Favard Classes and Hyperbolic Equations

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SUMMARY. - New regularity theorems are proved for the abstract Cauchy problem \( u'(t) = Au(t) + f(t), \ t \geq 0, \ u(0) = u_0 \) where \( A \) is a Hille-Yosida operator, by using properties of the Favard classes, matrix operators and extrapolation spaces.

0. Introduction

In a series of papers P. Grisvard (in collaboration with G. Da Prato) showed the importance of the interpolation spaces in the study of parabolic equations and even the necessity of their use to get the maximal regularity for the non homogeneous initial value problem:

\[
\begin{cases}
    u'(t) = A_0 u(t) + f(t), & t \geq 0 \\
    u(0) = u_0
\end{cases}
\]

(0.1)

where \( A_0 : D(A_0) \subset X_0 \to X_0 \) is the generator of an analytic semigroup in a Banach space \( X_0 \) (see the bibliographical references of [A. Lunardi 1995]). In this paper we will suppose that \( A_0 \) is the generator of a strongly continuous semigroup (and even only a Hille-Yosida operator) so that (0.1) is the abstract version of hyperbolic initial value problems.

The existence of a (classical solution) in the hyperbolic case has been proved in 1953 by R. Phillips when \( f \) is differentiable and by T.

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Kato when \( f \) is continuous with values in \( D(A_0) \) (see e.g. [A. Pazy 1993]).

We will show that these results can be improved by relaxing the conditions on \( A_0, u_0 \) and \( f \): the methods used are based on a regularizing property of certain interpolation spaces (the Favard classes), on the theory of Hille-Yosida operators (see [G. da Prato, E. Sinestrari 1987]), a method of homogenization of (0.1) (see [R. Nagel, E. Sinestrari 1994]) and the theory of extrapolation spaces as introduced in [R. Nagel 1983].

We will not consider applications, for which we refer to [R. Nagel, E. Sinestrari 1994, 1996; E. Sinestrari 1996].

This paper is a part of a work in progress with R. Nagel and is dedicated to the memory of P. Grisvard who introduced me to the methods of interpolation spaces in the study of abstract evolution equations.

1. Favard classes and regularity

Let us recall the definition and the main properties of the Favard classes (see [P. Butzer, H. Berens 1967]).

In this section \((E, \| \cdot \|)\) will denote a Banach space and \( \Lambda : D(\Lambda) \subset E \to E \) the generator of a strongly continuous semigroup \( e^{\Lambda t} \). For the problems considered in this paper it is not a restriction to suppose that the semigroup is bounded i.e., there exists \( M \geq 0 \) such that

\[
\| e^{\Lambda t} \|_{\mathcal{L}(E)} \leq M , \quad t \geq 0 .
\]

The Favard class of \( e^{\Lambda t} \) is the real interpolation space \((E, D(\Lambda))_{1,\infty}\) according to J.L. Lions i.e.

\[
F := \text{Fav}(e^{\Lambda t}) := \left\{ x \in E \ ; \ [x]_F := \sup_{t>0} \frac{\| e^{\Lambda t} x - x \|}{t} < \infty \right\}
\]

with norm

\[
\| x \|_F := \| x \| + [x]_F .
\]

We have

\[
D(\Lambda) \hookrightarrow F \hookrightarrow E
\]
where $D(\Lambda)$ is given the graph norm: this norm is equivalent to that induced by $F$ hence $D(\Lambda)$ is closed in $F$.

We will set $L^1([0, +\infty]; E) = L^1(E)$ and similarly for the Sobolev space $W^{1,1}(E)$, the space of continuous functions $C(E)$ and so on.

If $f \in L^1_{\text{loc}}(E)$ we set for each $t \geq 0$

$$(e^\Lambda \ast f)(t) := \int_0^t e^{\Lambda s} f(t-s) \, ds.$$  

If $A$ is a linear operator in a Banach space, $\rho(A)$ will denote its resolvent set.

