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Global Existence for a Quasilinear Maxwell System

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Summary. - *In this work we deal with quasilinear Maxwell system*

$$\begin{cases} \partial_t(\epsilon_0 E + \Phi(E)) = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases}$$

where $\epsilon_0 = diag(a^2, b^2, b^2)$ is a diagonal matrix and Φ is a smooth matrix such that $|\Phi|$ has polynomial growth near $E = 0$. Under suitable hypotheses on Φ, we establish a global existence result for small amplitude solutions. The main argument is the study of pseudo-differential equations obtained diagonalizing the system and using for these equations a particular von Wahl-type estimate described in our previous paper [5].

1. Introduction

The aim of this paper is to describe some global existence results on quasilinear Maxwell system

$$\begin{cases} \partial_t(\epsilon_0 E + \Phi(E)) = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases} \tag{1}$$

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where $E(x,t) = (E_1, E_2, E_3) \in \mathbb{R}^3$, $H(t,x) = (H_1, H_2, H_3) \in \mathbb{R}^3$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$. We take a diagonal matrix $\epsilon_0 = diag(a^2, b^2, b^2)$ and a smooth matrix Φ such that

$$|\Phi(E)| = O(|E|^p), \quad \text{near E} = 0, \quad p \in \mathbb{N}.$$

In this context the following two conservation laws naturally arise (see [7]):

$$\begin{cases} \operatorname{div}(\epsilon_0 E + \Phi(E)) = 0, \\ \operatorname{div} H = 0. \end{cases}$$

From physical point of view E, H represent respectively the electric and the magnetic vector fields. The most common model is isotropic linear Maxwell system where $\epsilon_0(E)$ is the identity matrix and $\Phi = 0$; here instead we are going to point our attention to some anisotropic media, namely the crystals. Moreover, a particular interest concerns some nonlinear models in electro-magnetics (see [2]); for this reason we consider a quasilinear perturbation to Maxwell system.

Dealing with a nonlinear perturbation having polynomial growth near $E = 0$, we treat with small amplitude solutions; then a global existence result will require p sufficiently large. More precisely, if $p \geq 4$ we find that there exists a unique global classical solution for (1) when considering small initial data which verify previous divergence free conditions.

The idea is to diagonalize the linear system associated to (1) and reduce it to single pseudo-differential equations. In [5] we studied scalar equations of type

$$u_t - \mathrm{i}\lambda(D)u = 0,$$

where $\lambda(D)$ is a first order pseudo-differential operator of convolution type, having symbol a real function homogeneous of degree one. We established a decay estimate for the solution of this equation when the manifold

$$\Sigma_\lambda = \{\lambda(\xi) = 1\}$$

is strictly convex. Our result may be regarded as a generalization of classical von Wahl estimate (cf. [8]) for the wave equations; in particular, taking initial data in $W^{k,1}$, the L^∞ norm of the solution decays like $t^{(n-1)/2}$.

It is worth to mention that the strict convexity of the manifold Σ_λ is not fulfilled by the characteristic roots of

$$\begin{cases} \epsilon_0 \partial_t E = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases}$$

if ϵ_0 has three different entries. This loss of convexity implies that the expected decay rate t^{-1} is not available and is replaced by a weaker $t^{-1/2}$. This phenomenon was analyzed by O. Liess in [4] and reflects the physical difference between uniaxial and biaxial crystals. In uniaxial crystals, there is a single axis along which light can propagate without exhibiting double refraction; along other axis a light beam splits into two different components which travel at different velocities. This corresponds to the choice $\epsilon_0 = diag(a^2, b^2, b^2)$. On the contrary, since biaxial crystals have low symmetry, conical refraction take place: a ray incident on a surface of the crystal in a certain direction splits into a family of rays which lie along a cone. To examine this case one takes $\epsilon_0 = diag(a^2, b^2, c^2)$ with $a \neq b \neq c \neq 0$.

The plan of the work is as follows: in section 2 we recall the decay estimate for the solution of the pseudo-differential equation established in [5]. In section 3 we apply this result to quasilinear Maxwell system in uniaxial crystals. Finally, in the last section we sketch the proof for a similar result in biaxial crystals.

Few remarks about the notations: we put $\|f\|_p := \|f\|_{L^p}$. The inner product of $x, y \in \mathbb{R}^n$ will be denoted by $\langle x, y\rangle$. With $\mathrm{Jac}_x f$ and $\nabla^2 f$ we mean respectively the Jacobian and the Hessian matrix of a smooth function $f(x)$. Our choice for the coefficients of Fourier transform is

$$\hat{f}(x) = \mathcal{F}(f) = (2\pi)^{-n/2} \int \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} f(\xi)\, \mathrm{d}\xi.$$

If f depends on (x, t) then we use Fourier transform with respect to the space variable only. Finally, given a space B of real functions defined on $\mathbb{R}^n$, considering a vector function $\Phi : \mathbb{R}^n \to \mathbb{R}^m$, by $\|\Phi\|_B$ we mean the norm of $|\Phi|$ in B.

2. A von Wahl type estimate

We consider the Cauchy problem

$$\begin{cases} u_t - \mathrm{i}\lambda(D)u = 0, \quad x \in \mathbb{R}^n,\, n \geq 2 \\ u(x,0) = g(x), \end{cases} \tag{2}$$

where $\lambda(D)$ is a first order pseudo-differential operator having symbol λ. We shall put

$$\Sigma_\lambda := \{\lambda(\xi) = 1\},$$

and require that Σ_λ is strictly convex in the sense of the following definition.

