A Note on Quasi-k-Spaces

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Summary. - We prove that for a regular Hausdorff space \( X \) the following conditions are equivalent: (1) \( X \) is locally compact, (2) for each quasi-k-space \( Y \), the product space \( X \times Y \) is also a quasi-k-space.

1. Introduction

Unless the contrary is explicitly stated, all topological spaces are assumed to be regular Hausdorff. Let \( \Sigma \) be a cover for a topological space \( X \) with topology \( \tau \). The family \( \Sigma(\tau) \) of those subsets of \( X \) which intersect each \( S \in \Sigma \) in an \( S \)-open set (i.e., open in \( S \) with the relative topology from \( \tau \)) is a topology for \( X \) finer than \( \tau \). Now to each space \( X \) and to each cover \( \Sigma \) for \( X \) we may associate the space \( \sigma(X) \), the same set of points topologized by \( \Sigma(\tau) \). Let us call a space a \( \Sigma \)-space whenever \( \sigma(X) = X \). If \( \Sigma \) is the cover of all countably compact (respectively, compact) subsets, \( \Sigma \)-spaces are called quasi-k-spaces (respectively, \( k \)-spaces).

The quasi-k-spaces and the \( k \)-spaces appear in several fields in General Topology and Functional Analysis. For instead, when studying compactness of function spaces in the topology of pointwise convergence [1] and in the theory of \( M \)-spaces introduced by K. Morita

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[4]. As it was showed by J. Nagata [6] a space $X$ is a quasi-$k$-space (respectively, a $k$-space) if and only if $X$ is a quotient space of a regular (respectively, paracompact) $M$-space.

In this note we are concerned with characterizing when a quasi-$k$-space satisfies that its product with every quasi-$k$-space is also a quasi-$k$-space. The similar question for $k$-spaces was solved by E. Michael [3]. He showed that $X \times Y$ is a $k$-space for every $k$-space $Y$ if and only if $X$ is locally compact. Our main result is to prove a similar one in the realm of quasi-$k$-spaces.

The terminology and notation are standard. If $X$ is a topological space, $Y$ is a set and $g : X \rightarrow Y$ is an onto mapping, the strongest topology on $Y$ making $g$ continuous is called the quotient topology on $Y$. When $Y$ is equipped with such a quotient topology, it is called a quotient space of $X$, and the inducing map $g$ is called a quotient map. We denote by $\bigoplus_{\alpha \in A} X_\alpha$ the disjoint topological sum of a family $\{X_\alpha\}_{\alpha \in A}$ of topological spaces. A subset $M$ of $X$ is said to be quasi-$k$-closed (in $X$) provided that $M \cap K$ is closed in $K$ for every countably compact subset $K$ of $X$. Obviously, the definition of quasi-$k$-space can be reformulated in the following way: A space $X$ is a quasi-$k$-space if every quasi-$k$-closed subset is closed. We remind the reader that a space $X$ is locally countably compact if each point has a countably compact neighborhood. In the category of regular spaces each such space $X$ has a base composed of countably compact neighborhoods at $x$ for every $x \in X$. For terminology and notation not defined here and for general background see [2].

2. The results

First we shall prove a characterization of quasi-$k$-spaces that it will be used in the sequel (compare with [5], Theorem 1.2).

**Theorem 2.1.** Let $X$ be a Hausdorff space. The following conditions are equivalent:

1. $X$ is a quasi-$k$-space;
2. $X$ is a quotient space of a disjoint topological sum of countably compact spaces;

3. $X$ is a quotient space of a locally countably compact space.

Proof. (1) $\implies$ (2) Let $\mathcal{K}$ be the family defined as

$$ \mathcal{K} = \{ K \subseteq X \mid K \text{ is countably compact} \} $$

and consider the space $Y = \bigoplus_{K \in \mathcal{K}} K$. We shall prove that $X$ is a quotient space of $Y$.

To see this, define the function $\varphi$ from $Y$ onto $X$ by the requirement that $\varphi(x)$ be $x$ whenever $x \in Y$. Beginning from the fact that $X$ is a quasi-$k$-space, it is a routine matter to check that $\varphi$ is a quotient map.

