Regularity Considerations
for Semilinear Parabolic Systems

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Dedicated to the Memory of Pierre Grisvard

0. Introduction, Notations

We consider semilinear parabolic systems

$$\partial_t u + A(t) u + M(t,x,u,Du,...,D^m u) = 0 \quad (0.1)$$

over $[0, +\infty) \times \overline{\Omega} \subset \mathbb{R}^{n+1}$. $A(t)$ is an elliptic system of order $2m$ satisfying the Legendre-Hadamard condition, the nonlinear term $M$ is subject to suitable growth conditions. $\Omega$ is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ on which the vector $u$ satisfies Dirichlet-0-conditions. Of course we prescribe the initial value

$$u(0, x) = \varphi(x), \quad x \in \overline{\Omega}. \quad (0.2)$$

In the first part we work within the class of Hölder-continuous vectors, this is $C^{\alpha/2m, \alpha}([0,T] \times \overline{\Omega})$. For simplicity we assume that $M(t,...)$ has the form $M(t,x,D^m u)$ and is quadratic in the $m$-th order derivatives $D^m u$. This is a direct approach to regularity and it yields the following result: If the maximal interval of existence $[0, T(\varphi))$ for $(0.1,0.2)$ is finite then the oscillation

$$\sup_{|t-s| \leq \delta} ||u(t) - u(s)||_{C^\alpha(\overline{\Omega})} \quad (0.3)$$

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for every $\delta > 0$ exceeds a certain value $\varepsilon_0 > 0$ which can be determined a-priori. Thus we improve on the results in [W1] in several respects. An important rôle in our considerations is played by the interpolation inequality

$$
\|u\|_{C^{\alpha/2m,0}(\Omega)} \leq c(T) \|\partial_t u\|_{C^{\alpha/2m,0}(\Omega)}^{(\alpha/2m)/(1+\alpha/2m)} \cdot ||u||^{1-(\alpha/2m)/(1+\alpha/2m)}_{C^0([0,T] \times \Omega)} + \|u\|_{C^0([0,T] \times \Omega)}.
$$

As it was brought to our attention by A. Lunardi (University of Parma) the constant $c(T)$ in (0.4) as $T \to 0$ blows up in a power-like way. We clarify its usage here in order to avoid non-controllable quantities. As an example we show that for a single second-order equation

$$
\partial_t u - a_{ij}(t,x) \partial_{x_i} \partial_{x_j} u + M(t,x,\nabla u) = 0
$$

with quadratic growth of $M$ with respect to $\nabla u$ we have an a-priori bound on $\|u(t)\|_{C^0(\Omega)}$ and that this is sufficient to ensure global (in time) classical solvability.

In the results previously described we considered solutions in classes of Hölder continuous vectors; the critical quantity is the oscillation (0.3), the critical growth of $M$ with respect to $D^m u$ is quadratic. This is different in the second part (Chapter 3) of the present paper. Here we switch over to weak solutions for which we have a reasonable notion of energy: $u \in L^\infty((0,T),L^2(\Omega)) \cap L^2((0,T),H^{m,2}(\Omega))$. In order to define weak solutions to systems like (0.1) different assumptions on the elliptic operator $A(t)$ are needed. Whereas in the first part it was sufficient to assume that the coefficient matrices in (0.1) are Hölder-continuous in $(t,x)$, we now suppose that $A(t)$ in (0.1) has divergence-structure. The regularity of the coefficient-matrices is of such a type that $A(t)u$ can be written down pointwise if $u$ permits it. For details we refer to [GW]. This assumption allows us to define the notion of a weak solution to (0.1) in the usual way and to ask for their regularity. The so called “controllable” growth conditions

$$
|M(t,,u,Du,...,D^m u)| \leq c \left( 1 + \sum_{\nu=0}^m |D^\nu u|^{\frac{\alpha+\nu}{\nu+2m}} \right)
$$

(0.5)
are “critical” with respect to the “energy class” \( u \in L^\infty((0, T), L^2(\Omega)) \cap L^2((0, T), H^{n,2}(\Omega)) \). It has been proved in [GW] that under this growth condition any weak solution is regular. A sign condition on \( M \) is not needed. Here we show, by means of a counterexample, that this result is optimal as it concerns the growth condition.

