Weierstrass Points, Inflection Points and Ramification Points of Curves

E. Ballico

Summary. - Let C be an integral curve of the smooth projective surface S and P ∈ C. Let π : X → C be the normalization and Q ∈ X with π(Q) = P. We are interested in the case in which Q is a Weierstrass point of X. We compute the semigroup N(Q, X) of non-gaps of Q when S is a Hirzebruch surface F_e, P ∈ C_{reg} and P is a total ramification point of the restriction to C of a ruling F_e → P^1. We study also families of pairs (X, Q) such that the first two integers of N(Q, X) are k and d. To do that we study families of pairs (P, C) with C plane curve, deg(C) = d, C has multiplicity d − k at P, C is unibranch at P and a line through P has intersection multiplicity d with C at P.

1. Introduction

We work over an algebraically closed field K with char(K) = 0. Let C be an integral Gorenstein projective curve with g := p_a(C) ≥ 2. Fix P ∈ C_{reg}. Since C is smooth at P, for every integer t the sheaf ω_C(−tP) is a line bundle. Hence, exactly as in the case of a smooth curve we may define the numerical semigroup of non-gaps of C at P, say N(P, C) ⊂ N, such that card(N \ N) = g ([15], [14]). P is not a Weierstrass point of C if and only if N \ N(P, C) = {1, . . . , g}. We just recall that in general N(P, C) is non a semigroup (for any conceivable definition of N(P, C)) without the assumption P ∈ C_{reg} ([14]). Reading [18], it seems obvious that the last assertion

(+) Author’s address: Dept. of Mathematic, University of Trento, 38050 Povo (TN), Italy; e-mail: ballico@science.unitn.it
of [18], Th. 1, p. 545, is not quite true as stated. Hence it seems natural to try to obtain a recipe for the construction of many triples $(g, k, d), k < d - 2k$, of a pair $(X, Q)$, where $X$ is a smooth curve of genus $g$ and $Q \in X$ is a Weierstrass point with $k$ and $d$ as first two positive integers of $N(Q, X)$. Even more: we want to construct nice families of such pairs. For any such pair $(Q, X)$ the complete linear system $|\mathcal{O}_X(Q)|$ is a base point free and induces a morphism $f : X \to \mathbb{P}^2$. Assume $f$ birational and set $C := f(X), P := f(Q)$. The reduction of the tangent cone of $C$ at $P$ is given by a unique line, $L$, which has intersection multiplicity $d$ with $C$ at $P$. Furthermore, $C$ is unibranch at $P$ and it has multiplicity $d - k$ at $P$ because every line $D \neq L$ with $P \in D$ has intersection multiplicity $d - k$. See Theorem 2.5 which clarifies the case in which $C$ is smooth outside $P$, except for a small number of nodes which are in general $\mathbb{P}^2$. According to [17], Remark 13. 12, our approach should be the classical approach considered in [4], p. 59, and in [13], p. 547. For the case in which $g$ is the first non gap and $g + x, 1 \leq x \geq g - 1$, is the only gap $> g$, see [17], Th. 14. 7. For the case in which then map $f : X \to C \subset \mathbb{P}^2$ is not birational, see Remark 2.3. In section 3 we compute $N(P, C)$ when $C$ is an integral curve contained in a Hirzebruch surface, $P \in C_{\text{reg}}$, and $P$ is a total ramification point of the restriction of a ruling of $F_e$ to $C$ (see Theorems 3.1 and 3.5). We solve the same problem for the curve $X$ which is a partial normalization of $C$ at a small number of nodes and cusps which are general points of $F_e$ (Theorem 3.3).

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2. Plane curves

In this section we consider the case of plane curves. Fix an integer $d \geq 4$ and an integer $y$ with $0 \leq y \leq d$. At the beginning of this section (Lemma 2.1 and Proposition 2.2) we consider the set-up of [9], i.e., we consider a pair $(C, P)$ such that $C$ is an integral plane curve of degree $d$, $P \in C_{\text{reg}}$ and such that the tangent line $T_PC$ of $C$ at $P$ has intersection multiplicity $y$ with $C$ at $P$. With the terminology of [9], $P$ is called a $(y - 2)$-inflection point of $P$. Set $V(y, d) := \{x \in \mathbb{N} : x = ay + b \text{ for some integers } a, b \text{ with } 0 \leq a \leq d - y \}$.
With the definition of the semigroup of non-gaps for smooth points of integral Gorenstein curves outlined in the introduction, the proofs of [9], Lemma 2.1 and Prop. 2.2, work verbatim and give the following lemma.

**Lemma 2.1.** Let \( u : X \to C \) be a partial normalization of an integral plane curve \( C \) and \( P \in C_{\text{reg}} \) such that the tangent line \( T_P C \) of \( C \) at \( P \) has intersection multiplicity \( y \) with \( C \) at \( P \). Then \( N(y, d) \subseteq N(P, C) \).

