ON $X-\vartheta$-SPLITTING AND $X-\vartheta$-JOINTLY CONTINUOUS TOPOLOGIES
ON FUNCTION SPACES (*)

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SOMMARIO. - In questo articolo definiamo una relazione su $\Theta(Y, Z)$, l'insieme di tutte le funzioni $\vartheta$-continue di uno spazio topologico $Y$ in uno spazio topologico $Z$. Studiamo inoltre la connessione di questa relazione con le nozioni di $X-\vartheta$-splitting e di topologie $X-\vartheta$-continue su questo insieme, in cui $X$ è lo spazio di Sierpinski oppure $X = D$.

SUMMARY. - In this paper we define a relation on the set $\Theta(Y, Z)$ of all $\vartheta$-continuous functions of a topological space $Y$ into a topological space $Z$ and we study the connection of this relation with the notions of $X-\vartheta$-splitting and $X-\vartheta$-jointly continuous topologies on this set, where $X$ is the Sierpinski space or $X = D$.

1. Introduction.

Let $Y, Z$ be topological spaces and let $f$ be a map of $Y$ into $Z$. Then $f$ is $\vartheta$-continuous at $y \in Y$ if for every open neighbourhood $V$ of $f(y)$ there exists an open neighbourhood $U$ of $y$ such that $f(\text{Cl}(U)) \subseteq \text{Cl}(V)$. (Let $Y$ be a topological space, then by $\text{Cl}(A)$ we denote the closure of $A$ in $Y$). The map $f$ is $\vartheta$-continuous on $Y$ if it is $\vartheta$-continuous at each point of $Y$. (See for example [F], [I-F] and [J]). A continuous function $f : Y \to Z$ is $\vartheta$-continuous, but the converse is true when $Z$ is regular, that is the closed neighbourhoods of any point form a local base. In what follows by $\Theta(Y, Z)$ we denote the set of all $\vartheta$-continuous maps of $Y$ into $Z$. If $\tau$ is a topology on the set $\Theta(Y, Z)$, then the corresponding topological space is denoted

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by $\Theta_{\tau}(Y, Z)$. Let $(Y, \tau)$ be a topological space. Then, by $N_Y(y)$, where $y \in Y$ we denote the family of all open neighbourhoods of $y$ in $Y$, that is $N_Y(y) = \{U \in \tau : y \in U\}$.

A point $x$ in a a space is in the $\vartheta$-closure of a subset $A$ of the space $x \in Cl_\vartheta(A)$ if each open subset $V$ about $x$ satisfies $A \cap Cl(V) \neq \emptyset$. A is $\vartheta$-closed if $Cl_\vartheta(A) = A$. (See for example [J]).

Let $X$ be a space and $F : X \times Y \to Z$ be a $\vartheta$-continuous map. If $F$ has $\vartheta$-continuous restrictions to $\{x\} \times Y$ for any $x \in X$, then by $F_x$, where $x \in X$, we denote the $\vartheta$-continuous map of $Y$ into $Z$, for which $F_x(y) = F(x, y)$, for every $y \in Y$. By $\tilde{F}$ we denote the map of $X$ into the set $\Theta(Y, Z)$, for which $\tilde{F}(x) = F_x$, for every $x \in X$.

Let $G$ be a map of the space $X$ into the set $\Theta(Y, Z)$. By $\tilde{G}$ we denote the map of the space $X \times Y$ into the space $Z$, for which $G(x, y) = G(x)(y)$, for every $(x, y) \in X \times Y$.

By $S$ we denote the Sierpinski space, that is, the set $\{0, 1\}$ equipped with the topology $\tau(S) = \{\emptyset, \{0, 1\}, \{1\}\}$, and by $D$ the set $\{0, 1\}$ with the trivial topology.

