GEOMETRICAL STRUCTURES ON DIFFERENTIABLE MANIFOLDS (*)

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SOMMARIO. - Si studiano le \((X,G)\)-varià e si danno alcuni esempi: quando il modello geometrico è la coppia \((G/H, H)\), si danno condizioni necessarie e sufficienti affinché ad una riduzione del fibrato degli \(r\)-getti su una varietà differenziabile \(M\) corrisponda una \((X,G)\)-struttura sopra \(M\).

SUMMARY. - We study \((X,G)\)-manifolds and we give examples: when the geometric model is the couple \((G/H, H)\), we give necessary and sufficient conditions ensuring that a reduction of the \(r\)-frames bundle on a differentiable manifold \(M\) gives rise to a \((X,G)\)-structure on \(M\).

1. Introduction.

The study of further structures on a differentiable manifold appears as one of the general frameworks in geometry.

Clearly, a very interesting situation is represented by those structures for which uniformization theorems are available. This is the case of \((X,G)\)-manifolds, i.e. those manifolds locally modelled on geometric spaces (see [9]). Typical examples are locally conformally flat manifolds (see [8], [12]), spherical manifolds (see [4]), quaternionic coordinate manifolds (see [13]) and Riemannian manifolds locally modelled on homogeneous space (see [2]).

In the present paper we investigate \((X,G)\)-structures and discuss several basic examples; moreover, when the model space is a homogeneous manifold, we describe \((X,G)\)-structures as special reductions of the bundle of \(r\)-frames (see Propositions 3.1 and 3.2).

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We recall some facts about \((X,G)\)-structures. Let \(X\) be a differentiable manifold and \(G\) be a formally analytic subgroup of \(\text{Diff}(X)\), i.e. such that if \(g \in G\) coincides with \(id_X\) in some open subset of \(X\), then \(g = id_X\). The couple \((X,G)\) is the geometric model. A \((X,G)\)-structure on a differentiable manifold is given by an open covering \(\{U_\alpha\}_{\alpha \in A}\) of \(M\) and diffeomorphisms \(\varphi_\alpha : U_\alpha \to X\) onto open sets of \(X\) such that, for every pair \((\alpha, \beta)\) with \(U_\alpha \cap U_\beta \neq \emptyset\), the change of coordinates map \(\varphi_\alpha \circ \varphi_\beta^{-1}\) is the restriction of an element of \(G\). A map \(f : M \to N\) between two \((X,G)\)-manifolds is a \((X,G)\)-map if for every \(p \in M\) there exist a local chart \((U, \varphi)\) around \(p\) and a local chart \((V, \psi)\) around \(f(p)\) for the \((X,G)\)-geometries of \(M\) and \(N\) respectively such that \(\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)\) is the restriction of an element of \(G\). A \((X,G)\)-map is a local diffeomorphism. Let \(M\) be a simply connected \((X,G)\)-manifold and \(p_0 \in M\) and \((U_0, \varphi_0)\) be a \((X,G)\)-chart around \(p_0\): we set \(\Phi = \varphi_0\) on \(U_0\). Then we can analytically continue \(\Phi\) on every curve for \(p_0\) and since \(M\) is simply connected, we get a \((X,G)\)-map \(\Phi : M \to X\), that is unique up to left composition with elements of \(G\). \(\Phi\) is the developing map of the \((X,G)\)-structure. If \(M\) is not simply connected, then we take the universal covering \(\tilde{M}\) of \(M\), that is still a \((X,G)\)-manifold: the developing map \(\Phi : \tilde{M} \to X\) induces a homomorphism \(\rho : \pi_1(M) \to G\), such that

\[
\Phi \circ [\gamma] = \rho([\gamma]) \circ \Phi,
\]

where \(\pi_1(M)\) is viewed as the group of the deck transformations of \(\tilde{M}\). The homomorphism \(\rho\) is called holonomy representation of the \((X,G)\)-structure.

Vice versa, \((X,G)\)-structures on \(M\) are determined by a homomorphism \(\rho : \pi_1(M) \to G\) and an equivariant immersion \(\Phi : M \to X\) (i.e. such that the (*) holds). A \((X,G)\)-structure on \(M\) is said to be complete if the developing map is a covering map on its image; it is said to be uniformizable if the developing map is injective. Note that in the latter case \(\rho\) is injective and \(\tilde{M} = \Phi(M)/\rho(\pi_1(M))\).