**Proposition 2.2.** Fix integers \( d, y \) with \( d \geq 4 \) and \( d - 2 \leq y \leq d \). Let \( C \) be an integral plane curve and \( P \in C_{\text{reg}} \) such that the tangent line \( T_P C \) of \( C \) at \( P \) has intersection multiplicity \( y \) with \( C \) at \( P \). Then \( N(y, d) = N(P, C) \).

Now we consider the problem described in the introduction. We want to construct for several triples \((g, k, d)\) good families of pairs \((X, Q)\) with \( X \) smooth genus \( g \) curve and \( Q \) Weierstrass point of \( X \) whose first non-gaps are \( k \) and \( d \). We will try to find a plane model \((C, P)\) for \((X, Q)\) such that \( C \setminus \{P\} \) has only nodes as singularities. We cannot apply [6], Th. 0. 2, because in our situation the two numerical assumptions of [6] Th. 0. 2, are never simultaneously satisfied, except in the trivial case \( k = d - 1 \), i.e. the case in which \( C \) is smooth at \( Q \); for this case, see appendix of [9] in which more is proved. However, we may easily adapt the proofs and ideas contained in [6] to cover our situation.

**Remark 2.3.** As recalled in the introduction the set-up considered in this section covers all pairs \((X, Q)\) with \( Q \) Weierstrass point on \( X \) whose first non-gaps are \( k \) and \( d \) with \( k < d < 2k \) and such that the complete linear system \( |dQ| \) on \( X \) induces a birational morphism \( f : X \to \mathbb{P}^2 \). Now we will show how to reduce to this case the construction of all the possible examples for which \( f \) is not birational. Start with a pair \((X', Q')\) with \( X' \) of genus \( g' \geq 3 \) and \( P' \) such that the first non-gaps are \( k' \) and \( d' \) with \( k' < d' < 2k' \). Take an integer \( t \geq 2 \) and a ramified degree \( t \) covering \( \pi : X \to X' \) which is totally ramified over \( P' \) and with \( X \) smooth of genus \( g \); set \( k := tk' \), \( d := td' \) and \( Q := \pi^{-1}(Q')_{\text{red}} \). Using Castelnuovo–Severi inequality (see e.g. [1], Ch. 3) we see that for many values of \( t, g, g', d' \) and
$k'$, the integers $k$ and $d$ must be the first non-gaps of $P$; for instance if $t = 2$ is true if $d \leq g - 2g'$. Viceversa, start with the integers $g$, $k$, $d$ and assume that $f : X \to \mathbb{P}^2$ is not birational. Let $g'$ be the genus of the normalization of $f(X)$. Since $h^0(f(X), O_f(X)(1)) = 3$ and $k < d < 2k$, it is easy to check that $g' \geq 3$.

We recall the ideas introduced in [10] to study equisingular deformations of plane curves. We need this reference for our problem because we need deformations of a unibranch point which preserve the condition “unibranch” and the multiplicity. We fix integers $k$ and $d$ with $3 \leq k < d$; in the application to gap-sequences it is sufficient to consider the case $d < 2k$. For simplicity we work over the field of complex numbers $\mathbb{C}$. We fix $P \in \mathbb{P}^2$ and the germ at $P$ (in the analytic category) of a curve $T$ with a unibranch singularity at $P$ with multiplicity $d - k$. Call $L$ the reduced tangent cone of $T$ at $P$ seen as a line in $\mathbb{P}^2$; we assume that $L$ has intersection multiplicity $d$ with $L$ at $P$. The construction of all such examples is obvious in terms of Puiseux expansions. The paper [10] contains the definition of a zero-dimensional subscheme $Z'$ of $\mathbb{P}^2$ (called the generalized singularity scheme) such that the condition $h^1(\mathbb{P}^2, I_{Z'}(d)) = 0$ gives the existence of a plane curve $U$ of degree $d$ with $P \in U$ and such that $U$ is topologically equivalent to $T$ at $P$. For any $Q \in \mathbb{P}^2$ and integer $x > 0$, let $xQ$ the fat point of order $x$ in $\mathbb{P}^2$ supported by $Q$, i. e. the $(x - 1)$-th infinitesimal neighborhood of $Q$ in $\mathbb{P}^2$. Hence $dP|L$ is the effective divisor of degree $d$ on $L$ given by the multiple of order $d$ of $P$. Taking $Z'' := Z' \cup (dP|L) \cup (d - k)Q$ instead of $Z'$ if $h^1(\mathbb{P}^2, I_{Z''}(d)) = 0$ we may obtain an irreducible plane curve $V$ of degree $d$ topologically equivalent to $T$ near $P$, $V$ and $L$ having intersection multiplicity $d$ at $P$ and such that $V$ has multiplicity at least $d - k$ at $P$. If $Z''$ does not contain $(d - k + 1)P$, then we may even find such $V$ with multiplicity $d - k$ at $P$. Take the germ $T$ at $P$ of any integral unibranch curve at $P$ whose ideal sheaf $I_T$ at $P$ contains $I_{Z''}$ and let $c$ be its conductor; since $T$ is Gorenstein at $P$, the partial normalization $T'$ of $T$ at $P$ has $\dim_K(O_{T'}/O_T) = \dim_K(O_{T,P}/c)$; we are interested mainly in the case in which $\dim_K(O_{T,P}/c)$ is as small as possible (compatibility with the order data). Let $Z$ be the minimal zero-dimensional subscheme of $\mathbb{P}^2$ containing both $Z''$ and the scheme $T/c$; since every deformation of $T$ preserving $c$ preserves
the geometric genus, if $h^3(P^2, I_Z(d)) = 0$ we may obtain an irreducible plane curve $V$ of degree $d$ topologically equivalent to $T$ near $P$, $V$ and $L$ having intersection multiplicity $d$ at $P$, such that $V$ has multiplicity at least $d - k$ at $P$ and such that the arithmetic genus of the partial normalization at $P$ is fixed and is given by $(d - 1)(d - 2)/2 - \dim_K(O_T, P)$. We will call any such $Z$ suitable for the pair $(k, d)$.

