A Fixed Point Theorem for Fuzzy Contraction Mappings

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Summary. - In this paper, we give a fixed point theorem for fuzzy contraction mappings in quasi-pseudo-metric spaces which is a generalization of the corresponding one for metric spaces given by S. Heilpern.

1. Introduction

S. Heilpern [1] introduced the concept of a fuzzy mapping, i.e., mapping from an arbitrary set to a certain subfamily of fuzzy sets in a metric linear space $X$. He proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multivalued mappings of Nadler [3] arising from the set-representation of fuzzy sets [4]. In this paper we extend the result of Heilpern to quasi-pseudo-metric spaces which are left K-sequentially complete.

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2. Preliminaries

The set of positive integers is denoted by $\mathbb{N}$. Recall that a nonnegative real valued function $d$ defined on a nonempty set $X$ is said to be a quasi-pseudo-metric provided it satisfies the following properties:

- for every $x$, $y$, $z \in X$,
  
  \[ d(x, z) \leq d(x, y) + d(y, z) \]
  
  \[ d(x, x) = 0. \]

The set $B_\varepsilon(x) = \{ y \in X : d(x, y) < \varepsilon \}$ is the $d$-ball with centre $x$ and radius $\varepsilon > 0$. The topology $T(d)$, having as a base the family of all $d$-balls $B_\varepsilon(x)$ with $x \in X$ and $\varepsilon > 0$, is the topology on $X$ induced by $d$. $(X, d)$ is called a quasi-pseudo-metric space, if $d$ is a quasi-pseudo-metric on $X$ and we will suppose it is endowed with the topology $T(d)$, in the following.

If $d$ is a quasi-pseudo-metric on $X$, then $d^{-1}$, defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$, is also a quasi-pseudo-metric on $X$. We will denote $B_{\varepsilon}^{-1}$ the $d^{-1}$-ball with centre $x$ and radius $\varepsilon > 0$. Only if confusion is possible we write $d$-closed or $d^{-1}$-closed, for example, to distinguish the topological concept in $(X, d)$ or $(X, d^{-1})$. We will denote $\min(d, d^{-1})$ by $d \wedge d^{-1}$. We will make use of the following notion, which has been studied by various authors under different names (see e.g. [2], [5]).

A sequence $(x_n)$ in a quasi-pseudo-metric space $(X, d)$ is called left $K$-Cauchy if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_r, x_s) < \varepsilon$ for all $r, s \in \mathbb{N}$ with $k \leq r \leq s$. A quasi-pseudo-metric space $(X, d)$ is said to be left $K$-sequentially complete if each left $K$-Cauchy sequence in $X$ converges (with respect to the topology $T(d)$).

Let $x$ be a point in $X$ and $A$ a nonempty subset of $X$. We define the distance $d(x, A)$ from $x$ to $A$ by

\[ d(x, A) = \inf \{ d(x, a) : a \in A \}. \]

Thus $d(x, A) = 0$ iff $x \in \overline{A}$, the closure of $A$ in $X$.

Now let $A$ and $B$ be nonempty subsets of $X$. We define the distance $d(A, B)$ from $A$ to $B$ by

\[ d(A, B) = \inf \{ d(x, y) : x \in A, y \in B \}. \]
and clearly \( d(A, B) \neq d(B, A) \) in general. Now, we define the Hausdorff separation of \( A \) from \( B \) by

\[
d_H(A, B) = \sup\{d(a, B) : a \in A\}.
\]

Thus we have \( d_H(A, B) \geq 0 \) with \( d_H(A, B) = 0 \) iff \( A \subseteq dB \). In addition, the triangle inequality

\[
d_H(A, C) \leq d_H(A, B) + d_H(B, C)
\]

holds for all nonempty subsets \( A, B, C \) of \( X \). In general, however \( d_H(A, B) \neq d_H(B, A) \).

We define the Hausdorff distance, deduced from the quasi-pseudometric \( d \), between nonempty subsets \( A \) and \( B \) of \( X \) by

\[
H(A, B) = \max\{d_H(A, B), d_H(B, A)\}.
\]

This is now symmetric in \( A \) and \( B \). Consequently, \( H(A, B) \geq 0 \) with \( H(A, B) = 0 \) iff \( dA = dB \), \( H(A, B) = H(B, A) \) and \( H(A, C) \leq H(A, B) + H(B, C) \) for any nonempty subsets \( A, B, C \) of \( X \). When \( d \) is a metric on \( X \), clearly \( H \) is the usual Hausdorff distance.

**Remark 2.1.** Given a quasi-pseudometric \( d : X \times X \rightarrow \mathbb{R}^+ \) let \( \rho = \max\{d, d^{-1}\} \). Then \( \rho \) is obviously a pseudometric on \( X \). Moreover, it is easy to notice that the Hausdorff distance \( H(A, B) = H_d(A, B) \) determined by \( d \) and the Hausdorff distance \( H_\rho(A, B) \) determined by the pseudometric \( \rho \) coincide.

