l_{00}(2)$ and $l_{32}(2)$, respectively, we have

$$\begin{aligned} l_{22}(2) &= -1 && (1)\\ l_{00}(2) &= -x_1 && (2)\\ l_{32}(2) &= -x_2. && (3)\end{aligned}$$

Applying Table 3, we obtain from (1) the values of the other two edges of the same triangle, i.e.

$$\begin{aligned} l_{20}(2) &= -{x_4}^{-1} && (4)\\ l_{21}(2) &= -x_3{x_4}^{-1}. && (5)\end{aligned}$$

In a similar way, from (2) and (3) we get

$$\begin{aligned} l_{01}(2) &= -x_1x_3, && (6)\\ l_{02}(2) &= -x_1x_4, && (7)\\ l_{30}(2) &= -x_2{x_4}^{-1} && (8)\\ l_{31}(2) &= -x_2x_3{x_4}^{-1}. && (9)\end{aligned}$$

Since $l_{02}(2)$ and $l_{30}(2)$ are glued to $l_{20}(1)$ and $l_{12}(1)$, respectively, we obtain

$$l_{20}(1) = x_1x_4 \tag{10}$$
$$l_{12}(1) = x_2{x_4}^{-1}. \tag{11}$$

Then, from (10) and (11), applying Table 3, we get

$$l_{21}(1) = x_1^2x_4, \tag{12}$$
$$l_{22}(1) = x_1x_2x_4, \tag{13}$$
$$l_{10}(1) = {x_4}^{-1} \tag{14}$$
$$l_{11}(1) = x_1{x_4}^{-1}. \tag{15}$$

From (13) and (14) it follows, respectively,

$$l_{12}(2) = -x_1x_2x_4 \tag{16}$$
$$l_{11}(2) = -{x_4}^{-1}. \tag{17}$$

Hence, from (17) we obtain

$$l_{10}(2) = -{x_3}^{-1}{x_4}^{-1} \tag{18}$$
$$l_{12}(2) = -{x_3}^{-1}. \tag{19}$$

Since (16) and (19) both represent $l_{12}(2)$, we get the first equation

$$A)\, x_1x_2x_4 = {x_3}^{-1}.$$

From (18) we have

$$l_{32}(1) = {x_3}^{-1}{x_4}^{-1}, \tag{20}$$

hence

$$l_{30}(1) = {x_2}^{-1}{x_3}^{-1}{x_4}^{-1} \tag{21}$$
$$l_{31}(1) = x_1{x_2}^{-1}{x_3}^{-1}{x_4}^{-1}. \tag{22}$$

Now we have to identify the edges of the boundary, i.e. $l_{21}(2) = -l_{11}(1)$, $l_{01}(2) = -l_{31}(1)$, $l_{20}(2) = -l_{21}(1)$, and $l_{31}(2) = -l_{30}(1)$.

This gives the equations

$$\begin{aligned}
&B)\ \ x_3{x_4}^{-1} = x_1{x_4}^{-1}\\
&C)\ \ x_1x_3 = x_1{x_2}^{-1}{x_3}^{-1}{x_4}^{-1}\\
&D)\ \ {x_4}^{-1} = x_1^2x_4\\
&E)\ \ x_2x_3{x_4}^{-1} = {x_2}^{-1}{x_3}^{-1}{x_4}^{-1}
\end{aligned}$$

which are, together with A), the equations of the second type. The manifold admits a complete hyperbolic structure if and only if the system of equations of the first and second type

$$\begin{cases}
x_1 + x_2 + 1 = 0\\
x_3 + x_4 + 1 = 0\\
x_1x_2x_4 = {x_3}^{-1}\\
x_3{x_4}^{-1} = x_1{x_4}^{-1}\\
x_1x_3 = x_1{x_2}^{-1}{x_3}^{-1}{x_4}^{-1}\\
{x_4}^{-1} = x_1^2x_4\\
x_2x_3{x_4}^{-1} = {x_2}^{-1}{x_3}^{-1}{x_4}^{-1}
\end{cases}$$

has a solution with $\operatorname{Im} x_{2r-1} > 0$ and $\operatorname{Im} x_{2r} < 0$. It is easy to check that the unique solution of the system, satisfying the above conditions, is given by $x_1 = x_3 = \dfrac{-1+i\sqrt{3}}{2}$ and $x_2 = x_4 = \dfrac{-1-i\sqrt{3}}{2}$.

