Irreducible Unitary Representations of a Diffeomorphisms Group of an Infinite-dimensional Real Manifold

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SUMMARY. - Groups of diffeomorphisms $\text{Diff}^1_{\beta,\gamma}(M)$ of infinite-dimensional real Banach manifolds $M$ are defined. Their structure is studied. Irreducible unitary representations of a group of diffeomorphisms associated with quasi-invariant measures on a Banach manifold are constructed.

1. Introduction

For a locally compact (finite-dimensional) manifold $M$ irreducible unitary representations of a group of diffeomorphisms were constructed in [13] with the help of a measure on $M$ induced by the Lebesgue measure on $\mathbb{R}^n$ and the Riemannian metric $g$ on $M$. Each group of diffeomorphisms is an infinite-dimensional manifold itself. Their structure for locally compact $M$ was investigated in [2,7].

This article is devoted to the definition of a group of diffeomorphisms of a Banach manifold and the construction its irreducible unitary representations. For this are used quasi-invariant Gaussian measures on $M$.

In Section 2 notations and definitions are given. Section 3 contains results about the structure of a group of diffeomorphisms. Irreducible unitary representations of a group of diffeomorphisms associated with a quasi-invariant measure on a Banach manifold are

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described in Section 4. There is the great difference in investigations between cases of finite-dimensional and infinite-dimensional $M$. The main results of the present paper are deduced for the first time and given below in Theorems 3.3, 4.1, 4.17, 4.18.

2. Notations and definitions

To avoid misunderstandings, we first present our notations and terminology.

**Definition 2.1.** Let $U$ and $V$ be open subsets in $l_2$. We consider a space of all infinitely many times Frechet (strongly) differentiable functions $f, g : U \to V$ fulfilling (i, ii) and with a finite metric $\rho_{\beta, \gamma}^f (f, h) < \infty$, where $h$ is some fixed smooth mapping $h : U \to V$ (that is of class $C^\infty$);

\[(i) \quad \rho_{\beta, \gamma}^f (f, g) := \sup_{x \in U, \ y \neq x, \ y \in U} \left( \sum_{n=0}^{\infty} \left[ d_{n, \beta, \gamma}^f (f, g) \right]^2 \right)^{1/2} < \infty; \]

\[d_{n, \beta, \gamma}^f (f, g) := \| x^\beta (f(x) - g(x)) \|_{l_{2, \gamma}}; \]

\[d_{n, \beta, \gamma}^f (f, g)^2 := \sum_{\alpha \cdot \gamma, |\alpha| \leq \beta} \| x^{\alpha \gamma} < x^{\beta + |\alpha|} |D_x^\alpha (f(x) - g(x))| \|_{l_{2, \gamma}}^2 + \]

\[+ \sum_{\alpha = [\alpha_1, \ldots, \alpha_n]} \left[ x^{\alpha} \right] < x^{\beta + |\alpha| + b} |D_x^\alpha (f(x) - g(x))| - D_y^\alpha (f(y) - g(y)) \|_{l_{2, \gamma}}^2 / |x^n - y^n|^{2b}, \]

for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $d_{n, \beta, \gamma}^t (f, g) = d_{n, \beta, \gamma}^t (f, g)(x, y)$, such that

\[(ii) \quad \lim_{R \to \infty} \rho_{\beta, \gamma}^h (f|U_R^c, h|U_R^c) = 0. \]

Here $x = (x^j : j \in \mathbb{N}, \ x^j \in \mathbb{R}) \in l_{2, \gamma}$ that is

\[\| x \|_{l_{2, \gamma}} = \left( \sum_{j=1}^{\infty} (x^j)^2 \right)^{1/2} < \infty, \]
$\infty > \gamma \geq 0$, $l_2 = l_{2,0}$ is the standard separable Hilbert space over $\mathbb{R}$ with the orthonormal base $\{e_n : n \in \mathbb{N}\}$, $U_R^c := \{x \in U : \|x\|_{l_2} > R\}$, $f(x) = (f^j(x) : j \in \mathbb{N}, f^j(x) \in \mathbb{R})$, $t \geq 0$, $[t]$ is the integral part of $t$ (the largest integer such that $[t] \leq t$, $b = \{t\} := t - [t]$, $0 \leq b < 1$ (for $b = 0$ the last term in the definition of $d^{n,\beta,\gamma}_R$ is omitted), $D_x^{e_j} := \partial/\partial x^j =: \partial_j$, $D_x^{e^j} f(x) := D_x^f (D_x f(x))$, $e_j = (0, \ldots, 1, 0, \ldots)$ with 1 in the $j$-th place, $\alpha = (\alpha^1, \ldots, \alpha^m)$, $\alpha^j \in \mathbb{N} \cup \{\infty\}$, $\alpha^0 = 0$, $\alpha^0 = \alpha^1 + \cdots + \alpha^n$, $\beta \in \mathbb{R}$, $\tilde{x} := \min(\langle x \rangle, \langle y \rangle)$, $\langle x \rangle := (1 + \|x\|^2_2)^{1/2}$, $f(x) - g(x) \in l_2$, $f|A$ denotes a restriction of $f$ on a subset $A \subset U$, $\tilde{n}^\alpha := 1^{\alpha^1} 2^{\alpha^2} \cdots n^{\alpha^n}$ for $n \in \mathbb{N}$.

We denote by $E_{\beta,\gamma}^{t,h}(U, V)$ the completion of such metric space, $E_{\beta,\gamma}^{t,h}(U, V) := \bigcap_{j=1}^{\infty} E_{\beta,\gamma}^{t,h}(U, V)$ with the topology given by the family $(\rho_{\beta,\gamma}^j : j \in \mathbb{N})$ in the latter case. For $V = l_2$ and $h(u) = 0$ it is the Banach space with $\|f - g\|_{E_{\beta,\gamma}^{t,h}(U, l_2)} := \rho_{\beta,\gamma}^t (f, g) = \rho_{\beta,\gamma}^t (f - g, 0)$ that is, the infinite-dimensional separable analog of the weighted Hölder space $C_{\beta}^t (U', \mathbb{R}^m)$ (compare with [5]) for open $U' \subset \mathbb{R}^k$, $k$ and $m \in \mathbb{N}$. When $\gamma = 0$ or $h(U) = 0$ we omit $\gamma$ or $h$ respectively. It is evident that each cylindrical function $g(P, x)$ is in $E_{\beta}^t(U, l_2)$ if $g \in C_{\beta}^t (U', \mathbb{R}^m)$, $P_k : l_2 \rightarrow \mathbb{R}^k$ is the orthogonal projection, $U = (P_k)^{-1}(U')$, $g(P, x) := (g^1(P, x), \ldots, g^m(P, x), 0, 0, \ldots)$. The spaces $E_{\beta}^{t,h}(U, V)$ differ from $E_{\beta}^{t}(U, V) =: E^t(U, V)$ for unbounded $U$ if $\beta > 0$.

**Definition 2.2.** Let $M$ be a manifold modelled on $l_2$ and fulfilling conditions (i–vi) below:

(i) an atlas $At(M) = \{(U_j, \phi_j) : j = 1, \ldots, k\}$ is finite, $k \in \mathbb{N}$ (or countable, $k = \infty$), $\phi_j : U_j \rightarrow l_2$ are homeomorphisms of $U_j$ onto $\phi_j(U_j)$, $0$, $U_j$ and $\phi_j(U_j)$ are open in $M$ and $l_2$, respectively, $(\phi_j \circ \phi_j^{-1} - \text{id}) \in E_{\omega,\delta}^{\infty} (\phi_i(U_i \cap U_j), l_2)$ for each $U_i \cap U_j \neq \emptyset$, where $\omega > 0$, $\gamma \geq 0$, $\text{id}$ is the identity mapping $\text{id}(x) = x$ for each $x$;

(ii) $TM$ is a Riemannian vector bundle with a projection $\pi : TM \rightarrow M$ and a metric $g_x$ in $T_x M$ induced by $\|\cdot\|_{l_2}$ with a RMZ-structure. This means that a connector $K$ and $g$ are such that $g_x(X, Y)$ is constant for each $C^\infty$-curve $c : I \rightarrow M$, $I = [0, 1] \subset \mathbb{R}$ and parallel translation along $c$ of $X$ and $Y \in \Xi(M)$,
\( \Xi(M) := \Xi_{TM}(M) \) is the algebra of infinitely differentiable vector fields on \( M \) (see 3.7 in [10]);

(iii) \( (M, g) \) is geodesically complete and supplied with the Levi-Civita connection and the corresponding covariant differentiation \( \nabla \) (see 1.1, 2.1 and 5.1 in [10]);

(iv) the charts \( (U_j, \phi_j) \) are natural with the natural (Gaussian) coordinates with locally convex \( \phi_j(U_j) \) and the exponential mapping \( \exp_p : V_p \to M \) corresponding to \( \nabla \), where \( V_p \) is open in \( T_pM \) for each \( p \in M \), each restriction \( \exp_p|V_p \) is the local homeomorphism (see Section III.8 in [15], Section 6, 7 in [10]) such that \( r_{inj} := \inf_{x \in M} r_{inj}(x) > 0 \), where \( r_{inj}(x) \) is a radius of injectivity for \( \exp_x \), \( r_{inj} \) is for entire \( M \);