**Theorem 1.1.** Let $(F_*, \| \cdot \|_*)$ be a Banach space such that

$$(1.3) \quad F_* \hookrightarrow F.$$  

If $f \in L^1_{\text{loc}}(F_*)$ then for each $t > 0$ we have

$$(1.4) \quad (e^\Lambda \ast f)(t) \in D(\Lambda)$$

$$(1.5) \quad \| (e^\Lambda \ast f)(t) \|_{D(\Lambda)} \leq c M \| f \|_{L^1([0, t]; F_*)}$$

$$(1.6) \quad e^\Lambda \ast f \in C(D(\Lambda))$$

where $c$ is independent on $f$ and $t$.

**Proof.** As for $t > 0$

$$\left\| \frac{1}{t} \int_0^t e^{\Lambda s} x \, ds \right\|_{D(\Lambda)} = \left\| \frac{1}{t} \int_0^t e^{\Lambda s} x \, ds \right\| + \left\| \frac{e^{\Lambda t} x - x}{t} \right\| \leq M \| x \| + \sup_{t > 0} \left\| \frac{e^{\Lambda t} x - x}{t} \right\|$$

we see that $F_* \hookrightarrow E$ satisfies (1.3) if and only if there exists $c > 0$ such that

$$(1.7) \quad \left\| \frac{1}{t} \int_0^t e^{\Lambda s} x \, ds \right\|_{D(\Lambda)} \leq c \| x \|_*, \quad t > 0, \ x \in F_*.$$  

Set for $t \geq 0, \ x \in E$

$$V(t)x := \int_0^t e^{\Lambda s} x \, ds.$$
If $\chi_A$ is the characteristic function of the set $A$, denote by $\Phi$ the set of the step functions $\varphi : \mathbb{R}_+ \to F_*$

\begin{equation}
(1.8) \quad \varphi := \sum_{i=0}^{n} x_i \chi_{[t_i, t_{i+1}]} \nonumber
\end{equation}

where $n \in \mathbb{N}$, $x_i \in F_*(i = 1, \ldots, n)$ and $0 = t_0 < t_1 < \cdots < t_{n+1}$.

Let us prove (1.4) and (1.5) for $f = \varphi$ given by (1.8). For $t \geq t_{n+1}$ we have

\[
(e^A \ast \varphi)(t) = \sum_{i=0}^{n} e^{A(t-t_{i+1})} \int_{t_i}^{t_{i+1}} e^{A(t_{i+1}-s)} x_i \, ds = 
\]

\[
= \sum_{i=0}^{n} e^{A(t-t_{i+1})} V(t_{i+1} - t_{i}) x_i .
\]

Hence from (1.7) we get $(e^A \ast \varphi)(t) \in D(A)$ and

\[
\| (e^A \ast \varphi)(t) \|_{D(A)} \leq \sum_{i=0}^{n} \| e^{A(t-t_{i+1})} \| \| V(t_{i+1} - t_{i}) x_i \|_{D(A)} \leq 
\]

\[
\leq cM \sum_{i=0}^{n} \| x_i \|_{*} (t_{i+1} - t_{i}) = cM \| \varphi \|_{L^1([0,t_{i+1}])} .
\]

If $t < t_{n+1}$, setting $\psi = \varphi \cdot \chi_{[0,t]}$ we have $\psi \in \Phi$ and so from the preceding result we see that each $f \in \Phi$ verifies (1.4) and (1.5): this can be proved also for $f \in L^1(F_*)$ because $\Phi$ is dense in $L^1(F_*)$.

To prove (1.6) let us choose $f \in L^1(F_*)$: given $T > 0$ there exists $\{f_n\}$ with $f_n \in C^1([0,T]; F_*)$ such that

\[
\lim_{n \to \infty} \| f - f_n \|_{L^1([0,T]; F_*)} = 0.
\]

As $e^A \ast f_n \in C([0,T]; D(A))$ (see corollary 2.5 of A. Pazy 1993) and

\[
\sup_{0 \leq t \leq T} \| (e^A \ast f)(t) - (e^A \ast f_n)(t) \|_{D(A)} \leq cM \| f - f_n \|_{L^1([0,T]; F_*)}
\]

we deduce for $n \to \infty$ that $e^A \ast f \in C([0,T]; D(A))$ and (1.6) is proved.