We introduce some notation. \( C^{\frac{\alpha}{2m}, \alpha}([T_1, T_2] \times \overline{\Omega}) \) is the subspace of \( C^0([T_1, T_2] \times \overline{\Omega}) \) whose members \( u \) have finite semi-norm

\[
[u]_{C^{\frac{\alpha}{2m}, \alpha}}^{[T_1, T_2] \times \overline{\Omega}} = \sup_{(t, x) \neq (t', x'), (t, x), (t', x') \in [T_1, T_2] \times \overline{\Omega}} \frac{|u(t', x') - u(t, x)|}{|t - t'|^{\alpha/2m} + |x - x'|^{\alpha}}
\]

\((0 \leq \alpha < 1)\). The norm of \( C^{\frac{\alpha}{2m}, \alpha}([T_1, T_2] \times \overline{\Omega}) \) is then given by

\[
\|u\|_{C^{\frac{\alpha}{2m}, \alpha}} = \|u\|_{C^0([T_1, T_2] \times \overline{\Omega})} + [u]_{C^{\frac{\alpha}{2m}, \alpha}}^{[T_1, T_2] \times \overline{\Omega}}.
\]

All coefficient-matrices of \( A(t) \) in (0.1) belong to this space. If we want to stress the underlying time-interval we also write \( \|u\|_{C^{\frac{\alpha}{2m}, \alpha}}^{[T_1, T_2]} \) for the norm of \( C^{\frac{\alpha}{2m}, \alpha}([T_1, T_2] \times \overline{\Omega}) \). Instead of \( \|\cdot\|_{0,0} \) we use the symbol \( \|\cdot\|_0 \). If no misunderstanding can arise \( \|\cdot\|_0 \) is also employed for the norm of \( C^0(\overline{\Omega}) \). Analogously to \( C^{\alpha/2m, \alpha}([T_1, T_2] \times \overline{\Omega}) \) we define \( C^{\gamma, k+\eta}([T_1, T_2] \times \overline{\Omega}) \) for \( 0 \leq \gamma < 1, k \in \mathbb{N} \cup \{0\}, 0 \leq \eta < 1 \) and the norms \( \|\cdot\|_{\gamma, k+\eta} \). Let

\[
w \in C^1([T_1, T_2] \times \overline{\Omega}),
\]

\[
\partial_t w \in C^{\frac{\alpha}{2m}, \alpha}([T_1, T_2] \times \overline{\Omega}),
\]

\[
w : [T_1, T_2] \to C^{2m+\alpha}(\overline{\Omega})
\]

with \( \sup_{T_1 \leq t \leq T_2} \|w(t)\|_{C^{2m+\alpha}(\overline{\Omega})} < +\infty \).

Then we set

\[
\|w\|_{[T_1, T_2]} = \|\partial_t w\|_{\frac{\alpha}{2m}, \alpha} + \sup_{T_1 \leq t \leq T_2} \|w(t)\|_{C^{2m+\alpha}(\overline{\Omega})}.
\]
If $T_1 = 0$ we also write $|||w|||_{T_2}$ instead of $|||\cdot|||_{[0,T_2]}$. Due to appropriate interpolation inequalities finiteness of $|||w|||_{[T_1,T_2]}$ implies finiteness of
\[
\sum_{|\tilde{t}|=\tilde{t}_j \leq 2^m} \| D^{[\tilde{t}]} w \|_{(2m-j+\alpha)/2m, \alpha}
\]
(cf. [W1]), together with the corresponding estimate.

