A Regularity Result for a Class of Anisotropic Systems

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SOMMARIO. - Si prova la regolarità parziale dei minimi del funzionale 
$I(u) = \int_{\Omega} G(Du)$, con $G$ integrando convesso a crescita anisotropa. 
Non si fanno ipotesi speciali sulla struttura di $G$.

SUMMARY. - We prove partial regularity of minimizers of the functional 
$I(u) = \int_{\Omega} G(Du)$, where $G$ is a convex integrand satisfying anisotropic 
growth condition. No special structure assumption is needed on $G$.

1. Introduction

In this paper we study the partial regularity of minimizers of integral functionals of the type

$$I(u) = \int_{\Omega} G(Du(x))dx$$

(1.1)

$u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$, $N \geq 1$, where $G$ is a $C^2$ convex integrand 
satisfying the growth condition:

$$C|\xi|^p \leq G(\xi) \leq L(1 + |\xi|^p)$$

(1.2)

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with $p > q$.

Few years ago it was observed that even in the scalar case, i.e. $N = 1$, minimizers of (1.1) may fail to be regular (see [M2], [G2]), when $p$ is too large with respect to $q$. On the other hand, one can prove regularity of scalar minimizers of (1.1) if $p$ is not too far away from $q$ (see e.g. [M3], [FS] and the references given in [M3]). More precisely, in [M3] it is shown that if one writes down the Euler equation for the functional $I$, under suitable assumptions on $p$ and $q$, the Moser iteration argument still works, thus leading to a sup estimate for the gradient $Du$ of the minimizer.

Clearly this approach can not be carried on in the vector valued case, i.e. when $N > 1$. As far as we know, the only regularity results for systems are proved under special structure assumptions (see [AF2], [M4]).

Namely, the model case covered in [AF2] is the functional

$$\int_{\Omega} |Du|^p + \sum_{\alpha=1}^k |D_{\alpha}u|^{p_{\alpha}}$$

with $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^N$, $N \geq 1$, $1 \leq k \leq n$, $2 \leq p < p_{\alpha}$, and $p_{\alpha}$ not too far from $p$, while in [M4], it is proved everywhere regularity of minimizers of (1.1) when $G(\xi) = f(|\xi|)$.

In this paper we prove that if $G$ satisfies (1.2) and the strong ellipticity assumption

$$\langle D^2 G(\xi) \eta, \eta \rangle \geq \gamma (1 + |\xi|^2)^{\frac{n-2}{2}} |\eta|^2$$

and

$$2 \leq q < p < \min \left\{ q + 1, \frac{qn}{n-1} \right\}, \quad (1.3)$$

a minimizer $u \in W^{1,q}(\Omega ; \mathbb{R}^N)$ of functional (1.1) is $C^{1,\alpha}$ for all $\alpha < 1$ in an open set $\Omega_0 \subset \Omega$ such that $\text{meas} (\Omega \setminus \Omega_0) = 0$.

We point out that a part from condition (1.3), no special structure assumption is needed on $G$.

The proof of our result goes through a more or less standard blow-up argument aimed to establish a decay estimate on the excess function for the gradient

$$U(x_0, r) = \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^q dx.$$  

The essential tool in the case we consider, is a lemma due to Fonseca and Maly (see [FM] and also Lemma 2.3 below) which makes possible to connect in the annulus $B_r \setminus B_s$ two $W^{1,q}$ functions $v$ and $w$ with a function $z \in W^{1,q}(B_r \setminus B_s)$ if $q < p < \frac{qn}{n-1}$.  

2. Statements and preliminary Lemmas

Let us consider the functional

\[ I(u) = \int_{\Omega} G(Du(x)) \, dx \]

where \( \Omega \) is a bounded open set of \( \mathbb{R}^n \), \( n \geq 2 \). Let \( G : \mathbb{R}^{nN} \to \mathbb{R} \), \( N \geq 2 \), satisfy the following assumptions:

\[ G \in C^2 \quad \text{(H1)} \]
\[ C|\xi|^p \leq G(\xi) \leq L(1 + |\xi|^p) \quad \text{(H2)} \]
\[ \langle D^2 G(\xi) \eta, \eta \rangle \geq \gamma (1 + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2 \quad \text{(H3)} \]

where \( 2 \leq q < p < \min \left\{ q + 1, \frac{qn}{n-1} \right\} \)

It is well known that

\[ |DG(\xi)| \leq c(1 + |\xi|^{p-1}). \quad \text{(H4)} \]

We say that \( u \in W^{1,q}(\Omega; \mathbb{R}^N) \) is a minimizer of \( I \) if

\[ I(u) \leq I(u + v) \]

for any \( v \in u + W^{1,q}_0(\Omega; \mathbb{R}^N) \).