$\square$
THEOREM 1.2. Let $F_*$ verify (1.3). Given $f \in L^1(F_*)$ and $u_0 \in D(\Lambda)$ there exists $u \in C(D(\Lambda))$, differentiable (in $E$) for $t \geq 0$ a.e. and solution of

\begin{equation}
\begin{cases}
  u'(t) = \Lambda u(t) + f(t), & t \geq 0 \text{ a.e.} \\
  u(0) = u_0
\end{cases}
\end{equation}

Proof. We can assume $u_0 = 0$. Setting $u(t) = \int_0^t e^{\Lambda (t-s)} f(s) \, ds$, $t \geq 0$ we have for $0 \leq t < t + h$

\begin{equation}
\frac{u(t + h) - u(t)}{h} = \frac{e^{\Lambda h} u(t) - u(t)}{h} +
\end{equation}

\begin{equation}
\frac{1}{h} \int_t^{t+h} e^{\Lambda (t+h-s)} [f(s) - f(t)] \, ds + \frac{1}{h} \int_0^h e^{\Lambda s} f(t) \, ds.
\end{equation}

Now by virtue of (1.4)

\begin{equation}
\lim_{h \to 0^+} \frac{e^{\Lambda h} u(t) - u(t)}{h} = \Lambda u(t)
\end{equation}

and $f \in L^1(E)$ implies

\begin{equation}
\lim_{h \to 0^+} \frac{1}{h} \int_t^{t+h} \|f(s) - f(t)\| \, ds = 0
\end{equation}

for $t \geq 0$ a.e.; letting $h \to 0^+$ in (1.10) we obtain

\begin{equation}
\frac{u'(t)}{t} = \Lambda u(t) + f(t), \quad t \geq 0 \text{ a.e.}
\end{equation}

For $0 \leq t - h < t$ we have

\begin{equation}
\frac{u(t - h) - u(t)}{-h} = \frac{e^{\Lambda h} u(t - h) - u(t - h)}{h} +
\end{equation}

\begin{equation}
\frac{1}{h} \int_{t-h}^t e^{\Lambda (t+h-s)} [f(s) - f(t)] \, ds + \frac{1}{h} \int_0^h e^{\Lambda s} f(t) \, ds.
\end{equation}

Now

\begin{equation}
\frac{e^{\Lambda h} u(t - h) - u(t - h)}{h} - \frac{e^{\Lambda h} u(t) - u(t)}{h} =
\end{equation}

\begin{equation}
= \frac{1}{h} \int_0^h e^{\Lambda s} [u(t - h) - u(t)] \, ds
\end{equation}
and so by virtue of (1.6) and (1.11)

\[ \lim_{h \to 0^+} \frac{e^{\lambda h}u(t-h) - u(t-h)}{h} = \Lambda u(t). \]

Proceeding as above we deduce from (1.13) for \( h \to 0^+ \)

(1.14) \quad \dot{u}(t) = \Lambda u(t) + f(t), \quad t \geq 0 \text{ a.e.}

From (1.12) and (1.14) the conclusion follows.

\[ \square \]

**Theorem 1.3.** Let \( F_s \) verify (1.3). Given \( f \in L^1(F_s) \cap C(E) \) and \( u_0 \in D(\Lambda) \) there exists a unique solution \( u \in C(D(\Lambda)) \cap C^1(E) \) of

(1.15) \quad \begin{cases}  
\dot{u}(t) = \Lambda u(t) + f(t), & t \geq 0 \\
 u(0) = u_0.
\end{cases}

**Proof.** By repeating the proof of the preceding theorem we can deduce now that (1.9) holds for every \( t \geq 0 \); hence \( \dot{u} \in C(E) \). The uniqueness follows from the fact that when \( f = u_0 = 0 \) the only solution \( u \in C(D(\Lambda)) \cap C^1(E) \) of (1.15) is zero.

\[ \square \]

2. **Hille-Yosida operators**

Let \((X, \| \cdot \|)\) be a Banach space. \( A : D(A) \subset X \to X \) is called a **Hille-Yosida operator** if there exist \( \omega \in \mathbb{R} \) and \( M \geq 1 \) such that if \( \lambda > \omega \) then \( \lambda \in \rho(A) \) and

\[ \| (\lambda - \omega)^n(\lambda - A)^{-n} \| \leq M \]

for each \( n \in \mathbb{N} \) (see [E. Sinestrari 1994]).

