Perturbation of Ornstein-Uhlenbeck Semigroups

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Dedicated to Pierre Grisvard

Introduction

In this paper we consider the Ornstein-Uhlenbeck process \( Z(t, x) \), solution of the following differential stochastic equation in a Hilbert space \( H \):

\[
dZ = AZdt + dW(t), \quad Z(0) = x.
\]

Here \( W \) is a cylindrical Wiener process on \( H \) and \( A \) is the infinitesimal generator of an exponentially stable analytic semigroup \( e^{tA} \) in \( H \). Under this hypothesis it is well known that the process \( Z(t, x) \) has a unique invariant measure \( \mu \), see e.g. [7].

Let us denote by \( A \) the infinitesimal generator of the transition semigroup

\[
R_t \varphi(x) = \mathbb{E}[\varphi(Z(t, x))], \quad t \geq 0,
\]

defined in the space \( L^2(H; \mu) \). \( A \) can be written formally as

\[
A \varphi = \frac{1}{2} \text{Tr} [D^2 \varphi(x)] + \langle Ax, D \varphi(x) \rangle.
\]

In G. Da Prato and J. Zabczyk see [9], it was proved that \( A \) is an \( m \)-dissipative operator on \( L^2(H; \mu) \). Moreover, in that paper we also

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studied perturbations of $\mathcal{A}$ of the form
\[ \langle F(x), D\varphi(x) \rangle, \tag{0.1} \]
where $F : H \to H$ is a continuous and bounded mapping.

The main result of the present paper is a precise characterization, under suitable assumptions, of the domain $D(\mathcal{A})$ of $\mathcal{A}$, as a subspace of $W^{2,2}(H; \mu)$.

We notice that the operator $\mathcal{A}$ has been extensively studied using the Theory of Dirichlet forms, see Z. M. Ma and M. Röckner [17]. Using this method one can show that, in several situations, the operator $\mathcal{A}$ is variational, and consequently one can conclude that $D(\mathcal{A})$ is a subspace of $W^{1,2}(H; \mu)$. Knowing that $D(\mathcal{A}) \subset W^{2,2}(H; \mu)$, will allow us to consider perturbations of $\mathcal{A}$ more general than (0.1).

Our method is based on a generalization of the well known L. Nirenberg’s proof about $H^2$ regularity of second order elliptic equations, see e.g. [2]. We establish a basic identity for functions belonging to $D(\mathcal{A})$, that, under suitable assumptions (see Hypotheses 1.1 and 3.1), yields a characterization of $D(\mathcal{A})$. These assumptions are in particular fulfilled when $\mathcal{A}$ is self-adjoint and when $H$ is finite-dimensional.

We notice that, when $\mathcal{A}$ is self-adjoint, a characterization of $D(\mathcal{A})$ could also be obtained by using the spectral decomposition of $\mathcal{A}$ written in terms of Hermite polynomial, see [7]. Moreover, when $H$ is finite-dimensional, our characterization coincides with that proved earlier by A. Lunardi, see [16], by a completely different method involving interpolatory arguments.

In section §1 we recall several known results, proved for instance in [7], about transition semigroups $\mathcal{R}_t$, $t \geq 0$, defined in space of continuous functions.

Section §2 is devoted to the description of the transition semigroup $\mathcal{R}_t$, $t \geq 0$, in $L^2(H; \mu)$. Here we recall several results proved earlier in [9] and [12], and we give some improvements that will be used later.

In §3 we present a characterization of the domain of $\mathcal{A}$. This characterization is exploited in §4 to study different perturbations of $\mathcal{A}$. 
1. Notation and setting of the problem

We are given a separable Hilbert space $H$ (norm $| \cdot |$, inner product $\langle \cdot, \cdot \rangle$), and a differential stochastic equation in $H$

$$
\begin{cases}
    dZ(t) = AZ(t)\,dt + dW(t) \\
    Z(0) = x \in H,
\end{cases}
$$

(1.1)

where $A : D(A) \subset H \to H$ is a linear operator and $W(t), t \geq 0$, is a cylindrical Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [7].

We shall assume that

**Hypothesis 1.1.**

(i) $A$ is the infinitesimal generator of an analytic semigroup $e^{tA}$ in $H$. There exist $M \geq 1$ and $\omega > 0$ such that

$$
\|e^{tA}\| \leq Me^{-\omega t}, \quad t \geq 0.
$$

(ii) For any $t > 0$, $e^{tA} \in L_2(H)$ (1) and, setting

$$
Q_t x = \int_0^t e^{sA} e^{sA^*} x \,ds, \quad x \in H,
$$

(1.2)

we have

$$
\text{Tr } [Q_t] < +\infty, \quad \forall \ t > 0.
$$

The following result is proved in [7].

**Proposition 1.1.** Assume that Hypothesis 1.1 holds.

(i) Problem (1.1) has a unique mild solution given by

$$
Z(t, x) = e^{tA}x + \int_0^t e^{(t-s)A}dW(s), \quad x \in H, \quad t \geq 0. \quad (1.3)
$$

---

$^1$\(L(H)\) is the Banach algebra of all linear bounded operators on $H$, endowed with the sup norm $\| \cdot \|$. By $L_1(H)$ (norm $\| \cdot \|_{L_1(H)}$) we mean the Banach space of all trace-class operators on $H$, and by $L_2(H)$ (norm $\| \cdot \|_{L_2(H)}$) the Hilbert space of all Hilbert-Schmidt operators in $H$. If $T \in L_1(H)$, the trace of $T$ is denoted by $\text{Tr } T$. 
Moreover $Z(t,x)$ is a Gaussian random variable $\mathcal{N}(e^{tA}x, Q_t)$, for all $t \geq 0$ and all $x \in H$. \(^{(2)}\)

(ii) There exists a unique probability measure $\mu$ on $(H, \mathcal{B}(H))$ that is invariant for the process $Z(t,x)$, that is such that

$$\int_H R_t \varphi(x) \mu(dx) = \int_H \varphi(x) \mu(dx), \forall \varphi \in C_b(H), \quad \text{for all } t \geq 0,$$

where $R_t$, $t \geq 0$ is the transition semigroup

$$R_t \varphi(x) = \int_H \varphi(y) \mathcal{N}(e^{tA}x, Q_t)(dy), \varphi \in C_b(H), \quad t \geq 0, \quad x \in H. \quad \text{(1.4)}$$

Moreover $\mu = \mathcal{N}(0, Q)$, where

$$Qx = \int_0^{+\infty} e^{tA} e^{tA^*} x dt, \quad x \in H. \quad \text{(1.5)}$$

One can easily check that $Q$ is a solution to the following Lyapunov equation

$$2\langle A^* x, Qx \rangle + \|x\|^2 = 0, \quad x \in D(A^*). \quad \text{(1.6)}$$

We end this section by recalling some properties of the semigroup $R_t$, $t \geq 0$, in the space $C_b(H)$.