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2. Examples.

In this section we give a list of examples of \((X, G)\)-manifolds.

1) Locally conformally flat manifolds. Let \(X = S^n\) be the unit sphere in \(\mathbb{R}^{n+1}\) and \(G = C_n\) be the conformal group of \(S^n\); we recall that an \(n\)-dimensional manifold \(M\) is called \textit{locally conformally flat} if there exists an atlas \(\{(U_{\alpha}, \varphi_{\alpha})\}_{\alpha \in A}\) such that, for every \(\alpha \in A\), \(\varphi_{\alpha} : U_{\alpha} \rightarrow S^n\) is an open diffeomorphism on the image, and if \(U_{\alpha} \cap U_{\beta} \neq \emptyset\) then the change of coordinates map is a conformal diffeomorphism (see [8], [12]). If \(n > 2\), by Liouville's Theorem it follows that \(\varphi_{\alpha} \circ \varphi^{-1}_{\beta}\) is the restriction of an element of \(C_n\). Thus a locally conformally flat manifold is a \((S^n, C_n)\)-manifold.

If \(M\) is compact and the conformal invariant \(d(M)\) (see [12] for the definition) is less than \(\frac{(n-2)^2}{2}\), then by a Theorem of [12] it follows that the developing map \(\Phi\) is injective.

2) Take \(X = \mathbb{C}\) and \(G = \text{Aut Hol}(\mathbb{C}) = \{f(z) = az + b | a, b \in \mathbb{C}, a \neq 0\}\); the compact \((X, G)\)-manifolds are the complex tori. In fact, let \(\mathbb{C}/\Gamma\) be a complex torus and \(\{(U_{\alpha}, \varphi_{\alpha})\}_{\alpha \in A}\) be the complex atlas which defines the complex structure on \(\mathbb{C}/\Gamma\); if \(U_{\alpha} \cap U_{\beta} \neq \emptyset\), then \(\varphi_{\alpha} \circ \varphi^{-1}_{\beta}\) is a translation. The converse is a consequence of the following

**Theorem.** ([3]) \textit{If \(M\) is compact and it is not a torus, then \(M\) cannot be covered by any system \((x_{\alpha}^{1}, x_{\alpha}^{2})\) of local coordinates such that \(\frac{\partial x_{\alpha}^{i}}{\partial x_{\beta}^{j}}\) is constant on \(U_{\alpha} \cap U_{\beta}\), for each pair of indices \((\alpha, \beta)\).}

Note that in this case the \((X, G)\)-structure is uniformizable and complete.

3) Fix \(X = \mathbb{R}^{n}\) and \(G = \text{Aff}(\mathbb{R}^{n}) = \mathbb{R}^{n} \rtimes \text{GL}(n, \mathbb{R})\), the affine transformations of \(\mathbb{R}^{n}\). In such a case the \((X, G)\)-manifolds are the locally flat manifolds (i.e., such that there exists a linear torsion free connection whose curvature vanishes).
4) Let $X$ be a differentiable manifold and $G = \{ e \}$ be the trivial subgroup of $\text{Diff}(X)$. If $M$ is a $(X, G)$-manifold it is possible to define a global map $\psi : M \rightarrow X$ in the following way: for every $p \in M$ we take a $(X, G)$-chart $(U_\alpha, \varphi_\alpha)$ around $p$ and we set $\psi(p) = \varphi_\alpha(p)$.

Since $G = \{ e \}$, the map $\psi$ is well defined.

If $M$ is compact, then $\psi : M \rightarrow X$ is a covering projection. In such a case the $(X, e)$-structure is complete but not necessarily uniformizable.

Vice versa, a covering space $(M, \psi)$ of $X$ is a $(X, G)$-manifold: this is immediate because $\psi$ is an equivariant immersion of $M$ in $X$.