(2) $\implies$ (3) It is clear.

(3) $\implies$ (1) Let $\varphi$ be a quotient map from a locally countably compact space $Y$ onto $X$. Since $\varphi$ is a quotient map, we need only show that $\varphi^{-1}(F)$ is closed in $Y$ whenever $F$ is quasi-$k$-closed in $X$. Suppose that there exists a quasi-$k$-closed subset (in $X$) $F$ such that $\varphi^{-1}(F)$ is not closed in $Y$. We shall see that this leads us to a contradiction. Choose $y \in \partial_Y \varphi^{-1}(F) \setminus \varphi^{-1}(F)$ and let $V$ be a countably compact neighborhood of $y$ in $Y$. $\varphi$ being continuous, $\varphi(V)$ is countably compact and, consequently, $\varphi(V) \cap F$ is closed in $\varphi(V)$. On the other hand, as $\varphi(y) \notin F$, we can find an open set $T$ such that $\varphi(y) \in T$ and

$$ T \cap (\varphi(V) \cap F) = \emptyset. $$

Thus,

$$ \varphi^{-1}(T) \cap \varphi^{-1}(\varphi(V) \cap F) = \emptyset. \quad (\star) $$

But, as $y \in \partial_Y \varphi^{-1}(F)$, there is $z \in Y$ satisfying

$$ z \in (\varphi^{-1}(T) \cap V) \cap \varphi^{-1}(F). $$
So, $\varphi(z) \in \varphi(V) \cap F$. Therefore,

$$z \in \varphi^{-1}(T) \cap \varphi^{-1}(\varphi(V) \cap F).$$

This is contrary to condition (\ast) and the proof is complete. \qed

Given an ordinal number $\alpha$, the symbol $W(\alpha)$ stands for the set of all ordinal numbers less than $\alpha$. When viewed as a topological space this has the usual order topology. As usual, $\omega$ denotes the first infinite ordinal number. If $X$ is a non locally compact space at a point $x_0$, E. Michael constructed in [3] a space $\mathcal{K}(X)$ associated with $X$ in the following way: let $\{U_i\}_{i \in I}$ be a base of non-compact closed neighborhoods of the point $x_0$. For each $i \in I$, since $U_i$ is not compact, there are a limit ordinal number $\eta(i)$ and a well-ordered (by inclusion) family $\{F^{i}_j\}_{j < \eta(i)}$ of closed subsets of $U_i$ such that

$$\bigcap \{F^{i}_j \mid j < \eta(i)\} = \emptyset.$$

Consider now the $k$-space $Z = \bigoplus_{i \in I} W(\eta(i) + 1)$. The Michael space $\mathcal{K}(X)$ is defined as the quotient space of $Z$ obtained by identifying all points $\{\eta(i)\}_{i \in I}$ with a point $y_0$. Since a quotient space of a $k$-space is also a $k$-space ([2], Theorem 3.3.23), $\mathcal{K}(X)$ is a $k$-space (and, a fortiori, a quasi-$k$-space). We need the following important property of $\mathcal{K}(X)$.

**Theorem 2.2.** Let $X$ be a non locally compact space at a point $x_0$. If $K$ is a countably compact subset of $\mathcal{K}(X)$, then $K$ meets at most finitely many elements of the family $\{W(\eta(i))\}_{i \in I}$.

**Proof.** Let $K$ be a subset of $\mathcal{K}(X)$ such that there exists a sequence $\{i_n\}_{n < \omega}$ in $I$ such that $K$ meets $W(\eta(i_n))$ for all $n < \omega$. We shall show that $K$ is not countably compact.

