The Representation of Weighted Quasi-Metric Spaces

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SUMMARY. - We show that every weighted quasi-metric space can be identified with a subspace of a space of some canonical type, which is constructed from a metric space.

We also present a very simple method to construct a weighted quasi-metric space, as the graph of a function defined on a metric space, and show that every weighted quasi-metric space arises in this way.

Similar results may be obtained if we drop the requirement that the weight function have nonnegative values.

In this note we continue the investigation carried out in [3]. In particular we fix our attention on a special class of quasi-metric spaces: the so-called weighted quasi-metric spaces (see [1]). The study of these spaces is motivated mainly because they are a useful tool in programming language semantics (see e.g. [2]).

We define a weighted quasi-metric space as a triple \((X, q, w)\), where \(X\) is a set, \(q\) and \(w\) are nonnegative real-valued functions defined on \(X \times X\) and \(X\) respectively, and the following conditions hold:

1. \(q(x, x) = 0\) for every \(x \in X\);
2. \(q(x, z) \leq q(x, y) + q(y, z)\) for every \(x, y, z \in X\);
3. if \(q(x, y) = q(y, x) = 0\) then \(x = y\);

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4. \( q(x, y) + w(x) = q(y, x) + w(y) \) for every \( x, y \in X \).

The function \( q \) is called \textit{quasi-metric}, and \( w \) is the \textit{weight function} (or simply the \textit{weight}).

Every metric space can be viewed as a weighted quasi-metric space, where the weight function is constant (usually zero). On the other hand, if \((X, q, w)\) is a weighted quasi-metric space, then (the restriction of) \( q \) is a metric on each fiber of \( w \).

We say that the weighted quasi-metric space \((X_0, q_0, w_0)\) is a \textit{subspace} of \((X, q, w)\) if \( X_0 \subseteq X \), \( q_0 \) is the restriction of \( q \) to \( X_0 \times X_0 \) and \( w_0 \) is the restriction of \( w \) to \( X_0 \).

If \((X, q, w)\) and \((E, r, v)\) are weighted quasi-metric spaces, a \textit{morphism} of \((X, q, w)\) into \((E, r, v)\) is a mapping \( \varphi \colon X \to E \) such that

\[
\forall x, y \in X \quad r(\varphi(x), \varphi(y)) \leq q(x, y),
\]

\[
\forall x \in X \quad v(\varphi(x)) \leq w(x).
\]

We say that the morphism \( \varphi \) is \textit{isometric} if in (1) we have equality; in this case \( w \) and \( v \circ \varphi \) differ by a constant.

An \textit{isomorphism} of a weighted quasi-metric space onto another one is a bijection between the underlying sets which preserves both the quasi-metric and the weight function. An \textit{embedding} of \((X', q', w')\) into \((X'', q'', w'')\) is an isomorphism of \((X', q', w')\) onto a subspace of \((X'', q'', w'')\).

A \textit{generalized weighted quasi-metric space} is like a weighted quasi-metric space, except that neither the quasi-metric nor the weight are required to be nonnegative; in this case they are called \textit{generalized quasi-metric} and \textit{generalized weight}, respectively. All the above considerations apply to generalized weighted quasi-metric spaces as well.

Given a weighted quasi-metric space \((X, q, w)\), define \( \bar{q} \colon X \times X \to [0, +\infty) \) as

\[
(x, y) \mapsto \frac{q(x, y) + q(y, x)}{2} = q(x, y) + \frac{1}{2} w(x) - \frac{1}{2} w(y);
\]

then \((X, \bar{q})\) is a metric space, which we call the \textit{symmetrization} of \((X, q, w)\). Let us observe that

\[
\forall x, y \in X \quad \left| \frac{1}{2} w(x) - \frac{1}{2} w(y) \right| \leq q(x, y).
\]
Hence, in any weighted quasi-metric space \((X, q, w)\), the weight is (uniformly) continuous as a function on the symmetrization \((X, \tilde{q})\) to \([0, +\infty]\) with the euclidean metric.

In the sequel, we will regard \([0, +\infty]\) as a generalized weighted quasi-metric space by endowing it with the generalized quasi-metric \(\delta: (\xi, \eta) \mapsto \eta - \xi\) and with the (generalized) weight \(\mu: \xi \mapsto 2\xi\).

**Proposition 0.1.** Let \((S, d)\) be a metric space. Consider the cartesian product \(S \times [0, +\infty]\) and denote by \(p\) and \(\pi\) its projections onto \(S\) and \([0, +\infty]\) respectively.

We can construct a generalized weighted quasi-metric space, having \(S \times [0, +\infty]\) as underlying set, by defining the generalized quasi-metric as

\[
Q: ((x, \xi), (y, \eta)) \mapsto d(x, y) + \eta - \xi
\]

and taking \(2 \cdot \pi\) as (generalized) weight function. The fibers of \(\pi\) are isomorphic to \(S\), and the fibers of \(p\) are isomorphic to \([0, +\infty[, \delta, \mu\).

**Proof.** It is clear that \(Q((x, \xi), (x, \xi)) = 0\) and that \(Q((x, \xi), (z, \zeta)) \leq Q((x, \xi), (y, \eta)) + Q((y, \eta), (z, \zeta))\), for every \((x, \xi), (y, \eta), (z, \zeta) \in S \times [0, +\infty]\). Now suppose that both \(Q((x, \xi), (y, \eta)) = 0\) and \(Q((y, \eta), (x, \xi)) = 0\); by adding these two equalities we get \(2d(x, y) = 0\), whence \(x = y\), so that \(\xi = \eta\), too. Finally, letting \(W(x, \xi) = 2 \cdot \pi(x, \xi) = 2\xi\), we have \(Q((x, \xi), (y, \eta)) + W(x, \xi) = d(x, y) + \xi + \eta = Q((y, \eta), (x, \xi)) + W(y, \eta)\) for each \((x, \xi), (y, \eta) \in S \times [0, +\infty]\). Hence \((S \times [0, +\infty[, Q, W)\) is a generalized weighted quasi-metric space. The statement about fibers is obvious. \(\square\)

We call the generalized weighted quasi-metric space constructed in the above proposition the bundle over \((S, d)\).