The following result is proved in [7].

**Proposition 1.2.** Assume that Hypothesis 1.1 holds.

(i) For all $t > 0$ we have $e^{tA}(H) \subset Q_t^{1/2}(H)$. Moreover the linear operator $\Gamma(t) := Q_t^{-1/2} e^{tA}$ belongs to $\mathcal{L}_2(H)$ and the following estimate holds

$$\|\Gamma(t)\| \leq t^{-1/2}, \quad t > 0. \quad \text{(1.7)}$$

---

\(^{(2)}\) For any $z \in H$, and any positive operator $L \in \mathcal{L}_1(H)$, we denote by $\mathcal{N}(z, L)$ the Gaussian measure on $(H, \mathcal{B}(H))$, (where $\mathcal{B}(H)$ is the family of all Borel subsets of $H$, with mean $z$ and covariance operator $L$.)

\(^{(3)}\) $C_b(H)$ is the Banach space of all uniformly continuous and bounded mappings from $H$ into $\mathbb{R}$, endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$. 
(ii) For all $t > 0$ and all $\varphi \in C_b(H)$, we have $R_t \varphi \in C^1_b(H)$ \footnote{$C^1_b(H)$ is the set of all functions in $C_b(H)$ that are uniformly continuous and bounded together with their Fréchet derivative.} and 
\[
\langle DR_t \varphi(x), h \rangle = \int_H \langle \Gamma(t)h, Q_t^{-1/2}y \rangle \varphi(e^{tA}x + y) \mathcal{N}(0, Q_t)(dy), \ h \in H.
\]

We notice that, when $A$ is not identically 0, the semigroup $R_t$, $t \geq 0$ is never strongly continuous, see [3]. Moreover its restriction to the “subspace of continuity”:

\[
\{ \varphi \in C_b(H) : t \to R_t \varphi \text{ is continuous in } C_b(H) \},
\]
is not an analytic semigroup, see [5].

Proceeding as in S. Cerrai [3], we define the infinitesimal generator $A$ of $R_t$, $t \geq 0$, through its resolvent, by setting
\[
R(\lambda, A)\varphi(x) = \int_0^{+\infty} e^{-\lambda t} R_t \varphi(x) dt, \ x \in H, \ \varphi \in C_b(H).
\]

To give a description of the infinitesimal generator $A$, it is convenient to introduce the space $\mathcal{E}$ of all finite linear combinations of the exponential functions $\varphi_h = e^{i(h, x)}$, $x \in H$, $h \in D(A^*)$.

2. Transition semigroup in $L^2(H; \mu)$

In this section we first recall the definition and some properties of the Sobolev spaces $W^{1,2}(H; \mu)$ and $W^{2,2}(H; \mu)$. Then we show, following [7], that the semigroup $R_t$, $t \geq 0$ can be uniquely extended as a contraction semigroup to $L^2(H; \mu)$, and we state several properties of it, needed in the sequel.

2.1 Sobolev spaces

First of all we remark that, as easily checked, the linear space $\mathcal{E}$ of exponential functions, as introduced in §1, is dense in $L^2(H; \mu)$. Moreover we denote by $\{e_k\}$ a complete orthonormal system in $H$ of
eigenvectors of $Q_\infty$, and by $\{\lambda_k\}$, the corresponding set of eigenvalues:

$$Qe_k = \lambda_k e_k, \ k \in \mathbb{N}.$$ 

For any $k \in \mathbb{N}$ we denote by $D_k \varphi$ the derivative of $\varphi$ in the direction of $e_k$, and we set $x_k = \langle x, e_k \rangle$, $x \in H$.

The following lemma and proposition are well known, see e.g. [12]. However, we give a sketch of proofs for the reader’s convenience.

**Lemma 2.1.** Let $\varphi, \psi \in \mathcal{E}$ and $h \in \mathbb{N}$. Then we have

$$\int_H D_h \varphi(x) \psi(x) \mu(dx) + \int_H D_h \psi(x) \varphi(x) \mu(dx) = \frac{1}{\lambda_h} \int_H x_h \varphi(x) \psi(x) \mu(dx). \quad (2.1)$$

**Proof.** Since $\mathcal{E}$ is dense in $L^2(H; \mu)$, it is enough to prove (2.1) for

$$\varphi(x) = e^{i(\alpha, x)}, \ \psi(x) = e^{i(\beta, x)}, \ \alpha, \beta \in H.$$ 

In this case we have (5):

$$\int_H D_h \varphi(x) \psi(x) \mu(dx) + \int_H D_h \psi(x) \varphi(x) \mu(dx) = i(\alpha_h - \beta_h) e^{-\frac{1}{2}(Q(\alpha, \alpha) + Q(\beta, \beta))}. \quad (2.2)$$

Moreover

$$\int_H x_h \varphi(x) \psi(x) \mu(dx) = \int_H x_h e^{i(\alpha - \beta, x)} \mu(dx)$$

$$= -i \frac{d}{d \lambda} \left. \int_H e^{i(\alpha - \beta + \lambda e_h, x)} \mu(dx) \right|_{\lambda=0}$$

$$= \left. -i \frac{d}{d \lambda} e^{-\frac{1}{2}(Q(\alpha - \beta + \lambda e_h, \alpha - \beta + \lambda e_h))} \right|_{\lambda=0}$$

$$= -i e^{\frac{1}{2}(Q(\alpha - \beta, \alpha - \beta))(\alpha_h - \beta_h) \lambda_h}. \quad (2.3)$$

Now the conclusion follows. \hfill \Box

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\(^5\) If $\nu = \mathcal{N}(0, Q)$ is a Gaussian measure on $H$, then the characteristic function of $\nu$ is defined as $F(h) = \int_H e^{i(h, x)} \nu(dx)$. One can easily show that $F(h) = e^{-\frac{1}{2}(Q(\nu, h))}$. 

From Lemma 2.1 we have

**Proposition 2.2.** For any \( h \in \mathbb{N} \) the linear operator

\[
D_h : \mathcal{E} \subset L^2(H; \mu) \rightarrow L^2(H; \mu), \quad \varphi \rightarrow D_h \varphi,
\]

is closable in \( L^2(H; \mu) \).

We shall still denote by \( D_h \) the closure of \( D_h \).

