Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces

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Summary. - We study integral vectorial functionals

\[ F(u, \Omega) = \int_{\Omega} f(x, u(x), Du(x)) \, dx \]

where \( f \) satisfies quasi-convexity assumption and its growth is controlled in terms of \( N \)-functions. We obtain semicontinuity results in the weak * topology of Orlicz-Sobolev spaces.

1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) with \( \partial \Omega \) lipschitzian, consider a function \( f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R} \) and the variational integral

\[ F(u, \Omega) = \int_{\Omega} f(x, u(x), Du(x)) \, dx \tag{1} \]

where \( u : \Omega \to \mathbb{R}^N \).

Assume that \( f = f(x, s, z) \) is a Carathéodory function, i.e. \( f \) is measurable in \( x \) for every \( (s, z) \in \mathbb{R}^N \times \mathbb{R}^{Nn} \) and continuous in \( (s, z) \) for almost every \( x \in \Omega \), and that it is also quasi-convex in \( z \), i.e. for every \( (x_0, s_0, z_0) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \) and \( \varphi \in C^1_{0}(\Omega, \mathbb{R}^N) \)

\[ f(x_0, s_0, z_0) |\Omega| \leq \int_{\Omega} f(x_0, s_0, z_0 + D\varphi(x)) \, dx. \]

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Moreover, suppose that $f$ satisfies the growth condition

$$-c_1 \{1 + \Phi_1(|s|) + \Phi_1(|z|)\} \leq f(x, s, z) \leq c_2 \{1 + \Phi_2(|s|) + \Phi_2(|z|)\} \quad (2)$$

for every $(x, s, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$, where $c_1, c_2$ are non-negative constants and $\Phi_i$, $i = 1, 2$, is a $N$-function, i.e. a positive continuous convex function such that $\Phi_i(0) = 0$, $\lim_{t \to 0} \Phi_i(t)/t = 0$ and

$$\lim_{t \to +\infty} \Phi_i(t)/t = +\infty$$

(see Section 2 for definitions).

If in (2) $\Phi_i(t) = t^p$, $i = 1, 2$, $p > 1$, Fusco [15] proved the weak semicontinuity of (1) in the ordinary Sobolev space $W^{1,p+\varepsilon}(\Omega, \mathbb{R}^N)$, $\varepsilon > 0$. This result was improved by Acerbi and Fusco in [2] showing weak semicontinuity of (1) in $W^{1,p}(\Omega, \mathbb{R}^N)$ if $f$ is non-negative and $\Phi_2(t) = t^p$, and by Marcellini in [20] under less restrictive growth conditions. If $\Phi_2(t) = t^p$ and if $f$ satisfies some additional structure conditions, the weak semicontinuity of (1) was proved by Marcellini [21] in $W^{1,q}(\Omega, \mathbb{R}^N)$ with $q > \frac{n+1}{n+1}p$, by Fonseca and Marcellini [14] for $q > p - 1$ and by Malý [19] for $q \geq p - 1$. Recently, Fonseca and Malý [13] and Malý [18] proved the lower semicontinuity of (1) for $q > \frac{n+1}{n+1}p$. Finally, if (1) is poli-convex and $n = N$, Dacorogna and Marcellini [6] proved a semicontinuity result for $q > n - 1$, while the borderline case $q = n - 1$ was established by Acerbi and Dal Maso [1] and by Dal Maso and Sbordone [8]. An elementary approach was found by Fusco and Hutchinson [16].

In this paper, we obtain, for quasi-convex integrals satisfying the non-standard growth condition (2), some semicontinuity results in the weak* topology of Orlicz-Sobolev space $W^{1,\Phi_2}(\Omega, \mathbb{R}^N)$ (see Section 2 for definitions).

In Section 2 we introduce the definitions and some properties of $N$-functions, Orlicz and Orlicz-Sobolev spaces.

In Section 3, Theorem 3.1, we show that if $f = f(z)$, $\Phi_2$ belongs to class $\Delta_2$ (see Section 2 for definitions) and $\Phi_1$ is suitably related to it, then (1) is sequentially lower semicontinuous in the weak* topology of the Orlicz-Sobolev space $W^{1,\Phi_2}(\Omega, \mathbb{R}^N)$. The proof generalizes the technique developed by Marcellini in [20]. Moreover, in this case, we prove an existence theorem.

In Section 4, Theorem 4.1, we consider functionals depending on $f = f(x, s, z)$ and satisfying (2) with $\Phi_1 = \Phi_2 = \Phi$. We succeed
in proving a semicontinuity result in $W^{1,\Gamma}(\Omega, \mathbb{R}^N)$ with $\Gamma$ a suitable $N$-function related to $\Phi$ following Marcellini and Sbordone [22].

Finally in Section 5, we exhibit some examples of non trivial applications of the semicontinuity Theorems 3.1, 4.1 and of the existence Theorem 3.3.

We observe that Ball in [5] considered some variational problems in the framework of Orlicz-Sobolev spaces obtaining some semicontinuity and existence results for povi-convex integrals.

2. N-Functions and Orlicz Spaces

In this section we recall some definitions and well known properties on $N$-functions and Orlicz spaces (see for references [3], [17], [25]). A continuous and convex function $\Phi : [0, +\infty) \to [0, +\infty)$ is called $N$-function if it satisfies

$$\Phi(0) = 0, \quad \Phi(t) > 0, \quad t > 0,$$

$$\lim_{t \to 0} \Phi(t)/t = 0, \quad \lim_{t \to +\infty} \Phi(t)/t = +\infty.$$ 

A $N$-function $\Phi$ has an integral representation

$$\Phi(t) = \int_0^t p(s) \, ds \quad t \in [0, +\infty),$$

where $p : [0, +\infty) \to [0, +\infty)$ is nondecreasing, right continuous and it satisfies

$$p(0) = 0, \quad p(s) > 0 \quad s > 0, \quad \lim_{s \to +\infty} p(s) = +\infty.$$ 

The function $p$ is the right derivative of $\Phi$.