When \( D(A) \) is dense then by virtue of Hille-Yosida theorem, \( A \) generates a strongly continuous semigroup: more generally we have the following result.
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Theorem 2.1. If $X_0 = (\overline{D(A)}, \| \cdot \|)$ and $A_0 : D(A_0) \subset X_0 \to X_0$ with

$$D(A_0) = \{ x \in D(A) : Ax \in X_0 \}$$

and $A_0 x = Ax$, then $A_0$ is the generator of a strongly continuous semigroup in $X_0$. In addition

(2.1) \[ D(A_0) \subset D(A) \hookrightarrow F. \]

For the proof of (2.1) see proposition 3.2 of [R. Nagel, E. Sinestrari 1994].

The Favard classes of a semigroup provide examples of Hille-Yosida operators as the following theorem shows.

Theorem 2.2. Let $\Lambda : D(\Lambda) \subset E \to E$ be the generator of a bounded semigroup and $F$ its Favard class and set $\Lambda_F : D(\Lambda_F) \subset F \to F$ with $D(\Lambda_F) = \{ x \in D(\Lambda), \Lambda x \in F \}$: then $\Lambda_F$ is a Hille-Yosida operator and

(2.2) \[ \overline{D(\Lambda_F)}^F = D(\Lambda). \]

Proof. By using (1.2), we see that if $\lambda \in \rho(\Lambda)$ then $\lambda \in \rho(\Lambda_F)$ and we have $(\lambda - \Lambda_F)^{-1} = (\lambda - \Lambda)^{-1}_{|F}$: and for each $n \in \mathbb{N}$,

$$\| (\lambda - \Lambda_F)^{-n} \|_{\mathcal{L}(F)} \leq \| (\lambda - \Lambda)^{-n} \|_{\mathcal{L}(E)};$$

as $\Lambda$ generates a semigroup the conclusion follows.

\[ \square \]

Remark 2.3. The restriction of $e^{\Lambda t}$ to $F$ is a semigroup which is not strongly continuous if $D(\Lambda) \neq F$: in fact we have

$$\lim_{t \to 0} \| e^{\Lambda t} x - x \|_F = 0$$

if and only if $x \in D(\Lambda)$ (see Corollary 3.1.8 of [P. Butzer, H. Berens 1967]).

Even if $\Lambda_F$ is only a Hille-Yosida operator we can obtain regularity results for the initial value problem

(2.3) \[ u'(t) = \Lambda u(t) + f(t), \quad t \geq 0; \quad u(0) = u_0. \]
THEOREM 2.4. Let $\Lambda$ be the generator of a strongly continuous semi-
group and $F$ its Favard class.

(i) If $f \in W^{1,1}(F)$, $u_0 \in D(\Lambda)$, $\Lambda u_0 \in F$ and $\Lambda u_0 + f(0) \in D(\Lambda)$ then
(2.3) has a unique solution $u \in C^1(F)$ such that $\Lambda u \in C(F)$.

(ii) If $f(t) \in D(\Lambda)$, $t \geq 0$ a.e.; $f, \Lambda f \in L^1(F)$, $u_0 \in D(\Lambda^2)$ then
there exists a unique $u \in W^{1,1}_{\text{loc}}(F)$ such that $\Lambda u \in C(F)$ and
satisfying (2.3) for $t \geq 0$ a.e.

(iii) If $f(t) \in D(\Lambda)$, $t \geq 0$ a.e., $f \in C(F)$; $\Lambda f \in L^1(F)$, $u_0 \in D(\Lambda^2)$
then (2.3) has a unique solution $u \in C^1(F)$ such that $\Lambda u \in C(F)$.

