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A Connected, not Separably Connected Metric Space

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Summary. - *A separably connected space is a topological space, where every two points may be joined by a separable connected subspace. We present an example of a connected, but not separably connected metric space and of a connected metric space, which contains no connected separable subspaces other than one-point ones.*

Recently, J. C. Candeal, C. Hervés and E. Indurain introduced a notion of separable connectedness as a natural generalization of path-connectedness [1].

□

Definition 0.1. *A topological space X is called separably connected, if for any two distinct points $x, y \in X$ there is a separable connected set $C(x,y) \subseteq X$ such that $\{x,y\} \subseteq C(x,y)$.*

The aim of this short note is to exhibit two examples of metric connected spaces which are not separably connected. The existence of such a space was a problem posed by Prof. E. Indurain during his lecture at the Second Italian-Spanish Conference on General Topology and its Applications held in Trieste in September 1999. The second example is stronger than the first one: It contains no non-degenerate connected separable subspaces.

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1. The space Z

Let us start with a nice connected space W:

$$W = \{f \in {}^{\omega_1}[0,1] \,:\, 0 < |\{\alpha < \omega_1 \,:\, f(\alpha) > 0\}| < \omega\}.$$

We equip the set W by ℓ_1-metric, i.e., for $f, g \in W$,

$$\varrho(f,g) = \sum_{\alpha<\omega_1} |f(\alpha) - g(\alpha)|.$$

For $\alpha < \omega_1$, let us denote by $W(\alpha)$ the subspace of W defined by

$$W(\alpha) = \{f \in W \,:\, \text{for some } \beta \geq \alpha, \beta < \omega_1,\ f(\beta) > 0\}.$$

With this notation, we have obviously $W = W(0)$.

Observation 1.1. *For every $\alpha < \omega_1$, the space $(W(\alpha), \varrho)$ is connected.*

Proof. It is enough to find for any two $f, g \in W(\alpha)$ a connected set $C \subseteq W(\alpha)$ containing both f and g. Put $K = \{\beta < \alpha : \max\{f(\beta), g(\beta)\} > 0\}$ and $L = \{\beta \geq \alpha : \max\{f(\beta), g(\beta)\} > 0\}$. Consider the set $C = \{h \in W \,:\, \{\beta < \omega_1 : h(\beta) > 0\} \subseteq K \cup L$ and there is some $\beta \in L$ with $h(\beta) > 0\}$. The set C obviously contains f and g. However, since both sets K, L are finite and disjoint and L is non-empty, the set C is clearly homeomorphic to some subset C' of a Euclidean cube, $(0,1)^{K \cup L} \subseteq C' \subseteq [0,1]^{K \cup L}$. The set C' is connected by ([2], 6.1.11). Thus C is connected. □

Next, choose and fix some family $\{A_\alpha : \alpha < \omega_1\}$ of subsets of $[0,1]$ with the following properties:

(i) for each $\alpha < \beta < \omega_1$, $A_\alpha \subseteq A_\beta$;

(ii) the set A_0 is dense in $[0,1]$;

(iii) for every $\alpha < \omega_1$, the set $A_{\alpha+1} \setminus A_\alpha$ is dense in $[0,1]$;

(iv) $\bigcup\{A_\alpha \,:\, \alpha < \omega_1\} = [0,1]$.

For the existence of such family, see e.g. ([2], 6.2.20).

For $t \in [0,1]$, define $\gamma(t) = \min\{\alpha < \omega_1 : t \in A_\alpha\}$. Now, let us define the subspace $Z \subseteq [0,1] \times W$ as follows:

$$Z = \bigcup\{\{t\} \times W(\gamma(t)) \;\; t \in [0,1]\}.$$

We claim that Z is as required.

THEOREM 1.2. *The space Z is metrizable, connected, but not separably connected.*

Proof. (a) Z is metrizable: Indeed, Z is a subspace of a product of two metric spaces. (b) Z is connected: Suppose not and choose a clopen set U such that $\emptyset \neq U \subsetneq Z$. By Observation, each set $\{t\} \times W(\gamma(t))$ is connected, so $U \cap \{t\} \times W(\gamma(t))$ cannot be a nonempty proper subset. Define $E = \{t \in [0,1] : \{t\} \times W(\gamma(t)) \subseteq U\}$, $F = \{t \in [0,1] : \{t\} \times W(\gamma(t)) \cap U = \emptyset\}$. Observe that both sets E, F are nonempty: If $(t,f) \in Z \setminus U$, then a connected set $\{t\} \times W(\gamma(t))$ meets a clopen set $Z \setminus U$, so $\{t\} \times W(\gamma(t)) \cap U = \emptyset$ and consequently $t \in F$. Similarly, using $U \neq \emptyset$ we get $E \neq \emptyset$. Trivially, $E \cup F = [0,1]$ and $E \cap F = \emptyset$.

