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On the Sum of Generators of Analytic Semigroups

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Summary. - *Let A and B be generators of analytic semigroups in a Banach space. Under some conditions on the commutator of the resolvents of A and B, already considered in the literature and not implying relative boundedness, we prove that the closure of $A+B$ (or a proper extension of it) also generates an analytic semigroup, and we characterize interpolation spaces related to it.*

1. Introduction

The present work has its origin in the paper [5] of Giuseppe Da Prato and Pierre Grisvard. In this paper the authors develop an investigation of (mainly) spectral properties of the sum of two linear operators in a Banach space, as a general framework for several problems on evolution equations. Their approach raised the interest of several researchers and gave origin to a variety of applications to different fields: see, among others, [14], [15], [13], [6], [17], [2], [1], [4].

When I started my research activity in this field, I had the fortune to meet personally Pierre Grisvard. I could appreciate his attitude to discuss and share mathematical ideas, and I received encouragement and constant help to enter this subject. Grisvard's interest in it lasted until his latest years when, in November 1993, he gave a course

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in Blaubeuren on applications of sums of operators to differential problems in polygonal domains.

Here we want to show how these techniques can be applied to produce results in perturbation theory for generators of analytic semigroups, and to study some related questions of interpolation theory. We continue, under different assumptions, the investigations carried out in [9].

We say that a linear operator A in a complex Banach space X, with (not necessarily dense) domain D_A, generates an analytic semigroup if its resolvent set contains a sector of the complex plane $\Sigma = \{z \in \mathbb{C} : z \neq \omega, |\arg(z-\omega)| < \pi - \vartheta\}$, for some $\omega \in \mathbb{R}, \vartheta \in (0, \pi/2)$, and its resolvent family satisfies the estimate $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq c|\lambda - \omega|^{-1}$ for some $c > 0$ and for every $\lambda \in \Sigma$. Under these conditions one can define in a usual way a semigroup $e^{tA} \in \mathcal{L}(X)$, not necessarily strongly continuous at $t = 0$ (see for instance [18]). We are interested in the additive perturbation problem: assuming that A is a generator and B is another linear operator in X, one looks for conditions implying that $A + B$, or some operator intrinsically related to $A + B$, is still a generator. We refer the reader to [16] and its bibliography for an exposition of the classical results, and to [12] for a more general study of the subject.

In the paper [5] the following assumptions are presented. In contrast to the classical approach to perturbative theory, they relate to commutativity properties of the operators A and B and do not require, in particular, the relative boundedness condition $D_B \subset D_A$.

Hypothesis *H.I. A and B are linear operators in X with domains D_A, D_B and there exist $\vartheta \in (0, \pi/2), c > 0$ such that*

$$\|(z-A)^{-1}\|_{\mathcal{L}(X)} + \|(z-B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{|z|},$$

for every $z \in \mathbb{C}$ with $|\arg(z)| < \pi - \vartheta$.

Hypothesis *H.IIa. We have $(A-v)^{-1}D_B \subset D_B$ and there exist $c > 0$, α, β such that $-1 \leq \alpha < \beta \leq 1$,*

$$\|[B; (A-v)^{-1}](B-z)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{|v|^{1-\alpha}|z|^{\beta}},$$

for every $z, v \in \mathbb{C}$ with $|\arg(z)|$, $|\arg(v)| < \pi - \vartheta$.

$[P;Q]$ denotes the commutator $PQ-QP$ of two linear operators in X. (In this section we do not state hypotheses in full generality: see below for precise statements.) Remark that $H.I$ simply states that *both* A and B generate bounded analytic semigroups.

Under Hypotheses $H.I$ and $H.IIa$, it has been shown in [9] that $A+B$, or some extension of it, also generates an analytic semigroup. Here we want to replace $H.IIa$ with the following

Hypothesis *H.IIb. There exist* $c>0$, α, β *such that* $0 \le \alpha < \beta \le 1$,

$$\|A(A-v)^{-1}[A^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \le \frac{c}{|v|^{1-\alpha}|z|^{1+\beta}},$$

for every $z,v \in \mathbb{C}$ *with* $|\arg(z)|,\ |\arg(v)| < \pi - \vartheta$.

Hypothesis $H.IIb$ is essentially taken from [14], where it is presented as an alternative (and independent) assumption to $H.IIa$. Such an assumption is used by the authors to find solutions (in various senses) of the equation

$$(A+B-\lambda)x = y$$

with x (unknown) and y (datum) in X, and $\lambda > 0$ sufficiently large. Their treatment, together with many examples and applications presented in [15], shows that it is fairly natural.

An intuitive meaning for assumptions such as $H.IIa$ or $H.IIb$ is the following. Consider the commutativity condition

Hypothesis *H.III*. $[(A-v)^{-1};(B-z)^{-1}]=0$.

If $H.III$ holds, the semigroups e^{tA} and e^{tB} commute, so that their product is still a semigroup. So in this case the simplest way to define a semigroup generated by $A+B$ is simply to identify it with $e^{tA}e^{tB}$ (a more precise characterization of the generator of this semigroup follows from our results below). So $H.IIa$ or $H.IIb$ can be considered as the statement that the commutator of A and B, though not zero, is suitably "small", so that our situation is a perturbation (and an extension) of the commutative case.

In the following, we give a short account of our results, sometimes omitting some details. First we can prove:

(i) Assume that H.I, H.IIb hold and suppose that $D_A \cap D_B$ is dense in X. Then $A+B$ is closable and its closure $\overline{A+B}$ generates an analytic semigroup.

In [5] the reader can find additional conditions implying that $A+B$ is closed; related closedness results are discussed in [2], [6]. If we drop the density assumption on $D_A \cap D_B$, we meet a difficulty, since we do not know whether $A+B$ is still closable. The closability of $A+B$ has been proved only when the stronger condition $H.III$ holds (see [1], [5], [13]). Following [5], the way we choose to overcome this difficulty is to identify $A+B$ with its graph in $X \times X$, and then take its closure in $X \times X$, which is always possible. It remains to prove that the object we construct in such a way still generates a semigroup in a suitable sense. We call *graph* any subspace of $X \times X$; it can be naturally identified with a multivalued linear operator in X. Several definitions of generation are available for multivalued operators. We adopt the definition of *analytic* semigroups generated by *linear subspaces* of $X \times X$ that can be found in [7]: see Subsection 2.3 below. Then we can prove:

(ii) Assume that H.I, H.IIb hold and suppose that D_A or D_B is dense in X. Then $\overline{A+B}$ generates an analytic semigroup, in the sense of graphs.

It is of some interest, in view of certain applications ([1], [13]), to drop the density assumption also for D_A and D_B. Then we meet situations where $\overline{A+B}$ is not a generator and has even empty resolvent set ([13]). Extending results of [1], [13], that only deal with hypothesis $H.III$, we can prove:

(iii) Assume H.I, H.IIb. Then there exists a graph $(A+B)^{\sim}$ that extends $A+B$ and generates an analytic semigroup.

More precise information about $(A+B)^{\sim}$ is also obtained (see Corollaries 5.5-5.9).

Finally, we come to some problem concerning interpolation theory. As a technical tool, we use several times the characterization of the real interpolation spaces $(X, D_A)_{\vartheta,p}$, $(X, D_B)_{\vartheta,p}$, given by Pierre Grisvard ([10]): see (2.1), (2.2) below. Another problem of interest to him was to investigate the validity of the equality

$$(X, D_A \cap D_B)_{\vartheta,p} = (X, D_A)_{\vartheta,p} \cap (X, D_B)_{\vartheta,p}$$

(see [11]). In the present framework, we will prove the following:

(iv) Assume H.I, H.IIb. Then for $\vartheta \in (0,1)$, $p \in [1,\infty]$*:*

$$D_A \cap D_B \subset D_{(A+B)^\sim} \subset (X, D_A)_{\vartheta,p} \cap (X, D_B)_{\vartheta,p} \ .$$

Moreover, for $\vartheta > 0$ *sufficiently small, we have*

$$(X, D_{(A+B)^\sim})_{\vartheta,p} = (X, D_A)_{\vartheta,p} \cap (X, D_B)_{\vartheta,p} \quad p \in [1,\infty]. \tag{1.1}$$

For this, we need to define a suitable norm in the domain $D_{(A+B)^\sim}$, and to extend Grisvard's characterization: see Proposition 2.3 below.

Now we sketch the proof of the main results and we present the plan of the paper. We will show that $(A+B)^\sim$ exists as the limit (in some sense), as $n \to \infty$, $m \to \infty$, of $A_m + B_n$, where $A_m, B_n \in \mathcal{L}(X)$ are the Yosida approximations of A and B.

Step 1. We prove the existence in $\mathcal{L}(X)$ of

$$\lim_{m\to\infty} \lim_{n\to\infty} (\lambda_0 - A_m - B_n)^{-1}, \tag{1.2}$$

for some $\lambda_0 > 0$ large.

Step 2. We prove an estimate (uniform with respect to m and n)

$$\|e^{t(A_m+B_n)}\|_{\mathcal{L}(X)} \leq \overline{M} e^{\overline{\omega} t}, \quad t \geq 0. \tag{1.3}$$

Step 3. We show that (1.2) and (1.3) imply the existence of $(A+B)^\sim$ and the convergence of $(\lambda - A_m - B_n)^{-1}$ to $(\lambda - (A+B)^\sim)^{-1}$, as $n \to \infty$, $m \to \infty$.

Step 3 follows from an extension of Trotter's theorem to semigroups generated by graphs: see Section 2., where notation and other preliminary facts on graphs are recalled. Step 1 is a consequence of some extensions of the results of [14] and [9]: Section 3. is devoted to recalling briefly all the material we need. Section 4. is devoted to proving Step 2: the estimate (1.3) is Proposition 4.4. In Section 5. we prove the main results announced earlier, whereas in Section 6. we prove (1.1). Finally, the appendix is devoted to the proof of a technical lemma.

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2. Notation and preliminaries

2.1 Notation

We define

$$\Sigma_\vartheta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \pi - \vartheta\}, \qquad \text{for } \vartheta \in [0,\pi), \omega \in \mathbb{R};$$

$$\Sigma'_{\vartheta,\omega} := \{z \in \mathbb{C} : z \neq \omega, |\arg(z-\omega)| < \vartheta\}, \quad \text{for } \vartheta \in (0,\pi]$$

(the argument $\arg(z)$ is assumed to take values in $(-\pi,\pi]$). We denote by X a complex Banach space, by $\mathcal{L}(X)$ the space of linear bounded operators on X. $\| \ \|$ denotes the norm in X and, if no confusion arises, also in $\mathcal{L}(X)$. To denote the norm in another Banach space Y we write $\| \ \|_Y$. $\mathcal{L}(X,Y)$ is the space of linear bounded operators from X to Y. Let C be a linear (in general unbounded) operator in X; we denote by D_C its domain, by $\sigma(C)$ its spectrum, by $\rho(C)$ its resolvent set, by $\mathcal{R}(C)$ its image, by $\overline{C}$ its closure (if defined). 1 also denotes the identity operator; accordingly, for any scalar λ, the operator $\lambda 1$ is denoted by λ and the resolvent family of C by $(\lambda - C)^{-1}$, for $\lambda \in \rho(C)$.

Given operators A and B in X, the operators $A+B, A^n$ etc. always have their natural domains and definitions; for example

$$D_{A+B} = D_A \cap D_B, \quad (A+B)x = Ax + Bx \text{ for } x \in D_{A+B};$$

$$D_{A^n} = \{x \in X : x \in D_{A^{(n-1)}}, A^{(n-1)}x \in D_A\}, \quad A^n x = AA^{(n-1)}x.$$

We set $[A;B] := AB - BA$, with natural domain.

For integers $m, n \geq 1$ we set $A_m := mA(m-A)^{-1}$, $B_n := nB(n-B)^{-1}$; these operators are called Yosida approximations of A and B.

If γ is a path in the complex plane, we shorten the symbol $\dfrac{1}{2\pi i}\displaystyle\int_\gamma$ by writing simply $\displaystyle⨍_\gamma$.

2.2 Interpolation spaces

Let A be a closed linear operator in X. We endow D_A, D_{A^2} with the norms

$$\|x\|_{D_A} := \|x\| + \|Ax\|, \quad \|x\|_{D_{A^2}} := \|x\| + \|Ax\| + \|A^2 x\|.$$

One can define the real interpolation spaces

$$(X, D_A)_{\vartheta,p} \quad \text{and} \quad (X, D_{A^2})_{\vartheta,p}\ ,$$

for $\vartheta \in (0,1)$, $p \in [1,\infty]$ (see e.g. [19]). We will denote them by $D_A(\vartheta,p)$, $D_{A^2}(\vartheta,p)$, respectively. We have the continuous inclusions

$$D_{A^2}(\frac{1}{2},\infty) \subset D_A(\eta,r) \subset D_A(\vartheta,p) \subset D_A(\vartheta,q) \subset D_A(\mu,s) \subset X$$

for $0<\mu<\vartheta<\eta<1$, $1 \le p \le q \le \infty$, $1 \le r \le \infty$, $1 \le s \le \infty$.

Assume in addition that $\rho(A) \supset (0,\infty)$ and there exists $c_A > 0$ such that $\|(\lambda - A)^{-1}\| \le c_A \lambda^{-1}$, for every $\lambda > 0$. The following properties are proved in [10].

(*i*) Let $p \in [1,\infty)$ and $\overline{\lambda} > 0$. Then x belongs to $D_A(\vartheta,p)$ if and only if

$$\|x\|'_{D_A(\vartheta,p)} := \|x\| + \left(\int_{\overline{\lambda}}^{\infty} \|\lambda^{\vartheta} A(A-\lambda)^{-1}x\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

(*ii*) Let $p = \infty$ and $\overline{\lambda} > 0$. Then x belongs to $D_A(\vartheta,\infty)$ if and only if

$$\|x\|'_{D_A(\vartheta,\infty)} := \|x\| + \sup_{\lambda > \overline{\lambda}} \|\lambda^{\vartheta} A(A-\lambda)^{-1}x\| < \infty. \tag{2.2}$$

Moreover, $\|x\|'_{D_A(\vartheta,p)}$ and $\|x\|_{D_A(\vartheta,p)}$ are equivalent norms, $\vartheta \in (0,1)$, $p \in [1,\infty]$.

If one also assumes that there exists $\vartheta_A \in (0,\pi)$ such that $\rho(A) \supset \Sigma_{\vartheta_A}$ and $\|(\lambda - A)^{-1}\| \le c_A|\lambda|^{-1}$, for every $\lambda \in \Sigma_{\vartheta_A}$, then it follows easily from (2.2) and the resolvent identity that for every $\vartheta \in (0,1)$ there exists $c > 0$ such that

$$\|A(A-z)^{-1}y\| \le c|z|^{-\vartheta}\|y\|_{D_A(\vartheta,\infty)} \qquad z \in \Sigma_{\vartheta_A}. \tag{2.3}$$

2.3 Graphs

Here we recall some definitions and basic properties about *graphs* (sometimes called *multivalued linear operators*, or *linear relations*).

We call *graph* in X a *subspace* of $X \times X$. A graph A can be identified with a multivalued function $x \mapsto Ax := \{y \in X : (x, y) \in A\}$ defined on X (possibly $Ax = \emptyset$). Every linear operator in X will be identified with a graph in a natural way. The *closure* $\overline{A}$ of a graph A is its closure in $X \times X$ (with the usual topology); it is still a graph. A is said to be *closed* if $A = \overline{A}$. Define

$$A - \lambda := \{(x, y) : (x, y + \lambda x) \in A\}; \qquad A^{-1} := \{(x, y) : (y, x) \in A\}$$

for any scalar λ. The *resolvent set* $\rho(A)$ of A is the set of all scalars such that $(\lambda - A)^{-1}$ is the graph of an everywhere defined bounded linear operator in X; we identify them and we simply write $(\lambda - A)^{-1} \in \mathcal{L}(X)$; $(\lambda - A)^{-1}$ will be called *resolvent operator* of A.

The following definition, even in greater generality, can be found in [7].