REMARK 6.3. *The complex parameters z_1, z_2, z_3 and w_1, w_2, w_3 of the Matveev-Fomenko manifold are related to x_1, x_2, x_3, x_4 in the following way:*

$$\begin{aligned}
z_1 &= -x_2\\
z_2 &= -x_1x_2^{-1}\\
z_3 &= -{x_1}^{-1}\\
w_1 &= -x_4\\
w_2 &= -x_3x_4^{-1}\\
w_3 &= -{x_3}^{-1}.
\end{aligned}$$

In the general case, we have

$$\begin{aligned} z_1(r) &= -x_{2r} \\ z_2(r) &= -x_{2r-1}{x_{2r}}^{-1} \\ z_3(r) &= -{x_{2r-1}}^{-1} \end{aligned}$$

where $z_i(r)$ denotes the parameter z_i of the tetrahedron r.

Now we describe a partial criterion of hyperbolicity, due to Matveev, whose equations can be directly read off from a special spine representing the bordered manifold M. In Section 3 we have constructed a special spine of M starting from a tetrahedra decomposition. The dihedral angles α_{1i}, α_{2i} and α_{3i} of the tetrahedra are the opposite angles in the special spine (see Figures 5 and 6). So we have a collection of variables α_{1i}, α_{2i}, α_{3i} which satisfy the system of equations

$$\alpha_{1i} + \alpha_{2i} + \alpha_{3i} = \pi, \qquad i = 1, 2, \ldots, n, \tag{$*$}$$

where n is the number of vertices of the special spine or, equivalently, the number of tetrahedra. Since each 2-component of the special spine is a 2-cell, with angles $\alpha_{j_1 i_1}, \alpha_{j_2 i_2}, \ldots, \alpha_{j_r i_r}$, we obtain a system of equations

$$\alpha_{j_1 i_1} + \alpha_{j_2 i_2} + \cdots + \alpha_{j_r i_r} = 2\pi \tag{$**$}$$

with as many equations as the number of 2-cells of the special spine.

THEOREM 6.4. *Let M be a compact connected irreducible* 3*-manifold whose boundary is the disjoint union of tori. If the system of equations* $(*) + (**)$ *has a non-negative solution, then M is hyperbolic, i.e. it admits a hyperbolic (in general non complete) structure.*

Now we apply this result to the Matveev-Fomenko manifold. We denote by α_i, $i = 1, 2, \ldots, 6$, the angles of the special spine (see Figure 6): α_1, α_2, α_3 (resp. α_4, α_5, α_6) correspond to dihedral angles of the first (resp. second) tetrahedron represented by the top (resp. bottom) vertex of the spine in Figure 6 (see also Figure 5). Then the system of equations $(*)$ is

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = \pi & (I) \\ \alpha_4 + \alpha_5 + \alpha_6 = \pi & (II). \end{cases}$$

Since the spine shown in Figure 6 has exactly two 2-cells, the system of equations $(**)$ is

$$\begin{cases} \alpha_3 + \alpha_5 + \alpha_2 + \alpha_6 + \alpha_2 + \alpha_5 = 2\pi & (III) \\ \alpha_3 + \alpha_4 + \alpha_1 + \alpha_6 + \alpha_1 + \alpha_4 = 2\pi & (IV). \end{cases}$$

We can observe that the equations $(**)$ can be read in Figure 12 around the points A and B, respectively, by setting $\alpha_1 = \arg z_1$, $\alpha_2 = \arg z_3$, $\alpha_3 = \arg z_2$, and $\alpha_4 = \arg w_1$, $\alpha_5 = \arg w_3$, $\alpha_6 = \arg w_2$. Let us consider the system of equations $(*) + (**)$. Since

$$(I) + (II): \sum_{i=1}^{6} \alpha_i = 2\pi \quad \text{and} \quad (III) + (IV): 2\sum_{i=1}^{6} \alpha_i = 4\pi,$$

we can eliminate, for example, equation (IV) because it is a consequence of the other ones. Therefore, the system of equations $(*) + (**)$ is equivalent to

$$\begin{cases} \alpha_3 = \pi - \alpha_1 - \alpha_2 \\ \alpha_6 = \pi - \alpha_4 - \alpha_5 \\ \alpha_2 - \alpha_1 + \alpha_5 - \alpha_4 = 0. \end{cases} \qquad (***)$$

A non-negative solution of this system is given by $\alpha_i = \frac{\pi}{3}$, $i = 1, 2, \ldots, 6$.

