τ-PARACOMPACTNESS AND REGULARITY (*)

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Sommario. - *Sia τ un numero cardinale infinito. Uno spazio $T_1 X$ si dice τ-paracompatto se ogni ricoprimento aperto $\mathcal{U}$ di X tale che* card$(\mathcal{U}) \leq \tau$ *ha un raffinamento aperto localmente finito. In questa nota si forniscono alcune condizioni equivalenti alla regolarità nell'ambito degli spazi τ-paracompatti. Come corollario si ottiene il seguente risultato di Aull: ogni spazio T_2 numerabilmente paracompatto e numerabile di 1° tipo è regolare.*

Summary. - *Let τ be an infinite cardinal number. A T_1-space X is called τ-paracompact if every open cover $\mathcal{U}$ of X such that* card$(\mathcal{U}) \leq \tau$ *has a locally finite open refinement. In this note we give some conditions which are equivalent to regularity in the realm of τ-paracompact spaces. As a corollary we obtain the following well-known result of Aull: every Hausdorff countably paracompact first countable space is regular.*

Let τ be an infinite cardinal number. A T_1-space X is τ-paracompact if any open cover $\mathcal{U}$ of X with cardinality $\leq \tau$ has a locally finite open refinement [3]. The $\aleph_0$-paracompact spaces are called countably paracompact. A space X is τ-pseudonormal if given any two closed sets C, F, one of which has cardinality $\leq \tau$, there exist disjoint open sets U, V such that $C \subset U$, $F \subset V$. The $\aleph_0$-pseudonormal spaces are called pseudonormal [4]. X is a C_τ-space if for each closed set F and for each $x \in X - F$ there exists a family $\mathcal{G}$ of open neighbourhoods of x such that card$(\mathcal{G}) \leq \tau$ and $\bigcap\{\bar{G} : G \in \mathcal{G}\} \subset X - F$. Obviously every regular space is a $C_{\aleph_0}$-space. Clearly the C_τ-property is hereditary, moreover we have the following

Proposition 1. *The C_τ-property is productive.*

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Proof. Let $\{X_i : i \in I\}$ be a family of C_τ-spaces and let $X = \prod\{X_i : i \in I\}$. Let F be a closed set of X and $x = \{x_i\}_{i\in I} \in X - F$. Take a basic open set $U = \prod\{U_i : i \in I\}$ of X such that $x \in U \subset X - F$. Let $I_0 = \{i \in I : U_i \neq X_i\}$, for each $i \in I_0$ take a family $\{G_\beta(x_i)\}_{\beta<\tau}$ of open sets of X_i such that $x_i \in \bigcap\{G_\beta(x_i) : \beta < \tau\}$ and $\bigcap\{\overline{G_\beta(x_i)} : \beta < \tau\} \subset U_i$. For each $\beta < \tau$ and $i \in I$ let $A_i(\beta) = G_\beta(x_i)$ if $i \in I_0$ and $A_i(\beta) = X_i$ otherwise. Now let $H_\beta = \Pi\{A_i(\beta) : i \in I\}$ for every $\beta < \tau$. Then $\{H_\beta\}_{\beta<\tau}$ is a family of open sets of X such that $x \in \bigcap\{H_\beta : \beta < \tau\}$ and $\bigcap\{\bar{H}_\beta : \beta < \tau\} \subset \Pi\{U_i : i \in I\} \subset X - F$. Hence X is a C_τ-space.

Remark 2. Let X be a Hausdorff space, let $\mathcal{V}$ be a collection of open neighbourhoods of x in X, let $x \in X$. Then $\mathcal{V}$ is a closed pseudobase for x if $\bigcap\{\bar{V} : V \in \mathcal{V}\} = \{x\}$. Now, for each $x \in X$, let $\psi_c(x, X) = \min\{\text{card}(\mathcal{V}) : \mathcal{V}$ is a closed pseudobase for $x\}$. The closed pseudocharacter of X is defined as follows: $\psi_c(X) = sup\{\psi_c(x, X) : x \in X\} + \omega$. If $\psi_c(X) = k$ then X is also said to be of H-pseudocharacter k [7]. A Hausdorff space with countable closed pseudocharacter is also called an E_1-space (see e.g. [5]). Obviously if X is a Hausdorff space such that $\psi_c(X) \leq \tau$ then it is a C_τ-space. In particular every E_1-space (and, a fortiori, every first countable Hausdorff space) is a $C_{\aleph_0}$-space.