5) **Spherical manifolds.** A connected real hypersurface $M$ in the complex manifold $N$ of complex dimension $(n + 1)$ is said to be spherical if, at every point $p \in M$, there exists a local holomorphic coordinate system $(z_1, \ldots, z_{n+1})$ of $N$ such that $M$ is defined by

$$|z_1|^2 + \ldots + |z_{n+1}|^2 = 1$$

(see [4]). For example, the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ is a spherical manifold. Let $B_{n+1}$ be the unit ball in $\mathbb{C}^{n+1}$; we recall that the group $\text{SU}(n + 1, 1)$ acts transitively on $B_{n+1}$ and on $S^{2n+1}$ by the fractional linear transformations

$$z \mapsto \frac{Az + B}{Cz + D},$$

where $A \in M_{n,n}(\mathbb{C})$, $B \in M_{n,1}(\mathbb{C})$, $C \in M_{1,n}(\mathbb{C})$, $D \in \mathbb{C}$ satisfy the following identities:

$$^t \bar{A} A - ^t \bar{C} C = I_n, \quad ^t \bar{A} B = ^t \bar{C} D, \quad \bar{D} D - ^t \bar{B} B = 1.$$

Further the automorphisms group of $B_{n+1}$, $\text{Aut}(B_{n+1})$ and the CR-automorphisms group of $S^{2n+1}$, $\text{Aut}_{\text{CR}}(B_{n+1})$ are given by the quotient

$$\text{SU}(n + 1, 1)/\text{center}.$$

We have the following

**Theorem.** ([1]) Let $f$ be a biholomorphic map from a connected neighbourhood $U$ of $p \in S^{2n+1}$. If $f(U \cap S^{2n+1}) \subset S^{2n+1}$, then $f$ is the restriction to $U$ of a fractional linear transformation.
Let $M$ be a spherical manifold and $\mathcal{U} = (U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be a spherical atlas: if $U_\alpha \cap U_\beta \neq \emptyset$, then the change coordinate map
\[ \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \rightarrow \varphi_\alpha(U_\alpha \cap U_\beta) \]
is a local biholomorphism from an open set in $\mathbb{C}^{n+1}$ intersecting $S^{2n+1}$ to $S^{2n+1}$. By the previous Theorem it follows that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the restriction of a linear fractional transformation. Therefore spherical manifolds are $(S^{2n+1}, \text{Aut}_{CR}(S^{2n+1}))$-manifolds.

6) Let $X = S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$ and $G = \mathbb{Z}_m$ be the $m$-cyclic group generated by $g = e^{2\pi im}$, acting on $S^{2n+1}$ by scalar multiplication; then the lens space is defined as $L^{2n+1}_{(m)} = S^{2n+1}/\mathbb{Z}_m$. Set $\rho = \text{id}_{\mathbb{Z}_m}$ and $\Phi = \text{id}_{L^{2n+1}_{(m)}} \mathbb{Z}_m$ being isomorphic to $\pi_1(L^{2n+1}_{(m)})$; therefore it follows that $L^{2n+1}_{(m)}$ is a $(X, G)$-manifold and, by definition, is both uniformizable and complete.

7) COORDINATE QUATERNIONIC MANIFOLDS. We recall the definition of quaternionic structure in the sense of Sommese (see [13]). A quaternionic manifold is a differentiable manifold with an open cover $\{U_i\}$ of $M$ and diffeomorphisms $\varphi_i : U_i \rightarrow \mathbb{H}^n$ such that $\varphi_i \circ \varphi_j^{-1}$ is a quaternionic map with respect to the standard right quaternionic structure on $\mathbb{H}^n$. By Proposition I of [13] it follows that the change of coordinates map is the restriction of a quaternionic affine map and therefore coordinate quaternionic manifolds are $(\mathbb{H}^n, \text{Aff}(\mathbb{H}^n))$-manifolds.

By a result of [5] it follows that the compact $(\mathbb{H}, \text{Aff}(\mathbb{H}))$-manifolds are uniformizable.