DEFINITION 2.1. *Let A be a graph in a complex Banach space X. We say that A generates an analytic semigroup in X if there exist $K > 0$, $\omega \in \mathbb{R}$, $\vartheta \in (\pi/2, \pi)$ such that*

$$\Sigma'_{\vartheta,\omega} \subset \rho(A) \quad \textit{and} \quad \|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{K}{|\lambda - \omega|}, \quad \textit{for } \lambda \in \Sigma'_{\vartheta,\omega}. \tag{2.4}$$

In this case we write $A \in AG(K, \omega, \vartheta)$ and we define the semigroup generated by A by the formula

$$e^{tA} := \int_\gamma e^{t\lambda}(\lambda - A)^{-1} d\lambda, \quad \textit{for } t > 0, \qquad e^{0A} := 1. \tag{2.5}$$

Note that, although A is a graph, $e^{tA} \in \mathcal{L}(X)$, since $(\lambda - A)^{-1} \in \mathcal{L}(X)$. In the definition we fix any number ϑ_0 satisfying $\pi/2 < \vartheta_0 < \vartheta$, and we set

$$\begin{aligned} \gamma \;:=\; & \{z \in \mathbb{C} : z = \omega + re^{-i\vartheta_0}, r \in [1, \infty)\} \cup \\ & \cup \{z \in \mathbb{C} : z = \omega + e^{i\varphi}, \varphi \in [-\vartheta_0, \vartheta_0]\} \cup \\ & \cup \{z \in \mathbb{C} : z = \omega + re^{i\vartheta_0}, r \in [1, \infty)\}, \end{aligned}$$

oriented with increasing imaginary part. More generally, without affecting the value of the integral, γ can be taken to be any piecewise differentiable path lying in $\Sigma'_{\vartheta,\omega}$ and joining $\infty e^{-i\vartheta_0}$ with $\infty e^{+i\vartheta_0}$.

We recall some properties of a graph $A \in AG(K, \omega, \vartheta)$. Detailed proofs can be found for example in [7], [9].

1. A is a closed graph and there exists a constant $M \geq 1$ such that $\|e^{tA}\| \leq Me^{\omega t}$, for $t \geq 0$;
2. The mapping $t \mapsto e^{tA}$ is infinitely differentiable in $(0, \infty)$ with respect to the norm of $\mathcal{L}(X)$ and, for every $t > 0$ and every integer $n \geq 1$,
$$\frac{d^n}{dt^n} e^{tA} = \int_\gamma \lambda^n e^{t\lambda} (\lambda - A)^{-1} d\lambda; \tag{2.6}$$
3. if $0 \in \rho(A)$, $A^{-1}e^{tA} \to A^{-1}$, for $t \to 0$, in the norm of $\mathcal{L}(X)$;
4. if $A \in AG(K, \omega, \vartheta)$ and $0 \in \rho(A)$ then, for every $t > 0$ and every integer $n \geq 1$,
$$e^{tA} = A^{-n} \frac{d^n}{dt^n} e^{tA} = A^{-n} \int_\gamma \lambda^n e^{t\lambda} (\lambda - A)^{-1} d\lambda. \tag{2.7}$$

The next Proposition is an approximation result for the class $AG(K, \omega, \vartheta)$. It is a simple extension of Trotter's theorem ([16], Theorem 3.4.3). The proof can be found in [9]. Similar convergence results are obtained in [3].

PROPOSITION 2.2. *Let A_n be a sequence of graphs in a complex Banach space X. Suppose*

(i) there exist $K > 0$, $\omega \in \mathbb{R}$, $\vartheta \in (\pi/2, \pi)$ such that, for every n,
$$A_n \in AG(K, \omega, \vartheta); \tag{2.8}$$

(ii) there exists $\lambda_0 \in \Sigma'_{\vartheta,\omega}$ such that $(\lambda_0 - A_n)^{-1}$ converges in $\mathcal{L}(X)$ for $n \to \infty$.

Then there exists a unique graph A such that for all $\lambda \in \Sigma'_{\vartheta,\omega}$
$$(\lambda - A_n)^{-1} \to (\lambda - A)^{-1} \quad \text{in } \mathcal{L}(X), \text{ for } n \to \infty. \tag{2.9}$$
Moreover, $A \in AG(K, \omega, \vartheta)$. Finally, for every compact interval $I \subset (0, \infty)$,
$$\lim_{n \to \infty} e^{tA_n} = e^{tA}, \tag{2.10}$$
in $\mathcal{L}(X)$, uniformly for $t \in I$.

Let A be a closed graph and define $D_A = \{x \in X : \text{there exists } y \in X, (x,y) \in A\}$. It is readily verified that D_A, endowed with the norm

$$\|x\|_{D_A} := \|x\| + [x]_{D_A} := \|x\| + \inf\{\|w\| : (x,w) \in A\}, \qquad (2.11)$$

is a Banach space, continuously embedded in X. One can then define the real interpolation spaces $(X, D_A)_{\vartheta,p}$, $\vartheta \in (0,1)$, $p \in [1,\infty]$, that we will denote by $D_A(\vartheta,p)$. The following Proposition, proved in [9], extends (2.1), (2.2).

PROPOSITION 2.3. *Suppose A is a graph satisfying*

$$\rho(A) \supset (\omega,\infty), \qquad \|t(t-A)^{-1}\|_{\mathcal{L}(X)} \leq K, \text{ for } t > \omega, \qquad (2.12)$$

for some constants $\omega \in \mathbb{R}$, $K > 0$, and fix $\lambda_0 > \max(\omega, 0)$. Then x belongs to $D_A(\vartheta,p)$ if and only if $\|x\|'_{D_A(\vartheta,p)} < \infty$, where

$$\|x\|'_{D_A(\vartheta,p)} := \begin{cases} \|x\| + \left(\displaystyle\int_{\lambda_0}^{\infty} \|\lambda^{\vartheta}(x - \lambda(\lambda - A)^{-1}x)\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}}, & p < \infty; \\ \|x\| + \displaystyle\sup_{\lambda > \lambda_0} \|\lambda^{\vartheta}(x - \lambda(\lambda - A)^{-1}x)\|, & p = \infty. \end{cases}$$

Moreover, $\|x\|'_{D_A(\vartheta,p)}$ and $\|x\|_{D_A(\vartheta,p)}$ are equivalent norms for $\vartheta \in (0,1)$, $p \in [1,\infty]$.

3. Some preliminary results

This section mainly contains generalizations of results of [8] to the case where neither D_A nor D_B is dense in X. This generality is needed in the sequel. The results concern the equation

$$(A - \lambda)x + (B - \lambda)x = y, \qquad (3.1)$$

with unknown x, for given $y \in X$ and $\lambda > 0$ sufficiently large. We present existence results (Proposition 3.5) and a representation formula $x = U_\lambda y$ for the solution (Proposition 3.10), and we show regularity properties of the operator U_λ (Proposition 3.12 and Corollary

3.13). As a technical tool, analogous results are proved for the Yosida approximations A_m, B_n.

To conform to the setting of [14] and [8], and to allow greater generality, in this section the following assumptions hold. Starting from the next section, the more restrictive Hypotheses 3, 4 will be assumed.

Recall the notation $\Sigma_\vartheta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \pi - \vartheta\}$.

Hypothesis 1. *A and B are linear operators in a complex Banach space X with domains D_A, D_B and there exist $\vartheta_A, \vartheta_B \in (0, \pi)$, $c_A, c_B > 0$ such that*

$$\rho(A) \supset \Sigma_{\vartheta_A}, \qquad \|(z-A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_A}{|z|}, \text{ for all } z \in \Sigma_{\vartheta_A}, \tag{3.2}$$

$$\rho(B) \supset \Sigma_{\vartheta_B}, \qquad \|(z-B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_B}{|z|}, \text{ for all } z \in \Sigma_{\vartheta_B}, \tag{3.3}$$

$$\vartheta_A + \vartheta_B < \pi. \tag{3.4}$$

Hypothesis 2. *There exist an integer $k \geq 1$, and real numbers $c_{AB} > 0$, $\lambda_0 > 0$, $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k$ such that*

$$0 \leq \alpha_i < \beta_i \leq 1, \qquad i = 1 \dots k,$$

$$\|(A-\lambda_0)(A-v)^{-1}[(A-\lambda_0)^{-1}; (B-z)^{-1}]\|_{\mathcal{L}(X)} \leq \\ \leq c_{AB} \sum_{i=1}^{k} \frac{1}{|v|^{1-\alpha_i}|z|^{1+\beta_i}}, \tag{3.5}$$

for all $v \in \Sigma_{\vartheta_A}$, $z \in \Sigma_{\vartheta_B}$. We also assume

$$\delta := \min_{1 \leq i \leq k} (\beta_i - \alpha_i) \in (0, 1). \tag{3.6}$$

Recall the notation $A_m := mA(m-A)^{-1}$, $B_n := nB(n-B)^{-1}$.

Lemma 3.1. *The operators A_m and B_n satisfy Hypotheses 1, 2 with constants independent from n and m.*

Proof. The proof is essentially contained in [5], formula 6.11 and [14], Lemma 3.1. □

Lemma 3.2. *For every* $\overline{\lambda} > 0$ *there exists* $c(\overline{\lambda}) > 0$ *such that*

$$\|(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \leq$$
$$\leq c(\overline{\lambda}) \sum_{i=1}^{k} \frac{1}{|\lambda+v|^{1-\alpha_i}|z|^{1+\beta_i}},$$

for all $\lambda > \overline{\lambda}$, $v \in \Sigma_{\vartheta_A}$, $z \in \Sigma_{\vartheta_B}$.

Proof. See [14], Lemma 1.2. □

Next we define several operators in $\mathcal{L}(X)$ and list some of their properties. They are used to construct the operator U_λ that gives a representation of the solutions of equation (3.1).

Let θ_0 be a number such that

$$\begin{cases} \theta_B < \theta_0 < \pi - \theta_A, & \theta_0 > \pi/2 \quad \text{if } \theta_A < \pi/2, \\ \theta_B < \theta_0 < \pi - \theta_A, & \theta_0 < \pi/2 \quad \text{if } \theta_A \geq \pi/2, \end{cases} \tag{3.7}$$

and let γ_0 be the path defined as

$$\begin{aligned} \gamma_0 \ := \ & \{\lambda \in \mathbb{C} : \lambda = re^{-i\vartheta_0}, r \in [0,\infty)\} \cup \\ & \cup \{\lambda \in \mathbb{C} : \lambda = re^{i\vartheta_0}, r \in [0,\infty)\}, \end{aligned} \tag{3.8}$$

oriented with increasing imaginary part. Define

$$S_\lambda := -\int_{\gamma_0} (A-\lambda-z)^{-1}(B-\lambda+z)^{-1}\, dz, \tag{3.9}$$

$$S'_\lambda := -\int_{\gamma_0} (B-\lambda+z)^{-1}(A-\lambda-z)^{-1}\, dz, \tag{3.10}$$

$$J_\lambda := -\int_{\gamma_0} z(A-\lambda)^{-1}(A-\lambda-z)^{-1}[(A-\lambda)^{-1};(B-\lambda+z)^{-1}]\, dz, \tag{3.11}$$

$$\begin{aligned} M_\lambda \ := \ & -\int_{\gamma_0} z(A-\lambda)^{-1}(A-\lambda-z)^{-1} \\ & [(A-\lambda)^{-1};(B-\lambda+z)^{-1}]z(A-\lambda-z)^{-1}\, dz. \end{aligned} \tag{3.12}$$

We call $S_{\lambda,n}$, $S'_{\lambda,n}$, $J_{\lambda,n}$, $M_{\lambda,n}$ (respectively $S_{m,\lambda}$, $S'_{m,\lambda}$, $J_{m,\lambda}$, $M_{m,\lambda}$; resp. $S_{m,\lambda,n}$, $S'_{m,\lambda,n}$, $J_{m,\lambda,n}$, $M_{m,\lambda,n}$) the integrals defined as S_λ, S'_λ, J_λ, M_λ by replacing A, B with A, B_n (resp. A_m, B; resp. A_m, B_n).

The next Lemma is proved in [9], Lemmas 3.1-3.5.

LEMMA 3.3. *For any* $\rho \in (0,\delta)$ *and* $\overline{\lambda} > 0$ *there exists* $c = c(\rho, \overline{\lambda}) > 0$ *such that, for every* $\lambda > \overline{\lambda}$,

$$\|S_\lambda\| + \|S'_\lambda\| \le c\lambda^{-1}, \quad \|J_\lambda\| + \|M_\lambda\| \le c\lambda^{-\delta}.$$

The same estimates hold with S_λ, S'_λ, J_λ, M_λ *replaced by* $S_{\lambda,n}$, $S'_{\lambda,n}$, $J_{\lambda,n}$, $M_{\lambda,n}$ *or by* $S_{m,\lambda}$, $S'_{m,\lambda}$, $J_{m,\lambda}$, $M_{m,\lambda}$ *or by* $S_{m,\lambda,n}$, $S'_{m,\lambda,n}$, $J_{m,\lambda,n}$, $M_{m,\lambda,n}$.

Moreover,

$$\|(\lambda - B_n)^{-\rho}\| + \|(\lambda - B_n)^{1-\rho} S'_{\lambda,n}\| + \|(\lambda - B_n)^{1-\rho} S'_{m,\lambda,n}\| \le c\lambda^{-\rho},$$

$$\|J_{\lambda,n}(\lambda - B_n)^{\rho}\| + \|J_{m,\lambda,n}(\lambda - B_n)^{\rho}\| \le c\lambda^{\rho-\delta}.$$

Finally, the following limits exist in the norm of $\mathcal{L}(X)$:

$$K_{m,\lambda} := \lim_{n\to\infty} J_{m,\lambda,n}(\lambda - B_n)^{\rho}, \quad K_\lambda := \lim_{n\to\infty} J_{\lambda,n}(\lambda - B_n)^{\rho}. \tag{3.13}$$

$$(\lambda - B)^{1-\rho} S'_\lambda := \lim_{n\to\infty} (\lambda - B_n)^{1-\rho} S'_{\lambda,n}, \tag{3.14}$$

and we have

$$(\lambda - B)^{1-\rho} S'_\lambda = \int_{\gamma_2} v^{1-\rho}(B - \lambda + v)^{-1}(A - \lambda - v)^{-1}\, dv. \tag{3.15}$$

The path γ_2 is defined as in (3.8), with θ_0 replaced by θ_2, where θ_2 satisfies $\theta_B < \theta_2 < \theta_0$.

Fix any $\rho \in (0,\delta)$ and define

$$\begin{aligned} U_{m,\lambda,n} := (\lambda - B_n)^{\rho-1}\Big\{1 + (\lambda - B_n)^{-\rho} J_{m,\lambda,n}(\lambda - B_n)^{\rho}\Big\}^{-1} \\ (\lambda - B_n)^{1-\rho} S'_{m,\lambda,n} + (\lambda - B_n)^{-1}\Big\{1 + J_{m,\lambda,n}\Big\}^{-1} M_{m,\lambda,n} \end{aligned} \tag{3.16}$$

$$\begin{aligned} U_{\lambda,n} := (\lambda - B_n)^{\rho-1}\Big\{1 + (\lambda - B_n)^{-\rho} J_{\lambda,n}(\lambda - B_n)^{\rho}\Big\}^{-1} \\ (\lambda - B_n)^{1-\rho} S'_{\lambda,n} + (\lambda - B_n)^{-1}\Big\{1 + J_{\lambda,n}\Big\}^{-1} M_{\lambda,n} \end{aligned} \tag{3.17}$$

$$U_\lambda := (\lambda - B)^{\rho-1}\Big\{1 + (\lambda - B)^{-\rho}K_\lambda\Big\}^{-1}(\lambda - B)^{1-\rho}S'_\lambda + \\ + (\lambda - B)^{-1}\Big\{1 + J_\lambda\Big\}^{-1}M_\lambda \tag{3.18}$$

In these formulae, fractional powers of $\lambda - B_n$ are defined using functional calculus for bounded operators, $(\lambda - B)^{-\alpha}$ is defined as $\displaystyle\int_{\gamma_0} z^{-\alpha}(\lambda - z - B)^{-1}dz$ for $\alpha \in (0,1)$, $(\lambda - B)^{1-\rho}S'_\lambda$ is defined in (3.14) and K_λ is defined in (3.13). Notice that the operators in curly brackets are invertible, for $\lambda > 0$ sufficiently large, by Lemma 3.3.

The next Lemma is proved in [9], Lemma 3.6, Corollary 3.7, formula (4.10).

LEMMA 3.4. *There exist constants $\lambda^* > 0$ and $c > 0$, depending only on the constants in Hypotheses 1, 2, such that, for $\lambda > \lambda^*$,*

$$\|U_{m,\lambda,n}\| + \|U_{m,\lambda}\| + \|U_{\lambda,n}\| + \|U_\lambda\| \le c\lambda^{-1}. \tag{3.19}$$

We have, in the norm of $\mathcal{L}(X)$,

$$U_{m,\lambda} = \lim_{n\to\infty} U_{m,\lambda,n}\,, \quad U_{\lambda,n} = \lim_{m\to\infty} U_{m,\lambda,n}\,, \\ U_\lambda = \lim_{m\to\infty} U_{m,\lambda} = \lim_{n\to\infty} U_{\lambda,n}\,. \tag{3.20}$$

We have

$$U_\lambda = S'_\lambda + (\lambda - B)^{-1}Q_\lambda, \tag{3.21}$$

for some bounded operator Q_λ satisfying $\|Q_\lambda\| \le c\lambda^{-\delta}$.