(v) \( M \) is Hilbertian at infinity, that is, there exists \( \bar{M} \subset M \) with \( M \setminus \bar{M} =: M_e \) equal to finite (or countable) disjoint union of connected open components \( \Omega_a, a = 1, \ldots, p \), such that \( \phi_a^{-1}(\Omega_a) = l_2 \setminus B_a \), where \( B_a \) are closed balls in \( l_2 \), each \( \Omega_a \) is with a metric \( c \) induced by \( \phi_a^{-1} \) and the standard metric in \( l_2 \). Let a metric \( g \) for \( M \) be elliptic, that is, there exists \( \lambda > 0 \) such that \( \lambda c(\xi, \xi) \leq g(\xi, \xi) \) for each \( \xi \in T_xM \) and \( x \in M \), where \( M := \{ x \in M : d_M(x, x_0) \leq R \} \), \( x_0 \) is some fixed point in \( M \), \( d_M \) is the distance function on \( M \) induced by \( g \), \( \infty > R > 0 \) (see for comparison the finite-dimensional case of \( M \) in [5]);

(vi) \( M \) contains a sequence of \( M_k \) and \( N_k \). They are supposed to be closed \( E_{\omega, n}^\infty \)-submanifolds with finite dimensions \( dim_{\mathbb{R}} M_k = k \) for \( M_k \) and codimensions \( codim_{\mathbb{R}} N_k = k \) for \( N_k \), \( k = k(n) \in \mathbb{N} \), \( k(n) < k(n + 1) \) for each \( n \), \( M_k \subset M_l \) and \( N_k \subset N_l \) for each \( k < l \), \( M = M_k \setminus N_k \), \( M_k \cap N_k = \partial M_k \cap \partial N_k \) for each \( k \) such that \( \bigcup_k M_k \) is dense in \( M \), \( At(M) \) and \( M \) are foliated in accordance with this decompositions. These means that (\( \alpha \) \( \phi_{i,j} := \phi_i \circ \phi_j^{-1} : (x^l : l \in \mathbb{N}) \to (x^l : l \in \mathbb{N}) \)) for each \( n \in \mathbb{N} \), \( k = k(n) \), when \( M \) is without boundary, \( \partial M = \emptyset \). If \( \partial M \neq \emptyset \) there is the following additional condition: (\( \beta \) for each boundary component \( M_0 \) of \( M \) and \( U_i \cap M_0 \neq \emptyset \) we have \( \phi_i : U_i \cap M_0 \to \)}
We denote by $/\mathcal{H}_1$, where $H_1 = \{(x_j : j \in \mathbb{N}) | x^l \geq 0\}$. If $U_i \cap M_0 \neq \emptyset$ and $U_j \cap M_0 \neq \emptyset$ we have both images in $H_1$ (or in $H_t$ with $l > 1$), then the foliation is called transverse (tangent respectively) to $M_0$.

Then the equivalence relation of $E^{\infty}_{\omega, \gamma}$-atlases that produces foliated $M$ (see also [12] for finite-dimensional $C^r$-manifolds) is as usually considered.

**Definition 2.3.** Let $M$ and $\tilde{M}$ be two manifolds as in 2.2 with a smooth mapping (for example, an embedding) $\theta : \tilde{M} \hookrightarrow M$, $\omega$ and $\tilde{\omega} \geq \max(0, \beta)$, $\beta \in \mathbb{R}$, $t \in \mathbb{R}^+$ := $[0, \infty)$, $\infty > \gamma \geq 0$, $\delta$ and $\tilde{\delta} \geq \gamma$. We denote by $E_{\beta, \gamma}^{\theta, \theta}(M, M)$ a space of functions $f : M \to M$ with $f_{ij} := f_{i} \circ f_{j}^{-1}|(\phi_{j}(U_j) \cap \phi_{j}(f^{-1}(U_i)))$, $(f_{ij} - \theta_{ij}) \in E_{\beta, \delta}^{\theta, \theta}(\phi_{j}(U_j) \cap \phi_{j}(f^{-1}(U_i)))$, $\phi_{j}(U_i))$ for each $i$ and $j$.

When $At(M)$ is finite it is metrizable by a metric (i) $\rho_{\beta, \gamma}^{t, \theta}(f, \theta) := \sum_{i,j} \rho_{\beta, \gamma}^{t, \theta}(f_{ij}, \theta_{ij})$ with (ii) $\lim_{R \to \infty} \rho_{\beta, \gamma}^{t, \theta}(f | M_{\beta, \gamma}^{R}, \theta) = 0$. For infinite countable $At(M)$ we denote by $E_{\beta, \gamma}^{\theta, \theta}(M, M)$ the strict inductive limit $str-\text{ind}-\lim E_{\beta, \gamma}^{\theta, \theta}(U^{E}, M), \Pi_{E}^{F}, \Sigma$, where $E \in \Sigma$, $\Sigma$ is the family of all finite subsets of $\mathbb{N}$ directed by the inclusion $E < F$ if $E \subset F$, $U^{E} := \bigcup_{j \in E} U_{j}$, $(\tilde{U}_{j}, \tilde{\phi}_{j})$ are charts of $At(M)$, $\Pi_{E}^{F} : E_{\beta, \gamma}^{\theta, \theta}(U^{E}, M) \hookrightarrow E_{\beta, \gamma}^{\theta, \theta}(U^{F}, M)$ and $\Pi_{E} : E_{\beta, \gamma}^{\theta, \theta}(\tilde{M}, M)$ are uniformly continuous embeddings (isometric for $0 \leq t < \infty$). Evidently, $E_{\beta, \gamma}^{\theta, \theta}(M, M)$ is the space of functions $f$ of the class $E_{\beta, \gamma}^{\theta}$ with supports $\text{supp}(f) := d\{x \in \tilde{M} : f(x) \neq 0\} \subset U^{E}(f)$, $E(f) \in \Sigma$ and $0 \in W \subset E_{\beta, \gamma}^{\theta, \theta}(M, M)$ is open if and only if $\Pi_{E}^{-1}(W) \cap E_{\beta, \gamma}^{\theta, \theta}(U^{E}, M)$ is open for each $E \in \Sigma$.

Let $Hom(M)$ be a group of homeomorphisms of $M$ and $Diff_{\beta, \gamma}^{t}(M) := \{f \in Hom(M) : f \text{ and } f^{-1} \in E_{\beta, \gamma}^{\theta, \theta}(M, M)\}$ be a group of homeomorphisms (diffeomorphisms for $t \geq 1$) of class $E_{\beta, \gamma}^{t}$. When $At(M)$ is finite it is metrizable with the right-invariant metric

$$(iii) \quad d(f, g) := \rho_{\beta, \gamma}^{t}(g^{-1} f, \text{id}),$$

where $\theta$ is the identity map for $\tilde{M} = M$, $\theta = \text{id}$ (in this case the index $\theta$ is omitted), $\beta \geq 0$ (see also [14] for finite-dimensional $M$, correctness of this definition is proved in Theorem 3.1). Henceforth, we omit tilde in $\tilde{E}$. 

IRREDUCIBLE UNITARY REPRESENTATIONS etc.
Definition 2.4. A Riemannian metric $g$ for $M$ Hilbertian at infinity is called regular Hilbertian asymptotically, if there exist $\delta > 0$, $t' > 1$, $\beta' > 0$, $\infty > \gamma' > 0$ such that $(g - e)_{x}(\xi, \xi) \in E^{t', \gamma'}_{\beta'}(M, \mathbb{R})$ by $x$ for each $\xi \in TM$, $\xi = (\xi_{x} : x \in M)$, $\|\xi_{x}\|_{L_{1}} \leq 1$ for each $x \in M$, sup$_{x \in TM}$, $\|\xi_{x}\|_{L_{1}} \| (g - e)_{x}(\xi, \xi) \|_{E^{t', \gamma'}_{\beta'}(M, \mathbb{R})} \leq \delta$. For spaces $E^{t', \gamma'}_{\beta'}(M, N)$ with $M = N$ or $N$ being a Banach space over $\mathbb{R}$ we assume that $\omega \geq \max(0, \beta)$ and $\beta' \geq \max(0, \beta)$, $t' > t + 1$, $\gamma' \geq \gamma$ in 2.2, 2.4.

**Definition 2.5.1.** Let $X$ be separable BS over $\mathbb{R}$. Suppose that $F_{n} \subset F_{n+1} \subset \cdots \subset X$, $dim_{\mathbb{R}}F_{n} = n$, is a sequence of finite-dimensional subspaces. Let $\{z_{n} : n \in \mathbb{N}\}$ be a sequence of linearly independent vectors in $X$ with $\|z_{n}\|_{X} = 1$, span$\{z_{1}, \ldots, z_{n}\} = F_{n}$ for each $n$. For open $U$ and $V$ in $X$ we consider a space of all infinitely many times Frechét differentiable functions $f, g : U \rightarrow V$ fulfilling (i, ii) in 2.1 and with $\rho_{\beta, \gamma}(f, h) < \infty$, where $h : U \rightarrow V$ is some fixed smooth (of class $C^{\infty}$) mapping $h : U \rightarrow V$, $D^{\alpha}_{x}$ for $\alpha = (\alpha^{1}, \ldots, \alpha^{n})$ is the operator of differentiation by $(x^{1}, \ldots, x^{n}) \in F_{n}$, but with $U^{C}_{R} := \{x \in U : \|x\|_{X} > R\}$ and $< x > = (1 + \|x\|_{X}^{2})^{1/2}$. We denote by $E^{t, h}_{\beta, \gamma}(U, V)$ the completion of such metric space and consider $E^{t, h}_{\beta}(U, V)$ as in 2.3.