Let $\mathcal{A}$ be a family of spaces. A topology $\tau$ on the set $\Theta(Y, Z)$ is called $\mathcal{A}-\vartheta$-splitting (respectively, $\mathcal{A}-\vartheta$-jointly continuous) (see [G]) if and only if for every element $X$ of $\mathcal{A}$, the $\vartheta$-continuity of a map $F : X \times Y \to Z$ (respectively, a map $G : X \to \Theta_{\tau}(Y, Z)$) implies the $\vartheta$-continuity of the map $\tilde{F} : X \to \Theta_{\tau}(Y, Z)$ (respectively, of the map $\tilde{G} : X \times Y \to Z$).

Obviously, if $\mathcal{A}$ is the family of all spaces, then the notions $\mathcal{A}-\vartheta$-splitting and $\mathcal{A}-\vartheta$-jointly continuous coincide with the notions $\vartheta$-splitting and $\vartheta$-jointly continuous, respectively. (See [C1] and [C2]). Also, these notions coincide with the notions $\mathcal{A}$-splitting and $\mathcal{A}$-jointly continuous topologies, respectively if $Z$ is a regular space. (See [G-I-P1]).

If $\mathcal{A} = \{X\}$, then instead of "$\mathcal{A}-\vartheta$-splitting" and "$\mathcal{A}-\vartheta$-jointly continuous" we write "$X-\vartheta$-splitting" and "$X-\vartheta$-jointly continuous".

In the present paper we define a relation on the set $\Theta(Y, Z)$ of all $\vartheta$-continuous functions of a topological space $Y$ into a topological space $Z$ and we study the connection of this relation with the notions of $X-\vartheta$-splitting and $X-\vartheta$-jointly continuous topologies on this set, where $X$ is the Sierpinski space or $X = D$. 

2. The relation "\(\prec\)" on \(\Theta(Y, Z)\).

2.1. Definition and notations.

For every space \(Y\) with a topology \(\tau\) we define a relation "\(\overset{\tau}{\prec}\)" on \(Y\) as follows: if \(x, y \in Y\), then we write \(x \overset{\tau}{\prec} y\) if and only if \(x \in \text{Cl}_\vartheta(\{y\})\) and \(y \in \text{Cl}_\vartheta(\{x\})\), that is for every \(U \in \mathcal{N}_Y(x)\) we have \(y \in \text{Cl}(U)\) and for every \(V \in \mathcal{N}_Y(y)\) we have \(x \in \text{Cl}(V)\). Clearly this relation is reflexive and symmetric. Also, if the space \(Y\) is regular, then the relation "\(\overset{\tau}{\prec}\)" is an equivalence relation.

We define a relation \(\prec\) on \(\Theta(Y, Z)\) as follows: if \(g, f \in \Theta(Y, Z)\), then we write \(g \prec f\) if and only if \(g(y) \overset{\tau}{\prec} f(y)\), for every \(y \in Y\), where \(\tau\) is the topology of the space \(Z\).

2.2. Theorem.

The following propositions are true:

1. If a topology \(\tau\) on \(\Theta(Y, Z)\) is \(S - \vartheta\)-splitting then from the condition \(g \prec f\) it follows that \(g \overset{\tau}{\prec} f\), where \(f, g \in \Theta(Y, Z)\) and \(Z\) regular.

2. If from the condition \(g \prec f\) it follows that \(g \overset{\tau}{\prec} f\), then the topology \(\tau\) on \(\Theta(Y, Z)\) is \(S - \vartheta\)-splitting.

Proof. (1) Let \(\tau\) be an \(S - \vartheta\)-splitting topology on \(\Theta(Y, Z)\) and let \(g \prec f\), where \(g, f \in \Theta(Y, Z)\). We prove that \(g \overset{\tau}{\prec} f\).

Let \(F : S \times Y \to Z\) be a map for which \(F(1, y) = f(y)\) and \(F(0, y) = g(y)\), where \(y \in Y\). We prove that \(F\) is \(\vartheta\)-continuous.