**Proof.** Let \( \{ \varphi_n \} \) be a sequence in \( \mathcal{E} \) and let \( g \in L^2(H; \mu) \) such that

\[
\varphi_n \rightarrow 0, \quad D_h \varphi_n \rightarrow g, \quad \text{in} \ L^2(H; \mu), \quad \text{as} \ n \rightarrow \infty.
\]

We have to show that \( g = 0 \).

By using (2.1) with \( \varphi = \varphi_n \) and with \( \psi \) being any element in \( \mathcal{E} \), we have in fact

\[
\int_H D_h \varphi_n(x) \psi(x) \mu(dx) + \int_H D_h \psi(x) \varphi_n(x) \mu(dx) = \\
= \frac{1}{\lambda_h} \int_H x_h \varphi_n(x) \psi(x) \mu(dx).
\]

Letting \( n \) tend to \( \infty \) we have by the hypothesis

\[
\int_H g(x) \psi(x) \mu(dx) = 0,
\]

that yields \( g = 0 \) due to the density of \( \mathcal{E} \) and the arbitrariness of \( \psi \).

This completes the proof. \( \square \)

We can now define Sobolev spaces. We denote by \( W^{1,2}(H; \mu) \) the linear space of all functions \( \varphi \in L^2(H; \mu) \) such that \( D_k \varphi \in L^2(H; \mu) \) for all \( k \in \mathbb{N} \) and

\[
\int_H |D \varphi(x)|^2 \mu(dx) = \sum_{k=1}^{\infty} \int_H |D_k \varphi(x)|^2 \mu(dx) < +\infty.
\]

\( W^{1,2}(H; \mu) \), endowed with the inner product

\[
\langle \varphi, \psi \rangle_1 = \int_H \varphi(x) \psi(x) \mu(dx) + \int_H \langle D \varphi(x), D \psi(x) \rangle \mu(dx),
\]
is a Hilbert space. We recall that the embedding of $W^{1,2}(H;\mu)$ into $L^2(H;\mu)$ is compact, see [6], [19], [7].

In a similar way we can define the Sobolev space $W^{2,2}(H;\mu)$ consisting of all functions $\varphi \in W^{1,2}(H;\mu)$ such that $D_h D_k \varphi \in L^2(H;\mu)$ for all $h, k \in \mathbb{N}$ and $D^2 \varphi(x) \in L^2(H)$ for all $x \in H$.

$W^{2,2}(H;\mu)$, endowed with the inner product

$$
\langle \varphi, \psi \rangle_2 = \langle \varphi, \psi \rangle_1 + \sum_{h,k=1}^{\infty} \int_H D_h D_k \varphi(x) \cdot D_h D_k \psi(x) \mu(dx)
$$

is a Hilbert space. Notice that, when $H$ is infinite-dimensional, the embedding of $W^{2,2}(H;\mu)$ into $W^{1,2}(H;\mu)$ is not compact, see [9].

Now from Lemma 2.1 and Proposition 2.2 the following integration by parts formula follows, see [12].

**Proposition 2.3.** Let $\psi_1, \psi_2 \in W^{1,2}(H;\mu)$ and $\alpha \in H$. Then we have

$$
\int_H \langle D \psi_1(x), Q \alpha \rangle \cdot \psi_2(x) \mu(dx) + \int_H \langle D \psi_2(x), Q \alpha \rangle \cdot \psi_1(x) \mu(dx) =
$$

$$
= \int_H \psi_1(x) \psi_2(x) \langle \alpha, x \rangle \mu(dx).
$$

We finish this subsection by proving some useful properties of the spaces $W^{1,2}(H;\mu)$ and $W^{2,2}(H;\mu)$.

**Proposition 2.4.** ([12]) Let $\zeta \in W^{1,2}(H;\mu)$ and $\alpha \in H$. Then the function

$$
x \rightarrow \langle x, \alpha \rangle \zeta(x),
$$

belongs to $L^2(H, \mu)$ and the following inequality holds.

$$
\int_H |\langle \alpha, x \rangle|^2 \zeta^2(x) \mu(dx) \leq 2|Q^{1/2} \alpha|^2 \int_H \zeta^2(x) \mu(dx) +
$$

$$
+ 16|Q \alpha|^2 \int_H |D \zeta(x)|^2 \mu(dx).
$$
Proof. It is enough to prove (2.5) when $\zeta \in \mathcal{E}$. We apply the integration by parts formula (2.4) with

$$\psi_1(x) = \langle \alpha, x \rangle, \psi_2(x) = \zeta^2(x).$$

Since

$$D\psi_1(x) = \alpha, \ D\psi_2(x) = 2\zeta(x)D\zeta(x), \ x \in H,$$

we obtain, using Hölder’s inequality

$$\int_H |\langle \alpha, x \rangle|^2 \zeta^2(x)\mu(dx) =$$

$$= \int_H \langle Q\alpha, \alpha \rangle \zeta^2(x)\mu(dx) + 2 \int_H \langle \alpha, x \rangle \langle D\zeta(x), Q\alpha \rangle \zeta(x)\mu(dx)$$

$$\leq |Q|^{1/2} \|\alpha\|_2 \|\zeta\|_{L^2(\mu, H)}^2 +$$

$$+ 2 \left[ \int_H |\langle \alpha, x \rangle|^2 \zeta^2(x)\mu(dx) \right]^{1/2} \left[ \int_H |\langle Q\alpha, D\zeta(x) \rangle|^2 \mu(dx) \right]^{1/2}$$

$$\leq |Q|^{1/2} \|\alpha\|_2 \|\zeta\|_{L^2(\mu, H)}^2 +$$

$$+ \frac{1}{2} \int_H |\langle \alpha, x \rangle|^2 \zeta^2(x)\mu(dx) + 8 \int_H |\langle Q\alpha, D\zeta(x) \rangle|^2 \mu(dx),$$

that yields (2.5). \qed

By Proposition 2.4 it follows the result.

**Corollary 2.5.** Let $\zeta \in W^{1,2}(H, \mu)$. Then the function

$$H \rightarrow \mathbb{R}, \ x \rightarrow |x|\zeta(x),$$

belongs to $L^2(H, \mu)$ and the following estimate holds

$$\int_H |x|^2 \zeta^2(x)\mu(dx) \leq 2 \text{Tr } Q \int_H \zeta^2(x)\mu(dx) +$$

$$+ 16 \text{Tr } [Q^2] \int_H |D\zeta(x)|^2 \mu(dx). \quad (2.6)$$

**Proof.** Let $k \in \mathbb{N}$; setting in (2.5) $\alpha = e_k$, we find

$$\int_H x_k^2 \zeta^2(x)\mu(dx) \leq 2\lambda_k \int_H \zeta^2(x)\mu(dx) + 16\lambda_k^2 \int_H |D\zeta(x)|^2 \mu(dx).$$

Summing up on $k$, the inequality (2.6) follows. \qed
We now consider functions \( \zeta \) in \( W^{2,2}(H,\mu) \).