Remark 1. If \( u \) is a local minimizer of \( I \) and \( \phi \in C^1_0(\Omega; \mathbb{R}^N) \) from the minimality condition one has for any \( \varepsilon > 0 \)

\[ 0 \leq \int_{\Omega} \left[ G(Du + \varepsilon D\phi) - G(Du) \right] \, dx \]
\[ = \varepsilon \int_{\Omega} dx \int_0^1 \frac{\partial G}{\partial \xi_\alpha}(Du + \varepsilon t D\phi) D_\alpha \phi \, dt \]

Dividing this inequality by \( \varepsilon \), and letting \( \varepsilon \) go to zero, from (H4) and the assumption \( p \leq q + 1 \) we get

\[ \int_{\Omega} \frac{\partial G}{\partial \xi_\alpha}(Du) D_\alpha \phi \, dx \geq 0 \]

and therefore by the arbitrariness of \( \phi \) the usual Euler-Lagrange system holds:

\[ \int_{\Omega} \frac{\partial G}{\partial \xi_\alpha}(Du) D_\alpha \phi \, dx = 0 \quad \forall \phi \in C^1_0(\Omega; \mathbb{R}^N) \]
We prove the following

**Theorem 2.1.** Let $G$ be as above and let $u \in W^{1,q}(\Omega; \mathbb{R}^N)$ be a minimizer of $I$. Then there exists an open subset $\Omega_0$ of $\Omega$ such that

$$\text{meas}(\Omega \setminus \Omega_0) = 0$$

and

$$u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N) \quad \text{for all} \quad \alpha < 1.$$  

In the following, we will denote by $u$ a $W^{1,q}(\Omega; \mathbb{R}^N)$ minimizer of $\int_\Omega G(Du)dx$ and assume that $G$ satisfies (H1), (H2), (H3). We set for every $B_r(x_0) \subset \Omega$

$$U(x_0, r) = \int_{B_r(x_0)} |Du - (Du)_{x_0,r}|^2 + |Du - (Du)_{x_0,r}|^q dx,$$

where

$$\int_{B_r(x_0)} g = (g)_{x_0,r} = \frac{1}{\text{meas}(B_r(x_0))} \int_{B_r(x_0)} g.$$

The next Lemma can be found in [FM], (Lemma 2.2), in a slightly different form.

**Lemma 2.1.** Let $v \in W^{1,q}(B_1(0))$ and $0 < s < r < 1$. There exists a linear operator $T : W^{1,q}(B_1(0)) \to W^{1,q}(B_1(0))$ such that

$$Tv = v \quad \text{on} \quad (B_1 \setminus B_s) \cup B_s$$

and for all $\mu > 0$, for all $p < q \frac{n}{n-1}$

$$||Tv||_{W^{1,2}(B_s \setminus B_s)} + \mu ||Tv||_{W^{1,p}(B_s \setminus B_s)}$$

$$\leq C \bigg\{ (r - s)^\sigma \left[ \sup_{t \in (s, r)} (t - s)^{-\frac{1}{2}} ||v||_{W^{1,2}(B_s \setminus B_s)} + \right.$$  

$$+ \sup_{t \in (s, r)} (r - t)^{-\frac{1}{2}} ||v||_{W^{1,2}(B_s \setminus B_s)} \bigg] +$$

$$+ \mu (r - s)^\tau \left[ \sup_{t \in (s, r)} (t - s)^{-\frac{1}{q}} ||v||_{W^{1,q}(B_s \setminus B_s)} + \right.$$  

$$+ \sup_{t \in (s, r)} (r - t)^{-\frac{1}{q}} ||v||_{W^{1,q}(B_s \setminus B_s)} \bigg\}$$

where $C = C(n, p, q) > 0$, $\sigma = \sigma(n) > 0$ and $\tau = \tau(n, p, q) > 0.$
Let us recall an elementary lemma also proved in [FM].

**Lemma 2.2.** Let \( \psi \) be a continuous nondecreasing function on an interval \([a, b], a < b\). There exist \( a' \in [a, a + \frac{1}{4}(b - a)] \), \( b' \in [b - \frac{1}{4}(b - a), b] \) such that \( a \leq a' < b' \leq b \) and

\[
\frac{\psi(t) - \psi(a')}{t - a'} \leq 3 \frac{\psi(b) - \psi(a)}{b - a}
\]

and

\[
\frac{\psi(b') - \psi(t)}{b' - t} \leq 3 \frac{\psi(b) - \psi(a)}{b - a}
\]

(2.3)

for all \( t \in (a', b') \).

Finally the next result is a straightforward generalisation to our case of Lemma 2.4 in [FM]. We give the proof here for completeness.

**Lemma 2.3.** Let \( v, w \in W^{1,q}(B_1(0)) \) and \( \frac{1}{4} < s < r < 1 \). Fix \( q < p < \frac{m^2}{n+1} \), for all \( \mu > 0 \) and \( m \in \mathbb{N} \) there exist a function \( z \in W^{1,q}(B_1(0)) \) and \( \frac{1}{4} < s' < s'' < r' < r < 1 \) with \( r', s' \) depending on \( v, w \) and \( \mu \), such that

\[
z = v \quad \text{on} \quad B_{s'}, \quad z = w \quad \text{on} \quad B_1 \setminus B_{s'},
\]

(2.4)

and

\[
||z||_{W^{1,q}(B_{r'} \setminus B_{s'})} + \mu ||z||_{W^{1,p}(B_{r'} \setminus B_{s'})}
\]

\[
\leq C \frac{(r - s)^\rho}{m^\rho} \left[ \int_{B_{r'} \setminus B_{s'}} (1 + |Dv|^2 + |Dw|^2 + |v|^2 + |w|^2 + m^2 \frac{|v - w|^2}{(r - s)^2}) + m^q \right]
\]

(2.5)

\[
+ \frac{r - s}{m^{1 - \frac{q}{2}}} \left[ \int_{B_{r'} \setminus B_{s'}} (1 + |Dv|^q + |Dw|^q + |v|^q + |w|^q + m^q \frac{|v - w|^q}{(r - s)^{q/2}}) \right]
\]

where \( C = C(n, p, q) > 0 \) and \( \rho = \rho(p, q, n) > 0 \).