What is important in the definition of a $N$-function is the behaviour at infinity, in fact, a continuous convex function $Q : [0, +\infty] \to [0, +\infty]$ satisfying

$$Q(t)/t \to +\infty \quad t \to +\infty$$

is such that there exist a $N$-function $\Phi$ and $t_0 > 0$ such that for every $t \geq t_0$ we get

$$Q(t) = \Phi(t).$$
Such a function $Q$ is called principal part of the $N$-function $\Phi$.

Let $\Phi$ be a $N$-function, for $t \geq 0$ consider the function

$$
\Psi(t) = \max_{s>0}\{st - \Phi(s)\},
$$

it is easy to show that $\Psi$ is a $N$-function, $\Psi$ is called the complementary $N$-function of $\Phi$. By the very definition of $\Psi$ it is obvious that the pair $\Phi, \Psi$ satisfies Young’s inequality:

$$
st \leq \Phi(s) + \Psi(t) \quad s, t \in \mathbb{R},
$$

with equality holding if $s = p(t)$ or $t = q(s)$, where $q$ is the right derivative of $\Psi$.

In the sequel we will deal with a particular class of $N$-functions. We say that a $N$-function $\Phi$ belongs to the class $\Delta_2$, denoted by $\Phi \in \Delta_2$, if there exist $k > 1$ and $t_0 \geq 0$ such that

$$
tp(t) \leq k \Phi(t) \quad t \geq t_0.
$$

(3)

It is not difficult to check that definition (3) is equivalent to the classical one, i.e. $\Phi \in \Delta_2$ if and only if there exist $k > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0$

$$
\Phi(2t) \leq 2^k \Phi(t).
$$

For related properties of $N$-functions of class $\Delta_2$ see [7].

Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$, the Orlicz class $K^\Phi(\Omega, \mathbb{R}^N)$ is the set of all (equivalence classes modulo equality a.e. in $\Omega$ of) measurable functions $u : \Omega \to \mathbb{R}^N$ satisfying

$$
\int_\Omega \Phi(|u(x)|) \, dx < +\infty.
$$

The Orlicz space $L^\Phi(\Omega, \mathbb{R}^N)$ associated with the $N$-function $\Phi$ and the open set $\Omega$, is the linear hull of $K^\Phi(\Omega, \mathbb{R}^N)$. The equality $K^\Phi(\Omega, \mathbb{R}^N) = L^\Phi(\Omega, \mathbb{R}^N)$ holds if and only if $\Phi \in \Delta_2$.

The functional $\| u \|_{\Phi, \Omega} : L^\Phi(\Omega, \mathbb{R}^N) \to \mathbb{R}$, simply denoted by $\| u \|_\Phi$, defined by

$$
\| u \|_{\Phi, \Omega} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\},
$$

(4)
is a norm and \( L^\Phi(\Omega, \mathbb{R}^N) \) is a Banach space with respect to it.

In the sequel we will denote with \( s - L^\Phi(\Omega, \mathbb{R}^N) \) the norm convergence in \( L^\Phi(\Omega, \mathbb{R}^N) \).

Many relevant properties of the Orlicz spaces are related to class \( \Delta_2 \), for instance:

**Proposition 2.1.** If \( \Phi \in \Delta_2 \) and \( \{u_a\}_{a \in I} \subset L^\Phi(\Omega, \mathbb{R}^N) \) then

\[
\sup_I \| u_a \|_{\Phi, \Omega} < +\infty \text{ if and only if } \sup_I \int_\Omega \Phi(|u_a(x)|) \, dx < +\infty.
\]

The closure in the norm topology of \( C_0^\infty(\Omega, \mathbb{R}^N) \) in \( L^\Phi(\Omega, \mathbb{R}^N) \) is denoted by \( E^\Phi(\Omega, \mathbb{R}^N) \). We have \( E^\Phi(\Omega, \mathbb{R}^N) \subseteq K^\Phi(\Omega, \mathbb{R}^N) \subseteq L^\Phi(\Omega, \mathbb{R}^N) \), with equality holding if and only if \( \Phi \in \Delta_2 \).

Moreover the Orlicz space associated with a \( N \)-function \( \Phi \) is separable if and only if the generating \( N \)-function belongs to class \( \Delta_2 \). The separability result is a consequence of the following approximation theorem which generalizes an analogous property of \( L^p \) spaces (see [12]).

**Theorem 2.2.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), and let \( \Phi \in \Delta_2 \). For a natural number \( r \), let \( \{Q_{i,r}\} \) be a family of open cubes satisfying

\[
\text{diam } Q_{i,r} \leq \frac{1}{r}; \quad Q_{i,r} \cap Q_{j,r} = \emptyset \quad i \neq j; \quad \bigcup_i Q_{i,r} = \Omega.
\]

For \( u \in L^\Phi(\Omega, \mathbb{R}^N) \) define the functions

\[
u_r(x) = \sum_i \left\{ \frac{1}{|Q_{i,r}|} \int_{Q_{i,r}} u(y) \, dy \right\} \chi_{Q_{i,r}}(x).
\]

Then \( \{u_r\} \subset E^\Phi(\Omega, \mathbb{R}^N) \) and moreover \( \{u_r\} \to u \) \( s-L^\Phi(\Omega, \mathbb{R}^N) \).

In the sequel we will use the following result (see [17]).

