ADDENDUM to the paper
The Bounded-Open Topology
and its Relatives

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Summary. - Addendum to the paper The Bounded-Open Topology
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In this short note, we would like to provide detailed clarification
of a few claims and rectification of two minor errors made in the
aforesaid paper cited as [1]. Throughout this note $X$ stands for a
Tychonoff space and $C(X)$ is the set of all real-valued continuous
functions on $X$ while $C^*(X) = \{ f \in C(X) : f \text{ is bounded} \}$. A
subset $A \subset X$ is called bounded if $f(A)$ is a bounded subset of $\mathbb{R}$
(the real line with the usual topology) for each $f \in C(X)$. Let $\mathcal{G}$ be
a collection of some bounded subsets of $X$ satisfying the condition:
if $A, B \in \mathcal{G}$, there exists a set $C \in \mathcal{G}$ such that $A \cup B \subseteq C$ holds. To
define the $\mathcal{G}$-open topology on $C(X)$, we take the subbasic open sets of
the form

$$[A, V] = \{ f \in C(X) : f(A) \subseteq V \}$$

where $A \in \mathcal{G}$ and $V$ is open in $\mathbb{R}$.

We denote the space $C(X)$ with $\mathcal{G}$-open topology by $C_{\mathcal{G}}(X)$. At the
end of the first paragraph of [1, p. 64], it has been noted that for
the point-open and compact-open topologies, $V$ can always be taken

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as a bounded open interval. In general for \( \mathcal{G} \)-open topologies this property can actually be assumed if we put a mild restriction on \( \mathcal{G} \). First we need the following result.

**Lemma 1.1.** Let \( f \in [A, V] \) where \( A \in \mathcal{G} \) and \( V \) is open in \( \mathbb{R} \). Then there exist bounded subsets \( A_i \) of \( X \) and bounded open intervals \( W_i \) in \( \mathbb{R} \) \( (1 \leq i \leq n) \) such that \( f \in \bigcap_{i=1}^{n} [A_i, W_i] \subseteq [A, V] \).

**Proof.** Let \( z \in \overline{f(A)} \). There exists \( \epsilon_z > 0 \) such that \( \overline{f(A)} \subseteq [z - \epsilon_z, z + \epsilon_z] \subseteq V \). Since \( f(A) \) is compact, there exist \( i = 1, 2, \ldots, n \) such that \( \overline{f(A)} \subseteq \bigcup_{i=1}^{n} (z_i - \epsilon_z, z_i + \epsilon_z) \subseteq \bigcup_{i=1}^{n} [z_i - 2\epsilon_z, z_i + 2\epsilon_z] \subseteq V \).

Let \( V_i = (z_i - \epsilon_z, z_i + \epsilon_z) \), \( W_i = (z_i - 2\epsilon_z, z_i + 2\epsilon_z) \) and \( A_i = \text{cl}_X(A \cap f^{-1}(V_i)) \) for \( i = 1, 2, \ldots, n \). It is routine to check that \( A = \bigcup_{i=1}^{n} A_i \), and \( f \in \bigcap_{i=1}^{n} [A_i, W_i] \subseteq [A, V] \).

A family \( \mathcal{G} \) of bounded subsets of \( X \) is said to be hereditary with respect to closed domains if it satisfies the following condition: whenever \( A \in \mathcal{G} \) and \( B \) is a closed subdomain of \( A \), then \( B \in \mathcal{G} \) as well.

**Corollary 1.2.** Suppose \( \mathcal{G} \) is a family of bounded subsets of \( X \) hereditary with respect to closed domains.

Then the collection \( \{ [A, V] : A \in \mathcal{G}, V \text{ a bounded open interval in } \mathbb{R} \} \) forms a subbase for \( C_\mathcal{G}(X) \).

Now we are going to give a clarification on Theorem 2.5 in [1]. The proof of this result (with a correction yet to be made) given in [1] can only work if we put a restriction on \( \mathcal{G} \).