Let \(F(1, y) = f(y)\) and \(U \in \mathcal{N}_Z(f(y))\). Since \(f\) is \(\vartheta\)-continuous, there exists an open neighbourhood \(V\) of \(y\) in \(Y\) such that \(f(\text{Cl}(V)) \subseteq \text{Cl}(U)\). We prove that \(F(\text{Cl}(S \times V)) \subseteq \text{Cl}(U)\).

Indeed, if \((1, y_1) \in \text{Cl}(S \times V) = S \times \text{Cl}(V)\), then \(F(1, y_1) = f(y_1) \in \text{Cl}(U)\). If \((0, y_1) \in \text{Cl}(S \times V)\), then \(F(0, y_1) = g(y_1)\). Since \(g \prec f\) we have \(g(y_1) \in \text{Cl}_\vartheta(\{f(y_1)\})\) and \(f(y_1) \in \text{Cl}_\vartheta(\{g(y_1)\})\). Hence, \(g(y_1) \in \text{Cl}(U)\).
Now, let \( F(0, y) = g(y) \) and \( U \in N_Z(g(y)) \). Since \( g \) is \( \vartheta \)-continuous there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( g(\text{Cl}(V)) \subseteq \text{Cl}(U) \). As the above we can prove that \( F(\text{Cl}(S \times V)) \subseteq \text{Cl}(U) \). Thus, the map \( F \) is \( \vartheta \)-continuous.

Since \( \tau \) is \( S \)-\( \vartheta \)-splitting the map \( \hat{F} : S \to \Theta_\tau(Y, Z) \) is \( \vartheta \)-continuous. We have that \( \hat{F}(1) = f \) and \( \hat{F}(0) = g \).

Let \( W \) be an open neighbourhood of \( g \) in \( \Theta_\tau(Y, Z) \). Since \( \hat{F} \) is \( \vartheta \)-continuous there exists an open neighbourhood \( V \) of \( 0 \) in \( S \) such that \( \hat{F}(\text{Cl}(V)) \subseteq \text{Cl}(W) \). Obviously, \( \text{Cl}(V) = S \). Hence we have \( \hat{F}(S) \subseteq \text{Cl}(W) \). Thus, \( \hat{F}(1) = f \in \text{Cl}(W) \) and \( g \in \text{Cl}_0(\{f\}) \). Similarly we can prove that \( f \in \text{Cl}_0(\{g\}) \). Hence \( g \prec f \).

(2) Let \( \tau \) be a topology on \( \Theta(Y, Z) \) such that from the condition \( g \prec f \) it follows that \( g \prec f \). We prove that \( \tau \) is \( S \)-\( \vartheta \)-splitting.

Let \( F : S \times Y \to Z \) be a \( \vartheta \)-continuous map. Consider the map \( \hat{F} : S \to \Theta_\tau(Y, Z) \). Let \( \hat{F}(1) = f \) and \( \hat{F}(0) = g \). We prove that \( g \prec f \).

Indeed, let \( y \in Y \) and let \( U \in N_Z(g(y)) \). We must prove that \( f(y) \in \text{Cl}(U) \). Since \( F \) is \( \vartheta \)-continuous and \( F(0, y) = g(y) \) there exists an open neighbourhood \( W = O \times V \) of \( (0, y) \) in \( S \times Y \) such that

\[
F(\text{Cl}(O \times V)) = F(S \times \text{Cl}(V)) \subseteq \text{Cl}(U).
\]

Hence \( F(1, y) = f(y) \in \text{Cl}(U) \). Similarly we can prove that if \( U \) is an open neighbourhood of \( f(y) \) in \( Z \), then \( g(y) \in \text{Cl}(U) \). Thus, \( g \prec f \).