**Proposition 2.3.** Let \( \Phi, \Gamma \) be \( N \)-functions such that

\[
\lim_{t \to +\infty} \frac{\Phi(t)}{\Gamma(t)} = +\infty,
\]
and let $H$ be a mean bounded family of functions in $L^\Phi(\Omega, \mathbb{R}^N)$, i.e.
$$\sup_H \int_\Omega \Phi(|u(x)|) \, dx < +\infty,$$
then the set of functions $G = \left\{ \Gamma(|u|) : u \in H \right\}$ has equi-absolutely continuous integrals on $\Omega$.

The *Orlicz-Sobolev space* $W^{1,\Phi}(\Omega, \mathbb{R}^N)$ consists of all functions $u$ in $L^\Phi(\Omega, \mathbb{R}^N)$ whose distributional derivatives belong to $L^\Phi(\Omega, \mathbb{R}^N)$. As in the case of ordinary Sobolev spaces, $W_0^{1,\Phi}(\Omega, \mathbb{R}^N)$ is taken to be the closure in the norm topology of $C_0^\infty(\Omega, \mathbb{R}^N)$ in $W^{1,\Phi}(\Omega, \mathbb{R}^N)$.

The following embedding theorem holds (see [3], [4], [10]).

**Theorem 2.4.** Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $\partial \Omega$ Lipschitzian, let $\Phi \in \Delta_2$, then the embedding
$$W^{1,\Phi}(\Omega, \mathbb{R}^N) \rightarrow L^\Phi(\Omega, \mathbb{R}^N)$$
is compact.

We now introduce the weak * convergence in $L^\Phi(\Omega, \mathbb{R}^N)$. The space $L^\Phi(\Omega, \mathbb{R}^N)$ can be regarded as the dual space of $E^\Psi(\Omega, \mathbb{R}^N)$ (see [3], [17], [25]), so it is possible to characterize the convergence of sequences in the weak * topology of $L^\Phi(\Omega, \mathbb{R}^N)$ in the following way: $\{u_r\} \rightarrow u$ *w*-t$\Phi(\Omega, \mathbb{R}^N)$ if and only if for every $\nu \in E^\Psi(\Omega, \mathbb{R}^N)$
$$\lim_{r} \int_\Omega u_r(x)\nu(x) \, dx = \int_\Omega u(x)\nu(x) \, dx.$$ Since this, weak * convergence is often called *$E^\Psi$-convergence*.

By means of the Hahn-Banach theorem we characterize the weak * convergence in the space $W^{1,\Phi}(\Omega, \mathbb{R}^N)$: $\{u_r\} \rightarrow u$ *w*-t$W^{1,\Phi}(\Omega, \mathbb{R}^N)$ if and only if $\{u_r\}$ and $\{D_i u_r\}$, $1 \leq i \leq n$, converge to $u$, $D_i u$ *w*-t$L^\Phi(\Omega, \mathbb{R}^N)$, respectively. Finally if $\Phi \in \Delta_2$ we get $[L^\Phi(\Omega, \mathbb{R}^N)]^\prime \sim L^\Psi(\Omega, \mathbb{R}^N)$.

3. **Semicontinuity theorem: the case** $f = f(z)$

**Theorem 3.1.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ with $\partial \Omega$ Lipschitzian, let $u : \Omega \rightarrow \mathbb{R}^N$, consider the functional
$$F(u, \Omega) = \int_\Omega f(Du(x)) \, dx,$$
where $f : \mathbb{R}^{\mathbb{N}_n} \rightarrow \mathbb{R}$ is a quasi-convex function such that for every $z \in \mathbb{R}^{\mathbb{N}_n}$ we have

$$-c\{1 + \Phi_1(||z||)\} \leq f(z) \leq c\{1 + \Phi(||z||)\}$$

(5)

with $c$ positive constant, $\Phi \in \Delta_2$ and $\Phi_1$ $N$-function such that

$$\lim_{t \to +\infty} \frac{\Phi(t)}{\Phi_1(t)} = +\infty.$$ 

(6)

Then $F$ is sequentially lower semicontinuous in $\ast w$-$W^{1,\Phi}(\Omega, \mathbb{R}^N)$, i.e. for every sequence $\{u_r\} \rightarrow u$ $\ast w$-$W^{1,\Phi}(\Omega, \mathbb{R}^N)$ we have

$$\liminf_r \int_{\Omega} f(Du_r) \, dx \geq \int_{\Omega} f(Du) \, dx.$$

In the sequel we will use the following result which generalizes a proposition given by Marcellini [20].

**Proposition 3.2.** Let $g : \mathbb{R}^{\mathbb{N}_n} \rightarrow \mathbb{R}$ be a function separately convex in each variable, such that there exist a $N$-function $\Gamma \in \Delta_2$ and a positive constant $c$ such that for every $z \in \mathbb{R}^{\mathbb{N}_n}$

$$|g(z)| \leq c\{1 + \Gamma(||z||)\}.$$ 

(7)

Then $g$ is continuous, besides, denoted by $h$ the right derivative of $\Gamma$, we have

$$|g(z) - g(w)| \leq c_1\{1 + h(1 + |z| + |w|)|z - w|$$

(8)

for every $z, w \in \mathbb{R}^{\mathbb{N}_n}$ with $c_1$ positive constant.