The conditions of completeness must be read on the polygon representing L in Figure 12. For the completeness of the structure we require that the gluings of the corresponding edges of L are made by translations, expressed in terms of α_i. For example, let us consider the gluing of the edge b with its corresponding edge (labeled with the same letter). Then we require that the algebraic sum of the angles encountered through the path from b to its corresponding edge is 0 and that the edge b does not change its lenght through this path. This happens if the quadrilaters $ABB'A'$ are parallelograms, i.e. if their opposite angles are equal. These facts produce the following conditions:

$$\begin{aligned} &(b) & \alpha_3 - \alpha_4 + \alpha_1 - \alpha_6 = 0 \\ &(m) & \alpha_3 - \alpha_4 + \alpha_1 - \alpha_6 = 0 \\ &(h) & -\alpha_2 + \alpha_4 - \alpha_2 + \alpha_4 = 0 \\ &(a) & -\alpha_1 + \alpha_5 - \alpha_1 + \alpha_5 = 0 \end{aligned}$$

and

$$\begin{aligned}
&\alpha_2 = \alpha_6 \\
&\alpha_3 + \alpha_5 = \alpha_4 + \alpha_1 \\
&\alpha_3 = \alpha_5 \\
&\alpha_1 + \alpha_4 = \alpha_2 + \alpha_6.
\end{aligned}$$

It is easy to check that the unique solution of system (∗∗∗), satisfying conditions above, is $\alpha_i = \dfrac{\pi}{3}$, $i = 1, 2, \ldots, 6$.

7. Hyperbolic manifolds of low complexity

We illustrate the constructions and the algorithms discussed above for some examples of hyperbolic manifolds. It was shown in [16] that among all irreducible atoroidal 3-manifolds of complexity less or equal to 3 (with torus boundary), there are exactly two hyperbolic manifolds M_1 and M_2 of complexity 2 and nine hyperbolic manifolds of complexity 3. The manifold M_1 is the Matveev-Fomenko manifold considered in the previous sections, which can be obtained as the complement of a certain knot in the lens space $L(5,1)$; the manifold M_2 is the complement in $\mathbb{S}^3$ of the figure eight knot (for knot theory we refer, for example, to [21]). The hyperbolic 3-manifolds of complexity 3 are homeomorphic to the complements of certain knots embedded in the standard 3-sphere, in the real projective 3-space, and in the lens spaces $L(3,1)$, $L(5,1)$, $L(6,1)$, $L(7,2)$, and $L(9,2)$.
It is well known that M_2 can be constructed by gluing two tetrahedra with truncated vertices according to Figure 15a. A triangulation of the torus boundary of M_2 is shown in Figure 15b (compare with [1] and [20]). It is also known that M_2 admits a complete hyperbolic structure corresponding to the complex parameters

$$z = w = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

We can construct the graph of the gluing of M_2 and the special spine corresponding to it (see Figure 16). In particular, we observe that the special spine of Figure 16 is equivalent in the sense of Theorem 2.2 to the special spine shown in [16].