A fuzzy set on \( X \) is an element of \( I^X \) where \( I = [0,1] \). The \( \alpha \)-level set \( A_\alpha \) of a fuzzy set \( A \) on \( X \) is defined as

\[
A_\alpha = \{x \in X : A(x) \geq \alpha\} \text{ for each } \alpha \in [0,1],
\]

\[
A_0 = d(\{x \in X : A(x) > 0\}).
\]

For \( x \in X \) we denote by \( \{x\} \) the characteristic function of the ordinary subset \( \{x\} \) of \( X \).

**Definition 2.2.** Let \( (X, d) \) be a quasi-pseudo-metric space. We define the family of fuzzy sets on \( X \), \( W^*(X) \), as follows:

\[
W^*(X) = \{A \in I^X : A_1 \text{ is nonempty } d-closed \text{ and } d^{-1}-compact\}.
\]
For a metric linear space \((X,d)\), in [1] it is defined the family
\(W(X)\) of fuzzy sets on \(X\), as follows, \(A \in W(X)\) iff \(A_\alpha\) is compact
and convex in \(X\) for each \(\alpha \in [0,1]\) and \(\sup_{x \in X} A(x) = 1\). Clearly,
\(A_\alpha\) is closed for \(\alpha \in [0,1]\) and it is easy to verify that \(A_1\) is nonempty.
Then, in a metric linear space \((X,d)\) we have the following inclusions:
\(W(X) \subset W^*(X) \subset I^X\).

For working with a similar notation to [1] we introduce the next
definition.

**Definition 2.3.** Let \((X,d)\) be a quasi-pseudo-metric space and let
\(A, B \in W^*(X)\), \(\alpha \in [0,1]\). Then we define:

\[ p_\alpha(A,B) = \inf\{d(x,y) : x \in A_\alpha, y \in B_\alpha\} = d(A_\alpha, B_\alpha) \]

\[ D_\alpha(A,B) = H(A_\alpha, B_\alpha) \]

where \(H\) is the Hausdorff distance deduced from the quasi-pseudo-metric \(d\) on \(X\);

\[ D(A,B) = \sup_\alpha D_\alpha(A,B) \]

Notice that \(p_\alpha\) is non-decreasing function of \(\alpha\), and then \(p_1(A,B) = d(A_1,B_1)\).

The following definition is more general than the one given in [1].

**Definition 2.4.** Let \(X\) and \(Y\) be an arbitrary set and a quasi-pseudo-metric space, respectively. \(F\) is said to be a fuzzy mapping if
\(F\) is a mapping from the set \(X\) into \(W^*(Y)\).

**Definition 2.5.** Let \(A, B \in I^X\). As usual in fuzzy theory, we
denote \(A \subseteq B\) when \(A(x) \leq B(x)\), for each \(x \in X\). We say \(x\) is a
**fixed point** of the mapping \(F : X \rightarrow I^X\), if \(\{x\} \subseteq F(x)\).

We will use the following three lemmas, whose proofs we omit, given for a quasi-pseudo-metric space \((X,d)\). They were given in metric version (the first one modified) by Heilpern [1], for the family
\(W(X)\).

**Lemma 2.6.** Let \(x \in X\) and \(A \in W^*(X)\). Then \(\{x\} \subseteq A\) if and only if \(p_1(x,A) = 0\).
Lemma 2.7. \( p_0(x,A) \leq d(x,y) + p_0(y,A) \), for any \( x,y \in X \), \( A \in W^*(X) \).

Lemma 2.8. \( \text{If } \{x_0\} \subset A \text{ then } p_0(x_0,B) \leq D_0(A,B) \) for each \( A,B \in W^*(X) \).

We will need the following lemma.

Lemma 2.9. \( \text{Suppose } K \neq \emptyset \text{ is compact in the quasi-pseudo-metric space } (X,d^{-1}). \text{ If } z \in X, \text{ then there exists } k_0 \in K \text{ such that } d(z,K) = d(z,k_0). \)

Proof. Let \( A \) be a nonempty subset of \( X \). From \( d(z,x) \leq d(z,y) + d(y,x) \) whenever \( x,y,z \in X \), we conclude, taking the infimum of the last expression for \( z \in A, \) that

\[
d(A,x) \leq d(A,y) + d(y,x) \tag{1}
\]

We will see that \( d(A,x) \) is a \( d^{-1}\)-lower-continuous (lsc) function of \( X \). Let \( x_0 \in X \) and \( \varepsilon > 0 \). By (1) we have \( d(A,y) \geq d(A,x_0) - d(y,x_0) \) and then for \( y \in B_{\varepsilon}^{-1}(x_0) \) we have \( d(A,y) > d(A,x_0) - \varepsilon \) and so \( d(A,x) \) is a \( d^{-1}\)-lsc function.