**Proposition 2.6.** Let \( \zeta \in W^{2,2}(H,\mu) \) and \( \alpha \in H \). Then the function \( x \to |\langle x, \alpha \rangle|^2 \zeta(x) \) belongs to \( L^2(H;\mu) \) and

\[
\int_H |\langle x, \alpha \rangle|^4 \zeta^2(x) \mu(dx) \leq 4 \left( |Q^{1/2} \alpha|^4 + 8 |\alpha|^2 |Q \alpha|^2 \right) \int_H \zeta^2(x) \mu(dx) + 96 |Q \alpha|^2 |Q^{1/2} \alpha|^2 \int_H |D \zeta(x)|^2 \mu(dx) + 512 |Q \alpha|^4 \int_H \|D^2 \zeta(x)\|^2_{L^2(H)} \mu(dx) \tag{2.7}
\]

**Proof.** Setting \( \eta(x) = \langle x, \alpha \rangle \zeta(x) \), we have by Proposition 2.4 that \( \eta \in L^2(H;\mu) \) and

\[
\int_H \eta^2(x) \mu(dx) \leq 2 |Q^{1/2} \alpha|^2 \int_H \zeta^2(x) \mu(dx) + 16 |Q \alpha|^2 \int_H |D \zeta(x)|^2 \mu(dx). \tag{2.8}
\]

Moreover, for any \( i \in \mathbb{N} \), we have

\[ D_i \eta(x) = \alpha_i \zeta(x) + \langle x, \alpha \rangle D_i \zeta(x). \]

Thus, by Proposition 2.4, \( D_i \eta \in L^2(H;\mu) \) and

\[
\int_H |D_i \eta(x)|^2 \mu(dx) \leq 2 |\alpha_i|^2 \int_H \zeta^2(x) \mu(dx) + 2 \int_H \langle x, \alpha \rangle^2 |D_i \zeta(x)|^2 \mu(dx) \leq 2 |\alpha_i|^2 \int_H \zeta^2(x) \mu(dx) + 4 |Q^{1/2} \alpha|^2 \int_H |D \zeta(x)|^2 \mu(dx) + 32 |Q \alpha|^2 \int_H |D D_i \zeta(x)|^2 \mu(dx). 
\]
Summing up on $i$ we have
\[
\int_H |D\eta(x)|^2 \mu(dx) \leq 2|\alpha|^2 \int_H \zeta^2(x) \mu(dx) + \\
+ 4|Q^{1/2}\alpha|^2 \int_H |D\zeta(x)|^2 \mu(dx) \\
+ 32|Q\alpha|^2 \int_H \|D^2\zeta(x)\|^2_{L_2(H)} \mu(dx). \tag{2.9}
\]
This shows that $\eta \in W^{1,2}(H;\mu)$. Now, applying once again Proposition 2.4, we have that $g = \langle x, \alpha \rangle \eta \in L^2(H;\mu)$ and
\[
\int_H |\langle x, \alpha \rangle|^4 \zeta^2(x) \mu(dx) \leq 2|Q^{1/2}\alpha|^2 \int_H \eta^2(x) \mu(dx) + \\
+ 16|Q\alpha|^2 \int_H |D\eta(x)|^2 \mu(dx). \tag{2.10}
\]
By substituting (2.8) and (2.9) into (2.10) we obtain the conclusion (2.7). \hfill \Box

In a similar way we prove the following result.

**Proposition 2.7.** Let $\zeta \in W^{2,2}(H,\mu)$. Then the function $x \to (1 + |x|^2)\zeta(x)$ belongs to $L^2(H;\mu)$ and
\[
\int_H (1 + |x|^2)^2 \zeta^2(x) \mu(dx) \leq \\
[32 \text{Tr } Q^2 + (1 + 2 \text{Tr } Q)^2] \int_H \zeta^2(x) \mu(dx) + \\
+ 48 \text{Tr } [Q^2](1 + 2 \text{Tr } Q) \int_H |D\zeta(x)|^2 \mu(dx) + \\
+ 512 (\text{Tr } [Q^2])^2 \int_H \|D^2\zeta(x)\|^2_{L_2(H)} \mu(dx). \tag{2.11}
\]

**Proof.** Setting $\rho(x) = \sqrt{1 + |x|^2} \zeta(x)$, we have by (2.6) that $\rho \in L^2(H;\mu)$ and
\[
\int_H \rho^2(x) \mu(dx) = \int_H \zeta^2(x) \mu(dx) + \int_H |x|^2 \zeta^2(x) \mu(dx) \leq \\
\leq (1 + 2 \text{Tr } Q) \int_H \zeta^2(x) \mu(dx) + 16 \text{Tr } [Q^2] \int_H |D\zeta(x)|^2 \mu(dx). \tag{2.12}
\]
For any $i \in \mathbb{N}$ we have

$$D_i \rho(x) = x_i (1 + |x|^2)^{-1/2} \zeta(x) + (1 + |x|^2)^{1/2} D_i \zeta(x),$$

so that

$$\int_H |D_i \rho(x)|^2 \mu(dx) \leq 2 \int_H \frac{x_i^2}{1 + |x|^2} \zeta^2(x) \mu(dx) + 2 \int_H |D_i \zeta(x)|^2 \mu(dx) + 2 \int_H |x|^2 |D_i \zeta(x)|^2 \mu(dx).$$

Consequently, by (2.6) it follows that $D_i \rho \in L^2(H; \mu)$ and

$$\int_H |D_i \rho(x)|^2 \mu(dx) \leq 2 \int_H \frac{x_i^2}{1 + |x|^2} \zeta^2(x) \mu(dx) + 2 \int_H |D_i \zeta(x)|^2 \mu(dx) + 4 \text{Tr} \, Q \int_H |D_i \zeta(x)|^2 \mu(dx) + 32 \text{Tr} \, [Q^2] \int_H |DD_i \zeta(x)|^2 \mu(dx).$$

Summing up on $i$ we obtain

$$\int_H |D \rho(x)|^2 \mu(dx) \leq 2 \int_H \zeta^2(x) \mu(dx) + (2 + 4 \text{Tr} \, Q) \int_H |D \zeta(x)|^2 \mu(dx) + 32 \text{Tr} \, [Q^2] \int_H \|D^2 \zeta(x)\|^2_{\mathcal{L}_2(H)} \mu(dx), \quad (2.13)$$

that yields $\rho \in W^{1,2}(H; \mu)$. Finally by (2.6) it follows

$$\int_H (1 + |x|^2)^2 \zeta^2(x) \mu(dx) \leq \int_H \rho^2(x) \mu(dx) + \int_H |x|^2 \rho^2(x) \mu(dx) \leq (1 + 2 \text{Tr} \, Q) \int_H \rho^2(x) \mu(dx) + 16 \text{Tr} \, [Q^2] \int_H |D \rho(x)|^2 \mu(dx). \quad (2.14)$$
By substituting (2.12) and (2.13) into (2.14) we complete the proof. □

2.2 Transition semigroup

The following result was proved in [7], see also [8]. We give however a sketch of the proof for the reader’s convenience.