8) Let $X = S^n = O(n+1)/O(n)$ be the unit sphere in $\mathbb{R}^{n+1}$ and $H = O(n)$ be the orthogonal group as a subgroup of $O(n+1)$; let $\mathbb{Z}_2$ be the cyclic group of order two generated by $a$ and $\mathbb{P}^n(\mathbb{R}) = S^n/\mathbb{Z}_2$ be the real projective space. $\mathbb{P}^n(\mathbb{R})$ is a $(S^n, O(n))$-manifold. It is sufficient to give the holonomy representation $\rho : \pi_1(\mathbb{P}^n(\mathbb{R})) \rightarrow O(n)$ and the equivariant immersion $\Phi : S^n \rightarrow S^n$, $S^n$ being the universal covering of $\mathbb{P}^n(\mathbb{R})$. Since $\pi_1(\mathbb{P}^n(\mathbb{R}))$ is isomorphic to $\mathbb{Z}_2$, we set
\[ \rho(e) = I, \quad \rho(a) = -I \]
and $\Phi = \text{id}_{S^n}$, where $I$ is the identity in $O(n)$. 

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9) Let \( X = S^6 = \{ x \in \text{Im Cay} \mid \| x \| = 1 \} \) and \( G_2 = \text{Aut (Cay)} \). We recall that

\[
G_2 = \{ g \in O(7) : g^*(\omega) = \omega \},
\]

where \( \omega \in \otimes^3(\text{Im Cay})^* \) is given by

\[
\omega(x, y, z) = \langle x, yz \rangle.
\]

**Remark 2.1.** If \( \Gamma \subset O(7) \) is a group acting freely on \( S^6 \), then \( \Gamma \cong \mathbb{Z}_2 \). In fact, let \( g \in \Gamma \); \( g \) has at least one real eigenvalue \( \lambda \) that is 1 or \(-1\). If \( \lambda = 1 \) (= \(-1\)) then \( g \) (respectively \( g^2 \)) has fixed points and consequently \( g = I \) (\( g^2 = I \)). Therefore, if \( g \neq I \) then all the eigenvalues of \( g \) are \(-1\) and since \( g \) is diagonalizable, \( g = -I \).

Since \( \Gamma \not\subset G_2 \), the previous remark implies that the only compact \((S^6, G_2)\)-manifold is \( S^6 \).

3. \((X, G)\)-structures as special reductions.

Let \( G \) be a Lie group and \( H \) be a closed subgroup. In this section we consider the \((X, G)\)-manifolds whose geometric model is given by an \( n \)-dimensional homogeneous space \( X = G/H \) and by the subgroup \( H \). Let us denote by \( o \) the origin of \( X \), (i.e. the coset \( H \)) and fix a linear frame \( u_o \in L(X)_o \); we assume that the *linear isotropy representation of \( H \), \( \alpha : H \rightarrow \text{GL}(n, \mathbb{R}) \) defined by*

\[
\alpha(h) = u_o^{-1} \circ h_* \circ u_o
\]

\( h_* \) being the differential of \( h \) in \( o \), is faithful.

**Remark 3.1.** If the subgroup \( H \) is compact, then this hypothesis is satisfied. In such a case the Lie algebra \( g \) of \( G \) admits an \( \text{ad}(H) \)-invariant scalar product which corresponds to a \( G \)-invariant metric on the homogeneous space \( X = G/H \). If \( h \in \text{Ker}(\alpha) \), then we have \( h_*[o] = id_{T_oX} \) and \( h(o) = o \); \( h \) being in \( H \). Therefore \( h \) fixes the geodesics starting from \( o \in X \). Let \( N \) be a normal neighbourhood of \( o \) in \( X \) and \( U = \{ x \in N : h(x) = x \} \neq \emptyset \); \( U \) is open and closed in \( X \) and consequently \( h = e \), i.e. \( \alpha \) is faithful.
Vice versa: if the linear isotropy representation of $H$ is faithful and $G$ admits a bi-invariant Riemannian metric, then $H$ is compact. This fact is a consequence of the following

**Theorem.** ([10]) Let $G$ be a connected Lie group; $G$ has a bi-invariant metric if and only if

$$G = \mathbb{R}^s \times K,$$

where $K$ is a compact Lie group.

In particular we have that

$$H = \mathbb{R}^p \times K'.$$

By the faithfulness of the linear isotropy representation, the factor $\mathbb{R}^p$ cannot occur in the last decomposition.