**Proof.** As in Lemma 2.4 in [FM], choose \( m \in \mathbb{N} \) and set

\[
f = 1 + |Dv|^2 + |Dw|^2 + |v|^2 + |w|^2 + m^2 \frac{|v - w|^2}{(r - s)^2} + \mu^q \left[ 1 + |Dv|^q + |Dw|^q + |v|^q + |w|^q + m^q \frac{|v - w|^q}{(r - s)^{q/2}} \right].
\]
We may find $k \in \{1, ..., m\}$ such that
\[
\int_{B_\frac{k(r-s)}{m} \setminus B_{s + \frac{(k-1)(r-s)}{m}}} f dx \leq \frac{1}{m} \int_{B_r \setminus B_s} f dx,
\]
Set, for $t \in [s + \frac{(k-1)(r-s)}{m}, s + \frac{k(r-s)}{m}]$,
\[
\psi(t) = \int_{B_1 \setminus B_s} f dx
\]
which is a continuous nondecreasing function. By Lemma 2.2, there exists $[s', r'] \subset [s + \frac{(k-1)(r-s)}{m}, s + \frac{k(r-s)}{m}]$ such that
\[
\frac{r - s}{m} \geq r' - s' \geq \frac{r - s}{3m}
\]
and
\[
\int_{B_1 \setminus B_{s'}} f dx \leq 3 \frac{(t - s')m}{r - s} \int_{B_{s + \frac{k(r-s)}{m} \setminus B_{s + \frac{(k-1)(r-s)}{m}}}} f dx
\]
\[
\leq 3 \frac{t - s'}{r - s} \int_{B_r \setminus B_s} f dx,
\]
\[
\int_{B_{r'} \setminus B_1} f dx \leq 3 \frac{r' - t}{r - s} \int_{B_r \setminus B_s} f dx
\]
for all $t \in (s', r')$. Set
\[
u(x) = \begin{cases} v(x) & \text{if } x \in B_{s'} \\ \frac{(r'-|x|)v(x) + (|x|-s')u(x)}{r'-s'} & \text{if } x \in B_{r'} \setminus B_{s'} \\ w(x) & \text{if } x \in B_1 \setminus B_{r'}.
\end{cases}
\]
A direct computation shows that
\[
|u|^2 + |Du|^2 + mu^q(|u|^q + |Du|^q) \leq Cf.
\]
If we apply Lemma 2.1 to the function $u$, we then find $z \in W^{1, q}(B_1)$ satisfying (2.4). Moreover, from (2.6) and (2.7) one readily checks
that
\[
||z||_{W^{1,2}(B_r \setminus B_s)} + \mu ||z||_{W^{1,p}(B_r \setminus B_s)} \\
\leq c \left\{ \frac{(r' - s')^\sigma}{(r' - s')^{\frac{\sigma}{2}}} \left| B_r \setminus B_{s'} \right|^\frac{1}{2} \left( \int_{B_r \setminus B_{s'}} f \right)^\frac{1}{2} + \right. \\
+ \frac{(r' - s')^{\tau}}{(r' - s')^{\frac{\tau}{2}}} \left| B_r \setminus B_{s'} \right|^\frac{1}{2} \left( \int_{B_r \setminus B_{s'}} f \right)^\frac{1}{q'} \right\} \\
\leq c \left\{ (r' - s')^\sigma \left( \int_{B_r \setminus B_{s'}} f \right)^\frac{1}{2} + (r' - s')^{\tau} \left( \int_{B_r \setminus B_{s'}} f \right)^\frac{1}{q'} \right\},
\]
from which (2.5) follows choosing \( \rho = \min \{ \sigma, \tau \} \).

3. Proof of Theorem 1

As usual, to get the partial regularity result stated in Theorem 1, we need a decay estimate for the excess function \( U(x_0, r) \) defined in section 2.

**Proposition 3.1.** Fix \( M > 0 \). There exists a constant \( C_M > 0 \) such that for every \( 0 < \tau < \frac{1}{4} \), there exists \( \epsilon = \epsilon(\tau, M) \) such that, if
\[
|\{(Du)_{x_0, r}\}| \leq M \quad \text{and} \quad U(x_0, r) \leq \epsilon
\]
then
\[
U(x_0, \tau r) \leq C_M \tau^2 U(x_0, r).
\]

**Proof.** Fix \( M \) and \( \tau \). We shall determine \( C_M \) later.

We argue by contradiction. We assume that there exists a sequence \( B_{r_n}(x_h) \) satisfying
\[
B_{r_n}(x_h) \subset \Omega, \quad |\{(Du)_{x_n, r_h}\}| \leq M, \quad \lim_{h} U(x_h, r_h) = 0,
\]
but
\[
U(x_h, \tau r_h) > C_M \tau^2 U(x_h, r_h).
\]
(3.1)