Proof. As $\Lambda_F$ is a Hille-Yosida operator we can use theorems 8.1 and
8.3 of [G. Da Prato, E. Sinestrari 1987] taking into account also
(2.2).

\hfill \Box

3. Homogeneization

In this section we exhibit a method for reducing the non homogene-
ous problem

\begin{equation}
\begin{cases}
u'(t) = \Lambda u(t) + f(t) \\
u(0) = u_0
\end{cases}
\end{equation}

with $\Lambda : D(\Lambda) \subset E \to E$ generator of a strongly continuous semi-
group $e^{\Lambda t}$ to a homogeneous one in a suitable product space (see [R.
Nagel, E. Sinestrari 1994]).

Let us consider in the Banach space $L^1(E) = L^1([0, +\infty]; E)$ the
left translations semigroup

$$(S(t)f)(s) := f(t + s); \ t, s \geq 0$$

for $f \in L^1(E)$; it is known that its generator is the a.e. derivative
with domain $W^{1,1}(E)$.

In the next theorem we show the homogeneization procedure and
a method to obtain regularity results for (3.1) (in particular the
Phillips theorem):
Theorem 3.1. In the Banach space

\[ Z := E \oplus L^1(E) \]

the semigroup

\[ G(t) \begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} e^{A}x + \left( e^{A} \ast f \right)(t) \\ S(t)f \end{pmatrix} \]

has the generator \( A : D(A) \subset Z \to Z \) defined by

\[ D(A) := D(\Lambda) \oplus W^{1,1}(E) \]

\[ \mathcal{A} \begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} \Lambda x + f(0) \\ f' \end{pmatrix} . \]

Hence given \( f \in W^{1,1}(E) \) and \( u_0 \in D(\Lambda) \), problem (3.1) has a unique solution \( u \in C^1(E) \cap C(D(\Lambda)) \), given by the first component of \( U(t) = G(t) (u_0) \).

Proof. The first part is a direct consequence of the definition of \( G(t) \).

To prove the last part let us observe that \( (u_0, f) \in D(A) \): hence the homogeneous problem in \( Z \)

\[ U'(t) = AU(t), \quad t \geq 0 ; \quad U(0) = U_0 \]

has a unique solution \( U \in C^1(Z) \cap C(D(A)) \) given by \( U(t) = G(t) (u_0) \). Now setting \( U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \), equation (3.6) is equivalent to the system

\[ \begin{cases} u'(t) = \Lambda u(t) + v(t)(0), & t \geq 0 \\ v'(t) = D_s v(t), & t \geq 0 . \end{cases} \]

But \( v(t) = S(t)f \) and so \( v(t)(0) = f(t) \); hence \( u \) satisfies (3.1).

\[ \square \]

An advantage of the homogenization is the possibility of using the semigroup theory’s classical procedures (restrictions, perturbations and so on) applied to \( G(t) \) to get regularity results for the solution \( u \) of (3.1)

Let us show a restriction procedure.
Theorem 3.2. Let \( D \hookrightarrow E \) and \( \mathcal{F} \hookrightarrow L^1(E) \) verify the following properties:

\[
\begin{align*}
(3.7) & \quad e^{\Lambda t} \text{ is a semigroup on } D \\
(3.8) & \quad S(t) \text{ is a semigroup on } \mathcal{F} \\
(3.9) & \quad \text{given } t \geq 0 \text{ and } f \in \mathcal{F}, \text{ we have } (e^{\Lambda} * f)(t) \in D \\
(3.10) & \quad \text{given } t \geq 0, \text{ there is } c(t) > 0 \text{ such that} \\
& \quad \|(e^{\Lambda} * f)(t)\|_D \leq c(t)\|f\|_\mathcal{F}, \forall f \in \mathcal{F} \\
(3.11) & \quad \text{given } f \in \mathcal{F}, \text{ we have } \lim_{t \to 0} \|(e^{\Lambda} * f)(t)\|_D = 0
\end{align*}
\]

Then the restriction of \( G(t) \) to the space
\[
(3.12) \quad Z_* := D \oplus \mathcal{F}
\]
is a strongly continuous semigroup with generator \( \mathcal{A}_* \), restriction of \( \mathcal{A} \) to
\[
(3.13) \quad D(\mathcal{A}_*) := \left\{ \begin{pmatrix} x \\ f \end{pmatrix} ; \ x \in D \cap D(\Lambda) ; \ f \in \mathcal{F} \cap W^{1,1}(E), \ f' \in \mathcal{F}, \ \Lambda x + f(0) \in D \right\}.
\]

Hence given \( \begin{pmatrix} u_0 \\ f \end{pmatrix} \in D(\mathcal{A}_*) \), problem (3.1) has a unique solution \( u \in C^1(D) \).