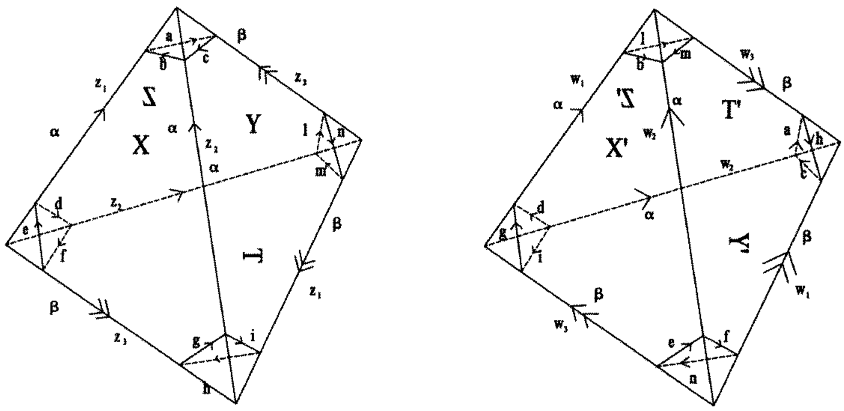


Figure 15a: The side pairing of the complement in $\mathbb{S}^3$ of the figure-eight knot.

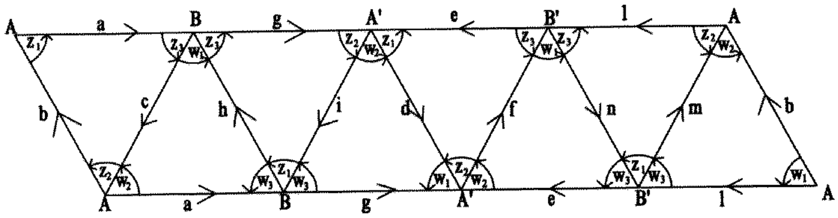


Figure 15b: The torus boundary of the manifold M_2 in Figure 15a.

The gluing and the corresponding graph are described by the 5-tuples of integers: $(1,0,2,3,1)$, $(1,1,2,1,0)$, $(1,2,2,2,0)$, $(1,3,2,0,1)$. The torus boundary of M_2 is encoded by the twelve 6-tuples:

$$\begin{array}{ll}
(1,2,0,2,0,2) & (1,0,0,2,0,0)\\
(1,1,1,2,2,1) & (1,3,1,2,1,2)\\
(1,3,2,2,1,0) & (1,1,2,2,3,1)\\
(1,3,0,2,3,0) & (1,1,0,2,3,2)\\
(1,0,1,2,2,2) & (1,2,1,2,1,1)\\
(1,2,2,2,0,1) & (1,0,2,2,2,0).
\end{array}$$

Starting from an arbitrary edge of the triangulation of the boundary

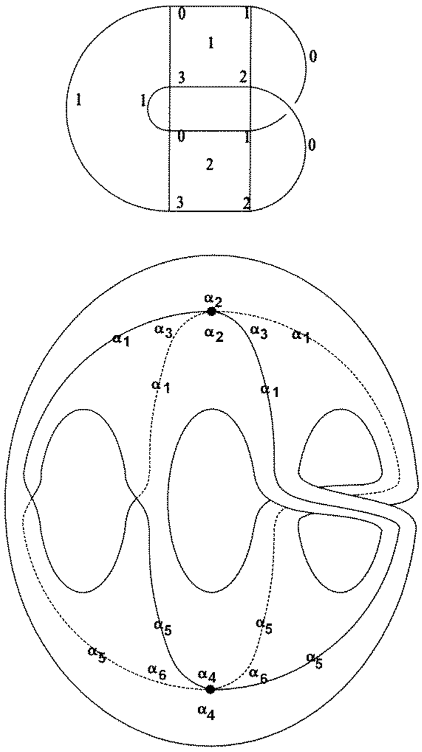


Figure 16: The graph of the gluing and a special spine of the complement of the figure-eight knot.

we obtain a system of hyperbolicity equations of the first and second type which is equivalent to the following system:

$$\begin{cases} x_1 + x_2 + 1 = 0 \\ x_3 + x_4 + 1 = 0 \\ x_1 = x_2 x_4 \\ x_3 = x_1 \\ x_2^2 = x_4^2. \end{cases}$$

The unique solution, satisfying the conditions $\operatorname{Im} x_{2r-1} > 0$ and $\operatorname{Im} x_{2r} < 0$, $r = 1, 2$, is given by $x_1 = x_3 = \dfrac{-1 + i\sqrt{3}}{2}$ and $x_2 =$

$x_4 = \dfrac{-1 - i\sqrt{3}}{2}.$
From the 2-cells of the special spine in Figure 16 we can read off the hyperbolicity equations in terms of angles α_i, $i = 1, 2, \ldots, 6$, i.e.