1. General Theory for Semilinear Parabolic Systems in Hölder Spaces under Homogeneous Dirichlet-Conditions

We carry over the assumptions in [W1]: instead of equations $\partial_t u + A(t)u = f$, $u(0) = \varphi$, we can as well treat systems where the $a_\delta(t,x)$ are $N \times N$-matrices. We then assume Legendre-Hadamard’s condition to be fulfilled, this is ($c_0$ is some positive constant)
\[
\text{Re}(-1)^m \sum_{|\tilde{a}|=2m} a_\delta(t,x) \xi^\delta \zeta^* \geq c_0 |\xi|^{2m} |\zeta|^2,
\]
$\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{C}^N$, $x \in \overline{\Omega}$, $t \geq 0$.

For simplicity we assume the ellipticity condition to be valid for all $t \geq 0$. The following quantities are assumed to be given:
\[
m, n, N, c_0, \Omega, \| a_\delta \|_{2m, \alpha}, \alpha.
\]

Dependence of constants on these quantities is not explicitly mentioned. In contrast to that, dependence of the constants on the time interval $[0,T]$, the initial value $\varphi$ and the right-hand side $f$ is mentioned. We are going to consider semilinear problems
\[
\partial_t u + A(t)u + M(t,, D^m u) = 0,
\]
$u(0) = \varphi$, $(A(0)\varphi + M(0,, D^m \varphi))|\partial \Omega = 0,$
\[
\frac{\partial^j}{\partial y^j} \varphi = 0, 0 \leq j \leq m - 1, \frac{\partial^j u}{\partial y^j} = 0 \text{ on } \partial \Omega, 0 \leq j \leq m - 1.
\]

Therefore we fix our assumptions on $M$:

A1. Let
\[
|M(t', x', p') - M(t, x, p)| \leq c(T) \cdot |p' - p| \cdot (|p'| + |p|) + c(T)(1 + |p'|^2 + |p|^2) \cdot (|t' - t|^m + |x' - x|^\alpha),
\]
$0 \leq t', t \leq T$, $x', x \in \overline{\Omega}$, $p', p \in \mathbb{R}^{N\cdot m}$, $T \geq 0$. 

$s_m$ is the number of multiindices $\vec{\alpha}$ of $\mathbb{R}^n$ with $|\vec{\alpha}| = m$. $c(.)$ depends monotonically non decreasing on $T \geq 0$. (If $w \in C^0([0, T], C^m(\Omega))$ has the property $D^m w \in C^{\alpha/2m, \alpha}(0, T)$ we arrive at

$$\|M(\cdot, \cdot, D^m w)\|_{[0, T]}^{\alpha/2m, \alpha} \leq c(T)(\|D^m w\|_{[0, T]}^{\alpha/2m, \alpha} \|D^m w\|_0^{[0, T]} + 1).$$

A2. Let $w_i \in C^0([0, T], C^m(\Omega))$, $D^m w_i \in C^{\alpha/2m, \alpha}(0, T \times \Omega)$, \(i = 1, 2\), $w_1(0) = w_2(0)$. Then we suppose that

$$\|M(\cdot, \cdot, D^m w_2) - M(\cdot, \cdot, D^m w_1)\|_{[0, T]}^{\alpha/2m, \alpha} \leq \lambda(T, D) \cdot |||w_2 - w_1|||_T,$$

where $\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $D \geq |||w_2|||_T + |||w_1|||_T$, $\lambda(T, D) \to 0$ as $T \to 0$ for every $D \geq 0$. (This requires a condition on $\partial M/\partial p$ analogous to the one for $M$ in A1, but somewhat weaker.)

As a consequence of Assumption A2 we have

**Theorem 1.1.** Let $\varphi \in C^{2m+\alpha}(\Omega)$, let

$$(A(0)\varphi + M(0, \cdot, D^m \varphi))|\partial \Omega = 0, \quad \frac{\partial^j \varphi}{\partial \nu^j} = 0 \text{ on } \partial \Omega, \quad 0 \leq j \leq m - 1.$$ 

Then there exists a $T(\varphi)$, $0 < T(\varphi) \leq +\infty$, such that there is a unique $u$ with $|||u|||_T < +\infty$ for every $T < T(\varphi)$,

$$\partial_t u + A(t) u + M(t, \cdot, D^m u) = 0, \quad 0 \leq t \leq T < T(\varphi),$$

$$u(0) = \varphi,$$

$$\frac{\partial^j u}{\partial \nu^j}(t) = 0 \text{ on } \partial \Omega, \quad 0 \leq t \leq T < T(\varphi).$$