Now we are ready to state and prove results on the equation (3.1).

PROPOSITION 3.5. *Assume that Hypotheses 1, 2 hold. Then there exists $\lambda^* > 0$ such that for $\lambda > \lambda^*$, $\theta \in (0,\delta)$, $p \in [1,\infty]$ and for every $y \in D_A(\theta,p)$ (or $y \in D_B(\theta,p)$) there exists a unique $x \in D_A \cap D_B$ satisfying the equation*

$$(A + B - 2\lambda)x = y.$$

Moreover, $S_\lambda y \in D_A$ and x is given by the formula

$$x = (A - \lambda)^{-1}(1 + J_\lambda)^{-1}(A - \lambda)S_\lambda y.$$

Proof. See [14], Theorem 4.1. □

The next Lemma is proved in [9], Corollary 3.7.

LEMMA 3.6. *There exists a constant $\lambda^* > 0$ such that, whenever $x, y \in X$ satisfy*

$$(A_m - \lambda)x + (B_n - \lambda)x = y$$

for some $\lambda > \lambda^*$ and some m, n, then

$$x = U_{m,\lambda,n}y.$$

From now on, λ^* is the constant appearing in Proposition 3.5 and Lemmas 3.4, 3.6.

LEMMA 3.7. *If $x \in D_B$, $y \in X$ satisfy*

$$(A_m - \lambda)x + (B - \lambda)x = y$$

for some $\lambda > \lambda^*$ and some m, then

$$x = U_{m,\lambda}y.$$

Proof. Since A_m is bounded, by Proposition 3.5 x is given by

$$x = (A_m - \lambda)^{-1}(1 + J_{m,\lambda})^{-1}(A_m - \lambda)S_{m,\lambda}y.$$

Similarly, for every n, the equation $(A_m - \lambda)x_n + (B_n - \lambda)x_n = y$ has a unique solution x_n given by

$$x_n = (A_m - \lambda)^{-1}(1 + J_{m,\lambda,n})^{-1}(A_m - \lambda)S_{m,\lambda,n}y.$$

It is readily verified that $J_{m,\lambda,n} \to J_{m,\lambda}$ and $S_{m,\lambda,n} \to S_{m,\lambda}$ as $n \to \infty$, in the norm of $\mathcal{L}(X)$, so we have $x_n \to x$. Since, by Lemma 3.6, $x_n = U_{m,\lambda,n}y$, we conclude that $x = U_{m,\lambda}y$, by (3.20). □

LEMMA 3.8. *For $\lambda > \lambda^*$ and for every m,*

$$\rho(A_m + B) \supset (2\lambda^*, \infty), \quad (A_m + B - 2\lambda)^{-1} = U_{m,\lambda}.$$

Proof. If $\lambda_0 > \lambda^*$ is so large that

$$\|(2\lambda_0 - B)^{-1} A_m\| \le c\lambda_0^{-1} \|A_m\| < 1,$$

then $2\lambda_0 \in \rho(A_m + B)$, since

$$(2\lambda_0 - A_m - B)^{-1} = \sum_{k=0}^{\infty} \left((2\lambda_0 - B)^{-1} A_m \right)^k (2\lambda_0 - B)^{-1}.$$

By Lemma 3.7, $(2\lambda_0 - A_m - B)^{-1} = U_{m,\lambda_0}$, and the Lemma is proved for $\lambda = \lambda_0$. The estimate $\|U_{m,\lambda}\| \le c\lambda^{-1}$, $\lambda > \lambda^*$, of Lemma 3.4 allows to deduce the general case by a standard argument (see e.g. [5], Theorem 2.1 or [14], Prop. 3.1). □

LEMMA 3.9. *For $\lambda > \lambda^*$,*

$$\lim_{m \to \infty} U_{m,\lambda} A(A - m)^{-1} = 0, \tag{3.22}$$

in the norm of $\mathcal{L}(X)$.

Proof. $U_{\lambda,m} A(A-m)^{-1} = (U_{\lambda,m} - U_\lambda) A(A-m)^{-1} + U_\lambda A(A-m)^{-1}$. Since $\|A(A - m)^{-1}\|_{\mathcal{L}(X)} \le 1 + c_A$, by (3.20) it suffices to prove $\lim_{m \to \infty} U_\lambda A(A - m)^{-1} = 0$. By (3.18) and Lemma 3.3 it is enough to show

$$\lim_{m \to \infty} (\lambda - B)^{1-\rho} S'_\lambda A(A - m)^{-1} = 0, \quad \lim_{m \to \infty} M_\lambda A(A - m)^{-1} = 0. \tag{3.23}$$

By (3.15), (3.12) we have

$$(\lambda - B)^{1-\rho} S'_\lambda A(A - m)^{-1} = \\ \int_{\gamma_2} z^{1-\rho} (B - \lambda + z)^{-1} (A - \lambda - z)^{-1} A(A - m)^{-1} dz,$$

$$M_\lambda A(A-m)^{-1} = -\int_{\gamma_0} z(A-\lambda)(A-\lambda-z)^{-1} \cdot$$
$$[(A-\lambda)^{-1};(B-\lambda+z)^{-1}]z(A-\lambda-z)^{-1}A(A-m)^{-1}dz.$$

Note first that for $z \in \gamma_2$ (respectively, $z \in \gamma_0$), by Hypotheses 1, 2 and Lemma 3.2,

$$\left\| z^{1-\rho}(B-\lambda+z)^{-1}(A-\lambda-z)^{-1}A(A-m)^{-1} \right\| \leq$$
$$\leq |z|^{1-\rho}\frac{c_B}{|z-\lambda|}\frac{c_A}{|z+\lambda|}(1+c_A),$$

$$\left\| z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1};(B-\lambda+z)^{-1}] \cdot \right.$$
$$\left. z(A-\lambda-z)^{-1}A(A-m)^{-1} \right\| \leq$$
$$\leq \sum_{i=1}^{k} \frac{c(\lambda^*)|z|}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}\frac{c_A|z|}{|z+\lambda|}(1+c_A).$$

and the right-hand sides are integrable over γ_2 and γ_0 respectively. Moreover, for fixed $z \in \gamma_2$ (resp. $z \in \gamma_0$),

$$\left\| z^{1-\rho}(B-\lambda+z)^{-1}A(A-\lambda-z)^{-1}(A-m)^{-1} \right\| \leq$$
$$\leq |z|^{1-\rho}\frac{c_B}{|z-\lambda|}(1+c_A)\frac{c_A}{m} \quad \to \quad 0,$$

$$\left\| z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1};(B-\lambda+z)^{-1}] \cdot \right.$$
$$\left. zA(A-\lambda-z)^{-1}(A-m)^{-1} \right\| \leq$$
$$\leq \sum_{i=1}^{k} \frac{c(\lambda^*)|z|}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}|z|(1+c_A)\frac{c_A}{m} \quad \to \quad 0,$$

as $m \to \infty$. So (3.23) holds by the dominated convergence Theorem.

□

Proposition 3.10. *Assume that Hypotheses 1, 2 hold. Then there exists $\lambda^* > 0$, depending only on the constants in Hypotheses 1, 2, with the following property:*

if $\lambda > \lambda^*$, $x \in D_A \cap D_B$ and $y \in X$ satisfy

$$(A-\lambda)x + (B-\lambda)x = y,$$

then x is given by

$$x = U_\lambda y. \tag{3.24}$$

In particular,

$$\|x\| \le c(\lambda^*)\lambda^{-1}\|y\| \qquad \lambda > \lambda^*, \tag{3.25}$$

with $c(\lambda^*) > 0$ independent of λ .

Proof. Let λ^* be the same as before. Since $(A-\lambda)x + (B-\lambda)x = y$ we have

$$(A_m - \lambda)x + (B-\lambda)x = y + (A_m - A)x.$$

From Lemma 3.8, and taking into account that

$$(A_m - A)x = (m(m-A)^{-1} - 1)Ax = A(m-A)^{-1}Ax\,,$$

we obtain

$$x = U_{\lambda,m}y + U_{\lambda,m}A(m-A)^{-1}Ax.$$

The conclusion follows from (3.20) and Lemma 3.9. □

Corollary 3.11. *Assume that Hypotheses 1, 2 hold and suppose that $D_A \cap D_B$ is dense in X. Then $A+B$ is closable.*

Proof. By (3.25), $\|x\| \le c(\lambda^*)\lambda^{-1}\|(A+B-2\lambda)x\|$, $\lambda > \lambda^*$, $x \in X$. The conclusion follows from [5], Theorem 2.1. □

The next Proposition and its Corollary show that U_λ has some regularizing effect: its image in contained in subspaces of X related to the operators A and B.

PROPOSITION 3.12. *Assume that Hypotheses 1, 2 hold. Then for* $\lambda > \lambda^*$,

$$U_\lambda \in \mathcal{L}(X, D_{A^2}(1/2,\infty)), \quad \|U_\lambda\|_{\mathcal{L}(X,D_{A^2}(1/2,\infty))} \le c(\lambda^*), \tag{3.26}$$

$$U_\lambda \in \mathcal{L}(X, D_{B^2}(1/2,\infty)), \quad \|U_\lambda\|_{\mathcal{L}(X,D_{B^2}(1/2,\infty))} \le c(\lambda^*). \tag{3.27}$$

Proof. It is proved in [9], Theorem 4.6, that (3.26) holds under the additional assumption that D_B is dense in X, and with $c(\lambda^*)$ depending only on the constants in Hypotheses 1, 2. Applying this to the operators A and B_n we obtain

$$\|U_{\lambda,n}\|_{\mathcal{L}(XD_{A^2}(1/2,\infty))} \le c(\lambda^*), \quad \lambda > \lambda^*,$$

i.e.

$$\|A^2(A-t)^{-2}U_{\lambda,n}y\| \le c(\lambda^*)t^{-1}\|y\|, \quad y \in X,\ t > 0,$$

with $c(\lambda^*)$ independent from n. Letting $n \to \infty$ we have, by (3.20),

$$\|A^2(A-t)^{-2}U_\lambda y\| \le c(\lambda^*)t^{-1}\|y\|, \quad y \in X,\ t > 0.$$

Now (3.26), is proved.

(3.27) is proved in [9], Theorem 4.6. □

COROLLARY 3.13. *For* $\lambda > \lambda^*$ *the following statements hold.*

(*i*) *The operator*

$$(A + B - 2\lambda) : D_A \cap D_B \to \mathcal{R}(A + B - 2\lambda)$$

is bijective and its inverse has a bounded extension given by

$$U_\lambda : X \to \mathcal{R}(U_\lambda) \subset D_{A^2}(1/2,\infty) \cap D_{B^2}(1/2,\infty).$$

In particular

$$D_A \cap D_B \subset \mathcal{R}(U_\lambda) \subset D_{A^2}(1/2,\infty) \cap D_{B^2}(1/2,\infty), \quad \lambda > \lambda^*. \tag{3.28}$$

(*ii*) $\mathcal{R}(U_\lambda^2) \subset D_A \cap D_B$ *and* $(A + B - 2\lambda)U_\lambda^2 = U_\lambda$.

(iii) $\mathcal{R}(U_\lambda^{n+1}) \subset D_{(A+B)^n}$ *and* $(A+B-2\lambda)^n U_\lambda^{n+1} = U_\lambda,\ n \geq 1.$

Proof. (i) is a direct consequence of Propositions 3.10 and 3.12.

To prove (ii), for $y \in X$ set $z := U_\lambda y$. Then $z \in D_{A^2}(1/2,\infty) \subset D_A(\theta,p)$, by Proposition 3.12. By Propositions 3.5 and 3.10, the equation

$$(A+B-2\lambda)x = z$$

has a unique solution $x \in D_A \cap D_B$ given by $x = U_\lambda z$. Therefore $x = U_\lambda^2 y \in D_A \cap D_B$ and $(A+B-2\lambda)U_\lambda^2 y = z = U_\lambda y$.

(iii) can be proved in a similar way, using induction on n. □

4. Estimates on approximating semigroups

In this section we state the assumptions needed for the perturbation theorems of the following section and we prove some preliminary result. The following hypotheses are assumed to hold from now on. They correspond to the Hypotheses *H.I*, *H.IIb* mentioned in the Introduction.

Recall the notation $\Sigma_\vartheta := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \pi - \vartheta\}$.

HYPOTHESIS 3. *A and B are linear operators in a complex Banach space X with domains D_A, D_B and there exist $\vartheta_A, \vartheta_B \in (0,\pi/2)$, $c_A, c_B > 0$ such that*

$$\rho(A) \supset \Sigma_{\vartheta_A} \cup \{0\}, \qquad \|(z-A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_A}{|z|},\ \text{for all } z \in \Sigma_{\vartheta_A}, \tag{4.1}$$

$$\rho(B) \supset \Sigma_{\vartheta_B} \cup \{0\}, \qquad \|(z-B)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c_B}{|z|},\ \text{for all } z \in \Sigma_{\vartheta_B}. \tag{4.2}$$

HYPOTHESIS 4. *There exist an integer $k \geq 1$, and real numbers $c_{AB} > 0$, $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k$ such that*

$$0 \leq \alpha_i < \beta_i \leq 1, \qquad i = 1 \dots k,$$

$$\|A(A-v)^{-1}[A^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \leq c_{AB} \sum_{i=1}^k \frac{1}{|v|^{1-\alpha_i}|z|^{1+\beta_i}}, \tag{4.3}$$

for all $v \in \Sigma_{\vartheta_A}$, $z \in \Sigma_{\vartheta_B}$. *We also assume*

$$\delta := \min_{1 \le i \le k} (\beta_i - \alpha_i) \in (0,1). \tag{4.4}$$

REMARK 4.1. 1. Hypothesis 3 implies $A \in AG(c_A, 0, \pi - \vartheta_A)$, $B \in AG(c_B, 0, \pi - \vartheta_B)$ (see Definition 2.1).

2. It is easy to verify that Hypotheses 3, 4 imply Hypotheses 1, 2. Therefore we can use the results of the previous section. This will be done only in Sections 5., 6..

3. (4.4) causes no loss of generality, since the inequality (4.3) turns out to be relevant for large values of $|v|$ and $|z|$.

LEMMA 4.2. *The operators* A_m *and* B_n *satisfy Hypotheses 3, 4 with constants independent from* n *and* m.

Proof. The proof is essentially contained in [5], formula 6.11 and [14], Lemma 3.1. □

REMARK 4.3. By this lemma, we can modify the values of the constants $\vartheta_A, \vartheta_B, c_A$, etc. in (4.1)-(4.4) in such a way that Hypotheses 3, 4 hold also for A_m and B_n uniformly with respect to m and n. We assume once and for all that this has been done. So from now on (4.1)-(4.4) also hold for A_m and B_n, and the constants ϑ_A, ϑ_B , c_A, c_B, c_{AB}, k, $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ are independent from m and n.

Recall that for integers $m, n \ge 1$ we set $A_m := mA(m - A)^{-1}$, $B_n := nB(n - B)^{-1}$. The operators A_m, B_n are approximations of A, B. The rest of this section is devoted to proving Proposition 4.4, the only result that will be needed in the sequel.

PROPOSITION 4.4. *There exist* $\overline{M} \ge 1$, $\overline{\omega} \ge 0$ *such that, for every* $t \ge 0$ *and for every* m, n,

$$\|e^{t(A_m + B_n)}\| \le \overline{M} e^{\overline{\omega} t}. \tag{4.5}$$

The idea of the proof is to show that $V(t) := e^{t(A_m+B_n)}$ satisfies an equation of Volterra type (equation (4.6) below, with $P = A_m$, $Q = B_n$). The estimates (4.10), (4.11) on the integral kernels of the Volterra equation, independent of m, n, allow to use a fixed point argument to deduce (4.5).