**Definition 2.5.2.** Let $M$ be a paracompact separable metrizable manifold modelled on $X$ [17] and fulfilling (i, ii) below:

(i) an atlas $\text{At}(M) = [(U_{j}, \phi_{j}) : 1 \leq j < k + 1]$ is finite, $k \in \mathbb{N}$ (or countable $k = \omega_{0}$), $\phi_{j} : U_{j} \rightarrow X$ are homeomorphisms of $U_{j}$ onto $\phi_{j}(U_{j}) \ni 0$, $U_{j}$ and $\phi_{j}(U_{j})$ are open in $M$ and $X$ respectively, $(\phi_{j} \circ \phi_{i}^{-1} - id) \in E^{\omega, \delta}_{\omega, \delta}(\phi_{j}(U_{i} \cap U_{j}), X)$ for each $U_{i} \cap U_{j} \neq \emptyset$, where $\omega > 0$, $\gamma \geq 0$, $id(x) = x$ is the identity mapping, $\omega_{0}$ is the initial number of cardinality $\aleph_{0}$ [9];

(ii) $M$ contains a sequence of $M_{k}$ and $L_{k}$ submanifolds. They are of class $E^{\infty, \gamma}_{\omega, \gamma}$ with $dim_{\mathbb{R}}M_{k} = k$ for $M_{k}$ and $codim_{\mathbb{R}}L_{k} = k$ for $L_{k}$, $k = k(n) \in \mathbb{N}$, $k(n) < k(n + 1)$ for each $n$, $M_{k} \subset M_{l}$ and $L_{k} \subset L_{l}$ for each $k < l$, $M = M_{k} \cup L_{k}$, $M_{k} \cap L_{k} = \partial M_{k} \cap \partial L_{k}$ for each $k$ such that $\bigcup_{k}M_{k}$ is dense in $M$. Moreover, $M$ and $At(M)$ are foliated. That is, they fulfil $(\alpha, \beta)$:
(α) \( \phi_{i,j} : \phi_i \circ \phi_j^{-1} |_{U_i \cap U_j} \rightarrow X \) are of the form \( \phi_{i,j}((x^l : l \in \mathbb{N})) = (\alpha_{i,j,k}(x^1, \ldots, x^k), \gamma_{i,j,k}(x^l : l > k)) \) for each \( n \in \mathbb{N}, k = k(n), \) when \( M \) is without a boundary, \( \partial M = \emptyset. \)

If \( \partial M \neq \emptyset \) then:

(β) for each boundary component \( M_0 \) of \( M \) and \( U_i \cap M_0 \neq \emptyset \) we have \( \phi_i : U_i \cap M_0 \rightarrow H_i \), where \( H_i = \{ x \in X : x^l \geq 0 \}, \)
\( x^l = P_{z_i}(x) \) is the projector of \( X \) onto \( \mathbb{R}_{\geq 0} \) along \( X \oplus \mathbb{R}_{\geq 0} \) (see [22]).

**Definition 2.5.3.** Analogously to Definition 2.3 we consider spaces \( E_{\beta,\gamma}^f(M, M) \) and \( Diff_{\beta,\gamma}^f(M) \) for \( M \) and \( \hat{M} \) as in 2.5.2.

Then \( Diff_{\beta,\gamma}^f(M) \) is defined as \( \bigcap_{l \in \mathbb{N}} Diff_{l,\gamma}^f(M) \) and \( Diff_{\beta,\gamma}^f(M) = \bigcap_{l \in \mathbb{N}} Diff_{l,\beta,\gamma}^f(M) \) with the corresponding standard topologies of projective limits [9,22].

**Definition 2.6.** Let \( G \) be a topological group. A Radon measure \( \mu \) on \( Af(G, \mu) \) (or \( \nu \) on \( Af(M, \nu) \)) is called left-quasi-invariant relative to a dense subgroup \( G' \) of \( G \), if \( \mu_\phi(\ast) \) (or \( \nu_\phi(\ast) \)) is equivalent to \( \mu(\ast) \) (or \( \nu(\ast) \)) respectively for each \( \phi \in G' \). Henceforth, we assume that a quasi-invariance factor \( q_\mu(\phi, g) = \mu_\phi(dg) / \mu(dg) \) (or \( q_\nu(\phi, x) \)) is continuous by \( (\phi, g) \in G' \times G \) (or \( (G' \times M) \)), \( \mu : Af(G, \mu) \rightarrow [0, \infty], \mu(V) > 0 \) (or \( \nu : Af(M, \nu) \rightarrow [0, \infty], \nu(V) > 0 \)) for some (open) neighbourhood \( V \subset G \) (or \( M \)) of the unit element \( e \in G \) (or a point \( x \in M \)), \( \mu(G) < \infty \) (or \( \nu(M) \leq \infty \)) and is \( \sigma \)-finite respectively, where \( \mu_\phi(E) := \mu(\phi^{-1}E) \) for each \( E \in Af(G, \mu) \), \( Af(G, \mu) \) is the completion of \( Bf(G) \) by \( \mu \), \( Bf(G) \) is the Borel-\( \sigma \)-field on \( G \) [6].

Let \((M, F)\) be a space \( M \) of measures on \((G, Bf(G)) \) (or \((M, Bf(M)) \)) with values in \( \mathbb{R} \) and \( G' \) be a dense subgroup in \( G \) such that a topology \( T \) on \( M \) is compatible with \( G' \), that is, \( \mu \rightarrow \mu_h \) (or \( \nu \rightarrow \nu_h \)) is the homeomorphism of \((M, F)\) onto itself for each \( h \in G' \). If \( T \) be the topology of convergence for each \( E \in Bf(G) \) (or \( Bf(M) \)) and \( W \) be a neighbourhood of the identity \( e \in G \) such that \( J \) is dense in \( W \), where \( J := \{ h : h \in G' \cap W =: W' \} \), there exists \( b \in (-1, 1) \) and \( g(b) = h \) with \([g(c) : c \in (-1, 1)] \subset W'\), \( g(c_1 + c_2) = g(c_1)g(c_2) \), \( g(0) = e \) are one parameter subgroups, \( c_1, c_2 \in \mathbb{R} \). We assume also that for each \( f \in W' \) there are \( g(b_1), \ldots, g(b_k) \in J \) such that \( f = g(b_1) \ldots g(b_k) \). A measure \( \mu \in M \) (or \( \nu \in M \)) is
called differentiable along \( g(b) \) in a point \( g(c) \) if \( \mu(g(b)^{-1}E) - \mu(E) = (b - c)(\mu'(g(c); E) + \alpha(g(b); E)) \) and there exists \( \lim_{b \to c} \alpha(g(b); E) = 0 \) and \( \mu'(g(c); E) \) is continuous by \( g(c) \) for each \( E \in Bf(G) \), where \( b \) and \( c \in \mathbb{R} \), \( \mu'(g(c); E) \) is called the derivative (by Lagrange) along \( g(b) \) in \( g(c) \) (analogously for \( \nu \) on \( M \)). Let by induction \( \lambda(s) = \mu^{(j-1)}(g(c_1), \ldots, g(c_{j-1}); s) \) and there exists \( \lambda'(g(c_j); E) \), then it is denoted \( \mu^{(j)}(g(c_1), \ldots, g(c_j); E) \) and is called the \( j \)-th derivative (by Lagrange) of \( \mu \) along \( (g(b_1), \ldots, g(b_j)) \) in \( (g(c_1), \ldots, g(c_j)) \), where \( j \in \mathbb{N} \).

**Lemma 2.7.** Let \( M \) be a \( E_{\omega, \delta}^{\infty} \)-domain in \( X \). Then there exists a Hilbert space \( Y \) such that \( Y \subset X \), \( Y \) is dense in \( X \), \( \|x\|_Y \geq \|x\|_X \) for each \( x \in X \) and \( Diff_{\beta, \gamma}^{t, \delta}(N) \) is a dense subgroup in \( Diff_{\beta, \gamma}^{t, \delta}(M) \), where \( N = M \cap Y, \infty \geq t \geq 0, \gamma \geq \gamma' \geq \gamma + 2, \omega \geq \beta', \delta \geq \gamma' \).

**Proof.** In view of Theorem I.4.4 [16] for BS \( X \) there exists a Hilbert space \( Y, Y \subset X \), \( \|x\|_Y \geq \|x\|_X \) for each \( x \in X \). We take \( \{F_n : n \in \mathbb{N}\} \) in \( X \) and an orthonormal base \( \{e_n : n \in \mathbb{N}\} \) in \( Y \) with \( e_1 = z_1, e_i = \sum_{j=1}^{i-1} b_{i,j} z_j \) are chosen by induction, \( b_{i,j} \neq 0 \). Since \( \|\sum_{i=1}^{\infty} x^i z_i \|_Y \leq \sum_{i=1}^{\infty} |x^i| \times \|z_i\|_Y \), \( \|\sum_{i=m}^{n} x^i z_i \|_X \leq (\sum_{i=m}^{n} |x^i|)^{1/2} \leq (n-m)^{1/2}, \sum_{i=1}^{\infty} (\sum_{m=1}^{\infty} m^{d}) < \infty \) for each \( d < -2 \), then there is a Hilbert space \( Y_0 \) with an injection \( T : Y_0 \to X \) being a nuclear operator [20,22], \( T x = \sum_{i=1}^{\infty} (x, y_i) y_i \), where \( x \in Y_0 \), \( (\ast, \ast)_{Y_0} \) is an inner product in \( Y_0 \), \( \{y_i\} \) is a base in \( Y_0 \) such that \( \sum_{i=1}^{\infty} |y_i| \leq 1 \). Moreover, we can choose \( e_i = b_{i,j} z_j \). Let \( Y_0 \subset Y \subset X, \|x\|_{Y_0} \geq \|x\|_Y \geq \|x\|_X \) for each \( x \in Y_0 \). Then from Definition 2.1 of \( \rho_{\beta, \gamma}^t \) and \( b_{\gamma}, \) also from the consideration of multipliers \( \tilde{n}^{\alpha, \gamma}, m\tilde{n}^{\alpha, \gamma} \), it follows that each \( g \in Diff_{\beta, \gamma}^{t, \delta}(N) \) belongs to \( Hom(M) \), since \( F_n \subset Y \subset X, t' \geq 1, \ast x \ast Y \geq \ast x \ast X \) for each \( x \in Y \). Therefore, \( g \) has the unique continuous extension \( \tilde{g} \) on \( M \) such that \( \tilde{g} \in Diff_{\beta, \gamma}^{t, \delta}(M) \), since \( N \) is dense in \( M \) and we can choose for each \( 0 < \epsilon \) the space \( Y_0 \) with \( |y_i| \leq i^{-2-\epsilon} \) for each \( i \in \mathbb{N} \).