**Definition 2.4.** Let $Z$ be a zero-dimensional subscheme of $P^2$ and $L \subset P^2$ a line. Set $Z\{1\} := Z$, $Z\{1\} := Z \cap L$ and call $Z\{2\}$ the residual scheme $\text{Res}_L(Z)$ of $Z$ with respect to $L$. Define inductively $Z(i)$ and $Z\{i\} + 1$ using the formulas $Z(i) := Z\{i\} \cap L$ and $Z\{i\} + 1 := \text{Res}_L(Z\{i\})$. Set $z(i) := h^0(Z(i), O_{Z(i)})$. Notice that $z(i + 1) = z(i)$ and $z(i + 1) \leq z(i)$ for every $i \geq 1$. We will call the non-increasing sequence $\{z(1), z(2), \ldots\}$ (resp. $\{z(1)^*, z(2)^*, \ldots\}$) the first (resp. second) associated sequence of $Z$ with respect to $L$.

**Theorem 2.5.** Fix integers $d, k, x$ with $d > k \geq 3$. Fix $P \in P^2$ and a zero-dimensional scheme $Z$ suitable for the pair $(k, d)$, say with respect to the line $L$. Let $\delta := h^0(Z, O_Z)$ be the length of $Z$ and let $\{z(1)^*, z(2)^*, \ldots\}$ be the first associated sequence of $Z$ with respect to $L$. Assume $z(1) = d$, $z(i) \leq \max\{d - i - 1, 0\}$ for every $i \geq 2$ and $0 \leq 3x \leq d(d + 3)/2 - 3 - z$. Then for $x$ general points $Q_1, \ldots, Q_x$ of $P^2$ there exists an integral degree $d$ curve $C \subset P^2$ with $P \in C$, $C$ unibranch at $P$, $L$ intersecting $C$ with multiplicity $d$, $Z \subset C$, $Q_i \in C$ for every $i$, $Q_i$ with ordinary nodes at each $Q_i$ and $C$ smooth outside $\{P, Q_1, \ldots, Q_x\}$. Furthermore, fixing $Z$ and varying $Q_1, \ldots, Q_x$ in a Zariski open subset of symmetric product $S^x(P^2)$ we obtain an irreducible family $M(Z, x)$ of dimension $d(d + 3)/2$ of plane curves with geometric genus $(d - 1)(d - 2)/2 - \delta - x$, where $\delta$ is the arithmetic genus of the singularity $(C, P)$.

**Proof.** Step 1) Here we will show that the union of $Z$ and the first infinitesimal neighborhoods $2Q_i$, $1 \leq i \leq x$, of $x$ general points $Q_i$ of $P^2$ imposes $z + 3x$ independent conditions to the linear system of degree $d$ plane curves containing them. Since $z(1) = d$ is the maximal length of a subscheme of $Z$ contained in a line through $P$, we have $Z \subseteq dP$. Hence the case $x = 0$ is obvious. Hence we will assume $x > 0$. Take a general $Q \in L$ and let $t$ the length 2 zero-dimensional
subscheme of $L$ with $t_{\text{red}} = \{Q\}$; the scheme $t$ is the second simple residue of $Q$ with respect to $L$ in the sense of [2], Definition 2.2. Take $x - 1$ general points $Q_i$, $1 \leq i \leq x - 1$, of $\mathbb{P}^2$ and let $W$ be the union of $Z\{2\}$, the schemes $2Q_i$, $1 \leq i \leq x - 1$, and $t$. By [2], Lemma 2.3 it is sufficient to prove the vanishing of $h^1(\mathbb{P}^2, \mathcal{I}_W(d-1))$. Notice that $\text{length}(W \cap L) = z(2) + 2 \leq d - 2$. Now we continue making again an application of Horace Lemma with respect to the line $L$.