Set
\[
a_h = (u)_{x_n, r_h} \quad A_h = (Du)_{x_n, r_h} \quad \lambda^2_h = U(x_h, r_h).
\]
Step 1. [Blow up.] We rescale the function $u$ in each $B_{r_h}(x_h)$ to obtain a sequence of functions on $B_1(0)$. Set

$$v_h(y) = \frac{1}{\lambda_h r_h} [u(x_h + r_h y) - a_h - r_h A_h y],$$

then

$$Dv_h(y) = \frac{1}{\lambda_h} [Du(x_h + r_h y) - A_h].$$

Clearly we have

$$(v_h)_{0,1} = 0 \quad (Dv_h)_{0,1} = 0.$$ 

Moreover,

$$\int_{B_1(0)} (1 + \lambda_h^{q-2} |Dv_h|^{q-2}) |Dv_h|^2 \, dy = 1. \quad (3.2)$$

Passing possibly to a subsequence we may suppose that

$$v_h \rightharpoonup v \quad \text{weakly in } W^{1,2}(B_1; \mathbb{R}^N) \quad (3.3)$$

and, since $\forall h \quad |A_h| \leq M$,

$$A_h \rightarrow A. \quad (3.4)$$

Step 2. Now we show that

$$\int_{B_1(0)} \frac{\partial^2 G}{\partial \xi_\alpha^i \partial \xi_\beta^j} (A) D_\beta v^i D_\alpha \phi^j \, dy = 0 \quad \forall \phi \in C_0^1 (B_1; \mathbb{R}^N). \quad (3.5)$$

Since we assume $p - 1 \leq q$ we can write the usual Euler-Lagrange system for $u$ (see Remark 1). Then, rescaling in each $B_{r_h}(x_h)$, we get for any $\phi \in C_0^1 (B_1; \mathbb{R}^N)$ and any $1 \leq i \leq N$

$$\int_{B_1(0)} \frac{\partial G}{\partial \xi_\alpha^i} (A_h + \lambda_h Dv_h) D_\alpha \phi^i \, dy = 0.$$

Then

$$\frac{1}{\lambda_h} \int_{B_1(0)} [\frac{\partial G}{\partial \xi_\alpha^i} (A_h + \lambda_h Dv_h) - \frac{\partial G}{\partial \xi_\alpha^i} (A_h)] D_\alpha \phi^i \, dy = 0. \quad (3.6)$$
Let us split
\[ B_1 = E^+_h \cup E^-_h \]
\[ = \{ y \in B_1 : \lambda_h |Dv_h(y)| > 1 \} \cup \{ y \in B_1 : \lambda_h |Dv_h(y)| \leq 1 \}, \]
then by (3.2) we get
\[ |E^+_h| \leq \int_{E^+_h} \lambda_h^2 |Dv_h|^2 dy \leq \lambda_h^2 \int_{B_1(0)} |Dv_h|^2 dy \leq c \lambda_h^2. \quad (3.7) \]

Now, by (H4) and Hölder inequality, we observe that
\[ \frac{1}{\lambda_h} \int_{E^+_h} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy \]
\[ \leq \frac{c}{\lambda_h} |E^+_h| + c \lambda_h^{p-2} \int_{E^+_h} |Dv_h|^{p-1} dy \]
\[ \leq c \lambda_h + c \left( \int_{E^+_h} \lambda_h^{p-2} |Dv_h|^q dy \right)^{\frac{p-1}{q}} \lambda_h^{\frac{2(p-2)}{q}} |E^+_h|^{\frac{2(p+1)}{q}} \leq c \lambda_h \]
where we used the assumption \( p - 1 \leq q. \)

From this it follows that
\[ \lim_{h \to 0} \frac{1}{\lambda_h} \int_{E^+_h} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy = 0. \quad (3.8) \]

On \( E^-_h \) we have
\[ \frac{1}{\lambda_h} \int_{E^-_h} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy \]
\[ = \int_{E^-_h} \int_0^1 D^2 G(A_h + s \lambda_h Dv_h) Dv_h D\phi ds dy \]
\[ = \int_{E^-_h} \int_0^1 [D^2 G(A_h + s \lambda_h Dv_h) - D^2 G(A_h)] Dv_h D\phi ds dy + \]
\[ + \int_{E^-_h} D^2 G(A_h) Dv_h D\phi dy. \]

Note that (3.7) ensures that \( \chi_{E^+_h} \to \chi_{B_1} \) in \( L^r(B_1) \) for all \( r < \infty \) and by (3.2) we have, passing possibly to a subsequence,
\[ \lambda_h Dv_h(y) \to 0 \quad \text{a.e. in } B_1. \]
Then, by (3.3), (3.4) and the uniform continuity of $D^2 G$ on bounded sets, we get
\[
\lim_h \frac{1}{\lambda_h} \int_{E_h} [DG(A_h + \lambda_h Dv_h) - DG(A_h)] D\phi dy
= \int_{B_1} D^2 G(A) Dv D\phi dy.
\]
By (3.6), (3.8) and the above equality, we obtain that $v$ satisfies equation (3.5), which is elliptic by (H3). We have for any $0 < \tau < 1$
\[
\iint_{B_r} |Dv - (Dv)_\tau|^2 dy \leq c\tau^2 \iint_{B_1} |Dv - (Dv)_1|^2 dy \leq c\tau^2. \quad (3.9)
\]
Moreover we have
\[
v \in C^\infty(B_1; \mathbb{R}^N). \quad (3.10)
\]
and
\[
\lambda_h^{-1} (v_h - v) \rightharpoonup 0 \quad \text{weakly in } W^{1,q}_0(B_1; \mathbb{R}^N)
\]

