An Example of Two Compact Spaces with Different Topological Dimensions

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SUMMARY. - In this paper we give two compact spaces $X$, $Y$ with $\dim(X) = \dim(Y) = 1$, $\text{ind}(X) = \text{ind}(Y) = \text{Ind}(X) = \text{Ind}(Y) = 3$, where $\dim$ is the covering dimension, $\text{ind}$ and $\text{Ind}$ are the small and large inductive dimensions respectively.

1. Introduction

Compact spaces with different topological dimensions have been studied. Filippov found a compact space $T_{mn}$ with $\dim(T_{mn}) = 1$, $\text{ind}(T_{mn}) = m$, $\text{Ind}(T_{mn}) = n$ for each $m, n$ such that $m \leq n \leq 2m - 1$ (see [5]). In this paper we prove the existence of two non homeomorphic compact spaces $X$, $Y$ with $\dim(X) = \dim(Y) = 1$, $\text{ind}(X) = \text{ind}(Y) = 3$, $\text{Ind}(X) = \text{Ind}(Y) = 3$. In order to get these spaces, we take into account the space $T_{23}$. In order to evaluate the covering dimension of $X$, $Y$, we use the local dimension $\text{loccdim}$ defined in [1], and we give a slight modification of the $\text{Ind}$-dimension, the $\text{Indc}$-dimension, to state the equalities $\text{Ind}(X) = \text{Ind}(Y) = 3$.

We ensure that the spaces $X$, $Y$ are not homeomorphic by defining a topological dimension, named $K$, such that $K(X) \neq K(Y)$. Since $X$ and $Y$ are topologically distinct, at least one of them is not homeomorphic to the space $T_{33}$, another compact space 1-dimensional

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1991 AMS Subject Classification: 54F45
Keywords: compact space, one-point compactification, topological dimensions.
for the \textit{dim}-dimension and \textit{three}-dimensional for the dimensions \textit{ind}, \textit{Ind}.

\section{Basic concepts and notations}

In this paper we consider only Hausdorff spaces. We begin by defining an inductive dimension by means of separating points:

\begin{definition}
Let $X$ be a topological space, $n = 0, 1, 2, \ldots$. The following conditions define inductively the dimension $K$:

\begin{enumerate}
\item $K(X) = -1 \iff X = \emptyset$.
\item If $|X| = 1$, then $K(X) = 0$.
\item On the assumption that $|X| > 1$, $K(X) \leq n$ if for every pair $x, y$ of distinct points of $X$ there exist two open sets $U, V \subset X$ such that $x \in U$, $y \in V$, $U \cap V = \emptyset$, $X = U \cup V \cup L$ with $K(L) < n$ where $L$ is $X - (U \cap V)$ (we said that $L$ is a separation of $X$ between $x, y$).
\item $K(X) = n$ if $K(X) \leq n$ and the inequality $K(X) < n$ does not hold.
\item $K(X) = \infty$ if $K(X) > n$ for every $n$ (this dimension is similar to the one defined in [6]).
\end{enumerate}

As an immediate consequence of the definition $K(X) = K(Y)$ for homeomorphic spaces $X, Y$:

\begin{theorem}
If $Y$ is a subspace of a space $X$ then $K(Y) \leq K(X)$.
\end{theorem}

\begin{theorem}
For every regular space $X$, $K(X) \leq \text{ind}(X)$.
\end{theorem}

We also define the $\text{Indc}$-dimension:

\begin{definition}
Let $X$ be a topological space $n = 0, 1, 2, \ldots$ the following conditions define inductively the dimension $\text{Indc}$:

\begin{enumerate}
\item $\text{Indc}(X) = -1 \iff X = \emptyset$
\end{enumerate}
\end{definition}
2. Indc($X$) $\leq n$ for every compact space $C$ included in $X$, for every open subset of $X$, $V$, there exists an open subset of $X$, $U$, with $C \subset U \subset V$, Indc($Bd(U)$) $< n$. ($Bd(U)$ is the boundary of $U$).