**Definition 2.8.** Let \( M \) be a \( E_{\omega, \delta}^{\infty} \)-manifold as in 2.5 that has a locally finite partition of unity of the same class of smoothness. Henceforward, we suppose that there exists \( E_{\omega, \delta}^{\infty} \)-submanifold \( N \) in \( M \); \( N \) is modelled on a Hilbert space \( Y \), where \( Y \) is as in 2.7 with \( Diff_{\omega, \delta}^{\infty}(Y) \subset Diff_{\omega, \delta}^{\infty}(X) \) for the corresponding \( \delta' \geq \delta \), where \( M \)
and $N$ are separable. Also let $N$ satisfy conditions in 2.2 and 2.4 such that $M_k \subset N$, $N_k \subset N$, $N_k$ is dense in $L_k$ for each $k \in \mathbb{N}$.

**Corollary 2.9.** Let $M$ be a Banach $E^{\infty}_{\omega,\delta}$-manifold and $N$ be a Hilbert $E^{\infty}_{\omega,\delta'}$-manifold such that they satisfy 2.8. Then $\text{Diff}^{t}_{\delta,\gamma}(N)$ is a dense subgroup of $\text{Diff}^{t}_{\delta,\gamma}(M)$, if $\delta' \geq \delta \geq \gamma' > \gamma + 2$, $t' \geq 1$, $\infty \geq t \geq t' \geq 0$ and $\omega \geq \beta$.

**Proof.** For charts $(V_j, \psi_j)$ of $N$ with $V_j \cap V_i \neq \emptyset$ a mapping $\psi_j \circ \psi_i^{-1}$ is in the class of smoothness $E^{\infty}_{\omega,\delta'}$. In view of Definitions 2.5, 2.8 and Lemma 2.7 $\text{Diff}^{t}_{\delta,\gamma}(N)$ is a dense subgroup of $\text{Diff}^{t}_{\delta,\gamma}(M)$. 

3. Structure of groups of diffeomorphisms

**Theorem 3.1.** Let $G = \text{Diff}^{t}_{\delta,\gamma}(M)$ be defined as in 2.5, 2.8. Then it is a separable topological group. If $\text{At}(M)$ is finite, $G$ is metrizable by a left-invariant metric $d$.

**Proof.** Let at first $\text{At}(M)$ be finite. If $f$ and $g \in G$ then $f \circ g^{-1} \in G$ due to Theorem 2.5 [1] and Ch. 5 in [21] about differentiation and difference quotients of composite functions and inverse functions, since $\phi_i \circ \phi_j^{-1} \in E^{\infty}_{\omega,\delta}$ for each $i$ and $j$. At first we have $d(f, id) > 0$ for $f \neq id$ in $G$, since there are $i$ and $j$ such that $f_{i,j} \neq id_{i,j}$. Then $d(hf, hg) = d(g^{-1}h^{-1}hf, id) = d(g^{-1}f, id) = d(f, g)$, hence $d$ is left-invariant, where $f, g \in G$. Therefore, $d(f^{-1}, id) = d(id, f)$, in view of 2.1 and 2.3(i,iii) we have that $d(id, f) = d(f, id)$, hence $d(f, g) = d(g, f)$.

It remains to verify, that the composition map $(f, g) \rightarrow f \circ g$ from $G \times G \rightarrow G$ and the inversion map $f \rightarrow f^{-1}$ are continuous relative to $d$. Let $W = \{f \in G : d^{t}_{\delta,\gamma}(f, id) < 1/2 \}$ and $f, g \in W$. We have $f_{i,j} \circ g_{j,t} = id_{i,t}$, $(f_{i,j} \circ g_{j,t} - f_{i,t}) + (f_{i,t} - id_{i,t})$ for corresponding domain as an intersection of domains of $f_{i,j} \circ g_{j,t}$ and $f_{i,t}$. Hence, using induction by $p = 1, 2, \ldots, \lfloor t \rfloor + 1$ and the Cauchy inequality we have that there are constants $\infty > C_1 > 0$, $\infty > C_2 > 0$ such that $d(f \circ g, id) \leq C_1 d(f, id) + d(g, id)$ and $d(f^{-1}, id) \leq C_2 d(f, id)$, since $\lim_{n \rightarrow \infty} d^{t}_{n,\delta,\gamma}(f_{i,j}, id_{i,t}) = 0$, $\lfloor t \rfloor + 1$ and $\text{At}(M)$ are finite, $r_{inj} > 0$ and $g$ satisfies 2.4 [8].
Indeed, in normal local coordinates $x$ (omitting indices $(i, j)$ for $f_{i, j}$), $M \ni x = (x^i : j \in \mathbb{N})$, $f = (f^j : C \to \mathbb{R}|j \in \mathbb{N})$, $C$ open in $X$, using the Cauchy inequality we get: $\sum_{i \in \mathbb{N}}((f \circ g)^i - x^i|\gamma|^2)^2 \leq 2(\sum_{i}(\gamma)|\gamma|)^2\times (\sum_{i}|g^j - x^i|\gamma|2)^{1/2} + \sum_{i, j}(\partial_j(f \circ g)^i - \delta_j^i)^2 \leq a + b + 2(\gamma)^2 + 2(\gamma)^2/b^{1/2}$, where $a = \sum_{i, j}(\partial_j(f \circ g)^i - \delta_j^i)^2$, $b = \sum_{i, j}(\partial_j(f \circ g)^i - \delta_j^i)^2$, $\delta_i^j = 1$ for $i = j$ and $\delta_i^j = 0$ for each $i \neq j$, $f \circ g = f \circ g(x)$, $f, g \in G$.

Then we can proceed by induction for finite products of $D_g^i(f \circ g)^i$ and $D_g^i(f \circ g)^i$, because $D_g^i(f \circ g)^i = 0$ for $|\alpha| > 1$. For $f = g^{-1}$ we can express recurrently $(D_g^i(f \circ g)^i)$ by $(D_g^i(f \circ g)^i)$ with $\xi^i \leq \alpha^i$ for each $i$, since $|\alpha| \leq t$. Analogously, for difference quotients, since $(1 + \zeta)b = 1 + \sum_{m=1}^{\infty}(\zeta)^m$ for $0 < b < 1$ and $0 < |\zeta| < 1$, $\zeta \in \mathbb{R}$ and $(1 + \zeta)b = 1 + b\zeta$, and $z : \mathbb{R} \to \mathbb{R}$, $\lim_{\zeta \to 0}(z(\zeta)/\zeta) = 0$ [21]. For countable infinite $At(M)$ for each $f, g \in G$ there are $E(f), E(g)$ and $E(g^{-1}) \in \Sigma$ such that $\supp(f) \subset U^E(f)$, etc., consequently, $\{f(\supp(f)) \cup g^{-1}(\supp(g^{-1})) \subset U^E \text{ for some } F \in \Sigma, \text{ whence } g^{-1} \circ f \in G \text{ and there is } E \in \Sigma \text{ with } \supp(g^{-1} \circ f) \subset U^E\}$. If $(f, g) \in \alpha$ and $(g, f) \in \alpha$ are two nets converging in $G$ to $f$ and $g$ respectively, so for each neighbourhood $W \subset G$ there exist $E \in \Sigma$ and $\beta \in \alpha$ such that $g^{-1} \circ f \in W$ and $\supp(g^{-1} \circ f) \subset U^E$ for each $\gamma \in \beta$, where $\alpha$ is a limit ordinal.

In view of the Stone-Weierstrass Theorem and 2.1(i, ii) in each $E_{\beta, \gamma}(U, V)$ for open $U$ and $V$ in $X$ are dense cylindrical polynomial functions with rational coefficients, consequently, $G$ is separable, since $E_{\beta, \gamma}(U, V)$ is dense in $E_{\beta, \gamma}(U, V)$.

Due to conditions 2.2(i, vi) and 2.5.2 for each open submanifold $V \subset M$ with $V \supset M_k$ and $\epsilon > 0$ every $f \in \text{Diff} f_{\beta}^k(M_k)$ has an extension $\hat{f}$ onto $M$ such that $\hat{f} \in \text{Diff} f_{\beta}^k(M)$ with $\rho_{\beta}^k(\hat{f})(M \setminus M_k) \cap U^E(id) < \epsilon$. \hfill \Box

**Lemma 3.2.** Let $M$ be a manifold defined in 2.2, 2.4 with submanifolds $M_k$ and $N_k$, $k = k(n)$, $n \in \mathbb{N}$. Then there exist connections $\nabla$ induced on $M_k$ by $\nabla$ are the Levi-Civita connections, where $\nabla$ is the Levi-Civita connection on $M$.