**Step 3. [Upper bound.]** We set
\[
G_h(\xi) = \frac{1}{\lambda_h^2} [G(A_h + \lambda_h \xi) - G(A_h) - \lambda_h D G(A_h) \xi]
\]
and for every $r < 1$
\[
I_{h,r}(w) = \int_{B_r} G_h(Dw) dy.
\]

Note that by the strong ellipticity assumption (H3) it follows that $G_h(\xi) \geq 0$, for any $\xi$. Fix $\frac{1}{4} < s < 1$. Passing to a subsequence we may always assume that
\[
\lim_h [I_{h,s}(v_h) - I_{h,s}(v)]
\]
eexists. We shall prove that
\[
\lim_h [I_{h,s}(v_h) - I_{h,s}(v)] \leq 0. \quad (3.11)
\]
Consider $r > s$ and fix $m \in \mathbb{N}$. Observe that, since $v \in W^{1,q}(B_1)$ and $v_h \in W^{1,q}(B_1)$, Lemma 2.3, with $\mu = \lambda_h^{-1}$, implies that there exist $z_h \in W^{1,q}(B_1)$ and $\frac{1}{4} < s_h < r_h < r < 1$ such that
\[
z_h = v \quad \text{on } B_{s_h} \quad z_h = v_h \quad \text{on } B_1 \setminus B_{r_h}
\]
and

\[
||z_h||_{W^{1,2}(B_{r,s} \setminus B_{r,h})} + \lambda_h^{\frac{p-2}{p}} ||z_h||_{W^{1,p}(B_{r,s} \setminus B_{r,h})} \\
\leq C \frac{(r-s)^p}{m^p} \left[ \int_{B_{r} \setminus B_s} (1 + |Dv|^2 + |Dv_h|^2 + |v|^2 + |v_h|^2 + \\
+ m^2 \frac{|v - v_h|^2}{(r-s)^2}) + \lambda_h^{\frac{p-2}{p}} \int_{B_{r} \setminus B_s} (1 + |Dv|^q + |Dv_h|^q + |v|^q + |v_h|^q + \\
+ m^q \frac{|v - v_h|^q}{(r-s)^q}) \right]^{\frac{1}{p}} \\
\]  

(3.12)

Since by (3.10), $Dv$ is locally bounded on $B_1$ we get

\[
I_{h,s}(v_h) - I_{h,s}(v) \\
\leq I_{h,r_h}(v_h) - I_{h,r_h}(v) + I_{h,r_h}(v) - I_{h,s}(v) \\
= I_{h,r_h}(v_h) - I_{h,r_h}(v) + \int_{B_{r_h} \setminus B_s} G_h(Dv) \\
\leq I_{h,r_h}(z_h) - I_{h,r_h}(v) + c(r-s) \\
\leq c \int_{B_{r_h} \setminus B_s} [G_h(Dz_h) - G_h(Dv)] + c(r-s). \\
\]  

(3.13)

where we used the minimality of $v_h$. As $|G_h(\xi)| \leq c(|\xi|^2 + \lambda_h^{p-2} |\xi|^p)$ (see [AF], Lemma II.3), we get by (3.12)

\[
I_{h,r_h}(z_h) - I_{h,r_h}(v) \\
\leq c \int_{B_{r_h} \setminus B_s} |Dz_h|^2 + \lambda_h^{p-2} |Dz_h|^p \\
\leq C \frac{(r-s)^{2p}}{m^{2p}} \left[ \int_{B_{r} \setminus B_s} (1 + |Dv|^2 + |Dv_h|^2 + |v|^2 + |v_h|^2 + \\
+ m^2 \frac{|v - v_h|^2}{(r-s)^2}) + \lambda_h^{\frac{p-2}{p}} \int_{B_{r} \setminus B_s} (1 + |Dv|^q + |Dv_h|^q + |v|^q + |v_h|^q + \\
+ m^q \frac{|v - v_h|^q}{(r-s)^q}) \right]^{\frac{2p}{p}} \\
+ C \frac{(r-s)^{2p}}{m^{2p}} \left[ \lambda_h^{\frac{p-2}{p}} \int_{B_{r} \setminus B_s} (1 + |Dv|^q + |Dv_h|^q + |v|^q + |v_h|^q + \\
+ m^q \frac{|v - v_h|^q}{(r-s)^q}) \right]^{\frac{2p}{q}} \\
= J_{h,1} + J_{h,2}. \\
\]

Since $v_h \rightarrow v$ in $L^2(B_1; \mathbb{R}^N)$ we have, using (3.2)

\[
\limsup_{h \rightarrow \infty} J_{h,1} \leq C m^{-2}. 
\]
Moreover, since
\[
\frac{g(p-2)}{p} \int_{B_1} |Dv_h|^q = \lambda_h^{q-2} \int_{B_1} |Dv_h|^q \leq C \lambda_h^{2(p-2)}
\]
and
\[
\lambda_h^{\frac{q(p-2)}{p}} \int_{B_1} |v_h - v|^q \leq c \lambda_h^{\frac{q(p-2)}{p}} \int_{B_1} |Dv_h|^q \leq c \lambda_h^{2(p-2)}
\]
we have
\[
\lim_{h} J_{h,2} = 0.
\]
Hence we conclude letting first \( m \to \infty \) and then \( r \to s \) in (3.13).