**Proof.** For $z, w \in \mathbb{R}^{\mathbb{N}_n}$ consider the vectors

$$a^k = (w_1, \ldots, w_k, z_{k+1}, \ldots, z_{N_n}) \quad 0 \leq k \leq Nn,$$

then using the convexity of $g$ in each variable, we have for $t \geq 1$

$$g(a^{k+1}) - g(a^k) \leq \frac{g(a^k + t(a^{k+1} - a^k)) - g(a^k)}{t}.$$
By the very definition of $a^k$ it follows that, for every $k$ and $t$, we have
\[ |a^k + t(a^{k+1} - a^k)| \leq 1 + |z| + |w| + t|z - w|, \]
so if we choose \( \tilde{t} = \frac{1 + |z| + |w|}{|z - w|} > 1 \) we get
\[ |a^k + \tilde{t}(a^{k+1} - a^k)| \leq 2(1 + |z| + |w|). \]
By (7) we have
\[ |g(a^k)| \leq c(1 + \Gamma(1 + |z| + |w|)). \]
and also, using assumption $\Gamma \in \Delta_2$ we get
\[ |g(a^k + \tilde{t}(a^{k+1} - a^k))| \leq c_1(1 + \Gamma(1 + |z| + |w|)). \]
Thus we have
\[ g(a^{k+1}) - g(a^k) \leq c_1 \frac{1 + \Gamma(1 + |z| + |w|)}{1 + |z| + |w|}|z - w| \]
\[ \leq c_1(1 + h(1 + |z| + |w|))|z - w|, \]
adding up on $k$ we get the inequality
\[ g(w) - g(z) \leq c_2(1 + h(1 + |z| + |w|))|z - w|, \]
reversing the role of $z$ and $w$ we get (8).

Proof of Theorem 3.1. We assume first that $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ is an affine function, i.e. there exists $z_0 \in \mathbb{R}^N$ such that for every $x \in \mathbb{R}^n$ it holds $Du(x) \equiv z_0$.

Denote with \{\{u_r\}\} a sequence such that \{\{u_r\}\} $\to u * w - W^{1,\Phi}(\Omega, \mathbb{R}^N)$. If $u, u_r$ have the same boundary values, i.e. $(u_r - u) \in W^{1,\Phi}_0(\Omega, \mathbb{R}^N)$, for every $r$, the result follows easily by quasi-convexity. In fact, by (5), the functional $F$ is continuous in $s-W^{1,\Phi}(\Omega, \mathbb{R}^N)$, then the quasi-convexity inequality holds for test functions in $W^{1,\Phi}_0(\Omega, \mathbb{R}^N)$ and so we get semicontinuity inequality.

In the general case we change the boundary data of $u_r$ using a method developed by De Giorgi [9]. Let $\Omega_0$ be an open set compactly
contained in \( \Omega \) and fix \( k = \frac{1}{2} \operatorname{dist}(\overline{\Omega}_0, \partial \Omega) \), for \( h \in \mathbb{N} \) define the open sets
\[
\Omega_i = \left\{ x \in \Omega : \operatorname{dist}(x, \Omega_0) < \frac{i}{k} \right\} \quad 1 \leq i \leq h
\]
and consider a family of functions \( \phi_i \in C_0^\infty(\Omega_i) \) such that
\[
0 \leq \phi_i \leq 1; \quad \phi_i \equiv 1 \quad \Omega_{i-1}; \quad \phi_i \equiv 0 \quad \Omega \setminus \Omega_i; \quad |D\phi_i| \leq \frac{h + 1}{k}.
\]
For every \( r \), let \( \nu_r = u_r - u \), then \( \{\nu_r\} \to 0 \ast w^{-1, \Phi}(\Omega, \mathbb{R}^N) \), now define the functions
\[
\nu_{i,r}(x) = \phi_i(x)\nu_r(x),
\]
since \( \nu_{i,r} \in W_0^{1, \Phi}(\Omega, \mathbb{R}^N) \) for every \( i \) and \( r \) we have
\[
F(u, \Omega) \leq F(u + \nu_{i,r}, \Omega) = \int f(z_0 + D\nu_{i,r}) \, dx
= \int_{\Omega_{i-1}} f(Du_r) \, dx + \int_{\Omega_i \setminus \Omega_{i-1}} f(z_0 + D\nu_{i,r}) \, dx + \int_{\Omega_i} f(z_0) \, dx
= \int_{\Omega} f(Du_r) \, dx - \int_{\Omega \setminus \Omega_{i-1}} f(Du_r) \, dx + \int_{\Omega_i \setminus \Omega_{i-1}} f(z_0 + D\nu_{i,r}) \, dx
+ |\Omega \setminus \Omega_0| f(z_0) - \int_{\Omega_i \setminus \Omega_0} f(Du) \, dx.
\]
(9)
Since \( \{\nu_r\} \) is weakly * convergent, then \( \{D\nu_r\} \) is bounded in norm \( L^\Phi(\Omega, \mathbb{R}^N) \), and then, by Proposition 2.1, there exists a positive constant \( c_1 \) such that
\[
\sup_{r} \int_{\Omega} \Phi(|D\nu_r|) \, dx \leq c_1.
\]
Therefore there is \( 0 \leq j \leq h \) such that
\[
\lim_{r} \sup_{\Omega_j \setminus \Omega_{j-1}} \int \Phi(|D\nu_r|) \, dx \leq \frac{c_1}{h}.
\]
Since the imbedding $W^{1,\Phi} \to L^\Phi$ is compact we obtain that $\{\nu_r\} \to 0$
$s$-$L^\Phi(\Omega, \mathbb{R}^N)$ and then

$$\lim_{r} \int_\Omega \Phi(|\nu_r|) \, dx = 0.$$  

Now we estimate the integrals in (9), we have

$$\int_{\Omega \setminus \Omega_{j-1}} f(z_0 + Du_{j,r}) \, dx \leq c \int_{\Omega \setminus \Omega_{j-1}} \{1 + \Phi(|z_0| + |\phi_j| |Du_r| + |D\phi_j| |\nu_r|)\} \, dx$$

(10)

$$\leq c_2 |\Omega \setminus \Omega_0| + c_4 \left( \frac{h+1}{k} \right) \int_\Omega \Phi(|\nu_r|) \, dx,$$

besides by (5) we get

$$- \int_{\Omega \setminus \Omega_{j-1}} f(Du_r) \, dx - \int_{\Omega \setminus \Omega_0} f(Du) \, dx \leq c_5 \left\{ \Phi_1(|Du_r|) + \Phi_1(|Du|) \right\} \, dx.$$  

(11)

Using Proposition 2.3 we obtain that the functions $\Phi_1(|Du_r|)$ have
equi-absolutely continuous integrals, so that the right term of (11) goes to zero if the measure of $\Omega \setminus \Omega_0$ does.