**Proposition 2.8.** (i) Assume that Hypothesis 1.1 holds. Then, for any \( t > 0 \), the operator \( R_t \), defined by (1.4), has a unique extension to a linear bounded operator in \( L^2(H; \mu) \), that we still denote by \( R_t \). Moreover \( R_t \), \( t \geq 0 \) is a strongly continuous semigroup of contractions in \( L^2(H; \mu) \), and

\[
R_t \varphi(x) = \int_H \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy),
\]

\( t \geq 0, \ x \in H, \ \varphi \in L^2(H; \mu). \)  \( \text{(2.15)} \)

(ii) \( \mathcal{E} \subset D(A) \) and

\[
A(e^{i(h, \cdot)})(x) = \left( \langle A^*h, x \rangle - \frac{1}{2}|h|^2 \right)e^{i(h, x)}, \ x \in H. \quad \text{(2.16)}
\]

Moreover, \( \mathcal{E} \) is a core for the infinitesimal generator \( A \) of \( R_t \), \( t \geq 0 \).

(iii) For all \( t > 0 \) and all \( \varphi \in L^2(H; \mu) \), one has \( R_t \varphi \in W^{1,2}(H; \mu) \) and

\[
\langle DR_t \varphi(x), h \rangle = \int_H \langle \Gamma(t)h, Q_t^{-1/2}y \rangle \varphi(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy).
\]

Consequently, \( R_t \) is compact on \( L^2(H; \mu) \) for all \( t > 0 \).

**Proof.** Let \( \varphi \in C_b(H) \), then by (1.4) and Hölder’s estimate, we have

\[
|R_t \varphi(x)|^2 \leq \int_H \varphi^2(e^{tA}x + y)\mathcal{N}(0, Q_t)(dy) = R_t(\varphi^2)(x).
\]
Using the invariance of $\mu$, it follows that
\[
\int_H |R_t \varphi(x)|^2 \mu(dx) \leq \int_H |\varphi(x)|^2 \mu(dx),
\]
that proves (i).

(ii) Notice first that, in view of (2.15), for all $h \in H$ we have
\[
R_t e^{i(h \cdot \gamma)}(x) = e^{i(e^{tA} h \cdot x) - \frac{1}{2} \langle Q_h h, h \rangle}.
\]
Thus, for any $t > 0$, $R_t$ maps $\mathcal{E}$ into itself. Since clearly $\mathcal{E} \subset D(\mathcal{A})$, we have that $\mathcal{E}$ is a core for $\mathcal{A}$, see [11, Theorem 1.9].

Let us prove (iii). Let $\varphi \in C_b(H)$ and $h \in H$. By (1.8) and the Hölder inequality we have
\[
|\langle DR_t \varphi(x), h \rangle|^2 \leq \\
\leq \int_H |\langle \Gamma(t) h, Q^{-1/2} \rangle|^2 \int_H |\varphi(e^{tA} x + y)|^2 N(0, Q_t)(dy) \\
= |\Gamma(t) h|^2 R_t(\varphi^2)(x).
\]
Integrating on $x$ and using the invariance of $\mu$, we find
\[
\int_H |\langle DR_t \varphi(x), h \rangle|^2 \mu(dx) \leq |\Gamma(t) h|^2 \int_H |\varphi(x)|^2 \mu(dx).
\]
Setting $h = e_k$, $k \in \mathbb{N}$, summing up on $k$, and recalling that by Proposition 1.2-(i), $\Gamma(t) \in \mathcal{L}_2(H)$, we obtain
\[
\int_H |DR_t \varphi(x)|^2 \mu(dx) \leq \text{Tr} \left[ \Gamma(t) \Gamma^*(t) \right] \int_H |\varphi(x)|^2 \mu(dx).
\]
The conclusion follows from the density of $C_b(H)$ in $L^2(H; \mu)$. \hfill \Box

The following propositions were proved in [12], see also [1]. Before stating it we need some preliminary results.

LEMMMA 2.9. For any $\varphi, \psi \in \mathcal{E}$ the following identity holds.
\[
\int_H (A \varphi)(x) \psi(x) \mu(dx) = \int_H \langle QD \psi(x), A^* D \varphi(x) \rangle \mu(dx). \tag{2.18}
\]
Proof. It is enough to prove (2.18) for
\[ \varphi(x) = e^{i(x, \alpha)}, \psi(x) = e^{i(x, \beta)}, \quad x \in H, \quad \alpha, \beta \in D(A^*). \]

In this case we have, by a simple computation,
\[
\int_H (A\varphi)(x)\psi(x)\mu(dx) = -\left(\langle A^*\alpha, Q(\alpha - \beta) \rangle + \frac{1}{2}|\alpha|^2 \right) e^{-\frac{1}{2}(Q(\alpha-\beta), \alpha-\beta)},
\]
and
\[
\int_H \langle QD\psi(x), A^*D\varphi(x) \rangle \mu(dx) = \langle A^*\alpha, Q\beta \rangle e^{-\frac{1}{2}(Q(\alpha-\beta), \alpha-\beta)}. \quad (2.20)
\]

Taking into account (2.19) and (2.20), we see that equality (2.18) is equivalent to
\[ 2\langle A^*\alpha, Q\alpha \rangle + |\alpha|^2 = 0, \]
that coincides with Lyapunov equation (1.6). \( \square \)

The lemma yields now the result

**Proposition 2.10.** For any \( \varphi \in D(A) \) and any \( \psi \in W^{1,2}(H; \mu) \) the following identity holds.
\[
\int_H (A\varphi)(x)\psi(x)\mu(dx) = \int_H \langle QD\psi(x), A^*D\varphi(x) \rangle \mu(dx). \quad (2.21)
\]

Finally, taking \( \phi = \psi \), and using again the Lyapunov equation we have

**Proposition 2.11.** Assume that Hypothesis 1.1 holds. Then for any \( \varphi \in D(A) \) one has \( \varphi \in W^{1,2}(H, \mu) \) and the following identity holds.
\[
\int_H (A\varphi)(x)\varphi(x)\mu(dx) = -\frac{1}{2} \int_H |D\varphi(x)|^2 \mu(dx). \quad (2.22)
\]

The following corollary is an immediate consequence of Proposition 2.11.
Corollary 2.12. Assume that Hypothesis 1.1 holds. Then for any \( \varepsilon > 0 \) one has
\[
\int_H |D\varphi(x)|^2 \mu(dx) \leq \varepsilon \int_H |A\varphi(x)|^2 \mu(dx) + 4 \int_H |\varphi(x)|^2 \mu(dx).
\]
(2.23)

Remark 2.13. If \( M = 1 \) (6), one can prove that the semigroup \( R_t t \geq 0 \) is analytic in \( L^2(H; \mu) \), see [12], [9].