LEMMA 4.5. *Assume $P, Q \in \mathcal{L}(X)$, $0 \in \rho(P)$, and define $V(t) := e^{t(P+Q)}$. Then*

$$\begin{aligned} V(t) = \; & e^{tQ}e^{tP} + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]\, ds \\ & - \int_0^t e^{(t-s)Q} P^2 e^{sP}[P^{-1}; e^{sQ}]\, ds \\ & + \int_0^t e^{(t-s)Q} P^2 e^{sP}[P^{-1}; e^{sQ}] V(s)\, ds \\ & + \int_0^t e^{(t-v)Q} \int_0^v \Big(P^3 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}] + \\ & \quad + P^2 e^{(v-s)P}[P^{-1}; Qe^{(v-s)Q}] \Big) \cdot (V(s) - V(v))\, ds\, dv \\ & - \int_0^t e^{(t-v)Q} \int_0^v P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}] QV(s) ds\, dv. \end{aligned} \tag{4.6}$$

Proof. Define $H(s) := e^{(t-s)P}e^{(t-s)Q}e^{s(P+Q)}$. Then

$$\begin{aligned} H'(s) &= -Pe^{(t-s)P}e^{(t-s)Q}e^{s(P+Q)} - e^{(t-s)P}Qe^{(t-s)Q}e^{s(P+Q)} \\ &\quad + e^{(t-s)P}e^{(t-s)Q}(P+Q)e^{s(P+Q)} \\ &= -Pe^{(t-s)P}e^{(t-s)Q}V(s) + e^{(t-s)P}e^{(t-s)Q}PV(s) \\ &= Pe^{(t-s)P}(P^{-1}e^{(t-s)Q} - e^{(t-s)Q}P^{-1})PV(s) \\ &= Pe^{(t-s)P}[P^{-1}; e^{(t-s)Q}]PV(s). \end{aligned}$$

The equality $H(t) = H(0) + \int_0^t H'(s) ds$ yields

$$V(t) = e^{tP}e^{tQ} + \int_0^t Pe^{(t-s)P}[P^{-1}; e^{(t-s)Q}]PV(s) ds. \tag{4.7}$$

Let us apply P to both sides of (4.7). Taking into account that $V'(t) = (P+Q)V(t)$ we obtain

$$V'(t) - QV(t) = Pe^{tP}e^{tQ} +$$

$$+\int_0^t P^2 e^{(t-s)P}[P^{-1}; e^{(t-s)Q}](V'(s) - QV(s))ds. \quad (4.8)$$

Now we apply $e^{(t-s)Q}$ to both sides of (4.8) and we integrate from 0 to t. Note that, integrating by parts,

$$\int_0^t e^{(t-s)Q}(V'(s) - QV(s))ds = [\, e^{(t-s)Q}V(s)\,]_{s=0}^{s=t}$$
$$+\int_0^t Qe^{(t-s)Q}V(s)ds - \int_0^t Qe^{(t-s)Q}V(s)ds = V(t) - e^{tQ},$$

and

$$\begin{aligned}\int_0^t e^{(t-s)Q}Pe^{sP}e^{sQ}ds &= \int_0^t e^{(t-s)Q}e^{sQ}Pe^{sP}ds + \\ &\quad + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]ds \\ &= e^{tQ}\int_0^t Pe^{sP}ds + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]ds \\ &= e^{tQ}(e^{tP} - 1) + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]ds.\end{aligned}$$

Then (4.8) becomes

$$V(t) - e^{tQ} = e^{tQ}(e^{tP} - 1) + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]ds$$
$$+\int_0^t e^{(t-v)Q}\int_0^v P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}](V'(s) - QV(s))dsdv,$$

i.e.

$$\begin{aligned}V(t) &= e^{tQ}e^{tP} + \int_0^t e^{(t-s)Q}[Pe^{sP}; e^{sQ}]ds \\ &\quad - \int_0^t e^{(t-v)Q}\int_0^v P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]QV(s)dsdv \\ &\quad + \int_0^t e^{(t-v)Q}\int_0^v P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]V'(s)dsdv. \quad (4.9)\end{aligned}$$

Now notice that

$$\int_0^v P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]V'(s)ds =$$

$$
\begin{aligned}
&= \Big[P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]V(s) \Big]_{s=0}^{s=v} - \\
&\quad - \int_0^v \frac{d}{ds} \, (P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]) \, V(s) ds \\
&= -P^2 e^{vP}[P^{-1}; e^{vQ}] - \\
&\quad - \int_0^v \frac{d}{ds} \, (P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]) \, (V(s) - V(v)) ds - \\
&\quad - \int_0^v \frac{d}{ds} \, (P^2 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}]) \, ds V(v) \\
&= -P^2 e^{vP}[P^{-1}; e^{vQ}] + \int_0^v \Big(P^3 e^{(v-s)P}[P^{-1}; e^{(v-s)Q}] + \\
&\quad + P^2 e^{(v-s)P}[P^{-1}; Q e^{(v-s)Q}] \Big) (V(s) - V(v)) ds + \\
&\quad + P^2 e^{vP}[P^{-1}; e^{vQ}] V(v).
\end{aligned}
$$

Substituting into (4.9) we finally obtain (4.6). □

LEMMA 4.6. *For every $\rho \in (0, \delta)$ (see (4.4)), there exists $k_1 = k_1(\rho) > 0$, such that for every $t \in (0,1]$ and for every m, n we have*

$$\|A_m^2 e^{tA_m}[A_m^{-1}; e^{tB_n}](-B_n)^\rho\| \le k_1 t^{-1+\delta-\rho}. \tag{4.10}$$

The proof of this Lemma is postponed to the appendix.

LEMMA 4.7. *Let $\alpha \in [0,1]$ Then there exist positive numbers k_2, k_3, k_4, $k_A(\alpha)$, $k_B(\alpha)$ such that for every $t \in (0,1]$ and for every m, n we have*

$$
\begin{aligned}
&i) && \|[A_m e^{tA_m}; e^{tB_n}]\| \le k_2 t^{-1+\delta}, \\
&ii) && \|A_m^2 e^{tA_m}[A_m^{-1}; e^{tB_n}]\| \le k_3 t^{-1+\delta}, \\
&iii) && \|A_m^3 e^{tA_m}[A_m^{-1}; e^{tB_n}]\| + \|A_m^2 e^{tA_m}[A_m^{-1}; B_n e^{tB_n}]\| \le k_4 t^{-2+\delta}, \\
&iv) && \|(-A_m)^\alpha e^{tA_m}\| \le k_A(\alpha) t^{-\alpha}, \\
&v) && \|(-B_n)^\alpha e^{tB_n}\| \le k_B(\alpha) t^{-\alpha}.
\end{aligned}
\tag{4.11}
$$

Proof. Let us choose ϑ_0 such that $\pi/2 < \vartheta_0 < \pi - \max(\vartheta_A, \vartheta_B)$. Let us define

$$\begin{aligned} \gamma := & \{\lambda \in \mathbb{C} : \lambda = re^{-i\vartheta_0}, r \in [1,\infty)\} \cup \\ & \cup \{\lambda \in \mathbb{C} : \lambda = e^{i\varphi}, \varphi \in [-\vartheta_0, \vartheta_0]\} \cup \\ & \cup \{\lambda \in \mathbb{C} : \lambda = re^{i\vartheta_0}, r \in [1,\infty)\}, \end{aligned} \tag{4.12}$$

oriented with increasing imaginary part. Fix $t \in (0,1]$. By our assumptions and a well known result (see [16], theorem 1.7.7 and formula (5.12)), we have, for $h = 0, 1, 2, \ldots$, and every n, m,

$$\begin{aligned} A_m^h e^{tA_m} &= \frac{1}{2\pi i}\int_\gamma \lambda^h e^{\lambda t}(\lambda - A_m)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\gamma t} (\lambda t^{-1})^h e^{\lambda}(\lambda t^{-1} - A_m)^{-1} t^{-1} d\lambda, \end{aligned} \tag{4.13}$$

$$\begin{aligned} B_n^h e^{tB_n} &= \frac{1}{2\pi i}\int_\gamma \lambda^h e^{\mu t}(\mu - B_n)^{-1} d\mu \\ &= \frac{1}{2\pi i}\int_{\gamma t} (\mu t^{-1})^h e^{\mu}(\mu t^{-1} - B_n)^{-1} t^{-1} d\mu. \end{aligned} \tag{4.14}$$

Since the integrands in (4.13), (4.14) are analytic, we can deform the path γt into γ without affecting the value of the integrals.

i) We have

$$\begin{aligned} &[A_m e^{tA_m}; e^{tB_n}] = \\ &= \frac{1}{2\pi i}\int_\gamma \frac{1}{2\pi i}\int_\gamma \lambda t^{-1} e^{\lambda}[(\lambda t^{-1} - A_m)^{-1}; (\mu t^{-1} - B_n)^{-1}] e^{\mu} t^{-2} d\lambda d\mu \\ &= -\frac{1}{2\pi i}\int_\gamma \frac{1}{2\pi i}\int_\gamma \lambda t^{-1} e^{\lambda} A_m(\lambda t^{-1} - A_m)^{-1}[A_m^{-1}; (\mu t^{-1} - B_n)^{-1}] \cdot \\ &\quad \cdot A_m(\lambda t^{-1} - A_m)^{-1} e^{\mu} t^{-2} d\lambda d\mu. \end{aligned}$$

By (4.3) we obtain

$$\begin{aligned} &\|[A_m e^{tA_m}; e^{tB_n}]\| \leq \\ &\leq c_{AB} \sum_{i=1}^{k} (2\pi)^{-2} \int_\gamma \int_\gamma t^{-1}|\lambda| e^{\mathrm{Re}(\lambda)} \frac{1}{|\lambda/t|^{1-\alpha_i}\, |\mu/t|^{\beta_i+1}} \cdot \\ &\quad \cdot (1 + c_A) e^{\mathrm{Re}(\mu)} t^{-2} |d\lambda|\, |d\mu| \\ &\leq k_2 t^{\delta-1}. \end{aligned}$$

ii) We have

$$\begin{aligned}
\left\| A_m^2 e^{tA_m}[A_m^{-1}; e^{tB_n}] \right\| &= \\
&= \left\| \int_\gamma\!\!\!\!\!- \int_\gamma\!\!\!\!\!- \lambda t^{-1} e^{\lambda} A_m (\lambda t^{-1} - A_m)^{-1} \cdot \right. \\
&\qquad \left. \cdot [A_m^{-1}; (\mu t^{-1} - B_n)^{-1}] e^{\mu} t^{-2} d\lambda d\mu \right\| \\
&\leq c_{AB} \sum_{i=1}^{k} (2\pi)^{-2} \int_\gamma \int_\gamma t^{-1} |\lambda| e^{\mathrm{Re}(\lambda)} \cdot \\
&\qquad \cdot \frac{1}{|\lambda/t|^{1-\alpha_i}\ |\mu/t|^{\beta_i+1}} e^{\mathrm{Re}(\mu)} t^{-2} |d\lambda|\ |d\mu| \\
&\leq k_3 t^{\delta-1}.
\end{aligned}$$

iii) can be proved in a similar way and *iv*), *v*) can be proved as in [16], Theorem 2.6.13, or using the formula, analogous to (4.13),

$$(-A_m)^{\alpha} e^{tA_m} = \int_\gamma\!\!\!\!\!- (-\lambda)^{\alpha} e^{\lambda t} (\lambda - A_m)^{-1} d\lambda,$$

and taking into account our assumptions and Lemma 4.2. □

Proof of Proposition 4.4. Fix m and n. Since $e^{t(A_m+B_n)}$, $t \geq 0$, is a semigroup, it suffices to show that there exist $T > 0, M_1 \geq 1$, independent of m, n, such that

$$\|e^{t(A_m+B_n)}\| \leq M_1, \quad t \in [0, T]. \tag{4.15}$$

This will be proved by means of equation (4.6) with $P = A_m$, $Q = B_n$. To this end let us define $g(t) := g_1(t) + g_2(t) + g_3(t)$, $\mathcal{Q} := \mathcal{Q}_1 + \mathcal{Q}_2 + \mathcal{Q}_3$, where

$$\begin{aligned}
g_1(t) &:= e^{tB_n} e^{tA_m}, \\
g_2(t) &:= \int_0^t e^{(t-s)B_n} [A_m e^{sA_m}; e^{sB_n}] ds, \\
g_3(t) &:= -\int_0^t e^{(t-s)B_n} A_m^2 e^{sA_m} [A_m^{-1}; e^{sB_n}] ds,
\end{aligned}$$

$$
\begin{aligned}
(\mathcal{Q}_1 f)(t) &:= \int_0^t e^{(t-s)B_n} A_m^2 e^{sA_m}[A_m^{-1}; e^{sB_n}]f(s)ds, \\
(\mathcal{Q}_2 f)(t) &:= \int_0^t e^{(t-v)B_n} \int_0^v \Big(A_m^3 e^{(v-s)A_m}[A_m^{-1}; e^{(v-s)B_n}] + \\
&\quad + A_m^2 e^{(v-s)A_m}[A_m^{-1}; B_n e^{(v-s)B_n}]\Big)(f(s)-f(v))dsdv, \\
(\mathcal{Q}_3 f)(t) &:= -\int_0^t e^{(t-v)B_n} \int_0^v A_m^2 e^{(v-s)A_m} \cdot \\
&\quad \cdot [A_m^{-1}; e^{(v-s)B_n}]B_n f(s)dsdv.
\end{aligned}
$$

Now, setting $V(t) := e^{t(A_m+B_n)}$, equation (4.6) becomes

$$V(t) = g(t) + (\mathcal{Q}V)(t). \tag{4.16}$$

For $T \in (0,1]$, $\rho \in (0,\delta)$, $\rho < \frac{1}{2}$ let us define the Banach space $F_{T,n,\rho}$ of the functions $f : (0,T] \to \mathcal{L}(X)$ such that the quantity

$$\|f\|_{T,n,\rho} := \sup_{0<t\le T} \|f(t)\| + \sup_{0<s<t\le T} s^{1-\rho}\frac{\|f(t)-f(s)\|}{(t-s)^{1-\rho}} \tag{4.17}$$

$$+ \sup_{0<t\le T} t^{1-\rho}\|(-B_n)^{1-\rho}f(t)\| \tag{4.18}$$

is finite. Note that (4.17) implies that for $f \in F_{T,n,\rho}$ we have

$$\|f(v)-f(s)\| \le \|f\|_{T,n,\rho}\left(\frac{v-s}{s}\right)^{1-\rho} \tag{4.19}$$

for $0 < s \le v \le T$. We will show that $g \in F_{T,n,\rho}$ and that $\mathcal{Q}$ is a bounded linear operator in $F_{T,n,\rho}$. More precisely we will prove

$$\|g\|_{T,n,\rho} \le C_1, \tag{4.20}$$

$$\|\mathcal{Q}\|_{\mathcal{L}(F_{T,n,\rho})} \le C_2(T), \tag{4.21}$$

with $C_2(T) \to 0$ for $T \to 0$ and $C_1, C_2(T)$ independent of m,n. Let us assume (4.20) and (4.21) for a moment. If T is so small that $C_2(T) \le 1/2$, then $1-\mathcal{Q}$ invertible in $\mathcal{L}(F_{T,n,\rho})$ and $\|1-\mathcal{Q}\|_{\mathcal{L}(F_{T,n,\rho})} \le 2$. Therefore (4.16) becomes $V = (1-\mathcal{Q})^{-1}g$ and we obtain $\|V\|_{F_{T,n,\rho}} \le 2C_1$. So by (4.17) $\|V(t)\| \le 2C_1$, $t \in (0,T]$, and (4.15) is proved.

To finish the proof we show that (4.20), (4.21) hold. We only use Lemmas 4.6 and 4.7 and formula (4.19). The constants k_i, $k_A(\cdot)$, $k_B(\cdot)$ below refer to these Lemmas.

(4.20) is easily established. For example, the inequalities

$$\| \int_0^t e^{(t-s)B_n}[A_m e^{sA_m}; e^{sB_n}]ds\| \le \int_0^t k_B(0)\frac{k_2}{s^{1-\delta}}ds \le k_B(0)k_2T^\delta\delta^{-1},$$

$$\begin{aligned}\left\|(-B_n)^{1-\rho}\int_0^t e^{(t-s)B_n}[A_m e^{sA_m}; e^{sB_n}]ds\right\| &\le \\ &\le \int_0^t \frac{k_B(1-\rho)}{(t-s)^{1-\rho}}\frac{k_2}{s^{1-\delta}}ds \\ &= \frac{k_B(1-\rho)k_2}{t^{1-\rho-\delta}}\int_0^1 \frac{ds}{(1-s)^{1-\rho}s^{1-\delta}} \\ &\le \frac{k_B(1-\rho)k_2}{t^{1-\rho}}T^\delta\int_0^1 \frac{ds}{(1-s)^{1-\rho}s^{1-\delta}},\end{aligned}$$

$$\begin{aligned}&\| \int_0^t e^{(t-v)B_n}[A_m e^{vA_m}; e^{vB_n}]dv - \int_0^s e^{(s-v)B_n}[A_m e^{vA_m}; e^{vB_n}]dv\| \\ &\le \int_0^s \|e^{(t-v)B_n} - e^{(s-v)B_n}\|\frac{k_2}{v^{1-\delta}}dv + \int_s^t k_B(0)\frac{k_2}{v^{1-\delta}}dv \\ &\le \int_0^s k_B(1)\log\frac{t-v}{s-v}\frac{k_2}{v^{1-\delta}}dv + k_B(0)k_2T^\delta\log\frac{t}{s} \\ &\le k_B(1)k_2\int_0^s\left(\frac{t-s}{s-v}\right)^{1-\rho}\frac{dv}{v^{1-\delta}} + k_B(0)T^\delta\left(\frac{t-s}{s}\right)^{1-\rho},\end{aligned}$$

show that $\|g_2\|_{T,n,\rho} \le c$, with c independent of m, n.