Definition 2.1. *(see [1]) Let $\lambda \geq 0$ be a smooth function. For any $a \in \mathbb{R}_+$ we consider the hypersurface*

$$\Sigma_{\lambda,a} := \{\lambda(x) = a\}.$$

We say that $\Sigma_{\lambda,a}$ is strictly convex if $\{\lambda(x) \leq a\}$ is a convex set and Gaussian curvature for $\Sigma_{\lambda,a}$ is strictly positive.

In other words $\Sigma_{\lambda,a}$ can be locally represented, as a graph of a convex function having non-degenerate Hessian.

Lemma 2.2. *Let $\Sigma_{\lambda,a}$ be as above. For any $\omega_0 \in \Sigma_{\lambda,a}$ there exist $j \in 1,\dots,n$ and a local chart (V,ψ) such that*

$$\omega_j = \psi(\omega_1,\cdots,\omega_{j-1},\omega_{j+1},\cdots,\omega_n) \quad \forall\omega \in V.$$

Since $\omega_0 \in V$, denoting by $\theta_0 = (\omega_{0_1},\cdots,\omega_{0_{j-1}},\omega_{0_{j+1}},\cdots,\omega_{0_n})$ one has $\psi(\theta_0) = 0$, and $\nabla\psi(\theta_0) = 0$. Finally,$\nabla^2\psi$ is positive defined.

This lemma in turn implies that $\{\lambda(x) \leq a\}$ is on one side of the tangent space to $\Sigma_{\lambda,a}$. The strict convexity for Σ_λ is the key assumption for the proof of the L^1-L^∞ estimate obtained in [5]:

Theorem 2.3. *Let $\lambda(D)$ be an elliptic pseudo-differential operator with real symbol $\lambda(\xi)$ homogeneous of degree 1 which satisfies either*

(i) *$\lambda(\xi) \geq 0$, and Σ_λ is a strictly convex and compact set of $\mathbb{R}^n$, or*

(ii) $\lambda(\xi) \leq 0$, *and* $\Sigma_{-\lambda}$ *is a strictly convex and compact set of* $\mathbb{R}^n$.

Let $u(x,t)$ *be the solution to the Cauchy Problem* (2) *with* $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. *Then:*

$$\|u(t)\|_\infty \leq Ct^{-\frac{n-1}{2}} \|g\|_{W^{[\frac{n}{2}]+h,1}} \quad \forall t \geq 1, \tag{3}$$

where $h = 1$ *if* n *is even,* $h = 2$ *if* n *is odd.*

We remark that the assumption $g \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ can be relaxed by the aid of a standard density argument. The proof is based of stationary phase method applied to the solution

$$u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle x-y,\xi\rangle + \mathrm{i}t\lambda(\xi)} g(y)\,\mathrm{d}\xi\,\mathrm{d}y.$$

After integration by parts with respect to the variable y, using polar coordinates

$$\begin{cases} \rho = \lambda(\xi), \\ \omega = \xi/\lambda(\xi), \end{cases}$$

we reduce to the estimate of the kernel

$$\int_0^{+\infty} \rho^{n-1-k} \int_{\Sigma_\lambda} \mathrm{e}^{\mathrm{i}\rho t\left(\frac{\langle x-y,\omega\rangle}{t}+1\right)} |\nabla\lambda(\omega)|^{-1}\,\mathrm{d}\sigma_{n-1}(\omega)\,\mathrm{d}\rho. \tag{4}$$

Hence we describe the surface Σ_λ by means of strictly convex charts given by Lemma 2.2 and we arrive to a sum of integrals over $\mathbb{R}^{n-1}$. To these integrals, we can apply stationary phase method with parameter ρt. In turn this gives the decay rate $t^{-(n-1)/2}$.

REMARK 2.4. *For the application to Maxwell system we need a suitable modification of the previous result. Let us consider the Cauchy problem*

$$\begin{cases} u_t - \mathrm{i}\lambda(D)u = 0, & x \in \mathbb{R}^n \\ u(x,0) = Qg(x), \end{cases}$$

where $Qg = \mathcal{F}^{-1}(M(\xi)\hat{g}(\xi))$. *If* $M(\xi)$ *is smooth and homogeneous of degree zero, then the inequality (3) still holds. Namely, with the same proof of the previous theorem one can see that it is possible to avoid* Q *in the norm on the right side of this formula.*

3. Quasilinear Maxwell system in uniaxial crystals

3.1. The linear Cauchy Problem

In this section we discuss linear Maxwell system in anisotropic media. We shall apply the obtained results to the quasilinear case. Let us consider

$$\begin{cases} \epsilon_0 \partial_t E = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases} \tag{5}$$

where $E(x,t) = (E_1, E_2, E_3)$, $H(t,x) = (H_1, H_2, H_3)$ and $x \in \mathbb{R}^3$, $t \in \mathbb{R}$ and ϵ_0 is a 3 by 3 matrix. For simplicity we can take ϵ_0 in diagonal form. Moreover for the reasons illustrated in the introduction, we take

$$\epsilon_0 = diag\,(a^2, b^2, b^2), \quad a \neq b \neq 0. \tag{6}$$

Finally, we denote by (E_0, H_0) the initial data

$$E_0 := E(0),\ H_0 := H(0).$$

Taking $\underline{u} = (\epsilon_0^{1/2} E, H)$, we can rewrite this system as

$$\partial_t \underline{u} - B\underline{u} = 0, \tag{7}$$

with

$$B := \begin{bmatrix} 0 & \epsilon_0^{-1/2} \operatorname{curl} \\ -\operatorname{curl} \epsilon_0^{-1/2} & 0 \end{bmatrix}. \tag{8}$$