We specialize $[(d - z(2) - 2)/2]$ of the points $Q_i$, say $Q_i$, $1 \leq i \leq [(d - z(2) - 2)/2]_i$, to general points $Q_i'$, $1 \leq i \leq [(d - z(2) - 2)/2]$ of $L$; if $d - z(2) - 2$ is even we call $W'$ the union of $Z\{2\}$, $2Q_i'$, $1 \leq i \leq [(d - z(2) - 2)/2]$ and $2Q_j$, $[(d - z(2) - 2)/2]_j < j \leq x$ with $Q_j'$ general in $\mathbb{P}^2$. Set $W'' := \text{Res}_L(W')$. Since $\text{length}(W' \cap L) = d$, we have $h^1(\mathbb{P}^2, \mathcal{I}_{W''}(d-1)) = h^1(\mathbb{P}^2, \mathcal{I}_{W''}(d-2))$ and hence we may continue the construction. If $d - z(2) - 2$ is odd instead of a general point $Q_\alpha$, $\alpha := [(d - z(2) - 2)/2] + 1$, of $\mathbb{P}^2$ we specialize it to a general $Q_\alpha' \in L$ and call $t'$ the length 2 zero-dimensional subscheme of $L$ with $t'_{\text{red}} = \{Q'\}$; set $W'(1) := (W'' \setminus 2Q_{x-1}) \cup t'$. By [2], Lemma 2.3, to prove the vanishing of $h^1(\mathbb{P}^2, \mathcal{I}_{W'(1)}(d-1))$ it is sufficient to prove the vanishing of $h^1(\mathbb{P}^2, \mathcal{I}_{W'(1)}(d-2))$. Then we continue to reduce the vanishing we need to the vanishing of some group $H^1(\mathbb{P}^2, \mathcal{I}_{A(t)})$ for some integer $t < d$ and some zero-dimensional scheme $A$ whose connected component, $B$, supported by $P$ is $Z\{d - t + 1\}$. Hence at each step we add at most one double point, $t''$, supported by a general point of $L$ and this length 2 zero-dimensional scheme gives no contribution when we take the residual scheme with respect to $L$. Since $z(d - t) \leq t + 1$ we are sure that at every step we may add on $L$ a scheme $t'$ of length 2 when $t + 2 - z(d - t - 1)$ is odd, and still have the vanishing of the cohomology group $H^1$ for $A \cap L$. If we finish the $x$ points before arriving to the case $t = 0$, we have won. Since $z + 3c \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) - (h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)))$, we are sure to finish the points $Q_i$.

Step 2) The generalized associated scheme of an ordinary double point at $Q$ is just ideal sheaf of $2Q$ ([10], Ex. 2 after Def. 2.3). Hence by [10] we obtain that the family of plane curves with $x + 1$ prescribed singularities (one at $P$ with $Z$ as associated scheme plus $x$ nodes in general position) is smooth of the expected dimension and its general member has only the expected singularities. □
Remark 2.6. Since in characteristic zero every smooth curve of genus at least 2 has only finitely many Weierstrass points, it is obvious that the “parametrization” of smooth curves with a Weierstrass point with first non-gaps $k$ and $d$ given by Theorem 2.5 is, up to an element of $\text{Aut}(\mathbb{P}^2)$, finite - to - one.

Example 2.7. Here we will compute the associated scheme in the particular case in which $d$ and $k$ are coprime. In this case the germ at $(0,0)$ of the curve $t \to (t^{d-k}, t^d)$ is the germ of a unibranch curve which has all the properties we want. We drop the condition $d < 2k$. We apply the standard blowing-up procedure to this curve whose local equation near $(0,0)$ is $y^{d-k} = x^d$. Set $x' = x$, $y' = yx$. In the first infinitesimal neighborhood we obtain an equation $y^{d-k} = x'^k$. Since $k$ and $d - k$ are coprime, we continue in the same way and find that the tree of the resolution is a Dynkin diagram of type $A_W$ for some $w$.

With the notations of [19], p. 15, the characteristic of this branch are the integers $(d - k, d)$ and $g = 1$. This example is important because any nearby germ with the same characteristic exponents is formally equivalent to it and in particular it has the same $\delta$-invariant, i.e. the conductor has the same co-length.

Example 2.8. Now we will construct in some cases plane curves with two associated schemes of type $(k, d)$, but with $2k < d$. More generally, we fix two integers $k, k'$ with $2 \leq k \leq k' \leq d - k$, take two distinct points $P, P'$, of $\mathbb{P}^2$, lines $L, L'$, on $\mathbb{P}^2$ with $P \in L$, $P' \in L'$, $P \not\in L'$, $P' \not\in L$ and look at integral plane curves, $C$, of degree $d$ passing through $Q$ (resp. $Q'$) with multiplicity $d - k$ (resp. $d - k'$), unibranch at $P$ and $P'$ and with $L$ (resp. $L'$) with intersection multiplicity $d$ with $C$ at $P$ (resp. $P'$). Let $Z$ (resp. $Z'$) be the associated scheme for the data $(k, d, P, L)$ (resp. $(k', d, P', L')$).

We want to prove (under suitable assumptions on $Z$ and $Z'$) that $h^1(\mathbb{P}^2, I_{Z\cup Z'}(d)) = 0$. Set $W := \text{Res}_L(Z)$ and $W' := \text{Res}_L(Z')$. We have $\text{length}(W) = \text{length}(Z) - d$ and $\text{length}(W') = \text{length}(Z') - d$.