**Proof.** For each chart $(U_j, \phi_j)$ we have $\phi_j(U_j) \subset l_2$ and in $l_2$ for each sequence of subspaces $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \cdots \subset l_2$ there are induced
embeddings $\phi_j^{-1}(\mathbb{R}^n) \cap U_j \leftrightarrow \phi_j^{-1}(\mathbb{R}^{n+1}) \cap U_j \leftrightarrow U_j$. The Levi-Civita connection and the corresponding covariant differentiation $\nabla$ for the Hilbertian manifold $M$ induces the Levi-Civita connection $\nabla'$ for each submanifold $M'$ embedded into $M$, if $M'$ is a totally geodesic submanifold. That is, for each $x \in M'$ and $X \in T_x M'$ there exists $\epsilon > 0$ such that a geodesic $\gamma = x_t \subset M$ defined by the initial condition $(x,X)$ lies in $M'$ for each $t$ with $|t| < \epsilon$ (Section 5 in [10], Section VII.8 in [15]). Then using Theorem 5 in Section 4.2 [17] and geodesic completeness of $M$ we can choose such $M' = M_k$ with dimensions $\dim(M_k) = k \in \mathbb{N}$ and $M_k(n) \hookrightarrow M_k(n+1) \hookrightarrow \cdots \hookrightarrow M$ with $\bigcup_k M_k$ dense in $M$. Each manifold $M_k$ was chosen Euclidean at infinity, since $M$ is Hilbertian at infinity. In view of Section VII.3 in [15] and 5.2, 5.4 in [10] $k(n+1) \nabla$ on $M_k(n+1)$ induces $k(n) \nabla$ on $M_k(n)$. The latter coincides with that of induced by $\nabla$ on $M$. Here each $M_k$ is geodesically complete, but normal coordinates are defined in $M_k$ in general locally as in $M$ also, since may be $r_{inj}(x) < \infty$ for $x \in M$, so that $At(M)$ induces $At(M_k)$ for each $k = k(n), n \in \mathbb{N}$.

**Theorem 3.3.** Let $M$ be a manifold fulfilling 2.2, 2.4 and $\text{Diff}_t^{l, \beta, \gamma}(M)$ be as in 2.3 with $t \geq 1$, $\beta \geq 0$, $\gamma \geq 0$. Then

(i) for each $E_{l, \beta, \gamma}(M, TM)$-vector field $V$ its flow $\eta_t$ is a one-parameter subgroup of $\text{Diff}_t^{l, \beta, \gamma}(M)$, the curve $t \rightarrow \eta_t$ is of class $C^1$, the mapping $\text{Exp} : T_x \text{Diff}_t^{l, \beta, \gamma}(M) \rightarrow \text{Diff}_t^{l, \beta, \gamma}(M)$, $V \rightarrow \eta_t$ is continuous and defined on a neighbourhood of the zero section in $T_x \text{Diff}_t^{l, \beta, \gamma}(M)$;

(ii) $T_x \text{Diff}_t^{l, \beta, \gamma}(M) = \{V \in E_t^{l, \beta, \gamma}(M, TM) | \pi \circ V = f\}$;

(iii) $(V, W) = \int_M g_{f(x)}(V_x, W_x)\mu(dx)$ is a weak Riemannian structure on a Banach manifold $\text{Diff}_t^{l, \beta, \gamma}(M)$, where $\mu$ is a measure induced on $M$ by $\phi_j$ and a Gaussian measure with zero mean value on $l_2$ produced by an injective self-adjoint operator $Q : l_2 \rightarrow l_2$ of trace class, $0 < \mu(M) < \infty$;

(iv) the Levi-Civita connection $\nabla$ on $M$ induces the Levi-Civita connection $\nabla$ on $\text{Diff}_t^{l, \beta, \gamma}(M)$. 

Proof. Let at first $At(M)$ be finite. In view of [12] we have that $Tf_j^t(M,N) = [g \in E^t_{\beta,\gamma}(M,TN') : \pi_N' \circ g = f]$, where $N'$ fulfils 2.5, 2.8, $\pi_N'^{-1} : TN' \rightarrow N'$ is the canonical projection. Therefore, $TE^t_{\beta,\gamma}(M,N') = E^t_{\beta,\gamma}(M,TN') = \bigcup_j T_j E^t_{\beta,\gamma}(M,N')$ and the following mapping $w:\exp : T\!\!\!\!E^t_{\beta,\gamma}(M,N') \rightarrow E^t_{\beta,\gamma}(M,N')$, $w(\exp) = \exp \circ g$ gives charts for $E^t_{\beta,\gamma}(M,N')$, since $TN'$ has an atlas of class $E^t_{\mu_\chi}$ with $\nu \geq \beta \geq 0$, $\chi \geq \gamma$. In view of Theorem 5 about differential equations on Banach manifolds in Section 4.2 [17] a vector field $V$ of class $E^t_{\beta,\gamma}$ on $M$ defines a flow $\eta_t$ of class $E^t_{\beta,\gamma}$, that is $d\eta_t/dt = V \circ \eta_t$ and $\eta_0 = e$. Then lightly modifying proofs of Theorem 3.1 and Lemmas 3.2, 3.3 in [7] we get that $\eta_t$ is a one-parameter subgroup of $Diff^t_{\beta,\gamma}(M)$, the curve $t \rightarrow \eta_t$ is of class $C^1$, the map $\Exp : T\!\!\!\!E^t_{\beta,\gamma}(M) \rightarrow Diff^t_{\beta,\gamma}(M)$ defined by $V \rightarrow \eta_t$ is continuous.

The curves of the form $t \rightarrow \tilde{E}(tV)$ are geodesics for $V \in T\!\!\!\!E^t_{\beta,\gamma}(M)$, $d\tilde{E}(tV)/dt$ is the map $m \rightarrow d(exp(tV(m)))/dt = \gamma'_m(t)$, where $\gamma_m(t)$ is the geodesic on $M$, $\gamma_m(0) = \eta(m)$, $\gamma'_m(0) = V(m)$. Indeed, this follows from the existence of solutions of corresponding differential equations in the Banach space $E^t_{\beta,\gamma}(M,TM)$ and then as in the proof of Theorem 9.1 [7].

From the definition of $\mu$ it follows that for each $x \in M$ there exists open neighbourhood $Y \ni x$ such that $\mu(Y) > 0$ [6]. In view of 2.2-4 there is the following inequality $\sup_x g_{f(x)}(V_x, V_x) \leq \infty$ and also for $W$. Consequently, $(V, V) > 0$ for each $V \neq 0$, since $V$ and $W$ are continuous vector fields and for some $x \in M$ and $Y \ni x$ with $\mu(Y) > 0$ we have $V_y = 0_y$ for each $y \in Y$. On the other hand $\sup_{x \in M} |g_{f(x)}(V_x, W_x)| < \infty$, hence $|(V, W)| < \infty$. From $g_{f(x)}(V_x, W_x) = g_{f(x)}(W_x, V_x)$ and bilinearity of $g$ by $(V_x, W_x)$ it follows that $(V, W) = (W, V)$ and $(aV, W) = (V, aW)$ for each $a \in \mathbb{R}$. Since $t \geq 1$, the scalar product (iii) gives a weaker topol-
ogy than the initial $E^t_{\beta,\gamma}$. For two Banach spaces $A$ and $B$ we have the following uniform linear isomorphism $E^t_{\beta,\gamma}(M, A \oplus B) = E^t_{\beta,\gamma}(M, A) \oplus E^t_{\beta,\gamma}(M, B)$, where $\oplus$ denotes the direct sum. Therefore, $E^t_{\beta,\gamma}(M, TM)$ is complemented in $E^t_{\beta,\gamma}(M, T(TM))$, since $TM$ and $T(TM) = TTM$ are the Banach foliated manifolds of class $E^{\infty}_{\beta,\gamma}$ with $\nu \geq \beta$, $\chi \geq \gamma \geq 0$. Then the right multiplication $\alpha_h(f) = f \circ h$, $f \to f \circ h$ is of class $C^\infty$ on $Diff^t_{\beta,\gamma}(M)$ for each $h \in Diff^t_{\beta,\gamma}(M)$.

Moreover, $Diff^t_{\beta,\gamma}(M)$ acts on itself freely from the right, hence we have the following principal vector bundle $\pi : TDiff^t_{\beta,\gamma}(M) \to Diff^t_{\beta,\gamma}(M)$ with the canonical projection $\pi$.

Analogously to [2,7,15] we get the connection $\tilde{\nabla} = \nabla \circ h$ on $Diff^t_{\beta,\gamma}(M)$. Then $(\nabla_X Y, Z) + (Y, \nabla_X Z) = \int_M [\langle \nabla_X Y_e, Z_e \rangle_{h(x)} + \langle Y_e, \nabla_X Z_e \rangle_{h(x)}] \mu(dx) = \int_M [Xg(Y_e, Z_e)]_{h(x)} \mu(dx) = \tilde{X}(Y, Z)$, since $Xg(Y, Z, g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (Satz 3.8 in [10]) and for each right-invariant vector field $V$ on $Diff^t_{\beta,\gamma}(M)$ there exists a vector field $X$ on $M$ with $V_\hbar = X \circ h$ for each $h \in Diff^t_{\beta,\gamma}(M)$, where $\tilde{X} := X \circ h$ (see also [13,19]). If $\nabla$ is torsion-free then $\tilde{\nabla}$ is also torsion-free. From this it follows that the existence of $E$ and $Diff^t_{\beta,\gamma}(M)$ is the Banach manifold of class $E^{\infty}_{\omega,\beta}$, since $exp$ and $M$ are of class $E^{\infty}_{\omega,\beta}$, $\alpha_h(f) = f \circ h$, $f \to f \circ h$ is a $C^\infty$ map with the derivative $\alpha_h : E^t_{\beta,\gamma}(M', TN) \to E^t_{\beta,\gamma}(M, TN)$ whilst $h \in E^t_{\beta,\gamma}(M, M')$, $\tilde{E}_h(V) := exp_{h(x)}(V(h(x)))$, $\tilde{V}_h = V \circ h$, $V \in \mathcal{E}(M)$, $\tilde{V} \in \tilde{\mathcal{E}}(Diff^t_{\beta,\gamma}(M))$.