If $T(\varphi) < +\infty$ then $|||u|||_T \to +\infty$ as $T \uparrow T(\varphi)$, $T(\varphi)$ is called the maximal interval of existence for the Problem (1.1,2,3). Let $F > 0$, let $\varphi$ fulfil the previous assumptions. Let $F \geq |||\varphi|||_{2m+\alpha}$. Then there is a finite interval $[0, T_1(F)]$, $T_1(F) > 0$, such that Problem (1.1,2,3) has a unique solution $u$ on $[0, T_1(F)]$ with $|||u|||_{T_1(F)} < +\infty$. $[0, T_1(F)]$ is called a first interval of existence.

**Proof.** In view of the linear estimates in [LSU, ch. VII], [W1, p. 437] being valid also for systems like ours the assertions of Theorem 1.1 can be easily shown to be true by making use of Banach’s fixed point theorem.  

$\square$
As for global existence we have

**Theorem 1.2.** Let \( \varphi \in C^{2m+\alpha}(\overline{\Omega}) \), \((A(0)\varphi + M(0,, D^m\varphi))|\partial\Omega = 0, \)
\( \partial^j \varphi|\partial \nu = 0 \) on \( \partial\Omega \), \( 0 \leq j \leq m - 1 \). Let \( F \geq \|\varphi\|_{2m+\alpha} \). Let \( T > T_1(F) > 0 \). Then there exists a constant

\[ \varepsilon_0 = \varepsilon_0(T, T_1(F)) > 0 \]

with the following property: Let \( u \) be a solution of Problem (1.1,2,3) on \([0, \tilde{T}]\) with \( |||u|||_{T} < +\infty \), for every \( \tilde{T}, 0 < \tilde{T} < T \). If for some \( \delta > 0 \) we have

\[ \|u(t + h) - u(t)\|_0 \leq \varepsilon_0 \]  \hspace{1cm} (1.4)

for all \( h, 0 \leq h \leq \delta, 0 \leq t \leq t + h < T \), then \( u \) can be continued into \( T \) such that \( |||u|||_T \) is finite and such that \( u \) solves Problem (1.1,2,3) on \([0,T]\). In particular we have \( T(\varphi) > T \).

**Proof.** Set

\[ v(t) = u(t) - u(t - \delta) \]

on \([\delta, \tilde{T}], T - \delta \leq \tilde{T} < T \). \( \delta \) is positive, \( < \frac{1}{4}T_1(F) \) and will be specified later on. We have

\[
\begin{align*}
\partial_t v + A(t)v &= -(A(t) - A(t - \delta)) u(t - \delta) - \\
&\quad - (M(t,, D^m u(t)) - M(t - \delta,, D^m u(t - \delta))), \\
&= -(A(t) - A(t - \delta)) u(t - \delta) - \\
&\quad - (M(t,, D^m u(t) - u(t - \delta)) + D^m u(t - \delta)) - \\
&\quad - M(t - \delta,, D^m u(t - \delta)).
\end{align*}
\]

Then (observe that \( |||\delta, \tilde{T}||| \geq T - 2\delta \geq T_1(F) - 2\delta > \frac{1}{2}T_1(F) \))

\[
|||v|||_{[\delta, \tilde{T}]} \leq A_1 
\]