So, by (10) and (11), (9) becomes

$$F(u, \Omega) \leq F(u_r, \Omega) + c_3 \left( \frac{h+1}{k} \right) \int_\Omega \Phi(|\nu_r|) \, dx + \frac{c_4}{h} +$$

$$+ c_5 \int_{\Omega \setminus \Omega_0} \left\{ \Phi_1(|Du_r|) + \Phi_1(|Du|) \right\} \, dx + c_6 |\Omega \setminus \Omega_0|,$$

the assertion follows passing to the limit as $|\Omega \setminus \Omega_0| \to 0$, $r \to +\infty$
and $h \to +\infty$.

Passing to the general case let $u \in W^{1,\Phi}(\Omega, \mathbb{R}^N)$ and $\{u_r\}$ be a
sequence such that $\{u_r\} \to u \ast_w W^{1,\Phi}(\Omega, \mathbb{R}^N)$. Consider a family
of open cubes \( \{Q_{i,m}\} \) as in Theorem 2.2, and define on every cube \( Q_{i,m} \) the functions
\[
\nu_{r,m} = u_r - u^+ < (Du)_{i,m}, x >
\]
where
\[
(Du)_{i,m} = \frac{1}{|Q_{i,m}|} \int_{Q_{i,m}} Du(y) \, dy.
\]
Then
\[
D\nu_{r,m} = Du_r - Du + (Du)_m
\]
where
\[
(Du)_m(x) = \sum_i (Du)_{i,m} \chi_{Q_{i,m}}(x).
\]
Fix \( 0 < \varepsilon < 1 \), we prove that for suitable \( m \) we have
\[
|F(u_r, \Omega) - F(\nu_{r,m}, \Omega)| \leq \varepsilon.
\]
Let \( p \) be the right derivative of \( \Phi \), by Proposition 3.2 and Young's inequality we get
\[
|F(u_r, \Omega) - F(\nu_{r,m}, \Omega)|
\leq \sum_i \int_{Q_{i,m}} |f(Du_r) - f(D\nu_{r,m})| \, dx
\leq c_1 \sum_i \int_{Q_{i,m}} \{1 + p(1 + |Du_r| + |D\nu_{r,m}|)|Du_r - D\nu_{r,m}| \, dx
\leq c_2 \int_{\Omega} \Phi(|Du - (Du)_m|) \, dx +
+ c_1 \varepsilon \sum_i \int_{Q_{i,m}} \{\Psi(1) + \Psi(p(1 + |Du_r| + |D\nu_{r,m}|)) \, dx
= I_1 + I_2.
\]
For a suitable \( m \), by Theorem 2.2, we obtain
\[
I_1 \leq \varepsilon.
\]
Since $\Phi \in \Delta_2$, there exist $k > 1$ and $t_0 \geq 0$ such that for every $t \geq t_0 \geq 0$: $\Phi(p(t)) \leq k\Phi(t)$, then

$$I_2 \leq c_2\varepsilon + c_3\varepsilon \int_{\Omega} \{\Phi(|Du_r|) + \Phi(|Du - (Du)_m|)\} \, dx,$$

therefore by Proposition 2.1 and Theorem 2.2 we have

$$|F(u_r, \Omega) - F(\nu_{r,\Omega})| \leq c_3\varepsilon.$$

In a similar way we can show that

$$\left| F(u, \Omega) - \int_{\Omega} f((Du)_m) \, dx \right| \leq \varepsilon.$$

Fix $M \in \mathbb{N}$ and set $\Omega_M = \bigcup_{i=1}^{M} Q_{i,m}$, since $\{\nu_{r,m}\} \to (Du)_{i,m, \mathbb{R}^N}$ for every $i$ by the first part of the proof we have

$$\int_{\Omega_M} f((Du)_m) \, dx \leq \liminf_{r} \int_{\Omega_M} f(D\nu_{r,m}) \, dx.$$

Using (5) and the convexity of $\Phi$, it is easy to prove that for suitable $M$ it holds

$$\int_{\Omega \setminus \Omega_M} f((Du)_m) \, dx \leq \varepsilon.$$

Moreover, as the integrals of functions $\Phi_1(|Du_r|)$ are equi-absolutely continuous, we get

$$\int_{\Omega \setminus \Omega_M} f(D\nu_{r,m}) \, dx \leq \varepsilon.$$

We can conclude that

$$F(u, \Omega) \leq \int_{\Omega} f((Du)_m) \, dx + \varepsilon$$

$$\leq \int_{\Omega_M} f((Du)_m) \, dx + 2\varepsilon \leq \liminf_{r} \int_{\Omega_M} f(D\nu_{r,m}) \, dx + 2\varepsilon$$

$$\leq \liminf_{r} \int_{\Omega} f(D\nu_{r,m}) \, dx + 3\varepsilon \leq \liminf_{r} \int_{\Omega} f(Du_r) \, dx + c_3\varepsilon.$$
Finally the semicontinuity follows. □

By the previous semicontinuity result, we are able to state the following existence theorem in the context of Orlicz-Sobolev spaces, using the Direct Methods of the Calculus of Variations.