3. Characterization of \( D(A) \)

In this section we want to characterize the domain of \( A \). From now on we shall assume that
\[
\{e_k\} \subset D(A).
\]
(3.1)

Then we set
\[
A_{h,k} = \langle Ae_k, e_h \rangle, \quad h, k \in \mathbb{N},
\]
and we write \( A \) on \( \mathcal{E} \) as
\[
(A\varphi)(x) = \frac{1}{2} \sum_{h=1}^{\infty} D_h^2 \varphi(x) + \sum_{h,k=1}^{\infty} A_{h,k} x_k D_h \varphi(x), \quad \varphi \in \mathcal{E}.
\]
(3.2)

We start with a basic identity.

Proposition 3.1. Assume that Hypotheses 1.1 and (3.1) hold. Let \( \varphi \in \mathcal{E} \) and let \( f = A\varphi \). Then the following identity holds:
\[
\frac{1}{2} \int_H \|D^2 \varphi(x)\|_{L^2(H)}^2 \mu(dx) - \int_H \langle D\varphi(x), A^* D\varphi(x) \rangle \mu(dx)
\]
\[
= 2 \int_H |f(x)|^2 \mu(dx) - 2 \int_H f(x) \langle Ax + \frac{1}{2} Q^{-1} x, D\varphi(x) \rangle \mu(dx).
\]
(3.3)

\( ^6 \)M is the constant in Hypothesis 1.1
Proof. By differentiating (3.2) with respect to $x_j$, we obtain

$$A(D_j \varphi)(x) + \sum_{h=1}^{\infty} A_{h,j} D_h \varphi(x) = D_j f(x).$$

Multiplying both sides by $D_j \varphi(x)$ and integrating with respect to $\mu$ it follows

$$\int_H A(D_j \varphi) D_j \varphi \mu(dx) + \sum_{h=1}^{\infty} \int_H A_{h,j} D_h \varphi D_j \varphi \mu(dx) = \int_H D_j \varphi D_j f(x) \mu(dx).$$

Recalling (2.22) we see that the above equality is equivalent to

$$\frac{1}{2} \int_H |D_D \varphi(x)|^2 \mu(dx) - \sum_{h=1}^{\infty} \int_H A_{h,j} D_h \varphi(x) D_j \varphi(x) \mu(dx) = - \int_H D_j \varphi(x) D_j f(x) \mu(dx).$$

By (2.1) we get

$$\frac{1}{2} \int_H |D_D \varphi(x)|^2 \mu(dx) - \sum_{h=1}^{\infty} \int_H A_{h,j} D_h \varphi(x) D_j \varphi(x) \mu(dx) = \int_H f(x) D_j^2 \varphi(x) \mu(dx) - \int_H \frac{x_j}{\lambda_j} f(x) D_j \varphi(x) \mu(dx).$$

Summing up on $j$ we find

$$\frac{1}{2} \int_H \|D^2 \varphi(x)\|_{L^2(H)}^2 \mu(dx) - \int_H \langle D \varphi(x), A^* D \varphi(x) \rangle \mu(dx) = \int_H f(x) \left\{ \text{Tr} \left[ D^2 \varphi(x) \right] - \langle Q^{-1} x, D \varphi(x) \rangle \right\} \mu(dx),$$

and the conclusion follows. □
In order to characterize $D(A)$ we need some further assumptions.

**Hypothesis 3.1.**

(i) $D(A) \cap Q(H)$ is dense in $H$ and the linear operator

\[
\begin{align*}
D(K) &= D(A) \cap Q(H), \\
Kx &= Ax + \frac{1}{2} Q^{-1}x, \quad x \in D(K),
\end{align*}
\]

is bounded in $H$.

(ii) There exists $\eta > 0$ such that

\[
\langle Ax, x \rangle \leq -\eta |x|^2, \quad x \in D(A).
\]

If Hypothesis 3.1 holds, we shall denote by $K$ the unique extension of the operator $K$ to $H$. Notice that if Hypothesis 1.1 holds with $M = 1$, then (3.5) holds with $\eta = \omega$.

In the following we denote by $H_A$ the Banach space obtained by taking the completion of $D(A)$ with respect to the norm

\[
|x|_{H_A}^2 = -\langle Ax, x \rangle, \quad x \in D(A).
\]

**Theorem 3.2.** Assume that Hypotheses 1.1, 3.1 and (3.1) hold. Let $A$ be the infinitesimal generator of the semigroup $R_t$, $t \geq 0$, defined by (2.15). Then we have

\[
D(A) = \left\{ \varphi \in W^{2,2}(H; \mu) : \\
|D\varphi(x)| \in H_A, \mu \text{ a.e., } |D\varphi(\cdot)|_{H_A} \in L^2(H; \mu) \right\}
\]

(3.6)

**Proof.** We first prove that

\[
D(A) \subset \left\{ \varphi \in W^{2,2}(H; \mu) : \\
D\varphi(x) \in H_A, \mu \text{ a.e. } |D\varphi(\cdot)|_{H_A} \in L^2(H; \mu) \right\}.
\]

(3.7)
For this, recalling that $D(A) \subset W^{1,2}(H; \mu)$ by Proposition 2.11, it suffices to prove that for any $\varphi \in D(A)$ the following estimate holds

\[
\frac{1}{4} \int_H \|D^2 \varphi(x)\|_{L^2(H)}^2 \mu(\,dx) + \int_H |D \varphi(x)|^2_{H, \mu}(\,dx) \\
\leq 2(1 + 128 \|K\|^2 \text{ Tr} [Q^2]) \int_H |f(x)|^2 \mu(\,dx) + \\
+ \frac{\text{ Tr } Q}{32 \text{ Tr } [Q^2]} \int_H |D \varphi(x)|^2 \mu(\,dx).
\]  

(3.8)