**Step 4. [Lower bound.]** We shall prove that, for a.e. \( \frac{1}{4} < r < \frac{1}{2} \), if \( t < r \) then
\[
\limsup_h \int_{B_t} |Dv - Dv_h|^2 (1 + \lambda_h^{q-2} |Dv - Dv_h|^{q-2})
\leq \lim_h [I_{h,r}(v_h) - I_{h,r}(v)].
\]
For any Borel set \( A \subset B_1 \), let us define
\[
\mu_h(A) = \int_A (|v_h|^2 + |Dv_h|^2) dx.
\]
Passing possibly to a subsequence, since \( \mu_h(B_1) \leq c \), we may suppose
\[
\mu_h \to \mu \quad \text{weakly * in the sense of measures},
\]
where \( \mu \) is a Borel measure over \( B_1 \). Then for a.e. \( r < 1 \)
\[
\mu(\partial B_r) = 0
\]
and let us choose such a radius \( r \). Consider \( \frac{1}{4} < t < s < r \), also such that \( \mu(\partial B_s) = 0 \), and fix \( m \in IN \). Observe that, as \( v_h \in W^{1,q}(B_1) \) Lemmas 2.3 implies that there exist \( z_h \in W^{1,q}(B_1) \) and \( \frac{1}{4} < s < s_h < r_h < r < 1 \) such that
\[
\begin{align*}
 z_h &= v_h \quad \text{on} \quad B_{s_h} \quad & z_h &= v_h \quad \text{on} \quad B_1 \setminus B_{r_h} \\
 r_h - s_h &\geq \frac{r - s}{3m}
\end{align*}
\]
and
\[
\|z_h\|_{W^{1,2}(B_{r,h} \setminus B_{s,h})} + \lambda_h^{\frac{p-2}{p}} \|z_h\|_{W^{1,p}(B_{r,h} \setminus B_{s,h})} \\
\leq C \left( \frac{r-s}{m} \right)^{\rho} \int_{B_r \setminus B_s} (1 + |Dv_h|^2 + |v_h|^2) + \lambda_h^{-\frac{p}{p-2}} \int_{B_r \setminus B_s} (1 + |Dv_h|^2 + |v_h|^2)^\frac{1}{2}
\]
(3.14)

Passing possibly to a subsequence, we may suppose that
\[
z_h \rightharpoonup v_{r,s} \quad \text{weakly in } W^{1,2}(B_1).
\]
and
\[
v_{r,s} = v \quad \text{in } (B_1 \setminus B_r) \cup B_s
\]
Moreover from (3.14) it is clear that
\[
\lambda_h^{\frac{p-2}{p}} \int_{B_1} |Dz_{r,s}|^2 \leq c
\]
(3.15)
Consider \(\zeta_h \in C_0^\infty(B_{r,h})\) such that \(0 \leq \zeta_h \leq 1\), \(\zeta_h = 1\) on \(B_{s,h}\) and
\[
|D\zeta_h| \leq \frac{C}{r_h-s_h}
\]
and set
\[
\psi_h = \zeta_h(z_h - v_{r,s}^\epsilon),
\]
where \(v_{r,s}^\epsilon = \rho_\epsilon * v_{r,s}\), and \(\rho_\epsilon\) is the usual sequence of mollifiers. Now, setting \(v^\epsilon = \rho_\epsilon * v\), we observe that
\[
I_{h,r,s}(v_h) - I_{h,r,s}(v) \\
= I_{h,r,s}(v_h) - I_{h,r,s}(z_h) + I_{h,r,s}(z_h) - I_{h,r,s}(v_{r,s}^\epsilon + \psi_h^\epsilon) + \\
+ I_{h,r,s}(\psi_h^\epsilon + v_{r,s}^\epsilon) - I_{h,r,s}(v_{r,s}^\epsilon) - I_{h,r,s}(\psi_h^\epsilon) + \\
I_{h,r,s}(v_{r,s}^\epsilon) - I_{h,r,s}(v^\epsilon) + I_{h,r,s}(\psi_h^\epsilon) \\
= R_{h,1} + R_{h,2} + R_{h,3} + R_{h,4} + R_{h,5}
\]
(3.16)
To bound \(R_{h,1}\) we observe that
\[
I_{h,r,s}(v_h) - I_{h,r,s}(z_h) = \int_{B_{r,h} \setminus B_{s,h}} G_h(Dv_h) - \int_{B_{r,h} \setminus B_{s,h}} G_h(Dz_h) + \\
\geq - \int_{B_{r,h} \setminus B_{s,h}} G_h(Dz_h)
\]
on the other hand we have
\[
\int_{B_{r_{h}} \setminus B_{s_{h}}} G_{h}(Dz_{h}) \leq \int_{B_{r_{h}} \setminus B_{s_{h}}} |Dz_{h}|^{2} + \lambda_{h}^{p-2} |Dz_{h}|^{p} \\
\leq c m^{-2} p \left[ \int_{B_{r_{h}} \setminus B_{s_{h}}} 1 + |Dv_{h}|^{2} + |v_{h}|^{2} + \\
+ \lambda_{h}^{p-2} \int_{B_{r_{h}} \setminus B_{s_{h}}} 1 + |Dv_{h}|^{2} + |v_{h}|^{2} \right]^{\frac{p}{2}}
\]
and then arguing as we did in Step 3 to bound $J_{h,1}$ we get
\[
\limsup_{h} \int_{B_{r_{h}} \setminus B_{s_{h}}} G_{h}(Dz_{h}) \leq C m^{-2} p
\]
hence, letting $h \to \infty$ we get
\[
\lim\inf_{h} R_{h,1} \geq -C m^{-2} p
\]  
(3.17)
We obtain that
\[
R_{h,2} = \int_{B_{r_{h}} \setminus B_{s_{h}}} G_{h}(Dz_{h}) - G_{h}(D\psi_{h}^{*} + Dv_{r,s}^{*}) \\
\geq - c \int_{B_{r_{h}} \setminus B_{s_{h}}} |D\psi_{h}^{*} + Dv_{r,s}^{*}|^{2} + \lambda_{h}^{p-2} |D\psi_{h}^{*} + Dv_{r,s}^{*}|^{p} \\
\geq - c \int_{B_{r_{h}} \setminus B_{s_{h}}} \left[ |Dz_{h}|^{2} + \lambda_{h}^{p-2} |Dz_{h}|^{p} + |Dv_{r,s}^{*}|^{2} + \\
+ \lambda_{h}^{p-2} |Dv_{r,s}^{*}|^{p} \right] - c \int_{B_{r_{h}} \setminus B_{s_{h}}} \left( m^{2} |z_{h} - v_{r,s}^{*}|^{2} + \\
+ m^{p} \lambda_{h}^{p-2} \frac{|z_{h} - v_{r,s}^{*}|^{p}}{(r-s)^{2}} \right)
\]  
(3.18)
where we used the bound $r_{h} - s_{h} \geq \frac{r-s}{3m}$. By (3.15), since $p < q^{*}$, we get
\[
\int_{B_{1}} \lambda_{h}^{p-2} |z_{h}|^{p} \leq c \lambda_{h}^{p-2} \left\{ \int_{B_{1}} |z_{h} - (z_{h})_{0,1}|^{p} + |(z_{h})_{0,1}|^{p} \right\} \\
\leq c \lambda_{h}^{p-2} \left\{ \left( \int_{B_{1}} |z_{h} - (z_{h})_{0,1}|^{2^{*}} \right)^{\frac{p}{2^{*}}} + \left( \int_{B_{1}} |z_{h}| \right)^{p} \right\}
\]
\[
\leq c \lambda_h^q \left\{ \left( \int_{B_1} |Dz_h|^q \right)^{\frac{q}{p}} + \left( \int_{B_1} |z_h|^2 \right)^{\frac{q}{2}} \right\} \\
\leq c \lambda_h^{2(p-1)/q} \left( \lambda_h^q \int_{B_1} |Dz_h|^q \right)^{1/q} + c \lambda_h^{q-2}.
\]