**Theorem 3.3.** Let \( f : \mathbb{R}^{Nn} \to \mathbb{R} \) be a quasi-convex function satisfying
\[
c_1 \{ \Phi(|z|) - 1 \} \leq f(z) \leq c_2 \{ \Phi(|z|) + 1 \}
\]
for every \( z \in \mathbb{R}^{Nn} \), where \( c_1 \) and \( c_2 \) are positive constants and \( \Phi \in \Delta_2 \).

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) with \( \partial \Omega \) lipschitzian, let \( \nu \) be a function in \( W^{1,\Phi}(\Omega, \mathbb{R}^N) \), consider the Dirichlet's class
\[
V = \nu + W^{1,\Phi}_0(\Omega, \mathbb{R}^N),
\]
then the problem \( m = \inf_V F(u, \Omega) \) has solution.

**Proof.** Functional \( F \) is lower bounded and coercive in the strong topology of \( V \).

In fact for every \( w \in L^\Phi(\Omega, \mathbb{R}^N) \) it holds
\[
\| w \|_\Phi \leq 1 + \int_\Omega \Phi(|w|) \, dx,
\]
then by (12) we have
\[
F(u, \Omega) \geq c_3 \{ \| Du \|_\Phi - 1 \},
\]
so \( F \) is lower bounded on \( V \), i.e. \( m > -\infty \).

On the other hand from
\[
\| Du \|_\Phi \geq \| D(u - \nu) \|_\Phi - \| D\nu \|_\Phi
\]
it follows
\[
F(u, \Omega) \geq c_4 \{ \| D(u - \nu) \|_\Phi - 1 \},
\]
and, as in \( W^{1,\Phi}_0(\Omega, \mathbb{R}^N) \) the norm of the gradient and the usual one are equivalent, \( F \) is coercive with respect to the strong topology of \( V \).
Let \( \{ u_r \} \) be a minimizing sequence of \( F \) on \( V \), i.e. \( \lim_{r} F(u_r, \Omega) = m \), then, by coercitivity of \( F \), \( \{ u_r \} \) is bounded in norm. Thus, there exist a subsequence of \( \{ u_r \} \), which we still denote by \( \{ u_r \} \), and a function \( u \in \nu + W^{1,k}(\Omega, \mathbb{R}^N) \) such that \( \{ u_r \} \to u * w^{-1,k}(\Omega, \mathbb{R}^N) \).

By Theorem 3.1 \( F \) is sequentially lower semicontinuous in \( * w^{-1,k}(\Omega, \mathbb{R}^N) \) then

\[
F(u, \Omega) \leq \liminf_{r} F(u_r, \Omega) = m,
\]

since this we get \( m = F(u, \Omega) \).

\[\square\]

4. Semicontinuity theorem: the general case

**Theorem 4.1.** Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) with \( \partial \Omega \) lipschitzian, let \( f(x, s, z) \), defined on \( \Omega \times \mathbb{R}^N \times \mathbb{R}^{N_n} \) with real values, be a Carathéodory function quasi-convex in \( z \) such that there exist positive constants \( c_0, c_1, c_2 \) and \( \Phi_1, \Phi_2 \) \( N \)-functions belonging to class \( \Delta_2 \) such that

\[
|f(x, s, z)| \leq c_0 + c_1 \Phi_1(|s|) + c_2 \Phi_2(|z|)
\]

(13)

for every \( (x, s, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N_n} \).

Then the functional

\[
F(u, \Omega) = \int_{\Omega} f(x, u(x), Du(x)) \, dx
\]

is sequentially lower semicontinuous in \( * w^{-1,\Gamma}(\Omega, \mathbb{R}^N) \) for every \( N \)-function \( \Gamma \in \Delta_2 \) such that

\[
\lim_{t \to +\infty} \frac{\Gamma(t)}{\Phi_i(t)} = +\infty \quad i = 1, 2.
\]

(14)

**Remark 4.2.** The assumptions \( \Gamma \in \Delta_2 \) and (14) imply that the following embeddings are compact

\[
W^{1,\Gamma}(\Omega, \mathbb{R}^N) \to L^{\Phi_i}(\Omega, \mathbb{R}^N) \quad i = 1, 2.
\]

**Remark 4.3.** If \( f = f(z) \) Theorem 4.1 is a consequence of Theorem 3.1.
The following result, due to Scorza Dragoni (see [11]), characterizes the Carathéodory functions.

**Proposition 4.4.** \( g : \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N} \rightarrow \mathbb{R} \) is a Carathéodory function if and only if for every compact subset \( C \subset \mathbb{R}^n \) and every \( \gamma > 0 \) there exists a compact subset \( C_{\gamma} \subset C \) such that \( |C \setminus C_{\gamma}| \leq \gamma \) and that the restriction of \( g \) to \( C_{\gamma} \times \mathbb{R}^N \times \mathbb{R}^{N} \) is continuous.

**Proof of Theorem 4.1.** Let \( \tau > 0 \), then there exist a positive integer \( m \) and a finite number of open cubes \( \{Q_{i,m}\} \), whose sides have length \( 1/m \), satisfying

\[
Q_{i,m} \subset \subset \Omega, \quad Q_{i,m} \cap Q_{j,m} = \emptyset, \quad i \neq j, \quad \left| \Omega \setminus \bigcup_{i \leq m} Q_{i,m} \right| \leq \tau.
\]

Let \( t > 0 \) and \( \nu \) be a function, define

\[
\Omega_{\nu,t} = \{ x \in \Omega : |\nu(x)| > t \}
\]

and set

\[
\nu_t(x) = \nu(x) \chi_{\Omega \setminus \Omega_{\nu,t}}(x).
\]

Fix \( m \) and \( i \), define

\[
\nu_{i,m} = \frac{1}{|Q_{i,m}|} \int_{Q_{i,m}} \nu(x) \, dx,
\]

then consider

\[
\nu_m(x) = \begin{cases} 
\sum_i \nu_{i,m} \chi_{Q_{i,m}}(x) & x \in Q_m \\
0 & x \in \Omega \setminus Q_m
\end{cases},
\]

and set \( \nu_{i,m} = (\nu_t)_m \).