In particular if \( A \) is the one-point set \( \{z\} \), the function \( d(z,k) \) is a \( d^{-1}\)-lsc function of \( k \in K \), and since \( K \) is \( d^{-1}\)-compact then there exists \( k_0 \in K \) such that \( d(z,k_0) = \min\{d(z,k) : k \in K\} \), i.e., \( d(z,k_0) = d(z,K) \).

\( \square \)

3. Fixed point theorem

Now, we prove a fixed point theorem for fuzzy contraction mappings in quasi-pseudo-metric spaces.

Theorem 3.1. \( \text{Let } (X,d) \text{ be a left } K\text{-sequentially complete quasi-pseudo-metric space, and } F \text{ be a fuzzy mapping from } X \text{ to } W^*(X) \)

satisfying the following condition: there exists \( q \in ]0,1[ \), such that

\[
D(F(x),F(y)) \leq q (d \wedge d^{-1})(x,y) \text{ for each } x,y \in X.
\]

Then there exists \( x^* \in X \) such that \( \{x^*\} \subset F(x^*) \).
Proof. Let \( x_0 \in X \) and \( \{x_1\} \subset F(x_0) \). By Lemma 2.9 there exists \( x_2 \in X \) such that \( \{x_2\} \subset F(x_1) \) and \( d(x_1, x_2) \leq d(x_1, (F(x_1))_1) \) since \( (F(x_1))_1 \) is \( d^{-1} \)-compact. We have
\[
d(x_1, x_2) \leq d(x_1, (F(x_1))_1) \leq H(x_1, (F(x_1))_1) \leq D(F(x_0), F(x_1)).
\]

Continuing in this way we produce a sequence \( (x_n) \) in \( X \) such that \( \{x_n\} \subset F(x_{n-1}) \) and \( d(x_n, x_{n+1}) \leq D(F(x_{n-1}), F(x_n)) \) for each \( n \in \mathbb{N} \). We will prove that \( (x_n) \) is a left \( K \)-Cauchy sequence.

\[
d(x_1, x_2) \leq D(F(x_0), F(x_1)) \leq q (d \wedge d^{-1})(x_0, x_1) \leq q d(x_0, x_1)
\]
and
\[
d(x_k, x_{k+1}) \leq D(F(x_k-1), F(x_k)) \leq q (d \wedge d^{-1})(x_k-1, x_k) \leq q d(x_{k-1}, x_k) \leq q^k d(x_0, x_1), \text{ for } k = 0, 1, 2 \ldots
\]

For \( n < m \) we have
\[
d(x_n, x_m) \leq \sum_{i=0}^{m-n-1} d(x_{n+i}, x_{n+i+1}) \leq \sum_{i=n}^{m-1} q^i d(x_0, x_1) \leq \frac{q^n}{1-q} d(x_0, x_1)
\]
whenever \( q \in [0, 1] \) and then \( (x_n) \) is a left \( K \)-Cauchy sequence, since \( q^n \) converges to 0 as \( k \to \infty \). Then, since \( X \) is left \( K \)-sequentially complete in \( X \), there exists \( x^* \in X \) such that \( \lim_n x_n = x^* \).

Now, by Lemma 2.7
\[
p_1(x^*, F(x^*)) \leq d(x^*, x_n) + p_1(x_n, F(x^*))
\]

Then by Lemma 2.8 (compare with [1]):
\[
p_1(x^*, F(x^*)) \leq d(x^*, x_n) + D_1(x_n, F(x^*)) \leq d(x^*, x_n) + D(F(x_{n-1}), F(x^*)) \leq d(x^*, x_n) + q (d \wedge d^{-1})(x_{n-1}, x^*) \leq d(x^*, x_n) + q d(x^*, x_{n-1}).
\]

Now, \( d(x^*, x_n) \) and \( d(x^*, x_{n-1}) \) converge to 0 as \( n \to \infty \). Hence, by Lemma 2.6 we conclude that \( \{x^*\} \subset F(x^*) \). \( \square \)
When \( d \) is a complete metric on \( X \), we get the following result of Heilpern [1]

**Corollary 3.2.** Let \( X \) be a complete metric linear space and \( F \) be a fuzzy mapping from \( X \) to \( W(X) \) satisfying the following condition: there exists \( q \in [0, 1] \) such that

\[
D(F(x), F(y)) \leq qd(x, y) \quad \text{for each } x, y \in X.
\]

Then, there exists \( x^* \in X \) such that \( \{x^*\} \subset F(x^*) \).

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**References**


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