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = \pi \\ \alpha_4 + \alpha_5 + \alpha_6 = \pi \\ 2\alpha_1 + \alpha_3 + 2\alpha_5 + \alpha_6 = 2\pi \\ 2\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_6 = 2\pi. \end{cases} \qquad (*) + (**)$$

For the completeness we add the conditions that the gluings of the corresponding edges of the boundary in Figure 15b are made by translations and that the opposite angles of quadrilaters are equal. Then we get the unique solution $\alpha_i = \dfrac{\pi}{3}$, for any $i = 1, 2, \ldots, 6$.

Now we consider the gluing 3-manifold M_3 of complexity four represented by the special spine depicted in Figure 17. This manifold was constructed as the complement of a certain knot in a homology sphere of Heegaard genus 2 (see [23]).
From the 2-cells of the special spine in Figure 17 we read off the hyperbolicity equations in terms of angles α_i, $i = 1, 2, \ldots, 12$, i.e.

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = \pi \quad (I) \\ \alpha_4 + \alpha_5 + \alpha_6 = \pi \quad (II) \\ \alpha_7 + \alpha_8 + \alpha_9 = \pi \quad (III) \\ \alpha_{10} + \alpha_{11} + \alpha_{12} = \pi \quad (IV) \\ \alpha_1 + \alpha_7 + 2\alpha_{10} = 2\pi \quad (V) \\ \alpha_5 + 2\alpha_8 + \alpha_{11} = 2\pi \quad (VI) \\ \alpha_3 + \alpha_5 + \alpha_1 + \alpha_6 + \alpha_9 + \alpha_{12} = 2\pi \quad (VII) \\ 2\alpha_2 + 2\alpha_4 + \alpha_7 + \alpha_3 + \alpha_{12} + \alpha_9 + \alpha_6 + \alpha_{11} = 2\pi \quad (VIII). \end{cases} \qquad (*) + (**)$$

As $(I)+(II)+(III)+(IV) : \sum_{i=1}^{12} \alpha_i = 4\pi$ and $(V)+(VI)+(VII)+(VIII) : 2\sum_{i=1}^{12} \alpha_i = 8\pi$, we can eliminate equation $(VIII)$. Thus

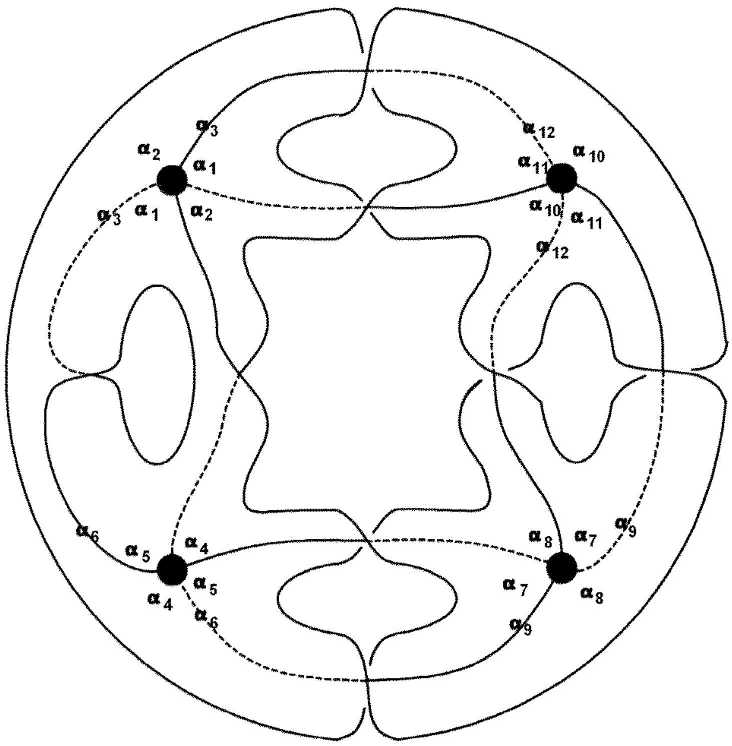


Figure 17: A special spine of the hyperbolic 3-manifold M_3.