The case of infinite $At(M)$ may be treated using the strict inductive limit topology. 

\textbf{Note 3.4.} For a manifold $N = \oplus \{M_j : j \in J\}$, $M_j = M$ for each $j$, $J \subset \mathbb{N}$, we have that $Diff^t_{\beta,\gamma}(N)$ is isomorphic to $S \oplus Diff^t_{\beta,\gamma}(M)$, where $S$ is a discrete symmetric group.

Henceforward, we assume that $M$ and $M_k$ are connected for each $k > n$ and some fixed $n \in \mathbb{N}$. For a finite-dimensional manifold $M$ a space $E^t_{\beta,\gamma}(M, \mathbb{R})$ (or $Diff^t_{\beta,\gamma}(M)$) is isomorphic with the usual weighted Hölder space $C^t_{\beta}(M, \mathbb{R})$ (or $Diff^t_{\beta}(M)$ correspondingly).
4. Irreducible unitary representations of a group of diffeomorphisms of a Banach manifold

Theorem 4.1. Let $M$ be a Banach manifold fulfilling 2.5, $G = \text{Diff}_\beta^l(M)$ be a group of diffeomorphisms as in 2.8 with $t \geq 1$, $\beta \geq \omega + \xi$ and $\gamma > 2(1 + \delta + \xi)$, where $\xi > 2$ for a Banach manifold, $\xi = 0$ for a Hilbert manifold. Then (for each $1 \leq l \leq \infty$) there exists a quasi-invariant (and $l$ times differentiable) measure $\nu$ on $M$ relative to $G$.

Proof. The exponential mapping $\exp$ is defined on a neighbourhood of the zero section of the tangent bundle $TM$ and $\exp$ is of class $\mathcal{E}^{\omega,\delta}_{\omega,\beta}$ due to 2.5 (see also [17]). For each $x \in N$ we have $T_xN = l_2$. Suppose $F$ is a nuclear (of trace class) operator on $l_2$ such that $F \in \mathcal{F}$ for each $i$, $\{e_i : i\}$ is the standard base in $l_2$, $1 - \gamma + 2\delta < b \leq c < -1$. Then there exists a $\sigma$-additive Gaussian measure $\lambda$ on $l_2$ with zero mean and a correlation operator equal $F$.

Then a Gaussian measure on $T_xN$ induces a Gaussian measure on $T_xM$ for $x \in N$ [16]. Therefore, $\exp_x$ induces a $\sigma$-additive measure $\nu$ on $W \ni x$, where $W = \exp_x(V)$, $0 \in V$ is open in $T_xM$, $0 < \mu(V) < \infty$, $\nu(C) = \mu(\exp_x^{-1}(C))$ for each $C \in Bf(W)$. The manifold $M$ is paracompact and Lindelöf [9], $GW = M$, hence there is a countable family $\{g_j : j \in \mathbb{N}\} \subset G$, $g_1 = e$, $W_1 = W$ and open $W_j \subset W$ such that $\{g_jW_j : j\}$ is a locally finite covering of $M$ with $W_1 = W$, $g_1 = id$. For $C \in Bf(M)$ let $\nu(C) := \sum_{j \in \mathbb{N}} \nu((g_j^{-1}C) \cap W_j)2^{-j}$ (without multipliers $2^{-j}$ the measure $\nu$ will be $\sigma$-finite, but not necessarily finite).

The following mapping $Y_g := (\exp \circ g \circ \exp_x^{-1})$ on $TM$ for each $g \in G$ satisfies conditions of Theorems 1.2 in Section 26 [23]. Indeed, $(\partial g^i/\partial x^j)_{i,j \in \mathbb{N}}$ in local natural coordinates $(x^i)$ is in the class $\mathcal{E}^{\beta+1,\gamma}_{\gamma,\gamma}$ (see 2.4, 2.8). In view of these theorems and [3,6,11] the measure $\nu$ is quasi-invariant and $l$ times differentiable, since the continuous extension of the operator $(Y_g^t - 1)F^{-1/2}Q$ from $T_xN$ onto $T_xM$ is of trace class on the Banach space $T_xM$ and $dg^t/\partial t = V \circ g^t$ (see the proof of Theorem 3.3 above and [20,22]), where $g^t = e^t$, $Qx = \sum_j x^j f^j e_j$, $x = \sum_j x_j$, $e_j \in l_2$, $x^t \in \mathbb{R}$. \hfill \Box

Definition 4.2.1. Let $M$ satisfy conditions in 2.5. For a given atlas $\mathcal{A}(M)$ we consider its refinement $\mathcal{A}'(M) = \{(U_j', \psi_j) : j \in \mathbb{N}\}$ of
the same class $E_{\omega,\lambda}^\infty$ such that $\{U'_j\}$ is a locally finite covering of $M$, for each $U'_j$ there is $i(j)$ with $U_{i(j)} \supset U'_j$, $exp^{-1}_x$ is injective on $U'_j$ for some $x \in U'_j$, $exp^{-1}_x(U'_j)$ is bounded in $T_x M$. Henceforward, $M$ will be supplied by such $At(M)$ and $Diff_{\beta,\gamma}(M)$ will be given relative to such atlas.

**Definition 4.2.2.** Let $\mu$ be a non-negative measure on $M$ quasi-invariant relative to $G = Diff_{\beta,\gamma}(M)$ (see Theorem 4.1) such that $\mu(M) = \infty$, $\mu$ is $\sigma$-finite and $\mu(U'_j) < \infty$ for each $j$. Then $\mu$ is considered on $Af(M, \mu)$. We consider $X = \prod_{i \in \mathbb{N}} M_i$, where $M_i = M$ for each $i$. Take $E_i \in Af(M_i, \mu)$, put $E = \prod_{i \in \mathbb{N}} E_i$, which is called a unital product subset of $X$ if it satisfies the following conditions:

$$(UPS1) \quad \sum_{i \in \mathbb{N}} |\mu(E_i) - 1| < \infty \text{ and } \mu(E_i) > 0 \text{ for each } i;$$

$$(UPS2) \quad E_i \text{ are mutually disjoint}.$$  

**Note 4.3.** In view of 4.2 the above definitions 4.2.1,2 and Lemmas 1.1, 1.2 [13] are valuable for the case considered here ($G, M, \mu$) for infinite-dimensional $M$. Henceforward, we denote by $G$ the connected component of $id \in Diff_{\beta,\gamma}(M)$ from 4.2.2. Further, the construction of irreducible unitary representations follows schemes of [13] for finite-dimensional $M$ and [18] for non-Archimedean Banach manifolds, so proofs are given briefly with emphasis on features of the case of the real Banach manifold $M$.

4.4. Let $E$ be cofinal with $E' (ERE')$ if and only if

$$(CF) \quad \sum_{i \in \mathbb{N}} \mu(E_i \cap E'_i) < \infty,$$

$E$ be strongly cofinal with $E' (E \setminus E' )$ if and only if

$$(SCF') \quad \text{there is } n \in \mathbb{N} \text{ such that } \mu(E_i \cap E'_i) = 0 \text{ for each } i > n,$$

where $E_i \cap E'_i = (E_i \setminus E'_i) \cup (E'_i \setminus E_i)$, $\Sigma(E) := \{E' : E'RE\}$.

Put $\nu_E(E') = \prod_{i \in \mathbb{N}} \mu(E'_i)$ for each $E' \in \Sigma(E)$. In view of the Kolmogorov’s Theorem [6] $\nu_E$ has the $\sigma$-additive extension onto the minimal $\sigma$-algebra $M(E)$ generated by $\Sigma(E)$. 

IRREDUCIBLE UNITARY REPRESENTATIONS etc.
The symmetric group of \( \mathbb{N} \) is denoted by \( \Sigma_{\infty} \), its subgroup of finite permutations of \( \mathbb{N} \) is denoted by \( \Sigma_{\infty} \). For \( g \in G \) there is \( gx = (gx_i : i \in \mathbb{N}) \), where \( x = (x_i : i \in \mathbb{N}) \in X \), for \( \sigma \in \Sigma_{\infty} \) let \( x\sigma = (x'_i : i \in \mathbb{N}) \), \( x'_i = x_{\sigma(i)} \) for each \( i \). Quite analogously to Lemma 1.3 \([13]\) we have the following Lemma 4.5 due to \( \text{supp}(g) \subset U^{E(g)} \) for some \( E(g) \in \Sigma \) and \( \mu(U^{E(g)}) < \infty \), where \( U^E = \bigcup_{j \in E} U_j \), \((U_j, \psi_j)\) are charts of \( M \).

**Lemma 4.5.** Let \( E \) be a unital product subset of \( X \). Then

(i) \((gE)RE \) for each \( g \in G \),

(ii) \( \Sigma(E) \) is invariant under \( G \) and \( \Sigma_{\infty} \).