Let $V$ and $V'$ be two neighbourhoods of $o$ and

$$f : V \longrightarrow M, \ f' : V' \longrightarrow M$$

be two diffeomorphisms onto their images such that $f(o) = f'(o) = p$; $f$ and $f'$ define the same $r$-jet at $p$ if they have the same partial derivatives up to the order $r$ at $o$. The equivalence class of $f$ is called an $r$-frame at $p$ and is denoted by $j^r_p(f)$. We set

$$G^r(n) = \{r \text{- frames at } o \in X\}$$

$$\Gamma_H^r = \{j^r_o(f) : f \in H\}$$

$$L^r(M)_p = \{r \text{- frames at } p \in M\}$$

$$L^r_G(M)_p = \{j^r_p(f) \in L^r(M)_p : f^{-1} \text{ is a } (X,G) \text{- chart around } p \in M\}$$

$$L^r(M) = \bigcup_{p \in M} L^r(M)_p$$

$$L^r_G(M) = \bigcup_{p \in M} L^r_G(M)_p.$$
\[ \pi : L^r(M) \rightarrow M \] be the projection defined by \( \pi(j^r_p(f)) = p \); then \((L^r(M), \pi, G^r(n))\) is a principal \( G^r(n)\)-bundle, called the bundle of \( r\)-frames. If \( n = 1 \), then \( L^1(M) \) is the bundle of linear frames. We remark that \( L^r_G(X) = G \) and that the subgroup \( \Gamma'_H \) is isomorphic to \( H \).

A \( H \)-reduction \( P \subset L^r(M) \) is said to be integrable if for every \( p \in M \) there exists a neighbourhood \( U \) of \( p \) and a diffeomorphism \( \varphi : U \rightarrow X \) onto its image such that

\[ \varphi_* : P|_U \rightarrow G|_{\varphi(U)}, \]

where \( \varphi_* (j^r_q(f)) = j^r_{\varphi(q)}(\varphi \circ f) \). We have the following

**Proposition 3.1.** Let \( M \) be a \((X, H)\)-manifold; then \( L^1_G(M) \) is an integrable \( H \)-reduction of \( L^1(M) \).

**Proof.** The subgroup \( H \) acts on \( L^1_G(M) \) on the right in the following way: for \( u = j^1_p(f) \in L^1_G(M) \) and \( a = j^1_b(h) \in H \), then

\[ ua = j^1_p(f \circ h). \]

Since an element \( h \in G \) belongs to \( H \) if and only if \( h(o) = o \), then \((f \circ h)^{-1} \) is a \((X, H)\)-chart such that \((f \circ h)(o) = p \). Let \( j^1_p(f), j^1_p(f') \) be in \( \pi^{-1}(p) \); by the definition of \((X, H)\)-manifold it follows that

\[ (f^{-1} \circ f')|_{f^{-1}(U \cap U')} = h|_{f^{-1}(U \cap U')} \]

where \( h \in H \). Thus \( j^1_p(f') = j^1_p(f \circ h) \), i.e. \( H \) is transitive on the fibre \( \pi^{-1}(p) \). Therefore \( L^1_G(M) \) is a subbundle of \( L(M) \) whose structural group is \( H \).

Let \( p \) be a point of \( M \), \((U_\alpha, \varphi_\alpha, V_\alpha)\) be a local \((X, H)\)-chart around \( p \) and \( j^1_q(f) \in L^1_G(M)|_{U_\alpha} \); set

\[ \varphi_{\alpha*}(j^1_q(f)) = j^1_{f^{-1}(q)}(\varphi_\alpha \circ f). \]

This definition does not depend on the local coordinates: if \((U_\beta, \varphi_\beta, V_\beta)\) is another local \((X, H)\)-chart around \( p \), we have

\[ \varphi_\alpha \circ \varphi^{-1}_\beta |_{\varphi_\beta(U_\alpha \cap U_\beta)} = h_{\alpha\beta} |_{\varphi_\beta(U_\alpha \cap U_\beta)}, \]
$h_{\alpha\beta} \in H$. Therefore, if $q \in U_\alpha \cap U_\beta$, we get

\[ \varphi_\alpha^*(\tilde{j}_q^1(f)) = j_{f^{-1}(q)}^1(\varphi_\alpha \circ f) = j_{f^{-1}(q)}^1(\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ f) = j_{f^{-1}(q)}^1(h_{\alpha\beta} \circ \varphi_\beta \circ f) = j_{f^{-1}(q)}^1(\varphi_\beta \circ f) = \varphi_{\beta*}(\tilde{j}_q^1(f)). \]