We apply first Horace Lemma with respect to $L$ (loosing one condition) and then Horace Lemma with respect to $L'$ (without loosing anything because $L'$ is in the base locus of $H^0(\mathbb{P}^2, I_{W\cup W'}(d - 1))$). Hence to prove the vanishing of $h^1(\mathbb{P}^2, I_{Z\cup Z'}(d)) = 0$ it is sufficient to prove the vanishing of $h^1(\mathbb{P}^2, I_{W\cup W'}(d - 2))$. Let $\{z(1), z(2), \ldots\}$ (resp. $\{z'(1), z'(2), \ldots\}$) be the first associated sequence of $Z$ (resp. $Z'$).
The proof of Theorem 2.5 works verbatim exploiting alternatively $L$ and $L'$ if for every integer $i \geq 2$ we have $z(i) \geq d - 2i$ and $z'(i) \geq d - 2i - 1$. However, if for some integers $k$, $k'$ we want to construct such integral curves with a few nodes, say at general point $Q_1, \ldots, Q_y$ with $y$ very small, it is better to use the following trick to check the vanishing of $h^1(P^2), I_{W(i-1)} \cup_2 Q_1, \ldots, \cup_2 Q_w (d - 2i)$. We apply Horace Lemma $[(d - 2)/2]$ times using a smooth conic passing through $P$, $P'$ and perhaps some of the points $Q_i$. Call $D$ a general conic through $P$ and $P'$, and define the schemes $W(i)$ (resp. $W'(i)$), $i \geq 1$, and the first associated $D$-sequence to $W$ (resp. $W'$) exploiting $D$ instead of the line $L$ (resp. $L'$) and denote it with $\{w(1), \ldots\}$ (resp. $\{w'(1), \ldots\}$). Hence $w(1) = w'(1) = d - 2$. Suppose that after $i - 1$ applications of Horace Lemma using a smooth conic through $P$ and $P'$ one need to check the vanishing $h^1(P^2, I_{W(i-1)} \cup_2 Q_1, \ldots, \cup_2 Q_w (d - 2i))$ for some non-negative integers $w$, $z$. By the generality of the points $Q_1, \ldots, Q_y$ it is sufficient to have $h^1(P^2, I_{W(i-1)} \cup_2 Q_1, \ldots, \cup_2 Q_w (d - 2i)) = 0$ and $\text{length}(W(i-1)) + \text{length}(W'(i-1)) + 3w + z \leq (d - 2i + 2)(d - 2i + 1)/2$.

Fix an integer $m$ with $0 \leq m \leq \min\{w, 5\}$ and take a general smooth conic $A$ through $P$, $P'$ and $m$ of the points $Q_1, \ldots, Q_w$. To apply another time Horace Lemma exploiting $A$ it is sufficient to have $2m + w(i - 1) + w'(i - 1) \leq 2(d - 2i)$.

3. Curves on Hirzebruch surfaces

In this section we fix an integer $e \geq 0$ and set $S := F_e$, where $F_e = P(O_{P^1} \oplus O_{P^1})$ is a Hirzebruch surface. Let $\pi : S \to P^1$ be the associated ruling. We take as a basis of $\text{Pic}(S) \cong Z^2$ a fiber, $f$, of $\pi$ and a section $h$ of $\pi$ with minimal self-intersection. Hence we have $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$. For the cohomological properties of the line bundles on $S$, see [11], pp. 379–381. We will use both the additive and multiplicative notation for line bundles, divisors and linear systems on $S$. We just observe that (as in [11], pp. 379–381) everything follows from the projection formula $\pi_* (O_S (ah + bf)) \cong \bigoplus_{0 \leq t \leq a} O_{P^1}(b - te)$ for all integers $a$, $b$ with $a \geq 0$. We have $\omega_S = -2h + (e - 2)f$. We are interested in integral curves $C \in |kh + xf|$, some $k \geq 2$, $x \geq k\epsilon$, $x > 0$, such that there are some points, say
$P_1, \ldots, P_z \in C_{reg}, z \geq 1$, such that the fiber of $f$ at each point $P_i$ has intersection multiplicity $k$ at $P_i$, i.e. such that every point $P_i$ is a total ramification point of $\pi$. By the adjunction formula we have $p_0(C) = kx - ek(k-1)/2 - x - k + 1$. Since the case $k = 2$ corresponds to hyperelliptic curves, we will assume $k \geq 3$.