4.6. In view of 2.6, 2.8, 4.2.1 and the proof of 4.1 we may choose \( \mu \) such that for each \( g \in G \) there is its neighbourhood \( W_g \) and there are constants \( 0 < C_1 < C_2 < \infty \) such that

\[
(i) \quad C_1 \leq q_{\mu}(f, z) \leq C_2
\]

for each \( x \in m \) and \( f \in W_g \) with \( \text{supp}(f) \subset U^{E(g)} \). Indeed, for each \( U_j \) there exists \( y \in U_j \) such that \( \exp_{y}^{-1} U_j \) is bounded in \( T_{y} M \). Hence for each fixed \( R, \infty > R > 0 \), for operators \( Y_{f} = U \) of non-linear transformations the term

\[
|\det((Y_{f})'(x))|^{-1} \exp\{\sum_{i=1}^{\infty}[2(x - Y_{f}^{-1}(x), e_{i})^2]/F_{1}\}
\]

is bounded (see \( f \) after (i)) for each \( x \in l_{2} \) with \( \|x\| < R \). For \( z \in M \setminus U^{E(g)} \) we have \( q_{\mu}(f, z) = 1 \). Therefore, we suppose further that \( \mu \) satisfies (i).

If \( S \in A f(M, \mu) \) and \( \mu(S) < \infty \) we may consider measures \( \mu_{k} = \mu \) on \( E_{k}', \nu_{k} = \mu_{k} \) on \( E_{k}' \setminus S \) and \( \nu_{k} = 0 \) on \( S \), suppose \( L_{n} = \prod_{i=1}^{n} M_{i} \), \( \mu_{L_{n}} = \bigotimes_{i=1}^{n} \mu_{i}, P_{n} : X \to L_{n} \) are projections, \( \rho_{k}(x) = \nu_{k}(dx)/\mu(dx) \). Then \( \rho_{k}(x) = 0 \) for each \( x \in S \). Using the analog of Lemma 16.1 \([23]\) for our case we obtain the analog of Lemmas 1.4, 1.6, 1.7 and Theorem 1.5 \([13]\), since \( M \) has a countable open base \( \{U_{j} : j \in \mathbb{N}\} \) there is \( E \in \Sigma \) such that \( U \subset U^{E} \).

4.7. The manifold \( M \) is Polish, hence \( M \) is the Radon space \([6]\) and for each unital product subset \( E \) for each \( i \) there is a compact \( E_{i} \subset M \) such that \( \mu(E_{i} \Delta E_{i}) < 2^{-i-1} \) and \( E_{i} \subset U^{h(i)} \) for corresponding \( h(i) \in \Sigma \). Since each open covering of \( E_{i} \) has a finite subcovering we may
choose $E_i^t \in \mathcal{A}(M, \mu)$ with finite number of connected components. As in Section 1.8 [13] we can construct $E^tRE$ such that $E^t_i$ are mutually disjoint.

**Proposition 4.8.** Each unital product subset $E$ is cofinal with $E^0$ satisfying the following conditions:

(UP3) the closure $cl(E_i^t)$ and $cl(\bigcup_{j \neq i} E_j^0)$ are mutually disjoint and $E_i^0$ is open for each $i$ and $\inf \inf_{x \in E_i^t, y \in \bigcup_{j \neq i} E_j^0} d_M(x, y) > 0$, $E_i^0 \subset U^{h(i)}$, $h(i) \in \Sigma$;

(UP4) $E_i^0$ and $E_{i,k}^0$ are connected and simply connected, there is $n \in \mathbb{N}$ such that for each $k > n$ and $i \in \mathbb{N}$ there exists $g \in G$ with $g(E_{i,k}^0) = B_{i,k}$ being an open ball in a coordinate neighbourhood of $M_k$ with $g(M \setminus M_k) = id$ and $\inf \inf_{x \in \partial M_k, y \in E_i^0} d_M(x, y) > 0$, $g(E_i^0) = B_{i,k}$, where $B := cl(B)$, $E_{i,k}^0 := E_i^0 \cap M_k$. For $i \neq j$, $E_i^0$ and $E_j^0$ can be connected by an open path $P_{i,j}$ such that $P_{i,j} \cap cl(\bigcup_{k \neq i,j} E_k^0) \neq \emptyset$.

**Proof.** In view of 3.4, $M$ and $M_k$ are connected for each $k > n$ and some fixed $n \in \mathbb{N}$. Then using 3.1, locally finite coverings of $M$ and $M_k$ [9] and shrinking slightly $E_i^0$ such that $\partial E_i^0$ are of class $E_i^{\infty \mu \delta}$ analogously to steps 1-4 [13] and using properties of $\mu$ we prove this proposition. Indeed, $\mu$ is approximable from beneath by the class of compact subsets [6].

4.9. Henceforth, $\Pi : \Sigma_\infty \to U(V(\Pi))$ denotes a unitary representation on a Hilbert space $V(\Pi)$ over $\mathbb{C}$, $H(\Sigma_\infty)$ denotes a Hilbert space that is the completion of $\bigcup_{E^t \in \Sigma(E)} H_E^{|E^t|}$ with the scalar product

$$<\phi_1, \phi_2> = \sum_{\sigma \in \Sigma_\infty} \int_{E^t_{1} \cap E^t_{2} \sigma} <\phi_1(x), \Pi(\sigma)^{-1}\phi_2(x\sigma^{-1})>_{V(\Pi)} \nu_E(dx),$$

where $H_E^{|E^t|} := L^2(E^t; M(E); \nu_E|E^t; V(\Pi))$ is a Hilbert space of functions on $E^t$ with values in $V(\Pi)$, $\Sigma := (\Pi; \mu, E)$; $E^tRE$, $E$ is a unital product subset of $X$. Then we define a representation

$$(i) \quad T_\Sigma(g) \phi(x) := \rho_E(g^{-1}|x)^{1/2} \phi(g^{-1}x),$$
where $\rho_E(g^{-1}|x) := (\nu_E)_g(dx)/\nu_E(dx)$, $(\nu_E)_g(C) := \nu_E(g^{-1}C)$ and $\rho_E(g|x) = \prod_{i\in\mathbb{N}} \rho_M(g;x_i)$, $\rho_M(g;x_i) := q_i(g^{-1};x_i)$ (see Section 2 [13] and 5.9 [18]).

**Proposition 4.10.** The formula 4.9(i) determines a strongly continuous unitary representation of $G$ (given by 4.2 and 4.3) on the Hilbert space $H(\Sigma)$.

**Proof.** The space $H(\Sigma)$ is isomorphic with the completion $H'(\Sigma)$ of $\bigcup_{E \in \Sigma(H)} H_{E}^{\Pi}$ with the scalar product $< f_1, f_2 >_{H'} := \int_{E} f_1(x), f_2(x) >_{\nu_E(dx)}$, where $f_i \in H_{E}^{\Pi}$, $E^{(i)} \in \Sigma(E)$, $F \in M(E)$, $E'$ for $\sigma \in \Sigma_\infty$ are disjoint and $\text{supp}(f_1(x), f_2(x)) \subset \bigcup_{E \in \Sigma_\infty} F_{E'}$. Here $H_{E'}^{\Pi}$ is a space of functions $f = Q_{\Pi}\phi$, where $\phi \in H_{E'}^{\Pi}$ and

(i) $Q_{\Pi}\phi := \sum_{E \in \Sigma}(R(\sigma)\Pi(\sigma))\phi$, $(Q_{\Pi}(\phi))(x\sigma) = \Pi(\sigma)^{-1}\phi(x)$;

(ii) $R(\sigma)\phi(x) := \phi(x\sigma)$;

(iii) $\Pi(\sigma)\phi(x) := \Pi(\sigma)(\phi(x))$, $\|f\|^2 = \int_{E} \|f(x)\|^2_{\nu_E(dx)} < \infty$,

since $E'\sigma$ for $\sigma \in \Sigma_\infty$ are disjoint for different $\sigma$. Therefore, as in 2.1 [13] we get

$$< T_{\Sigma}(g)f_1, f_2 > = < v_1, v_2 >_{\nu_{(\Pi)}} \times \prod_{i\in\mathbb{N}} \int_{[gE^{(1)}(i) \cap B_{i}^{(1)}]} \rho_M(g^{-1};x_i)^{1/2} \mu(dx_i),$$

for $f_j = Q_{\Pi}\phi_j$, $\phi_j = \chi_{B_{(1)}} \otimes v_j$, where $\chi_C$ is the characteristic function of $C$ (see also 4.6(i)).

Let us fix $J \in \Sigma$ and take $U^J = \bigcup_{j\in J} U_j \subset M$. As in the proof of Theorem 5.6(a) [19] (see 4.6(i)) we can find a neighbourhood $W \ni \text{id}$ in $G$ and $0 < c_1 < c_2 < \infty$ such that $c_1 \leq \rho_M(g^{-1};y) \leq c_2$ for each $y \in U^J$ and $\rho_M(g^{-1};y) = 1$ for each $y \notin U^J$ for each $g \in W$ with $\text{supp}(g) \subset U^J$. Hence for each $\epsilon > 0$ there exists $W \ni \text{id}$ such that $| < T_{\Sigma}(g)f_1, f_2 > - < f_1, f_2 > | < \epsilon$, consequently, due to the Banach-Steinhaus Theorem [36] there exists a neighbourhood $V \ni \text{id}$ such that $\| (T_{\Sigma}(g) - I)f_1 \| < \epsilon$ and $T_{\Sigma}$ is strongly continuous.
It is interesting to note that 4.10 may be proved from the inequality:

$$
\|T_\Sigma(f_1 - f_1)\|_{H'(\Sigma)}
\leq |v|^2 \int_F |f_1(x) - f_1(g^{-1}x)|^2 \rho_E(g^{-1}|x|^{1/2})^2 \nu_E(dx).
$$