In Fourier transform coordinates we get

$$\begin{cases} \hat{\underline{u}}' - \mathrm{i} B(\xi) \hat{\underline{u}} = 0, \\ \hat{\underline{u}}(0) = (\epsilon_0^{1/2} \hat{E}_0, \hat{H}_0), \end{cases}$$

where

$$\begin{aligned} B(\xi) :\ &= \begin{bmatrix} 0 & -\epsilon_0^{-1/2} \xi \wedge \\ \xi \wedge \epsilon_0^{-1/2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ U^T & 0 \end{bmatrix}, \\ U :\ &= \begin{bmatrix} 0 & -|a|^{-1}\xi_3 & |a|^{-1}\xi_2 \\ |b|^{-1}\xi_3 & 0 & -|b|^{-1}\xi_1 \\ -|b|^{-1}\xi_2 & |b|^{-1}\xi_1 & 0 \end{bmatrix}. \end{aligned}$$

We see that $B(\xi)$ is symmetric, hence it is diagonalizable with eigenvalues

$$\begin{aligned}
&\lambda_{1,2} = 0,\\
&\lambda_{3,4}(\xi) = \pm\sqrt{b^{-2}\xi_1^2 + a^{-2}\xi_2^2 + a^{-2}\xi_3^2},\\
&\lambda_{5,6}(\xi) = \pm b^{-2}|\xi|.
\end{aligned}$$

The most relevant point is that Σ_{λ_j} is strictly convex for $j \geq 3$. Let $\Lambda(\xi)$ be the diagonal matrix of these eigenvalues. Since the multiplicity of λ_j is constant, there exists a smooth matrix $Q(\xi)$ such that

$$Q^{-1}(\xi)\Lambda(\xi)Q(\xi) = B(\xi).$$

Denoting by $\hat{v} = Q(\xi)\underline{\hat{u}}$, we arrive at the diagonal system

$$\begin{cases} \hat{v}' = \mathrm{i}\Lambda(\xi)\hat{v},\\ \hat{v}(0) = Q(\xi)(\epsilon_0^{1/2}\hat{E}_0, \hat{H}_0). \end{cases} \tag{9}$$

If $v_1 = v_2 = 0$ then we can apply Theorem 2.3, obtaining a decay estimate for v, whence for u.

Now we prove that the condition $v_1 = v_2 = 0$ occurs whenever we associate to (5) the other two physical Maxwell conditions

$$\begin{cases} \operatorname{div} \epsilon_0 E_0 = 0,\\ \operatorname{div} H_0 = 0. \end{cases} \tag{10}$$

First of all, we have the conservation laws

$$\begin{cases} \operatorname{div} \epsilon_0 E_0 = \operatorname{div} \epsilon_0 E(t),\\ \operatorname{div} H_0 = \operatorname{div} H(t). \end{cases} \tag{11}$$

We notice that

$$\operatorname{div} \epsilon_0 E_0 = 0 = \operatorname{div} H_0 \iff \underline{\hat{u}}(0) \perp \operatorname{Ker} B(\xi).$$

In fact $\operatorname{div} \epsilon_0 E_0 = 0 = \operatorname{div} H_0$ is equivalent to $\langle \xi, \epsilon_0 \hat{E}_0\rangle = 0 = \langle \xi, \hat{H}_0\rangle$. On the other hand, $w \in \operatorname{Ker} B(\xi)$ if and only if $w = (w_1, w_2)$ such that $\epsilon_0^{1/2}\xi \wedge w_2 = 0 = \xi \wedge \epsilon_0^{-1/2} w_1$ that is $w_2 = \beta\xi$, $w_1 = \alpha\epsilon_0^{1/2}\xi$, for some $\alpha, \beta \in \mathbb{R}$. These informations are sufficient to obtain

$\langle \epsilon_0^{1/2}\hat{E}_0, w_1\rangle = \langle \hat{H}_0, w_2\rangle = 0$; this yields $(\epsilon_0^{1/2}\hat{E}_0, \hat{H}_0) \perp \operatorname{Ker} B(\xi)$. As a consequence, (11) gives

$$\underline{\hat{u}}(t) \perp \operatorname{Ker} B(\xi) \quad \text{for all } t \geq 0.$$

Since $Q(\xi)$ has the first two rows that span $\operatorname{Ker} B(\xi)$, one deduces $v_1 = v_2 = 0$. Moreover, denoting by $\pi(\lambda_j)$ the projection on the eigenspace corresponding to λ_j and using the identity

$$\mathbb{I} = \pi_{\operatorname{Ker} B} + \sum_{j=3}^{6} \pi(\lambda_j),$$

for any initial data which satisfy (10), one gets

$$\underline{\hat{u}}(t) = \sum_{j=3}^{6} \pi(\lambda_j) Q^{-1}(\xi)\hat{v}(t).$$

On the other hand, if $\hat{v}$ solves (9), then $\pi(\lambda_j)Q^{-1}(\xi)\hat{v}$ solves $\hat{w}' = \mathrm{i}\lambda_j(\xi)\hat{w}$ with initial data $\pi(\lambda_j)\hat{u}(0)$. Since λ_j are simple eigenvalues we deduce that $\pi(\lambda_j)$ as function of ξ is homogeneous of degree zero. Then we can combine Theorem 2.3 and Remark 2.4 obtaining the decay of the solution $\underline{u}$ from the decay of each component $\pi(\lambda_j)Q^{-1}(\xi)\hat{v}$. We summarize these properties in the following result.