By assumption \( g \prec f \). Let \( U \) be an open neighbourhood of \( g \) in \( \Theta_\tau(Y, Z) \). Since \( g \prec f \) we have that \( g \in \text{Cl}_0(\{f\}) \) and \( f \in \text{Cl}_0(\{g\}) \). Thus \( f \in \text{Cl}(U) \). Hence

\[
\hat{F}(\text{Cl}(S)) = \hat{F}(S) \subseteq \text{Cl}(U).
\]

Let \( U \) be an open neighbourhood of \( f \) in \( \Theta_\tau(Y, Z) \). Similarly we can prove that \( g \in \text{Cl}(U) \) and \( \hat{F}(\text{Cl}(S)) = \hat{F}(S) \subseteq \text{Cl}(U) \). Thus the the map \( \hat{F} \) is \( \vartheta \)-continuous and the topology \( \tau \) is \( S \)-\( \vartheta \)-splitting. \( \Box \)

2.2.1. Corollary.

If \( Z \) is a discrete space, then the discrete topology and, hence, every topology on \( \Theta(Y, Z) \) is \( S \)-\( \vartheta \)-splitting.
Proof. Indeed, suppose that $Z$ is a discrete space, then by the condition $g \prec f$, where $g, f \in \Theta(Y, Z)$, it follows that $g = f$. Hence, $g \prec f$, for every topology $\tau$ on $\Theta(Y, Z)$. Thus, by Theorem 2.2, every topology on $\Theta(Y, Z)$ is $S - \vartheta$-splitting.

\[ \Diamond \]

2.3. Theorem.

The following propositions are true:

(1) If a topology $\tau$ on $\Theta(Y, Z)$ is $S - \vartheta$-jointly continuous then from the condition $g \prec f$ it follows that $g \prec f$.

(2) If from the condition $g \prec f$ it follows that $g \prec f$ and $Z$ regular, then the topology $\tau$ on $\Theta(Y, Z)$ is $S - \vartheta$-jointly continuous.

Proof. (1) Let $\tau$ be an $S - \vartheta$-jointly continuous topology on $\Theta(Y, Z)$ and let $g \prec f$, where $g, f \in \Theta(Y, Z)$. We prove that $g \prec f$.

Let $G : S \to \Theta_{\tau}(Y, Z)$ be a map for which $G(1) = f$ and $G(0) = g$. We prove that $G$ is $\vartheta$-continuous. Let $U$ be an open neighbourhood subset of $f$ in $\Theta_{\tau}(Y, Z)$. Since $g \prec f$ we have that $g \in Cl(U)$. Hence

$$G(Cl(S)) = G(S) \subseteq Cl(U).$$

Similar if $V \in N_{\Theta_{\tau}(Y, Z)}(g)$, then

$$G(Cl(S)) = G(S) \subseteq Cl(V).$$

Hence, the map $G$ is $\vartheta$-continuous. Since $\tau$ is $S - \vartheta$-jointly continuous, the map $G : S \times Y \to Z$ is also $\vartheta$-continuous.

Let $y \in Y$ and let $W \in N_{Z}(g(y))$. We must prove that $f(y) \in Cl(W)$. Indeed, since the map $\tilde{G}$ is $\vartheta$-continuous at the point $(0, y) \in S \times Y$ there exists an open neighbourhood $V \times U$ of $(0, y)$ in $S \times Y$ such that

$$\tilde{G}(Cl(V \times U)) = \tilde{G}(S \times Cl(U)) \subseteq Cl(W).$$

Thus $\tilde{G}(1, y) = f(y) \in Cl(W)$.

Similar, if $W \in N_{Z}(f(y))$, then $g(y) \in Cl(W)$. Hence $g \prec f$. 


(2) Let \( \tau \) be a topology on \( \Theta(Y,Z) \) such that from the condition \( g \prec f \) it follows that \( g \prec f \). We prove that \( \tau \) is \( S - \vartheta \)-jointly continuous.

Let \( G : S \to \Theta_\tau(Y,Z) \) be a \( \vartheta \)-continuous map and let \( G(1) = f \) and \( G(0) = g \). We prove that \( g \prec f \).