Remark 3. If X is a Hausdorff space and $L(X) \leq \tau(L(X)$ is the Lindelöf degree of X, i.e. $L(X)$ is the smallest infinite cardinal k such that every open cover of X has a subcover of cardinality $\leq k$) then X is a C_τ-space. In fact let F be a closed set of X and $x \in X - F$. Since X is T_2 then for every $y \in F$ there exist open sets G_y, H_y of X such that $x \in G_y$, $y \in H_y$ and $G_y \bigcap H_y = \emptyset$. $\mathcal{U} = \{H_y\}_{y\in F} \cup \{X - F\}$ is an open cover of X so there is a set $\{y_\alpha\}_{\alpha<\tau} \subset F$ such that $X = \bigcup\{H_{y_\alpha} : \alpha < \tau\} \cup (X - F)$. Then $\mathcal{G} = \{G_{y_\alpha}\}_{\alpha<\tau}$ is a family of open neighbourhoods of x such that $\text{card}(\mathcal{G}) \leq \tau$ and $\bigcap\{\bar{G}_{y_\alpha} : \alpha < \tau\} \subset X - F$. Hence X is a C_τ-space. It is worth noting that if X is a first countable T_3-space such that $L(X) < b$ (b is the unbounding number, i.e. the smallest cardinality of an unbounded subset of $({}^\omega\omega, \leq^*)$) then X is pseudonormal [2].

Remark 4. A Hausdorff space X is functionally regular [6] if for each $x \in X$ and for each open neighbourhood V of x there exists a zero-set Z such that $x \in Z \subset V$. A functionally regular space need not be regular but it is a $C_{\aleph_0}$-space (if Z is a zero-set of a space X then there is a sequence $\{G_n\}_{n\in\omega}$ of open sets such that $Z = \bigcap\{G_n : n \in \omega\} = \bigcap\{\bar{G}_n : n \in \omega\}$).

THEOREM 5. *Let X be a τ-paracompact space. Then the following conditions are equivalent:*

i) *X is a C_τ-space.*

ii) *X is regular.*

iii) *X is τ-pseudonormal.*

Proof. ii) $\to$ i) and iii) $\to$ ii) are obvious. i) $\to$ ii) Let F be a closed set of X and $x \in X - F$. Since X is a C_τ-space there is a family $\mathcal{G} = \{G_\alpha\}_{\alpha<\tau}$ of open neighbourhoods of x such that $\bigcap\{\bar{G}_\alpha : \alpha < \tau\} \subset X - F$. So $\mathcal{U} = \{X - \bar{G}_\alpha : \alpha < \tau\} \cup \{X - F\}$ is an open cover of X with cardinality $\leq \tau$. X is τ-paracompact so there is a locally finite open cover $\mathcal{V} = \{V_\alpha : \alpha < \tau\} \cup \{V\}$ of X such that $V \subset X - F$ and $V_\alpha \subset X - \bar{G}_\alpha$ for each $\alpha < \tau$. Let $H = \bigcup\{V_\alpha : \alpha < \tau\}$, obviously $F \subset H$. Since $\mathcal{V}$ is locally finite we have $\bar{H} = \bigcup\{\bar{V}_\alpha : \alpha < \tau\}$, moreover $\bar{V}_\alpha \subset X - G_\alpha$ for every $\alpha < \tau$, so $x \notin \bar{H}$. Hence H and $G = X - \bar{H}$ are disjoint open sets such that $F \subset H$ and $x \in G$. ii) $\to$ iii) Let C and K be disjoint closed sets of X such that $\operatorname{card}(C) \leq \tau$. For each $c \in C$ there exist two disjoint open sets $U(c)$ and $V(c)$ of X such that $c \in U(c)$ and $K \subset V(c)$. $\mathcal{U} = \{U(c)\}_{c\in C} \cup \{X - C\}$ is an open cover of X with cardinality $\leq \tau$. X is τ-paracompact hence there is a locally finite open cover $\mathcal{W} = \{W(c)\}_{c\in C} \cup \{W\}$ of X such that $W \subset X - C$ and $W(c) \subset U(c)$ for every $c \in C$. Let $G = \cup\{W(c) : c \in C\}$, since $\mathcal{W}$ is locally finite we have $\bar{G} = \cup\{\overline{W(c)} : c \in C\}$, moreover $K \cap \overline{W(c)} = \emptyset$ for every $c \in C$, so $K \cap \bar{G} = \emptyset$. Therefore G and $H = X - \bar{G}$ are disjoint open sets such that $C \subset G$ and $K \subset H$.

COROLLARY 6 ([1]). *Every Hausdorff countably paracompact first countable space is regular.*

Remark 7. The fact that every T_3 countably paracompact space is pseudonormal was observed by Proctor [4].

A space X is called initially k-compact if every open cover $\mathcal{U}$ of X such that $card(\mathcal{U}) \leq k$ has a finite subcover. It is interesting to note that every initially k-compact space of H-pseudocharacter k is a regular space of character k [7].

Remark 8. A space X is a strongly C_τ-space if for each closed set B and for each open set G containing B there is a family $\mathcal{U}$ of open sets such that $\mathrm{card}(\mathcal{U}) \leq \tau$, $B \subset \bigcap\{U : U \in \mathcal{U}\}$ and $\bigcap\{\bar{U} : U \in \mathcal{U}\} \subset G$. Obviously every normal space is a strongly $C_{\aleph_0}$-space. Arguing as in theorem 5 one can show that a τ-paracompact space is normal iff it is a strongly C_τ-space.

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