Proposition 3.1. *Let us denote by* $\mathrm{e}^{tB}(E_0, H_0)$ *the solution of* (5) *with initial data* $(E_0, H_0) \in C_0^\infty(\mathbb{R}^3, \mathbb{R}^6)$. *Suppose* (E_0, H_0) *satisfy*

$$\begin{cases} \operatorname{div} \epsilon_0 E_0 = 0, \\ \operatorname{div} H_0 = 0. \end{cases}$$

Then for $t \geq 0$ *we have*

$$\begin{cases} \operatorname{div} \epsilon_0 E(t) = 0, \\ \operatorname{div} H(t) = 0, \end{cases}$$

and the following decay estimate holds:

$$\|\mathrm{e}^{tB}(E_0, H_0)\|_\infty \leq C(1+t)^{-1}\|(E_0, H_0)\|_{W^{3,1}}. \tag{12}$$

3.2. The quasilinear Cauchy Problem

We can apply the result of the previous section to the following quasilinear Maxwell system in anisotropic media:

$$\begin{cases} \partial_t(\epsilon_0 E + \Phi(E)) = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E. \end{cases} \tag{13}$$

with initial data (E_0, H_0). Here ϵ_0 is given by (6) and the nonlinear perturbation Φ is smooth function of E with polynomial growth near $E = 0$:

$$|\Phi(E)| = O(|E|^p) \quad p \in \mathbb{N}.$$

Our aim is to study the global existence theory for (13), whenever p is sufficiently large and data are small and with compact support.

Our first step consists in writing (13) in the form

$$u_t - Bu = F \tag{14}$$

with B a skew-adjoint operator. First we apply implicit function theorem to the function

$$M(E) = \epsilon_0 E + \Phi(E)$$

and obtain, in a neighborhood of $M = 0$, the rapresentation

$$E = \epsilon_0^{-1} M + \Psi(M)$$

with

$$|\Psi(M)| \simeq |M|^p.$$

Meanwhile, we observe that for any Φ such that $\mathrm{Jac}_E \Phi$ is symmetric, we get $\mathrm{Jac}_M \Psi$ is symmetric.

Since we deal with small amplitude solutions, it is now possible to rewrite (13) in the form

$$\begin{cases} \partial_t M = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} \epsilon_0^{-1} M + \operatorname{curl} \Psi(M). \end{cases} \tag{15}$$

with initial data $(\epsilon_0 E_0 + \Phi(E_0), H_0)$. Taking $u = (\epsilon_0^{-1/2} M, H) = (u_1, u_2)$, and

$$F(t) = (0, -\operatorname{curl}(\Psi(\epsilon_0^{-1/2} u_1(t))),$$

we obtain the expression (14) with B given by (8), which is a skew-adjoint operator on $L^2(\mathbb{R}^3, \mathbb{R}^6)$.
For this reason we can apply Duhamel's principle and write the solution in the form

$$u(t) = \mathrm{e}^{tB}(\epsilon_0 E_0 + \Phi(E_0), H_0) + \int_0^t \mathrm{e}^{(t-s)B} F(s)\,\mathrm{d}s. \tag{16}$$

If $(\epsilon_0 E_0 + \Phi(E_0), H_0)$ satisfies the divergence free conditions, then we apply the decay estimate. After this preparation we can state and prove our main result:

THEOREM 3.2. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth real function such that*

$$\Phi(E) = \nabla f(E);$$

and

$$|\Phi(E)| = O(|E|^p) \quad \text{near } E = 0 \;\; p \in \mathbb{N},\, p \geq 4.$$

Consider the quasilinear system

$$\begin{cases} \partial_t(\epsilon_0 E + \Phi(E)) = \operatorname{curl} H, \\ \partial_t H = -\operatorname{curl} E, \end{cases} \tag{17}$$

with $\epsilon_0 = diag\,(a^2, b^2, b^2)$, $a \neq b \neq 0$ and initial data (E_0, H_0) verifying

$$\begin{cases} \operatorname{div}(\epsilon_0 E_0 + \Phi(E_0)) = 0, \\ \operatorname{div} H_0 = 0. \end{cases} \tag{18}$$

There exists a small $0 < \varepsilon < 1$ such that if

$$\|E_0\|_{W^{6,1}} + \|E_0\|_{H^7} + \|H_0\|_{W^{6,1}} + \|H_0\|_{H^7} < \varepsilon, \tag{19}$$

then (17) admits a unique global solution $(E, H) : \mathbb{R}^3 \to \mathbb{R}^6$.
Moreover $u \in \mathcal{C}\left([0, +\infty); W^{6,1} \cap H^7\right)$.

We shall look for a global solution (M, H) to the system (15); clearly this will give the global solution for (17).

Fixing $k, h \in \mathbb{N}$, we introduce the following norms:

$$\begin{aligned}
\mathcal{E}_k(u) &= \sum_{|\alpha|=k} \frac{1}{2} \int_{\mathbb{R}^3} \langle \epsilon_0^{-1} D^\alpha M, D^\alpha M \rangle + |D^\alpha H|^2 \,\mathrm{d}x, \\
\mathcal{I}_k(u) &= \sum_{j=0}^{k} \mathcal{E}_j, \\
X_{h,k}(u)(t) &= \sup_{0 \le \tau \le t} \left\{ \sup_{|\alpha| \le h} (1+\tau) \|D^\alpha u(\tau)\|_\infty + \mathcal{I}_k(u)(\tau) \right\}.
\end{aligned}$$

Here and in what follows $\alpha = (\alpha_0, \alpha') = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a space-time multi-index and $D^\alpha = D_t^{\alpha_0} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{x_3}^{\alpha_3}$. Being ϵ_0 positive definite, the following result holds:

$$\|u\|_{H^k} \le \mathcal{I}_k(u).$$

We shall often omit to write u in the previous norms, when we deal with the expression (16).