To reach a contradiction, it is enough to show that both sets E, F are closed. Because of the symmetry, we shall give a proof for E only. Let $t \in \overline{E}$. Choose a sequence $\{t_n : n \in \omega\}$ of points in E with $\lim t_n = t$. Pick some $\alpha < \omega_1$ such that $\alpha > \gamma(t_n)$ for all $n \in \omega$ and $\alpha > \gamma(t)$.

According to the definition of $W(\alpha)$, we have that $W(\alpha) \subseteq W(\gamma(t_n))$ for all n and $W(\alpha) \subseteq W(\gamma(t))$ as well. Thus the set $\{t_n : n \in \omega\} \times W(\alpha)$ is a subset of U. Since U is closed, we conclude that $\{t\} \times W(\alpha) \subseteq U$, too. Since the set $\{t\} \times W(\gamma(t))$ is connected and meets the clopen set U, because $W(\gamma(t)) \supseteq W(\alpha)$, it follows that $\{t\} \times W(\gamma(t)) \subseteq U$ and therefore $t \in E$. Since $t \in \overline{E}$ was arbitrary, the set E is closed.

(c) Z is not separably connected: Let $h \in W$ be defined by $h(0) = 1$, $h(\alpha) = 0$ for $0 < \alpha < \omega_1$ and pick two distinct points $t < u$ from the set A_0. Since $h(0) \neq 0$, we have that both points (t,h) and (u,h) belong to Z. Suppose that a set $D \subseteq Z$ is separable and contains

$\{(t,h),(u,h)\}$. Pick a countable dense subset $\{d_n : n \in \omega\} \subseteq D$. We have that $d_n = (t_n, f_n)$, where $t_n \in [0,1]$ and $f_n \in W(\gamma(t_n)) \subseteq W$. Denote by S_n the support of f_n, $S_n = \{\alpha < \omega_1 : f_n(\alpha) \neq 0\}$. Since each set S_n is finite, there is some $\alpha < \omega_1$ such that for all $n \in \omega$, $S_n \subseteq [0,\alpha)$. The set $A_{\alpha+1} \setminus A_\alpha$ is dense in $[0,1]$, so there is some point $v \in A_{\alpha+1} \setminus A_\alpha$, $t < v < u$. For this v, we evidently have $\gamma(v) = \alpha + 1$.

Pick an arbitrary $f \in W$ so that $(v,f) \in Z$. Then $f \in W(\gamma(v))$ and thus there is some $\beta \geq \alpha + 1$ with $f(\beta) = r > 0$. Consider a neighborhood $G = [0,1] \times B_\varrho(f,r)$ of a point (v,f). (Here, $B_\varrho(f,r)$ denotes the open ball with the center f and radius r in the metric ϱ, i.e., $B_\varrho(f,r) = \{g \in W : \varrho(f,g) < r\}$.) If $n \in \omega$ is arbitrary, we have $f_n(\beta) = 0$ and so

$$\varrho(f,f_n) = \sum_{\alpha<\omega_1} |f(\alpha) - f_n(\alpha)| \geq |f(\beta) - f_n(\beta)| = |r - 0| = r,$$

so $d_n = (t_n, f_n)$ does not belong to G. Since n was arbitrary, we have that $G \cap D = \emptyset$. In particular, $(v,f) \notin D$. However, also $f \in W(\gamma(v))$ was arbitrary, so we may conclude: Whenever $(x,f) \in D$, then $x \neq v$. Let $D(+) = \{(x,f) \in D : x > v\}$ and $D(-) = \{(x,f) \in D : x < v\}$. Clearly, these two sets are both open in D and $D(+) \cup D(-) = D$. Since $(u,h) \in D(+)$ and $(t,h) \in D(-)$, the set D is not connected. The proof is complete. □

2. The space $Z^{\#}$

Since we do not want to repeat all steps from the previous proof, we request the reader to consider that easy modifications of some parts of the previous proof give the same results.

Remark 2.1. *To show that Z is connected, there was no need to have the first coordinate space just the closed unit interval* $[0,1]$. *This part of the proof will work verbatim for any connected metric space in the place of* $[0,1]$.

Remark 2.2. *To show that Z is not separably connected, we found just one point v between u and t with $\gamma(v)$ big enough. But what we* *really needed was to find a neighborhood N of a point u such that $t \notin \overline{N}$ and all points $v \in bd N$ have $\gamma(v)$ bigger than a bound given in advance.*

We plan to define the space $Z^{\#}$ as a subspace of a Tychonov product W^{ω}. That will automatically give a metrizability. The only problem is to find an analogue of sets A_{α}, since their role was vital for the previous construction to work.

Fix some Hamel basis H for the reals $\mathbb{R}$ over rationals $\mathbb{Q}$, $H = \{r_{\alpha} : \alpha < \mathfrak{c}\}$ with $r_0 = 1$. Given some $f \in W$, the set of its values is a finite set of real numbers, say $\{x_0, x_1, \dots, x_n\}$. Every real number x admits a unique representation $x = q_0 + q_1 \cdot r_{\alpha(1)} + \dots + q_k \cdot r_{\alpha(k)}$, where k is an integer and $\alpha(1), \dots, \alpha(k) < \mathfrak{c}$. Define A_{α} to be the set of all $f \in W$ such that for each $\beta < \mathfrak{c}$, if r_{β} appears in a representation of some value of f, then either $\beta < \alpha$ or $\omega_1 < \beta$.