3. Indc($X$) = $n$ if Indc($X$) $\leq n$ and the inequality Indc($X$) $< n$ does not hold.

4. Indc($X$) = $\infty$ if Indc($X$) $> n$ for every $n$.

We have the following result:

**Proposition 2.5.** Let $V'$ be an open subset of $X$, then Indc($V'$) $\leq$ Indc($X$).

**Proof.** We apply induction with respect to Indc($X$), assuming that Indc($X$) $< \infty$. If Indc($X$) = $-1$, then $X = V' = \emptyset$, so Indc($V'$) = $-1$.

Assume that the result is true for every space $X$ with Indc($X$) $< n$, $n \geq 0$.

Let $X$ be a space with Indc($X$) = $n$, $C$ a compact included in $V'$, $V$ an open subset of $V'$ such that $C \subset V$. As $V$ is an open subset of the normal space $X$, there exists an open subset of $X$, $U$ with $C \subset U \subset cl(U) \subset V$, Indc($Bd(U)$) $< n$, where $cl(U)$ is the closure of $U$ in $X$.

Since $cl(U)$ is included in $V$, $Bd_{V'}(U) = Bd(U)$, where $Bd_{V'}(U)$ is the boundary of $U$ in $V'$, and the Indc($Bd_{V'}(U)$) $< n$. This implies that Indc($V'$) $\leq n = Indc(X)$. $\square$

Since the unitary sets are compact spaces, we have the next proposition:

**Proposition 2.6.** For every regular space $X$, ind($X$) $\leq$ Indc($X$).

3. The spaces $X$, $Y$

In order to establish the spaces $X$, $Y$ with the required properties, we need some preliminary lemmas:

**Lemma 3.1.** For a compact space $X$ with $K(X) \leq 1$ we have that ind($X$) = $K(X)$.
Proof. For $K(X) = 0$, let $x$ be a point of $X$ and $F$ a closed subset of $X$ such that $x \notin F$. We can find, for each point $y$ in $F$, an open subset and closed subset $U(y)$ such that $x \notin U(y)$. Consequently, as $F$ is a compact space, there exists an open and closed subset of $X$ containing $F$, say $U$, such that $x \notin U$. This implies that $\text{ind}(X) = 0$.

For $K(X) = 1$, let $F$ be a closed subset of $X$, $x$ a point of $X$ such that $x \notin F$. Since $F$ is compact and $K(X) = 1$, we can find an open subset of $X$, $U$, with $F \subset U$, $x \notin U$, $Bd(U) \subset \{Bd(U_i)/i = 1, \ldots, n\}$, $K(Bd(U_i)) < 1$ for $i = 1, \ldots, n$. Since $Bd(U_i)$ are compact spaces, we get $\text{ind}(Bd(U_i)) = K(Bd(U_i)) < 1$ for $i = 1, \ldots, n$. Now, the subspace theorem and the sum theorem for compact spaces with $\text{ind}$-dimension $0$ yield $\text{ind}(Bd(U)) < 1$ (see [3], theorem 2.2.7). This implies that $\text{ind}(X) \leq 1$.

**Lemma 3.2.** Let $X$ be a locally compact, noncompact space, then $K(w(X)) \leq K(X) + 1$, $\text{ind}(w(X)) \leq \text{ind}(X) + 1$.

Proof. We have that $w(X) = X \cup \{p\}$, with $p \notin X$. Let $x, y$ be two distinct point of $w(X)$, and, say $x \in X$.

As $X$ is an open subset of the regular space $w(X)$, we can find and open subset of $w(X)$, say $U$, such that $x \in U \subset \text{cl}(U) \subset U$, $y \notin \text{cl}(U)$, where $\text{cl}(U)$ means the closure of $U$ in $w(X)$.

Then $w(X) - Bd(U) = U \cup (X - \text{cl}(U))$, where $U$, $X - \text{cl}(U)$ are disjoint open subsets of $w(X)$ such that $x \in U$, $y \in X - \text{cl}(U)$. Moreover, applying the Theorem 2.2 we have that $K(Bd(U)) \leq K(\text{cl}(U)) \leq K(X) < K(X) + 1$.

This implies that $K(w(X)) \leq K(X) + 1$.

The proof for the $\text{ind}$-dimension is analogous (we just have to substitute the point $y$ for a closed subset of $X$, say $F$).

