A Note on Fixed Fuzzy Points for Fuzzy Mappings

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SUMMARY. - We prove a fixed fuzzy point theorem for fuzzy contraction mappings (in the S. Heilpern's sense) over a complete metric space, and as a consequence we obtain a fixed point theorem in the context of intuitionistic fuzzy sets.

1. Introduction

After the introduction of the concept of a fuzzy set by Zadeh, several researches were conducted on the generalizations of the concept of a fuzzy set. The idea of intuitionistic fuzzy set is due to Atanassov [1], [2], [3] and recently Çoker [4] has defined the concept of intuitionistic fuzzy topological space which generalizes the concept of fuzzy topological space introduced by Chang [5]. Heilpern [6] introduced the concept of a fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [7]. In this paper we give a fixed fuzzy point theorem for fuzzy contraction mappings over a complete metric space, which is a generalization of the given by S. Heilpern for fixed points. Then, we introduce the concept

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of intuitionistic fuzzy mapping and give an intuitionistic version of Heilpern’s mentioned theorem.

2. Preliminaries

Let $X$ be a nonempty set and $I = [0, 1]$. A fuzzy set of $X$ is an element of $I^X$. For $A, B \in I^X$ we denote $A \subset B$ iff $A(x) \leq B(x)$, $\forall x \in X$.

**Definition 2.1.** (Atanassov [3]) An intuitionistic fuzzy set (i-fuzzy set, for short) $A$ of $X$ is an object having the form $A = [A_1, A_2]$ where $A_1, A_2 \in I^X$ and $A_1(x) + A_2(x) \leq 1$, $\forall x \in X$.

We denote $IFS(X)$ the family of all i-fuzzy sets of $X$.

**Remark 2.2.** If $A \in I^X$, then $A$ is identified with the i-fuzzy set $< A, 1 - A >$ denoted by $[A]$.

For $x \in X$ we write $\{x\}$ the characteristic function of the ordinary subset $\{x\}$ of $X$. For $\alpha \in ]0, 1]$ the fuzzy point $[8]$ $x_\alpha$ of $X$ is the fuzzy set of $X$ given by $x_\alpha(x) = \alpha$ and $x_\alpha(z) = 0$ if $z \neq x$. Now we give the following definition.

**Definition 2.3.** Let $x_\alpha$ be a fuzzy point of $X$. We will say $< x_\alpha, 1 - x_\alpha >$ is an i-fuzzy point of $X$ and it will be denoted by $[x_\alpha]$. In particular $[x] = < \{x\}, 1 - \{x\} >$ will be called an i-point of $X$.

**Definition 2.4.** (Atanassov [3]) Let $A, B \in IFS(X)$. Then $A \subset B$ iff $A_1 \subset B_1$ and $B_2 \subset A_2$.

**Remark 2.5.** Notice $[x_\alpha] \subset A$ iff $x_\alpha \subset A_1$.

Let $(X, d)$ be a metric linear space. Recall the $\alpha - \text{level}$ $A_\alpha$ of $A \in I^X$ is defined by $A_\alpha = \{x \in X : A(x) \geq \alpha\}$ for each $\alpha \in ]0, 1]$, and $A_0 = \text{cl}(\{x \in X : A(x) > 0\})$ where $\text{cl}(B)$ is the closure of $B$. Heilpern [6] called fuzzy mapping a mapping from the set $X$ into a family $W(X) \subset I^X$ defined as follows: $A \in W(X)$ iff $A_\alpha$ is compact and convex in $X$ for each $\alpha \in ]0, 1]$ and $\text{sup}\{A(x) : x \in X\} = 1$. In this context we give the following definitions.
Definition 2.6. Let \( A, B \in W(X) \), \( \alpha \in [0, 1] \). Define

\[
p_\alpha(A, B) = \inf \{ d(x, y) : x \in A_\alpha, \ y \in B_\alpha \},
\]

\[
D_\alpha = H(A_\alpha, B_\alpha),
\]

\[
D(A, B) = \sup_\alpha D_\alpha(A, B), \text{ where } H \text{ is the Hausdorff distance}.
\]

For \( x \in X \) we write \( p_\alpha(x, B) \) instead of \( p_\alpha(\{x\}, B) \).

Definition 2.7. Let \( X \) be a metric space and \( \alpha \in [0, 1] \). Consider the following family \( W_\alpha(X) \):

\[
W_\alpha(X) = \{ A \in I^X : A_\alpha \text{ is nonempty and compact} \}.
\]

Now, we define the family \( \Phi_\alpha(X) \) of \( \mu \)-fuzzy sets of \( X \) as follows:

\[
\Phi_\alpha(X) = \{ A \in IFS(X) : A^1 \in W_\alpha(X) \}
\]

Clearly, for \( \alpha \in I \), \( W(X) \subset \Phi_\alpha(X) \) in the sense of Remark 2.2.