**Theorem 3.1.** Fix integers $k, x$ with $k \geq 3$, $x > 0$, $x \geq ke$, an integral curve $C \in [kh + xf]$ and $P \in C_{reg}$ which is a total ramification point of $\pi$. If $e > 0$ assume that $P$ is not contained in the section $h$ of $\pi$ with minimal self-intersection. If $e = 0$ we have $h^0(C, \mathcal{O}_C(akP)) = a + 1$ if $0 \leq a \leq x$, $h^0(C, \mathcal{O}_C(tP)) = [t/k] + 1$ if $bk < t < (b + 1)k$ for some $b$ with $b = 1 \leq x$, while $h^0(C, \mathcal{O}_C(tP)) = t + 1 - g$ (i.e. $h^1(C, \mathcal{O}_C(tP)) = 0$) if $t > xk$, i.e. the semigroup $N(P, C)$ is the minimal one compatible with its first positive value, $k$, and the constraint card $(N \setminus N(P, C)) = g$. Assume $e > 0$; we have $h^1(C, \mathcal{O}_C(akP)) = 0$, i.e. $h^0(C, \mathcal{O}_C(akP)) = ak - kx + ek(k-1)/2 + x + k$ if and only if $\alpha \geq x - e - 1$; fix an integer $\beta \leq x - e - 2$; let $u$ be the minimal non-negative integer with $e(k - 2 - u) \leq x - e - 2 - \beta$; we have $h^0(C, \mathcal{O}_C(\beta kP)) = \beta - kx + ek(k-1)/2 + x + k + \sum_{0 \leq i \leq k-2-u} (x - e - 1 - \beta - ie)$; the non-gaps of $P$ in the interval $k\beta < t \leq k\beta + k$ are at last $h^0(C, \mathcal{O}_C((\beta + 1)kP)) - h^0(C, \mathcal{O}_C(\beta kP))$ integers in this interval.

**Proof.** It is sufficient to compute the value of $h^0(C, \omega_C(-tP))$ for every integer $t > 0$. For every integer $t > 0$, call $D(t)$ the zero-dimensional subscheme of $S$ corresponding to the positive Cartier divisor $tP$ on $C$. Since $h^0(S, K_S) = h^1(S, K_S) = 0$, the restriction map $H^0(S, \mathcal{O}_S((k-2)h + (x - e - 2)f)) \to H^0(C, \omega_C)$ is bijective. Hence it is sufficient to compute $h^0(S, \mathcal{O}_S((k-2)h + (x - e - 2)f) \oplus I_D(t))$ for every integer $t > 0$. Set $w := [t/k]$ and $z := t - kw$. Hence $0 \leq z \leq k$ and $h^0(S, \mathcal{O}_S((k-2)h + (x - e - 2)f) \oplus I_D(w)) = h^0(S, \mathcal{O}_S((k-2)h + (x - e - 2 - w)f) \oplus I_D(z))$. By the cohomology exact sequence of the restriction of $K_S$ to $\alpha$ fibers, $\alpha \geq 0$ we obtain $h^1(S, K_S(-\alpha f)) = 0$ for every positive integer $\alpha$. Since $h^0(S, K_S(-\alpha f)) = 0$ for every positive integer $\alpha$, by Serre duality on $C$ we obtain $h^1(C, \mathcal{O}(atP)) = 0$ if and only if $\alpha \geq x - e - 1$ and $h^1(C, \mathcal{O}(\beta tP)) = h^0(S, \mathcal{O}_S((k-2)h + (x - e - 2 - \beta)f)))$ for every positive integer $\beta$; let $u$ be the multiplicity of $h$ as base component of $(k-2)h + (x - e - 2 - \beta)f)$, i.e. let $u$ be the minimal non-negative integer with $e(k-2-u) \leq x - e - 2 - \beta$;
by the projection formula we obtain \( h^0(S, O_S((k-2)h + (x-e-2-\beta)f)) = h^0(S, O_S((k-2-u)h + (x-e-2-\beta)f)) = \sum_{0 \leq i \leq k-2-u}(x-e-1-\beta-ic). \) Hence we know \( h^0(C, O_C(wkP)) \) and \( h^0(C, O_C((w+1)kP)) \) and we have \( 1 \leq h^0(C, O_C((w+1)kP)) - h^0(C, O_C(wkP)) \leq k-1, \) until \( O_C((w+1)kP) \) is a special line bundle, i.e. for \( w \leq x-e-2. \) Furthermore since \( O_C((w+1)kP) \) is spanned, we know that \( h^0(C, O_C((wk+k-1)kP)) = h^0(C, O_C((w+1)kP)) - 1. \) For every integer \( q \) we have \( h^0(C, O_C(qP)) \leq h^0(C, O_C((q+1)P)) \leq h^0(C, O_C(qP)) + 1. \) To prove the theorem it is sufficient to check that the \( h^0(C, O_C((w+1)kP)) - h^0(C, O_C(wkP)) \) jumps for the values of \( h^0(C, O_C(qP)) \) in the interval \( wk \leq q < (w+1)k \) occur for \( q = wk \) and the last \( h^0(C, O_C((w+1)kP)) - h^0(C, O_C(wkP)) - 1 \) values of \( q. \) First we assume \( e > 0. \) Call \( m \geq 0 \) the multiplicity of \( h \) as base component of \( |(k-2)h + (x-e-2-w-1)f|, \) i.e. set \( m := 0 \) if \( x-e-2-w \geq ek \) and set \( m := k-2 - [(x-e-2-w-1)/e], \) otherwise. By Serre duality on \( C \) it is sufficient to note that the restriction of \( |(k-2-m)h + (x-e-2-w-1)| \) to any fiber of \( \pi \) (even to \( \pi^{-1}(\pi(P)) \)) is a complete linear system of degree \( (k-2-m) \), that \( h^0(C, O_C((w+1)kP)) - h^0(C, O_C(wkP)) = k-1-m \) and that for every integer \( z \) with \( 0 \leq z \leq t \) the zero-dimensional scheme \( D(z) \) is contained in the fiber \( \pi^{-1}(\pi(P)) \). The case \( e = 0 \) is similar and easier.