Then we consider restrictions $g|_{M_k}$ and properties of $(Y_q)'$ (or $g$ on $M \setminus M_k$) such that $\text{card}\{i : \text{supp}(g) \cap F_{i,k}\} < \aleph_0$ for each $k \in \mathbb{N}$. In view of Theorems 26.12 [23] for each sequence $g_n$ with $\lim_n g_n = \epsilon$ and for each $\epsilon > 0$ there is $m$ such that

$$
\int_F |f_1(x) - f_1(g_n^{-1}x)|^{1/2} \rho_{E}(g_n^{-1}|x|^{1/2})^2 \nu_E(dx) < \epsilon,
$$

for all $n > m$, since there is $E \in \Sigma$ with $\text{supp}(g_n) \subset U^E$ for every $n > m$. \hfill \Box

4.11. Let $E_1, \ldots, E_r$ be mutually disjoint open subsets of $M$, $H_1 := \bigotimes_{i=1}^r L^2(E_i)$, $L^2(E_i) := L^2(E_i; \mu|E_i)$, $G_1 := \prod_{i=1}^r G_{|E_i}$, $G_{|E_i} := \{g \in G : \text{supp}(g) \subset E_i\}$, denote by $G(E_i)$ the connected component of $\text{id} \in Diff\beta\gamma(E_i)$, also let $\{E_{i,j} : j \in J_i\}$ be the connected components of $E_i$. Then $G_{|E_{i,j}} = G(E_{i,j})$, since for each continuous mapping $F : [0,1] \to G$ we have by continuity that

1. $F(\epsilon)(E_{i,j}) \subset E_{i,j}$ for each $\epsilon \in [0,1] \subset \mathbb{R}$ and each $j \in J_i$.

Indeed, suppose $J$ is the connected subset of $[0,1]$ such that $0 \in J$ and for each $\epsilon \in J$ is satisfied (i). If $v = \sup(J) < 1$ then by continuity there is $w > v$ for which $[0,w]$ have the same properties as $J$. Hence the maximal such $J$ coincides with $[0,1]$.

We define and consider $G(E_i) := \prod_{i \in \mathbb{N}} G(E_i^i) := \{g = (g_i : i) : \text{supp}(g_i) \subset U^{E_i(g_i)} \text{ for each } i\}$. Therefore, $\prod_{i \in \mathbb{N}} G(E_{i,j}) = G_{|E_i}$. Then quite analogously to Lemma 3 [13] and Lemma 5.12 II [18] we get that the following representation $L_1$ of $G_1$ is irreducible: $L_1(g, f)(y) = \prod_{i=1}^r \rho_M(g_i^{-1}y)^{1/2} f(q^{-1}y)$ for $f \in H_1$, $g = (g_i : i) \in G_1$ and $y = (y_i : i) \in \prod_{i=1}^r E_i$, since $G_{|E_i}$ is dense in $G_i := G \cap \prod_{i \in J_i} G(E_{i,j})$ and $L_1$ is strongly continuous, $G_{|E_i} \subset \prod_{i \in J_i} G(E_{i,j})$. Indeed, in view of Proposition 4.8 $G_{|E_i}$ is connected, since $G$ is connected.
Then $L_1$ on $G_i$ is decomposable into irreducible components, since $L_1$ of $G(E_{i,j})$ on $L^2(E_{i,j})$ is irreducible. In view of strong continuity of $L_1$ on the dense subgroup $G_{E_i}$ it follows that its strongly continuous extension on $G_i$ is also unitary. Then the rest of Section 3.1 [13] may be transferred onto the case considered here.

Let $L_{E'}(g)f(x) = \rho_E(g^{-1}|x|^{1/2}f(g^{-1}x)$ for $g \in G(E'), f \in H_{E'} := L^2(E', \mu(E)|E', \nu_E|E'), x \in E'$. Then we get the following.

**Lemma 4.12.** Let $E' \in \Sigma(E)$ and $E'_i$ be open and connected. Then the unitary representation $L_{E'}$ of $G(E')$ on $H_{E'}$ is irreducible.

4.13. Let us consider

(i) $G((E')) := \{g \in G|\text{there is } k = k(n), n \in \mathbb{N} \text{ and } \sigma \in \Sigma_\infty, \text{such that } g(E'_{i,k}) = E'_{\sigma(i),k} \text{ for each } i \in \mathbb{N} \text{ and } g|M \setminus M_k = \text{id}\}$, where $E' = \prod_{i \in \mathbb{N}} E'_i \subset M$ satisfies $(UP3 - 4)$ and $E' \in \Sigma(E)$, $E_{i,k} = E_i \cap M_k$. In view of the foliated structure in $M$ this group is dense in

(ii) $\{g \in G: \text{supp}(g) \subset \bigcup_{i \in \mathbb{N}} E'_i\}$.

**Lemma 4.14.** Let $E' \in \Sigma(E)$ satisfy $(UP3 - 4)$. Then for any $\sigma \in \Sigma_\infty$ there is $n$ such that for each $k > n$ there exists $g \in G((E'))$ with $g(E'_{i,k}) = E'_{\sigma(i),k}$ for each $i$, moreover, $g|E'_i = \text{id}|E'_i$ if $\sigma(i) = i$.

**Proof.** It is quite analogous to that of Lemma 3.4 [13], since each $M_k$ is locally compact and connected, also due to properties of $\mu$ induced as the image of the Gaussian $\sigma$-additive measure. On the other hand, the latter is fully characterised by its weak distribution and is with the Radon property (see Lemma 2 and Theorem 1 in Section 2 [23]).

4.15. Let $E'$ be as in 4.12, $H^{II}_{E'} = L^2(E', \mu(E)|E', \nu_E|E'; \nu(V))$, $H^{II}_{E'} = Q_{II}H^{II}_{E'}$ (see the proof of 4.10). For each $g \in G((E'))$ there are $\sigma \in \Sigma_\infty$ and $k = k(n), n \in \mathbb{N} \text{ such that } g(E'_{i,k}) = E'_{\sigma(i),k}$ for each $i \in \mathbb{N}$ and $g|M \setminus M_k = \text{id}$. Suppose $f = Q_{II} \phi, \phi \in H^{II}_{E'}$ If $(\alpha) \phi$ depends only on $\{x = (x_i : i)|x_i \in E'_{i,k}\}$ then $(T_{\Sigma}(g)f)(x) = \rho_E(g^{-1}|x|^{1/2}\Pi(\sigma)\phi(g^{-1}x\sigma)$. If $(\beta) \phi$ depends only on $\{x = (x_i : i)|x_i \in E'_i \setminus M_k\}$ then $(T_{\Sigma}(g)f)(x) = f(x)$. Then if $\phi(x) = \phi_1(x) \times \phi_2(x)$, where $\phi_2(x) = \text{of type } (\alpha) \text{ or } (\beta)$ and $\phi_1$:
Proof. Consider \( F_{i,k} = F_i \cap M_k \) and measures \( \mu_k \) on \( M_k \) induced by \( \mu \) on \( M \) and the projection \( P_k : l_2 \to \mathbb{R}^k \) and choose \( F' \) such that

\[
|\mu_k(n+1)(F'_{i,k}(n+1) \Delta F_{i,k}(n+1) - \mu_k(F'_{i,k}(n) \Delta F_{i,k}(n))| < 3^{-i-2(k(n)+1)} \mu(F_i),
\]

for each \( k = k(n) \) and \( i, n \in \mathbb{N} \). Then use Theorem 3.1 [13]. \( \square \)

**Theorem 4.17.** The unitary representation \( T_{\Sigma} \) of \( G \) (defined in 4.2) on \( H(\Sigma) \) is irreducible.

**Proof.** Considering the sequences \( \{M_k : k\}, \{G_k((E')) : k\} \) and \( \{H_k : k\} \), using 4.2-4.16 and strong continuity of \( T_{\Sigma} \), we get from the proof of Theorem 4.1 [13] that \( T_{\Sigma} \) is irreducible. Indeed, we may consider \( \Delta = \{E' : E' = E^0, E' \text{ satisfies (UP3-4)}\} \) instead of \( \Delta \) in Section 4.3 [13]. \( \square \)

**Theorem 4.18.** Suppose \( T_{\Sigma_i} \) are unitary representations of \( G \) with parameters \( \Sigma_i = (\Pi_i; E^0) \). Then, \( (T_{\Sigma_i}, H(\Sigma_i)), i = 1, 2 \) are mutually equivalent if and only if there exists \( a \in \Sigma_\infty \) such that \( \Pi_1 = a\Pi_2 \) and \( E_1 \in \Sigma(E_2a^{-1}) \), where \( (a\Pi)(\sigma) := \Pi(a^{-1}\sigma a) \).

**Proof.** In view of 4.8 and 4.9 we may assume without loss of generality that \( E^i \) satisfies (UP3-4, UPS5) for \( i = 1 \) and 2. Then we consider \( G^{[1]} = G((E^{(1)})) \cap G((E^{(2)})) \subset G \) and \( G^{[2]} := \prod_{k \in \mathbb{N}} G(C_k) \),
where $C_k$ are all connected components of $E_{i,j}^{(1)} = E_{j,i}^{(2)}$ (with $E^{(2)}$ here instead of $F^{(2)}$ in [13]). Instead of equations (5.7) [13] we have corresponding expressions as intersections with $M_k$ in both sides for some $k = k(n), n \in \mathbb{N}$. Using the sequences $\{M_k\}, \{G_k((E'))\}$ and strong continuity of $T_\Sigma$ we get the statement of Theorem 4.18 analogously to Section 5 [13].

\textbf{Note 4.19.} The construction presented above of irreducible unitary representations is valid as well for each dense subgroup $G'$ of $\text{Diff}_0^\infty(M)$ such that the corresponding non-negative measure $\lambda$ on $M$ is left-quasi-invariant relative to $G'$ and satisfies 4.2 and 4.6.

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