Indeed, let \( U \) be an open neighbourhood of \( g \) in \( \Theta_\tau(Y,Z) \). Since \( G \) is \( \vartheta \)-continuous, there exists \( V \in \mathcal{N}_G(0) \) such that

\[
G(Cl(V)) = G(S) \subseteq Cl(U).
\]

Hence \( f \in Cl(U) \). Similar, if \( U \in \mathcal{N}_{\Theta_\tau(Y,Z)}(f) \), then \( g \in Cl(U) \).

Thus, \( g \prec f \). By assumption \( g \prec f \).

Consider the map \( \bar{G} : S \times Y \to Z \). Then we have \( \bar{G}(0,y) = g(y) \) and \( \bar{G}(1,y) = f(y) \). We prove that the map \( \bar{G} \) is \( \vartheta \)-continuous.

Indeed, let \( W \) be an open neighbourhood of \( g(y) \) in \( Z \). Since \( g \) is \( \vartheta \)-continuous there exists an open neighbourhood \( V \) of \( y \) in \( Y \) such that \( g(Cl(V)) \subseteq Cl(W) \). We prove that

\[
\bar{G}(Cl(S \times V)) = \bar{G}(S \times Cl(V)) \subseteq Cl(W).
\]

Let \( (0,y_1) \in Cl(S \times V) \). Then \( \bar{G}(0,y_1) = g(y_1) \in Cl(W) \). Now, let \( (1,y_1) \in Cl(S \times V) \). Then \( \bar{G}(1,y_1) = f(y_1) \). Since \( g \prec f \), we have that \( f(y_1) \in Cl(W) \). This means that the map \( \bar{G} \) is \( \vartheta \)-continuous at the point \( (0,y) \). Similarly the map \( \bar{G} \) is \( \vartheta \)-continuous at the point \( (1,y) \). Thus the map \( \bar{G} \) is \( \vartheta \)-continuous and the topology \( \tau \) is \( S - \vartheta \)-jointly continuous.

\[ \diamond \]

2.3.1. Corollary.

If \( Z \) is a regular space, then the discrete topology \( \tau \) on the set \( \Theta(Y,Z) \) is \( S - \vartheta \)-jointly continuous.

Proof. Let \( \tau \) be the discrete topology on \( \Theta(Y,Z) \). Then from the condition \( g \prec f \) it follows that \( g = f \) and hence \( g \prec f \). Thus by Theorem 2.3 the topology \( \tau \) is \( S - \vartheta \)-jointly continuous.

\[ \diamond \]
2.4. Theorem.

A topology \( \tau \) on \( \Theta(Y, Z) \), where \( Z \) is a regular space, is simultaneously \( S - \varphi \)-splitting and \( S - \varphi \)-jointly continuous if and only if the relations "\( \varphi \)" and "\( \zeta \)" coincide.

Proof. The proof of this theorem follows by Theorems 2.3 and 3.3.

2.5. Remarks.

(1) Relevant results for continuous functions there exist in [G-I-P2].
(2) The Theorems 2.2, 2.3 and 2.4 are also true if we replace the space \( S \) by the space \( D \).

2.6 Problems.

We give some problems concerning topologies on \( \Theta(Y, Z) \).
Let \( A \) be an arbitrary family of spaces.

(1) Does there exist a characterization of the \( A - \varphi \)-splitting and \( A - \varphi \)-jointly continuous topologies with the relations "\( \varphi \)" and "\( \zeta \)"?

(2) Does there exist the finest \( A - \varphi \)-splitting topology on \( \Theta(Y, Z) \)?
It is known that on the set \( C(Y, Z) \) of all continuous maps of a space \( Y \) into a space \( Z \) there exists the finest \( A \)-splitting topology. (See [G-I-P1]).

REFERENCES


[C2] Di Concilio Anna, Exponential law and $\vartheta$—continuous functions, Quaestiones Mathematicae, 8 (1985), 131-142.