where we used (3.14) to bound \( \left( \int_{B_1} |z_h|^2 \right)^{1/2} \). Therefore

\[
\limsup_{h \to \infty} S_{h,2} \leq c \frac{m^2}{(r-s)^2} \int_{B_2} |v_{r,s} - v_{r,s}^e|^2.
\]

To bound \( S_{h,1} \), observe that for every \( h \)

\[
\int_{B_{r,h} \setminus B_{r,s}} |Dv_{r,s}^e|^2 \\
\leq c \int_{B_r \setminus B_s} |Dv_{r,s}|^2 + c \int_{B_{r,s}} |Dv_{r,s} - Dv_{r,s}^e|^2 \\
\leq \liminf_j c \int_{B_r \setminus B_{s,j}} |Dz_j|^2 + c \int_{B_{r,s}} |Dv_{r,s} - Dv_{r,s}^e|^2 \\
= c \liminf_j \int_{(B_r \setminus B_{s,j}) \setminus (B_{r,s} \setminus B_{r,s})} |Dv_j|^2 + \\
+ c \limsup_j \int_{B_{r,s} \setminus B_{s,j}} |Dz_j|^2 + c \int_{B_{r,s}} |Dv_{r,s} - Dv_{r,s}^e|^2
\]

We control the second integral as usual using Lemma 2.3, while the first is less or equal than \( c \mu(B_r \setminus B_s) \).

Moreover we can estimate

\[
\int_{B_{r,h} \setminus B_{r,s}} |Dz_h|^2 + \lambda_h^{q-2} |Dz_h|^q
\]

as we did in Step 3 to bound \( J_{h,1} \). Hence

\[
\liminf_h R_{h,2} \geq - cm^{q-2} - c \mu(B_r \setminus B_s) + \\
- c \int_{B_{r,s}} |Dv_{r,s} - Dv_{r,s}^e|^2 + \\
- \frac{cm^2}{(r-s)^2} \int_{B_{r,s}} |v_{r,s} - v_{r,s}^e|^2
\]