\[
\begin{align*}
\leq c(T)\|u\|_{T - \delta} + \\
\quad + c(T) \left( \|D^m(u(,) - u(\cdot, - \delta)) + D^m u(\cdot, - \delta)||_{[\delta, \tilde{T}]} \right) \\
\quad \cdot \|D^m(u(,) - u(\cdot, - \delta)) + D^m u(\cdot, - \delta)||_{[\delta, \tilde{T}]} + \\
+ \|D^m u(\cdot)||_{[\alpha, \tilde{T}]} + D^m u(\cdot)\|_{[\delta, \tilde{T}]} + 2 + 
\end{align*}
\]
\begin{align*}
+ c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}),
\leq c(T)|||u|||_{T-\delta} + \\
+ c(T)||D^m(u(. - \delta))||_{[\delta,T]}^\frac{[\delta,T]}{2m\alpha} \\
\cdot\|D^m(u(. - \delta))\|_{0}^\frac{[\delta,T]}{2m\alpha} + \\
+ c(T)||D^m(u(. - \delta))||_{[\delta,T]}^\frac{[\delta,T]}{2m\alpha} \\
\cdot\|D^m(u(. - \delta))\|_{0}^\frac{[\delta,T]}{2m\alpha} + \\
+ c(T)||D^m(u(. - \delta))||_{[\delta,T]}^\frac{[\delta,T]}{2m\alpha} ||D^m(u(. - \delta))||_{0}^\frac{[\delta,T]}{2m\alpha} + \\
+ c(T)||D^m(u(. - \delta))||_{[\delta,T]}^\frac{[\delta,T]}{2m\alpha} ||D^m(u(. - \delta))||_{0}^\frac{[0,T-\delta]}{2m\alpha} + \\
+ 2c(T) + c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}),
\end{align*}

[W1, pp. 438, 439]

\begin{align*}
\leq c(T)|||u|||_{T-\delta} + \\
(0.4) \text{ on } [\delta,T] \\
+ c(T, T_1(F)) ||v||_{[\delta,T]} g(||v(.)||_{0}^\frac{[\delta,T]}{2m\alpha}) + \\
+ c(T, T_1(F)) \left(||v||_{[\delta,T]} h_1(||v(.)||_{0}^\frac{[\delta,T]}{2m\alpha}) + 1\right) \\
\cdot\|D^m(u(.))\|_{0}^\frac{[0,T-\delta]}{2m\alpha} + \\
+ c(T, T_1(F)) \left(||v||_{[\delta,T]} h_2(||v(.)||_{0}^\frac{[\delta,T]}{2m\alpha}) + 1\right) \\
\cdot\|D^m(u(.))\|_{0}^\frac{[0,T-\delta]}{2m\alpha} + \\
+ c(T)||D^m(u(.))||_{[\delta,T]}^\frac{[\delta,T]}{2m\alpha} ||D^m(u(.))||_{0}^\frac{[0,T-\delta]}{2m\alpha} + \\
+ 2c(T) + c(T)(\|\varphi\|_{2m+\alpha} + \|u(\delta)\|_{2m+\alpha}).
\end{align*}

Here \(\gamma_1, \gamma_2\) denote fixed positive numbers with \(\gamma_1, \gamma_2 \in (0, 1)\). \(h_1, h_2\) are some continuous functions from \(\mathbb{R}^+\) into itself. \(g\) is a fixed con-
timous function from $\mathbb{R}^+$ into itself with $g(r) \to 0$ as $r \to 0$. Now we choose $\varepsilon_0$ in such a way that

$$c(T, T_1(F))g(\varepsilon_0) \leq \frac{1}{2}.$$ 

Since $g$ is a function simply originating from the interpolation inequalities employed in [W1, p. 438] we have $\varepsilon_0 = \varepsilon_0(T, T_1(F))$. Assume now that (1.4) is valid. $\delta$ is taken from (1.4). Possibly we diminish it to satisfy $\delta < \frac{1}{4} T_1(F)$. Then $\|u(t)\|_0 \leq c(\delta; T)(1 + \|\varphi\|_0)$, $0 \leq t < T$. Employing the inequality $a^{1-\gamma}b \leq c(\gamma)(\rho a + (\frac{1}{\rho^{1-\gamma}b})^{1/\gamma})$, $a, b \geq 0$, $\rho > 0$, $\gamma \in (0, 1)$, we arrive at

$$\|u\|_{[\delta, T]} \leq c(T)\|u\|_{T-\delta} +$$

$$+ c(T, T_1(F), \gamma_1, \gamma_2, h_1, h_2, \delta, \|\varphi\|_{2m+\alpha}, \|u(\delta)\|_{2m+\alpha}, \|u\|_{T-\delta}).$$