Since $\mathcal{E}$ is a core for $A$, it is enough to prove (3.8) for all $\varphi \in \mathcal{E}$. Let $a > 0$ be a positive number to be fixed later. By (3.3) it follows

\[
\frac{1}{2} \int_H \|D^2 \varphi(x)\|^2_{L^2(H)} \mu(\,dx) + \int_H |D \varphi(x)|^2_{H,\mu}(\,dx) \leq \\
\leq (2 + 4a) \int_H |f(x)|^2 \mu(\,dx) + \frac{\|K\|^2}{a} \int_H |x|^2 |D \varphi(x)|^2 \mu(\,dx).
\]

Taking into account (2.6) we find

\[
\frac{1}{2} \int_H \|D^2 \varphi(x)\|_{L^2(H)}^2 \mu(\,dx) + \int_H |D \varphi(x)|^2_{H,\mu}(\,dx) \leq \\
\leq (2 + 4a) \int_H |f(x)|^2 \mu(\,dx) + \\
+ 2 \frac{\|K\|^2 \text{ Tr } Q}{a} \int_H |D \varphi(x)|^2 \mu(\,dx) + \\
+ 16 \frac{\|K\|^2 \text{ Tr } [Q^2]}{a} \int_H \|D^2 \varphi(x)\|^2_{L^2(H)} \mu(\,dx).
\]

Choosing finally $a$ such that

\[a = 64 \|K\|^2 \text{ Tr } Q^2 \]

(3.8) and consequently (3.7) follows.

We now prove that

\[D(A) \supset \left\{ \varphi \in W^{2,2}(H; \mu) : \right. \]

\[D \varphi(x) \in H_A, \ \mu \text{ a.e., } |D \varphi(\cdot)|_{H_A} \in L^2(H; \mu) \}.
\]

(3.9)
Let $\varphi \in \mathcal{E}$ and set
\[ L = \frac{1}{2} \int_H \| D^2 \varphi(x) \|^2_{L_2(H)} \mu(dx) + \int_H |D\varphi(x)|^2_{H,A} \mu(dx), \]
then from (3.3) we have
\[ 2 \int_H |A\varphi(x)|^2 \mu(dx) \leq L + 2\|K\| \int_H |A\varphi(x)| \cdot |x| \cdot |D\varphi(x)| \mu(dx) \]
\[ \leq L + \int_H |A\varphi(x)|^2 \mu(dx) + 4\|K\|^2 \int_H |x|^2 |D\varphi(x)|^2 \mu(dx), \]
and so
\[ \int_H |A\varphi(x)|^2 \mu(dx) \leq L + 4\|K\|^2 \int_H |x|^2 |D\varphi(x)|^2 \mu(dx). \]
By (2.6) it follows
\[ \int_H |A\varphi(x)|^2 \mu(dx) \leq L + 8 \text{Tr } Q \|K\|^2 \int_H |D\varphi(x)|^2 \mu(dx) \]
\[ + \ 64 \left(\text{Tr } Q^2\right) \int_H \| D^2 \varphi(x) \|_{L_2(H)}^2 \mu(dx), \]
Taking into account (2.23), for any $\varepsilon > 0$ we have
\[ \int_H |A\varphi(x)|^2 \mu(dx) \leq L + 8\varepsilon\|K\|^2 \text{Tr } Q \int_H |A\varphi(x)|^2 \mu(dx) \]
\[ + \frac{32\|K\|^2 \text{Tr } Q}{\varepsilon} \int_H |\varphi(x)|^2 \mu(dx). \]
Now choosing
\[ \varepsilon = \frac{1}{16 \text{Tr } Q \|K\|^2}, \]
we have
\[ \frac{1}{2} \int_H |A\varphi(x)|^2 \mu(dx) \leq L + 512 \left(\text{Tr } Q\right)^2 \|K\|^4 \int_H |\varphi(x)|^2 \mu(dx) \]
\[ + \ 64 \text{Tr } [Q^2] \int_H \| D^2 \varphi(x) \|_{L_2(H)}^2 \mu(dx). \]
(3.10)
From (3.10) and the density of \( \mathcal{E} \) it follows that if \( \varphi \) is such that \( L \) is bounded, then \( \varphi \in D(A) \). This proves the inclusion (3.9).

The proof is complete. \( \Box \)

**Remark 3.3.** It is well known that when \( A \) is a variational operator and \( D(A) = D(A^*) \), then \( H_A \) coincides with \( D_A \left( \frac{1}{2}, 2 \right) \), the real interpolation space consisting of all \( x \in H \) such that

\[
|x|_{D_A(\frac{1}{2}, 2)}^2 := \int_0^\infty |A e^{tA} x|^2 dt < +\infty,
\]

see [13]. Thus in this case, if Hypotheses 1.1, 3.1 and (3.1) hold, then the domain of \( A \) is given by

\[
D(A) = \left\{ \varphi \in W^{2,2}(H; \mu) : \ D\varphi(x) \in D_A \left( \frac{1}{2}, 2 \right), \ \mu \text{ a.e.,} \right\}.
\]

**Remark 3.4.** Assume that Hypotheses 1.1, and (3.1) hold and that \( A \) is self-adjoint. In this case from (1.5) we have

\[
Q x = \int_0^{+\infty} e^{2A t} x dt = -\frac{1}{2} A^{-1} x, \ x \in H,
\]

that obviously implies \( K = 0 \). Consequently Hypotheses 3.1 holds and, from Theorem 3.2 it follows that

\[
D(A) = \left\{ \varphi \in W^{2,2}(H; \mu) : \ D\varphi(x) \in D( (-A)^{1/2} ), \ \mu \text{ a.e.}, \right\}.
\]

**Remark 3.5.** Assume that \( H \) is finite-dimensional and that \( A \) is of negative type. Then Hypotheses 1.1, 3.1 and (3.1) obviously hold. Then from Theorem 3.2 it follows that

\[
D(A) = W^{2,2}(H; \mu).
\]

This result was earlier proved by a different method based on interpolation, by A. Lunardi, see [16].
4. Perturbation results

We assume here that \( A \) is self-adjoint and fulfills Hypotheses 1.1 and 3.1. We will be concerned with some perturbations of the operator \( \mathcal{A} \), the infinitesimal generator of the semigroup \( R_t, t \geq 0 \), in \( L^2(H; \mu) \), defined in §2. We recall that \( \mathcal{A} \) is \( m \)-dissipative and that the domain of \( \mathcal{A} \) is defined by (3.12). Then the graph norm of \( \mathcal{A} \) can be defined as

\[
\| \varphi \|_{D(\mathcal{A})}^2 = \| \varphi \|^2_{W^{2,2}(H; \mu)} + \| (-A)^{-1/2} D \varphi \|^2_{L^2(H; \mu)}, \quad \varphi \in D(\mathcal{A}).
\]  

(4.1)

4.1 Relatively bounded perturbations

Let \( F : H \rightarrow H \) be a Borel mapping such that

HYPOTHESIS 4.1. \((-A)^{-1/2} F \) is bounded.