the system $(*) + (**)$ is equivalent to the system

$$\begin{cases} \alpha_3 = \pi - \alpha_1 - \alpha_2 \quad (I) \\ \alpha_6 = \pi - \alpha_4 - \alpha_5 \quad (II) \\ \alpha_9 = -\alpha_1 + 2\alpha_2 + 2\alpha_4 - 2\alpha_5 - 3\alpha_8 + 3\pi \quad (III)' \\ \alpha_{12} = \alpha_1 - \alpha_2 - \alpha_4 + 2\alpha_5 + 3\alpha_8 - 3\pi \quad (IV)' \\ \alpha_7 = \alpha_1 - 2\alpha_2 - 2\alpha_4 + 2\alpha_5 + 2\alpha_8 - 2\pi \quad (V)' \\ \alpha_{11} = 2\pi - \alpha_5 - 2\alpha_8 \quad (VI) \\ \alpha_{10} = -\alpha_1 + \alpha_2 + \alpha_4 - \alpha_5 - \alpha_8 + 2\pi \quad (VII). \end{cases} \qquad (***)$$

A non-negative solution is given by $\alpha_1 = \alpha_5 = \epsilon\pi$, $\alpha_2 = \alpha_4 = \beta\pi$, $\alpha_8 = \psi\pi$, $\alpha_3 = \alpha_6 = (1 - \epsilon - \beta)\pi$, $\alpha_7 = (3\epsilon - 4\beta + 2\psi - 2)\pi$, $\alpha_9 =$

$(-3\epsilon+4\beta-3\psi+3)\pi$, $\alpha_{10} = (-2\epsilon+2\beta-\psi+2)\pi$, $\alpha_{11} = (2-\epsilon-2\psi)\pi$ and $\alpha_{12} = (3\epsilon-2\beta+3\psi-3)\pi$, where $\epsilon = 0.15$, $\beta = 0.05$ and $\psi = 0.90$.

Finally, we consider the four truncated tetrahedra shown in Figure 18, glued together along their faces. The gluing 3-manifold M_4 is homeomorphic to the complement in $\mathbb{S}^3$ of the Whitehead link (compare with [20]). The triangulations of the links L_1 and L_2 of the removed vertices v and w are drawn in Figure 19.

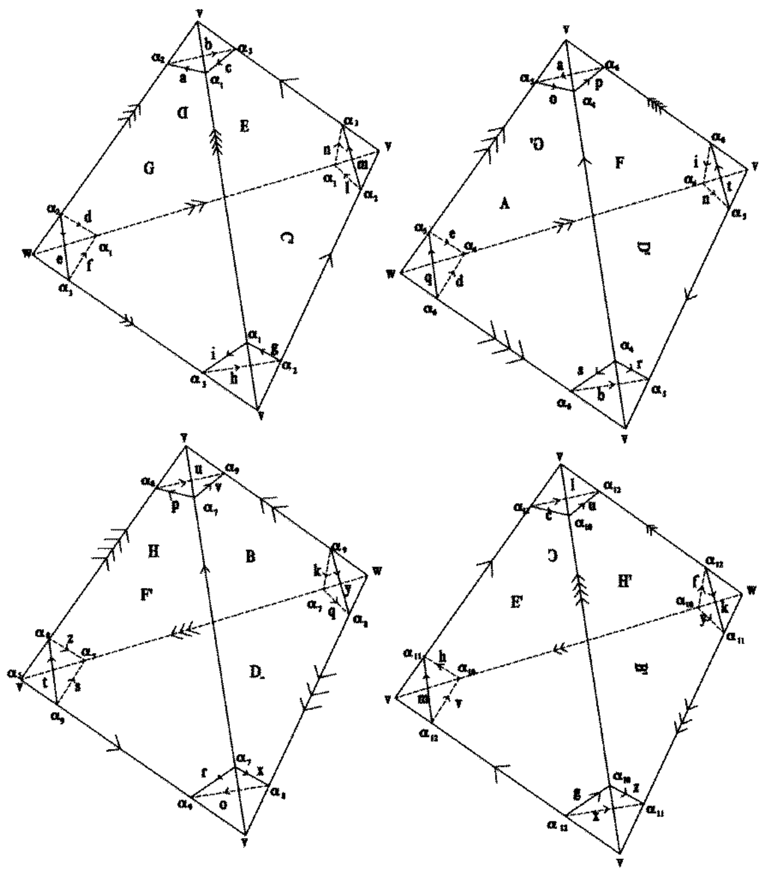


Figure 18: A gluing for the complement in $\mathbb{S}^3$ of the Whitehead link.