Letting $\tilde{T}$ tend to $T$ we arrive at the assertion. \qed

2. An Application to Second Order Equations

We now consider the previous problem for $m = 1$ and for a single equation. Then we have

$$\partial_t u - a_{ij}(t,x)\partial^2_{x_i, x_j} u + M(t, x, \nabla u) = 0, \quad u(0) = \varphi, \quad u(t) = 0 \text{ on } \partial \Omega, \; t \geq 0. \quad (2.1)$$

We omit the summation sign in the spatial elliptic part and set $A(t)u = -a_{ij}(t, x)\partial^2_{x_i, x_j} u$, thereby assuming that $A(t)$ only contains second order derivatives. The first compatibility condition reads

$$A(0)\varphi + M(0, x, \nabla \varphi) = 0 \text{ on } \partial \Omega. \quad (2.2)$$

Instead of (2.1) we consider the problems

$$\partial_t u_\sigma - a_{ij}(t,x)\partial^2_{x_i, x_j} u_\sigma + M(t, x, \nabla u_\sigma) - M(0, x, \sigma \nabla \varphi) + \sigma M(0, x, \nabla \varphi) = 0, \quad u_\sigma(0) = \sigma \varphi, \quad u_\sigma(t) = 0 \text{ on } \partial \Omega, \; t \geq 0, \quad (2.3)$$
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\[ 0 \leq \sigma \leq 1. \] Since

\[
A(0)\sigma \varphi + M(0, x, \nabla \varphi) - M(0, x, \sigma \nabla \varphi) = \sigma(A(0)\varphi + M(0, x, \nabla \varphi)), \]

the first order compatibility condition is fulfilled for all problems (2.3), provided it is so for (2.1). For \( \sigma = 1 \) the unique solution of (2.3, \( \sigma = 1 \)) is the function \( u \) under consideration, for \( \sigma = 0 \) the unique solution of (2.3, \( \sigma = 0 \)) is \( u_0 = 0 \). A minor generalisation of Theorem 1.1 shows that there is a joint first interval of existence \([0, T_1]\) for all problems (2.3), \( 0 \leq \sigma \leq 1 \). The maximum-principle furnishes

\[ [\text{LSU, p. 13}] \]

\[
\|u_{\sigma_2}(t) - u_{\sigma_1}(t)\|_0 \leq |\sigma_2 - \sigma_1| \cdot c(\|\varphi\|_0, \|\nabla \varphi\|_0) \cdot e^t. \]

Let us set \( \sigma_2 = \sigma_1 + \varepsilon \) for some \( \varepsilon > 0 \). Then

\[
\|u_{\sigma_2}(t + h) - u_{\sigma_2}(t)\|_0 \\
\leq \|u_{\sigma_2}(t + h) - u_{\sigma_1}(t + h)\|_0 + \|u_{\sigma_1}(t + h) - u_{\sigma_1}(t)\|_0 + \|u_{\sigma_1}(t) - u_{\sigma_2}(t)\|_0 \leq 2\varepsilon e^t c(\|\varphi\|_0, \|\nabla \varphi\|_0) + \|u_{\sigma_1}(t + h) - u_{\sigma_1}(t)\|_0. \]

Let \( T > 0 \). Let \( u_{\sigma_2}, u_{\sigma_1} \) solve (2.3, \( \sigma = \sigma_2 \)), (2.3, \( \sigma = \sigma_1 \)) resp. over any cylinder \([0, \bar{T}] \times \overline{\Omega}, 0 < \bar{T} < T\). Let

\[
2\varepsilon e^T c(\|\varphi\|_0, \|\nabla \varphi\|_0) \leq \frac{1}{2}\varepsilon_0(T, T_1), \]