Observe that we have again a sound definition of $\gamma(f)$ for $f \in W$, namely, $\gamma(f) = \min\{\alpha : f \in A_{\alpha}\}$.

The space $Z^{\#}$, a subspace of W^{ω}, is defined as follows:

$$Z^{\#} = \{F \in {}^{\omega}W : F(0) \in W \text{ and for each } n \in \omega, F(n+1) \in W(\gamma(F(n)))\}.$$

Theorem 2.3. *The space $Z^{\#}$ is metrizable and connected, but contains no separable connected subsets with more than one point.*

Proof. As a subspace of a countable product of metric spaces, $Z^{\#}$ is metrizable.

$Z^{\#}$ is connected: Select some $F \in Z^{\#}$. Define for $n < \omega$ the subspaces $Z^{\#}(n)$ by $Z^{\#}(n) = \{G \in Z^{\#} : \text{for all } k > n, G(k) = F(k)\}$.

We clearly have that $Z^{\#}(0)$ is homeomorphic to W, hence connected.

Proceeding by induction, assume $Z^{\#}(n)$ is connected. Then $Z^{\#}(n+1)$ is homeomorphic to $\{(t, f) : t \in Z^{\#}(n), f \in W(\gamma(t(n))\}$. By Remark 1 and the corresponding part of the proof of Theorem 1, $Z^{\#}(n+1)$ is connected.

We clearly have $Z^{\#}(0) \subseteq Z^{\#}(1) \subseteq Z^{\#}(2) \subseteq \dots$ and $Z^{\#} = \overline{\bigcup_{n \in \omega} Z^{\#}(n)}$. So $Z^{\#}$ is connected.

$Z^{\#}$ does not contain nontrivial separable connected subsets: Assume $D \subseteq X$ is separable, $|D| > 1$.

Let $m = \min\{n \in \omega : \text{there are } F, G \in D \text{ such that } F(n) \neq G(n)\}$. So there is some $\varphi \in {}^{m}W$ such that for every $F \in D$ and every $n < m$ one has $F(n) = \varphi(n)$. Choose a countable dense subset $\{d_k : k \in \omega\}$ of D. Each $d_k(m+1)$ is an element of W, hence a mapping from ω_1 to $[0,1]$ which vanishes at all except finitely many points from ω_1. Therefore there is some $\alpha < \omega_1$ such that for each $n < \omega$ and $\beta < \omega_1$, $d_n(m+1)(\beta) > 0$ implies $\beta < \alpha$. Consequently, whenever $H \in D$ and $\beta < \omega_1$ is such that $H(m+1)(\beta) > 0$, then $\beta < \alpha$. This follows from the fact that $H \in \overline{\{d_n : n \in \omega\}}$ and from the continuity of π_{m+1}, the $(m+1)$-st projection.

Choose two distinct points $t, u \in W$ such that for some $F, G \in D$, $t = F(m)$ and $u = G(m)$. According to the definition of m, this is possible. By previous, $\gamma(t) \leq \alpha$. Select $q \in \mathbb{Q}$ such that $0 < q \cdot r_{\alpha+1} = r < \varrho(t, u)$.

Observe now: There is no point $H \in D$ such that $H(m) \in bd\, B_\varrho(t, r)$. Indeed, we must have $\varrho(t, H(m)) = r$ in such a case, hence

$$\sum_{\beta < \omega_1} |t(\beta) - H(m)(\beta)| = r.$$

But every number from the left-hand side of the equality is in a vector subspace of $\mathbb{R}$ spanned by $\{r_\xi : \xi \neq \alpha + 1\}$, while r is not. Therefore, $D \cap \pi_m^{-1}[B_\varrho(t, r)]$ and $D \cap \pi_m^{-1}[W \setminus \overline{B_\varrho(t, r)}]$ are nonempty disjoint open sets in D which cover it. □

3. Concluding remarks

It should be clear that our examples can be easily modified (replace ω_1 by $\mathfrak{c}$ everywhere) to get a bit stronger property: Z is connected, but contains a pair of points $(t, h), (u, h)$ such that no set D of density $< cf\,\mathfrak{c}$ with $\{(t, h), (u, h)\} \subseteq D$ is connected; and similarly for $Z^{\#}$.

After completing the present paper, I have learned that another example of a connected, not separably connected metric space has been recently constructed also by Richard M. Aron and Manuel Maestre in their joint paper "Separable connectedness: A remark on a paper by J. Candeal, C. Hervés and E. Indurain", submitted to the Journal of Mathematical Economics. They used a different method and got a different space than our example Z. I would like to thank to Prof. C. Hervés for giving me this information.

References

[1] J. C. Candeal, C. Hervés, and E. Indurain, *Some results on representation and extension of preferences*, Journal of Math. Economics **29** (1998), 75 – 81.

[2] R. Engelking, *General topology*, PWN, Warszawa, 1977.

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