**Theorem 0.2.** Every weighted quasi-metric space \((X, q, w)\) is embeddable in the bundle over a suitable metric space \((S, d)\).

**Proof.** Take as \((S, d)\) the symmetrization \((X, \tilde{q})\) of \((X, q, w)\), and denote by \((T, Q, W)\) the corresponding bundle, so that, in particular, the set \(T\) is \(X \times [0, +\infty]\). Define a mapping \(\varphi: X \to T\) as follows:

\[
x \mapsto \left(x, \frac{1}{2}w(x)\right).
\]
Then, for every \( x, y \in X \), we have, according to (4):

\[
Q(\varphi(x), \varphi(y)) = \tilde{q}(x, y) + \frac{1}{2}w(y) - \frac{1}{2}w(x) = q(x, y).
\]

Clearly we also have \( W(\varphi(x)) = w(x) \) for each \( x \in X \). Thus \( \varphi \) is an embedding. \( \square \)

Now we present a simple construction that give rise to any weighted quasi-metric space starting from a metric space and a suitable non-negative function defined on it.

**Theorem 0.3.** Given a metric space \( (S, d) \) and a function \( f : S \to [0, +\infty] \) such that

\[
\forall s, t \in S \quad |f(s) - f(t)| \leq d(s, t),
\]

let \( G = \{(s, f(s)) \mid s \in S\} \) be the graph of \( f \); if \( \rho : G \times G \to [0, +\infty] \) is defined by

\[
((s, f(s)), (t, f(t))) \mapsto d(s, t) + f(t) - f(s),
\]

then \((G, \rho, 2 \cdot f)\) is a weighted quasi-metric space. Moreover, every weighted quasi-metric space can be constructed in this way.

**Proof.** Indeed \((G, \rho, 2 \cdot f)\) turns out to be a subspace of the bundle over \((S, d)\), and \( \rho \) is nonnegative on \( G \) by (5).

That every weighted quasi-metric space \((X, q, w)\) arises in this way follows from the proof of Theorem 0.2, taking into account the inequality (3). \( \square \)

In order to construct a \( T_1 \)-separated weighted quasi-metric space (i.e. one in which the distance between different points is always positive), it is necessary and sufficient that in the above theorem the function \( f \) satisfy a suitable strengthening of condition (5), namely the following:

\[
\forall s \neq t \in S \quad |f(s) - f(t)| < d(s, t).
\]

Recall that a *quasi-metric space* is a pair \((X, q)\) where \( X \) is a set and \( q \) is a quasi-metric. The *dual* of \((X, q)\) is \((X, \tilde{q})\), where \( \tilde{q} : (x, y) \mapsto q(y, x) \).
THE REPRESENTATION OF WEIGHTED etc.

We say that the quasi-metric space \((X,q)\) admits the (generalized) weight \(w\) if \((X,q,w)\) is a (generalized) weighted quasi-metric space; if \((X,q)\) admits a weight we also say that it is weightable.

Let \((X,q,\gamma)\) be a generalized weighted quasi-metric space; then \(\gamma'\) is another generalized weight for \((X,q)\) if and only if \(\gamma' - \gamma\) is constant; similarly \(\gamma\) is a generalized weight for \((X,\bar{q})\) if and only if \(\bar{\gamma} + \gamma\) is constant. Hence, given a quasi-metric space \((X,q)\) which admits a weight \(\gamma\), the dual space \((X,\bar{q})\) is weightable if and only if \(\gamma\) is bounded.

By replacing \([0, +\infty]\) with \(\mathbb{R}\) throughout, Theorems 0.2 and 0.3 readily extend to the case in which the function \(w\) is a generalized weight.

We conclude with a characterization of those quasi-metric spaces which admit a (generalized) weight.

**Proposition 0.4.** A quasi-metric space \((X,q)\) admits a generalized weight if and only if

\[
\forall x, y, z \in X \quad q(x, y) + q(y, z) + q(z, x) = q(x, z) + q(z, y) + q(y, x).
\]

**Proof.** Necessity is a straightforward consequence of condition 4. Let us prove sufficiency. Fix an element \(a \in X\), and for each \(x \in X\) put \(\gamma_a(x) = q(a, x) - q(x, a)\); then, from (6), it follows that

\[
\forall x, y \in X \quad q(x, y) + \gamma_a(x) = q(y, x) + \gamma_a(y),
\]

so that \(\gamma_a\) is a generalized weight for \((X,q)\). \(\square\)

**Corollary 0.5.** A quasi-metric space \((X,q)\) is weightable if and only if it satisfies (6) and, for some (equivalently, for each) \(a \in X\), the set

\[
T_a = \{ q(a, x) - q(x, a) \mid x \in X \}
\]

is bounded below.

**Proof.** Indeed, if \(w\) is a weight for \((X,q)\) then \(-w(a)\) is a lower bound for \(T_a\). Conversely, let \(\gamma_a\) be as in the proof of the previous proposition: a weight for \((X,q)\) can be defined by \(x \mapsto \gamma_a(x) - \ell_a\), where \(\ell_a\) is a lower bound for \(T_a\). \(\square\)
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