where \( \varepsilon_0(T, T_1) > 0 \) is the quantity constructed in Theorem 1.2. It can be chosen uniformly for \( \sigma \in [0, 1] \). If \( u_{\sigma_1} \) is uniformly continuous from \([0, T] \times \overline{\Omega} \) into \( C^0(\overline{\Omega}) \) and thus, according to Theorem 1.2, exists on \([0, T] \times \overline{\Omega}\) by continuation as the unique solution of (2.3, \( \sigma = \sigma_1 \)), Theorem 1.2 now shows: \( u_\sigma \) exists on \([0, T] \times \overline{\Omega}\) for all \( \sigma, \sigma_1 \leq \sigma \leq \sigma_1 + \left(\frac{\varepsilon_0(T, T_1)}{e^T} \right) \cdot \left(\frac{\varepsilon_{\sigma_1}(\|\varphi\|_0, \|\nabla \varphi\|_0)}{e^T}\right). \) Starting with \( \sigma_1 = 0 \) we exhaust \([0, 1]\) in finitely many steps. Since \( T \) can be chosen arbitrarily we end up with the global solution for (2.1).
3. On the Necessity of Controllable Growth Conditions in Regularity Theory

We consider the semilinear parabolic equation

\[ u_t + (-\Delta)^m u = M(t, x, u) \text{ in } [0, 1] \times \overline{B} \]  

(3.1)

with smooth initial and boundary values. \( B \subset \mathbb{R}^n \) denotes the (open) unit ball, \( M \) a H"older continuous nonlinear function. In [GW] the sufficiency of controllable growth conditions

\[ |M(t, x, u)| \leq c(1 + |u|)^{1+\frac{4m}{n}} \]  

(3.2)

for weak solutions \( u \in L^\infty((0, 1), L^2(B)) \cap L^2((0, 1), H^{m,2}(B)) \) of (3.1) to be smooth was shown. Here by means of a simple example we also demonstrate the necessity of (3.2). In [GW] for simplicity we considered homogeneous Dirichlet boundary data on \([0, 1] \times \partial B\). But by simply subtracting the data, smoothly extended to \([0, 1] \times \overline{B}\), it is sufficient to assume smooth initial and boundary data:

\[
\begin{cases}
\left( \frac{\partial}{\partial \nu} \right)^j u(t, \cdot)|_{\partial B} = \left( \frac{\partial}{\partial \nu} \right)^j \varphi(t, \cdot)|_{\partial B} \\
\text{for } j = 0, \ldots, m - 1, \ t \in [0, 1], \\
u(0, \cdot) = \varphi(0, \cdot),
\end{cases}
\]  

(3.3)

with some \( \varphi \in C^\infty([0, 1] \times \overline{B}) \).

For some \( \gamma > 0 \), to be specified below, we define on \([0, 1] \times \overline{B}\) :

\[ u(t, x) = (1 - t + |x|^{2m})^{-\frac{\gamma}{2m}}. \]

Obviously \( u \) is arbitrarily smooth in \([0, 1] \times \overline{B} \setminus \{(1,0)\} \) and develops a singularity in \((t, x) = (1, 0)\). We want to show that for any \( \delta > 0 \) there is some \( \gamma > 0 \) such that \( u \in L^\infty((0, 1), L^2(B)) \cap L^2((0, 1), H^{m,2}(B)) \) weakly solves the equation (3.1) with an appropriate nonlinearity \( M \), satisfying the growth condition

\[ |M(t, x, u)| \leq c(1 + |u|)^p \]

with “slightly supercritical” exponent: \( 1 + \frac{4m}{n} < p < 1 + \frac{4m}{n} + \delta \).
Let \( \frac{\partial}{\partial r} \) denote the radial derivative. By induction on \( j \) we find:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\gamma}{2m} (1 - t + |x|^{2m})^{-\frac{\gamma}{2m} - 1}, \\
\frac{\partial}{\partial r} \Delta^j u &= \sum_{k=1}^{2j+1} c_{jk} |x|^{2mk - 2j - 1} (1 - t + |x|^{2m})^{-\frac{\gamma}{2m} - k}, \\
\Delta^j u &= \sum_{k=1}^{2j} d_{jk} |x|^{2mk - 2j} (1 - t + |x|^{2m})^{-\frac{\gamma}{2m} - k},
\end{align*}
\]

(3.4)

\((t, x) \in [0, 1] \times \bar{B} \setminus \{(1, 0)\}, \ c_{jk}, d_{jk} \in \mathbb{R} \) are suitable numbers, depending on \( \gamma \) and \( m \).