Proof. Properties (3.7)–(3.11) guarantee that \( G(t) \in \mathcal{L}(Z_*) \) and that \( \lim_{t \to 0} \|G(t) \begin{pmatrix} x \\ f \end{pmatrix} - \begin{pmatrix} x \\ f' \end{pmatrix}\|_{Z_*} = 0 \) for each \( \begin{pmatrix} x \\ f \end{pmatrix} \in Z_* \); hence the part of \( \mathcal{A} \) in \( Z_* \) is the generator of the semigroup \( G(t)|_{Z_*} \). The last part of the theorem is proved as the corresponding one in theorem 3.1.

\[ \square \]

We can give now a regularity result for (3.1) as an application of this theorem and theorem 1.1.

Theorem 3.3. Let \( F_* \) be a Banach space such that \( F_* \hookrightarrow F \). Given \( f \in W^{1,1}(F_*) \) and \( u_0 \in D(\Lambda) \) such that \( \Lambda u_0 + f(0) \in D(\Lambda) \), there exists a unique solution \( u \in C^1(D(\Lambda)) \) of problem (3.1).
Proof. By virtue of theorem 1.1, conditions (3.9)–(3.11) are satisfied by setting \( D = D(\Lambda) \) and \( \mathcal{F} = L^1(F_*) \). Now

\[
D(A_{\Theta}) = \left\{ \left( \begin{array}{c} x \\ f \end{array} \right) \in D(\Lambda) \oplus W^{1,1}(F_*) \ ; \ Ax + f(0) \in D(\Lambda) \right\}
\]

and so the conclusion follows from the last part of theorem 3.2.

\[\square\]

Examples of perturbation methods and their applications to Volterra integrodifferential equations are given in [R. Nagel, E. Sinestrari 1994].

4. Extrapolation spaces

In this section we will recall some definitions and results about extrapolation spaces; for details see [R. Nagel, E. Sinestrari 1994].

We will assume that \( A_0 : D(A_0) \subset X_0 \to X_0 \) is the generator of a bounded semigroup \( T_0(t) \) in \((X_0, \| \cdot \|)\) such that \( A_0^{-1} \in \mathcal{L}(X_0) \).

**Definition 4.1.** The extrapolation space of \( X_0 \) (associated with the operator \( A_0 \)) is the completion of \( Y_0 := (X_0, \| \cdot \|_{-1}) \) where \( \| x \|_{-1} := \| A_0^{-1} x \|, \ x \in X_0 \). It will be denoted by \( X_{-1} \).

The extrapolated semigroup \( T_{-1} \) in \( X_{-1} \) is the (unique) continuous extension of \( T_0(t) : Y_0 \subset X_{-1} \to X_{-1}, t \geq 0 \).

The Favard class of \( T_{-1} \) will be denoted by \( F_{-1} \).

The following results are proved in [R. Nagel, E. Sinestrari 1994]

**Theorem 4.2.** The semigroup \( T_{-1} \) is strongly continuous and

\[
\| T_{-1}(t) \|_{\mathcal{L}(X_{-1})} = \| T_0(t) \|_{\mathcal{L}(X_0)}.
\]

If \( A_{-1} : D(A_{-1}) \subset X_{-1} \to X_{-1} \) is the generator of \( T_{-1} \), then

\[
D(A_{-1}) = X_0 \text{ with equivalent norms}
\]

\[
(4.1) \quad D(A_{-1}) = X_0 \text{ with equivalent norms}
\]

\[
(4.2) \quad \| A_{-1} x \|_{-1} = \| x \|, \ x \in X_0
\]
(4.3) $A_{-1}$ is an extension of $A_0$

(4.4) there exists $A_{-1}^{-1} \in \mathcal{L}(X_{-1})$

(4.5) $A_{-1}(F) = F_{-1}$ and $\|x\|_F = \|A_{-1}x\|_{F_{-1}}$, $x \in F$.