**Theorem 1.3.** Suppose \( \mathcal{G} \) is a family of bounded subsets of \( X \) hereditary with respect to closed domains. Then \( C_\mathcal{G}(X) \) is dense in \( C_\mathcal{G}(X) \).

**Proof.** Let \( \bigcap_{i=1}^{n} [A_i, V_i] \) be a basic open set in \( C_\mathcal{G}(X) \) containing \( f \) where \( V_i \)'s are bounded open intervals in \( \mathbb{R} \) and \( A_i \in \mathcal{G} \). Then there exists a bounded open interval \((a, b)\) such that \( \bigcup_{i=1}^{n} f^{-1}(A_i) \subseteq \bigcup_{i=1}^{n} V_i \subseteq (a, b) \). The rest of the proof is as mentioned in the proof of Theorem 2.5 in [1].

We really do not need \( \mathcal{G} \) to be hereditary with respect to the closed domain in order that \( C_\mathcal{G}(X) \) to be dense in \( C_\mathcal{G}(X) \), but we need to change the proof.
THEOREM 1.4. For any space $X$, $C^*(X)$ is dense in $C_{\mathcal{G}, u}(X)$. (Note $C_{\mathcal{G}, u}(X)$ is the space $C(X)$ equipped with the topology of uniform convergence on $\mathcal{G}$).

Proof. We show that $< f, A, \epsilon > \cap C^*(X) \neq \emptyset$ for all $f \in C(X)$, for all $A \in \mathcal{G}$ and for all $\epsilon > 0$. Let $f \in C(X)$. Then $f$ has a continuous extension $f^\nu$ from $\nu X$ (Hewitt-realcompactification of $X$) into $\mathbb{R}$. Since $A$ is bounded, $\text{cl}_{\beta X} A \subseteq \nu X$ ($\beta X$ is the Stone-Čech compactification of $X$). Let $A_1 = \text{cl}_{\beta X} A$ and $f_1 = f^\nu|_{A_1}$, the restriction of $f^\nu$ to $A_1$. Note $f_1(A_1)$ is compact in $\mathbb{R}$ and hence $f_1(A_1) \subseteq [a, b]$ for some closed bounded interval $[a, b]$ in $\mathbb{R}$. Now there exists a continuous function $F : \beta X \to [a, b]$ such that $F|_{A_1} = f_1$. Let $g = F|_X$. It is easy to check that $g \in C^*(X) \cap < f, A, \epsilon >$. $\square$

Since $C_{\mathcal{G}}(X) \leq C_{\mathcal{G}, u}(X)$ (see [1, Theorem 3.1]), we have the following result.

COROLLARY 1.5. For any space $X$, $C^*(X)$ is dense in $C_{\mathcal{G}}(X)$.

Now we would like to clarify the status of Proposition 2.1 in [1]. This result should be modified to the following version.

PROPOSITION 1.6. Suppose $\mathcal{G}$ is a family of bounded subsets of $X$ hereditary with respect to closed domains. Then $C_{\mathcal{G}}(X)$ is completely regular. If in addition $\mathcal{G}$ is a network, then $C_{\mathcal{G}}(X)$ is also Hausdorff.

Proof. The fact that $C_{\mathcal{G}}(X)$ is completely regular can be proved in a manner similar to the proof in [2, Lemma 5.1]. $\square$

Finally we would like to make a remark on the Examples 3.14, 3.15, 3.17 of [1]. The proof of the result that $X = \beta \mathbb{N} - \{ p \}$ (where $p$ belongs to $\mathbb{N}^* = \beta \mathbb{N} - \mathbb{N}$) is not normal assumes the Continuum Hypothesis (CH). See [3] for the details. But Jack Porter has recently communicated to the authors that Sapirovskii (around 1988 but reference unknown) proved the following without Continuum Hypothesis: there exists a point $p$ in $\beta \mathbb{N}$ such that $\beta \mathbb{N} - \{ p \}$ is not normal. Consequently, it should be emphasized that in these examples the CH can be avoided.
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