Choose, for each $n < \omega$, an $\alpha_n \in K \cap W(\eta(i_n))$. We shall prove that the sequence $\{\alpha_n\}_{n < \omega}$ does not admit any cluster point in $Y$. For this in turn, it suffices to check that the point $y_0$ is not a cluster.
point of \( \{ \alpha_n \}_{n<\omega} \). As \( \eta(i_n) \) is a limit ordinal, there exists an open set \( V_n \) for each \( n < \omega \) such that

\[
\alpha_n \notin V_n, \quad \eta(i_n) \in V_n.
\]

Let \( D_n \) be the set defined as follows:

\[
D_n = \{ \lambda \in W(\eta(i_n)) \mid \lambda \in V_n \} \cup \{ y_0 \}
\]

and consider the open neighborhood \( D \) of \( \{ y_0 \} \),

\[
D = \left( \bigcup_{n<\omega} D_n \right) \cup E
\]

where \( E = \bigcup \{ W(\eta(i)) : i \neq i_n \text{ for every } n < \omega \} \). It is clear that \( D \) does not meet \( \{ \alpha_n \}_{n<\omega} \) as was to be proved. \( \square \)

We determine next when a space \( X \) satisfies that \( X \times Y \) is a quasi-\( k \)-space for each quasi-\( k \)-space \( Y \). The following lemma is well-known; a proof can be extracted from [2], Corollary 3.10.14.

**Lemma 2.3.** The product space \( X \times Y \) of a locally compact space \( X \) and a locally countably compact space \( Y \) is locally countably compact.

**Theorem 2.4.** Let \( X \) be a regular Hausdorff space. The following assertions are equivalent:

1. \( X \) is locally compact;
2. If \( Y \) is a quasi-\( k \)-space, then so is \( X \times Y \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( Y \) be a quasi-\( k \)-space. According to Theorem 2.1, we can find a locally countably compact space \( Z \) such that \( Y \) is a quotient space of \( Z \). Let \( \varphi \) be a quotient map from \( Z \) onto \( Y \). By [7], Lemma 4 the function \( f \) from \( X \times Z \) onto \( X \times Y \) defined as

\[
f = \text{id}_X \times \varphi
\]
(where \(id_X\) stands for the identity map on \(X\)) is a quotient map. As, by Lemma 2.3, the space \(X \times Z\) is locally countably compact, the result holds by condition (3) in Theorem 2.1.

(2) \(\implies\) (1) Let \(X\) be a non locally compact space at \(x_0\). We shall construct a quasi-\(k\)-space \(Y\) such that \(X \times Y\) is not a quasi-\(k\)-space. To see this, let \(\{U_i\}_{i \in I}\) be a base for closed neighborhoods of the point \(x_0\) and consider, for each \(U_i, i \in I\), a family \(\{F^i_\alpha\}_{\alpha < \eta(i)}\) of nonempty closed sets of \(U_i\) satisfying the same conditions as in Michael’s construction. Let \(Y = \mathcal{K}(X)\) be the Michael space associated with this family. We shall prove that \(X \times Y\) is not a quasi-\(k\)-space. For this end, given \(i \in I\) and \(\mu \in W(\eta(i))\), let \(M^i_\mu\) be the closed set defined as \(M^i_\mu = \bigcap_{\lambda < \mu} F^i_\lambda\). Since the family \(\{F^i_\alpha\}_{\alpha < \eta(i)}\) is well-ordered by inclusion, the set \(M^i_\mu\) is nonempty. Now, for each \(i \in I\), let \(H_i = \bigcup_{\lambda < \eta(i)} \{M^i_\lambda \times \{\lambda\}\}\). Because \(\bigcap_{\mu < \eta(i)} M^i_\mu = \emptyset\), it is easy to check that each \(H_i\) is a closed set. We shall complete the proof by showing that \(H = \bigcup_{i \in I} H_i\) is a quasi-\(k\)-closed, non closed set in \(X \times Y\). In fact, for each \(i \in I\),

\[
H \cap \{X \times (W(\eta(i)) \cup \{y_0\})\} = H_i,
\]

and, by Theorem 2.2, \(H\) is quasi-\(k\)-closed. On the other hand, for each neighborhood \(U \times V\) of \((x_0, y_0)\), we can find \(i \in I\) such that \(U_i \subset U\) and, consequently, if \(\mu \in V \cap W(\eta(i))\),

\[
(U \times V) \cap H_i \neq \emptyset.
\]

Thus, \((x_0, y_0) \in \text{cl}_{X \times Y} H \setminus H\) as was to be proved.

\[\square\]

References

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