We start establishing an L^∞ estimate for u. Here the condition on p appears.

Proposition 3.3. *Suppose the assumptions of Theorem* 3.2 *are satisfied. Then we have*

$$\|D^\alpha u(t)\|_\infty \le C(1+t)^{-1} \left[\|u(0)\|_{W^{3+|\alpha|,1}} + X^{p-1}_{[(|\alpha|+4)/2],|\alpha|+4}(t) \right].$$

Proof. We can write $u = \underline{u} + (u - \underline{u})$ where $\bar{u}$ solves (7) with initial data $(M_0, H_0) = (\epsilon_0 E_0 + \Phi(E_0), H_0)$. In order to apply the decay estimate (12) we require that the conditions (18) are conserved in time. This follows directly from (17). Hence we get

$$\|D^\alpha \underline{u}\|_\infty \le C(1+t)^{-1} (\|E_0\|_{W^{3+|\alpha|,1}} + \|\Phi(E_0)\|_{W^{3+|\alpha|,1}} + \|H_0\|_{W^{3+|\alpha|,1}}).$$

At the same time to estimate the function $u - \underline{u}$ we need $\mathrm{div} F(s) = 0$ for all $s \ge 0$; this condition is trivially satisfied since F is expressed by $\mathrm{curl}\,\Psi$ hence

$$\|\mathrm{e}^{(t-s)B} F(s)\|_\infty \le C(1+t-s)^{-1} \|F(s)\|_{W^{3,1}}.$$

Finally, we find

$$\begin{aligned}\|D^\alpha u(t)\|_\infty &\leq C(1+t)^{-1}\|u(0)\|_{W^{3+|\alpha|,1}} + \\ &+ \int_0^t (1+t-s)^{-1}\|D^\alpha \mathrm{curl}\,\Psi(\epsilon_0^{1/2}u_1(s))\|_{W^{3,1}}\,\mathrm{d}s.\end{aligned}$$

It remains to estimate

$$\|D^\alpha \mathrm{curl}\,\Psi(\epsilon_0^{1/2}u_1)\|_{W^{3,1}} \simeq \|D^\alpha\Psi(M)\|_{W^{4,1}} \simeq \sum_{|\beta|\leq|\alpha|+4}\|D^\beta\Psi(M)\|_1$$

Using chain rule we find

$$\begin{aligned}&D^\beta\psi_j(M) = \\ &= \sum_{r=1}^{|\beta|}\sum_{k_1,\cdots,k_r=1}^{3}\sum_{\substack{\beta_1+\cdots+\beta_r=\beta\\ \beta_j\neq 0}} C_{k,\beta}\frac{\partial^r\Psi_j}{\partial M_{k_1}\cdots\partial M_{k_r}}D^{\beta_1}M_{k_1}\cdots D^{\beta_r}M_{k_r}\end{aligned} \tag{20}$$

where $C_{k,\beta}$ stands for $C_{k_1,\ldots,k_r,\beta_1,\ldots\beta_r}$. In the case $r=1$ we can use Cauchy-Schwartz inequality, having in mind that $|\Psi|\simeq|M|^p$; we obtain

$$\left\|\frac{\partial\Psi_j}{\partial M_{k_1}}D^\beta M_{k_1}\right\|_1 \leq \|M\|_\infty^{p-2}\|M\|_2\|D^\beta M_{k_1}\|_2.$$

Suppose $r\geq 2$; we see that

$$\begin{aligned}&\left\|\frac{\partial^r\Psi_j}{\partial M_{k_1}\cdots\partial M_{k_r}}D^{\beta_1}M_{k_1}\cdots D^{\beta_r}M_{k_r}\right\|_1 \leq \\ &\leq \|M\|_\infty^{p-r}\left\|D^{\beta_1}M_{k_1}\cdots D^{\beta_r}M_{k_r}\right\|_1.\end{aligned}$$

We apply the general inequality

$$\begin{aligned}&\sum_{\substack{\gamma_1+\ldots+\gamma_m=\beta\\ |\gamma_j|\geq 1}}|D^{\gamma_1}w|\cdots|D^{\gamma_m}w| \leq \\ &\leq \sum_{|\eta|=1}^{|\beta|-1}|D^\eta w|\sum_{\substack{\eta+\nu_1+\ldots+\nu_m=\\ \gamma_1+\ldots+\gamma_m}}|D^{\nu_1}w|\cdots|D^{\nu_m}w|\end{aligned} \tag{21}$$

with $\nu_i \leq [|\beta|/2]$, $|\eta| \geq \nu_i$. Taking higher derivatives in L^2 norm we find

$$\|D^{\beta_1} M_{k_1} \cdots D^{\beta_r} M_{k_r}\|_1 \leq C(1+t)^{-(r-2)} X^{r-2}_{[|\beta|/2],0} \mathcal{E}_{|\beta|-1}.$$

Finally, this gives

$$\|D^{\beta}\Psi(M)\|_1 \leq C(1+t)^{-(p-2)} X^{p-2}_{[|\beta|/2],0} \mathcal{I}_{|\beta|}.$$

Then we deduce

$$\|D^{\alpha}\mathrm{curl}\,\Psi(\epsilon_0^{1/2} u_1(s))\|_{W^{3,1}} \leq C(1+s)^{-(p-2)} X^{p-1}_{[(|\alpha|+4)/2],|\alpha|+4}(s).$$