The graph of the gluing which represents M_4 and the corresponding special spine are shown in Figure 20 and Figure 21, respectively.
From the 2-cells of the special spine in Figure 21 we read off the

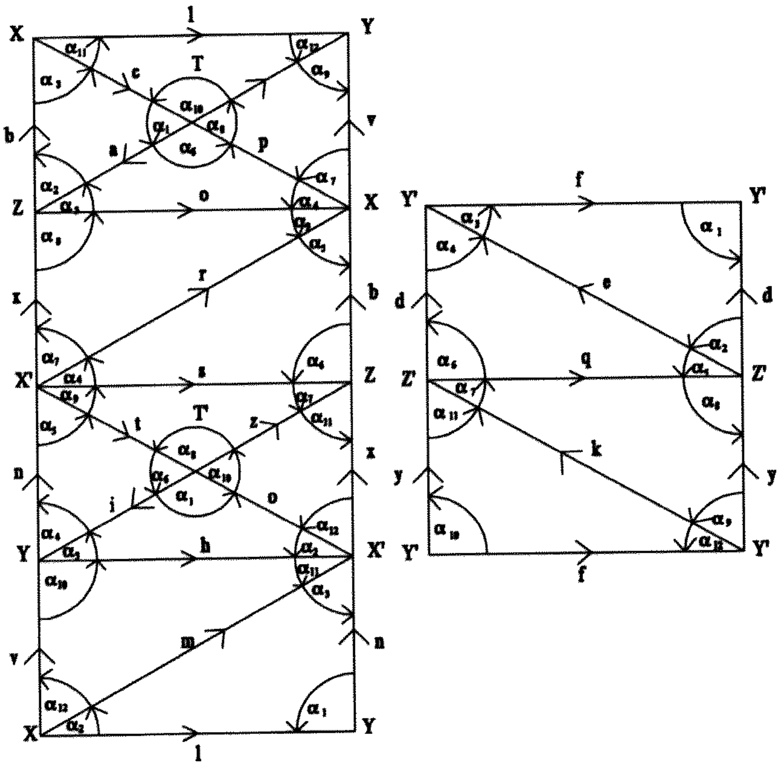


Figure 19: The triangulation of the boundary of the manifold in Figure 18.

hyperbolicity equations in terms of angles α_i, $i = 1, 2, \ldots, 12$, i.e.

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = \pi \quad (I) \\ \alpha_4 + \alpha_5 + \alpha_6 = \pi \quad (II) \\ \alpha_7 + \alpha_8 + \alpha_9 = \pi \quad (III) \\ \alpha_{10} + \alpha_{11} + \alpha_{12} = \pi \quad (IV) \\ \alpha_3 + \alpha_{11} + \alpha_2 + \alpha_{12} + \alpha_7 + \alpha_4 + \alpha_9 + \alpha_5 = 2\pi \quad (V) \\ \alpha_{12} + \alpha_9 + \alpha_{10} + \alpha_3 + \alpha_4 + \alpha_1 = 2\pi \quad (VI) \\ \alpha_6 + \alpha_2 + \alpha_5 + \alpha_8 + \alpha_{11} + \alpha_7 = 2\pi \quad (VII) \\ \alpha_1 + \alpha_{10} + \alpha_8 + \alpha_6 = 2\pi \quad (VIII). \end{cases} \qquad (*) + (**)$$