Let \( u \in W^{1,1}(\Omega, \mathbb{R}^N) \) and \( \{u_r\} \) be a sequence convergent to \( u \) in \( W^{1,1}(\Omega, \mathbb{R}^N) \).

By Remark 4.2, \( \{u_r\} \) converges to \( u \) in \( s-L^p(\Omega, \mathbb{R}^N) \) and then it is convergent to \( u \) almost everywhere in \( \Omega \). Moreover, by Proposition 2.3, the functions \( \Phi_1(|u|), \Phi_1(|u_r|), \Phi_2(|Du|) \) and \( \Phi_2(|Du_r|) \) have equi-absolutely continuous integrals on \( \Omega \).
Consider \( \{x_m\} \), \( x_m = [\text{Id}_{\mathbb{R}^n}]_m \), \( \{u_{t,m}\} \) and \( \{[Du]_{t,m}\} \), by Theorem 2.2 they are convergent almost everywhere in \( \Omega \) to \( \text{Id}_{\mathbb{R}^n}, u_t, [Du]_t \) respectively.

We get

\[
\int \Omega f(x, u, Du) \, dx
\]

\[
= \int_{\Omega \setminus Q_m} f(x, u, Du) \, dx + \quad (I_1)
\]

\[
+ \int_{Q_m} \{f(x, u, Du) - f(x, u_t, [Du]_t)\} \, dx + \quad (I_2)
\]

\[
+ \int_{Q_m} \{f(x, u_t, [Du]_t) - f(x_m, u_{t,m}, [Du]_{t,m})\} \, dx + \quad (I_3)
\]

\[
+ \int_{Q_m} \{f(x_m, u_{t,m}, [Du]_{t,m}) - f(x_m, u_{t,m}, [Du]_{t,m}) + 
\]

\[
+ D(u_r - u)) \} \, dx + \quad (I_4)
\]

\[
+ \int_{Q_m} \{f(x_m, u_{t,m}, [Du]_{t,m} + D(u_r - u)) + 
\]

\[
- f(x_m, u_{t,m}, [Du]_{t,m} + [Du]_t) \, dx + \quad (I_5)
\]

\[
+ \int_{Q_m} \{f(x_m, u_{t,m}, [Du]_{t,m} + [Du]_t) - [Du]_t + 
\]

\[
- f(x, u_t, [Du]_t)\} \, dx + \quad (I_6)
\]

\[
+ \int_{Q_m} \{f(x, u_t, [Du]_t) - f(x, [u_r]_t, [Du]_t)\} \, dx + \quad (I_7)
\]

\[
+ \int_{Q_m} \{f(x, [u_r]_t, [Du]_t) - f(x, u_r, Du_r)\} \, dx + \quad (I_8)
\]

\[
- \int_{\Omega \setminus Q_m} f(x, u_r, Du_r) \, dx \quad (I_9)
\]
\[ + \int_{\Omega} f(x, u_r, Du_r) \, dx. \]  

(I_{10})

By Proposition 2.3 it follows from (13) and (14) the equi-absolute continuity of integrals \( F(u, \Omega) \) and \( F(u_r, \Omega) \), thus if \( t \) is large enough we get

\[ I_2 + I_5 + I_8 \leq \varepsilon. \]

Fixing a suitable \( m \) and using (13) we get

\[ I_1 + I_9 \leq \varepsilon. \]

Consider \( I_3, I_6 \) and \( I_7 \), by Egorov theorem, Proposition 4.4 and equi-absolute continuity of integrals it follows that the sum of these addenda is less than \( \varepsilon \).

Finally \( I_4 \) has non positive inferior limit by Remark 4.3.

So we have

\[ \int_{\Omega} f(x, u, Du) \, dx \leq \varepsilon + \int_{\Omega} f(x, u_r, Du_r) \, dx \]

and finally the result follows passing to inferior limits for \( r \to +\infty \).

\[ \square \]

5. Examples

In this section we exhibit some examples of applications of the semi-continuity Theorems 3.1, 4.1 and of the existence Theorem 3.3. The first example deals with Theorem 4.1. We are interested in the case \( N = n = 2 \), completely solved in [1], [6] and [8] for positive poli-convex functionals, so we consider a suitable modification of a family of quasi-convex functions, introduced by Šverák [26], with sub-quadratic growth at infinity, which, then, are neither convex nor poli-convex.

Let \( A, B \in M^{2 \times 2} \) such that

\[ \text{rank} (A - B) \geq 2, \]

then \( K = \{A, B\} \) is compact and non convex.
For $p > 1$ define the function

$$d_p(z) = [d(z)]^p,$$

where $z \in M^{2 \times 2}$ and $d(z)$ denotes the distance of $z$ from $K$.

Šverák in [26] proved that the quasi-convex envelope $Qd_p$ of $d_p$ satisfies

$$Qd_p(z) > 0 \quad \text{for every } z \in M^{2 \times 2} \setminus K,$$

moreover if $1 < p < 2$ then $Qd_p$ is quasi-convex but not poli-convex.

For $1 < p < 2$ define the function $f_p : M^{2 \times 2} \to \mathbb{R}$ by

$$f_p(z) = d_p(z) \ln(e + d(z)).$$

Since $d_p \leq f_p$ we have $(Qf_p)^{-1}(0) = K$, then $Qf_p$ is not convex, and not even poli-convex since it has sub-quadratic growth at infinity.