Since for $p > 3$, the following inequality is fulfilled:

$$\int_0^t (1+t-s)^{-1}(1+s)^{-(p-2)}\,\mathrm{d}s \leq C(1+t)^{-1};$$

we get the conclusion. ■

Now we prove energy estimates. For the $\mathcal{E}_0$ we have to exploit conservation of energy, multiplying the first equation by $E = \epsilon_0^{-1} M + \Psi(M)$ and the second by H. We see that

$$\mathrm{div}(E \wedge H) = \langle E,\, \mathrm{curl}\ H\rangle - \langle H,\, \mathrm{curl}\ E\rangle. \tag{22}$$

Since g is compactly supported and the system has finite speed of propagation, for all $t \in \mathbb{R}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \langle \epsilon_0^{-1} M, M\rangle + |H|^2 + 2\tilde{\Psi}(M)\,\mathrm{d}x = 0 \tag{23}$$

where $\tilde{\Psi} := \mathbb{R}^3 \to \mathbb{R}$ satisfies $\nabla\tilde{\Psi} = \Psi$.

Proposition 3.4. *With the previous notations, one finds*

$$\begin{aligned}&\mathcal{E}_0(t) \leq \\ &\quad\leq C\left(\|M_0\|_2^2 + \|H_0\|_2^2 + \|M_0\|_{p+1}^{p+1}\right) + C(1+t)^{-(p-1)} X^{p-1}_{0,0}\mathcal{I}_0.\end{aligned}$$

Proof. It suffices to apply (23) and the evident relations

$$\begin{aligned}&\|\tilde{\Psi}(M)\|_1 \leq \|M\|_\infty^{p-1}\mathcal{E}_0(t),\\ &\left|\int_{\mathbb{R}^3} \langle \epsilon_0^{-1} M_0, M_0\rangle + |H_0|^2 + 2\tilde{\Psi}(M_0)\,\mathrm{d}x\right| \leq \\ &\quad\leq \|M_0\|_2^2 + \|H_0\|_2^2 + \|M_0\|_{p+1}^{p+1}.\end{aligned}$$

Here we have used the fact that $\mathrm{Jac}\Psi$ is symmetric. ∎

It remains to obtain higher energy estimates.

PROPOSITION 3.5. *Using the above notations, the following relation holds:*

$$\mathcal{I}_k(t) \le C(M_0, H_0) + C X_{k',0}^{p-1}(t)\mathcal{I}_k(t), \tag{24}$$

where $C(M_0, H_0)$ *is a suitable norm of the initial data and* $k' = \max\{1, [k/2]\}$.

Proof. We first derive the system applying D^α and then we multiply the first equation by $D^\alpha E = D^\alpha(\epsilon_0^{-1}M + \Psi(M))$ and the second by $D^\alpha H$. Using (22), we get

$$\begin{aligned}&\mathrm{div}(D^\alpha H \wedge D^\alpha(\epsilon_0^{-1}M + \Psi(M))) =\\ &\quad = \frac{1}{2}\partial_t\left(|D^\alpha H|^2 + \langle \epsilon_0^{-1}D^\alpha M, D^\alpha M\rangle\right) + \langle D^\alpha\Psi(M), \partial_t D^\alpha M\rangle.\end{aligned}$$

For any $F, G : \mathbb{R}^3 \to \mathbb{R}^3$, we denote $\langle F, G\rangle_2 = \int_{\mathbb{R}^3}\langle F, G\rangle\,\mathrm{d}x$, hence

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{|\alpha|}(t) = \langle D^\alpha\Psi(M), \partial_t D^\alpha M\rangle_2 =\\ &\quad = \sum_{j=1}^{3}\int_{\mathbb{R}^3} D^\alpha\Psi_j(M), \partial_t D^\alpha M_j\,\mathrm{d}x = I + II.\end{aligned}$$

Due to the expression (20), we can take

$$I = \langle \mathrm{Jac}_M\Psi\, D^\alpha M, \partial_t D^\alpha M\rangle_2.$$

We recall our key assumption: $\mathrm{Jac}_M\Psi$ is symmetric; this implies

$$\begin{aligned}&\langle \mathrm{Jac}_M\Psi\, D^\alpha M, \partial_t D^\alpha M\rangle_2 = \frac{1}{2}\partial_t\langle \mathrm{Jac}_M\Psi\, D^\alpha M, D^\alpha M\rangle_2 - A_\alpha(t),\\ &A_\alpha = \frac{1}{2}\langle \partial_t(\mathrm{Jac}_M\Psi)D^\alpha M, D^\alpha M\rangle_2.\end{aligned} \tag{25}$$

It is clear that $II = 0$ when $|\alpha| = 1$. Let us consider the case $|\alpha| \ge 2$.