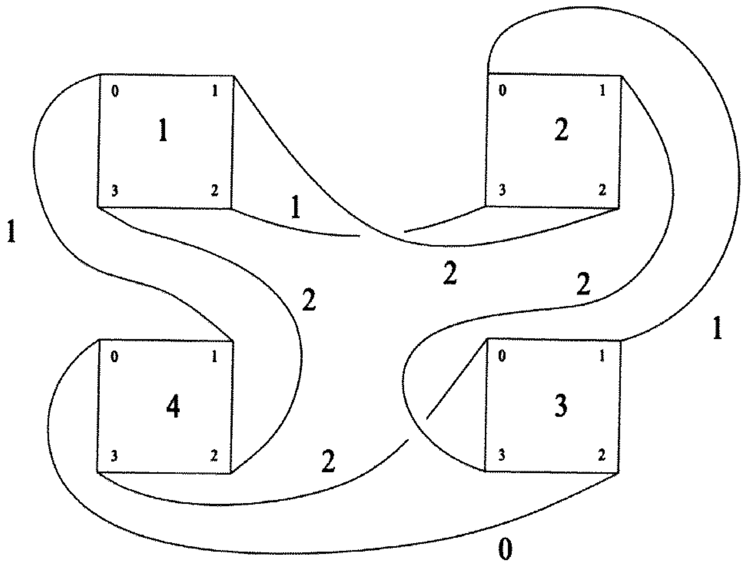


Figure 20: The graph of the gluing of the complement of the Whitehead link.

The system $(*) + (**)$ is equivalent to the system

$$\begin{cases} \alpha_3 = \pi - \alpha_1 - \alpha_2 \quad (I) \\ \alpha_6 = \pi - \alpha_4 - \alpha_5 \quad (II) \\ \alpha_9 = \pi - \alpha_7 - \alpha_8 \quad (III) \\ \alpha_{12} = \pi - \alpha_{10} - \alpha_{11} \quad (IV) \\ \alpha_2 - \alpha_4 + \alpha_7 + \alpha_8 + \alpha_{11} = \pi \quad (VI)' \\ \alpha_1 - \alpha_4 - \alpha_5 + \alpha_8 + \alpha_{10} = \pi \quad (VIII)'. \end{cases} \qquad (***)$$

A non-negative solution is given by $\alpha_1 = \alpha_6 = \alpha_8 = \alpha_{10} = \dfrac{\pi}{2}$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_7 = \alpha_9 = \alpha_{11} = \alpha_{12} = \dfrac{\pi}{4}$.

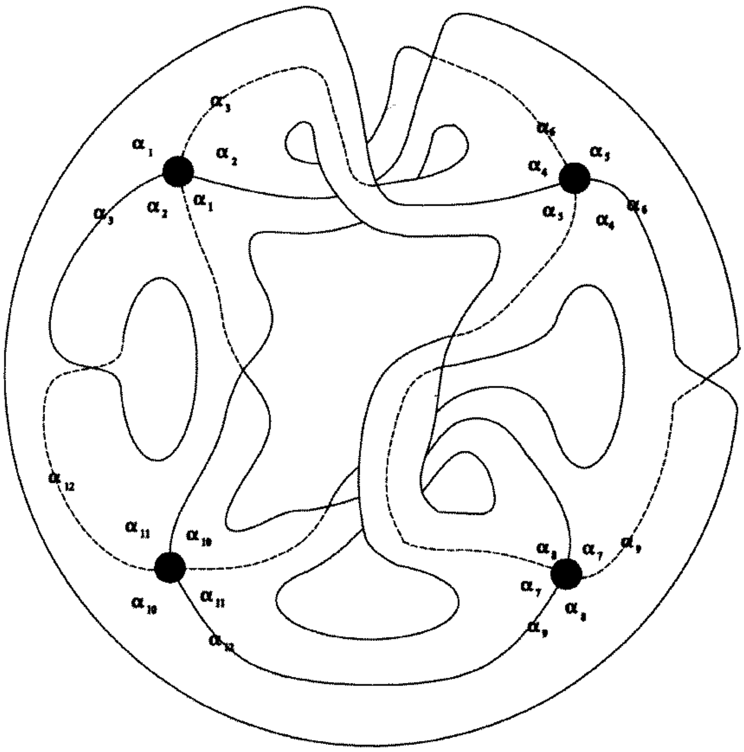


Figure 21: A special spine of the complement of the Whitehead link.

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