**Lemma 3.3.** Let $X$ be a locally compact, noncompact space with $K(X) = \text{ind}(X)$. Then $K(w(X)) = K(X)$.

Proof. We are going to see this lemma for every space $X$ under the hypothesis of the lemma with $K(X) < \infty$.

Let $x, y$ be two distinct point of $w(X)$. If, say, $x \in X$, we can find an open subset of $w(X)$, $V$, such that $x \in V$, $y \in H = w(X) - \text{cl}(V)$, $\text{cl}(V) \subset X$. Then $V$ is an open subset of $X$. 

Furthermore, as $\text{ind}(X) = K(X)$ and $V$ is an open subset of $X$, there exists an open subset of $X, U$, such that $x \in U \subset \text{cl}_X(U) \subset V$, $\text{ind}(\text{Bd}_X(U)) < K(X)$, and then $K(\text{Bd}_X(U)) \leq \text{ind}(\text{Bd}_X(U)) < K(X)$.

On the other hand, as $cl(U) \subset cl(V) \subset X$, we have that $\text{cl}_X(U) = cl(U), \text{Bd}_X(U) = Bd(U)$.

Then $w(X) = U \cup H' \cup Bd(U)$, where $U, H' = w(X) - cl(U)$ are open subsets of $w(X)$ such that $x \in U, y \in H'$ ($H$ is included in $H'$).

So we have that $Bd(U)$ is a separation of $w(X)$ between $x, y$ with $K(Bd(U)) = K(Bd_X(U)) < K(X)$, and then $K(w(X)) \leq K(X)$.

\begin{lemma}
\textbf{Lemma 3.4.} Let $X$ be a compact space with $\text{ind}(X) = 3$. If the equality $K(w(U)) = K(U)$ holds for every open and non-closed subset $U$ of $X$ such that $K(U) = 2$, then $K(X) = 3$.
\end{lemma}

\begin{proof}
As $\text{ind}(X) = 3$, we can find a point of $X, x$, and an open neighbourhood of $x, V(x)$ such that for every open neighbourhood of $x$ included in $V(x), U(x)$, we have that $\text{ind}(\text{Bd}(U(x))) \geq 2$.

If $K(\text{Bd}(U(x))) < 2$, since $\text{Bd}(U(x))$ is a compact space, by Lemma 3.1 we get $\text{ind}(\text{Bd}(U(x))) = K(\text{Bd}(U(x))) < 2$, a contradiction, so $K(\text{Bd}(U(x))) \geq 2$ for every open subset $U(x)$ of $X$ included in $V(x)$.

If we now consider $V(x), V(x)$ is an open subset of $X$ such that it’s not closed in $X$ (it has boundary), so it’s locally compact, non-compact space.

For every set $U(x)$ such that $U(x)$ is open in $V(x)$ and $cl_{V(x)}(U(x))$ is a compact space, $U(x)$ is an open subset of $X$ included in $V(x)$, and therefore $K(\text{Bd}(U(x))) \geq 2$.

On the other hand, as $cl_{V(x)}U(x)$ is a closed subset of $X$ and then $cl_{V(x)}U(x) = cl(U(x)), Bd_{V(x)}U(x) = Bd(U(x))$.

We have proved the following assertion: “If $U(x)$ is an open subset of $V(x)$ such that $cl_{V(x)}U(x)$ is a compact space, then $K(\text{Bd}_{V(x)}(U(x))) \geq 2$.

Now, let $L$ be a separation of $w(X(x))$ between $x, p$, we have that $w(V(x)) = U'(x) \cup V'(p) \cup L$, where $U'(x), V'(p)$ are disjoint open subsets of $w(V(x))$. \hfill \Box
Since \( p \not\in U'(x) \), the set \( U'(x) \) is an open subset of \( V(x) \) with \( cl_{V(x)}U'(x) \subset w(V(x)) - V'(p) \), so we have that \( cl_{V(x)}U'(x) \) is a compact space and then \( K(Bd_{V(x)}(U'(x))) \geq 2 \). If we apply the subspace theorem, we get: \( K(L) \geq K(Bd_{V(x)}(U'(x))) \geq 2 \), and then \( K(w(V(x))) > 2 \).