We may write

$$
\begin{aligned}
II = & \sum_{r=2}^{|\alpha|} \sum_{k_1,\dots,k_r=1}^{3} \sum_{\substack{\beta_1+\cdots+\beta_r=\alpha \\ 0<|\beta_j|<|\alpha|}} C_{k,\beta} \cdot \\
& \cdot \langle \frac{\partial^r \Psi}{\partial M_{k_1} \dots \partial M_{k_r}} D^{\beta_1} M_{k_1} \cdots D^{\beta_r} M_{k_r}, \partial_t D^\alpha M \rangle_2 = \\
& = \sum_{r=2}^{|\alpha|} \sum_{\substack{\beta_1+\cdots+\beta_r=\alpha \\ 0<|\beta_j|<|\alpha|}} \Big(R_{\alpha,r,\beta} + \frac{\mathrm{d}B_{\alpha,r,\beta}}{\mathrm{d}t} \Big) = R_\alpha + \frac{\mathrm{d}B_\alpha}{\mathrm{d}t},
\end{aligned}
$$

where

$$
\begin{aligned}
& B_{\alpha,r,\beta} = \\
& = \sum_{k_1,\dots,k_r=1}^{3} C_{k,\beta} \langle \frac{\partial^r \Psi}{\partial M_{k_1} \dots \partial M_{k_r}} D^{\beta_1} M_{k_1} \cdots D^{\beta_r} M_{k_r}, D^\alpha M \rangle_2, \\
& R_{\alpha,r,\beta} = \\
& = \sum_{k_1,\dots,k_r=1}^{3} \langle \frac{\partial^r \Psi}{\partial M_{k_1} \dots \partial M_{k_r}} \partial_t \big(D^{\beta_1} M_{k_1} \cdots D^{\beta_r} M_{k_r} \big), D^\alpha M \rangle_2 + \\
& + \sum_{k_1,\dots,k_{r+1}=1}^{3} C_{k,\beta} \cdot \\
& \cdot \langle \frac{\partial^{r+1} \Psi}{\partial M_{k_1} \dots \partial M_{k_r} \partial M_{k_{r+1}}} D^{\beta_1} M_{k_1} \cdots D^{\beta_r} M_{k_r} \partial_t M_{k_{r+1}}, D^\alpha M \rangle_2 .
\end{aligned}
$$

This representation and (25) allows us to integrate with respect to t avoiding the loss of one derivative, so that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{|\alpha|} + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \langle \mathrm{Jac}_M \Psi \, D^\alpha M, D^\alpha M \rangle_2 + \frac{\mathrm{d}}{\mathrm{d}t} B_\alpha - A_\alpha + R_\alpha = 0
$$

or equivalently

$$
\begin{aligned}
\mathcal{E}_{|\alpha|}(t) = & \mathcal{E}_{|\alpha|}(0) + \langle \mathrm{Jac}_M \Psi(M_0) \, D^\alpha M_0, D^\alpha M_0 \rangle_2 + B_\alpha(0) + \\
& - \langle \mathrm{Jac}_M \Psi(M(t)) \, D^\alpha M(t), D^\alpha M(t) \rangle_2 - B_\alpha(t) + \\
& - \int_0^t R_\alpha(\tau) \, \mathrm{d}\tau + \int_0^t A_\alpha(\tau) \, \mathrm{d}\tau .
\end{aligned}
$$

It is clear that

$$|\langle \mathrm{Jac}_M \psi(M(t)) D^\alpha M(t), D^\alpha M(t)\rangle_2| \leq$$
$$\leq C(1+t)^{-(p-1)} X_{0,0}^{p-1}(t)\mathcal{E}_{|\alpha|}(t),$$
$$|A_\alpha(\tau)| \leq C(1+\tau)^{-(p-1)} X_{1,0}^{p-1}(\tau)\mathcal{E}_{|\alpha|}(\tau).$$

In order to estimate $|B_\alpha(t)|$ and $|R_\alpha(\tau)|$ we use the relation (21), obtaining

$$|B_\alpha(t)| \leq C(1+t)^{-(p-1)} X_{[|\alpha|/2],0}^{p-1}(t)\mathcal{I}_{|\alpha|-1}^{1/2}(t)\mathcal{E}_{|\alpha|}^{1/2}(t),$$
$$|R_\alpha(\tau)| \leq C(1+\tau)^{-(p-1)} X_{[|\alpha|/2],0}^{p-1}(\tau)\mathcal{I}_{|\alpha|}^{1/2}(\tau)\mathcal{E}_{|\alpha|}^{1/2}(\tau).$$

Hence we have

$$\mathcal{I}_k(t) \leq C_k(M_0,H_0) + C(1+t)^{-(p-1)} X_{k',0}^{p-1}(t)\mathcal{I}_k(t) +$$
$$+C\left(\int_0^t \frac{\mathrm{d}\tau}{(1+\tau)^{p-1}}\right) X_{k',0}^{p-1}(t)\mathcal{I}_k(t) \leq$$
$$\leq C_k(M_0,H_0) + CX_{k',0}^{p-1}(t)\mathcal{I}_k(t)$$

with $k' = \max\{1,[k/2]\}$. In particular

$$C_k(M_0,H_0) \leq C\|H_0\|_{H^k}^2 +$$
$$+C\left[\|M_0\|_{H^k}^2 + \|M_0\|_{p+1}^{p+1} + \sum_{|\alpha|\leq[k/2]} \|D^\alpha M_0\|_\infty^{p-1}\|M_0\|_{H^k}^2\right].$$

with C independent of p and k. This gives the conclusion. ■

Now we are ready to prove the global existence result for (13) with small data, our strategy is to establish an estimate of the form

$$X_{h,k}(u)(t) \leq C\varepsilon + CX_{h,k}^{p-1}(u)(t) + CX_{h,k}^p(u)(t) \tag{26}$$

with ε small and suitable h,k.