In particular, with suitable numbers \( \tilde{d}_{mk} \in \mathbb{R} \), \( u \) is a classical solution on \([0, 1) \times \bar{B}\) of the following equation:

\[
u_t + (-\Delta)^m u = \sum_{k=0}^{2m-1} \tilde{d}_{mk} |x|^{2mk} (1 - t + |x|^{2m})^{-\frac{\gamma}{2m} - k - 1}
\]

\[
= \left( \sum_{k=0}^{2m-1} \tilde{d}_{mk} |x|^{2mk} (1 - t + |x|^{2m})^{-k + \varepsilon \frac{2}{2m}} \right) (1 - t + |x|^{2m})^{-\frac{\gamma}{2m} (1 + \frac{2m}{2m} + \varepsilon)}
\]

\[
=: \ g(t, x) u^p = g(t, x) |u|^{p-1} u,
\]

where the additional parameter \( \varepsilon > 0 \) will also be specified below and \( p = p(\gamma, \varepsilon) = 1 + \frac{2m}{\gamma} + \varepsilon \). The function

\[
g(t, x) := \sum_{k=0}^{2m-1} \tilde{d}_{mk} |x|^{2mk} (1 - t + |x|^{2m})^{-k + \varepsilon \frac{2}{2m}}
\]

is Hölder continuous on \([0, 1] \times \bar{B}\). We set

\[
M(t, x, u) = g(t, x) |u|^{p-1} u.
\]
Now we want to investigate the integrability properties of the solution $u$. We additionally assume
\begin{equation}
\gamma < \frac{n}{2}.
\end{equation}

For $t \in [0,1]$ we find
\begin{equation}
\|u(t)\|_{L^2(B)}^2 = \int_B (1 - t + |x|^{2m})^{-\frac{2\gamma}{m}} \, dx \leq \int_B |x|^{-2\gamma} < \infty
\end{equation}
uniformly on $[0,1]$. Moreover by Lebesgue’s theorem we see that
\begin{equation}
u \in C^0([0,1], L^2(B)).
\end{equation}

Observing the radial symmetry of $u$ and the estimates (3.7) we calculate by means of Fubini-Tonelli:
\begin{align}
\int_0^1 \|u(t)\|_{H^{m,2}}^2 \, dt & \leq c \int_0^1 (1 - t + |x|^{2m})^{-\frac{2\gamma}{m} - 1} \, dx \, dt \\
& \leq c \int_B |x|^{-2\gamma} \, dx < \infty,
\end{align}
\begin{equation}u \in L^2((0,1), H^{m,2}(B)).\end{equation}

Due to the properties (3.8) and (3.9) of $u$ and
\begin{align}
\int_0^1 \int_B |M(t,x,u)| \, dx \, dt & \leq c \int_0^1 (1 - t + |x|^{2m})^{-\frac{2\gamma}{m} - 1} \, dx \, dt \\
& \leq \int_B |x|^{-\gamma} \, dx < \infty,
\end{align}
we conclude that $u$ is a singular weak solution to (3.1) on $[0,1] \times \overline{B}$. Admissible testing functions are e.g. differentiable once with respect to $t$ and $2m$-times with respect to $x$.

To conclude we let $\gamma \nearrow \frac{n}{2}$ and $\varepsilon \searrow 0$ and find that
\begin{equation}p = p(\gamma, \varepsilon) = 1 + \frac{2m}{\gamma} + \varepsilon \searrow 1 + \frac{4m}{n}
\end{equation}
approaches the “critical exponent” in our regularity result [GW].
REGULARITY CONSIDERATIONS

REFERENCES