**Remark 3.2.** The proof of Theorem 3.1 works verbatim if \( C \) is any integral curve such that \( P \in C_{\text{reg}} \) and such that the fiber of \( \pi|C \) is just \( P \) with multiplicity \( k. \) Of course, here we use the notation of gap sequence at a smooth point of any Gorenstein curve. The power of this tool will be shown by the proof of Theorem 3.3 we assume \( \text{Sing}(C) = \{Q_1, \ldots, Q_y\} \), i.e. even in the case in which we are studying Weierstrass points on a smooth curve (the normalization of \( C \)).

Now we consider the case of the normalization of nodal or cuspidal curves. The case \( e = 0 \) was considered (at least for the nodes) in [5] and [7]. Again, the aim is to show that when we have very few nodes and cusps and/or their position is sufficiently general, then the situation is the best possible one with the numerical constraints we have and in particular the Weierstrass point has the a priori possible minimal weight. Since it comes for free, we will consider a more gen-
eral statement allowing the partial normalization of a very singular curve.

**Theorem 3.3.** Fix integers \( k, x, y \) with \( k \geq 3, x > 0, x \geq ke \) and \( y \geq 0 \). Fix \( y \) general points \( Q_1, \ldots, Q_y \) of \( S := F_c \). Assume the existence of an integral curve \( C \) with \( C \in [kh + xf] \) such that each \( Q_i \) is an ordinary node or an ordinary cusp of \( C \) and such that there is \( P \in C_{\text{reg}} \) which is a total ramification point of \( \pi \). If \( e > 0 \) assume that \( P \) is not contained in the section \( h \) of \( \pi \) with minimal self-intersection. Let \( X \) be the partial normalization of \( C \) at \( Q_1, \ldots, Q_y \). Set \( g := kx - ek(k - 1)/2 - k - x + 1 - y \). Let \( C' \in [kh + xf] \) be a smooth curve with \( P \in C \) and such that \( P \) is a total ramification point for \( \pi|C' \). Then for every integer \( t > 0 \) we have \( h^0(C, \omega_C(-tP)) = \max \{ h^0(C', \omega_C'(-tP)) - y, 0 \} \).

**Proof.** The adjoint linear system to \( X \) is given by the curves in \( |(x - 2)k + (x + e + 2)f| \) passing through the points \( Q_1, \ldots, Q_y \). Since these points are in general points of \( S \), they impose the maximal possible number of conditions (i.e., \( \min \{y, \dim(V)\} \)) to any complete linear system \( V \) on \( S \). Hence the proof of Theorem 3.1 and the claim that the same proof works for a singular curve (Remark 3.2) give the result.

For the existence of a curve \( C \) as in the statement of Theorem 3.3 and with \( \text{Sing}(C) = \{Q_1, \ldots, Q_y\} \), see Remark 3.4.

**Remark 3.4.** Modifying the proof of [3], Prop. 3.7 and Prop. 4.1, Theorem 3.3 may be applied to several cases in which \( C \) has an ordinary node at each point \( Q_1, \ldots, Q_y \) as unique singularities and hence in which \( X \) is smooth; the only difference in each cohomological computation comes from the condition “\( \pi|C \) with total ramification at \( P \)” for some case for \( e = 0 \), see [5] or [7]; Proof of Theorem 0.1; for similar cases in the plane, see [9] and [8]; anyway if \( y \) is very small with respect to \( x - ke \), the existence of such curve \( C \) is an easy exercise. Now we will consider the case in which in the statement of Theorem 3.1 we have \( e > 0 \) and \( P \in h \).

**Theorem 3.5.** Fix integers \( e, k, x \) with \( e > 0, k \geq 3, x > 0 \), \( x \geq ke \), an integral curve \( C \in [kh + xf] \), \( p \in h \) and \( P \in C_{\text{reg}} \) which is a total ramification point of \( \pi \). We have \( h^1(C, \mathcal{O}_C(\alpha kP)) = 0 \),
i.e. \( h^0(C, \mathcal{O}_C(\alpha kP)) = \alpha k - kx + ek(k - 1)/2 + x + k \) if and only if \( \alpha \geq x - e - 1 \); fix an integer \( \beta \leq x - e - 2 \); let \( u \) be the minimal non-negative integer with \( e(k - 2 - u) \leq x - e - 2 - \beta \); we have \( h^0(C, \mathcal{O}_C(\beta kP)) = \beta - kx + ek(k - 1)/2 + x + k + \sum_{0 \leq i \leq k - 2 - u} (x - e - 1 - \beta - ie) \); the non-gaps in the interval \( k\beta < t \leq k\beta + k \) are in the first \( h^0(C, \mathcal{O}_C((\beta + 1)kP)) - h^0(C, \mathcal{O}_C(\beta kP)) - 1 \) values of \( t \) and the integer \( k\beta + k \).