Let $s > 3/2$, $h,k \in \mathbb{N}$, $k \geq 2$ and $\varepsilon > 0$; suppose

$$\|H_0\|_{W^{3+h,1}} + \|H_0\|_{H^k}^2 + \tag{27}$$
$$+\|M_0\|_{W^{3+h,1}} + \|M_0\|_{H^k}^2(1+\|M_0\|_{H^{[k/2]+s}}^{p-1}) + \|M_0\|_{p+1}^{p+1} < \varepsilon,$$

from Proposition 3.3 and Proposition 3.5, we get

$$\begin{aligned} X_{h,0}(t) &\leq C\varepsilon + CX^{p-1}_{[(h+4)/2],h+4}(t), \\ \mathcal{I}_k(t) &\leq C\varepsilon + CX^{p}_{[\frac{k}{2}],k}(t). \end{aligned}$$

Since $X_{h,k}(t) \leq X_{h,0}(t) + \sup_{0\leq\tau\leq t} \mathcal{I}_k(\tau)$, we conclude that (26) holds when

$$\begin{cases} [(h+4)/2] \leq h, \\ h+4 \leq k, \\ [\frac{k}{2}] \leq h. \end{cases}$$

The smallest indexes we can take are $k = 7$ and $h = 3$. It remains to establish that the condition (19) for sufficiently small ε implies the condition (27); it suffices to use Sobolev embedding theorems and Moser type inequalities (see [5], [6])

$$\begin{aligned} \||f|^m\|_{H^k_p(\mathbb{R}^n)} &\leq C\|f\|_{H^k_p(\mathbb{R}^n)}\|f\|^{m-1}_{\infty(\mathbb{R}^n)}. \\ \||f|^m\|_{W^{k,1}} &\leq C\|f\|^{m-2}_{\infty}\|f\|^2_{H^k}. \end{aligned}$$

Finally, the contraction mapping argument gives the global existence result.

4. Quasilinear Maxwell system in biaxial crystals

Here we consider the case of biaxial crystals: this means that in (5) the matrix $\epsilon_0 = diag(a^2, b^2, c^2)$ has three different entries. After reduction to (7) one see that $B(\xi)$ has characteristic form

$$\lambda^2(\lambda^4 - \psi(\xi)\lambda^2 + \phi(\xi)|\xi|^2) = 0,$$

where

$$\begin{aligned} \psi(\xi) &= (|b|^{-1} + |c|^{-1})\xi_1^2 + (|a|^{-1} + |c|^{-1})\xi_2^2 + (|a|^{-1} + |b|^{-1})\xi_3^2, \\ \phi(\xi) &= |b|^{-1}|c|^{-1}\xi_1^2 + |a|^{-1}|c|^{-1}\xi_2^2 + |a|^{-1}|b|^{-1}\xi_3^2. \end{aligned}$$

As in the previous section we add the divergence free conditions (10) so that null eigenvalues do not influence the decay result. Taking $\{\lambda(\xi) = 1\}$ one gets the Fresnel surface with four singular points. Hence Σ_λ is not strictly convex. This is consequence of the variable multiplicity of the characteristic roots. It follows that our decay result is not available; conversely for the linear system, O. Liess in [4], established the following estimate:

LEMMA 4.1. *Considering the linear Maxwell system* (5) *with initial data* (E_0, H_0) *which satisfies* (10), *for great enough* $m \in \mathbb{N}$ *one can find a constant* C *so that*

$$\|(E(t), H(t))\|_\infty \leq Ct^{-1/2}\|(E_0, H_0)\|_{W^{1,m}}.$$

We want to combine this estimate with the technique used in the previous section to treat the nonlinear Maxwell system in biaxial crystals. Since we loose the optimal decay rate, we expect a nonlinear existence theorem with greater nonlinear exponent. More precisely, we obtain the next result.

THEOREM 4.2. *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *be a smooth real function such that* $\Phi(E) = \nabla f(E)$ *and*

$$|\Phi(E)| = O(|E|^p) \quad \text{near } E = 0 \;\; p \in \mathbb{N},\ p \geq 5.$$

Consider quasilinear system (17) *with* $\epsilon_0 = diag\,(a^2, b^2, c^2)$, $a \neq b \neq c \neq 0$. *Take initial data* (E_0, H_0) *verifying* (18). *There exists a small* $0 < \varepsilon < 1$ *such that if*

$$\|E_0\|_{W^{k,1}} + \|E_0\|_{H^k} + \|H_0\|_{W^{k,1}} + \|H_0\|_{H^k} < \varepsilon$$

for sufficiently large k *then* (17) *admits a unique global solution* $(E, H) : \mathbb{R}^3 \to \mathbb{R}^6$.

The main difference with Theorem 3.2 is the nonlinear exponent greater than in the previous case. Moreover, here we don't give the number of derivatives of data that we require to be small. This second fact is consequence of O. Liess result which does not point out this aspect. To examine the role of the hypothesis $p \geq 5$ we sketch the proof of this theorem. Here one has to consider the norm

$$X_{h,k}(u)(t) = \sup_{0 \leq \tau \leq t} \left\{ \sup_{|\alpha| \leq h} (1+\tau)^{1/2} \|D^\alpha u(\tau)\|_\infty + \mathcal{I}_k(u)(\tau) \right\}.$$

and the restriction on p appears in the proof of the following proposition.

Proposition 4.3. *Under the same assumptions of the previous theorem we get*

$$\|D^\alpha u(t)\|_\infty \le \\ \le C(1+t)^{-1/2}\left[\|u(0)\|_{W^{m+|\alpha|,1}} + X^{p-1}_{[(|\alpha|+m+1)/2],|\alpha|+m+1}(t)\right].$$

where m is the same regularity index which appears in Lemma 4.1

The difference with the previous section is that we need the estimate

$$\int_0^t (1+t-s)^{-1/2}(1+s)^{-(p-2)/2}\,\mathrm{d}s \le C(1+t)^{-1/2}$$

which holds for $p > 4$.

Energy inequality is still given by (24). Combining these estimates one can prove the existence theorem.

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