Let us prove (4.21) for $\mathcal{Q}_3$. Setting

$$(Hf)(v) = \int_0^v A_m^2 e^{(v-s)A_m}[A_m^{-1}; e^{(v-s)B_n}]B_n f(s)ds,$$

we have

$$\|(Hf)(v)\| \quad = \quad \left\|\int_0^v A_m^2 e^{(v-s)A_m}[A_m^{-1}; e^{(v-s)B_n}]\right. \cdot$$

$$\cdot (-B_n)^{\rho}(-B_n)^{1-\rho}f(s)ds\Big\| \le$$

$$\begin{aligned} &\le \int_0^v \frac{k_1}{(v-s)^{1-\delta+\rho}}\frac{1}{s^{1-\rho}}ds \sup_{0<t\le T} t^{1-\rho}\|(-B_n)^{1-\rho}f(t)\| \\ &\le v^{\delta-1}\int_0^1 \frac{k_1}{(1-s)^{1-\delta+\rho}}\frac{1}{s^{1-\rho}}ds\|f\|_{T,n,\rho} \\ &= cv^{\delta-1}\|f\|_{T,n,\rho}. \end{aligned}$$

Then the estimates

$$\left\|\int_0^t e^{(t-v)B_n}(Hf)(v)dv\right\| \le \int_0^t k_B(0)cv^{\delta-1}dv\|f\|_{T,n,\rho} \le \tilde{c}T^{\delta}\|f\|_{T,n,\rho},$$

$$\begin{aligned} &\left\|(-B_n)^{1-\rho}\int_0^t e^{(t-v)B_n}(Hf)(v)dv\right\| \le \\ &\le \int_0^t \frac{k_B(1-\rho)}{(t-v)^{1-\rho}}cv^{\delta-1}dv\|f\|_{T,n,\rho} \\ &= ct^{\delta+\rho-1}\int_0^1 \frac{k_B(1-\rho)}{(1-v)^{1-\rho}}v^{\delta-1}dv\|f\|_{T,n,\rho} \\ &\le \frac{\tilde{c}}{t^{1-\rho}}T^{\delta}\|f\|_{T,n,\rho}, \end{aligned}$$

$$\begin{aligned} &\|\int_0^t e^{(t-v)B_n}(Hf)(v)dv - \int_0^s e^{(s-v)B_n}(Hf)(v)dv\| \\ &\le \int_0^s \|e^{(t-v)B_n} - e^{(s-v)B_n}\|\,\|(Hf)(v)\|dv + \int_s^t k_B(0)\|(Hf)(v)\|dv \\ &\le \int_0^s k_B(1)\log\frac{t-v}{s-v}\frac{c}{v^{1-\delta}}dv\|f\|_{T,n,\rho} + k_B(0)\int_s^t cT^{\delta}v^{-1}dv\|f\|_{T,n,\rho} \\ &\le k_B(1)c\int_0^s \left(\frac{t-s}{s-v}\right)^{1-\rho}\frac{dv}{v^{1-\delta}}\|f\|_{T,n,\rho} + k_B(0)cT^{\delta}\log\frac{t}{s}\|f\|_{T,n,\rho} \\ &\le \tilde{c}\,T^{\delta}\left(\frac{t-s}{s}\right)^{1-\rho}\|f\|_{T,n,\rho}\,, \end{aligned}$$

show that (4.21) holds for $\mathcal{Q}_3$. To estimate $\mathcal{Q}_2$, we set

$$(Hf)(v) = \int_0^v \Big(A_m^3 e^{(v-s)A_m}[A_m^{-1}; e^{(v-s)B_n}] + \\ + A_m^2 e^{(v-s)A_m}[A_m^{-1}; B_n e^{(v-s)B_n}] \Big) (f(s) - f(v)) ds,$$

we notice that, by (4.19),

$$\begin{aligned} \|(Hf)(v)\| &\leq \int_0^v \frac{k_4}{(v-s)^{2-\delta}} \left(\frac{v-s}{s}\right)^{1-\rho} ds \|f\|_{T,n,\rho} \\ &= v^{\delta-1} \int_0^1 \frac{k_4}{(1-s)^{1-\delta+\rho} s^{1-\rho}} ds \, \|f\|_{T,n,\rho} \\ &= c v^{\delta-1} \|f\|_{T,n,\rho}, \end{aligned}$$

and we proceed as before. The estimate for $\mathcal{Q}_1$ is easier. Proposition 4.4 is proved. □

5. Perturbation theorems for generators of analytic semigroups

Throughout this section we assume that Hypotheses 3, 4 hold.

Recall the notation $\Sigma'_{\omega,\vartheta} := \{z \in \mathbb{C} : z \neq \omega, |\arg(z-\omega)| < \vartheta\}$.

LEMMA 5.1. *For every ϑ satisfying $\pi/2 < \vartheta < \pi - \max(\vartheta_A, \vartheta_B)$ there exist $K > 0$, $\omega \geq 0$, depending only on ϑ and on the constants in Hypotheses 3, 4, such that for every m, n,*

$$\rho(A_m + B_n) \supset \Sigma'_{\omega,\vartheta} \ ,$$

and

$$\|(\lambda - A_m - B_n)^{-1}\|_{\mathcal{L}(X)} \leq \frac{K}{|\lambda - \omega|} \quad \textit{for } \lambda \in \Sigma'_{\omega,\vartheta}. \tag{5.1}$$

Equivalently,

$$A_m + B_n \in AG(K, \omega, \vartheta), \tag{5.2}$$

for every m, n, with K, ω independent from m, n.

Proof. We only sketch the proof, since the argument is the same as in [9], Lemma 4.3. For any real number φ with $|\varphi| \le \vartheta - \pi/2$, put $A_{\varphi,m} := e^{i\varphi} A_m$, $B_{\varphi,n} := e^{i\varphi} B_n$. Then it can be shown that $A_{\varphi,m}$, $B_{\varphi,n}$ satisfy Hypotheses 3, 4 uniformly with respect to m, n, φ. Applying Proposition 4.4 to $A_{\varphi,m}$, $B_{\varphi,n}$ we obtain

$$\|e^{z(A_m+B_n)}\| = \|e^{te^{i\varphi}(A_m+B_n)}\| = \|e^{t(A_{\varphi,m}+B_{\varphi,n})}\| \le \overline{M} e^{t\overline{\omega}},$$

for all $z \in \mathbb{C}$ with $|\arg(z)| = |\varphi| \le \vartheta - \pi/2$, $|z| = t$.

Next we consider the well known equality

$$(\lambda - A_m - B_n)^{-1} = \int_0^\infty e^{-\lambda t} e^{t(A_m+B_n)} dt, \quad \lambda \in \mathbb{R}, \lambda > \overline{\omega}. \quad (5.3)$$

By shifting the path of integration from $[0, \infty)$ to

$$\gamma_1 := \{z = re^{i\vartheta_1}, r \ge 0\}$$

for suitable ϑ_1 we see that the integral in (5.3) extends to an analytic function of λ in $\Sigma'_{\omega,\vartheta}$, and the estimate (5.1) follows. □

LEMMA 5.2. *If ϑ satisfies $\pi/2 < \vartheta < \pi - \max(\vartheta_A, \vartheta_B)$, then there exist $K > 0, \omega \ge 0$ such that for every m,*

$$(A_m + B) \in AG(K, \omega, \vartheta). \quad (5.4)$$

Moreover, for every compact interval $I \subset (0, \infty)$,

$$\lim_{n\to\infty} e^{t(A_m+B_n)} = e^{t(A_m+B)}, \quad (5.5)$$

in $\mathcal{L}(X)$, uniformly for $t \in I$.

Proof. By Lemma 5.1 there exist $K > 0, \omega \ge 0$ such that $A_m + B_n \in AG(K, \omega, \vartheta)$, for all m, n. Let us fix m and show that there exists

$$\lim_{n\to\infty} (\lambda - A_m - B_n)^{-1} = (\lambda - A_m - B)^{-1}, \quad (5.6)$$

in the norm of $\mathcal{L}(X)$, for $\lambda \in \Sigma'_{\omega,\vartheta}$. Since Hypothesis 3 holds uniformly with respect to n, we can choose $\lambda_1 > 0$ so large that $\lambda_1 \in \Sigma^1_{\omega,\vartheta}$ and

$$\|(\lambda_1 - B_n)^{-1} A_m\| \le c_B \lambda_1^{-1} \|A_m\| < 1, \quad (5.7)$$

for every n. From the equality $\lambda_1 - A_m - B_n = (\lambda_1 - B_n)(1 - (\lambda_1 - B_n)^{-1} A_m)$ it follows that

$$(\lambda_1 - A_m - B_n)^{-1} = \sum_{h=0}^{\infty} (\, (\lambda_1 - B_n)^{-1} A_m \,)^h \, (\lambda_1 - B_n)^{-1} \tag{5.8}$$

and this series converges in $\mathcal{L}(X)$ uniformly with respect to n, by (5.7). Since $(\lambda_1 - B_n)^{-1} \to (\lambda_1 - B)^{-1}$ in $\mathcal{L}(X)$ for $n \to \infty$, $(\lambda_1 - A_m - B_n)^{-1}$ converges in $\mathcal{L}(X)$ for $n \to \infty$. By Proposition 2.2, there exists $C_m \in AG(K, \omega, \vartheta)$ such that, for all $\lambda \in \Sigma^1_{\omega,\vartheta}$,

$$\lim_{n\to\infty} (\lambda - A_m - B_n)^{-1} = (\lambda - C_m)^{-1}, \tag{5.9}$$

in $\mathcal{L}(X)$. Passing to the limit in (5.8) yields

$$(\lambda_1 - C_m)^{-1} = \sum_{h=0}^{\infty} ((\lambda_1 - B)^{-1} A_m)^h (\lambda_1 - B)^{-1},$$

and this equals $(\lambda_1 - A_m - B)^{-1}$, as it can be easily seen. So $C_m = A_m + B$.

Finally, (5.5) follows from (2.10). □

THEOREM 5.3. *Assume that Hypotheses 3, 4 hold and let ϑ satisfy*

$$\pi/2 < \vartheta < \pi - \max(\vartheta_A, \vartheta_B).$$

Then one can find $\omega \geq 0$ such that there exists a unique graph $(A + B)^\sim$ satisfying

$$\lim_{m\to\infty} (A_m + B - \lambda)^{-1} = ((A + B)^\sim - \lambda)^{-1}, \tag{5.10}$$

in $\mathcal{L}(X)$, for all $\lambda > \omega$. Moreover,

(i) $(A + B)^\sim$ is an extension of $A + B$;

(ii) there exists $K > 0$ such that $(A + B)^\sim \in AG(K, \omega, \vartheta)$;

(iii) (5.10) holds for every $\lambda \in \Sigma^1_{\omega,\vartheta}$;

(iv) for every compact interval $I \subset (0,\infty)$,

$$e^{t(A+B)^{\sim}} = \lim_{m\to\infty} e^{t(A_m+B)} = \lim_{m\to\infty}\lim_{n\to\infty} e^{t(A_m+B_n)}, \qquad (5.11)$$

in $\mathcal{L}(X)$, uniformly for $t \in I$.

Proof. We apply Proposition 2.2 to the operators $A_m + B$. By Lemma 5.2 there exist $K > 0$, $\omega \geq 0$ such that $A_m + B \in AG(K,\omega,\vartheta)$. Lemma 3.8 and (3.20) show that, for $\lambda_0 > 0$ sufficiently large, the limit $\lim_{m\to\infty}(A_m+B-\lambda_0)^{-1}$ exists in the norm of $\mathcal{L}(X)$. Therefore there exists a unique graph $(A+B)^{\sim}$ such that

$$\lim_{m\to\infty}(A_m + B - \lambda)^{-1} = ((A+B)^{\sim} - \lambda)^{-1}, \qquad (5.12)$$

in $\mathcal{L}(X)$ for all $\lambda \in \Sigma'_{\omega,\vartheta}$, and $(A+B)^{\sim} \in AG(K,\omega,\vartheta)$. (5.11) follows from (5.5) and (2.10). Now we only have to prove that $(A+B)^{\sim}$ is an extension of $A+B$. It suffices to show $(x,y) \in A+B-\lambda_0 \Rightarrow (x,y) \in (A+B)^{\sim} - \lambda_0$ i.e.

$$x \in D_A \cap D_B,\ y = (A+B-\lambda_0)x \quad \Rightarrow \quad x = ((A+B)^{\sim} - \lambda_0)^{-1} y.$$

Note that

$$y = (A+B-\lambda_0)x \quad \Rightarrow \quad y = (A_m+B-\lambda_0)x + (A-A_m)x \quad \Rightarrow$$
$$\Rightarrow \quad (A_m+B-\lambda_0)^{-1}y = x + (A_m+B-\lambda_0)^{-1}(A-A_m)x$$
$$\Rightarrow \quad (A_m+B-\lambda_0)^{-1}y = x + (A_m+B-\lambda_0)^{-1}A(A-m)^{-1}Ax.$$

By Lemma 3.8, $(A_m+B-\lambda_0)^{-1}y = x + U_{m,\lambda}A(A-m)^{-1}Ax$.

Then, letting $m \to \infty$ we obtain, by (5.12) and Lemma 3.9, $((A+B)^{\sim} - \lambda_0)^{-1}y = x$. □

REMARK 5.4. Under the assumptions of Theorem 5.3, there are situations where $A+B$ is a closable operator, but $\overline{A+B}$ has empty resolvent set (see [13]). So in general $(A+B)^{\sim} \neq \overline{A+B}$. The next Corollary gives a necessary and sufficient condition in order that $(A+B)^{\sim} = \overline{A+B}$. It is essentially due to [13], Lemma 2.3 (cfr. also [5]). We omit the proof, since it is the same as [9], Theorem 5.6.

Corollary 5.5. *Under the assumptions of Theorem 5.3, the following holds.*

(*i*) *If* $\mathcal{R}(A+B-\lambda_0)$ *is dense in* X *for a* $\lambda_0 > \omega$, *then we have* $(A+B)^\sim = \overline{A+B}$.

(*ii*) *Conversely, if* $(A+B)^\sim = \overline{A+B}$, *then* $\mathcal{R}(A+B-\lambda)$ *is dense in* X, *for every* $\lambda > \omega$.

Remark 5.6. If D_A or D_B is dense in X, then $\mathcal{R}(A+B-\lambda)$ is also dense, for λ large. Indeed, as a consequence of Proposition 3.5, we have $\mathcal{R}(A+B-\lambda) \supset D_A(\theta,p) \supset D_A$ and $\mathcal{R}(A+B-\lambda) \supset D_B(\theta,p) \supset D_B$.

The following Corollary exhibits an explicit formula for the resolvent of $(A+B)^\sim$.