We set

\[
a = \sup \text{ ess } \{ \| (-A)^{-1/2} F(x) \| : x \in H \}.
\]

Now we define a mapping \( \mathcal{F} \) in \( L^2(H; \mu) \) by setting

\[
\mathcal{F} \varphi(x) = \langle F(x), D \varphi(x) \rangle = - \langle (-A)^{-1/2} F(x), (-A)^{1/2} D \varphi(x) \rangle,
\]

\( \forall \varphi \in D(\mathcal{A}). \)

(4.2)

The following proposition concerns the operator \( \mathcal{A} + \mathcal{F} \), defined in \( D(\mathcal{A}) \).

PROPOSITION 4.1. Assume that Hypotheses 1.1, 3.1, and 4.1 hold, and let \( \mathcal{F} \) be defined by (4.2).

(i) If \( a < 1 \) then \( \mathcal{A} + \mathcal{F} \) is \( m \)-dissipative in \( L^2(H; \mu) \).

(ii) If \( a = 1 \) then \( \mathcal{A} + \mathcal{F} \) is closable and its closure is \( m \)-dissipative in \( L^2(H; \mu) \).
Proof. We first note that by (3.13) we have $D(\mathcal{F}) \subset D(\mathcal{A})$. Moreover for any $\varphi \in D(\mathcal{A})$ we have

\[
\|\mathcal{F}\varphi\|_{L^2(H;\mu)}^2 = \int_H |\langle F(x), D\varphi(x) \rangle|^2 \mu(dx)
\]

\[
= \int_H |\langle (-A)^{-1/2}F(x), (-A)^{1/2}D\varphi(x) \rangle|^2 \mu(dx)
\]

\[
\leq a^2 \int_H |\langle (-A)^{1/2}D\varphi(x) \rangle|^2 \mu(dx) \leq a^2 \|\mathcal{A}\varphi\|_{L^2(H;\mu)}^2.
\]

This implies that $\mathcal{F}$ is relatively bounded with respect to $\mathcal{A}$. By a well-known perturbation result, see e.g. [18], the conclusion follows.

\[\Box\]

Example 4.2. We take $H = L^2([0, 2\pi])$ and define a linear operator $A$ on $H$ by setting

\[
\begin{aligned}
D(A) &= \{ x \in H^2(0, 2\pi) : x(0) = x(2\pi), D_\xi x(0) = D_\xi x(2\pi) \}, \\
A x(\xi) &= D_\xi^2 x(\xi) - x(\xi), \quad \xi \in [0, 2\pi], \quad x \in D(A). 
\end{aligned}
\]

(4.3)

$A$ is clearly self-adjoint and fulfills Hypothesis 1.1 with $M = 1$ and $\omega = 1$, and Hypothesis 3.1, since the eigenvectors of $A$ are given by

\[
e_\xi(\xi) = \frac{1}{2\pi} e^{ik\xi}, \quad \xi \in [0, 2\pi], \quad k \in \mathbb{Z}.
\]

Let $L$ be a positive number, and set

\[
F(x)(\xi) = L \frac{d}{d\xi} \sin x(\xi), \quad \xi \in [0, 2\pi].
\]

(4.4)

Then

\[
(-A)^{1/2}F(x)(\xi) = L \sin x(\xi), \quad \xi \in [0, 2\pi].
\]

so that Hypothesis 4.1 holds. Thus by Proposition 4.1 it follows that if $L < 1$, then the operator $\mathcal{B}$:

\[
\mathcal{B}\varphi(x)(\xi) := A\varphi(x) + k \left( \frac{d}{d\xi} \sin x(\xi), D\varphi(x) \right), \quad \varphi \in D(A)
\]

is $m$-dissipative in $L^2(H;\mu)$, whereas if $L = 1$ then $\mathcal{B}$ is closable and its closure is $m$-dissipative in $L^2(H;\mu)$. 
4.2 Perturbation by a potential

We are given a nonnegative Borel function \( V : H \to \mathbb{R} \), and we define a mapping \( \mathcal{V} \) in \( L^2(H; \mu) \) by setting

\[
D(\mathcal{V}) = \{ \varphi \in L^2(H; \mu) : V \varphi \in L^2(H; \mu) \}
\]

\[
\mathcal{V} \varphi(x) = -V(x) \varphi(x), \ \forall \varphi \in D(\mathcal{V}).
\]

Next proposition concerns the operator \( A + \mathcal{V} \) with domain \( D(A) \).

**Proposition 4.3.** Let \( \mathcal{V} \) be defined by (4.5), and assume that there are numbers \( a > 0 \) and \( \beta \in [0, 1] \) such that

\[
V(x) \leq a|x|^{1+\beta}, \ x \in H.
\]

Then \( A + \mathcal{V} \) is self-adjoint in \( L^2(H; \mu) \).

**Proof.** Let \( \varepsilon > 0 \) to be chosen later, and let \( C(\varepsilon, \beta) > 0 \) such that

\[
a^2|x|^{2+2\beta} \leq \varepsilon |x|^\beta + C(\varepsilon, \beta), \ x \in H.
\]

Let \( \varphi \in D(A) \), then we have

\[
\int_H V^2(x)\varphi^2(x)\mu(dx) \leq \varepsilon \int_H |x|^4\varphi^2(x)\mu(dx) + C(\varepsilon, \beta) \int_H \varphi^2(x)\mu(dx).
\]

Consequently, in view of Proposition 2.7, we have \( \varphi \in D(\mathcal{V}) \) and

\[
\int_H V^2(x)\varphi^2(x)\mu(dx) \leq [32\varepsilon \text{ Tr } Q^2 + \varepsilon(1 + 2 \text{ Tr } Q)^2 + C(\varepsilon, \beta)] \int_H \varphi^2(x)\mu(dx) + \\
+ \varepsilon(48 \text{ Tr } [Q^2](1 + 2 \text{ Tr } Q) + 512 \text{ (Tr } [Q^2])^2)\|A\varphi\|^2_{L^2(\mu; H)}.
\]

So, by choosing \( \varepsilon \) sufficiently small, we see that \( \mathcal{V} \) is relatively bounded with respect to \( A \), and the conclusion follows by the quoted result in [18]. \[ \square \]

**Remark 4.4.** If (4.6) is fulfilled with \( \beta = 1 \), then the argument above works with \( a \) sufficiently small.
References