This shows that $L_G(M)$ is integrable.

\[ \Box \]

**Proposition 3.2.** Let $P$ be an integrable $H$-reduction of $L^2(M)$, the bundle of the 2-frames over $M$; then $M$ is a $(X, H)$-manifold.

**Proof.** We shall construct an atlas of $(X, H)$-geometry. Since $P$ is integrable, for every $p \in M$ there exists a neighbourhood $U_\alpha$ and a diffeomorphism $\varphi_\alpha : U_\alpha \rightarrow V_\alpha \subset X$, such that

\[ \varphi_\alpha^* : P|_{V_\alpha} \rightarrow L^2_G(X)|_{V_\alpha} = G|_{V_\alpha}. \]

Then if $(U_\beta, \varphi_\beta, V_\beta)$ is another diffeomorphism, for $q \in U_\alpha \cap U_\beta$, we obtain

\[ \varphi_\alpha^*(\tilde{j}_q^2(\varphi_\beta^{-1})) = j_x^2(\varphi_\alpha \circ \varphi_\beta^{-1}) = j_x^2(h_{\alpha\beta}^x), \]

where $x = \varphi_\beta(q), h_{\alpha\beta}^x \in H$. We shall prove that $h_{\alpha\beta}^x$ does not depend on $x$. The last relation implies that for every $x \in \varphi_\beta(U_\alpha \cap U_\beta)$ the change coordinate map $\varphi_\alpha \circ \varphi_\beta^{-1}$ and the linear transformation $h_{\alpha\beta}^x$ have the same partial derivatives up to the order 2; thus if $(U, \psi, V)$ is a local chart around $o$ the diffeomorphism $\varphi_\alpha \circ \varphi_\beta^{-1}$ is linear and consequently $h_{\alpha\beta}^x = h_{\alpha\beta}$. Therefore

\[ \varphi_\alpha \circ \varphi_\beta^{-1}|_{\varphi_\beta(U_\alpha \cap U_\beta)} = h_{\alpha\beta}|_{\varphi_\beta(U_\alpha \cap U_\beta)}, \]

$\mathcal{A} = (U_\alpha, \varphi_\alpha, V_\alpha)$ is an atlas of $(X, H)$-geometry and $M$ is a $(X, H)$-manifold.

\[ \Box \]

If we consider as the model space the couple $(G/H, H)$ such that the subgroup $H$ can be embedded into the group $G^r(n)$ via the $r$-representation of isotropy, (i.e. the elements of $H$ are known when we give the partial derivatives up to the order $r$ at the point $o$), then the previous Propositions can be generalized in the following way:

**Proposition 3.3.** If $M$ is a $(X, H)$-manifold, then $L'_G(M)$ is an integrable $H$-reduction of $L'(M)$. 

As for the case \( r = 1 \) an integrable \( H \)-reduction of \( \mathcal{L}^{r+1} \) determines a \((X, H)\)-structure on \( M \). We have the following

**Proposition 3.4.** If \( P \) is an integrable \( H \)-reduction of the bundle of \((r+1)\)-jets \( \mathcal{L}^{r+1} \), then \( M \) is a \((X, H)\)-manifold.