Corollary 5.7. *Under the assumptions of Theorem 5.3, there exists* $\lambda^* > 0$ *such that for all* $\lambda > \lambda^*$,

$$((A+B)^\sim - \lambda)^{-1} = U_\lambda, \tag{5.13}$$

where U_λ *is given by (3.18).*

Proof. Let λ^* be as in Section 3. We can suppose $\lambda^* > \omega$. Then by Lemma 3.8 $(A_m + B - \lambda)^{-1} = U_{m,\lambda}$. Now let $m \to \infty$ and recall (3.20) and (5.10). □

The uniqueness of $(A+B)^\sim$ is discussed in the following

Corollary 5.8. *Under the assumptions of Theorem 5.3, suppose there exist* $\vartheta_1 \in (\pi/2, \pi)$, $K_1 > 0$, $\omega_1 \geq 0$ *and a graph*

$$C \in AG(K_1, \omega_1, \vartheta_1)$$

such that C *is an extension of* $A+B$. *Then*

(*i*) *if* $y \in \overline{\mathcal{R}(A+B-\lambda)}$ *for every* λ *in some interval* $(\lambda_0, \lambda_1) \subset \mathbb{R}$ *with* $\lambda_1 > \lambda_0 > \max(\omega, \omega_1)$ *we have*

$$\begin{aligned} &((A+B)^\sim - \mu)^{-1} y = (C-\mu)^{-1} y, \quad \mu \in \Sigma'_{\vartheta,\omega} \cap \Sigma'_{\vartheta_1,\omega_1}, \\ &e^{tC} y = e^{t(A+B)^\sim} y, \quad t > 0; \end{aligned} \tag{5.14}$$

(ii) if $\mathcal{R}(A+B-\lambda)$ *is dense in* X *for a* $\lambda > \max(\omega, \omega_1)$ *we have*

$$(A+B)^{\sim} = C.$$

Proof. The proof is the same as in [9], Theorem 5.5. □

COROLLARY 5.9. *Under the assumptions of Theorem 5.3, we have, for* $t > 0$ *and for every integer* $n \geq 1$,

$$\mathcal{R}(e^{t(A+B)^{\sim}}) \subset D_{(A+B)^n}; \tag{5.15}$$

$$\frac{d^n}{dt^n} e^{t(A+B)^{\sim}} = (A+B)^n e^{t(A+B)^{\sim}}. \tag{5.16}$$

Moreover, with continuous embedding,

$$D_A \cap D_B \subset D_{(A+B)^{\sim}} \subset D_{A^2}(1/2, \infty) \cap D_{B^2}(1/2, \infty). \tag{5.17}$$

REMARK 5.10. Concerning (5.17), we recall that a norm can be defined in $D_{(A+B)^{\sim}}$ as in Section 2., formula (2.11). We also set $\|x\|_{D_A \cap D_B} := \|x\|_{D_A} + \|x\|_{D_B}$,

$$\|x\|_{D_{A^2}(1/2,\infty) \cap D_{B^2}(1/2,\infty)} := \|x\|_{D_{A^2}(1/2,\infty)} + \|x\|_{D_{B^2}(1/2,\infty)}.$$

REMARK 5.11. From general results on graphs generating analytic semigroups (see [7], Theorem 3.2), instead of (5.15) and (5.16) (say for $n = 1$) one would only obtain the differential inclusion

$$\frac{d}{dt} e^{t(A+B)^{\sim}} - (A+B)^{\sim} e^{t(A+B)^{\sim}} \ni 0, \quad t > 0.$$

Proof of Corollary 5.9. We only sketch the proof, since it is similar to [9], Theorem 5.4. Choose $\lambda_1 > 0$ sufficiently large. Then $0 \in \rho((A+B)^{\sim} - \lambda_1)$ and $(A+B)^{\sim} - \lambda_1$ generates $S_1(t) := e^{-t\lambda_1} e^{t(A+B)^{\sim}}$. By (2.7) and Corollary 5.7

$$S_1(t) = ((A+B)^{\sim} - \lambda_1)^{-n-1} \frac{d^{n+1}}{dt^{n+1}} S_1(t) = (U_{\lambda_1})^{n+1} \frac{d^{n+1}}{dt^{n+1}} S_1(t).$$

By Corollary 3.13 we have $\mathcal{R}(S_1(t)) \subset D_{(A+B)^n}$ (i.e. (5.15)) and

$$(A+B-\lambda_1)^n S_1(t) = U_{\lambda_1} \frac{d^{n+1}}{dt^{n+1}} S_1(t).$$

Using the integral in (2.6) and the resovent identity (the latter being valid for graphs, too), one proves

$$\frac{d^n}{dt^n} S_1(t) = (A+B-\lambda_1)^n S_1(t).$$

(5.16) follows from this.

Finally, note that by Corollary 5.7 we have $\mathcal{R}(U_\lambda) = D_{(A+B)^\sim}$. (5.17) is then a consequence of (3.28). □

Under some density assumptions for the domains of A and B the conclusions of Theorem 5.3 can be strengthened and made more precise. Recall that $D_A \subset D_A(\theta,p)$, $D_B \subset D_B(\theta,p)$.

THEOREM 5.12. *Assume that Hypotheses 3, 4 hold and suppose that there exist $\vartheta \in (0,1)$ and $p \in [1,\infty]$ such that $D_A(\vartheta,p)$ or $D_B(\vartheta,p)$ is dense in X. Then $\overline{A+B}$ generates an analytic semigroup.*

More precisely, for every ϑ satisfying

$$\pi/2 < \vartheta < \pi - \max(\vartheta_A, \vartheta_B)$$

there exist $K > 0$ and $\omega \geq 0$ such that

$$\overline{A+B} \in AG(K,\omega,\vartheta). \tag{5.18}$$

($\overline{A+B}$ is a graph, in general.)

$\overline{A+B}$ is the unique graph that simultaneously extends $A+B$ and generates an analytic semigroup.

Proof. By Proposition 3.5, for $\lambda > 0$ large we have

$$D_A(\vartheta,p) \subset \mathcal{R}(A+B-\lambda) \quad \text{and} \quad D_B(\vartheta,p) \subset \mathcal{R}(A+B-\lambda)\,,$$

so that $\mathcal{R}(A+B-\lambda)$ is dense in X. Then (5.18) follows from Theorem 5.3 and Corollary 5.5. Uniqueness follows from Corollary 5.8. □

THEOREM 5.13. *Assume that Hypotheses 3, 4 hold and suppose that $D_A \cap D_B$ is dense in X. Then $A+B$ is a closable operator and the (single-valued) operator $\overline{A+B}$ generates a strongly continuous analytic semigroup. More precisely, for every ϑ satisfying*

$$\pi/2 < \vartheta < \pi - \max(\vartheta_A, \vartheta_B).$$

there exist $K > 0$, $\omega \geq 0$ such that $\overline{A+B} \in AG(K, \omega, \vartheta)$.

Proof. Since $D_A \cap D_B$ is dense, $A+B$ is closable by Corollary 3.11. So $(A+B)^\sim = \overline{A+B}$ is a single-valued operator. Since $D_{\overline{A+B}} \supset D_A \cap D_B$ and $D_A \cap D_B$ is dense, then $D_{\overline{A+B}}$ is also dense. By a well known result (see e.g. [16], Theorem 1.7.7), the semigroup generated by $\overline{A+B}$ is strongly continuous. □

6. Some results on interpolation spaces

This section is devoted to giving a characterization of the real interpolation spaces between $D_{(A+B)^\sim}$ and X, where $(A+B)^\sim$ is defined in (5.10). We extend some results of [13]. We assume Hypotheses 3, 4 throughout.

LEMMA 6.1. *Let δ be as in (4.4) and S_λ, S'_λ be given in (3.9), (3.10). Then for every $\overline{\lambda} > 0$, $\vartheta \in (0, \delta)$ there exist $c, c_p > 0$ such that for every $y \in D_A(\vartheta, p) \cap D_B(\vartheta, p)$ we have $S_\lambda y \in D_A$, $S'_\lambda y \in D_B$ and*

$$\left(\int_{\overline{\lambda}}^{\infty} \|\lambda^\vartheta A S_\lambda y\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \leq c_p \|y\|_{D_A(\vartheta,\infty)}, \qquad \text{if } p \in [1,\infty), \tag{6.1}$$

$$\left(\int_{\overline{\lambda}}^{\infty} \|\lambda^\vartheta B S'_\lambda y\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \leq c_p \|y\|_{D_B(\vartheta,\infty)}, \qquad \text{if } p \in [1,\infty), \tag{6.2}$$

$$\lambda^\vartheta \|A S_\lambda y\| \leq c \|y\|_{D_A(\vartheta,\infty)}, \qquad \text{for } \lambda > \overline{\lambda},\ \text{if } p = \infty, \tag{6.3}$$

$$\lambda^\vartheta \|B S'_\lambda y\| \leq c \|y\|_{D_B(\vartheta,\infty)}, \qquad \text{for } \lambda > \overline{\lambda},\ \text{if } p = \infty \tag{6.4}$$

Proof. Let $\lambda > \overline{\lambda}$; let γ_0 be the path (already defined in Section 3.)

$$\begin{aligned}\gamma_0 \quad := & \quad \{\lambda \in \mathbb{C} : \lambda = re^{-i\vartheta_0}, r \in [0,\infty)\} \cup \\ & \quad \cup \{\lambda \in \mathbb{C} : \lambda = re^{i\vartheta_0}, r \in [0,\infty)\},\end{aligned}$$

oriented with increasing imaginary part, where θ_0 satisfies $\pi/2 < \theta_0 < \pi - \theta_A$. Then

$$\begin{aligned}AS_\lambda y &= -\int_{\gamma_0} A(A-z-\lambda)^{-1}(B-\lambda+z)^{-1}y \\ &= -\int_{\gamma_0} [A(A-z-\lambda)^{-1}; (B-\lambda+z)^{-1}]ydz \\ &\quad -\int_{\gamma_0} (B-\lambda+z)^{-1}A(A-z-\lambda)^{-1}ydz \\ &= -\int_{\gamma_0} (z+\lambda)(A-\lambda)(A-z-\lambda)^{-1} \cdot \\ &\quad \cdot [(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-z-\lambda)^{-1}ydz \\ &\quad -\int_{\gamma_0} (B-\lambda+z)^{-1}A(A-z-\lambda)^{-1}ydz \\ &= \int_{\gamma_0} I_1(z,\lambda)dz + \int_{\gamma_0} I_2(z,\lambda)dz.\end{aligned}$$

(The estimates below show that these integrals are absolutely convergent.) By Lemma 3.2 we have, for $\lambda > \overline{\lambda}$,

$$\begin{aligned}\left\|\int_{\gamma_0} I_1(z,\lambda)dz\right\| &\le \\ &\le \frac{1}{2\pi}\int_{\gamma_0} |z+\lambda| \sum_{i=1}^{k} \frac{c(\overline{\lambda})}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}} \cdot \\ &\quad \cdot \left(1 + \frac{c_A|z|}{|z+\lambda|}\right)\|y\||dz| \\ &\le c\lambda^{-\delta}\|y\| \int_{\gamma_0} \sum_{i=1}^{k} \frac{|v+1|}{|v+1|^{1-\alpha_i}|v-1|^{1+\beta_i}}|dv| \\ &= c\lambda^{-\delta}\|y\|.\end{aligned}$$

Therefore to prove (6.1) it suffices to show that

$$\left(\int_{\overline{\lambda}}^{\infty} \|\lambda^{\vartheta} \fint_{\gamma_0} I_2(z,\lambda)dz\|^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}} \leq c\|y\|_{D_A(\vartheta,p)}. \tag{6.5}$$

Since $D_A(\vartheta,p) \subset D_A(\vartheta,\infty)$ we have

$$\|I_2(z,\lambda)\| \leq \frac{c_B}{|z-\lambda|}\frac{1}{|z+\lambda|^{\vartheta}} c\|y\|_{D_A(\vartheta,p)}, \quad z \in \Sigma_{\vartheta_A} \cap (-\Sigma_{\vartheta_B}),$$

(see (2.3)). Therefore $\fint_{\gamma_0} I_2(z,\lambda)dz$ can be computed over $\gamma_0 - \lambda$ instead of γ_0. By the change of variable $v = z + \lambda$ we obtain

$$\begin{aligned}\fint_{\gamma_0} I_2(z,\lambda)dz &= \fint_{\gamma_0-\lambda} I_2(z,\lambda)dz \\ &= -\fint_{\gamma_0} (B-2\lambda+v)^{-1}A(A-v)^{-1}ydv.\end{aligned}$$

Let us set $\varphi(r) := \max\left(\|A(A-re^{i\vartheta_0})^{-1}y\|, \ \ \|A(A-re^{-i\vartheta_0})^{-1}y\| \right)$. Then by the elementary inequality $2|v-2\lambda| \geq |v|+2\lambda$, $\lambda > \overline{\lambda}$, $v \in \gamma_0$, we have

$$\|\fint_{\gamma_0} I_2(z,\lambda)dz\| \leq 2\frac{1}{2\pi}\int_0^{\infty} \frac{2c_B}{r+2\lambda}\varphi(r)dr.$$

Since $y \in D_A(\vartheta,p)$ we have

$$\left(\int_0^{\infty} |r^{\vartheta}\varphi(r)|^p \frac{dr}{r}\right)^{\frac{1}{p}} \leq c\|y\|_{D_A(\vartheta,p)}$$

Then we obtain

$$\begin{aligned}\left(\int_{\overline{\lambda}}^{\infty} \left\|\lambda^{\vartheta} \fint_{\gamma_0} I_2(z,\lambda)dz\right\|^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}} &\leq \\ &\leq c\left(\int_0^{\infty} |\lambda^{\theta}\int_0^{\infty} \frac{1}{r+2\lambda}\varphi(r)dr|^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}} \\ &= c\left(\int_0^{\infty} |\int_0^{\infty} \frac{(\lambda r^{-1})^{\vartheta}}{1+2\lambda r^{-1}} r^{\vartheta}\varphi(r)\frac{dr}{r}|^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}.\end{aligned}$$

In the space $L^p((0,\infty);\frac{dr}{r})$ this is the norm of the multiplicative convolution of the functions $r \mapsto r^\vartheta \varphi(r)$ and $r \mapsto \frac{r^\vartheta}{1+2r}$, with respect to the measure $\frac{dr}{r}$. By Young's Theorem we obtain

$$\begin{aligned}
\left(\int_{\overline{\lambda}}^\infty \left\| \lambda^\vartheta \int_{\gamma_0} I_2(z,\lambda) dz \right\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} &\leq \\
&\leq c \left(\int_0^\infty \frac{r^\vartheta}{1+2r} \frac{dr}{r} \right) \left(\int_0^\infty |r^\vartheta \varphi(r)|^p \frac{dr}{r} \right)^{\frac{1}{p}} \\
&\leq c \|y\|_{D_A(\vartheta,p)}.
\end{aligned}$$

Now (6.5) is proved, and so is (6.1).

Now we prove (6.2).

$$\begin{aligned}
BS'_\lambda y &= -\int_{\gamma_0} B(B-\lambda+z)^{-1}(A-z-\lambda)^{-1} y \\
&= -\int_{\gamma_0} [B(B-\lambda+z)^{-1};(A-z-\lambda)^{-1}] y dz \\
&\quad -\int_{\gamma_0} (A-z-\lambda)^{-1} B(B-\lambda+z)^{-1} y dz \\
&= +\int_{\gamma_0} (\lambda-z)(A-\lambda)(A-z-\lambda)^{-1} \cdot \\
&\quad \cdot [(A-\lambda)^{-1};(B-\lambda+z)^{-1}](A-\lambda)(A-z-\lambda)^{-1} y dz \\
&\quad -\int_{\gamma_0} (A-z-\lambda)^{-1} B(B-\lambda+z)^{-1} y dz \\
&= \int_{\gamma_0} I_1(z,\lambda) dz + \int_{\gamma_0} I_2(z,\lambda) dz.
\end{aligned}$$

As before we have $\left\| \int_{\gamma_0} I_1(z,\lambda) dz \right\| \leq c\lambda^{-\delta}\|y\|$, $\lambda > \overline{\lambda}$, and to prove (6.2) it suffices it verify that

$$\left(\int_{\overline{\lambda}}^\infty \left\| \lambda^\vartheta \int_{\gamma_0} I_2(z,\lambda) dz \right\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \leq c\|y\|_{D_B(\vartheta,p)}.$$

This can be proved as (6.5), i.e. computing $\int_{\gamma_0} I_2(z,\lambda)dz$ over $\gamma_0+\lambda$, setting $v = z-\lambda$ and using Young's Theorem.

Finally, (6.3) and (6.4) (i.e. the case $p = \infty$) can be proved with obvious modifications. $\square$

THEOREM 6.2. *Assume Hypotheses 3, 4, and let $(A+B)^\sim$ be defined by (5.10). Then*

$$D_{(A+B)^\sim}(\vartheta, p) = D_A(\vartheta, p) \cap D_B(\vartheta, p), \qquad \vartheta \in (0, \delta),\ p \in [1, \infty]. \tag{6.6}$$

REMARK 6.3. δ is the same as in (4.4). $D_{(A+B)^\sim}(\vartheta, p)$ is the real interpolation space $(X, D_{(A+B)^\sim})_{\vartheta,p}$ defined in Subsection 2.3 and $D_A(\vartheta, p) \cap D_B(\vartheta, p)$ is endowed with the norm

$$\|x\|_{D_A(\vartheta,p)\cap D_B(\vartheta,p)} := \|x\|_{D_A(\vartheta,p)} + \|x\|_{D_B(\vartheta,p)}.$$

The equality (6.6) is understood in the sense that for any fixed ϑ and p the two spaces coincide with equivalence of norms.

