Rend. Istit. Mat. Univ. Trieste
Vol. XXX, 181–183 (1998)

A Short Proof of Regularity for Solutions to Semilinear Elliptic Problems with Exponential Critical Growth

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Summary. - *We show that the weak solutions of the elliptic semilinear Dirichlet problem* (P) *are classical solutions. The proof's simplicity is based on the fact that the nonlinearity is of exponential type, in contrast to nonlinearities of polynomial type.*

We are interested in the solutions the elliptic Dirichlet boundary value problem

$$\begin{cases}(-\Delta)^m u = g(x,u) & \text{in } \Omega \subset \mathbb{R}^{2m},\\ u = Du = \cdots = D^{m-1}u = 0 & \text{on } \partial\Omega;\end{cases} \tag{P}$$

where Ω is a bounded domain and m is a positive integer. The function g is a C^1-class function of *critical* or *subcritical growth* for problem (P) in the sense of [3]. That is g must satisfy

$$\limsup_{t\to\infty} \frac{\sup_{x\in\Omega} |g(x,t)|}{\exp(bt^2)} < \infty \tag{1}$$

for some constant $b > 0$. Under extra assumptions on g, in [3] is proved the existence of one (nontrivial) weak solution to the problem (P). A function u is a defined to be a weak solution of (P) if $g(x,u) \in H^{-m}(\Omega)$ and the following holds

$$(\nabla^m u, \nabla^m v) = \langle g(x,u), v\rangle, \text{ for all } v \in H_0^m(\Omega). \tag{2}$$

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Here we use the convention that

$$\nabla^m \varphi = \begin{cases} \operatorname{grad} \Delta^{(m-1)/2} \varphi & \text{if } m \text{ is odd,} \\ \Delta^{m/2} \varphi & \text{if } m \text{ is even.} \end{cases} \tag{3}$$

THEOREM 1. *A weak solution u of* (P) *is a classical solution of class $C^{2m,\alpha}$ for some $\alpha \in (0,1)$.*

In order to prove this result we first observe that by the Trudinger embedding [4, 1] we know that

$$\int_\Omega \exp(pb\, |v(x)|^2)\, \mathrm{d}\, x < \infty \tag{4}$$

for all $p \geq 1$ whenever $v \in H_0^m(\Omega)$. Let now u be any solution of (P), by condition (1) it follows that $f(\cdot) := g(\cdot, u(\cdot)) \in L^p(\Omega)$ for any $p \geq 1$. By the L^p-regularity theory it is well known that

$$\|u\|_{W^{2m,p}(\Omega')} \leq C \left(\|f\|_{L^p(\Omega')} + \|u\|_{L^p(\Omega')} \right) \tag{5}$$

for any fixed compact subdomain $\Omega' \subset \Omega$. In view of the Morrey embedding $W^{2m,p}(\Omega') \hookrightarrow C^{0,\alpha}(\Omega')$, for sufficiently big p, we conclude that $u \in C^{0,\alpha}(\Omega')$. So f itself is in $C^{0,\alpha}(\Omega')$ since g is in $C^1(\mathbb{R} \times \Omega')$. By the C^α-regularity theory of Schauder it follows that $u \in C^{2m,\alpha}(\Omega')$.

If we assume furthermore that the domain Ω is smooth enough we can prove in the same fashion that $u \in C^{2m,\alpha}(\Omega)$.

Notice that this proof works only in the case of exponential critical growth. In the case of polynomial critical growth, like in [2], the semilinear term $g(u)$ is guaranteed to be in L^1 but not in $L^{1+\epsilon}$ with $\epsilon > 0$ and the argument above is not applicable.

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Received April 25, 1999.