For the applications we will need the following

**Theorem 4.3.** Let $A : D(A) \subset X \rightarrow X$ be a Hille-Yosida operator and $T_0(t)$ the strongly continuous semigroup generated by $A_0 : D(A_0) \subset X \rightarrow X$ with $X_0 = \overline{D(A)}$, $A_0 = A|_{D(A_0)}$ and $D(A_0) := \{x \in D(A), Ax \in \overline{D(A)}\}$ (see Theorem 2.1). We have

$$D(A_0) \subset D(A) \hookrightarrow F \hookrightarrow X_0 \subset X \hookrightarrow F_{-1} \hookrightarrow X_{-1}$$

and $A_{-1}$ is an extension of $A$.

The inclusion $X \hookrightarrow F_{-1}$ is important because it allows us to apply theorem 3.3 to find a new proof of a theorem of [G. DA PRATO, E. SINESTRARI 1985, 1987].

**Theorem 4.4.** Let $A : D(A) \subset X \rightarrow X$ be a Hille-Yosida operator. Given $f \in W^{1,1}(X)$ and $u_0 \in D(A)$ such that $Au_0 + f(0) \in \overline{D(A)}$ there exists a unique $u \in C^1(X) \cap C(D(A))$, solution of

$$\begin{cases} u'(t) = Au(t) + f(t), & t \geq 0 \\ u(0) = u_0 \end{cases}$$

**Proof.** As $X \hookrightarrow F_{-1}$ we can use theorem 3.3 with $E = X_{-1}$, $\Lambda = A_{-1}$ and $X = F\ast$. Hence, given $f \in W^{1,1}(X)$ there is a solution $u \in C^1(X_0)$ if $u_0 \in X_0$ and $A_{-1}u_0 + f(0) \in X_0$: as $f(0) \in X$ this implies $A_{-1}u_0 \in X$ and so $u_0 \in D(A)$.

As in many applications to differential operators $A$ in non reflexive Banach spaces $X$, the restriction of $A_{-1}$ to $F$, $F$ and $F_{-1}$ can be characterized, it is interesting to use the extrapolation space restricted to $F_{-1}$ as in the following results.
Theorem 4.5. (i) If \( f \in W^{1,1}(F_{-1}) \), \( u_0 \in F \) and \( A_{-1}u_0 + f(0) \in X_0 \) there exists a unique \( u \in C^1(F_{-1}) \) such that \( A_{-1}u \in C(F_{-1}) \) and solution of

\[
\begin{align*}
\frac{du}{dt}(t) &= A_{-1}u(t) + f(t), & t \geq 0 \\
u(0) &= u_0
\end{align*}
\]

(ii) If \( f \in L^1(F) \) and \( u_0 \in D(A_0) \) there exists a unique solution \( u \in W^{1,1}_{\text{loc}}(F_{-1}) \cap C(F) \) of (4.8) for \( t \geq 0 \) a.e.

(iii) If \( f \in L^1(F) \cap C(F_{-1}) \) and \( u_0 \in D(A_0) \) there exists a unique solution \( u \in C^1(F_{-1}) \cap C(F) \) of (4.8).

Proof. We can use the fact that the part of \( A_{-1} \) in \( F_{-1} \) is a Hille-Yosida operator, and apply theorem 2.4 taking into account that \( A_{-1}u_0 \in F_{-1} \) or \( u_0 \in D(A_0^2) \) are equivalent to \( u_0 \in F \) or \( u_0 \in D(A_0) \) respectively and \( A_{-1}f \in L^1(F_{-1}) \) or \( A_{-1}f \in C(F_{-1}) \) are equivalent to \( f \in L^1(F) \) or \( f \in C(F) \).

\[ \square \]

References