To finish this Section, we give a description of the group \( G^2(n) \). We may suppose \( X = \mathbb{R}^n \). By definition

\[
G^2(n) = \{ j^2_0(f) \mid f : U \rightarrow \mathbb{R}^n \text{ is a diffeomorphism } f(0) = 0 \}
\]

and the group operation is defined by \( j^2_0(f) j^2_0(f') = j^2_0(f \circ f') \). Every 2-frame \( u = J^2_0(f) \) has a unique polynomial representation given by

\[
g(x) = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} u^j_i x^j + \sum_{j,k=1}^{n} u^i_{jk} x^j x^k \right) e_i
\]

\( \{e_1, \ldots, e_n\} \) being the canonical basis of \( \mathbb{R}^n \), \( x = \sum_{i=1}^{n} x^i e_i \) and \( u^i_j = u^i_{jk} \). The \( (u^j_i, u^i_{jk}) \) define a coordinate system in \( G^2(n) \). Therefore, we may identify every 2-jet \( u = J^2_0(f) \) with the couple \((A, \alpha)\), where \( A \) is the Jacobian matrix \( (u^j_i) \) and \( \alpha \) is the Hessian matrix, i.e. \( \alpha \) is a bilinear form on \( \mathbb{R}^n \times \mathbb{R}^n \) taking its values in \( \mathbb{R}^n \). Thus, the product expression has the following form

\[
(A, \alpha) (B, \beta) = (AB, \gamma)
\]

where \( AB \) denotes the matrices product and \( \gamma \) is defined by \( \gamma(x, y) = \alpha(Bx, By) + A\beta(x, y) \). The identity element is the couple \((I, 0)\) and the inverse of \((A, \alpha)\) has the following representation

\[
(A, \alpha)^{-1} = (A^{-1}, \beta),
\]

\( \beta \) being defined by \( \beta(x, y) = -A^{-1} \alpha(A^{-1} x, A^{-1} y) \).

### 4. The Riemannian case.

In this section we take as the model space a simply connected Riemannian homogeneous space \((X, k)\), \( k \) being an invariant metric on \( X \). We recall the well known
THEOREM. ([11]) The group $\text{Iso}(M)$ of isometries of a Riemannian manifold $M$ is a Lie transformation group with respect to the compact-open topology. For each $x \in M$, the isotropy subgroup $\text{Iso}_x(M)$ is compact. If $M$ is compact, $\text{Iso}(M)$ is also compact.

Therefore $X = G/H$, where $G = \text{Iso}(X)$ and $H$ is the isotropy group at the origin $o$ of $X$; moreover, the linear isotropy representation of $H$

$$\alpha : H \rightarrow \text{GL}(n, \mathbb{R})$$

is faithful, $H$ being compact and $\alpha(H) \subset O(n, \mathbb{R})$.

Let $M$ be a $(X, H)$-manifold; by Proposition 3.1 it follows that the bundle $L^1(M) = L(M)$ reduces to $H \subset O(n, \mathbb{R})$ and this gives a Riemannian structure on $M$. Since the reduction is integrable, the $(X, H)$-manifold $M$ is locally isometric to the model space $X$. In particular $M$ is locally homogeneous.

Let us consider now a Riemannian manifold $(M, g)$ locally isometric to the model space $(X, k)$. We recall the following result

THEOREM. Let $M$ and $M'$ be connected and simply connected, complete analytic Riemannian manifolds. Then every isometry between connected open subsets of $M$ and $M'$ can be uniquely extended to an isometry between $M$ and $M'$ (see [7]).

Since a Riemannian homogeneous space is analytic and complete, the previous Theorem implies that if $f : V \rightarrow X$, $f' : U' \rightarrow X$ are two local isometries onto their images, with $U \cap U' \neq \emptyset$, then the local isometry of $X$

$$(f' \circ f^{-1})|_{f(U \cap U')} : f(U \cap U') \rightarrow f'(U \cap U')$$

can be extended to a global isometry. Thus we have the following

PROPOSITION 4.1. If $(M, g)$ is locally isometric to a simply connected Riemannian homogeneous space $(X = G/H, k)$, then $M$ is a $(X, G)$-manifold.

We recall that if $(M, g)$ is a connected Riemannian manifold, then any isometry $f : M \rightarrow M$ is determined by the value which $f$ and its differential $df$ take in $p \in M$. Therefore in the case of
a Riemannian homogeneous model, Propositions 3.1 and 3.2 can be collected in the following

**Proposition 4.2.** Let \((X = G/H, k)\) be a homogeneous Riemannian manifold; \(M\) is a \((X, H)\)-manifold if and only if there exists an integrable \(H\)-reduction of the bundle of the linear frames on \(M\).

**References**