We will use the following lemmas which are adequate modifications of the ones given in [6] for the family \( W(X) \), when \((X, d)\) is a metric space.

Lemma 2.8. Let \( x \in X \) and \( A \in W_\alpha(X) \). Then \( x_\alpha \subset A \) if \( p_\alpha(x, A) = 0 \).

Lemma 2.9. \( p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A) \), for \( x, y \in X \), \( A \in W_\alpha(X) \).

Lemma 2.10. If \( x_\alpha \subset A \), then \( p_\alpha(x, B) \leq D_\alpha(A, B) \), for each \( A, B \in W_\alpha(X) \).

3. Fixed fuzzy point theorem

In mathematical programming, problems are expressed as optimizing some goal function given certain constraints and there are real-life problems that consider multiple objectives. Generally, it is very difficult to get a feasible solution that carries us to the optimum of all the objective functions. A possible method of resolution that is quite useful is one using Fuzzy Sets. The idea is to relax the pretenses of
optimization by means of a subjective gradation which can be modelled into fuzzy membership functions \( \mu_i \). If \( F = \cap \mu_i \) the objective will be to search \( x \) such that \( \max F = F(x) \). If \( \max F = 1 \), then there exists \( x \) such that \( F(x) = 1 \), but if \( \max F = \alpha, \alpha \in [0,1] \) the solution of the multiobjective optimization is a fuzzy point \( x_\alpha \) and \( F(x) = \alpha \).

In a more general sense than the one given by Heilpern, a mapping \( F : X \rightarrow I^X \) is a fuzzy mapping over \( X \) ([9]) and \( (F(x))(x) \) is the fixed degree of \( x \) for \( F \). In this context we give the following definition.

**Definition 3.1.** Let \( x_\alpha \) be a fuzzy point of \( X \). We will say that \( x_\alpha \) is a fixed fuzzy point of the fuzzy mapping \( F \) over \( X \) if \( x_\alpha \subset F(x) \) (i.e., the fixed degree of \( x \) is at least \( \alpha \)). In particular, and according to [6], if \( \{x\} \subset F(x) \) we say that \( x \) is a fixed point of \( F \).

The next proposition is a generalization of [6] Theorem 3.1.

**Theorem 3.2.** Let \( \alpha \in [0,1] \) and let \( (X,d) \) be a complete metric space. Let \( F \) be a fuzzy mapping from \( X \) into \( W_\alpha(X) \) satisfying the following condition: there exists \( q \in [0,1] \) such that

\[
D_\alpha(F(x), F(y)) \leq qd(x,y), \text{ for each } x, y \in X.
\]

Then there exists \( x \in X \) such that \( x_\alpha \) is a fixed fuzzy point of \( F \). In particular if \( \alpha = 1 \), then \( x \) is a fixed point of \( F \).

**Proof.** Let \( x_0 \in X \). Since \( (F(x_0))_\alpha \neq \emptyset \), then there exists \( x_1 \in X \) such that \( x_1 \in (F(x_0))_\alpha \). There exists \( x_2 \in (F(x_0))_\alpha \). Since \( (F(x_1))_\alpha \) is a nonempty compact subset of \( X \), then there exists \( x_2 \in (F(x_1))_\alpha \) such that

\[
d(x_1, x_2) = d(x_1, (F(x_1))_\alpha) \leq D_\alpha(F(x_0), F(x_1))
\]

by Lemma 2.10. By induction we construct a sequence \( (x_n) \) in \( X \) such that \( x_n \in (F(x_{n-1}))_\alpha \) and \( d(x_n, x_{n+1}) \leq D_\alpha(F(x_{n-1}), F(x_n)) \).

In a similar way to the proof of [6] Theorem 3.1., it is proved that \( (x_n) \) is a Cauchy sequence. Suppose \( (x_n) \) converges to \( x \in X \). Now, by Lemmas 2.9, 2.10 we have
\[ p_\alpha(x, F(x)) \leq d(x, x_n) + p_\alpha(x_n, F(x)) \leq d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \]
\[ \leq d(x, x_n) + qd(x_{n-1}, x) \]

In consequence \( p_\alpha(x, F(x)) = 0 \) and by Lemma 2.8, \( x_\alpha \subset F(x) \). \( \square \)

Remark 3.3. Theorem 3.1 of [6] states the existence of a fixed point of the fuzzy mapping \( F : X \rightarrow W(X) \) whenever \( D(F(x), F(y)) \leq qd(x, y) \) for each \( x, y \in X \), being \( q \in [0, 1] \). Now, in the light of the above theorem, it is clear that the condition \( D(F(x), F(y)) \leq qd(x, y) \) can be weakened to \( D_1(F(x), F(y)) \leq qd(x, y) \).