Proof. **Step 1.** Let us show the inclusion

$$D_{(A+B)^\sim}(\vartheta, p) \subset D_A(\vartheta, p) \cap D_B(\vartheta, p). \tag{6.7}$$

In the Corollary 5.9 we proved the continuous inclusion

$$D_{(A+B)^\sim} \subset D_{A^2}(1/2, \infty) \cap D_{B^2}(1/2, \infty)\,.$$

So the identity operator

$$\mathrm{id}\ : D_{(A+B)^\sim} \to D_{A^2}(1/2, \infty)$$

is continuous. Since evidently id $: X \to X$ is continuous, by interpolation id $: D_{(A+B)^\sim}(\vartheta, p) = (X, D_{(A+B)^\sim})_{\vartheta,p} \to (X, D_{A^2}(1/2, \infty))_{\vartheta,p}$ is also continuous. By a result of [10] (Cor. I.3.2 and Rem. I.3.1) we have $(X, D_{A^2}(1/2, \infty))_{\vartheta,p} = D_A(\vartheta, p)$. So we have $D_{(A+B)^\sim}(\vartheta, p) \subset D_A(\vartheta, p)$. In a similar way one can show $D_{(A+B)^\sim}(\vartheta, p) \subset D_B(\vartheta, p)$ and (6.7) is proved.

Step 2. Let us prove the inclusion

$$D_{(A+B)^\sim}(\vartheta, p) \supset D_A(\vartheta, p) \cap D_B(\vartheta, p). \tag{6.8}$$

We fix $\lambda^* > 0$ sufficiently large and $\lambda > \lambda^*$. For $y \in D_A(\vartheta, p) \cap D_B(\vartheta, p)$, let $x \in D_A \cap D_B$ be the unique solution of the equation $(A - \lambda)x + (B - \lambda)x = y$. Such an x exists by Proposition 3.5, and $x = U_\lambda y$, by Proposition 3.10.

By Corollary 5.7, $x = ((A + B)^\sim - 2\lambda)^{-1} y$. So we have

$$y - 2\lambda(2\lambda - (A + B)^\sim)^{-1} y = y + 2\lambda x = Ax + Bx = AU_\lambda y + BU_\lambda y.$$

By Proposition 2.3 it suffices to show

$$\left(\int_{\lambda^*}^\infty \|\lambda^\vartheta A U_\lambda y\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \leq c_p \Big(\|y\|_{D_A(\vartheta,p)} + \|y\|_{D_B(\vartheta,p)} \Big), \quad (6.9)$$

$$\left(\int_{\lambda^*}^\infty \|\lambda^\vartheta B U_\lambda y\|^p \frac{d\lambda}{\lambda} \right)^{\frac{1}{p}} \leq c_p \Big(\|y\|_{D_A(\vartheta,p)} + \|y\|_{D_B(\vartheta,p)} \Big), \quad (6.10)$$

for $p \in [1, \infty)$, with obvious modifications for the case $p = \infty$. By Proposition 3.5, the vector x introduced above is also given by the formula $x = (A - \lambda)^{-1}(1 + J_\lambda)^{-1}(A - \lambda)S_\lambda y$. Taking into account the identity $(1 + J_\lambda)^{-1} = 1 - J_\lambda(1 + J_\lambda)^{-1}$ we obtain

$$U_\lambda y = x = S_\lambda y - (A - \lambda)^{-1} J_\lambda (1 + J_\lambda)^{-1} (A - \lambda) S_\lambda y.$$

By Lemma 3.3 we have, for every $\lambda > \lambda^*$,

$$\begin{aligned} \|AU_\lambda y\| &\leq \|AS_\lambda y\| + \|A(A - \lambda)^{-1}\| \|J_\lambda\| \cdot \\ &\quad \cdot \|(1 + J_\lambda)^{-1}\| (\|AS_\lambda y\| + \|\lambda S_\lambda y\|) \\ &\leq \|AS_\lambda y\| (1 + \|A(A - \lambda)^{-1}\| \, \|J_\lambda\| \, \|(1 + J_\lambda)^{-1}\|) \\ &\quad + \|A(A - \lambda)^{-1}\| \|J_\lambda\| \|(1 + J_\lambda)^{-1}\| \|\lambda S_\lambda y\| \\ &\leq \|AS_\lambda y\| (1 + (1 + c_A) c \lambda^{-\delta}) + (1 + c_A) c \lambda^{-\delta} \|y\| \\ &\leq c \|AS_\lambda y\| + c \lambda^{-\delta} \|y\|. \end{aligned}$$

Then (6.9) follows from (6.1) and (6.3). To prove (6.10) recall that by (3.21) $U_\lambda = S'_\lambda + (\lambda - B)^{-1} Q_\lambda$, where $Q_\lambda \in \mathcal{L}(X)$ and $\|Q_\lambda\| \leq c(\lambda^*)\lambda^{-\delta}$ with $c(\lambda^*)$ independent from λ. Therefore we obtain

$$\begin{aligned} \|BU_\lambda y\| &\leq \|BS'_\lambda y\| + \|B(\lambda - B)^{-1} Q_\lambda y\| \\ &\leq \|BS'_\lambda y\| + (1 + c_B) \|Q_\lambda\| \|y\| \\ &\leq \|BS'_\lambda y\| + c\lambda^{-\delta} \|y\|, \quad \forall \lambda \geq \lambda^*. \end{aligned}$$

(6.10) follows from (6.2) and (6.4), and step 2 is finished. $\square$

7. Appendix

This section is devoted to proving Lemma 4.6. In the sequel we will use only Hypotheses 3, 4 and the Remark 4.3, the well-known formulae (4.13), (4.14) and the material introduced in Section 2..

For convenience, we state again Lemma 4.6. As before,

$$A_m := mA(m-A)^{-1},\ B_n := nB(n-B)^{-1},$$

for every pair of integers $m, n \geq 1$.

LEMMA 7.1. *Assume Hypotheses 3, 4. Then for every $\rho \in (0,\delta)$ (see (4.4)) there exists $k_1 = k_1(\rho) > 0$, such that*

$$\|A_m^2 e^{tA_m}[A_m^{-1}; e^{tB_n}](-B_n)^\rho\| \leq k_1 t^{-1+\delta-\rho}, \quad t \in (0,1],\ m,n \geq 1.$$

For the proof we need another preliminary lemma. It consists simply in a convenient splitting of an expression that we will meet later.

LEMMA 7.2. *Let us define*

$$\begin{aligned} T_{m,t,n}(\xi,\lambda,\mu) \quad := \quad & -e^\lambda \lambda t^{-1} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ & \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}] \cdot \\ & \cdot e^\mu (\xi t^{-1})^\rho (B_n + \xi t^{-1})^{-1} t^{-3}. \end{aligned} \tag{7.1}$$

Then $T_{m,t,n}(\xi,\lambda,\mu) := \sum_{i=1}^{5} T_{m,t,n}^{(i)}(\xi,\lambda,\mu)$, *where*

$$\begin{aligned} T_{m,t,n}^{(1)}(\xi,\lambda,\mu) \quad := \quad & e^\lambda \lambda t^{-1} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ & \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}](B_n - \mu t^{-1} + \xi t^{-1})^{-1} \cdot \\ & \cdot (B_n + \xi t^{-1})^{-1} \mu t^{-1} (\xi t^{-1})^\rho e^\mu t^{-3}, \end{aligned}$$

$$\begin{aligned} T_{m,t,n}^{(2)}(\xi,\lambda,\mu) \quad := \quad & -e^\lambda \lambda t^{-1} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ & \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}](\xi t^{-1})^{\rho-1} e^\mu t^{-3}, \end{aligned}$$

$$T_{m,t,n}^{(3)}(\xi,\lambda,\mu) \quad := \quad e^\lambda \lambda t^{-1} A_m (A_m - \lambda t^{-1})^{-1} \cdot$$

$$
\begin{aligned}
&\cdot[A_m^{-1};(B_n-\mu t^{-1}+\xi t^{-1})^{-1}]\cdot\\
&\cdot(\xi t^{-1})^{\rho-1}e^{\mu}t^{-3},
\end{aligned}
$$

$$
\begin{aligned}
T^{(4)}_{m,t,n}(\xi,\lambda,\mu) \;:=\;& e^{\lambda}\lambda t^{-1}(B_n-\mu t^{-1})^{-1}A_m(A_m-\lambda t^{-1})^{-1}\cdot\\
&\cdot[A_m^{-1};(B_n-\mu t^{-1}+\xi t^{-1})^{-1}](\xi t^{-1})^{\rho}e^{\mu}t^{-3},
\end{aligned}
$$

$$
\begin{aligned}
T^{(5)}_{m,t,n}(\xi,\lambda,\mu) \;:=\;& e^{\lambda}(\lambda t^{-1})^2A_m(A_m-\lambda t^{-1})^{-1}\cdot\\
&\cdot[A_m^{-1};(B_n-\mu t^{-1})^{-1}]A_m(A_m-\lambda t^{-1})^{-1}\cdot\\
&\cdot[A_m^{-1};(B_n-\mu t^{-1}+\xi t^{-1})^{-1}](\xi t^{-1})^{\rho}e^{\mu}t^{-3},
\end{aligned}
$$

provided all the inverse operators exist and belong to $\mathcal{L}(X)$.

Proof. It consists in algebraic manipulation. To simplify the notation we write P,Q,a,b,c instead of $A_m,B_n,\lambda t^{-1},\mu t^{-1},\xi t^{-1}$ respectively.

$$
\begin{aligned}
T_{m,t,n}(\xi,\lambda,\mu) \;=&\\
=\;& -aP(P-a)^{-1}[P^{-1};(Q-b)^{-1}](Q+c)^{-1}c^{\rho}e^{\lambda}e^{\mu}t^{-3}\\
=\;& -aP(P-a)^{-1}[P^{-1};(Q-b)^{-1}](Q-b+c)^{-1}c^{\rho}e^{\lambda}e^{\mu}t^{-3}\\
&+aP(P-a)^{-1}[P^{-1};(Q-b)^{-1}]\cdot\\
&\quad\cdot(Q-b+c)^{-1}(Q+c)^{-1}bc^{\rho}e^{\lambda}e^{\mu}t^{-3}\\
=\;& -aP(P-a)^{-1}Q(Q-b)^{-1}[P^{-1};Q^{-1}]Q(Q-b)^{-1}\cdot\\
&\quad\cdot(Q-b+c)^{-1}c^{\rho}e^{\lambda}e^{\mu}t^{-3}\\
&+T^{(1)}_{m,t,n}(\xi,\lambda,\mu)\\
=\;& -aP(P-a)^{-1}Q(Q-b)^{-1}[P^{-1};Q^{-1}]Q(Q-b)^{-1}\cdot\\
&\quad\cdot c^{\rho-1}e^{\lambda}e^{\mu}t^{-3}\\
&+aP(P-a)^{-1}Q(Q-b)^{-1}[P^{-1};Q^{-1}]Q(Q-b+c)^{-1}\cdot\\
&\quad\cdot c^{\rho-1}e^{\lambda}e^{\mu}t^{-3}\\
&+T^{(1)}_{m,t,n}(\xi,\lambda,\mu)\\
=\;& T^{(2)}_{m,t,n}(\xi,\lambda,\mu)\\
&+aP(P-a)^{-1}Q(Q-b+c)^{-1}[P^{-1};Q^{-1}]\cdot\\
&\quad\cdot Q(Q-b+c)^{-1}c^{\rho-1}e^{\lambda}e^{\mu}t^{-3}\\
&+aP(P-a)^{-1}Qc(Q-b+c)^{-1}(Q-b)^{-1}[P^{-1};Q^{-1}]\cdot
\end{aligned}
$$

$$
\begin{aligned}
& \cdot Q(Q-b+c)^{-1}c^{\rho-1}e^{\lambda}e^{\mu}t^{-3} \\
& + T^{(1)}_{m,t,n}(\xi,\lambda,\mu) \\
= \;& T^{(2)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) \\
& + aP(P-a)^{-1}(Q-b)^{-1}[P^{-1};(Q-b+c)^{-1}]\cdot \\
& \cdot c^{\rho}e^{\lambda}e^{\mu}t^{-3} + T^{(1)}_{m,t,n}(\xi,\lambda,\mu) \\
= \;& T^{(2)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) \\
& + a(Q-b)^{-1}P(P-a)^{-1}[P^{-1};(Q-b+c)^{-1}]\cdot \\
& \cdot c^{\rho}e^{\lambda}e^{\mu}t^{-3} \\
& + a[P(P-a)^{-1};(Q-b)^{-1}][P^{-1};(Q-b+c)^{-1}]c^{\rho}e^{\lambda}e^{\mu}t^{-3} \\
& + T^{(1)}_{m,t,n}(\xi,\lambda,\mu) \\
= \;& T^{(2)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) + T^{(4)}_{m,t,n}(\xi,\lambda,\mu) \\
& + a^2[(P-a)^{-1};(Q-b)^{-1}][P^{-1};(Q-b+c)^{-1}]\cdot \\
& \cdot c^{\rho}e^{\lambda}e^{\mu}t^{-3} + T^{(1)}_{m,t,n}(\xi,\lambda,\mu) \\
= \;& T^{(2)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) + T^{(4)}_{m,t,n}(\xi,\lambda,\mu) \\
& + a^2P(P-a)^{-1}[P^{-1};(Q-b)^{-1}]P(P-a)^{-1}\cdot \\
& \cdot [P^{-1};(Q-b+c)^{-1}]c^{\rho}e^{\lambda}e^{\mu}t^{-3} \\
& + T^{(1)}_{m,t,n}(\xi,\lambda,\mu) \\
= \;& \sum_{i=1}^{5} T^{(i)}_{m,t,n}(\xi,\lambda,\mu).
\end{aligned}
$$

□

Proof of Lemma 7.1. Let $\rho \in (0,\delta)$. Let ϑ_0 satisfy

$$\pi/2 < \vartheta_0 < \pi - \max(\vartheta_A, \vartheta_B). \tag{7.2}$$

Let us define the path

$$
\begin{aligned}
\gamma_0 \;:=\; & \{t\exp(-i\vartheta_0) : t \in [1,\infty)\} \cup \{\exp(i\vartheta) : \vartheta \in [-\vartheta_0, \vartheta_0]\} \cup \\
& \cup \{t\exp(i\vartheta_0) : t \in [1,\infty)\},
\end{aligned}
\tag{7.3}
$$

oriented with increasing imaginary part.

For every $R > 1$ let γ_0^R be the intersection of γ_0 with the disc of radius R centered at the origin of the complex plane, namely

$$\begin{aligned}\gamma_0^R \quad := \quad & \{t\exp(-i\vartheta_0) : t \in [0,R]\} \cup \{\exp(i\vartheta) : \vartheta \in [-\vartheta_0, \vartheta_0]\} \cup \\ & \cup \{t\exp(i\vartheta_0) : t \in [0,R]\}.\end{aligned} \tag{7.4}$$

Let us fix once and for all $t \in (0,1], m, n \geq 1$. Then let us choose $R > 1$ and define $J_{m,t,n} := A_m^2 e^{tA_m}[A_m^{-1}; e^{tB_n}]$. By (4.13), (4.14),

$$\begin{aligned}J_{m,t,n} \quad &= \quad \int_{\gamma_0}\int_{\gamma_0} \lambda e^{\lambda t} A_m (A_m - \lambda)^{-1} [A_m^{-1}; (B_n - \mu)^{-1}] e^{\mu t} d\lambda d\mu \\ &= \quad \int_{\gamma_0 t}\int_{\gamma_0 t} \lambda t^{-1} e^{\lambda} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ &\qquad \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}] e^{\mu} t^{-2} d\lambda d\mu.\end{aligned}$$

Since the integrand is analytic we can deform $\gamma_0 t$ into γ_0 without affecting the integral. We obtain

$$\begin{aligned}J_{m,t,n} \quad = \quad & \int_{\gamma_0}\int_{\gamma_0} \lambda t^{-1} e^{\lambda} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ & \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}] e^{\mu} t^{-2} d\lambda d\mu.\end{aligned} \tag{7.5}$$

Let us define

$$\begin{aligned}J_{m,t,n}^R \quad := \quad & \int_{\gamma_0^R}\int_{\gamma_0^R} \lambda t^{-1} e^{\lambda} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ & \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}] e^{\mu} t^{-2} d\lambda d\mu.\end{aligned} \tag{7.6}$$

Let ϑ_1 satisfy

$$\vartheta_0 < \vartheta_1 < \pi. \tag{7.7}$$

Now let us define

$$\gamma_1 := \{t\exp(-i\vartheta_1) : t \in [0,\infty)\} \cup \{t\exp(i\vartheta_1) : t \in [0,\infty)\}, \tag{7.8}$$

oriented with increasing imaginary part. Finally for r, s with $0 < s < r$ let us define

$$\gamma_1' := \gamma_{11}^{rs} + \gamma_{12}^{s} + \gamma_{13}^{rs} + \gamma_{14}^{r} \tag{7.9}$$

where

$$\begin{aligned}\gamma_{11}^{rs} &:= \{t\exp(-i\vartheta_1) : t \in [s,r]\}\\ \gamma_{12}^{s} &:= \{s\exp(i\vartheta) : \vartheta \in [-\vartheta_1,\vartheta_1]\}\\ \gamma_{13}^{rs} &:= \{t\exp(i\vartheta_1) : t \in [s,r]\}\\ \gamma_{14}^{r} &:= \{r\exp(i\vartheta) : \vartheta \in [-\vartheta_1,\vartheta_1]\},\end{aligned}$$

oriented *clockwise*. In the following we will let $s \to 0, r \to \infty$, so that γ_1' "tends" to γ_1 (including orientation).