On the other hand, if we assume that \( K(X) \leq 2 \), we have that \( K(V(x)) \leq K(X) \leq 2 \), \( K(w(V(x))) > 2 \). Since \( K(w(V(x))) \leq K(V(x)) + 1 \) (Lemma 3.2), we have that \( K(V(x)) = 2 \), \( K(w(V(x))) > 2 \), a contradiction with the hypothesis.

Consequently, \( K(X) = 3 \). \( \square \)

**Lemma 3.5.** Let \( Y \) be a compact space with \( \text{Ind}(Y) = 3 \). If the equality \( \text{ind}(w(U)) = \text{ind}(U) \) holds for every open, non-closed subset \( U \) of \( Y \) with \( \text{ind}(U) = 2 \), then \( \text{ind}(Y) = 3 \).

**Proof.** The inequality \( \text{ind}(Y) \leq \text{Ind}(Y) \) for every normal space \( Y \) is known (see [3], Theorem 1.6.3). Consequently, we only need to prove the inequality \( \text{ind}(Y) > 2 \).

As \( \text{Ind}(Y) > 2 \), we can find a closed subset of \( Y \), say \( F \), and an open subset of \( Y \) including \( F \), say \( V \), such that \( \text{Ind}(Bd(U)) \geq 2 \) for every set \( U \) open in \( Y \) with \( F \subset Y \subset V \). Since \( Bd(U) \) is a compact space, the inequality \( \text{ind}(Bd(U)) < 2 \) would imply that \( \text{Ind}(Bd(U)) = \text{ind}(Bd(U)) < 2 \) (see [3], Theorems 2.4.2, 2.4.3), so we have that \( \text{ind}(Bd(U)) \geq 2 \).

If we take now the open and non-closed subset of \( X \), \( V \) following the argument of the proof of the Lemma 3.4 we can see that every partition \( L \) of \( w(V) \) between \( F \), \( p \) has \( \text{ind}(L) \geq 2 \). As \( F \) is a closed subset of \( w(V) \) (it’s a compact space included in \( w(V) \)), this implies that \( \text{ind}(w(V)) > 2 \).

If we now assume that \( \text{ind}(Y) \leq 2 \), we get that \( \text{ind}(V) \leq 2 \), \( \text{ind}(w(V)) > 2 \). Since \( \text{ind}(w(V)) \leq \text{ind}(V) + 1 \) (Lemma 3.2) we that \( \text{ind}(V) = 2 \), \( \text{ind}(w(V)) > 2 \), a contradiction with the hypothesis. Then we have that \( \text{ind}(Y) = 3 \). \( \square \)

As we have the equivalence, for a compact space \( X \), between closed subset of \( X \) and compact space included in \( X \), we have the next lemma:

**Lemma 3.6.** If \( X \) is a compact space, then \( \text{Ind}(X) = \text{Ind}(X) \).
LEMMA 3.7. Let $X$ be a compact space, if there exists a partition $L$ of $X$ between $C$, $F$ with $\text{Ind}(L) < n$ for all disjoint compact spaces $C$, $F$ included in $X$, then $\text{Ind}(X) \leq n$.

Proof. Let $C$ be a compact in $X$ and let $V$ be an open subset of $X$ such that $C \subset V$. Since $C$, $X - V$, are disjoint compact spaces, we can find a partition $L$ of $X$ between $C$, $X - V$ with $\text{Ind}(L) < n$. So we have that there exists an open subset of $X$, say $U$, such that $C \subset U$, $\text{Bd}(U) \subset L$, $U \cap (X - V) = \emptyset$. Applying the Lemma 3.6 and the closed subspace theorem for the Ind-dimension, we have that $\text{Ind}(\text{Bd}(U)) \leq \text{Ind}(L) < n$, and the $\text{Ind}(X) \leq n$. \qed

LEMMA 3.8. For every locally compact, noncompact space $X$,

$$\text{Ind}(w(X)) = \text{Ind}(X)$$

Proof. The relation $\text{Ind}(X) \leq \text{Ind}(w(X))$ is a consequence of the Proposition 2.5, so we only need to prove that $\text{Ind}(w(X)) \leq \text{Ind}(X)$.

Let $C$, $F$ be disjoint compact spaces in $w(X)$. At least one of them, say $C$, is included in $X$. We can find an open subset of $X$, $V$, with $C \subset V \subset X$, $cl_X(V) \cap F = \emptyset$, $cl_X(V)$ being a compact space (see [2], Theorem 3.3.2). Since $cl_X(V)$ is a compact space, we have that $cl_X(V) = cl(V)$.