The next example illustrates when the above theorem has certain advantages if compared with Heilpern’s (Theorem 3.1 of [6]).

Example 3.4. Take \( a, b, c \in ]-\infty, +\infty[ \), such that \( a < b < c \). Let \( X = \{a, b, c\} \) and \( d : X \times X \rightarrow [0, +\infty[ \) the Euclidean metric. Let \( \alpha \in ]0, 0.5[ \) and suppose \( F : X \rightarrow I^X \) defined by

\[
F(a)(x) = \begin{cases} 
1 & \text{if } x = a \\
2\alpha & \text{if } x = b \\
\alpha/2 & \text{if } x = c 
\end{cases}
\]

\[
F(b)(x) = \begin{cases} 
\alpha & \text{if } x = b \\
\alpha/2 & \text{if } x = c 
\end{cases}
\]

\[
F(c)(x) = \begin{cases} 
1 & \text{if } x = a \\
\alpha & \text{if } x = b \\
0 & \text{if } x = c 
\end{cases}
\]

Then \( F(a)_1 = F(b)_1 = F(c)_1 = \{a\}, F(a)_\alpha = F(b)_\alpha = F(c)_\alpha = \{a, b\}, \)

\( F(a)_{1/2} = F(b)_{1/2} = \{a, b, c\} \) and \( F(c)_{1/2} = \{a, b\} \).

Consequently

\[
D_1(F(x), F(y)) = H(F(x)_1, F(y)_1) = 0, \quad D_\alpha(F(x), F(y)) = H(F(x)_\alpha, F(y)_\alpha) = 0, \quad \forall x, y \in X .
\]
By Theorem 3.2 there exists a fixed fuzzy point \( x_1 \) (a fixed point) and a fixed fuzzy point \( x_\alpha \) of the fuzzy mapping \( F \). We can see by the definition of \( F \) that \( a \) is a fixed point and \( b_\alpha \) is a fuzzy point. Nevertheless Heilpern’s theorem is not useful in this example because \( D_\alpha(F(a), F(c)) = H(\{a, b, c\}, \{a, b\}) \geq d(a, c) \) and then \( D(F(a), F(c)) = \sup_r H(F(a)_r, F(c)_r) \geq d(a, c) \).

4. Fixed i-fuzzy point

**Definition 4.1.** An i-fuzzy mapping over \( X \) is a mapping \( F \) from \( X \) into \( IFS(X) \). We will say that \([x_\alpha]\) is a fixed i-fuzzy point of \( F \) if \([x_\alpha] \subset F(x) \). In particular we will say that \([x]\) is a fixed i-point of \( F \) if \([x] \subset F(x) \).

**Definition 4.2.** Let \((X, d)\) be a metric space and \(0 < \alpha \leq 1\). For \( A, B \in \Phi_\alpha(X) \) we define \( D^*_\alpha(A, B) = \max\{D_\alpha(A^1, B^1), D_\alpha(1 - A^2, 1 - B^2)\} \). Clearly, \( D^*_\alpha \) is a pseudometric on \( \Phi_\alpha(X) \).

**Remark 4.3.** Let \( A, B \in W_\alpha(X) \). If we consider \([A], [B] \in \Phi_\alpha(X) \) then

\[
D^*_\alpha([A], [B]) = \max\{D_\alpha(A, B), D_\alpha(1 - (1 - A), 1 - (1 - B))\} = D_\alpha(A, B)
\]

Now, as a consequence of Theorem 3.2 we have the following corollary.

**Corollary 4.4.** Let \((X, d)\) be a complete metric space and let \( F \) be an i-fuzzy mapping from \( X \) into \( \Phi_\alpha(X) \), \( \alpha \in [0, 1] \), satisfying the following condition: There exists \( q \in [0, 1] \) such that

\[
D^*_\alpha(F(x), F(y)) \leq qd(x, y), \text{ for each } x, y \in X
\]

Then there exists \( x \in X \) such that \([x_\alpha]\) is a fixed i-fuzzy point of \( F \). In particular if \( \alpha = 1 \), then \([x]\) is a fixed i-point of \( F \).

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