Since $\sigma(-B_n)$ lies in the right half plane, it lies to the right of γ_1; since it is compact, if we require

$$0 < s < \text{ dist } (\sigma(-B_n), 0), \qquad r > \|B_n\|, \tag{7.10}$$

then γ_1' surrounds $\sigma(-B_n)$. Therefore

$$(-B_n)^\rho = -\int_{\gamma_1'} (B_n + \xi)^{-1}\xi^\rho d\xi = -\int_{\gamma_1' t} (B_n + t^{-1}\xi)^{-1}(t^{-1}\xi^\rho)t^{-1}d\xi.$$

So we obtain (see (7.1), (7.6)),

$$\begin{aligned}J^R_{m,t,n}(-B_n)^\rho &= -\int_{\gamma_0^R}\int_{\gamma_0^R}\int_{\gamma_1' t} \lambda t^{-1}e^\lambda A_m(A_m - \lambda t^{-1})^{-1}\cdot\\ &\qquad \cdot[A_m^{-1};(B_n - \mu t^{-1})^{-1}]e^\mu t^{-2}(B_n + t^{-1}\xi)^{-1}\cdot\\ &\qquad \cdot(t^{-1}\xi^\rho)t^{-1}d\xi d\lambda d\mu\\ &= \int_{\gamma_0^R}\int_{\gamma_0^R}\int_{\gamma_1' t} T_{m,t,n}(\xi,\lambda,\mu)d\xi d\lambda d\mu. \end{aligned}\tag{7.11}$$

Recall the notation $\Sigma_{\vartheta_B} := \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \pi - \vartheta_B\}$. Let us define

$$V_t := \bigcup_{\mu\in\gamma_0} \{-(\mathbb{C}\backslash\Sigma_{\vartheta_B}) + \mu t^{-1}\}.$$

We claim the following:

$$V_t \text{ lies in the open region to the right of } \gamma_1. \tag{7.12}$$

Indeed, notice that $-(\mathbb{C}\backslash\Sigma_{\vartheta_B}) = \{0\} \cup \{w \in \mathbb{C} : |\arg(w)| \leq \vartheta_B\}$. Pick such a w. If $\mu \in \gamma_0$, then $\mu \neq 0$, and $|\arg(\mu t^{-1})| = |\arg(\mu)| \leq \vartheta_0$, whence $w + \mu t^{-1} \neq 0$, and, by (7.2), $|\arg(w + \mu t^{-1})| \leq \vartheta_0 < \vartheta_1$. Now the claim (7.12) is proved.

Let us define

$$V_{R,t,n} := \bigcup_{\mu \in \gamma_0^R} \{(\sigma(-B_n) + \mu t^{-1}\}.$$

Since $\sigma(-B_n) \subset -(\mathbb{C}\backslash\Sigma_{\vartheta_B})$, therefore $V_{R,t,n}$ is contained in V_t , and, by (7.12), it also lies in the open region to the right of γ_1. Since $\sigma(-B_n)$ and γ_0^R are compact, we have

$$\begin{aligned} m_{R,t,n} &:= \text{dist } (V_{R,t,n}, 0) > 0, \\ M_{R,t,n} &:= \max\{|w| : w \in V_{R,t,n}\} < \infty. \end{aligned}$$

Therefore if s and r, in addition to (7.10), satisfy

$$s < m_{R,\lambda,n} \quad \text{and} \quad r > M_{R,\lambda,n}, \tag{7.13}$$

we conclude that the closed path γ_1' surrounds $V_{R,t,n}$. Therefore:

1. We have

$$\mu t^{-1} - \xi t^{-1} \in \Sigma_{\vartheta_B}, \qquad \text{for } \mu \in \gamma_0,\ \xi \in \gamma_1. \tag{7.14}$$

Indeed, if $\mu t^{-1} - \xi t^{-1} \in \mathbb{C}\backslash\Sigma_{\vartheta_B}$, we would have

$$\xi t^{-1} \in -(\mathbb{C}\backslash\Sigma_{\vartheta_B}) + \mu t^{-1}$$

for a $\mu \in \gamma_0$; this contradicts (7.12), since $\xi t^{-1} \in t^{-1}\gamma_1 = \gamma_1$.

2. We have

$$\begin{aligned} B_n - \mu t^{-1} + \xi t^{-1} &\text{ has a bounded inverse} \\ &\text{for every } \xi \in t\gamma_1',\ \mu \in \gamma_0^R. \end{aligned} \tag{7.15}$$

Indeed, if $\xi t^{-1} \in \sigma(-B_n) + \mu t^{-1}$ for a $\mu \in \gamma_0^R$ we would have a contradiction, since $\xi t^{-1} \in \gamma_1'$ and $\gamma_1' \cap \{\sigma(-B_n) + \mu t^{-1}\} = \emptyset$, for every $\mu \in \gamma_0^R$, by (7.13).

Now we exploit the formula (7.11). By (7.15) we can apply Lemma 7.2 obtaining

$$J_{m,t,n}^R(-B_n)^\rho = \int_{\gamma_0^R}\int_{\gamma_0^R}\int_{\gamma_1' t} \sum_{i=1}^{5} T_{m,t,n}^{(i)}(\xi,\lambda,\mu)\,d\xi\, d\lambda\, d\mu.$$

Now observe that $\int_{\gamma_1' t} T^{(2)}_{m,t,n}(\xi,\lambda,\mu)d\xi = 0$, for every $\lambda, \mu \in \gamma_0^R$, by analyticity. So

$$J^R_{m,t,n}(-B_n)^\rho = \int_{\gamma_0^R}\int_{\gamma_0^R}\int_{\gamma_1' t} T'_{m,t,n}(\xi,\lambda,\mu)d\xi d\lambda d\mu,$$

where

$$\begin{aligned} T'_{m,t,n}(\xi,\lambda,\mu) \quad &:= \quad T^{(1)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) + \\ &\quad + T^{(4)}_{m,t,n}(\xi,\lambda,\mu) + T^{(5)}_{m,t,n}(\xi,\lambda,\mu). \end{aligned}$$

For fixed $\lambda, \mu \in \gamma_0^R$ let us show (cfr. (7.9))

$$\lim_{s\to 0}\int_{\gamma_{12}^s t} T'_{m,t,n}(\xi,\lambda,\mu)d\xi = 0, \tag{7.16}$$

and

$$\lim_{r\to \infty}\int_{\gamma_{14}^r t} T'_{m,t,n}(\xi,\lambda,\mu)d\xi = 0. \tag{7.17}$$

(7.16) is obvious, since, for $\xi \to 0$,

$$\begin{aligned} &T^{(3)}_{m,t,n}(\xi,\lambda,\mu) = O(|\xi|^{\rho-1}), \\ &T^{(1)}_{m,t,n}(\xi,\lambda,\mu) + T^{(4)}_{m,t,n}(\xi,\lambda,\mu) + T^{(5)}_{m,t,n}(\xi,\lambda,\mu) = O(|\xi|^{\rho}). \end{aligned}$$

To prove (7.17) observe that for $|\xi| \to \infty$,

1. $T^{(1)}_{m,t,n}(\xi,\lambda,\mu) + T^{(3)}_{m,t,n}(\xi,\lambda,\mu) = O(|\xi|^{-2+\rho})$;
2. We have
$$\begin{aligned} T^{(4)}_{m,t,n}(\xi,\lambda,\mu) \quad &= \quad e^\lambda \lambda t^{-1}(B_n - \mu t^{-1})^{-1} A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ &\quad \cdot B_n(B_n - \mu t^{-1} + \xi t^{-1})^{-1}[A_m^{-1}; B_n^{-1}] \cdot \\ &\quad \cdot B_n(B_n - \mu t^{-1} + \xi t^{-1})^{-1}(\xi t^{-1})^\rho e^\mu t^{-3} \\ &= \quad O(|\xi|^{-2+\rho}); \end{aligned}$$
3. We have
$$\begin{aligned} T^{(5)}_{m,t,n}(\xi,\lambda,\mu) \quad &= \quad e^\lambda (\lambda t^{-1})^2 A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ &\quad \cdot [A_m^{-1}; (B_n - \mu t^{-1})^{-1}] A_m (A_m - \lambda t^{-1})^{-1} \cdot \\ &\quad \cdot B_n(B_n - \mu t^{-1} + \xi t^{-1})^{-1}[A_m^{-1}; B_n^{-1}] \cdot \\ &\quad \cdot B_n(B_n - \mu t^{-1} + \xi t^{-1})^{-1}](\xi t^{-1})^\rho e^\mu t^{-3} \\ &= \quad O(|\xi|^{-2+\rho}). \end{aligned}$$

Now (7.16) and (7.17) are proved.

By the analyticity of $\xi \mapsto T'_{m,t,n}(\xi,\lambda,\mu)$ we have

$$J^R_{m,t,n}(-B_n)^\rho = \int_{\gamma_0^R}\int_{\gamma_0^R}\lim_{\substack{r\to\infty\\ s\to 0}}\int_{\gamma_1' t} T'_{m,t,n}(\xi,\lambda,\mu)d\xi d\lambda d\mu.$$

So, by (7.16), (7.17) and setting $s = r^{-1}$,

$$J^R_{m,t,n}(-B_n)^\rho = \int_{\gamma_0^R}\int_{\gamma_0^R}\left\{\lim_{r\to\infty}\int_{\gamma_{11}^{rr^{-1}}t+\gamma_{13}^{rr^{-1}}t} T'_{m,\lambda,n}(\xi,\lambda,\mu)d\xi\right\}d\lambda d\mu.$$

In this integral ξ takes values in γ_1. From Hypotheses 3, 4 and the Remark 4.3, we deduce the following estimates on $T^{(1)}_{m,t,n}$, $T^{(3)}_{m,t,n}$, $T^{(4)}_{m,t,n}$, $T^{(5)}_{m,t,n}$. They hold for $\lambda,\mu \in \gamma_0$ and for $\xi\in\gamma_1$, by (7.14). For simplicity of notation and without loss of generality let us assume that Hypothesis 4 holds with $k=1, \alpha := \alpha_1, \beta := \beta_1$.

$$\begin{aligned}
\|T^{(1)}_{m,t,n}(\xi,\lambda,\mu)\| &\le \\
&\le |e^\lambda\lambda|t^{-1}\frac{c_{AB}}{|\lambda t^{-1}|^{1-\alpha}|\mu t^{-1}|^{1+\beta}}\cdot \\
&\quad\cdot\frac{c_B}{|\mu t^{-1}-\xi t^{-1}|}\frac{c_B}{|\xi t^{-1}|}|\mu t^{-1}||\xi t^{-1}|^\rho|e^\mu|t^{-3} \\
&= |e^\lambda|\frac{c_{AB}}{|\lambda|^{-\alpha}|\mu|^\beta}\frac{c_Bc_B}{|\mu-\xi|}|\xi|^{\rho-1}|e^\mu|t^{-1+\delta-\rho} = \\
&=: \varphi_1(\xi,\lambda,\mu)t^{-1+\delta-\rho}. \qquad (7.18)\\
\|T^{(3)}_{m,t,n}(\xi,\lambda,\mu)\| &\le \\
&\le |e^\lambda\lambda|t^{-1}\frac{c_{AB}}{|\lambda t^{-1}|^{1-\alpha}|\mu t^{-1}-\xi t^{-1}|^{1+\beta}}\cdot \\
&\quad\cdot|\xi t^{-1}|^{\rho-1}|e^\mu|t^{-3} \\
&= |e^\lambda|\frac{c_{AB}}{|\lambda|^{-\alpha}|\mu-\xi|^{\beta+1}}|\xi|^{\rho-1}|e^\mu|t^{-1+\delta-\rho} \\
&=: \varphi_3(\xi,\lambda,\mu)t^{-1+\delta-\rho}. \qquad (7.19)\\
\|T^{(4)}_{m,t,n}(\xi,\lambda,\mu)\| &\le \\
&\le |e^\lambda\lambda|t^{-1}\frac{c_B}{|\mu t^{-1}|}\frac{c_{AB}}{|\lambda t^{-1}|^{1-\alpha}|\mu t^{-1}-\xi t^{-1}|^{1+\beta}}|\xi t^{-1}|^\rho|e^\mu|t^{-3}
\end{aligned}$$

$$= |e^{\lambda}|\frac{c_B}{|\mu|}\frac{c_{AB}}{|\lambda|^{-\alpha}|\mu-\xi|^{\beta+1}}|\xi|^{\rho}|e^{\mu}|t^{-1+\delta-\rho}$$

$$=: \varphi_4(\xi,\lambda,\mu)t^{-1+\delta-\rho}. \tag{7.20}$$

$$\|T^{(5)}_{m,t,n}(\xi,\lambda,\mu)\| \le$$

$$\le |e^{\lambda}\lambda^2|t^{-2}\frac{c_{AB}}{|\lambda t^{-1}|^{1-\alpha}|\mu t^{-1}|^{1+\beta}}\cdot$$

$$\cdot\frac{c_{AB}}{|\lambda t^{-1}|^{1-\alpha}|\mu t^{-1}-\xi t^{-1}|^{1+\beta}}|\xi t^{-1}|^{\rho}|e^{\mu}|t^{-3}$$

$$= |e^{\lambda}|\frac{c_{AB}}{|\lambda|^{-\alpha}|\mu|^{\beta+1}}\frac{c_{AB}}{|\lambda|^{-\alpha}|\mu-\xi|^{\beta+1}}|\xi|^{\rho}|e^{\mu}|t^{-1+2\delta-\rho}$$

$$=: \varphi_5(\xi,\lambda,\mu)t^{-1+2\delta-\rho}. \tag{7.21}$$

It is easy to verify that the functions $\varphi_i(\xi,\lambda,\mu)$ are integrable over $\gamma_1\times\gamma_0\times\gamma_0$. Therefore $T'_{m,t,n}(\xi,\lambda,\mu)$ is absolutely integrable over $\gamma_1\times\gamma_0\times\gamma_0$, the integral $\displaystyle\int_{\gamma_1} T'_{m,t,n}(\xi,\lambda,\mu)d\xi$ exists as a Bochner integral for every $\lambda,\mu\in\gamma_0$, and

$$\int_{\gamma_1} T'_{m,\lambda,n}(\xi,\lambda,\mu)d\xi = \lim_{r\to\infty}\int_{\gamma_{11}^{rr^{-1}}t+\gamma_{13}^{rr^{-1}}t} T'_{m,\lambda,n}(\xi,\lambda,\mu)d\xi,$$

for every $\lambda,\mu\in\gamma_0$, and therefore

$$J^R_{m,t,n}(-B_n)^{\rho} = \int_{\gamma_0^R}\int_{\gamma_0^R}\int_{\gamma_1} T'_{m,t,n}(\xi,\lambda,\mu)d\xi d\lambda d\mu.$$

Letting $R\to\infty$ we have, by (7.5), (7.6),

$$J_{m,t,n}(-B_n)^{\rho} = \int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_1} T'_{m,t,n}(\xi,\lambda,\mu)d\xi d\lambda d\mu.$$

From (7.18)-(7.21) it follows that

$$\|J_{m,t,n}(-B_n)^{\rho}\| \le$$

$$\le \frac{t^{-1+\delta-\rho}}{(2\pi)^3}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_1}(\varphi_1+\varphi_3+\varphi_4)(\xi,\lambda,\mu)|d\xi||d\lambda||d\mu| +$$

$$+\frac{t^{-1+2\delta-\rho}}{(2\pi)^3}\int_{\gamma_0}\int_{\gamma_0}\int_{\gamma_1}\varphi_5(\xi,\lambda,\mu)|d\xi||d\lambda||d\mu|.$$

Since $t \in (0,1]$ we finally obtain

$$\|J_{m,t,n}(-B_n)^\rho\|_{\mathcal{L}(X)} \leq kt^{-1+\delta-\rho}.$$

The Lemma 7.1 is now proved. □

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