On the other hand, provided that $\text{Ind}(X)$ is a finite number, there exists an open subset of $X$, say $U$, such that $C \subset U \subset V$, $\text{Ind}(\text{Bd}_X(U)) < \text{Ind}(X)$. Since $\text{Bd}(U) = \text{Bd}_X(U)$, we have that $L = \text{Bd}(U)$ is a partition between $C$, $F$ with $\text{Ind}(L) < \text{Ind}(X)$. Because of Lemma 3.7, this implies that $\text{Ind}(w(X)) \leq \text{Ind}(X)$. \qed

LEMMA 3.9. If $X$ is a locally compact, noncompact space with

$$\dim(w(X)) < \infty$$

then $\dim(w(X)) \leq \text{locdim}(X)$

Proof. Assume that $\text{locdim}(X) = n$, then we can find, for each $x \in X$, an open neighbourhood of $x$, $U(x)$ with $\dim(cl_X(U(x))) \leq n$.

As $cl_X(U(x)) \subset cl(U(x)) \subset cl_X(U(x)) \cup \{p\}$ we have that

$$\dim(cl(U(x))) \leq \dim(cl_X(U(x))) \leq n \quad (\text{see [1], [2.2]})$$
Consequently, if we assume that $\dim(w(X)) > n$, $C = \{x \in w(X) / w(X) \text{ is } \dim(w(X))-\text{dimensional at } x\} \subset \{p\}$ and then $\dim(C) \leq 0$.

On the other hand, we know that $C$ is a $\dim(w(X))-\text{dimensional space}$ (see [1], [3.7]), so we have a contradiction and $\dim(w(X)) \leq n$. 

**Example 3.10.** The compact space $T_{23}$ has $\dim(T_{23}) = 2$, $\text{ind}(T_{23}) = 3$. So there exists an open subset of $T_{23}$, say $V$, with $\text{ind}(V) = 2 < \text{ind}(w(V)) = 3$ (Lemma 3.5). Now, Lemma 3.1 implies that $K(w(V)) \geq 2$. If $K(w(V)) = 3$, then $K(V) = 2$ (Lemma 3.2 and Theorem 2.3), and the Lemma 3.3 yields $K(w(V)) = K(V) = 2$, a contradiction.

So we have that $K(w(V)) = 2 = \text{ind}(w(V)) = 3$. In order to prove that $\text{Ind}(w(V)) = 3$, we are going to see that $\text{Ind}(w(V)) \leq 3$:

Since $\text{Indc}(T_{23}) = \text{Ind}(T_{23}) = 3$ (Lemma 3.6), we have the inequality $\text{Indc}(V) \leq \text{Indc}(T_{23}) = 3$ (Proposition 2.5), so we get $\text{Ind}(w(V)) = \text{Indc}(w(V)) \leq \text{Indc}(V) \leq 3$ (Lemma 3.8).

Now, we are going to see that $\dim(w(V)) = 1$.

As $\text{locdim}(T_{23}) \leq \dim(T_{23}) = 1$ ([1], [1.7]), we have that $\text{locdim}(V) \leq \text{locdim}(T_{23}) \leq 1$ ([1], [4.1]), and then $\dim(w(V)) \leq \text{locdim}(V) \leq 1$ (Lemma 3.9).

Since $w(V)$ is a compact space, the equality $\dim(w(V)) = 0$ is not possible (it would imply that $\text{ind}(w(V)) = 0$: see [3], Theorem 3.1.30).

Now, as $Y = w(V)$ is a compact space with $K(w(V)) = 2 < \text{ind}(w(V)) = 3$, we can apply the Lemma 3.4 to find an open, non-closed subset $U$ included in $Y$ with $K(U) = 2 < K(w(U)) = 3$. The same reasoning for $X = w(U)$ as the one we have made for $Y$ implies that $\dim(X) = 1$, $\text{ind}(X) = 3$, $\text{Ind}(X) = 3$, but $X$, $Y$ are non-homeomorphic spaces because we have that $K(X) = 3$, $K(Y) = 2$.

**References**


Received February 9, 1999.