Moreover, since $d(z)$ has linear growth at infinity, we obtain

$$0 \leq Qf_p(z) \leq c_1 \{1 + |z|^p \ln(e + |z|)\}. \quad (15)$$

Let $a : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a non-negative measurable function belonging to $L^\infty(\Omega)$, define the function $g_p(x, s, z) = a(x, s)Qf_p(z)$. Then, by (15), $g_p$ satisfies growth conditions of type (13) with the $N$-function $\Phi_p(t) = t^p \ln(e + t) \in \Delta_2$, thus by Theorem 4.1 the functional

$$G_p(u, \Omega) = \int_\Omega g_p(x, u, Du) \, dx \quad (16)$$

is sequentially lower semicontinuous in $\text{sw-}W^{1, \Phi_{a,p}}(\Omega, \mathbb{R}^2)$ with $\Phi_{a,p}(t) = t^p \ln^\alpha(e + t)$, $\alpha > 1$.

Finally, observe that applying Theorem 2.4 of [2], Proposition 1 of [15] and Theorem 1.1 of [20] we obtain the weak lower semicontinuity of (16) in $W^{1,p+\varepsilon}(\Omega, \mathbb{R}^2)$ for every $\varepsilon > 0$, which is a proper subspace of $W^{1,\Phi_{a,p}}(\Omega, \mathbb{R}^2)$ for every $\alpha > 1$, $\varepsilon > 0$.

The following example is obtained by applying a result of Zhang who developed in [27] a method to construct quasi-convex functions with linear growth at infinity from known quasi-convex functions.

Consider, as before, $A, B \in M^{N \times n}$ such that

$$\text{rank } (A - B) \geq 2,$$
and set $K = \{ A, B \}$, then $K$ is compact and non convex.

Let $z \in M^{N \times n}$ and denote with $d(z)$ the distance of $z$ from $K$, in [27] Zhang proved that the quasi-convex envelope $Qd$ of $d$ satisfies

$$Qd(z) > 0 \quad \text{for every } z \in M^{N \times n} \setminus K.$$ 

Thus $Qd$ is a quasi-convex function with linear growth at infinity, i.e. there exist $c_i$, $1 \leq i \leq 4$, non negative constants satisfying for every $z \in M^{N \times n}$

$$-c_1 + c_2 |z| \leq Qd(z) \leq c_3 + c_4 |z|, \quad (17)$$

and $Qd$ is not convex since $(Qd)^{-1}(0) = K$.

Consider $\Phi_{a,b}(t) = t^{a+b \sin((\pi t))}$, which is the principal part of a $N$-function of class $\Delta_2$ if $a > 1 + b \sqrt{2}$, then the function

$$h_{a,b}(z) = (\Phi_{a,b} \circ Qd)(z)$$

is quasi-convex but not convex. In fact $h_{a,b}$ is the composition of a $N$-function with a quasi-convex function, then, since $\Phi_{a,b}(t) = 0$ if and only if $t = 0$, it follows $(h_{a,b})^{-1}(0) = K$.

Since $\Phi_{a,b} \in \Delta_2$, by (17), there exist $c_5, c_6$ non negative constants such that for every $z \in M^{N \times n}$

$$0 \leq h_{a,b}(z) \leq \Phi_{a,b}(c_3 + c_4 |z|) \leq c_5 + c_6 \Phi_{a,b}(|z|). \quad (18)$$

By (18) we get that $h_{a,b}$ satisfies

$$0 \leq h_{a,b}(z) \leq c_5 + c_6 |z|^{a+b}, \quad (19)$$

moreover by (17) and the continuity of $Qd$ it is easy to show that the power $a + b$ in (19) is sharp.

By Theorem 3.1, it follows the sequential lower semicontinuity of the functional

$$H_{a,b}(u, \Omega) = \int_\Omega h_{a,b}(Du) \, dx \quad (20)$$

in $\text{w} \text{-} W^{1, \Phi_{a,b}}(\Omega, \mathbb{R}^N)$. Moreover, Theorem 3.3 gives the existence of minimizers for the Dirichlet problem $\min_V H_{a,b}(u, \Omega)$, where $V = \nu + W^{1, \Phi_{a,b}}(\Omega, \mathbb{R}^N)$ and $\nu \in W^{1, \Phi_{a,b}}(\Omega, \mathbb{R}^N)$. 

We remark that Theorem 3.1 applied to (20) gives a different result with respect to semicontinuity theorems known in ordinary Sobolev spaces. Let \( N = n \geq 3 \), Theorems 2.4 of [2] and 1.1 of [20], assure the weak lower semicontinuity of (20) in \( W^{1,a+b}(\Omega,\mathbb{R}^N) \) which is a proper subspace of \( W^{1,p}(\Omega,\mathbb{R}^N) \). Moreover, the results in [13] and [18] give the lower semicontinuity of (20) in \( w-W^{1,p}(\Omega,\mathbb{R}^N) \) for \( p > \frac{N}{N-1}(a+b) \), and taking in account [19] we get semicontinuity for \( p \geq a+b-1 \). Let \( a+b < N \), then \( \frac{N}{N-1}(a+b) > a+b-1 \), if we assume \( a+b-1 > a-b \), i.e. \( b > \frac{a}{2} \), since \( \Phi_{a,b}(t) = t^{a-b} \) for infinite \( t \in \mathbb{R} \), we can conclude that Theorem 3.1 states semicontinuity in a different space with respect to previous results. We observe explicitly that there exist positive constants \( a, b \) satisfying \( a > 1 + b\sqrt[4]{2} \), \( a+b < N \) and \( b > \frac{1}{2} \), e.g. for \( N = 3 \) take \( a = 2 \) and \( b = \frac{5}{8} \).

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