(3.19)
To bound $R_{h,3}$ we observe that

$$G_h(A + B) - G_h(A) - G_h(B) = \int_0^1 \int_0^1 D^2 G_h(sA + tB) AB ds dt$$

and

$$D^2 G_h(s Dv^\varepsilon_{r,s} + t D\psi^\varepsilon_h) = D^2 G(A_h + s \lambda_h Dv^\varepsilon_{r,s} + t \lambda_h D\psi^\varepsilon_h)$$

is bounded and converges to $D^2 G(A)$ a.e. Since

$$R_{h,3} = \int_{B_r} dx \int_{[0,1] \times [0,1]} D^2 G(A_h + s \lambda_h Dv^\varepsilon_{r,s} + t \lambda_h D\psi^\varepsilon_h) Dv^\varepsilon_{r,s} D\psi^\varepsilon_h ds dt$$

and we may suppose that $\psi^\varepsilon_h \rightharpoonup \psi^\varepsilon$ weakly in $W^{1,2}(B_1)$, where

$$\int_{B_1} |D\psi^\varepsilon|^2 \leq c \frac{m^2}{(r-s)^2} \int_{B_{\frac{r}{2}}} |v^\varepsilon_{r,s} - v^\varepsilon_{r,s}|^2 +$$

$$+ c \int_{B_{\frac{r}{2}}} |Dv^\varepsilon_{r,s} - Dv^\varepsilon|^2$$

we get easily

$$\limsup_h |R_{h,3}| \leq c(M)||Dv^\varepsilon_{r,s}||_{L^2(B_{\frac{r}{2}})} ||D\psi^\varepsilon||_{L^2(B_{\frac{r}{2}})}.$$  (3.20)

To bound $R_{h,4}$ we observe that

$$n R_{h,4} = \int_{B_r \setminus B_{r,\varepsilon}} [G_h(Dv^\varepsilon_{r,s}) - G_h(Dv^\varepsilon)]$$

$$\geq - \int_{B_r \setminus B_{r,\varepsilon}} G_h(Dv^\varepsilon)$$

$$\geq - c|B_r \setminus B_{r,\varepsilon}|.$$  (3.21)

Then

$$\liminf_h R_{h,4} \geq - c|B_r \setminus B_{r,\varepsilon}|.$$  (3.22)

Moreover (H3) implies

$$|R_{h,5}| = I_{h, r_h}(\psi^\varepsilon_h)$$

$$= \int_{B_{r_h}} G_h(D\psi^\varepsilon_h)$$

$$\geq \gamma \int_{B_r} (1 + \lambda_h^{q-2} |Dv^\varepsilon - Dv_h|^{q-2}) |Dv^\varepsilon - Dv_h|^2$$

$$\geq \gamma \int_{B_r} |Dv^\varepsilon - Dv_h|^2,$$  (3.23)
for \( \epsilon \) small enough.

Passing to a subsequence we may suppose that

\[
\limsup_h R_{h,5} = \lim_h R_{h,5}.
\]

Therefore returning to the (3.16), from (3.17), (3.19), (3.21), (3.22) and (3.23) we get

\[
\liminf_h [I_{h,r}(v_h) - I_{h,r}(v^\epsilon)] \\
\geq \gamma \limsup_h \int_{B_r} (1 + \lambda_h^{q-2} |Dv - Dv_h|^{q-2}) |Dv - Dv_h|^2 + 
- c|B_r \setminus B_{s-\epsilon}| - c\mu(B_r \setminus B_s) - c||Dv_h^\epsilon||_{L^q(B_r/2)}||Dv^\epsilon||_{L^q(B_r)} + 
- cm^{-2\rho} - \int_{B_r} |Dv_{r,s} - Dv_{r,s}^\epsilon|^2 - c\frac{m^2}{(r - s)^2} \int_{B_r} |v_{r,s} - v_{r,s}^\epsilon|^2.
\]

Passing to the limit as \( \epsilon \to 0^+ \) we get easily

\[
\liminf_h [I_{h,r}(v_h) - I_{h,r}(v)] \\
\geq \gamma \limsup_h \int_{B_r} (1 + \lambda_h^{q-2} |Dv - Dv_h|^{q-2}) |Dv - Dv_h|^2 + 
- c|B_r \setminus B_s| - c\mu(B_r \setminus B_s) - cm^{-2\rho}
\]

then passing to the limit as \( m \to \infty \) and \( s \to r \) we get

\[
\limsup_h \int_{B_r} |Dv - Dv_h|^2 (1 + \lambda_h^{q-2} |Dv - Dv_h|^q) \leq \liminf_h [I_{h,r}(v_h) - I_{h,r}(v)].
\]

**Step 5.** [Conclusion.] From the two previous steps we conclude that, for any \( B_r \), with \( 0 < \tau < \frac{1}{4} \)

\[
\lim_h \int_{B_r} |Dv - Dv_h|^2 (1 + \lambda_h^{q-2} |Dv - Dv_h|^q) = 0.
\]

Now, from this equality and by (3.9) we get

\[
\lim_h \frac{U(x_h, \tau r_h \lambda_h^2)}{\lambda_h^2} = \lim_h \frac{1}{\lambda_h^2} \int_{B_{\tau r_h}(x_h)} (|Du - (Du)_{\tau r_h}|^2 + |Du - (Du)_{\tau r_h}|^q) dx
\]
\[ \lim_{h \to 0} \int_{B_r} \left( |Du - (Du)_\tau|^2 + \lambda_h^{q-2} |Du - (Du)_\tau|^q \right) dy = \int_{B_r} |Dv - (Dv)_\tau|^2 dy \leq C_M^* \tau^2 \]

which contradicts (3.1) if we choose \( C_M = 2C_M^* \).

\[ \square \]

The proof of Theorem 1 follows by proposition 3.1 by a standard iteration argument, see [G1].

**Remark 2.** Notice that the proof of Proposition 3.1 and of Theorem 1 still works if, beside assuming \( p < \frac{nq}{n-q-2} \), we have \( p \leq q + 1 \).

**References**


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