**Proof.** We use the notations of the proof of 3.1. To prove 3.5 it is sufficient to check that the \( h^0(C, \mathcal{O}_C((w + 1)kP)) - h^1(C, \mathcal{O}_C(\omega kP)) \) jumps for the values of \( h^0(C, \mathcal{O}_C(qP)) \) in the interval \( wk \leq t < (w + 1)k \) occurs for the value \( t = wk \) (which is true because \( \mathcal{O}_C(\omega kP) \) is spanned) and for the next \( h^0(C, \mathcal{O}_C((w + 1)kP)) - h^1(C, \mathcal{O}_C(\omega kP)) - 1 \) values of \( t \). The latter assertion is true because \( u \) contains \( D(u) \).

\[ \square \]

**Theorem 3.6.** Fix integers \( k, x \) and \( z \) with \( z \geq 1 \), \( k \geq 2 \), \( x > 0 \), \( x \geq ke \). Let \( C | kh + x | f | be an integral projective curve and \( P_1, \ldots, P_Z \) \( C_{reg} \) total ramification points of \( \pi \). Thus \( g := p_0(C) = kx + ke - k - x + 1 \). Call \( M(P_1, \ldots, P_z, k, x) \) the subset of \( |kh + x| \) parametrizing the curves, \( D \), containing the effective divisor \( \sum_{1 \leq i \leq z} kP_i \) of \( C \), i.e. containing each \( P_i \) and such that \( \pi | D \) has total ramification at each \( P_i \). Call \( M'(P_1, \ldots, P_z, k, x) \) the subset of \( M(P_1, \ldots, P_z, k, x) \) formed by the curves, \( D \), with \( P_i \in D_{reg} \) for every \( i \). Let \( N_C \) be the normal bundle of \( C \) in \( S \). Hence \( N_C = \mathcal{O}_C(kh + x + f) \) and \( \deg(N_C) = C^2 = 2g - 2 - KS \cdot C = 2g - 2 + 2k + 2x + ke \). Hence we have the following remark.

**Remark 3.7.** If \( zk < 2k + 2x + ke \) we have \( h^1(C, N_C(-\sum_{1 \leq i \leq z} kP_i)) = 0 \).

**Remark 3.8.** By Remark 3.7 and [17], Th. 1.5 if \( zk < 2k + 2x + ke \) the scheme \( M'(P_1, \ldots, P_z, k, x) \) is smooth at \( P \) of dimension \( h^0(C, N_C(-\sum_{1 \leq i \leq z} kP_i)) = g - 1 + 2x + 2k + ke - zk = kx + 2k + ke \) for all integers \( i \) with \( 1 \leq i \leq w \) is smooth and of the expected dimension. Hence we obtain the following result.
Proposition 3.9. Fix integers $k$, $x$ and $w$ with $z \geq 1$, $k \geq 2$, $x \geq ke$ and $zk < 2k + 2x + ke$. Let $C \in \{kh + xf\}$ be an integral projective curve and $P_1, \ldots, P_z \in C_{reg}$ total ramification points of $\pi$. For every integer $w$ with $1 \leq w \leq z$ there exists a generically smooth irreducible open subset $M'(w)$ of $M'(P_1, \ldots, P_w, k, x)$ such that $\dim(M'(w)) = kx = 2ke + k(e - w - 1)$, $M'(w + 1)$ is contained in the closure of $M'(w)$ if $w < z$ and $C \in M'(z)$.

Now we some integers $e$, $k$, $x$, and $z$ we will construct the data $(C, P_1, \ldots, P_z)$ we were looking for.

Proposition 3.10. Fix integers $e$, $k$, $x$, $z$, with $e \geq 0$, $k \geq 3$, $x > 0$, $x \geq ke + z$. Fix $z$ distinct points $Q_1, \ldots, Q_z$ of $\mathbb{P}^1$ and $P_i \in S$ with $\pi(P_i) = Q_i$ for every $i$. Then there exists a smooth curve $C \in \{kh + xf\}$ such that every integer $i$ $P_i \in C$ and $P_i$ is a total ramification point of $\pi|C$.

Proof. Call $Z$ the zero-dimensional subscheme of $S$ contained in $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q)$, with $D_{red} = \{P_1, \ldots, P_z\}$ and such that each connected component of $Z$ has length $k$. By assumption the lineal system $\{kh + (x - z)f\}$ is a base point free. Hence the sheaf $\mathcal{I}_Z \mathcal{O}_S(kh + bf)$ is spanned outside $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q)$. Since it is easy to check that no line of $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q)$ is in the base locus of $\mathcal{I}_Z \mathcal{O}_S(kh + bf)$, by Bertini theorem a general curve $C \in \{kh + bf\}$ containing $D$ is smooth outside $\{P_1, \ldots, P_z\}$. Taking reducible curves union of $\bigcup_{1 \leq i \leq z} \pi^{-1}(Q)$ and of a curve $C' \in \{kh + (x - z)f\}$ with $P_i \notin C'$ for every $i$, we obtain the smoothness of a general curve $C \in \{kh + xf\}$ containing $D$.

References


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